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# Local structure of topological charge fluctuations in QCD

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We consider the lattice topological charge density introduced by Hasenfratz, Laliena, and Niedermayer and propose its eigenmode expansion as a tool to investigate the structure of topological charge fluctuations in QCD. The resulting effective density is built from local chiralities studied previously. At every order of the expansion, the density exactly sums up to the global topological charge, and the leading term describes the maximally smooth space-time distribution of charge relevant for propagating light fermions. We use this framework to demonstrate our previous suggestion that the bulk of the topological charge in QCD does not effectively appear in the form of quantized unit lumps. Our conclusion implies that it is unlikely that the mixing of "would-be" zero modes associated with such lumps is the prevalent microscopic mechanism for spontaneous chiral symmetry breaking in QCD. We also present the first results quantitatively characterizing the space-time behavior of effective densities. For coherent fluctuations contained in spherical regions, we find a continuous distribution of associated charges essentially ending at $\approx 0.5$.

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Some intriguing effects in QCD, such as the large $\eta'$ mass and $\theta$ dependence, are related to vacuum fluctuations of topological charge [1]. Understanding the *local* structure of topological charge fluctuations is thus of great interest for building a detailed picture of how these phenomena arise in terms of fundamental degrees of freedom. Another important phenomenon possibly related to topological charge fluctuations is spontaneous chiral symmetry breaking (S$\chi$SB). This is based on the proposition that the effective low-energy structure of topological charge fluctuations in QCD is such that in a typical configuration *most* of the topological charge is concentrated in $N_L$ typically nonoverlapping, four-dimensional space-time regions $\mathcal{L}_i$, each containing a sign-coherent lump of approximately unit topological charge (i.e., the generalization of the instanton liquid picture). *If true*, this would imply an appealing microscopic explanation for the origin of Dirac near-zero modes [2] and hence the origin of S$\chi$SB [3]. The connection to fermions arises by associating with each lump $\mathcal{L}_i$ a localized chiral mode $\chi^i$ which "would be" a zero mode in the absence of other lumps, and a formation of a topological subspace of low-lying eigenmodes $\psi^i \approx \sum_j a_{ij}\chi^j, i=1,\ldots,N_L$. Verifying this "topological mixing" scenario has long been hindered by inherent difficulties in interpreting the *local* behavior of topological charge density for typical configurations of lattice-regularized theory. We have argued [4,5] that meaningful information can be extracted indirectly by studying the local chirality of low-lying modes. The flat behavior of the associated $X$ distribution observed in the initial study suggested that topological mixing was not the origin of the near-zero modes, and also indicated the lack of extended self-dual excitations [4]. However, later studies revealed the double-peaked behavior [6] which is qualitatively *consistent* with both these aspects, but *not sufficient* for their demonstration [5]. Being agnostic about self-duality, we observed in Ref. [5] that even with the double-peak structure present, the patterns in low eigenmodes of the overlap operator are inconsistent with the presence of a distinctive topological subspace, and hence with the topological mixing scenario. In this work, we propose a framework generalizing the local chirality method, and use it to demonstrate this conclusion *directly*. The basis of our approach is the topological charge density $q_x = \frac{1}{2}\mathrm{tr}\gamma_5 D_{x,x}$ [7] associated with $\gamma_5$-Hermitian $D$ satisfying $\{D,\gamma_5\} = D\gamma_5 D$ [Ginsparg-Wilson (GW) fermions]. Note that since $\mathrm{tr}\gamma_5 = 0$, one can also write $q_x = -\mathrm{tr}\gamma_5(C - \frac{1}{2}D_{x,x})$ with arbitrary constant $C$. We choose $C=1$ (see also Ref. [8]), which guarantees that the eigenmode expansion of $q_x$ is properly normalized and satisfies the index theorem for arbitrary truncation (see below). In this case we have

$$q_x = -\mathrm{tr}\gamma_5(1 - \tfrac{1}{2}D_{x,x}) = -\sum_{\lambda}\left(1 - \frac{\lambda}{2}\right)c_x^{\lambda}, \qquad (1)$$

where $c_x^{\lambda} = \psi_x^{\lambda +}\gamma_5\psi_x^{\lambda}$ is the local chirality of the mode with eigenvalue $\lambda$. For low-lying truncation ($|\lambda| \approx 0$) this is effectively the sum of individual local chiralities. Very recently, Gattringer offered evidence for a large degree of self-duality in the vicinity of peaks of an eigenmode-filtered action density [9]. If the extended nature of self-dual regions is confirmed, then combined with our conclusions one is led to a picture of inhomogeneous, but essentially continuous topological charge fluctuations at low energy, with a significant level of self-duality in the vicinity of local maxima. The origin of the double-peaked behavior of local chirality would then be the *local* attraction of spinorial components by an (anti-)self-dual field, as argued in [4], with topological mixing not playing a role. In the last part of this work, we describe the first attempt to characterize the typical amounts of topological charge associated with coherent fluctuations, as well as some interesting quantitative results that emerged.

 

(i) In the usual discussion of topological mixing, there is a paradox rooted in the negativity of the Euclidean topological charge-density correlator at nonzero distance,

$$\langle q(x)q(0)\rangle \leqslant 0, \quad |x|>0. \tag{2}$$

This follows from reflection positivity and the fact that the operator $q(x)$ is reflection-odd [10,11]. One consequence is that it is impossible that in typical configurations *most* of topological charge is concentrated in coherent four-dimensional lumps of finite physical size (e.g., instantons). Indeed, if this were the case and the typical size of such lumps was $r_c$, then the average correlator would be positive over $r_c$, thus contradicting (2). This means that lumps $\mathcal{L}_i$ cannot be identified in $q(x)$.

These considerations do not rule out topological mixing as an effective low-energy scenario for S$\chi$SB, but emphasize the necessity of some short-distance filtering when studying these issues. This can be achieved by replacing the local operator $q(x)$ by a *nonlocal* one. While the use of nonlocal operators is generally unacceptable, here they will only serve as a filter to smear the singular contact term in Eq. (2) into a finite positive core. The dominance of coherent structures is then possible. Following up on our reasoning in Refs. [4,5], we propose that the low-energy truncation of the Dirac eigenmode expansion for $q(x)$ (see below) can serve as the physically motivated nonlocal operator needed. We stress the physical motivation because, for the problem at hand, the underlying issue is whether the light fermion effectively feels a collection of coherent unit lumps as it moves through the vacuum, or whether there is some other dynamics governing its propagation. This is to be decided by the fermion.

The desired fermion filtering can be realized starting from lattice-regularized theory. Among the lattice topological charge-density operators considered, the ones associated with GW fermionic kernels are perhaps theoretically most appealing [7]. These are constrained so that they sum up to integer global charge defined by counting the exact zero modes of the GW kernel used. The continuum gauge-fermion correspondence (index theorem) for smooth gauge fields is thus extended to the lattice by construction [7]. Before defining the filtered densities starting from Eq. (1), it is useful to recall that the spectrum of $D$ contains $N_0$ zero modes with global chirality $+1$ ($N_0^+$) or $-1$ ($N_0^-$), $N_2$ modes at $\lambda=2$ with global chirality $+1$ ($N_2^+$) or $-1$ ($N_2^-$), and $N_p$ pairs of complex modes with zero global chirality. The dimension of Dirac space is $N=N_0+N_2+2*N_p$, and the topological charge is $Q=\Sigma_x q_x = N_0^- - N_0^+ = N_2^+ - N_2^-$. We now associate with $q_x$ the set of related densities

$$q_x^{(k)} = -\sum_{i=1}^{N_0} c_x^{0,i} - \sum_{j=1}^{k} (2-\mathrm{Re}\lambda_j) c_x^{\lambda_j} \tag{3}$$

representing the truncated eigenmode sum that includes all zero modes and the $k$ lowest-lying complex pairs. Complex eigenvalues $\lambda_j$ enter the sum ordered on the upper (lower) half of the spectral circle. Eigenmode-filtered densities $q^{(k)}$ are real and $\Sigma_x q_x^{(k)} = Q = N_0^- - N_0^+$, thus leading to identical global fluctuations satisfying the index theorem. We have $q_x^{(N_p)} = q_x$, and since $q^{(0)}$ can be identically zero for $Q=0$ configurations, we consider $q^{(1)}$ to be the leading order in the expansion. The infrared eigenmodes of $D$ are significantly smoother than the underlying gauge field [5], and $q^{(1)}$ is expected to be maximally smooth. As $k$ increases, $q^{(k)}$ gradually incorporates more short-distance structure. This does not mean that there is a strict new cutoff present in filtered densities. However, ultraviolet fluctuations irrelevant for propagation of light quarks are filtered out.

TABLE I. Ensembles of Wilson gauge configurations.

| Ensemble | $\beta$ | $a$ (fm) | $V$ | Configs. | Eigenpairs |
|---|---|---|---|---|---|
| $\mathcal{E}_1$ | 6.00 | 0.093 | $14^4$ | 12 | 2 |
| $\mathcal{E}_2$ | 6.20 | 0.068 | $20^4$ | 8 | 2 |
| $\mathcal{E}_3$ | 6.55 | 0.042 | $32^4$ | 5 | 2 |
| $\mathcal{E}_4$ | 5.91 | 0.110 | $12^4$ | 6 | 9 |
| $\mathcal{E}_5$ | 6.20 | 0.068 | $20^4$ | 3 | 10 |

We now use the overlap Dirac operator [12] (see details in Ref. [5]) on Wilson gauge backgrounds (see Table I) to demonstrate the basic properties of filtered densities discussed above. We have calculated the full density $q$ for configuration $\mathcal{C}_2$ from ensemble $\mathcal{E}_4$. In Fig. 1, we show its correlator $C_q(r)$ (normalized at the origin) and compare it to $C_{q^{(1)}}$ and $C_{q^{(16)}}$ for filtered densities. Note that $C_q$ has a very short-ranged positive core as expected from Eq. (2). This provides indirect evidence that the locality of the nonultralocal operator $q$ is quite good. As for the correlators $C_{q^{(k)}}$, their range is evidently larger. The shape stabilizes at about $k=6$ and changes very slowly from then on. To further characterize the roughness of filtered densities, we calculate

$$(G^{(k)})^2 \equiv \sum_{x,\mu} (q_{x+\mu}^{(k)} - q_x^{(k)})^2 \tag{4}$$

and plot $G^{(k)}$ as a function of $k$. As expected, $q^{(k)}$ becomes rougher as $k$ increases. While the *physical size* of the positive core in $C_q$ is expected to go to zero in the continuum limit, this is not necessarily so for $C_{q^{(k)}}$. To see that, we have calculated $C_{q^{(2)}}$ for ensembles $\mathcal{E}_1$–$\mathcal{E}_3$. The size of the positive core in the average correlator was determined as $\langle r\rangle$ over the probability distribution given by $C_{q^{(2)}}(r)$ in the range from zero up to the maximal distance where $C_{q^{(2)}}(r)$ is manifestly positive (with errors taken into account). As can be seen in Fig. 1, the size scales well, indicating that if the dominance of coherent four-dimensional structures in $q^{(2)}$ can be established, these structures can survive the continuum limit.

(ii) Introduction of fermion-filtered densities allows us to study the relevance of topological mixing *consistently*. If the topological subspace of low-lying modes $\psi^i \approx \Sigma_j a_{ij}\chi^j$, $i=1,\ldots,N_L$ is formed, then

$$\sum_{x\in\mathcal{L}_i}\sum_{j=1}^{N_L} \psi_x^{j+}\gamma_5\psi_x^j \approx s_i, \quad \gamma_5\chi^i = s_i\chi^i, \quad s_i=\pm 1. \tag{5}$$

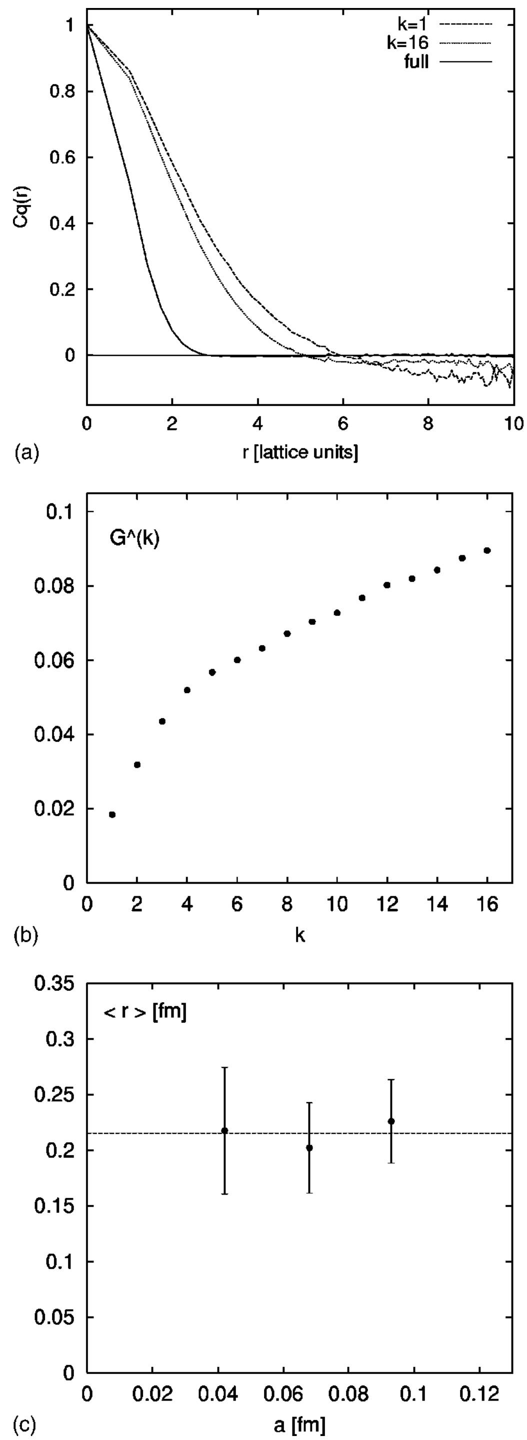


FIG. 1. (a) $C_q$, $C_{q^{(1)}}$, and $C_{q^{(16)}}$ for configuration $\mathcal{C}_2$ $(Q=0)$ from $\mathcal{E}_4$. (b) Roughness $G^{(k)}$ of $q^{(k)}$ for the same configuration. For the full density, $G^{(N_p)}=0.87$. (c) Size of the positive core of the average $C_{q^{(2)}}$ correlator for ensembles $\mathcal{E}_1$–$\mathcal{E}_3$. The Sommer parameter was used to set the scale.

This implies that the structure of quantized unit lumps $\mathcal{L}_i$ will be revealed in $q^{(k)}$ when all modes belonging to the topological subspace are included. Indeed, from Eq. (5) we have

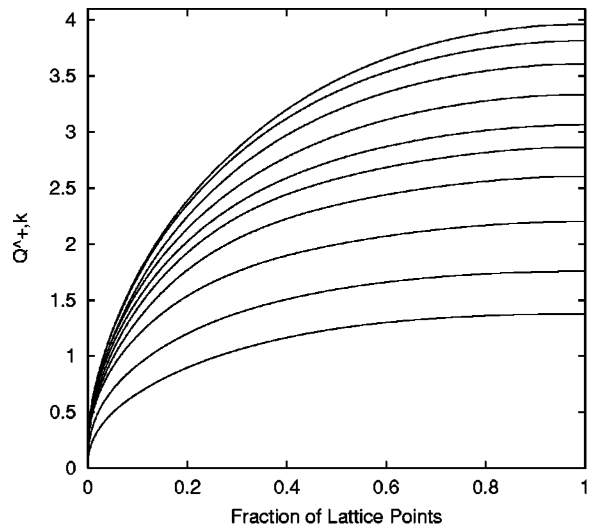


FIG. 2. The function $Q^{+,k}(f)$ for configuration $\mathcal{C}_2$ $(Q=1)$ of ensemble $\mathcal{E}_5$. The lowest curve corresponds to $k=1$.

$$\sum_{x\in\mathcal{L}_i} q_x^{(k)} \equiv Q_i \approx \pm 1, \quad 2k \approx N_L - N_0 \tag{6}$$

since Re $\lambda \approx 0$ for modes in topological subspace. Moreover, the value of pure gauge topological susceptibility $(\approx 1\,\mathrm{fm}^{-4})$ constrains $N_L$ in lump-dominated configurations of volume V fm$^4$ to be $N_L \approx V$. This implies that the lumpy structure should typically be saturated in $q^{(k)}$ with $k \lesssim V/2$. Consequently, $q^{(2)}$ is expected to be well sufficient for all ensembles in Table I.

We now ask whether the subset $\cup_i \mathcal{L}_i$ of the lattice containing most of the topological charge can be identified using $q^{(k)}$. A simple way to proceed is to order lattice points by the magnitude of $q_x^{(k)}$ and compute the running sum of positive $[Q^{+,k}(f)]$ and negative $[Q^{-,k}(f)]$ charge as the fraction $f$ of the highest points included increases. Functions $Q^{+,k}$ $(Q^{-,k})$ should stabilize to a constant ($\approx$ integer-valued) plateau at a well-defined value of $f$ corresponding to the fraction of volume occupied by $\cup_i \mathcal{L}_i$. In Fig. 2, we show the behavior of $Q^{+,k}$ for configuration $\mathcal{C}_2$ $(Q=1)$ from $\mathcal{E}_5$. We find no sign of plateaus for any $k$, and nothing special happens around $k=2$. Such behavior is characteristic for *all* configurations from the ensembles in Table I. The same conclusion applies to functions $Q^k(f)$ monitoring total charge. The smooth monotonic behavior exhibited in Fig. 2 excludes the possibility that the bulk of the topological charge is effectively concentrated in a small subvolume $\cup_i \mathcal{L}_i$ of typically isolated lumps.

For another quantitative test, consider a configuration with topological charge $Q$ and *assume* it is dominated by $|Q|+\xi$ unit lumps (antilumps) and $\xi$ antilumps (lumps). If the number of zero modes is minimal, i.e., $|Q|=N_0$ (true for all our configurations), then the dimension of the topological subspace is $N_L=|Q|+2\xi$ and we should have $\Sigma_x |q_x^{(\xi)}| \equiv Q^{\xi,\mathrm{abs}} \approx N_L = |Q|+2\xi$. In fact, the quantity $|Q|+2k - Q^{k,\mathrm{abs}}$ would be close to zero for all $k \leqslant \xi$ (since each mode

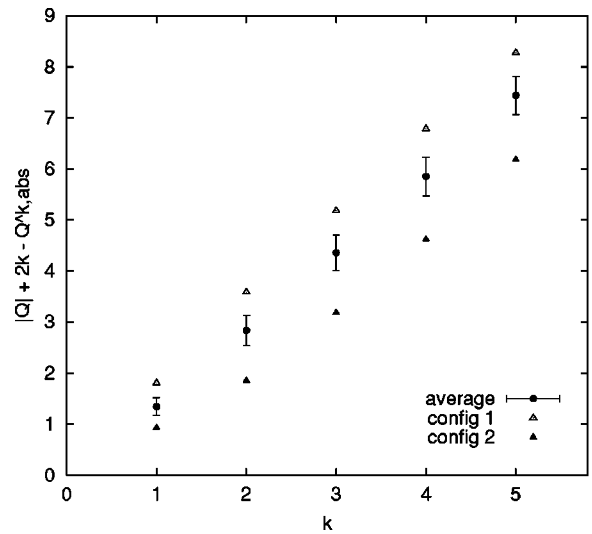


FIG. 3. $|Q|+2k-Q^{k,\text{abs}}$ for ensemble $\mathcal{E}_4$.

in the topological subspace contributes approximately unity to $Q^{k,\text{abs}}$), and it would start to increase rapidly for $k>\xi$. In a theory where the topological mixing scenario is relevant, monitoring the $k$ dependence of $|Q|+2k-Q^{k,\text{abs}}$ could thus serve as a procedure for determining the dimension of the topological subspace for a given configuration. We have computed this $k$ dependence for all of our configurations. The minimal value is always achieved at $k=1$ and is of order 1 rather than close to zero. Moreover, in ensembles $\mathcal{E}_4,\mathcal{E}_5$, where a wide range of $k$ is available, we observe a monotonic (linear) increase for every configuration. This robust behavior is illustrated in Fig. 3, confirming again that there are no signs of a distinctive topological subspace and hence no signs of dominance by unit lumps.

(iii) The above arguments do not depend on specific properties of $\mathcal{L}_i$, such as their shape, volume, or particular field content. The results show that the topological charge is effectively carried by the bulk of the lattice, which is inconsistent with dominance by unit-quantized lumps. At the same time, we find inhomogeneous behavior with noticeable peaks in $q^{(k)}$, accompanied by visible coherence (see $C_{q^{(k)}}$ in Fig. 1). As a first step toward understanding this structure, we now provide a simple characteristic of the typical values of charge associated with such coherent fluctuations. Given an arbitrary density $q$, we consider the sets $\mathcal{I}^l$ of *centers* of coherent behavior with lattice resolution $\sqrt{l}$, namely

$$\mathcal{I}^l \equiv \{x : |q_y| < |q_x|, q_x q_y > 0 \, \forall y; 0 < |x-y|^2 \leqslant l\}.$$

The elements of $\mathcal{I}^l$ are local maxima of $|q|$ over distance $\sqrt{l}$, for which the sign of $q$ is coherent over at least the same distance. Obviously, $\mathcal{I}^1 \supset \mathcal{I}^2 \supset \mathcal{I}^3 \supset \cdots$. To every $x \in \mathcal{I}^l$ we assign a radius $R_x$ defined as the maximal distance from $x$ over which the density is still coherent, i.e., $R_x \geqslant \sqrt{l}$. The charge $Q_x$ is also assigned by summing the density over the sphere of radius $R_x$ centered at $x$. However, to be able to interpret $Q_x$ as a charge corresponding to an individual fluctuation, there should typically be no other centers within $R_x$.

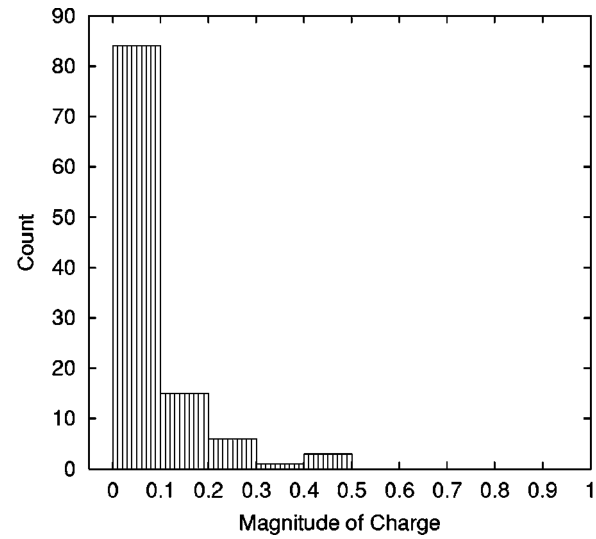


FIG. 4. Distribution of charges in coherent fluctuations for ensemble $\mathcal{E}_4$.

This allows for fixing the resolution in a self-consistent way by choosing the smallest $l$ with this condition satisfied.

To carry out the above procedure meaningfully for $q^{(k)}$, one should work in the range of $k$ where coherent fluctuations and their charges are relatively stable with respect to a change of $k$. For our ensembles, this is best satisfied for $\mathcal{E}_4$, where $q^{(9)}$ is available. In physical terms, this corresponds to including eigenmodes with imaginary parts up to $\approx 500$ MeV. The fraction of overlapping centers drops dramatically (to 5%) at $l=3$, which is the value we have used. The corresponding distribution of charges for nonoverlapping centers is shown in Fig. 4. A property which is insensitive to the choice of $k$ and $l$ is that the distribution effectively ends at about 0.5. This appears to hold also in the case of full density $q_x$ (with only two configurations available). Interestingly, this behavior might be compatible with the presence of center vortices in the QCD vacuum as the recent discussion of topology in the field of an idealized vortex suggests [13] (the possible manifestation of center vortices in topological charge fluctuations was also recently discussed in Ref. [14]).

To summarize, we have proposed that the low-energy behavior of the topological charge density can be studied by using a suitable nonlocal realization of $q(x)$ as a filter for short-distance fluctuations. If one is interested in the aspects of topological charge affecting the low-momentum propagation of light quarks, then the appropriate nonlocal realization is naturally available through the low-eigenmode expansion of $q(x)$. Such fermion filtering can be explicitly realized starting from lattice-regularized theory. We have proposed the expansion of $q_x$ associated with GW fermions as an ideal tool for this purpose. This provided us with a consistent framework to test whether the propagation of light fermion is *effectively* driven by the dominance of unit topological lumps in the QCD vacuum. We find that this is not the case, implying that the picture of S$\chi$SB based on the mixing of corresponding topological "would-be" zero modes is not accurate. We emphasize that we have not ruled out the logical possibility that some unit-quantized structures with well-defined boundaries can occur. However, we have shown that the *bulk* of the topological charge does not come in that form. A first attempt to characterize the inhomogeneous nature of topological charge fluctuations in filtered densities resulted in very interesting results indicating that patterns of local behavior can provide us with detailed information about dynamically important structures in the QCD vacuum.

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