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David Michael Roberts and Danny Stevenson

Abstract. In this paper we study the classifying theory of principal bundles in the parametrized setting, motivated by recent interest in higher gauge theory. Using simplicial techniques, we construct a product-preserving classifying space functor for groups in the category of spaces over a fixed space B. Additionally, we prove that the fiberwise geometric realization functor sends a large class of simplicial parametrized principal bundles to ordinary parametrized principal bundles. As an application we show that the fiberwise geometric realization of the universal simplicial principal bundle for a simplicial group G in the category of spaces over B gives rise to a parametrized principal bundle with structure group $|G|$.

Contents

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1. Introduction

The construction of a classifying space for a topological group is conveniently done using simplicial techniques, namely via the geometric realization of a certain simplicial space. Similarly a model for the universal principal bundle can be constructed as the geometric realization of a certain simplicial principal bundle. The utility of these constructions rests in part on the fact that the classifying space functor so obtained is product-preserving. The aim of this paper is to extend these constructions to the parametrized setting of [MaS06], in which the category of topological spaces is replaced by a suitable category of spaces over a fixed space B.

We recall the setting for parametrized homotopy theory from [MaS06]. Let $\mathscr{K}$ denote the category of k-spaces [V71] and let $\mathscr{U}$ denote the full subcategory of compactly generated spaces (i.e., weakly Hausdorff k-spaces). Let B be an object of $\mathscr{U}$ which will remain fixed throughout the paper. We will work in the category $\mathscr{K}_{/B}$ of spaces over B; an object of $\mathscr{K}_{/B}$ is thus a space X together with a map $X \to B$ (the *structure map*), while a morphism is a map of the underlying spaces which is compatible with the structure maps. There is a natural homotopy theory associated to the category $\mathscr{K}_{/B}$, this is described by the f-model structure of May and Sigurdsson, recalled in Theorem 5 below.

We will be interested in groups in the category $\mathscr{K}_{/B}$. In fact there are four notions of (internal) group that will play a role in this paper: groups in the category $\mathscr{K}$ (which we will refer to as *groups*), groups in the category $s\mathscr{K}$ of simplicial objects in $\mathscr{K}$ (which we will refer to as *simplicial groups*), groups in the category $\mathscr{K}_{/B}$ (which we will refer to as *parametrized groups*) and groups in the category $s\mathscr{K}_{/B}$ (which we will refer to as *simplicial parametrized groups*). In each case, it should be clear from the ambient category with respect to which we are working internal to, which of these labels for group objects applies. For each of the four notions of group, we have a corresponding notion of principal bundle. For example, we have parametrized principal bundles (Definition 12) and simplicial parametrized principal bundles (Definition 13).

The main result of this paper is the construction of a product-preserving classifying space functor for parametrized groups, together with a corresponding classification theorem for parametrized principal bundles. If G is a parametrized group, we will denote by BG the fiberwise geometric realization of the standard simplicial model (see [Ma75, Se68]; see also the description following Definition 18) for the classifying space of G. Our main result states that BG is a classifying space for parametrized principal G-bundles. One advantage of this result over previous constructions of classifying spaces in the parametrized setting (see for instance [CJ98]) is that the classifying space functor $B(-)$ so defined is product-preserving.

Theorem 1. *Let M be a paracompact space over B and let G be a well-sectioned fibrant parametrized group. Then there is a bijection*

$$H^1(M,G)_{\mathscr{K}_{/B}} \simeq [M,BG]_{\mathscr{K}_{/B}}.$$

Here if X and Y are spaces over B, we denote by $[X,Y]_{\mathscr{K}_{/B}}$ the set of fiberwise homotopy classes of maps from X to Y (see Section 2), and $H^1(M,G)_{\mathscr{K}_{/B}}$ denotes the set of isomorphism classes of parametrized principal G-bundles on M (see Section 4). We now explain the hypotheses in the theorem above. The assumption that G is well-sectioned (Definition 6) is the parametrized analog of the notion of well-pointed group, which is a standard hypothesis to impose in the analogous construction of a classifying space for a group in the topological setting. One new feature here is that we must also impose a fibrancy condition on our parametrized groups; namely we must require that they are fibrant objects in the model structure of Theorem 5, and we then refer to fibrant parametrized groups. This is necessary so that, among other things, the projection maps of principal bundles are fibrations.

There is a fiberwise geometric realization functor $|-|\colon s\mathscr{K}_{/B} \to \mathscr{K}_{/B}$ sending simplicial parametrized spaces to parametrized spaces. We shall see, in Lemma 8, that in analogy with the corresponding results for ordinary geometric realization, the fiberwise geometric realization of a simplicial parametrized group G is a parametrized group $|G|$. More generally, we shall prove the following technical theorem which asserts that fiberwise geometric realization sends a large class of simplicial parametrized principal bundles to ordinary parametrized principal bundles; this theorem is a key ingredient in the proof of Theorem 1.

Theorem 2. *Let G be a fibrant simplicial parametrized group and let M be a proper simplicial object in $\mathscr{K}_{/B}$. If P is a simplicial principal bundle over M with structure group G such that $P_n \to M_n$ is a numerable, parametrized principal G_n-bundle in $\mathscr{K}_{/B}$ for all $n \geq 0$, then the induced map*

$$|P| \to |M|$$

on fiberwise geometric realizations is the projection map for a locally trivial parametrized principal $|G|$-bundle $|P|(|M|,|G|)$ in $\mathscr{K}_{/B}$. Moreover, if the bundle $P_n \to M_n$ is trivial for all $n \geq 0$, then $|P| \to |M|$ is numerable.

Here by a fibrant *simplicial* parametrized group, we mean one for which the parametrized groups of n-simplices are fibrant for all $n \geq 0$. By a *proper* simplicial object in $\mathscr{K}_{/B}$ we just mean the obvious generalization of the classical notion of proper simplicial space [Ma72] to the parametrized setting (see Definition 20). A standard argument (see Appendix A) shows that every *good* simplicial object in $\mathscr{K}_{/B}$, i.e., one whose degeneracy maps are cofibrations in the f-model structure, is automatically proper.

A prime example of a simplicial principal bundle is the classical notion of *principal twisted cartesian product* internal to the category $\mathscr{K}_{/B}$

of parametrized spaces (see Section 5). In particular, if G is a simplicial parametrized group then we may consider the universal principal twisted cartesian product $WG \to \overline{W}G$. The following proposition gives a criterion on G which ensures that $\overline{W}G$ is proper and hence satisfies the hypotheses of Theorem 2.

Proposition 3. *Let G be a well-sectioned simplicial parametrized group. Then the following statements are true:*

(1) *G is a good simplicial group in $\mathscr{K}_{/B}$.*
(2) *$\overline{W}G$ is proper in $s\mathscr{K}_{/B}$.*
(3) *$|G|$ is a well-sectioned group in $\mathscr{K}_{/B}$.*

Using Proposition 3 we can easily establish that the hypotheses of Theorem 2 are met for the universal principal twisted cartesian product

$$WG \to \overline{W}G.$$

Hence we obtain the following result.

Proposition 4. *Let G be a well-sectioned fibrant simplicial parametrized group. Then the fiberwise geometric realization $|WG| \to |\overline{W}G|$ of the universal G-bundle $WG \to \overline{W}G$ is a numerable parametrized principal $|G|$ bundle. Moreover $|WG|$ is a fiberwise contractible group in $\mathscr{K}_{/B}$ containing $|G|$ as a closed subgroup.*

Our motivation comes from recent interest in higher principal bundles or gerbes [B06, JaL06, Mu96, R10, Sch11, St04, W11]. Recall that for a paracompact space M, there is a bijection between $H^3(M, \mathbb{Z})$ and the set of equivalence classes of S^1-*bundle gerbes* on M. An S^1-bundle gerbe on M is, roughly speaking, a principal bundle on M where the structure group S^1 is replaced by the simplicial topological group $\overline{W}S^1$. From another point of view, $H^3(M, \mathbb{Z})$ parametrizes the set of isomorphism classes of principal $K(\mathbb{Z}, 2)$ bundles on M. The process of passing from a simplicial principal bundle for $\overline{W}S^1$ to a principal $K(\mathbb{Z}, 2)$ bundle can be viewed as an instance of our Theorem 2 (recall the geometric realisation of $\overline{W}S^1$ is a $K(\mathbb{Z}, 2)$).

Our interest lies in a generalization of this, namely when the simplicial group $\overline{W}S^1$ is replaced by an arbitrary simplicial parametrized topological group G (subject to quite minor topological conditions) and we consider simplicial principal bundles with structure group G on M. The resulting set of equivalence classes is isomorphic to the *nonabelian cohomology set* $H^1(M, G)$. In this case the process of geometric realization produces an ordinary principal $|G|$ bundle from a simplicial principal G bundle and therefore gives rise to a map $H^1(M, G) \to H^1(M, |G|)$. In [St12b], based on the results of this paper, the second author proves that this map is an isomorphism provided that M is paracompact and G satisfies some mild topological conditions.

In outline then this paper is as follows. In Section 2 we review the homotopy theory of parametrized spaces from [MaS06]. In Section 3 we specialize our discussion to parametrized groups and we follow this in Section 4 with a discussion of principal bundles in the parametrized setting. In Section 5 we consider simplicial parametrized bundles and in particular the classical notion of principal twisted cartesian product. Section 6 contains the detailed statements of our main results and the proof of Theorem 1. Sections 7 and 8 contain the proofs of Theorems 2 and 3 respectively, while Appendix A is devoted to a discussion of the relation between good and proper simplicial objects.

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2. Parametrized spaces

In this section we recall some of the basic notions of parametrized homotopy theory from [MaS06]; in particular we recount some of the details of the f-model structure on the category $\mathscr{K}_{/B}$ of spaces over B.

Recall from [MaS06] that $\mathscr{K}_{/B}$ is a *topological bicomplete* category, in the sense that $\mathscr{K}_{/B}$ is enriched over $\mathscr{K}$, the underlying category is complete and cocomplete, and that it is tensored and cotensored over $\mathscr{K}$. For any space K and space X over B the tensor $X \otimes K$ is defined to be the space $X \times K$ in $\mathscr{K}$, considered as a space over B via the obvious map $X \times K \to B$. In the sequel, we will often denote the tensor $X \otimes K$ simply as $X \times K$. Similarly, the cotensor X^K is defined to be the space $\mathrm{Map}_B(K, X)$ given by the pullback square

$$\begin{tikzcd} \mathrm{Map}_B(K,X) \arrow[r] \arrow[d] & \mathrm{Map}(K,X) \arrow[d] \\ B \arrow[r] & \mathrm{Map}(K,B) \end{tikzcd}$$

in $\mathscr{K}$, where the map $B \to \mathrm{Map}(K, B)$ is the conjugate of the projection $B \times K \to B$. Recall also (see [MaS06]) that $\mathscr{K}_{/B}$ is cartesian closed under the fiberwise cartesian product $X \times_B Y$ and the fiberwise mapping space $\mathrm{Map}_B(X, Y)$ over B. The definition of the fiberwise mapping space $\mathrm{Map}_B(X, Y)$ is rather subtle and we will not give it here, we instead refer the reader to Definition 1.3.7 of [MaS06].

Since $\mathscr{K}_{/B}$ is a topological bicomplete category there is a natural notion of geometric realization for simplicial objects in $\mathscr{K}_{/B}$ — the notion of *fiberwise geometric realization*. If X is a simplicial object in $\mathscr{K}_{/B}$, i.e., a parametrized simplicial space, then the fiberwise geometric realization $|X|$ of X is defined by the usual coend formula:

$$|X| = \int^{[n]\in\Delta} X_n \times \Delta^n.$$

In other words, one regards X as a simplicial object in $\mathscr{K}$ and computes the ordinary geometric realization, and then one equips this with the induced map to B. In particular, $|X|$ is obtained as a quotient from the coproduct $\sqcup_{n\geq 0} X_n \times \Delta^n$.

It follows from the nonparametrized case that fiberwise geometric realization gives rise to a co-continuous functor $|\cdot|\colon s\mathscr{K}_{/B} \to \mathscr{K}_{/B}$. Since ordinary geometric realization commutes with finite limits, fiberwise geometric realization also commutes with finite limits in $\mathscr{K}_{/B}$, and moreover is compatible with the topological structures on $s\mathscr{K}_{/B}$ and $\mathscr{K}_{/B}$ in the sense that $|X \times K| = |X| \times K$ for any space K in $\mathscr{K}$. Note also that the fiberwise geometric realization of a level-wise closed inclusion is a closed inclusion.

In [MaS06] several model structures on $\mathscr{K}_{/B}$ are introduced. The model structure on $\mathscr{K}_{/B}$ that we will be interested in has its origins in the work [ScV02] of Schwänzl and Vogt. In [ScV02] (see also [C06] and [MaS06]) the authors consider a topological bicomplete category $\mathscr{C}$ and define three classes of morphisms: h-equivalences, h-fibrations and $\bar{h}$-cofibrations. A morphism $f\colon X \to Y$ in $\mathscr{C}$ is an h-equivalence if and only if it is a homotopy equivalence, defined in terms of the cylinder object $X \times I$ where I denotes the unit interval. A morphism $f\colon X \to Y$ is called an h-fibration if and only if it has the RLP (right lifting property) with respect to all morphisms of the form $Z \times \{0\} \to Z \times I$, while f is called an $\bar{h}$-cofibration if and only if $X \times I \cup_{X\times\{0\}} Y \times \{0\} \to Y \times I$ has the LLP (left lifting property) with respect to all h-fibrations in $\mathscr{C}$.

In [ScV02] the $\bar{h}$-cofibrations are called *strong cofibrations* and the following alternative characterization of them is given: a morphism $f\colon X \to Y$ is an $\bar{h}$-cofibration if and only if it has the LLP with respect to all h-fibrations which are also h-equivalences — i.e., the h-acyclic h-fibrations. When $\mathscr{C} = \mathscr{K}$, the class of strong cofibrations equals the class of closed cofibrations. Under suitable hypotheses on $\mathscr{C}$ (see Theorem 4.2 of [C06] and Theorem 4.2.12 of [MaS06]; see also [BR13]) these three classes of morphisms equip $\mathscr{C}$ with the structure of a proper, topological model category. This model structure is sometimes called the *h-model structure*.

If we specialize to the case when $\mathscr{C} = \mathscr{K}_{/B}$, it turns out (see [BR13, C06, MaS06]) that the required hypotheses are satisfied and the above notions of h-equivalence, h-fibration and $\bar{h}$-cofibration equip $\mathscr{K}_{/B}$ with the structure of a model category. This model structure is called the f-model structure (for fiberwise) and the weak equivalences, fibrations and cofibrations are labelled accordingly. A precise statement is the following.

Theorem 5 (May–Sigurdsson [MaS06], Theorem 5.2.8)**.** *$\mathscr{K}_{/B}$ has the structure of a proper, topological model category for which*

- *the weak equivalences are the f-equivalences,*
- *the fibrations are the f-fibrations,*
- *the cofibrations are the $\bar{f}$-cofibrations.*

Recall that a model category $\mathscr{C}$ is said to be *topological* if it is a $\mathscr{K}$-model category in the sense of Definition 4.2.18 of [H99], for the monoidal model structure on $\mathscr{K}$ given by the above h-model structure (observe that this coincides with the classical Strøm model structure [C06, MaS06, Str72] on $\mathscr{K}$).

To be completely explicit, we explain the labels on the three classes of maps in the above theorem. A map $g\colon X \to Y$ in $\mathscr{K}_{/B}$ is called an *f-equivalence* if it is a fiberwise homotopy equivalence. This needs the notion of homotopy over B, which is formulated in terms of $X \times I$. A map $g\colon X \to Y$ in $\mathscr{K}_{/B}$ is called an *f-fibration* if it has the fiberwise covering homotopy property, i.e., if it has the RLP property with respect to all maps of the form $i_0\colon Z \to Z \times I$ for all $Z \in \mathscr{K}_{/B}$. A map $g\colon X \to Y$ in $\mathscr{K}_{/B}$ is called an *$\bar{f}$-cofibration*, or a *strong cofibration* if it has the LLP property with respect to all f-acyclic f-fibrations. There is also the notion of an *f-cofibration*: this is a map $g\colon X \to Y$ which satisfies the LLP property with respect to all maps of the form $p_0\colon \mathrm{Map}_B(I, Z) \to Z$ for some $Z \in \mathscr{K}_{/B}$. Every $\bar{f}$-cofibration $g\colon X \to Y$ in $\mathscr{K}_{/B}$ is an f-cofibration. The converse is not true in general. However May and Sigurdsson prove (see Theorems 4.4.4 and 5.2.8 of [MaS06]) that if $g\colon X \to Y$ is a *closed* f-cofibration then g is an $\bar{f}$-cofibration.

Moreover, in analogy with the standard characterization of closed Hurewicz cofibrations in terms of NDR pairs, May and Sigurdsson give a criterion (see Lemma 5.2.4 of [MaS06]) which detects when a closed inclusion $i\colon A \to X$ in $\mathscr{K}_{/B}$ is an $\bar{f}$-cofibration. Such an inclusion $i\colon A \to X$ in $\mathscr{K}_{/B}$ is an $\bar{f}$-cofibration if and only if (X, A) is a *fiberwise NDR pair* in the sense that there is a map $u\colon X \to I$ for which $A = u^{-1}(0)$ and a homotopy $h\colon X \times I \to X$ over B such that $h_0 = id$, $h_t|_A = id_A$ for all $0 \leq t \leq 1$ and $h_1(x) \in A$ whenever $u(x) < 1$.

3. Parametrized groups

In this section we study the four classes of groups described in the introduction: groups, parametrized groups, simplicial groups and simplicial parametrized groups, corresponding to group objects in $\mathscr{K}$, $\mathscr{K}_{/B}$, $s\mathscr{K}$ and $s\mathscr{K}_{/B}$ respectively.

As a group object in $\mathscr{K}_{/B}$, a parametrized group has a natural structure as an *ex-space*, i.e., a space X over B equipped with a section of the structure map $X \to B$ (see Section 1.3 of [MaS06] for more details). For such a parametrized group G, the structure as an ex-space arises from the canonical section of the structure map of G given by the identity section. In the context of parametrized spaces, ex-spaces are the analog of pointed spaces in the nonparametrized setting. The analog of a well-pointed, or nondegenerately based space, is the notion of a well-sectioned ex-space, i.e., one for which the distinguished section of the structure map is an $\bar{f}$-cofibration. In particular the ex-space analog of a well-pointed group is the notion of a well-sectioned parametrized group in the sense of the following definition.

Definition 6. Let G be a parametrized group. We say that G is *well-sectioned* if the identity section $1_G\colon B \to G$ is an $\bar{f}$-cofibration. We say that a simplicial parametrized group is *well-sectioned* if the parametrized group of n-simplices is well-sectioned for every $n \geq 0$.

We shall also need to impose a fibrancy condition on parametrized groups. Accordingly, we make the following definition.

Definition 7. Let G be a parametrized group. We say that G is *fibrant*, if G is f-fibrant considered as an object of $\mathscr{K}_{/B}$. We say that a simplicial parametrized group is *fibrant* if it is level-wise fibrant in the sense that G_n is fibrant for all $n \geq 0$.

We shall see that in order to obtain a notion of parametrized principal G-bundle (Definition 12 below) that is well-behaved homotopically in the sense that is a f-fibration, then we need to impose the condition that G is fibrant (see Theorem 14 below).

Recall from Section 2 above, that the fiberwise geometric realization functor $|\cdot|\colon s\mathscr{K}_{/B} \to \mathscr{K}_{/B}$ preserves products. Hence we have the following obvious Lemma.

Lemma 8. *The fiberwise geometric realization functor $|-|\colon s\mathscr{K}_{/B} \to \mathscr{K}_{/B}$ sends group objects in $s\mathscr{K}_{/B}$ to group objects in $\mathscr{K}_{/B}$, in other words, if G is a simplicial parametrized group then $|G|$ is a parametrized group.*

If G is a parametrized group then there is a natural notion of a G-space over B and a G-map between G-spaces over B. A G-space over B is a space X over B equipped with an action of G, i.e., a map $X \times_B G \to X$ of spaces over B making the usual diagrams commute, and a G-map from X to Y is a map $X \to Y$ in $\mathscr{K}_{/B}$ compatible with the respective G-actions. We write $G\mathscr{K}_{/B}$ for the category consisting of G-equivariant objects and G-maps between them. We have the following lemma.

Lemma 9. *The category $G\mathscr{K}_{/B}$ is a topological bicomplete category.*

Proof. To construct limits in $G\mathscr{K}_{/B}$ one first constructs the corresponding limit in $\mathscr{K}_{/B}$ and then equips it with the induced G-action. To construct colimits in $G\mathscr{K}_{/B}$ one first constructs the colimit in $\mathscr{K}_{/B}$ and then one observes that, since $G\times_B(-)$ is a left adjoint and therefore preserves colimits, the colimit in $\mathscr{K}_{/B}$ comes equipped with a natural G-action. The category $G\mathscr{K}_{/B}$ is naturally enriched over $\mathscr{K}$; if X and Y are objects of $G\mathscr{K}_{/B}$ then the space of morphisms $G\mathscr{K}_{/B}(X,Y)$ is given by the equalizer diagram

$$G\mathscr{K}_{/B}(X,Y) \to \mathscr{K}_{/B}(X,Y) \rightrightarrows \mathscr{K}_{/B}(X \times_B G, Y)$$

in $\mathscr{K}$, where the last two maps are induced by the actions of G on X and Y respectively, i.e., the maps which send a map $f\colon X \to Y$ in $\mathscr{K}_{/B}$ to the compositions

$$X \times_B G \xrightarrow{f\times_B 1_G} Y \times_B G \to Y \quad \text{and} \quad X \times_B G \to X \xrightarrow{f} Y.$$

If $X \in G\mathscr{K}_{/B}$ and $K \in \mathscr{K}$ then the tensor $X \otimes K$ is the usual one in $\mathscr{K}_{/B}$ equipped with the G-action where G acts trivially on the K factor. The cotensor in $G\mathscr{K}_{/B}$ is the usual cotensor in $\mathscr{K}_{/B}$ equipped with an action of G described as follows. The commutative diagram

$$\begin{tikzcd} G \arrow[r] \arrow[d] & \mathrm{Map}(K,G) \arrow[d] \\ B \arrow[r] & \mathrm{Map}(K,B), \end{tikzcd}$$

where the top horizontal map is the adjoint of the projection $G \times K \to G$ in $\mathscr{K}$, shows that there is a natural morphism $G \to \mathrm{Map}_B(K,G)$ in $\mathscr{K}_{/B}$. The action of G on $\mathrm{Map}_B(K,X)$ is given by the following composite:

$$\mathrm{Map}_B(K,X) \times_B G \to \mathrm{Map}_B(K,X) \times_B \mathrm{Map}_B(K,G) \to \mathrm{Map}_B(K,X),$$

where the second map is induced by the action of G on X via the identification

$$\mathrm{Map}_B(K,X) \times_B \mathrm{Map}_B(K,G) \cong \mathrm{Map}_B(K, X \times_B G).$$

One can check that this gives a G-action as claimed. To check that we have required adjunction homeomorphisms, observe that we have the following isomorphisms of diagrams in $\mathscr{K}$:

$$\begin{tikzcd} \mathscr{K}_{/B}(X \times K, Y) \arrow[r, phantom, "\cong"] \arrow[d, shift left] \arrow[d, shift right] & \mathscr{K}_{/B}(X, Y^K) \arrow[r, phantom, "\cong"] \arrow[d, shift left] \arrow[d, shift right] & \mathscr{K}(X, \mathscr{K}_{/B}(X,Y)) \arrow[d, shift left] \arrow[d, shift right] \\ \mathscr{K}_{/B}((X \times_B G) \times K, Y) \arrow[r, phantom, "\cong"] & \mathscr{K}_{/B}(X \times_B G, Y^K) \arrow[r, phantom, "\cong"] & \mathscr{K}(K, \mathscr{K}_{/B}(X \times_B G, Y)) \end{tikzcd}$$

where we have used the fact that we have an isomorphism

$$(X \times K) \times_B G \cong (X \times_B G) \times K.$$

Therefore, on forming equalizers we get the required natural isomorphisms

$$G\mathscr{K}_{/B}(X \times K, Y) \cong G\mathscr{K}_{/B}(X, Y^K) \cong \mathscr{K}(K, G\mathscr{K}_{/B}(X,Y)),$$

using the fact that $\mathscr{K}(K, -)$ preserves equalizers. □

Let G continue to denote a group object in $\mathscr{K}_{/B}$. In $G\mathscr{K}_{/B}$ there are natural notions of f-equivalence, f-fibration, f-cofibration and $\bar{f}$-cofibration. Thus a map $g\colon X \to Y$ in $G\mathscr{K}_{/B}$ is an f-cofibration if it has the LLP in $G\mathscr{K}_{/B}$ with respect to G-maps of the form $p_0\colon \mathrm{Map}_B(I,Z) \to Z$ for all Z in $G\mathscr{K}_{/B}$. Similarly, we say that a map $g\colon X \to Y$ in $G\mathscr{K}_{/B}$ is an f-equivalence if it is a fiberwise G-homotopy equivalence. A map $g\colon X \to Y$ in $\mathscr{K}_{/B}$ is an f-fibration if it has the RLP in $G\mathscr{K}_{/B}$ with respect to G-maps of the form $i_0\colon Z \to Z \times I$ for all Z in $G\mathscr{K}_{/B}$. A map $g\colon X \to Y$ in $G\mathscr{K}_{/B}$ is an $\bar{f}$-cofibration if it has the LLP in $G\mathscr{K}_{/B}$ with respect to all f-acyclic f-fibrations in $G\mathscr{K}_{/B}$.

Just as above, there is a criterion to detect when an inclusion $i\colon A \to X$ in $G\mathscr{K}_{/B}$ is an $\bar{f}$-cofibration in $G\mathscr{K}_{/B}$. We have the following result which will play a key role in the proof of Theorem 2.

Lemma 10. *An inclusion $i\colon A \to X$ in $G\mathscr{K}_{/B}$ is an $\bar{f}$-cofibration if and only if $i(A)$ is closed in X and there is a representation of (X, A) as a G-fiberwise NDR pair.*

Here by a representation of (X, A) as a G-fiberwise NDR pair we understand, in analogy with [St68], that there is a pair (u, h) of maps with $u\colon X \to I$ and $h\colon X \times I \to X$ which represent (X, A) as a fiberwise NDR pair and which satisfy $u(xg) = u(x)$ for all $x \in X$ and $g \in G$, and $h(xg, t) = h(x, t)g$ for all $(x, t) \in X \times I$ and $g \in G$.

Proof. We will explain how to adapt Steps 1–3 in the proof of Theorem 4.4.4 of [MaS06] to our setting. Step 3 adapts in a straightforward way to show that $i(A)$ is closed in X: factor the inclusion $i\colon A \to X$ as $A \to E \to X$ where $E = A \times I \cup X \times (0, 1]$ and where $i_0\colon A \to E$ is given by $i_0(a) = (a, 0)$. Analogous to the corresponding statement in [MaS06], the projection $\pi\colon E \to X$ is an f-acyclic f-fibration in $G\mathscr{K}_{/B}$. Therefore there exists a map $\lambda\colon X \to E$ extending i, i.e., $\lambda \circ i = i_0$. Let $\psi\colon E \to I$ denote the projection onto the second factor and note that $\psi^{-1}(0) = i_0(A)$, so that $i_0(A)$ is closed in E. Therefore $\lambda^{-1}(i_0(A)) = i(A)$ is closed in X (since λ is injective). Standard arguments now show that (X, A) has a representation as a G-fiberwise NDR pair.

Next we explain how Steps 1 and 2 can be adapted to show that if (X, A) has a representation as a G-fiberwise NDR pair then $i\colon A \to X$ is an $\bar{f}$-cofibration. The usual argument shows that $X \times \{0\} \cup A \times I$ is a retract of $X \times I$ in $G\mathscr{K}_{/B}$. Hence $i\colon A \to X$ is a closed f-cofibration in $G\mathscr{K}_{/B}$ and so $Mi \to X \times I$ is the inclusion of a strong deformation retraction (see Lemma 4.2.5 of [MaS06]), where Mi is the mapping cylinder of i. The map u in a representation (u, h) of (X, A) as a G-fiberwise NDR pair can be used to show that there exists a map $\psi\colon X \times I \to I$ such that $\psi^{-1}(0) = Mi$. The analogue of Theorem 3 of [Str66] for the category $G\mathscr{K}_{/B}$ then shows that $Mi \to X \times I$ has the LLP with respect to all f-fibrations and hence $i\colon A \to X$ is an $\bar{f}$-cofibration. □

Finally, let us note ([ScV02] Lemma 2.6) that since $\bar{f}$-cofibrations in $G\mathscr{K}_{/B}$ are defined by a left lifting property, the following is true.

Lemma 11. *If $X_0 \to X_1 \to \cdots \to X_n \to \cdots$ is a sequence of $\bar{f}$-cofibrations in $G\mathscr{K}_{/B}$, then $X_n \to X$ is an $\bar{f}$-cofibration in $G\mathscr{K}_{/B}$ for all $n \geq 0$, where $X = \operatorname{colim} X_n$.*

4. Parametrized principal bundles

In this section we study the notion of a principal bundle in $\mathscr{K}_{/B}$ for a parametrized group G, in other words the notion of a parametrized principal bundle. In particular we study some homotopy-theoretic properties of parametrized principal bundles for the homotopy theory of Theorem 5. The notion of parametrized principal bundle was introduced in [CJ98], we re-phrase it in the following way.

Definition 12 ([CJ98])**.** Let G be a parametrized group. A *parametrized principal G bundle* in $\mathscr{K}_{/B}$ consists of a G-space P in $\mathscr{K}_{/B}$ together with a map $\pi\colon P \to M$ such that:

(i) π admits local sections (in $\mathscr{K}_{/B}$).

(ii) The square

$$\begin{tikzcd} P \times_B G \arrow[r] \arrow[d, "p_1"'] & P \arrow[d, "\pi"] \\ P \arrow[r, "\pi"'] & M \end{tikzcd} \tag{1}$$

is a pullback in $\mathscr{K}_{/B}$, where the horizontal map $P \times_B G \to P$ is the action of G on P and the map p_1 is projection onto the first factor.

The condition (ii) implies that the action of G on P is principal with M as its space of orbits and that the action of G preserves the fibers of π. The condition that $\pi\colon P \to M$ admits local sections means that for every point of m of M there is an open neighborhood $U_m \subset M$ of m together with a fiberwise map $s\colon U_m \to P$ which is a section of π.

We use the standard terminology: P is the *total space*, M is the *base space* and G is the *structure group* of a parametrized principal bundle, which we shall sometimes denote by $P(M,G)$. A *morphism* $P(M,G) \to P'(M',G')$ of parametrized principal bundles consists of a triple of maps $f\colon M \to M'$, $\bar{f}\colon P \to P'$ and $\alpha\colon G \to G'$ in $\mathscr{K}_{/B}$, where α is a homomorphism of parametrized groups and $\bar{f}$ is equivariant for α. Parametrized principal bundles, together with the morphisms between them, form the category of parametrized principal bundles.

We make the following definition.

Definition 13. A *simplicial parametrized principal bundle* is a simplicial object in the category of parametrized principal bundles.

Thus if $P(M,G)$ is a simplicial parametrized principal bundle with projection map $\pi\colon P \to M$, then each map $\pi_n\colon P_n \to M_n$ is a parametrized principal G_n-bundle and the face and degeneracy maps are morphisms of parametrized principal bundles.

It is worth reformulating this definition in a slightly different way. A simplicial parametrized principal bundle consists of a simplicial parametrized group G, a simplicial parametrized space P equipped with an action of G

in $s\mathscr{K}_{/B}$ and a map $\pi\colon P \to M$ which satisfies the analogs of (i) and (ii) in Definition 12 above. Thus the diagram analogous to (1) is a pullback in $s\mathscr{K}_{/B}$ and the map π admits local sections level-wise in the sense that $\pi_n\colon P_n \to M_n$ admits local sections for all $n \geq 0$. Note that we do not require any compatibility between these local sections and the face and degeneracy maps for the simplicial spaces.

Recall from Section 2 that the fiberwise geometric realization functor $|\cdot|\colon s\mathscr{K}_{/B} \to \mathscr{K}_{/B}$ preserves finite limits. It follows that if $P(M,G)$ is a simplicial parametrized principal bundle then there is an induced action of $|G|$ (see Lemma 8) on $|P|$ in $\mathscr{K}_{/B}$ such that the diagram

$$\begin{tikzcd} {|P| \times_B |G|} \arrow[r] \arrow[d, "p_1"'] & {|P|} \arrow[d, "|\pi|"] \\ {|P|} \arrow[r, "|\pi|"'] & {|M|} \end{tikzcd}$$

is a pullback in $\mathscr{K}_{/B}$. In Theorem 2 we will give conditions on M and G which ensure that the map $|\pi|\colon |P| \to |M|$ has local sections and hence that $|P|(|M|,|G|)$ is a parametrized principal bundle.

If we consider morphisms of parametrized principal bundles with fixed structure group G and fixed base space M (both parametrized, of course), then, just as for ordinary principal bundles, every such morphism is an isomorphism. We denote the set of isomorphism classes of parametrized principal G-bundles on M by $H^1(M,G)_{\mathscr{K}_{/B}}$.

Every parametrized principal G-bundle $\pi\colon P \to M$ is a parametrized fiber bundle in the sense that each point of M has an open neighborhood U such that the restriction of P to U is isomorphic to the trivial parametrized G-bundle $U \times_B G$. If G is fibrant, such a trivial parametrized fiber bundle is an f-fibration in the sense of Theorem 5. When B is a point it is a well known theorem that every numerable fiber bundle $E \to M$ is a Hurewicz fibration. There is an obvious extension of this notion to the notion of a numerable parametrized fiber bundle: a parametrized fiber bundle is *numerable* if it is fiberwise locally trivial relative to a numerable open cover of the base space. We have the following theorem from [CJ98].

Theorem 14 ([CJ98])**.** *Let $p\colon E \to M$ be a map in $\mathscr{K}_{/B}$. Suppose that $p^{-1}V_i \to V_i$ is an f-fibration for each open set V_i in a numerable covering $(V_i)_{i\in I}$ of M. Then p is an f-fibration. In particular, if G is a fibrant parametrized group, then any parametrized principal G-bundle $\pi\colon P \to M$ in $\mathscr{K}_{/B}$ over a paracompact base space M, or more generally any numerable parametrized principal G bundle in $\mathscr{K}_{/B}$, is an f-fibration.*

This theorem has the following important corollary. In the parametrized context, principal G-bundles P_0 and P_1 on M are said to be *fiberwise concordant* if there exists a parametrized principal G-bundle P on $M \times I$ together with fiberwise isomorphisms $P_0 \cong P|_{M\times\{0\}}$ and $P_1 \cong P|_{M\times\{1\}}$. The fiberwise concordance relation is clearly an equivalence relation. When B is a point it is well known that there is a bijection between the set of isomorphism classes of numerable principal G bundles on M and concordance classes of principal G bundles on M. From Theorem 14, we see that in the parametrized setting there is an analogous bijection.

Corollary 15. *Let M be a paracompact space in $\mathscr{K}_{/B}$ and let G be a fibrant parametrized group. Then there is a bijection between $H^1(M,G)_{\mathscr{K}_{/B}}$ and the set of fiberwise concordance classes of parametrized principal G-bundles on M.*

Proof. To prove that there is such a bijection one needs to know that fiberwise concordant bundles are isomorphic. For this, it is enough to prove that there is an isomorphism $P \cong P_0 \times I$, when P is a parametrized principal G-bundle on $M \times I$, and P_0 denotes the restriction to $M \times \{0\}$. Consider the bundle $P \times_G (P_0 \times I)$ on $M \times I$. There is a section of this bundle over the closed subspace $M \times \{0\}$ of $M \times I$. We want to know that this section extends to a section defined over $M \times I$. Since $P \times_G (P_0 \times I)$ is a fiberwise locally trivial bundle on $M \times I$, it is an f-fibration. Therefore the required extension of the section exists, since the inclusion $M \times \{0\} \subset M \times I$ is an f-acyclic $\bar{f}$-cofibration. It follows that the set of fiberwise concordance classes of G-bundles on M is isomorphic to $H^1(M,G)_{\mathscr{K}_{/B}}$. $\square$

We shall also need the following result, related to Theorem 12 of [Str68].

Proposition 16. *Let $\pi\colon P \to M$ be a numerable parametrized principal G bundle for a fibrant parametrized group G and suppose that $A \subset M$ is a closed inclusion which is an $\bar{f}$-cofibration in $\mathscr{K}_{/B}$. Then the closed inclusion $P|_A \subset P$ is an $\bar{f}$-cofibration in $G\mathscr{K}_{/B}$.*

Proof. The proof of the analogous result in [Str68] can be adapted to this setting as follows. Choose a representation (u,h) of (M,A) as a fiberwise NDR pair in $\mathscr{K}_{/B}$. Next observe that in the diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d, "i_0"'] & P \arrow[d, "\pi"] \\ P \times I \arrow[r, "h(\pi\times 1)"'] \arrow[ur, dotted, "\bar{h}"] & M \end{tikzcd}$$

the indicated lifting $\bar{h}$ can be found, and moreover can be chosen to be G-equivariant, in light of the proof of Corollary 15 above. To finish the proof, we need to show that we can choose $\bar{h}$ so that $\bar{h}(x,t) = x$ for any $x \in P|_A$. Consider the associated bundle $\mathrm{Aut}_0(P \times I) = (P \times I) \times_G G$ on $M \times I$, where the action of G on itself is conjugation. Note that sections of $\mathrm{Aut}_0(P \times I)$ are bundle automorphisms of $P \times I$ covering the identity on $M \times I$. Since $\pi\bar{h} = h(\pi \times 1)$ and $\bar{h}$ is equivariant, it follows that $\bar{h}$

restricts to a section of $\mathrm{Aut}_0(P \times I)$ over $A \times I \subset M \times I$. Similarly the restriction of $\bar{h}$ to $P \times \{0\}$ defines a section of $\mathrm{Aut}(P \times I)$ over $M \times \{0\}$. Since $\mathrm{Aut}_0(P \times I) \to M \times I$ is a locally trivial, numerable, parametrized bundle, and $(A \times I) \cup (M \times \{0\}) \subset M \times I$ is a closed $\bar{f}$-cofibration, it follows that we can find the indicated lifting in the diagram

$$\begin{tikzcd}
(A \times I) \cup (M \times \{0\}) \arrow[r] \arrow[d] & \mathrm{Aut}_0(P \times I) \arrow[d] \\
M \times I \arrow[r, "1"'] \arrow[ur, dotted, "\bar{k}"] & M \times I.
\end{tikzcd}$$

Now define $\tilde{h} = \bar{h}\bar{k}^{-1}$. Then $\tilde{h}\colon P \times I \to P$ is G-equivariant and satisfies $\pi\tilde{h} = h(\pi \times 1)$. If we set $\tilde{u} = u\pi$ then it is easily checked that $(\tilde{u}, \tilde{h})$ is a representation of $(P, P|_A)$ as a G-equivariant NDR pair. □

5. Simplicial principal bundles and twisted cartesian products

In this section we recall the notion of principal twisted Cartesian product defined internally to a category $\mathscr{C}$ with finite limits, and we recall the definition of the universal simplicial G-bundle $WG \to \overline{W}G$ associated to a group object G in $\mathscr{C}$. Recall the following classical definition (see for instance [Ma67]).

Definition 17. Let G be a group in *sSet*. A *principal twisted cartesian product* with structure group G in *sSet* consists of a G-simplicial set P and a map $\pi\colon P \to M$ such that π has a pseudo-cross section and the diagram analogous to (1) above is a pullback.

By a pseudo-cross section (Definition 18.5 of [Ma67]) we mean a collection of maps $\sigma_n\colon M_n \to P_n$ for all $n \geq 0$ such that $\sigma_i s_i = s_i \sigma_i$ for all $0 \leq i \leq n+1$, $n \geq 0$, and $\sigma_i d_i = d_i \sigma_i$ for all $0 < i \leq n$ and $n \geq 0$. We note that a pseudo-cross section can be conveniently reformulated in terms of Illusie's décalage functor (see [Du75] and also the discussion below) and that this leads to a simple description of the classifying theory of principal twisted cartesian products (see [St12a]).

It is clear from the preceding discussion that we may replace the category *Set* of sets with any category $\mathscr{C}$ with finite limits and obtain the notion of principal twisted cartesian product *internal to the category* $s\mathscr{C}$ of simplicial objects in $\mathscr{C}$. Of particular interest for us will be the case where $\mathscr{C} = \mathscr{K}_{/B}$; note that principal twisted cartesian products in this case are examples of simplicial parametrized principal bundles (Definition 13).

The data of a principal twisted cartesian product may be conveniently reformulated in terms of *twisting functions*, as we now recall. A family of maps $t_n\colon M_n \to G_{n-1}$ defined for $n \geq 1$ is called a *twisting function* if the identities (T) on page 71 of [Ma67] are satisfied, when interpreted internally in the obvious fashion. Every principal twisted cartesian product determines a unique twisting function, and conversely a twisting function determines a principal twisted cartesian product

$$M \times_t G$$

in which the object of n-simplices is the product $(M \times_t G)_n = M_n \times G_n$, and where the face and degeneracy maps are defined as in Definition 18.3 of [Ma67]. In particular the description in terms of twisting functions explains the origin of the terminology 'twisted cartesian product'.

If G is a simplicial group internal to $\mathscr{C}$ (for instance a simplicial group or a simplicial parametrized group), then the universal G bundle $WG \to \overline{W}G$ has a convenient description via twisting functions.

Definition 18. Let $\mathscr{C}$ be a category with finite limits and let G be a group in $s\mathscr{C}$. The *classifying complex* $\overline{W}G$ is defined to be the simplicial object of $\mathscr{C}$ with $(\overline{W}G)_0 = 1$, the terminal object of $\mathscr{C}$, and

$$(\overline{W}G)_n = G_{n-1} \times \cdots \times G_0$$

for $n \geq 1$, with face and degeneracy maps defined by the following formulae:

$$\begin{aligned} d_i(g_{n-1}, \ldots, g_0) &= (d_i(g_{n-1}), \ldots, (d_i(g_i))g_{i-1}, \ldots, g_{i-2}, \ldots, g_0) \\ s_i(g_{n-1}, \ldots, g_0) &= (s_i(g_{n-1}), \ldots, s_i(g_i), 1, g_{i-1}, \ldots, g_0), \end{aligned}$$

if $(g_{n-1}, \ldots, g_0) \in (\overline{W}G)_n$.

When $\mathscr{C} = Set$ is the category of sets and so G is an ordinary simplicial group, this is the traditional classifying complex construction introduced in [K58]. In the next section we shall make a more careful study of this construction in the case when $\mathscr{C} = \mathscr{K}_{/B}$. Note that when G is group in $\mathscr{C}$ and we abusively denote by G the constant simplicial group in $\mathscr{C}$ with all face and degeneracy maps equal to the identity, then $\overline{W}G$ reduces to the familiar description in terms of the nerve of the one-object groupoid G in $\mathscr{C}$. Therefore, in this case we have the identification

$$(\overline{W}G)_n = G \times \cdots \times G \quad (n \text{ factors}) \tag{2}$$

with face and degeneracy maps defined by the usual formulae:

$$d_i(g_0, \ldots, g_{n-1}) = \begin{cases} (g_1, \ldots, g_{n-1}) & \text{if } i = 0, \\ (g_0, \ldots, g_{i-1}g_i, \ldots, g_{n-1}) & \text{if } 1 \leq i \leq n-1, \\ (g_0, \ldots, g_{n-2}) & \text{if } i = n. \end{cases} \tag{3}$$

$$s_i(g_0, \ldots, g_{n-1}) = (g_0, \ldots, g_{i-1}, 1, g_i, \ldots, g_n). \tag{4}$$

Alternatively, we may think of the one-object groupoid G as the action groupoid $1//G$ in $\mathscr{C}$, associated to the trivial action of G on the terminal object 1 of $\mathscr{C}$. Although it will not play an important role in this paper, we mention in passing a very useful conceptual approach to the classifying complex construction due to Duskin.

For every $n \geq 0$, we may form the simplicial object $N(1//G_n)$ which is the nerve of the action groupoid $1//G_n$ associated to the group G_n; in this way we obtain a bisimplicial object $N(1//G)$ in $\mathscr{C}$. In the paper [AM69], Artin and Mazur introduced the construction of the *total simplicial set* $T(X)$ associated to a bisimplicial set X. This construction makes sense in any category $\mathscr{C}$ with finite limits and defines a functor

$$T\colon ss\mathscr{C} \to s\mathscr{C},$$

called the total simplicial object functor. It is not hard to show, using explicit formulas for face and degeneracy maps, that there is an isomorphism

$$\overline{W}G = T(N(1//G))$$

of simplicial objects in $\mathscr{C}$. Besides the conceptual understanding that this observation brings to the classifying complex construction, it also gives a useful perspective on the construction of the universal principal twisted cartesian product over $\overline{W}G$.

The right action of G on itself defines an action groupoid $G//G$ in $\mathscr{C}$; there is a natural functor $G//G \to 1//G$ and hence a simplicial map

$$T(N(G//G)) \to T(N(1//G)) \tag{5}$$

on taking nerves and applying the total simplicial object functor. It is straightforward to see that there is a canonical action of the simplicial group G on $T(N(G//G))$ such that the diagram analogous to (1) above is a pullback. With a little more work, exploiting the close relationship between the functor T and Illusie's décalage functor, one may show that the map (5) has a pseudo-cross section, and hence has a natural structure as a principal twisted cartesian product. It is not hard to show that the principal twisted cartesian product (5) is equal to the universal twisted cartesian product (see pages 88–89 of [Ma67])

$$WG \to \overline{W}G$$

defined in terms of the canonical twisting function t on $\overline{W}G$ defined by

$$t_n\colon (\overline{W}G)_n \to G_{n-1}, \quad t_n(g_{n-1}, \ldots, g_0) = g_{n-1}.$$

We summarize the preceding discussion in the following lemma.

Lemma 19. *Let $\mathscr{C}$ be a category with finite limits and let G be a group in $s\mathscr{C}$. Then there is a canonical principal twisted cartesian product $WG \to \overline{W}G$ with structure group G. Moreover WG has a natural structure as a group in $s\mathscr{C}$ containing G as a subgroup.*

The only statement in Lemma 19 that has not been discussed above is the statement regarding the group structure on WG; this is a simple consequence of the description of WG in terms of the total simplicial object functor. We refer to [R13] for further discussion of this.

Finally, we note that there is another useful perspective on the universal principal twisted cartesian product $\pi\colon WG \to \overline{W}G$; the map π is equal to the canonical map $\mathrm{Dec}_0\overline{W}G \to \overline{W}G$, where $\mathrm{Dec}_0\colon s\mathscr{C} \to s\mathscr{C}$ is the décalage functor. In this description the pseudo-cross section appears as a certain monadic structure on the functor Dec_0.

Recall (see for example [Du75, I72, St12a]), that Dec_0 is the functor which shifts degrees up by one so that if X is a simplicial object in $\mathscr{C}$ then $\mathrm{Dec}_0(X)_n = X_{n+1}$ with the first face and degeneracy map at each level forgotten or 'stripped away'. In other words Dec_0 is the functor induced by restriction along the functor $\sigma_0\colon \Delta \to \Delta$, where σ_0 is defined by $\sigma_0([n]) = \sigma([0],[n])$, where $\sigma\colon \Delta \times \Delta \to \Delta$ is ordinal sum, i.e.,

$$\sigma([m],[n]) = [m+n+1].$$

Observe that the first face map at every level defines a simplicial map $d_{\mathrm{first}}\colon \mathrm{Dec}_0 X \to X$ for any simplicial object X in $\mathscr{C}$ which in degree n is given by $d_0\colon X_{n+1} \to X_n$.

6. Geometric realization of simplicial principal bundles

In this section we show that fiberwise geometric realization of a large class of simplicial parametrized principal bundles gives parametrized principal bundles. We discuss sufficient conditions on a simplicial parametrized group G to ensure that G is good and $\overline{W}G$ is proper (see Definition 20 below).

Recall from Section 4 that if $P(M,G)$ is a simplicial parametrized principal bundle, then after taking fiberwise geometric realizations there is a principal action of $|G|$ on $|P|$ with $|M|$ as the space of orbits. To prove that $|\pi|\colon |P| \to |M|$ is the projection map in a parametrized principal bundle all that remains is to prove that $|\pi|$ admits local sections.

We will show that a sufficient condition for this is that the following hold:

(a) The group G is fibrant in the sense of Definition 7.
(b) M satisfies a cofibrancy condition.

This latter condition is the parametrized analog of May's notion of *proper* simplicial space introduced in [Ma72]. In fact this notion, and the allied notion of a *good* simplicial space [Se74], makes sense in any topological bicomplete category.

Definition 20. Let $\mathscr{C}$ be a bicomplete topological category. A simplicial object X in $\mathscr{C}$ is called *proper* if the latching maps $L_n X \to X_n$ are $\bar{h}$-cofibrations for all $n \geq 0$; X is called *good* if all of the degeneracy morphisms $s_i\colon X_n \to X_{n+1}$ are $\bar{h}$-cofibrations.

In particular, specialized to the case where $\mathscr{C} = \mathscr{K}_{/B}$, we obtain the notion of a proper simplicial parametrized space. With these definitions understood, we re-state Theorem 2 from the Introduction.

Theorem 2. *Let G be a fibrant simplicial parametrized group and let M be a proper simplicial object in $\mathscr{K}_{/B}$. If P is a simplicial principal bundle over M with structure group G such that $P_n \to M_n$ is a numerable, parametrized principal G_n-bundle in $\mathscr{K}_{/B}$ for all $n \geq 0$, then the induced map*

$$|P| \to |M|$$

on fiberwise geometric realizations is the projection map for a locally trivial parametrized principal $|G|$-bundle $|P|(|M|,|G|)$ in $\mathscr{K}_{/B}$. Moreover, if the bundle $P_n \to M_n$ is trivial for all $n \geq 0$, then $|P| \to |M|$ is numerable.

Since the proof of Theorem 2 is somewhat technical we have deferred it to Section 7. We discuss some consequences. Observe that, subject to the hypotheses above, if $P \to M$ is a principal twisted cartesian product with structure group G, then $|P| \to |M|$ is a numerable parametrized principal $|G|$ bundle. An example of special interest is the universal principal twisted cartesian product $WG \to \overline{W}G$ (Lemma 19); in order to apply Theorem 2 in this case we need to investigate sufficient conditions for $\overline{W}G$ to be proper.

In principle, it is easier to check that a simplicial object is good than it is to check that it is proper. In Appendix A we give a proof, in the setting of a topological bicomplete category, of Proposition 26, which says that every good simplicial object is proper. This fact is standard for simplicial spaces (see for instance [GaL82]; we show that the proof given in op. cit. carries through to this more general setting). Therefore, we search for a condition on the simplicial parametrized group G which ensures that $\overline{W}G$ is proper.

Recall (Definition 6) the notion of a well-sectioned simplicial parametrized group. We will say that a simplicial parametrized group G is a *good* simplicial group if the object in $s\mathscr{K}_{/B}$ underlying G is good. We recall the statement of Proposition 3 from the Introduction; it gives a condition on G which ensures that G is good, and that $\overline{W}G$ is good and hence proper.

Proposition 3. *Let G be a well-sectioned simplicial parametrized group. Then the following statements are true:*

(1) *G is a good simplicial group in $\mathscr{K}_{/B}$.*
(2) *$\overline{W}G$ is proper in $s\mathscr{K}_{/B}$.*
(3) *$|G|$ is a well-sectioned group in $\mathscr{K}_{/B}$.*

We have deferred the proof of Proposition 3 to Section 8. Note that there is a partial converse to the first statement: if G is a good simplicial group in $\mathscr{K}_{/B}$ such that G_0 is well-sectioned, then G_n is well-sectioned for every $n \geq 0$.

Combining Theorem 2, Proposition 3 and Lemma 19 we obtain Proposition 4 from the Introduction.

Proposition 4. *Let G be a well-sectioned fibrant simplicial parametrized group. Then the fiberwise geometric realization $|WG| \to |\overline{W}G|$ of the universal G-bundle $WG \to \overline{W}G$ is a numerable parametrized principal $|G|$ bundle. Moreover $|WG|$ is a fiberwise contractible group in $\mathscr{K}_{/B}$ containing $|G|$ as a closed subgroup.*

Now we turn to the statement and proof of the main result of this paper. Let G denote a parametrized group. In [CJ98] (see pages 37–39) a construction of a universal parametrized principal G-bundle is given, based on the Milnor construction of a universal bundle, using infinite joins. This model of the universal bundle is very useful as it makes almost no assumptions on G. We will impose a mild restriction on G — we will require that G is well-sectioned — and build a model with more convenient properties.

If G is well-sectioned, Proposition 4 specializes, with G regarded as a constant simplicial parametrized group, to the statement that

$$|WG| \to |\overline{W}G|$$

is a numerable parametrized principal G-bundle. Here $\overline{W}G$ is the simplicial parametrized space whose n-simplices are described in (2) and whose face and degeneracy maps are described in (3) and (4). In the remainder of this section we shall write

$$BG := |\overline{W}G| \quad \text{and} \quad EG := |WG|,$$

since, as we will see, the parametrized G-bundle $EG \to BG$ is a model for the universal parametrized G-bundle. Firstly, let us note that if H is another parametrized group, then there is a canonical isomorphism $\overline{W}(G \times_B H) = \overline{W}G \times_B \overline{W}H$ and hence a canonical isomorphism $B(G \times H) = BG \times_B BH$, since the fiberwise geometric realization functor preserves finite limits. Thus by construction the classifying space functor $B(-)$ is product-preserving.

Recall Theorem 1 from the Introduction:

Theorem 1. *Let M be a paracompact space over B and let G be a well-sectioned fibrant parametrized group. Then there is a bijection*

$$H^1(M, G)_{\mathscr{K}_{/B}} \simeq [M, BG]_{\mathscr{K}_{/B}}.$$

We now turn to the proof of this theorem.

Proof. We make use of the fact that $H^1(M, G)_{\mathscr{K}_{/B}}$ is isomorphic to the set of fiberwise concordance classes of fiberwise principal G bundles on M (Corollary 15). We define a map

$$\begin{aligned} [M, BG]_{\mathscr{K}_{/B}} &\to H^1(M, G)_{\mathscr{K}_{/B}} \\ [f] &\mapsto [f^*EG] \end{aligned} \tag{6}$$

for $f\colon M \to BG$. It is easy to verify that this map is well defined. To prove that it is a bijection we construct an inverse. For this we need some preparation.

Suppose that P is a parametrized principal G-bundle on M. Recall that the *Čech nerve* $\check{C}(P)$ of $P \to M$ is the augmented simplicial object

$$\cdots \; P \times_M P \times_M P \;\substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longleftarrow}\; P \times_M P \;\substack{\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longrightarrow}\; P \longrightarrow M \tag{7}$$

in $\mathscr{K}_{/B}$ where the face and degeneracy maps are given by omission and inclusions by diagonals. Since $\check{C}(P)$ is augmented over M it follows on taking fiberwise geometric realizations that we obtain a map

$$|\check{C}(P)| \to M$$

in $\mathscr{K}_{/B}$. The Čech nerve $\check{C}(Y)$ can of course be defined for any map $\pi\colon Y \to M$. It is a well known fact (essentially going back to [Se68]) that if π admits local sections and M is paracompact then the map $|\check{C}(Y)| \to M$ is a homotopy equivalence. The following lemma is a straightforward variation on this result whose proof we leave to the reader.

Lemma 21. *If $\pi\colon Y \to M$ is a map in $\mathscr{K}_{/B}$ which admits local sections and M is paracompact then the canonical map*

$$|\check{C}(Y)| \to M$$

is a fiberwise homotopy equivalence.

With these preparations out of the way, we can return to the problem of defining an inverse for the map $[M, BG]_{\mathscr{K}_{/B}} \to H^1(M, G)_{\mathscr{K}_{/B}}$. Let π be the projection map $P \to M$ of the bundle. Since G acts principally on P, there exist maps

$$P \times_M P \to G, \quad P \times_M P \times_M P \to G \times G, \quad \dots \quad \text{etc.}$$

which fit together to give a simplicial map $\check{C}(P) \to \overline{W}G$. Observe that there is another simplicial map $\check{C}(\pi^* P) \to WG$ defined in an analogous fashion which forms part of a pullback diagram

$$\begin{tikzcd} \check{C}(\pi^* P) \arrow[r] \arrow[d] & WG \arrow[d] \\ \check{C}(P) \arrow[r] & \overline{W}G \end{tikzcd}$$

in $s\mathscr{K}_{/B}$. On taking fiberwise geometric realizations we obtain a map

$$|\check{C}(P)| \to |\overline{W}G| =: BG \tag{8}$$

in $\mathscr{K}_{/B}$. Let $\sigma\colon M \to |\check{C}(P)|$ denote a homotopy inverse to the map $|\check{C}(P)| \to M$ from Lemma 21. Composing σ with the map (8) gives a map $M \to BG$. It is clear that this map respects the relation of concordance (recall that we are identifying $H^1(M, G)_{\mathscr{K}_{/B}}$ with the set of fiberwise concordance classes using Corollary 15) to give a map

$$H^1(M, G)_{\mathscr{K}_{/B}} \to [M, BG]_{\mathscr{K}_{/B}}. \tag{9}$$

We need to prove that this map is the inverse of the map (6). We first examine the composite $H^1(M, G)_{\mathscr{K}_{/B}} \to [M, BG]_{\mathscr{K}_{/B}} \to H^1(M, G)_{\mathscr{K}_{/B}}$. To show that this is the identity we need to show that the pullback of $EG \to BG$ under the map $|\check{C}(P)| \to BG$ (equation (8) above) is fiberwise isomorphic to $q^* P$ where we use q to denote the map $|\check{C}(P)| \to M$. For it then follows that the pullback of $EG \to BG$ under the composite map $M \to |\check{C}(P)| \to BG$ is isomorphic to $\sigma^* q^* P \cong P$. Observe that on taking fiberwise geometric realizations we obtain the commutative diagram

$$\begin{tikzcd} P \arrow[d] & {|\check{C}(\pi^*P)|} \arrow[l] \arrow[r] \arrow[d] & {|WG|} \arrow[d] \\ M & {|\check{C}(P)|} \arrow[l] \arrow[r] & {|\overline{W}G|} \end{tikzcd}$$

in $\mathscr{K}_{/B}$ in which each square is a pullback. Hence it follows that the composite $H^1(M,G)_{\mathscr{K}_{/B}} \to [M,BG]_{\mathscr{K}_{/B}} \to H^1(M,G)_{\mathscr{K}_{/B}}$ is the identity.

Now we examine the composite map

$$[M,BG]_{\mathscr{K}_{/B}} \to H^1(M,G)_{\mathscr{K}_{/B}} \to [M,BG]_{\mathscr{K}_{/B}}.$$

To prove that this is the identity it is sufficient to prove the following: in the diagram

$$\begin{tikzcd} {|\check{C}(EG)|} \arrow[r] \arrow[d] & {|\overline{W}G| = BG} \\ BG & \end{tikzcd} \tag{10}$$

the two maps $|\check{C}(EG)| \to BG$ are fiberwise homotopic. Here the horizontal map is the fiberwise geometric realization of the map (8) (in the special case of $P = EG$) and the vertical map is the canonical map obtained by the augmentation of the Čech nerve $\check{C}(EG)$ of $EG \to BG$.

The existence of this fiberwise homotopy can be understood as a simple fact about the *total décalage* functor; therefore we shall need a short interlude to discuss this latter object. Recall (see for example [Du75, St12a]) that $\mathrm{Dec}\colon s\mathscr{K}_{/B} \to ss\mathscr{K}_{/B}$ is the functor induced by restriction along the ordinal sum functor $\sigma\colon \Delta \times \Delta \to \Delta$ defined above. Thus $\mathrm{Dec}\,X$ is the bisimplicial parametrized space whose columns form the simplicial object

$$\mathrm{Dec}_0 X \leftleftarrows \mathrm{Dec}_1 X \;\cdots\; \mathrm{Dec}_2 X \cdots$$

where $\mathrm{Dec}_n X = (\mathrm{Dec}_0)^{n+1} X$. In particular the 0-skeleton of $\mathrm{Dec}\,X$ is $\mathrm{Dec}_0 X$. Ordinal sum with the empty set defines canonical natural transformations $p_1 \to \sigma$ and $p_2 \to \sigma$, where $p_1, p_2\colon \Delta \times \Delta \to \Delta$ denote the projections onto the first and second factors. Hence the total décalage $\mathrm{Dec}\,X$ of a simplicial parametrized space X comes equipped with row and column augmentations $\mathrm{Dec}\,X \to p_1^* X$ and $\mathrm{Dec}\,X \to p_2^* X$ respectively. On taking diagonals and fiberwise geometric realizations, we obtain a diagram

$$\begin{tikzcd} {|d\mathrm{Dec}\,X|} \arrow[r, "q_2"] \arrow[d, "q_1"'] & X \\ X & \end{tikzcd} \tag{11}$$

A further useful property of the total décalage is that the fiberwise geometric realization $|d\mathrm{Dec}\, X|$ is isomorphic to X, for $d\mathrm{Dec}\, X$ is easily seen to be equal to the edge-wise subdivision of X as defined in [BöHM93].

We return to the problem at hand. Recall, see the remarks following Lemma 19, that $WG = \mathrm{Dec}_0\overline{W}G$. It follows, by an adjointness argument using the fact that $\mathrm{sk}_0\mathrm{Dec}\,\overline{W}G = WG$, that there is a canonical map of bisimplicial parametrized spaces

$$\mathrm{Dec}\,\overline{W}G \to \check{C}(WG) = \mathrm{cosk}_0(WG). \tag{12}$$

This map is easily checked to be an isomorphism and moreover the diagram (11) is equal to the diagram (10) above with $X = \check{C}(WG)$. Note that we also obtain the not-so-obvious fact that $|\check{C}(EG)|$ is isomorphic to BG.

Thus to prove that the two maps in (10) are fiberwise homotopic, it suffices to prove that the two maps in (11) are fiberwise homotopic. We shall prove that if X is a simplicial parametrized space, then there is a canonical simplicial homotopy $d\mathrm{Dec}\, X \otimes \Delta[1] \to X$ from q_1 to q_2. Taking fiberwise geometric realizations then gives the required fiberwise homotopy.

By adjointness, exhibiting such a simplicial homotopy, is equivalent to exhibiting a map $d\mathrm{Dec}\, X \to X^{\Delta[1]}$ such that the diagram

$$\begin{tikzcd} & X \\ d\mathrm{Dec}\, X \arrow[ur, "q_1"] \arrow[r] \arrow[dr, "q_2"'] & X^{\Delta[1]} \arrow[u] \arrow[d] \\ & X \end{tikzcd} \tag{13}$$

commutes, where $X^{\Delta[1]}$ denotes the usual simplicial path space of X, and the two projections $X^{\Delta[1]} \to X$ are induced by the inclusions $0, 1\colon \Delta[0] \to \Delta[1]$. To be more concrete, $X^{\Delta[1]}$ is the simplicial parametrized space whose space of n-simplices is the generalized matching object

$$(X^{\Delta[1]})_n = M_{\Delta[n]\times\Delta[1]}X$$

(see VII 1.21 of [GoJ99]). It is an easy calculation to see that the object of n-simplices of $d\mathrm{Dec}(X)$ is given by

$$(d\mathrm{Dec}(X))_n = M_{\Delta[n]\star\Delta[n]}X = M_{\Delta[2n+1]}X,$$

where $\Delta[n] \star \Delta[n]$ denotes the *join* of $\Delta[n]$ with itself [EP00]. To construct the map $d\mathrm{Dec}\, X \to X^{\Delta[1]}$ it suffices to construct a simplicial map

$$\Delta[n] \times \Delta[1] \to \Delta[2n+1],$$

natural in $[n]$, such that the diagram

$$\begin{tikzcd} & & \Delta[n] \arrow[dll, "\tilde{q}_1"'] \arrow[d] \\ \Delta[n] \star \Delta[n] = \Delta[2n+1] & & \Delta[n] \times \Delta[1] \arrow[ll] \\ & & \Delta[n] \arrow[ull, "\tilde{q}_2"] \arrow[u] \end{tikzcd}$$

commutes, where the two maps $\tilde{q}_1, \tilde{q}_2 \colon \Delta[n] \to \Delta[2n+1]$ are induced by $\sigma([n], \emptyset) \to \sigma([n],[n])$ and $\sigma(\emptyset,[n]) \to \sigma([n],[n])$ respectively. The required homotopy is the nerve of the canonical natural transformation $\alpha \colon \tilde{q}_1 \to \tilde{q}_2$ defined by $\alpha(i) \colon i \to n+i+1$. It is easy to see that this map is natural in n in the appropriate sense. This finishes the proof of Theorem 1. $\square$

7. Proof of Theorem 2

In this Section, we prove Theorem 2. First recall the statement of this theorem.

Theorem 2. *Let G be a fibrant simplicial parametrized group and let M be a proper simplicial object in $\mathscr{K}_{/B}$. If P is a simplicial principal bundle over M with structure group G such that $P_n \to M_n$ is a numerable, parametrized principal G_n-bundle in $\mathscr{K}_{/B}$ for all $n \geq 0$, then the induced map*

$$|P| \to |M|$$

on fiberwise geometric realizations is the projection map for a locally trivial parametrized principal $|G|$-bundle $|P|(|M|,|G|)$ in $\mathscr{K}_{/B}$. Moreover, if the bundle $P_n \to M_n$ is trivial for all $n \geq 0$, then $|P| \to |M|$ is numerable.

The proof of Theorem 2 is a variation on the approach of the papers [Ma75, Mc69, St68] (which deal with the case where G is a constant simplicial group) to the case where G is an arbitrary group in $s\mathscr{K}_{/B}$. We note that an important ingredient in [Ma75, Mc69, St68] is the notion of an equivariant NDR pair, a notion which we have already explained (see Section 3 above) has a straightforward generalization to the parametrized setting.

Proof of Theorem 2. Let $n \geq 0$ be an integer. Recall that the n^{th} skeleton $\mathrm{sk}_n M$ of M comes equipped with a map $\mathrm{sk}_n M \to M$ and that there are natural maps $\mathrm{sk}_n M \to \mathrm{sk}_m M$ whenever $m \leq n$. Recall also that $M = \mathrm{colim}_n \, \mathrm{sk}_n M$ and that there is a pushout diagram of the form

$$\begin{tikzcd} (M_n \otimes \partial\Delta[n]) \cup (L_n M \otimes \Delta[n]) \arrow[r] \arrow[d] & \mathrm{sk}_{n-1} M \arrow[d] \\ M_n \otimes \Delta[n] \arrow[r] & \mathrm{sk}_n M \end{tikzcd} \tag{14}$$

(see for instance Proposition VII 1.7 of [GoJ99]), where $\Delta[n]$ denotes the simplicial n-simplex and $\partial\Delta[n]$ denotes its boundary.

We use the n^{th} skeletons of M to define a filtration

$$|P|_0 \subset |P|_1 \subset \cdots \subset |P|_n \subset \cdots \subset |P|$$

of $|P|$ as follows. The canonical maps $\mathrm{sk}_n M \to M$ induce by pullback simplicial principal bundles with structure group G on each of the simplicial spaces $\mathrm{sk}_n M$. Let $|P|_n = |\mathrm{sk}_n M \times_M P|$. Observe that $|P|_n \subset |P|_{n+1}$ and $|P|_n \subset |P|$ are closed inclusions for all $n \geq 0$. For convenience of notation we will also denote $|\mathrm{sk}_n M|$ by $|M|_n$, but note the potential confusion with $|P|_n$: we remind the reader that this does *not* denote the geometric realization of the n-skeleton of P. Recall that $M = \mathrm{colim}_n\, \mathrm{sk}_n M$ and hence $|M| = \mathrm{colim}_n\, |M|_n$ in $\mathscr{K}_{/B}$. We claim that $P = \mathrm{colim}_n(\mathrm{sk}_n M \times_M P)$. This is easy to see in the special case that P is trivial. We can reduce the general statement to this special case, since P is a colimit of trivial bundles and colimits commute amongst themselves.

The map $|P| \to |M|$ is a quotient map, since the map

$$\sqcup_{n\geq 0} P_n \times \Delta^n \to \sqcup_{n\geq 0} M_n \times \Delta^n$$

is a quotient map, and both of the maps

$$\sqcup_{n\geq 0} P_n \times \Delta^n \to |P| \quad \text{and} \quad \sqcup_{n\geq 0} M_n \times \Delta^n \to |M|$$

are quotient maps. Since the diagram

$$\begin{tikzcd} |P|_n \arrow[r] \arrow[d] & |P| \arrow[d] \\ |M|_n \arrow[r] & |M| \end{tikzcd}$$

is a pullback, we see that $|P|_n \to |M|_n$ is also a quotient map ($|M|_n \to |M|$ is a closed inclusion, and quotient maps pullback along closed inclusions to quotient maps). In particular $|M|_n$ has the quotient topology induced by the map $|\pi|\colon |P|_n \to |M|_n$.

The main step in our proof is to prove that $(|P|_n, |P|_{n-1})$ is a $|G|$-fiberwise NDR pair in $\mathscr{K}_{/B}$ for all $n \geq 1$, so that we can apply the method of [Ma75, Mc69, St68]. As a first step in this direction we have the following lemma.

Lemma 22. *For every $n \geq 1$ we have a pushout diagram in $s\mathscr{K}_{/B}$ of the form*

$$\begin{tikzcd} ((M_n \otimes \partial\Delta[n]) \cup (L_n M \otimes \Delta[n])) \times_M P \arrow[r] \arrow[d] & \mathrm{sk}_{n-1} M \times_M P \arrow[d] \\ (M_n \otimes \Delta[n]) \times_M P \arrow[r] & \mathrm{sk}_n M \times_M P. \end{tikzcd} \tag{15}$$

Proof. Observe that the canonical map from the pushout to $\mathrm{sk}_n M \times_M P$ is a continuous bijection in each degree. Therefore it suffices to show that for each $m \geq 0$ the induced map

$$((M_n \otimes \Delta[n])_m \times_{M_m} P_m) \sqcup ((\mathrm{sk}_{n-1}M)_m \times_{M_m} P_m) \to (\mathrm{sk}_n M)_m \times_{M_m} P_m \tag{16}$$

is a quotient map. The map (16) is the map of fiberwise principal bundles induced by pullback along the quotient map

$$(M_n \otimes \Delta[n])_m \sqcup (\mathrm{sk}_{n-1}M)_m \to (\mathrm{sk}_n M)_m.$$

Therefore to prove the lemma it suffices to establish the following claim: if $P \to M$ is a fiberwise principal bundle and $f\colon N \to M$ is a quotient map in $\mathscr{K}_{/B}$, then $f^*P \to P$ is also a quotient map. To see this observe that since P can be constructed as a quotient of a coproduct of spaces of the form $U \times_B G$, and f^*P can be constructed as a quotient of a coproduct of spaces of the form $f^{-1}U \times_B G$, it suffices to prove that $f^{-1}U \times_B G \to U \times_B G$ is a quotient map for any open set $U \subset M$. Since the functor $(-) \times_B G$ preserves colimits this follows from the fact that $f^{-1}U \to U$ is a quotient map, since $U \subset M$ is open. $\square$

Continuing the proof of Theorem 2, the second step is to show that in the diagram (15) the realization of the left hand vertical map is an $\bar{f}$-cofibration in $|G|\mathscr{K}_{/B}$. For this we will need the hypotheses that each $P_n \to M_n$ is a numerable principal G_n bundle, and that M is proper.

Lemma 23. *For every $n \geq 1$, the map*

$$|(M_n \otimes \partial\Delta[n]) \cup (L_n M \otimes \Delta[n]) \times_M P| \to |M_n \otimes \Delta[n] \times_M P| \tag{17}$$

is an $\bar{f}$-cofibration in $|G|\mathscr{K}_{/B}$ and hence $(|P|_n, |P|_{n-1})$ is a $|G|$-fiberwise NDR pair in $\mathscr{K}_{/B}$ for all $n \geq 1$.

Proof. Using the fact that geometric realization commutes with pullbacks, we obtain a pullback diagram

$$\begin{tikzcd}
|((M_n \otimes \partial\Delta[n]) \cup (L_n M \otimes \Delta[n])) \times_M P| \arrow[r] \arrow[d] & |M_n \otimes \Delta[n] \times_M P| \arrow[d] \\
(M_n \times \partial\Delta^n) \cup (L_n M \times \Delta^n) \arrow[r] & M_n \times \Delta^n.
\end{tikzcd}$$

Since M is proper, the closed inclusion $L_n M \subset M_n$ is an $\bar{f}$-cofibration and standard results show that this induces a closed inclusion

$$(M_n \times \partial\Delta^n) \cup (L_n M \times \Delta^n) \to M_n \times \Delta^n$$

which is also an $\bar{f}$-cofibration. Therefore if we can show that

$$|M_n \otimes \Delta[n] \times_M P| \to M_n \times \Delta^n$$

is a numerable fiberwise principal $|G|$ bundle in $\mathscr{K}_{/B}$, then we may use Proposition 16 to deduce that the closed inclusion (17) is an $\bar{f}$-cofibration in $\mathscr{K}_{/B}$. It then follows from Lemma 22 that $|P|_{n-1} \to |P|_n$ is an $\bar{f}$-cofibration, since these are preserved under pushout. Finally, it follows from Lemma 10 that $(|P|_n, |P|_{n-1})$ is a $|G|$-fiberwise NDR pair.

Since we have shown that $|P| \to |M|$ satisfies the condition (ii) of Definition 12 for the group $|G|$, and this condition is stable under pullback, $|M_n \otimes \Delta[n] \times_M P| \to M_n \times \Delta^n$ also satisfies the condition (ii). We thus only need to show that this map satisfies the condition (i). That is, it admits local sections relative to a numerable open cover of $M_n \times \Delta^n$. For this, consider the commutative diagram

$$\begin{tikzcd} P_n \times \Delta^n \arrow[r] \arrow[d] & {|P|} \arrow[d] \\ M_n \times \Delta^n \arrow[r] & {|M|} \end{tikzcd}$$

where the horizontal maps are the canonical ones into the colimits defining $|P|$ and $|M|$. The map $P_n \times \Delta^n \to |P|$ factors through $|M_n \otimes \Delta[n] \times_M P|$ and hence $|M_n \otimes \Delta[n] \times_M P| \to M_n \times \Delta^n$ admits local sections relative to a numerable open cover of $M_n \times \Delta^n$ since the principal G_n bundle $P_n \times \Delta^n \to M_n \times \Delta^n$ does by hypothesis. □

We now proceed in our proof of Theorem 2 in analogy with the arguments in [Ma75, Mc69, St68]. Since $(|P|_n, |P|_{n-1})$ is a fiberwise $|G|$-equivariant NDR pair for every $n \geq 1$ and $|P| = \operatorname{colim}_n |P|_n$, we see that $(|P|, |P|_n)$ is a fiberwise $|G|$-equivariant NDR pair for every $n \geq 0$ (by Lemma 10 and Lemma 11). For any $n \geq 0$ let $h_n\colon |P| \times I \to |P|$ and $u_n\colon |P| \to I$ be a representation of $(|P|, |P|_n)$ as a fiberwise $|G|$-equivariant NDR pair. Define functions $\hat{\rho}_n\colon |P| \to I$ for every $n \geq 1$ by

$$\hat{\rho}_n(x) = (1 - u_n(x))u_{n-1}(h_n(x, 1)).$$

The functions $\hat{\rho}_n$ are easily seen to be $|G|$-invariant and hence descend to functions $\rho_n\colon |M| \to I$. Let $U_n = \hat{\rho}_n^{-1}(0, 1]$ and let $V_n = \rho_n^{-1}(0, 1]$ so that $U_n = |\pi|^{-1} V_n$ (and hence U_n is $|G|$-invariant). Following [Ma75, Mc69] let $r_n\colon |P| \to |P|$ denote the map $r_n(|x, t|) = h_n(|x, t|, 1)$. Then we have (see [Ma75, Mc69]) the following chain of inclusions

$$|P|_n \setminus |P|_{n-1} \subset U_n \subset r_n^{-1}(|P|_n \setminus |P|_{n-1}).$$

Observe that we have a commutative diagram

$$\begin{tikzcd} U_n \arrow[r, "r_n"] \arrow[d, "|\pi|"'] & {|P|_n \setminus |P|_{n-1}} \arrow[r] \arrow[d] & {|M_n \otimes \Delta[n] \times_M P|} \arrow[d] \\ V_n \arrow[r] & {|M|_n \setminus |M|_{n-1}} \arrow[r] & {|M_n \otimes \Delta[n]|} \end{tikzcd} \tag{18}$$

in which the top horizontal maps are $|G|$-equivariant. The lower right hand map in this diagram arises as follows: after taking geometric realizations in (14), we see that there is an isomorphism

$$|M|_n \setminus |M|_{n-1} = |M_n \otimes \Delta[n]| \setminus |(M_n \otimes \partial\Delta[n]) \cup (L_n M \otimes \Delta[n])|$$

and hence a natural inclusion $|M|_n \setminus |M|_{n-1} \subset |M_n \otimes \Delta[n]|$.

After the previous Lemma 23 we observed that

$$|M_n \otimes \Delta[n] \times_M P| \to |M_n \otimes \Delta[n]|$$

is a numerable fiberwise principal $|G|$-bundle in $\mathscr{K}_{/B}$ and hence is locally trivial. Using local sections of this map, we can find an open cover $(V_{n,i})$ of V_n and $|G|$-equivariant maps $\zeta_{n,i} \colon U_{n,i} \to |G|$, where $U_{n,i} = |\pi|^{-1} V_{n,i}$. Then we can define $|G|$-invariant maps $\hat{\sigma}_{n,i} \colon U_{n,i} \to U_{n,i}$ by $\hat{\sigma}_{n,i}(x) = x\zeta_{n,i}(x)^{-1}$. Since $\hat{\sigma}_{n,i}$ is $|G|$-invariant, it descends to define a unique map $\sigma_{n,i} \colon V_{n,i} \to U_{n,i}$ so that the diagram

$$\begin{tikzcd} U_{n,i} \arrow[r, "\hat{\sigma}_{n,i}"] \arrow[d, "|\pi|"'] & U_{n,i} \\ V_{n,i} \arrow[ur, "\sigma_{n,i}"'] & \end{tikzcd}$$

commutes. The set $V_{n,i}$ has the quotient topology induced by $|\pi|$ and hence $\sigma_{n,i}$ is continuous. Clearly $\sigma_{n,i}$ is a section of $|\pi|$. Thus we have proven that there exist trivializations of $|\pi| \colon |P| \to |M|$ over the open subsets $V_{n,i}$.

It remains to prove the statement regarding the numerability of the bundle $|P| \to |M|$. We argue as follows. From the proof above we obtain the commutative diagram (18). In this case the bundle

$$|M_n \otimes \Delta[n] \times_M P| \to |M_n \otimes \Delta[n]|$$

is trivial, and therefore we can define $|G|$-equivariant maps $\zeta_n \colon U_n \to |G|$. In exactly the same way as above we can use the maps ζ_n to define $|G|$-invariant maps $\hat{\sigma}_n \colon U_n \to U_n$ which descend to sections $\sigma_n \colon V_n \to U_n$ of $|\pi|$. The problem now is to show that the open cover (V_n) is numerable. To do this we use the functions $\rho_n \colon V_n \to I$ constructed earlier. The collection of functions (ρ_n) may not be locally finite, this can be fixed however using the method of Dold [D63, Proof of Proposition 6.7]; one defines new functions $\phi_n \colon U_n \to I$ with $\mathrm{supp}(\phi_n) \subset U_n$ by

$$\phi_n(x) = \max\left(0, \rho_n(x) - n\sum_{i=1}^{n-1} \rho_i(x)\right).$$

Then one can check as in [D63] that the collection of functions (ϕ_n) is locally finite. It is now clear how to form a partition of unity from the ϕ_n. This ends the proof of Theorem 2. □

8. Proof of Proposition 3

Recall the statement of Proposition 3.

Proposition 3. *Let G be a well-sectioned simplicial parametrized group. Then the following statements are true:*

1. *G is a good simplicial group in $\mathscr{K}_{/B}$.*
2. *$\overline{W}G$ is proper in $s\mathscr{K}_{/B}$.*
3. *$|G|$ is a well-sectioned group in $\mathscr{K}_{/B}$.*

Proof. We prove statement (1). We need to show that $s_i \colon G_n \to G_{n+1}$ is an $\bar{f}$-cofibration for all $0 \leq i \leq n$ and all $n \geq 0$. Since s_i is a section of the corresponding face operator d_i, we can identify s_i with the map

$$G_n \to G_n \times_B \ker(d_i)$$

which sends $g \mapsto (g, 1)$. Therefore, by Lemma 24 below, to prove that s_i is an $\bar{f}$-cofibration it is sufficient to prove that $\ker(d_i)$ is well sectioned. For this, we observe that $\ker(d_i)$ is a retract of G_{n+1} by the map

$$G_{n+1} \to \ker(d_i)$$

sending g to $g s_i d_i(g)^{-1}$. Therefore the section $B \to \ker(d_i)$ is an $\bar{f}$-cofibration since it is a retract of the map $B \to G_{n+1}$ which is an $\bar{f}$-cofibration by hypothesis.

We prove statement (2). From what we have just proved, we have that each degeneracy map of G is an $\bar{f}$-cofibration. Lemma 24 below implies that the degeneracies of $\overline{W}G$ are $\bar{f}$-cofibrations and hence Proposition 29 in Appendix A implies that $\overline{W}G$ is proper.

Finally we prove statement (3). Since G is well-sectioned, the simplicial object G is proper, and hence the inclusion $|G|_n \subset |G|_{n+1}$ is an $\bar{f}$-cofibration for all $n \geq 0$ (with the notation of the proof of Theorem 2). This follows from the fact that $|G|_n \subset |G|_{n+1}$ is a pushout of

$$|G_n \otimes \partial\Delta[n]) \cup (L_n G \otimes \Delta[n])| \to |G_n \otimes \Delta[n]|,$$

which is an $\bar{f}$-cofibration using Proposition 3 and the fact that $\mathscr{K}_{/B}$ is a topological model category. Therefore the inclusion $|G|_n \subset |G|$ is an $\bar{f}$-cofibration for all $n \geq 0$ (by the nonequivariant version of Lemma 11). Since $|G|_0$ is well-sectioned and the composite of two $\bar{f}$-cofibrations is an $\bar{f}$-cofibration, it follows that $|G|$ is well-sectioned. □

To complete the proof of Proposition 3 we need to give the proof of the following lemma.

Lemma 24. *Suppose that $A_1 \to X$ and $A_2 \to Y$ are $\bar{f}$-cofibrations in $\mathscr{K}_{/B}$. Then $A_1 \times_B A_2 \to X \times_B Y$ is also an $\bar{f}$-cofibration.*

Proof. It is clearly sufficient to prove that if $A \to X$ is an $\bar{f}$-cofibration and Y is any space over B, then $A \times_B Y \to X \times_B Y$ is an $\bar{f}$-cofibration, in other words it has the LLP with respect to all f-acyclic f-fibrations $U \to V$. By adjointness, this is equivalent to checking that $A \to X$ has the LLP against all maps of the form $\mathrm{Map}_B(Y, U) \to \mathrm{Map}_B(Y, V)$ where $U \to V$ is an f-acyclic f-fibration.

By an adjointness argument, the functor $\mathrm{Map}_B(Y,-)\colon \mathscr{K}_{/B}\to\mathscr{K}_{/B}$ preserves f-fibrations. It also preserves fiberwise homotopies: if $g_0,g_1\colon X\to Z$ are fiberwise homotopic through a fiberwise homotopy $h\colon X\times I\to Z$, then the maps $\mathrm{Map}_B(Y,g_0)$ and $\mathrm{Map}_B(Y,g_1)$ are fiberwise homotopic through the fiberwise homotopy $\tilde{h}\colon \mathrm{Map}_B(Y,X)\times I\to \mathrm{Map}_B(Y,Z)$ defined as the composite

$$\mathrm{Map}_B(Y,X)\times I\to \mathrm{Map}_B(Y,X\times I)\xrightarrow{\mathrm{Map}_B(Y,h)}\mathrm{Map}_B(Y,Z), \tag{19}$$

where the first map is the adjoint of the canonical map

$$Y\times_B \mathrm{Map}_B(Y,X)\times I\to X\times I.$$

One can check that $\tilde{h}$ so defined does give such a fiberwise homotopy as claimed. It follows that the functor $\mathrm{Map}_B(Y,-)$ preserves f-equivalences, and hence f-acyclic f-fibrations, which proves the lemma. $\square$

Appendix A. Good implies proper

Our goal in this section is to prove that a good simplicial object X in a topological bicomplete category $\mathscr{C}$ is automatically proper, provided that a generalization of Lillig's union theorem on cofibrations [L73] holds in $\mathscr{C}$, and an assumption on colimits in the slice categories $\mathscr{C}_{/X_n}$ is met. We begin by making the following definition.

Definition 25. Let $\mathscr{C}$ be a topological bicomplete category. We say that $\mathscr{C}$ satisfies the *Lillig condition* if the following is true: Given a pullback diagram in $\mathscr{C}$,

$$\begin{tikzcd} A_3 \arrow[r] \arrow[d] & A_2 \arrow[d] \\ A_1 \arrow[r] & X \end{tikzcd}$$

such that the morphisms $A_1\to X$, $A_3\to X$ and $A_2\to X$ are $\bar{h}$-cofibrations, then the canonical map $A_1\cup_{A_3}A_2\to X$ is an $\bar{h}$-cofibration.

When $\mathscr{C}=\mathscr{K}$ this is Lillig's union theorem [L73]. We will prove shortly that a reworking of the proof in [L73] shows that the Lillig condition holds when $\mathscr{C}=\mathscr{K}_{/B}$; we do not know if this condition holds more generally.

With this definition understood we can turn to our main goal in this appendix, which is the proof of the following proposition.

Proposition 26. *Let $\mathscr{C}$ be a topological bicomplete category and let X be a good simplicial object in $\mathscr{C}$. Suppose that the following two conditions are satisfied:*

(1) *$\mathscr{C}$ satisfies the Lillig condition of Definition 25.*

(2) *$s_k\colon X_n\to X_{n+1}$ is properly extensive for all $n\geq 0$ and all $0\leq k\leq n$.*

Then X is proper.

Here we say that a map $f\colon X \to Y$ in $\mathscr{C}$ is *properly extensive* if the pullback functors $f^*\colon \mathscr{C}_{/Y} \to \mathscr{C}_{/X}$ commutes with finite colimits. The proof of Proposition 26 that we shall give is based on the proof of Corollary 2.4(b) of [GaL82]. We begin with some preparation.

Recall (Definition 20) that a proper simplicial object X in a topological bicomplete category $\mathscr{C}$ is one for which the latching maps $L_nX \to X_n$ are $\bar{h}$-cofibrations for all $n \geq 0$. We need to examine the notion of latching object in a little more detail. Recall (see for example Remark VII 1.8 of [GoJ99]), that L_nX may also be described as the coequalizer

$$\coprod_{0\leq i<j\leq n-2} X_{n-2} \rightrightarrows \coprod_{0\leq l\leq n-1} X_{n-1} \to L_nX \tag{20}$$

where the two maps defining the coequalizer arise from the simplicial identity $s_is_{j-1} = s_js_i$ if $i < j$ (see for example V Lemma 1.1 and VII Remark 1.8 of [GoJ99]). It is well known that $L_0X = \emptyset$, $L_1X = X_0$ and $L_2X = X_1 \cup_{X_0} X_1$.

It will be convenient to introduce a family of *partial latching objects* $L_{n,k}X$ associated to the simplicial object X for $k = 0, 1, \ldots, n$. For $0 \leq k \leq n$ we define $L_{n,k}X$ by the coequalizer

$$\coprod_{0\leq i<j\leq k-1} X_{n-2} \rightrightarrows \coprod_{l=0}^{k-1} X_{n-2} \to L_{n,k}X$$

where the restrictions of the two displayed maps to the summand labelled by the pair (i, j) are given by the composites

$$X_{n-2} \xrightarrow{s_i} X_{n-1} \xrightarrow{\mathrm{in}_j} \coprod_{l=0}^{k-1} X_{n-1}$$

$$X_{n-2} \xrightarrow{s_j} X_{n-1} \xrightarrow{\mathrm{in}_i} \coprod_{l=0}^{k-1} X_{n-1}$$

and where $\mathrm{in}_i, \mathrm{in}_j$ denote the inclusions into the summands labelled by i and j. Note that there are isomorphisms $L_{n,0}X \simeq \emptyset$, $L_{n,n}X \simeq L_nX$. Note also that there is a canonical map $L_{n,k}X \to X_n$ induced by the degeneracies $s_i\colon X_{n-1} \to X_n$ for $0 \leq i \leq k-1$. These partial latching objects are precisely the objects $L_{n,k}X$ defined on pages 362–363 of [GoJ99]. We have the following result:

Lemma 27 ([GoJ99], chapter VII Proposition 1.27)**.** *Let X be a simplicial object in $\mathscr{C}$. Then for any $0 \leq k \leq n-1$ there is a pushout diagram*

$$\begin{tikzcd} L_{n-1,k}X \arrow[r] \arrow[d] & X_{n-1} \arrow[d] \\ L_{n,k}X \arrow[r] & L_{n,k+1}X. \end{tikzcd}$$

Proof. The lemma follows from the statements (i)–(iii) below, together with the fact that colimits commute amongst themselves.

(i) The diagram

$$\begin{tikzcd}
\coprod_{0\le i<j\le k-1} X_{n-3} \arrow[r] \arrow[d, "s_k"'] & \coprod_{0\le i<j\le k-1} X_{n-3} \sqcup \coprod_{l=0}^{k-1} X_{n-2} \arrow[d] \\
\coprod_{0\le i<j\le k-1} X_{n-2} \arrow[r] & \coprod_{0\le i<j\le k} X_{n-2}
\end{tikzcd}$$

is a pushout.

(ii) The diagram

$$\begin{tikzcd}
\coprod_{l=0}^{k-1} X_{n-2} \arrow[r] \arrow[d, "s_{k+1}"'] & \coprod_{l=0}^{k-1} X_{n-2} \sqcup X_{n-1} \arrow[d] \\
\coprod_{l=0}^{k-1} X_{n-1} \arrow[r] & \coprod_{l=0}^{k} X_{n-1}
\end{tikzcd}$$

is a pushout.

(iii) The diagram

$$\coprod_{0\le i<j\le k-1} X_{n-3} \sqcup \coprod_{l=0}^{k-1} X_{n-2} \rightrightarrows \coprod_{l=0}^{k-1} X_{n-2} \sqcup X_{n-1} \to X_{n-1}$$

is a coequalizer, where the two displayed maps are defined to be the corresponding maps in the coequalizer defining $L_{n-1,k}X$ on the first summand $\coprod_{0\le i<j\le k-1} X_{n-3}$, and are defined to be the composites

$$X_{n-2} \xrightarrow{\mathrm{in}_i} \coprod_{l=0}^{k-1} X_{n-2} \to \coprod_{l=0}^{k-1} X_{n-2} \sqcup X_{n-1}$$

$$X_{n-2} \xrightarrow{s_i} X_{n-1} \to \coprod_{l=0}^{k-1} X_{n-2} \sqcup X_{n-1}$$

on the summand X_{n-2} labelled by i in $\coprod_{i=0}^{k-1} X_{n-2}$ (in this case it is straightforward to check that the universal property for a coequalizer is satisfied). $\square$

Next, we need a lemma asserting that under certain hypotheses on colimits in $\mathscr{C}$, a canonical square built out of the partial latching objects is a pullback square.

Lemma 28. *Suppose that $s_k\colon X_n \to X_{n+1}$ is properly extensive for all $n \geq 0$ and for all $0 \leq k \leq n$. Then for every $0 \leq k \leq n$ the diagram*

$$\begin{tikzcd} L_{n,k}X \arrow[r] \arrow[d] & X_n \arrow[d, "s_k"] \\ L_{n+1,k}X \arrow[r] & X_{n+1} \end{tikzcd}$$

is a pullback.

Proof. Under the hypothesis in the statement of the lemma, we have a coequalizer diagram

$$X_n \times_{X_{n+1}} L_{n+1,k}X \longrightarrow \coprod_{0\leq i<j\leq k-1} X_{n-1} \times_{X_{n+1}} X_{n-1} \rightrightarrows \coprod_{l=0}^{k-1} X_n \times_{X_{n+1}} X_n$$

The result then follows from the well-known fact that the diagrams

$$\begin{tikzcd} X_{n-1} \arrow[r, "s_i"] \arrow[d, "s_{j-1}"'] & X_n \arrow[d, "s_j"] \\ X_n \arrow[r, "s_i"'] & X_{n+1} \end{tikzcd}$$

are pullbacks for $i < j$. $\square$

We can now give the proof of Proposition 26.

Proof of Proposition 26. We will prove by induction on $n \geq 0$ that the maps $L_{n,k}X \to X_n$ are $\bar{h}$-cofibrations for all $0 \leq k \leq n$. The base case is the statement that $L_{0,0}X \to X_0$ is an $\bar{h}$-cofibration. But $L_{0,0}X = \emptyset$ and hence the statement is true in this case, since every object of $\mathscr{C}$ is $\bar{h}$-cofibrant.

Now we make the inductive assumption that the maps $L_{n-1,k}X \to X_{n-1}$ are $\bar{h}$-cofibrations for all $0 \leq k \leq n-1$. We will prove by induction on k that $L_{n,k}X \to X_n$ is an $\bar{h}$-cofibration for all $0 \leq k \leq n$.

To start the induction, we again observe that $L_{n,0}X = \emptyset$ and hence $L_{n,0}X \to X_n$ is an $\bar{h}$-cofibration. Assume then that $L_{n,k}X \to X_n$ is an $\bar{h}$-cofibration for $k \geq 0$ and consider the diagram

(21)
$$\begin{tikzcd} L_{n-1,k}X \arrow[r] \arrow[d] & X_{n-1} \arrow[d] \arrow[ddr, "s_k"] & \\ L_{n,k}X \arrow[r] \arrow[drr] & L_{n,k+1}X \arrow[dr] & \\ & & X_n \end{tikzcd}$$

for $0 \leq k \leq n-1$. Since the inner square in (21) is a pushout, it follows from the assumption that $L_{n-1,k}X \to X_{n-1}$ is an $\bar{h}$-cofibration for $0 \leq k \leq n-1$ that $L_{n,k}X \to L_{n,k+1}X$ is an $\bar{h}$-cofibration. By hypothesis, $L_{n,k}X \to X_n$ is an $\bar{h}$-cofibration and $s_k\colon X_{n-1} \to X_n$ is an $\bar{h}$-cofibration since X is good. By Lemma 28 the outer square in (21) is a pullback. Therefore, since $\mathscr{C}$ satisfies the Lillig condition of Definition 25 we conclude that $L_{n,k+1}X \to X_n$ is an $\bar{h}$-cofibration, completing the inductive step. Therefore $L_{n,n}X \to X_n$ is an $\bar{h}$-cofibration, i.e., $L_nX \to X_n$ is an $\bar{h}$-cofibration, completing the original inductive step. Hence X is proper. $\square$

As an application, we prove the following result, which we need in the proof of Proposition 3 above.

Proposition 29. *Let $\mathscr{C} = \mathscr{K}_{/B}$. Then any good simplicial object in $\mathscr{K}_{/B}$ is proper.*

Proof. We verify that the two conditions from Proposition 26 are satisfied; we deal with Condition (2) first. We need to know that the functor

$$s_n^*\colon (\mathscr{K}_{/B})_{/X_{n+1}} \to (\mathscr{K}_{/B})_{/X_n},$$

i.e., restriction along the closed inclusion $s_n\colon X_n \to X_{n+1}$, preserves finite colimits. In other words, since $(\mathscr{K}_{/B})_{/X} \cong \mathscr{K}_{/X}$ for any object X in $\mathscr{K}_{/B}$, we have to show that $s_n^*\colon \mathscr{K}_{/X_{n+1}} \to \mathscr{K}_{/X_n}$ preserves finite colimits.

A colimit in $\mathscr{K}_{/X_{n+1}}$ is constructed as a quotient of a coproduct in $\mathscr{K}$ and then equipped with the canonical map to X_{n+1}. Therefore it is sufficient to prove two things: firstly that restriction along X_n preserves coproducts in $\mathscr{K}_{/X_{n+1}}$ and secondly that if $q\colon Y \to Z$ is a quotient map in $\mathscr{K}_{/X_{n+1}}$ then in the pullback diagram

$$\begin{tikzcd}
X_n \times_{X_{n+1}} Y \arrow[r] \arrow[d] & Y \arrow[d, "q"] \\
X_n \times_{X_{n+1}} Z \arrow[r] & Z
\end{tikzcd}$$

in $\mathscr{K}$ the map $X_n \times_{X_{n+1}} Y \to X_n \times_{X_{n+1}} Z$ is a quotient map. The first of these things is easy to prove, for the second it is enough to prove that $X_n \times_{X_{n+1}} Z \to Z$ is a closed inclusion, since quotient maps restrict to quotient maps along closed subspaces. This is clear however, since $s_n\colon X_n \to X_{n+1}$ is a closed inclusion, and closed inclusions pull back along arbitrary maps to closed inclusions.

For the Lillig condition, suppose that

$$\begin{tikzcd}
A_3 \arrow[r] \arrow[d] & A_2 \arrow[d] \\
A_1 \arrow[r] & X
\end{tikzcd}$$

is a pullback diagram in $\mathscr{K}_{/B}$ as in Definition 25 above, i.e., the maps $A_1 \to X$, $A_2 \to X$ and $A_3 \to X$ are $\bar{f}$-cofibrations. From the pushout-product theorem (see [ScV02]) it follows that

$$A_1 \cup_{A_3} A_3 \otimes I \cup_{A_3} A_2 \to X \otimes I \tag{22}$$

is an $\bar{f}$-cofibration. This map fits into the commutative diagram

$$\begin{tikzcd} A_1 \cup_{A_3} A_3 \otimes I \cup_{A_3} A_2 \arrow[r] \arrow[d] & A_1 \cup_{A_3} A_2 \arrow[d] \\ X \otimes I \arrow[r] & X. \end{tikzcd}$$

The pushout of (22) along $A_1 \cup_{A_3} A_3 \otimes I \cup_{A_3} A_2 \to A_1 \cup_{A_3} A_2$ can be identified with a map

$$A_1 \cup_{A_3} A_2 \to X \otimes I \cup_{A_3 \otimes I} A_3$$

which is also an $\bar{f}$-cofibration. Therefore, to prove that $A_1 \cup_{A_3} A_2 \to X$ is an $\bar{f}$-cofibration it suffices to prove that $A_1 \cup_{A_3} A_2 \to X$ is a retract of

$$A_1 \cup_{A_3} A_2 \to X \otimes I \cup_{A_3 \otimes I} A_3.$$

Suppose (u_1, h_1) and (u_2, h_2) are representations of (X, A_1) and (X, A_2) as fiberwise NDR pairs. As in [L73] define a map $u\colon X \to X \otimes I \cup_{A_3 \otimes I} A_3$ by

$$u(x) = \begin{cases} [x, u_1(x)/(u_1(x) + u_2(x))] & \text{if } x \notin A_3, \\ [x, 0] & \text{if } x \in A_3. \end{cases}$$

Then it is easy to check that $u(x) = [x, 0]$ if $x \in A_1$ and $u(x) = [x, 1]$ if $x \in A_2$. This map exhibits $A_1 \cup_{A_3} A_2 \to X$ as a retract, as required. $\square$

We do not know if the Lillig condition holds more generally; the proof we have given (which is a re-working of Lillig's original proof) uses crucially the characterization of $\bar{f}$-cofibrations in terms of fiberwise NDR pairs. We note that the result is false in general if $\bar{f}$-cofibrations are replaced by f-cofibrations.

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