GROUP DUALITIES, T-DUALITIES, AND TWISTED K-THEORY

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Abstract. This paper explores further the connection between Langlands duality and T-duality for compact simple Lie groups, which appeared in work of Daenzer-Van Erp and Bunke-Nikolaus. We show that Langlands duality gives rise to isomorphisms of twisted K-groups, but that these K-groups are trivial except in the simplest case of SU(2) and SO(3). Along the way we compute explicitly the map on H^3 induced by a covering of compact simple Lie groups, which is either 1 or 2 depending in a complicated way on the type of the groups involved. We also give a new method for computing twisted K-theory using the Segal spectral sequence, which is better adapted to many problems involving compact groups. Finally we study a duality for orientifolds based on complex Lie groups with an involution.

1. Introduction

This paper was motivated by a number of rather diverse sources, especially the very intriguing papers [20, 18], which point out an interesting connection between Langlands duality (which appears in representation theory and number theory) and T-duality (a relationship between string theories on different spacetime manifolds, especially when these are torus bundles over a common base). A second source comes from the extensive literature on WZW (Wess-Zumino-Witten) models in physics, which appear both as conformal field theories and as string theories (sigma models, to be precise) whose underlying spacetime manifold is a Lie group, usually compact and simple. In string theories in general, D-brane charges are expected to take their values in twisted K-theory of spacetime, so the study of WZW models led to the study of twisted K-theory of compact Lie groups, first by physicists (e.g., [53, 48, 26, 12, 14, 15, 32]) and later by mathematicians (Hopkins, unpublished, but quoted in [48], and Douglas [23]). The third motivation comes from the problem of trying to understand exactly what one should expect from a "duality theory" for

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WZW models. All three of these topics come together in the problems discussed in this paper.

Section 2 deals with the connection between Langlands duality and T-duality. The papers [20, 18] show that on any compact simple Lie group G not of type B_n or C_n $(n \geq 3)$[1], there is a class $h \in H^3(G)$ for which the "T-dual" of the torus bundle $G \to G/T$ is the Langlands dual $G^\vee$ with a dual class $h^\vee \in H^3(G^\vee)$. However, these papers give a description of h which is rather indirect, and which doesn't make it completely clear how it relates to the generator of $H^3(G)$. (The group $H^3(G)$ is infinite cyclic except for the exceptional case of $\mathrm{PSO}(2n)$ for n divisible by 2; even in this case, $H^3(G)$ has an infinite cyclic summand which is almost unique.) Secondly, we realized that there doesn't seem to be an explicit description in the literature of the pull-back map from $H^3(G)$ to $H^3(\widetilde{G})$, $\widetilde{G}$ the universal cover of G, so we provide this in Section 2.1, Theorems 1 and 3. The results are surprisingly intricate. The following Sections 2.2 and 2.3 deal with making the choice of h and $h^\vee$ more precise, to the point where the twisted K-theory $K^\bullet(G, h)$[2] can be computed explicitly in Theorem 11. Section 2.4 then revisits the topic of computing twisted K-theory $K^\bullet(G, h)$, for arbitrary choices of h. This has been the subject of an extensive literature, most notably [48, 53, 14, 23], and the results are rather complicated and hard to understand. However, this is an important problem because of the connection, discovered by physicists, between these twisted K-groups and fusion rings and representations of loop groups. We therefore present in Section 2.4 an easier way of computing these twisted K-groups in some cases. Theorems 14, 16, and 17 recover some of the results of [14, 23] via much more elementary methods, and Theorems 18 and 20 illustrate how the same methods can be used in the case of non-simply connected Lie groups, where the literature is still very incomplete. Section 2.5 applies this to level-rank duality. Finally, Section 3 and Theorem 22 deal with a duality theory for orientifolds based on complex Lie groups with a group involution (either holomorphic or anti-holomorphic).

2. Langlands duality and the twisted K-theory of simple compact Lie groups

2.1. The canonical class in H^3.

In this subsection, we consider what at first sight appears to be a rather trivial and also somewhat esoteric question. Suppose G is a connected simple compact Lie group, not isomorphic to $\mathrm{PSO}(2n)$ with n even, with universal cover $\widetilde{G}$. Since it is a classical fact that $\widetilde{G}$ is 2-connected, with $\pi_3(G) = \pi_3(\widetilde{G}) \cong \mathbb{Z}$, it follows that $H^3(G)$ and $H^3(\widetilde{G})$ are both infinite cyclic, i.e., isomorphic to $\mathbb{Z}$.[3] Let $\pi\colon \widetilde{G} \to G$ be the covering map; its degree d divides the

[1] These are the only cases where the flag manifolds for G and its Langlands dual are different.

[2] This notation will always mean complex topological K-theory twisted by the class $h \in H^3$.

[3] Throughout the paper, homology and cohomology groups will be taken with integer coefficients unless specified otherwise. The statement about $H^3(G)$ follows from considering the Serre spectral sequence for the fibration $\widetilde{G} \to G \to K(\pi_1(G), 1)$ and using the fact that $\pi_1(G)$ is cyclic,

determinant of the Cartan matrix of G, and is equal to it in case $G = \widetilde{G}/Z(\widetilde{G})$ is the adjoint group for its Lie algebra. Thus if G is of adjoint type, d is $n+1$ for type A_n ($\widetilde{G} = \mathrm{SU}(n+1)$), 2 for type C_n ($\widetilde{G} = \mathrm{Sp}(n)$), 2 for type B_n ($G = \mathrm{SO}(2n+1)$), 4 for type D_n ($G = \mathrm{PSO}(2n)$, $n \geq 3$), 1 for types G_2, F_4, and E_8, 3 for type E_6 and 2 for type E_7.

The fact that $H^3(G)$ and $H^3(\widetilde{G})$ are both infinite cyclic means that $\pi^*\colon H^3(G) \to H^3(\widetilde{G})$ sends a generator of $H^3(G)$ to some positive multiple of a generator of $H^3(\widetilde{G})$. What multiple is this? The following theorem largely answers this question. Note incidentally that this question was taken up in [24, Appendix 1], where it was pointed out that there is a natural exact sequence

$$0 \to H_3(\widetilde{G}) \xrightarrow{\pi_*} H_3(G) \to \pi_1(G) \to 0.$$

However, this doesn't completely settle things since there is an extension problem — it is not clear when this exact sequence splits, or when it "partially splits."

Theorem 1. *Let G be a connected simple compact Lie group of rank n, with universal cover $\widetilde{G}$, with G not isomorphic to* $\mathrm{PSO}(2n)$ *with n even, and let d be the degree of the covering $\pi\colon \widetilde{G} \to G$. Think of $\pi^*\colon H^3(G) \to H^3(\widetilde{G})$ as a map $\mathbb{Z} \to \mathbb{Z}$.*

1. *If G is of type A_n and if $n+1 = 2q$ with q odd, then π^* is multiplication by* 2 *if d is even and the identity if d is odd. If $n+1$ is either odd or divisible by* 4*, then π^* is the identity.*
2. *If G is of type C_n and π is not an isomorphism, i.e., $\widetilde{G} = \mathrm{Sp}(n)$ and $G = \mathrm{PSp}(n)$, then π^* is the identity if n is even and is multiplication by* 2 *if n is odd.*
3. *If $G = \mathrm{SO}(2n)$ and $\widetilde{G} = \mathrm{Spin}(2n)$ with $n \geq 3$, or if $G = \mathrm{SO}(2n+1)$ and $\widetilde{G} = \mathrm{Spin}(2n+1)$ with $n \geq 2$, or if $G = \mathrm{PSO}(2n)$ and $\widetilde{G} = \mathrm{Spin}(2n)$ with $n \geq 3$ odd (so G is of type D_n or B_n, though we are excluding one other possibility for G in type D_n when n is even), then π^* is the identity.*
4. *If G is the adjoint group of E_6, then π^* is the identity, but if G is the adjoint group of E_7, then π^* is multiplication by $d = 2$.*

Proof. First recall that (by [1, Ch. 4], for example) there is a transfer map $\pi_*\colon H^3(\widetilde{G}) \to H^3(G)$ such that $\pi_* \circ \pi^*$ is multiplication by d. So if d is prime, that means there are only two possibilities: either π^* is multiplication by d or else it is the identity.

(1) One case is obvious — if G has rank 1, so that $\widetilde{G} \cong \mathrm{SU}(2) \cong \mathrm{Sp}(1) \cong \mathrm{Spin}(3)$, then the only nontrivial covering map is $\pi\colon \mathrm{SU}(2) \to \mathrm{SO}(3)$, and this can be identified with the 2-to-1 covering map $S^3 \to \mathbb{RP}^3$. But a covering map of

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so that we know its integral cohomology. There is one exceptional case: if $G = \mathrm{PSO}(2n)$ with n even, then $\pi_1(G) \cong (\mathbb{Z}/2)\times(\mathbb{Z}/2)$ and the same spectral sequence shows that $H^3(G) \cong \mathbb{Z}\oplus(\mathbb{Z}/2)$. The torsion generator of $H^3(G)$ is killed on passage to the double cover $\mathrm{SO}(2n)$.

connected compact oriented k-manifolds induces multiplication by the degree of the covering on H^k, so in this case π^* sends the generator of $H^3(\mathbb{RP}^3)$ to *twice* a generator of $H^3(S^3)$.

For the remaining cases, observe that the $(n+1)$-fold covering map $\mathrm{SU}(n+1) \to \mathrm{PSU}(n+1)$ factors through G. So when $\pi^* = 1$ for the adjoint group $\mathrm{PSU}(n+1)$, then the same will be true for G as well. Next suppose that $n+1 = 2q$ with q odd and we know that $\pi^* = 2$ when $G = \mathrm{PSU}(n+1)$. Of the two coverings $\mathrm{SU}(n+1) \to G$ and $G \to \mathrm{PSU}(n+1)$, exactly one is of even degree, and the π^* maps for these two coverings must multiply together to give 2. Since π^* for any covering divides the order of the covering, the only possibility is that π^* is 2 for the even covering and 1 for the odd covering. So the result is reduced to the case $G = \mathrm{PSU}(n+1)$.

Now we apply a result of Baum and Browder [6, Corollary 4.4]: if $f\colon\ \mathrm{U}(n+1) \to \mathrm{PSU}(n+1)$ is the quotient map (recall that $\mathrm{PSU}(n+1) = \mathrm{U}(n+1)/Z(\mathrm{U}(n+1))$), then f^*, the induced map on cohomology with $\mathbb{F}_p$ coefficients, for any prime p dividing $n+1$, has kernel equal to the ideal generated by the generator y of $H^2(\mathrm{PSU}(n+1), \mathbb{F}_p)$. Also note that the diagram

$$\begin{tikzcd} \mathrm{SU}(n+1) \arrow[r, "\iota"] \arrow[dr, "\pi"] & \mathrm{U}(n+1) \arrow[d, "f"] \\ & \mathrm{PSU}(n+1) \end{tikzcd}$$

commutes, and since $\iota^*\colon\ H^3(\mathrm{U}(n+1)) \to H^3(\mathrm{SU}(n+1))$ is an isomorphism, this identifies the kernel of $\pi^*\colon\ H^3(\mathrm{PSU}(n+1), \mathbb{F}_p) \to H^3(\mathrm{SU}(n+1), \mathbb{F}_p)$ as well.

We recall from [10, Théorème 11.4] and [6, Corollary 4.4] that if $n+1 = p^r m$, with p prime and $\gcd(m,p) = 1$, then

$$H^\bullet(\mathrm{PSU}(n+1), \mathbb{F}_p) \cong \mathbb{F}_p[y]/(y^{p^r}) \otimes \bigwedge(x_1, x_3, \cdots, \widehat{x_{2p^r-1}}, \cdots, x_{2n+1}), \tag{1}$$

with $\beta x_1 = y$ (β the Bockstein) and with the x_j's except for x_1 all reductions of integral classes. If $p = 2$ and $r = 1$, one has the additional relation $x_1^2 = y$. So, in all cases,

$$H^3(\mathrm{PSU}(n+1), \mathbb{F}_p) = \mathbb{F}_p\, x_1 y + \mathbb{F}_p\, x_3 \text{ (if present)}.$$

We have $\beta(x_3) = 0$ and $\beta(x_1 y) = y^2$.

There are a few cases to consider. If p is odd, then $p^r \geq 3$ and $2p^r - 1 > 3$. So x_3 is not omitted, $y^2 \neq 0$, and $x_1 y$ is not the reduction of an integral class. Hence no integral class in $H^3(\mathrm{PSU}(n+1))$ reduces to something nonzero in the kernel of f^* on mod p cohomology. Hence $\pi^*\colon\ H^3(\mathrm{PSU}(n+1)) \to H^3(\mathrm{SU}(n+1))$ is never divisible by an odd prime.

Next, suppose $n+1 = 2q$ with q odd. If we choose $p = 2$, then $r = 1$, so x_3 is missing in (1), and $y^2 = 0$. So $x_1 y$ is the reduction of the generator of $H^3(\mathrm{PSU}(n+1))$ in this case, and it lies in the kernel of f^*. Hence π^* is divisible by 2. Since we have already shown it is not divisible by any odd prime, it is exactly 2.

Finally, consider the case $p = 2$ when $n+1$ is divisible by 4. In this case, $p^r \geq 4$ so, again, $2p^r - 1 > 3$. So x_3 is present in (1), and $y^2 \neq 0$, so that $x_1 y$ is not the reduction of an integral class. Hence the kernel of f^* does not contain the nonzero reduction of any integral cohomology class, and π^* cannot be divisible by 2. Since it is also not divisible by any odd prime, $p^* = 1$ in this case.

Note that since $A_3 = D_3$, $\mathrm{PSU}(4) = \mathrm{PSO}(6)$. So $\pi^* = 1$ in the case $\widetilde{G} = \mathrm{Spin}(6)$, $G = \mathrm{PSO}(6)$.

(2) The case $G = \mathrm{PSp}(n)$, $\widetilde{G} = \mathrm{Sp}(n)$, $n \geq 2$, is handled using [10, Théorème 11.3], which gives

$$H^\bullet(\mathrm{PSp}(n), \mathbb{F}_2) \cong \mathbb{F}_2[a]/(a^{4s}) \otimes \bigwedge(x_3, \cdots, \widehat{x_{4s-1}}, \cdots, x_{4n-1}), \tag{2}$$

with a of degree 1 and s the largest power of 2 dividing n. Note that $4s = 4$ if n is odd and $4s \geq 8$ if n is even. The Bockstein $\beta = \mathrm{Sq}^1$ sends a to a^2, and all torsion is of order 2, so the integral cohomology is

$$H^\bullet(\mathrm{PSp}(n), \mathbb{Z}) \cong \mathbb{Z}[a^2]/(2a^2, a^{4s}) \otimes \bigwedge(x_3, \cdots, x_{4n-1}), \tag{3}$$

with x_{4s-1} reducing mod 2 to a^{4s-1}. Consider the commuting diagram of fibrations

$$\begin{tikzcd} \mathrm{Sp}(n-1) \arrow[r] \arrow[d, equal] & \mathrm{Sp}(n) \arrow[r] \arrow[d, "\pi"] & S^{4n-1} \arrow[d] \\ \mathrm{Sp}(n-1) \arrow[r] & \mathrm{PSp}(n) \arrow[r] & \mathbb{RP}^{4n-1}. \end{tikzcd} \tag{4}$$

From (3), $H^4(\mathrm{PSp}(n))$ vanishes if and only if $s = 1$, i.e., if and only if n is odd. However, in the spectral sequence for the bottom row of (4), $E_2^{4,0} = \mathbb{F}_2\, a^4$. So $d_4(x_3)$ will be 0 if n is even and must equal a^4 if n is odd. Comparing with the spectral sequence for the top row of (4), we see that we get a diagram

$$\begin{tikzcd} H^3(\mathrm{Sp}(n-1)) \arrow[r, equal] \arrow[d, equal] & E_\infty^{0,3} & H^3(\mathrm{Sp}(n)) \arrow[l, "\cong"'] \\ H^3(\mathrm{Sp}(n-1)) \arrow[r, equal] & E_\infty^{0,3} & H^3(\mathrm{PSp}(n)) \arrow[l, "\cong"'] \arrow[u, "\pi^*"'] \end{tikzcd}$$

for n even, whereas for n odd, the image under π^* of $H^3(\mathrm{PSp}(n))$ in $H^3(\mathrm{Sp}(n-1)) \cong H^3(\mathrm{Sp}(n))$ is of index two. This is consistent, incidentally, with what happened for $\mathrm{Sp}(1) = \mathrm{SU}(2)$ and for $\mathrm{PSp}(1) = \mathrm{SO}(3)$.

(3) Now consider the orthogonal and spinor groups. Here we proceed by induction on the dimension. Recall that $\mathrm{Spin}(4)$ is not simple and splits as a product of two copies of $\mathrm{SU}(2)$, while $\mathrm{Spin}(3) = \mathrm{SU}(2)$ was already dealt with above, so we start the induction with $\mathrm{Spin}(5)$. This is the same as $\mathrm{Sp}(2)$ and $\mathrm{SO}(5)$ is the same as $\mathrm{PSp}(2)$, so we've already handled this case. We could get the case of $\mathrm{Spin}(6)$ by the induction below, but instructive to give a separate argument just

as a check. Indeed, $\mathrm{Spin}(6) = \mathrm{SU}(4)$, and its cohomology is torsion-free, though $\mathrm{SO}(6)$ is a double cover of $\mathrm{PSU}(4)$. But as in (4), we have a commutative diagram of fibrations

$$\begin{tikzcd} \mathrm{SU}(3) \arrow[r] \arrow[d, equal] & \mathrm{SU}(4) \arrow[r] \arrow[d, "\pi"] & S^7 \arrow[d] \\ \mathrm{SU}(3) \arrow[r] & \mathrm{SO}(6) \arrow[r] & \mathbb{RP}^7. \end{tikzcd}$$

(The action of $SO(6)$ on $\mathbb{RP}^7$ is by dividing out the action of $\mathrm{SU}(4)$ on S^7 by ± 1.) As before, we get a commuting diagram of Serre spectral sequences. Along the vertical axis we have the cohomology ring of $\mathrm{SU}(3)$, which is $\bigwedge(x_3, x_5)$. Along the horizontal axis we have $\bigwedge(x_7)$ in the case of $\mathrm{SU}(4)$ and $H^\bullet(\mathbb{RP}^7)$, generated by a torsion-free class x_7 with $x_7^2 = 0$ and a 2-torsion class y_2 with $y_2^4 = 0$, in the case of $\mathrm{SO}(6)$. The spectral sequence for $\mathrm{SU}(4)$ collapses, while the one for $\mathrm{SO}(6)$ has a potential differential d_4 sending the generator x_3 in $E_2^{0,3}$ to y_2^2. However, this would kill off the torsion in $H^4(\mathrm{SO}(6))$, and Borel proves in [10, Théorème 8.6] that $H^\bullet(\mathrm{SO}(n))$ has 2-torsion in degree 4 as long as $n \geq 5$. So the differential d_4 must vanish on x_3 and we conclude that π^* is an isomorphism via a diagram chase as before.

Now we're ready for the inductive step. Assume $n \geq 6$ and we already know

$$\pi^* \colon\ H^3(\mathrm{SO}(n-1)) \to H^3(\mathrm{Spin}(n-1))$$

is an isomorphism, and consider the commutative diagram of fibrations

$$\begin{tikzcd} \mathrm{Spin}(n-1) \arrow[r] \arrow[d, "\pi"] & \mathrm{Spin}(n) \arrow[r] \arrow[d, "\pi"] & S^{n-1} \arrow[d, equal] \\ \mathrm{SO}(n-1) \arrow[r] & \mathrm{SO}(n) \arrow[r] & S^{n-1}. \end{tikzcd}$$

Examination of the Serre spectral sequences shows that $H^3 = E_\infty^{0,3}$, for both $\mathrm{SO}(n)$ and $\mathrm{Spin}(n)$. By inductive hypothesis, π^* is an isomorphism on $E_2^{0,3}$, and there is no room for differentials affecting the vertical axis until we get to H^4 (since the non-zero columns are distance $n - 1 \geq 5$ apart.) So $E_2^{0,3} = E_\infty^{0,3}$ in both spectral sequences and we're done.

To get the case of $G = \mathrm{PSO}(2n)$ with n odd, recall that we already did this case when $n = 3$, since $\mathrm{PSO}(6) = \mathrm{PSU}(4)$. However, by [43, Corollary 1.8], the inclusion $\mathrm{PSO}(6) \to \mathrm{PSO}(2n)$ induces an isomorphism on cohomology in degree ≤ 3 for any n odd. Similarly for the inclusion $\mathrm{Spin}(6) \to \mathrm{Spin}(2n)$, by what we've already done. So the case of $\mathrm{PSO}(2n)$ follows from the known case of $\mathrm{PSO}(6)$.

(4) To handle the case where G is the adjoint group of E_6, since $d = 3$ in this case, we need to study the cohomology of G and $\widetilde{G}$ with $\mathbb{F}_3$ coefficients. Fortunately this is known. By [45] or by [31],

$$H^\bullet(\widetilde{G}, \mathbb{F}_3) = \mathbb{F}_3[e_8]/(e_8^3) \otimes \bigwedge(e_3, e_7, e_9, e_{11}, e_{15}, e_{17}), \tag{5}$$

where $e_7 = \mathcal{P}^1 e_3$, $e_8 = \beta e_7$, and $e_{15} = \mathcal{P}^1 e_{11}$, whereas by [42],

$$H^\bullet(G, \mathbb{F}_3) = \mathbb{F}_3[e_2, e_8]/(e_2^9, e_8^3) \otimes \bigwedge(e_1, e_3, e_7, e_9, e_{11}, e_{15}). \tag{6}$$

(One can also read off (5) and (6) from [46].) To understand what's going on here, the comments in [31] are useful. To a good degree of approximation, $\widetilde{G}$ looks like $K(\mathbb{Z}, 3)$, whose cohomology is known and agrees with (5) in low degree. Now G, even though it is not simply connected, is a simple space and so has a Postnikov tower, the bottom of which looks like a fibration

$$\begin{tikzcd} K(\mathbb{Z}, 3) \arrow[r] & X \arrow[d] \\ & K(\mathbb{Z}/3, 1). \end{tikzcd} \tag{7}$$

The k-invariant of (7) is identified with the differential d_4 of the associated Serre spectral sequence, taking the canonical class in $H^3(K(\mathbb{Z}, 3))$ to the k-invariant living in $H^4(K(\mathbb{Z}/3, 1)) = \mathbb{F}_3 e_2^2$, with notation consistent with (6). Since $e_2^2 \neq 0$ in $H^4(G, \mathbb{F}_3)$, the k-invariant must vanish, meaning that $X \simeq K(\mathbb{Z}, 3) \times K(\mathbb{Z}/3, 1)$. Indeed, the low-dimensional cohomology of G indeed looks like that of $K(\mathbb{Z}, 3) \times K(\mathbb{Z}/3, 1)$. So to compute π^*, we can replace $\widetilde{G}$ by $K(\mathbb{Z}, 3) \times E(\mathbb{Z}/3)$ and G by $K(\mathbb{Z}, 3) \times B(\mathbb{Z}/3)$ without changing the low-degree cohomology. Since H^3 lives on the $K(\mathbb{Z}, 3)$ factor, we see that π^* is the identity on cohomology with coefficients in $\mathbb{F}_3$ and thus on integral cohomology as well.

The case of the adjoint group G of E_7 is handled similarly, even though the eventual result is different. This time π is a covering of degree 2 and so we have to look at cohomology with coefficients in $\mathbb{F}_2$. For $\widetilde{G}$ this turns out to be ([40, Proposition 5.1] or [31, Proposition 2.30]):

$$H^\bullet(\widetilde{G}, \mathbb{F}_2) = \mathbb{F}_2[x_3, x_5, x_9]/(x_3^4, x_5^4, x_9^4) \otimes \bigwedge(x_{15}, x_{17}, x_{23}, x_{27}). \tag{8}$$

And for the adjoint group G we get ([40, Theorem 5.3] or [44]):

$$H^\bullet(G, \mathbb{F}_2) = \mathbb{F}_2[x_1, x_5, x_9]/(x_1^4, x_5^4, x_9^4) \otimes \bigwedge(x_6, x_{15}, x_{17}, x_{23}, x_{27}). \tag{9}$$

We again have an approximation of G by a two-stage Postnikov system

$$\begin{tikzcd} K(\mathbb{Z}, 3) \arrow[r] & X \arrow[d] \\ & K(\mathbb{Z}/2, 1). \end{tikzcd} \tag{10}$$

The k-invariant of (10) is identified with the differential d_4 of the associated Serre spectral sequence, taking the canonical class x_3 in $H^3(K(\mathbb{Z}, 3))$ to the k-invariant living in $H^4(K(\mathbb{Z}/2, 1)) = \mathbb{F}_2 x_1^4$, with notation consistent with (9). But since $x_1^4 = 0$ in $H^\bullet(G, \mathbb{F}_2)$, this time the k-invariant is non-zero, meaning that $d_4(x_3) =$

x_1^4. The edge homomorphism $H^3(X, \mathbb{F}_2) \to H^3(K(\mathbb{Z}, 3), \mathbb{F}_2)$ is therefore 0. But this map must also coincide with the map $\pi^*\colon H^3(G, \mathbb{F}_2) \to H^3(\widetilde{G}, \mathbb{F}_2)$ (since $K(\mathbb{Z}, 3)$ is a good approximation to $\widetilde{G}$ and X is a good approximation to G), so we conclude that π^* is *not* the identity on H^3 with coefficients in $\mathbb{F}_2$, hence is not an isomorphism on integral cohomology. □

Remark 2. For G of types G_2, F_4, or E_8, we automatically have $d = 1$ and there is nothing to prove. So more work remains only in certain cases in type D_n (n even). Just to fill in a little more about the case of D_n (n even), we have the following.

Theorem 3. *Let $G = \mathrm{PSO}(2n)$ with n even, $n \geq 4$, and let $\pi\colon \widetilde{G} = \mathrm{Spin}(2n) \to G$ be its universal covering map. Recall that in this case that $H^3(G) \cong \mathbb{Z} \oplus (\mathbb{Z}/2)$. Then $\pi^*\colon H^3(G) \to H^3(\widetilde{G})$ kills the torsion and on the torsion-free part is an isomorphism for n divisible by 4, multiplication by 2 for $n \equiv 2 \pmod 4$.*

Proof. Since we have already shown in Theorem 1(3) that the covering $\mathrm{Spin}(2n) \to \mathrm{SO}(2n)$ induces an isomorphism on H^3, it suffices to study the behavior of the covering map $\mathrm{SO}(2n) \to \mathrm{PSO}(2n)$ on the torsion-free part of $H^3(\mathrm{PSO}(2n))$. For this we use the fibration

$$\mathrm{SO}(2n-1) \to \mathrm{PSO}(2n) \to \mathbb{RP}^{2n-1} \tag{11}$$

together with Borel's calculation of $H^\bullet(\mathrm{PSO}(2n), \mathbb{F}_2)$ in [10, Théorème 11.5]. In low degrees, the E_2 term for the Serre spectral sequence of (11) looks like Figure 1. Now the ×'s in positions $(2, 0)$, $(0, 2)$, and $(1, 2)$ must survive to E_∞ since

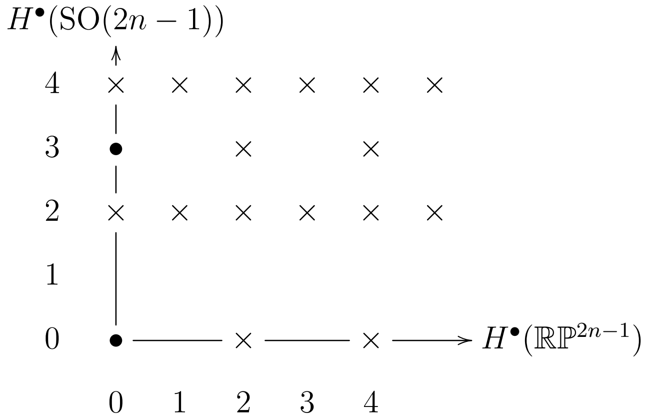

FIGURE 1. The Serre SS for integral cohomology of $\mathrm{PSO}(2n)$. The symbol • denotes a copy of $\mathbb{Z}$; × denotes a copy of $\mathbb{Z}/2$.

$\pi_1(\mathrm{SO}(2n)) \cong (\mathbb{Z}/2) \times (\mathbb{Z}/2)$ and this gives rise to two independent torsion classes in $H^2(G)$ and a torsion class in $H^3(G)$. So the only differential that can affect

$H^3(G)$ is d_4: $H^3(\mathrm{SO}(2n-1)) \to H^4(\mathbb{RP}^{2n-1})$. If this differential is 0, then the torsion-free generator of $H^3(G)$ will pull back to the generator of $H^3(\mathrm{SO}(2n-1))$, and hence will also lift under π^* to the generator of $H^3(\mathrm{SO}(2n))$. However, if this d_4 is non-zero, then the map $H^3(G) \to H^3(\mathrm{SO}(2n-1))$ will have image of index 2, and so π^* will be multiplication by 2, as well. To ascertain whether or not d_4 is non-zero, we need only compare the rank of $H^4(G)$ (which will be an elementary abelian 2-group) with the rank of E_2 in total degree 4, which is 3. If they are equal, d_4 must vanish; if they are unequal, d_4 will be nontrivial.

By [10, Théorème 11.5],

$$H^\bullet(\mathrm{PSO}(2n), \mathbb{F}_2) \cong (\mathbb{F}_2(a)/(a^s)) \otimes A, \quad A = \langle x_1, x_2, \cdots, \widehat{x_{s-1}}, \cdots, x_{2n-1} \rangle ,$$

where a is in degree 1, $s \geq 4$ is the largest power of 2 dividing $2n$, and in the algebra A, $x_1^2 = x_2$, $x_2^2 = x_4$, etc., and the monomials in the x's with no x_j repeated are a basis for A over $\mathbb{F}_2$. If $n \equiv 0 \pmod 4$, then $s \geq 8$ and $a^4 \neq 0$, so we have three independent classes in $H^4(\mathrm{PSO}(2n), \mathbb{F}_2)$ which are reductions of integral classes, namely a^4, $a^2x_2 = (ax_1)^2$, and $x_4 = x_2^2 = x_1^4$. However, if $n \equiv 2 \pmod 4$, then $s = 4$ and $a^4 = 0$, so there are only two independent classes in $H^4(\mathrm{PSO}(2n), \mathbb{F}_2)$ which are reductions of integral classes, namely $a^2x_2 = (ax_1)^2$ and $x_4 = x_2^2 = x_1^4$. That concludes the proof. □

2.2. **T-dualities from Langlands duality.** Now we return to the main subject of Section 2, the connection between Langlands duality and T-duality. We make no claims for originality here — we are just restating the results of [20] and [18] in a form that will be convenient for later calculations. Let G be a connected compact simple Lie group, and let T be a maximal torus in G. Associated to T we have the *weight lattice*, the character group $\mathrm{Hom}(T, \mathbb{T})$ of T, which can also be identified with $H^1(T)$, and the *coweight lattice* $H_1(T)$, which can be identified with the set of cocharacters $\mathrm{Hom}(\mathbb{T}, T)$. The weight lattice is also the set of allowable weights for irreducible representations of G. Inside the weight lattice is the set of *roots* $\Phi(G, T)$ (these are exactly the non-zero weights of the adjoint representation), which span a sublattice, called the *root lattice*. The weight lattice and the root lattice coincide exactly when G is an adjoint group. The triple $(H^1(T), H_1(T), \Phi(G, T))$ determines G up to isomorphism. The *Langlands dual* $G^\vee$ of G is the connected compact simple Lie group obtained by interchanging $H^1(T)$ and $H_1(T)$ and replacing the roots $\Phi(G, T)$ by the *coroots* $\Phi(G, T)^\vee$. Langlands duality performed twice brings one back to the starting point, and the duality interchanges simply connected groups and adjoint groups. The Langlands dual $G^\vee$ is locally isomorphic to G if G is of type A, D, E, F, or G, while Langlands duality interchanges types B and C. The group $\mathrm{SO}(2n)$ of type D_n is neither simply connected nor an adjoint group, and is in fact self-dual. See [20, §2] for a very clear explanation of all of this, as well as for additional references. The following is the main theorem of [20] and [18] (though the case of type $B_2 = C_2$ is never mentioned there explicitly).

Theorem 4 (Daenzer-van Erp, Bunke-Nikolaus)**.** *Let G be a connected compact simple Lie group of rank n with maximal torus T, and let $G^\vee$ be its Langlands dual. Assume G is not of type B_n or C_n with $n \geq 3$; this guarantees that G and $G^\vee$ are locally isomorphic (i.e., have isomorphic Lie algebras). Then there are canonical choices $h \in H^3(G)$ and $h^\vee \in H^3(G^\vee)$ for which (G, h) and $(G^\vee, h^\vee)$ are T-dual as principal $\mathbb{T}^n$-bundles over the flag manifold G/T, in a sense which we will make precise below.*

Remark 5. Theorem 4 actually requires some further interpretation, because the term "T-duality" has various meanings in the literature. For example the definition used in [20] is much weaker than the one used in [18], which comes from [19]. One could also work just as well with the definition in [51]. However, the definition used in [20] is not strong enough for getting the results on twisted K-theory which are our main interest here. Just to fix terminology for what follows, we can phrase the definition of T-duality as follows. A set of *T-duality data* over a a space X begins with a pair (E, h) consisting of a principal $\mathbb{T}^n$-bundle $p\colon E \to X$ and a class $h \in H^3(E)$. However, this by itself is not enough for a (classical) T-dual to be defined; one needs two additional things. First, the class h must pull back to 0 on each torus fiber. In the case where $E = G$ is a connected compact simple Lie group and $X = G/T$, this condition is automatic [18, Corollary 8.3].[4] Then to get a well-defined T-dual, one needs a choice of a lifting $X \to \widetilde{R}$ of the classifying map $X \to R$ of the pair (E, h) over X. The classifying spaces R and $\widetilde{R}$ are discussed in [51, §5]. The space $\widetilde{R}$ is a two-stage Postnikov system

$$\begin{tikzcd} K(\mathbb{Z},3) \arrow[r] & \widetilde{R} \arrow[d] \\ & K(\mathbb{Z}^n,2) \times K(\mathbb{Z}^n,2), \end{tikzcd} \tag{12}$$

with k-invariant $x_1\, y_1 + \cdots + x_n\, y_n$ (where x_j and y_j are the canonical classes in H^2 of the first and second $K(\mathbb{Z}^n, 2)$ factors, respectively), and R is a more complicated space that has $\widetilde{R}$ as its universal covering. In (12), the bundle projection projected into the first factor corresponds to the characteristic class $c(p)$ of the torus bundle $E \xrightarrow{p} X$, while the bundle projection projected into the second factor corresponds (when $\dim T = 1$ — this case is easier to explain) to the push-forward of the class h under the Gysin map $p_!\colon H^3(E) \to H^2(X)$. T-duality interchanges these. In our situation, $X = G/T$ will be simply connected, so existence of a lift from R to $\widetilde{R}$ is automatic, but we will see that there is a canonical lift.

[4]For another proof, note that it suffices to check this for cohomology with coefficients in $\mathbb{R}$. But $H^3(G, \mathbb{R}) \cong H^3_{\mathrm{deR}}(G, \mathbb{R})$ is generated by the 3-form $(X, Y, Z) \mapsto \langle X, [Y, Z]\rangle$, $X, Y, Z \in \mathfrak{g}$ (the Lie algebra) and $\langle\ ,\ \rangle$ the Killing form. Since the bracket vanishes on the Lie algebra of T, this form restricts to 0 in $H^3_{\mathrm{deR}}(T, \mathbb{R})$.

Finally, there is one additional complication. In the case of Langlands duality, G and $G^\vee$ are really torus bundles over two different spaces, G/T and $G^\vee/T^\vee$. If G is of type B or C with rank $n \geq 3$, then these spaces are not even homotopy equivalent, although they are homotopy equivalent after inverting 2 [36]. However in all cases except these, there is a diffeomorphism $G/T \to G^\vee/T^\vee$ coming from an isomorphism of root systems $\Phi \to \Phi^\vee$. (This diffeomorphism can be taken to be the identity in types A, D, and E, but it can't be the identity in types $B_2 = C_2$, G_2, or F_4, since long roots have short coroots and *vice versa*.)

To finish making Theorem 4 precise, we need to make explicit the choice of h and the choice of the lift $X \to R$ of the classifying map of (G, h) over $X = G/T$. The following is a slight reformulation of the recipe given in [18, Theorem 8.5].

Theorem 6 (Bunke-Nikolaus)**.** *Let G, T be as in Theorem 4, and let $\pi\colon \widetilde{G} \to G$ be the universal cover of G, $\widetilde{T}$ the inverse image of T in $\widetilde{G}$ (a maximal torus in $\widetilde{G}$). Let $(G^\vee, T^\vee)$ be the Langlands dual of (G, T) and let $\pi^\vee\colon \widetilde{G^\vee} \to G^\vee$ be its universal cover, $\widetilde{T^\vee}$ the inverse image of $T^\vee$ in $\widetilde{G^\vee}$. Then Theorem 4 holds for the following explicit choice of h and a lift. Recall that $\widetilde{G}$ is 2-connected and that $G/T \cong \widetilde{G}/\widetilde{T}$ is a complex projective algebraic variety with a cell decomposition with only even-dimensional cells, indexed by elements of the Weyl group $W = N_G(T)/T$. Thus the cohomology of G/T is torsion-free and concentrated in even degrees, and the sum of the Betti numbers is $|W|$, while $\widetilde{G}$ has no cohomology in degrees 1 and 2 and $H^3(\widetilde{G}) \cong \mathbb{Z}$. The group G in general has torsion in its cohomology in degree 2 but also has $H^3(G) \cong \mathbb{Z}$. (The relationship between $H^3(G)$ and $H^3(G)$ was discussed above in Section 2.1.[5]) In the Serre spectral sequence for the fibration $\widetilde{T} \to \widetilde{G} \to \widetilde{G}/\widetilde{T} = G/T$, the differential d_2 gives a transgression isomorphism $\psi\colon H^1(\widetilde{T}) \xrightarrow{\cong} H^2(G/T)$. (Restricted to $H^1(T)$, its image has finite index, with the quotient identified to $H^2(G) \cong \pi_1(G)$, or more precisely its dual group, via the universal coefficient theorem.) Since $H^2(T)$ is naturally identified with $\bigwedge^2(H^1(T))$, and similarly $H^2(\widetilde{T})$ with $\bigwedge^2(H^1(\widetilde{T}))$, the differential d_2 sends $H^2(T)$, respectively $H^2(\widetilde{T})$, to $H^2(G/T) \otimes H^1(T)$ (resp., $H^2(G/T) \otimes H^1(\widetilde{T})$) via $a \wedge b \mapsto \psi(a) \otimes b - \psi(b) \otimes a$.*

The principal T-bundle $p\colon G \to G/T$ has a classifying map $G/T \to BT$, determined up to homotopy by the induced map on cohomology $H^2(BT) \to H^2(G/T)$[6]*, and since $H^2(BT)$ is canonically isomorphic to $H^1(T)$, we can think of this as a characteristic class $c(p) \in \mathrm{Hom}(H^1(T), H^2(G/T))$. (We use the letter c since this is precisely the Chern class when $n = 1$.) Fix an isomorphism $\widetilde{G^\vee} \to \widetilde{G}$ (possible since we are excluding types B and C with $n \geq 3$) and let $\phi\colon H^2(G^\vee/T^\vee) \to$*

[5]As explained there, a slight adjustment is needed in case D_n for n even when $G = \mathrm{PSO}(2n)$, since in this case $H^3(G) \cong \mathbb{Z} \oplus \mathbb{Z}/2$. The torsion summand is harmless here.

[6]This is because BT is a $K(\mathbb{Z}^n, 2)$ space.

$H^2(G/T)$ be the induced isomorphism. Then $c(p) = \psi \circ \pi^*\colon\ H^1(T) \to H^2(G/T)$, and $\phi \circ c(p^\vee) = \phi \circ \psi^\vee \circ (\pi^\vee)^*\colon\ H^1(T^\vee) \to H^2(G/T)$. Let $h \in H^3(G)$ be the image in $E_\infty^{2,1} \cong H^3(G)$ of the composite $\phi \circ c(p^\vee)\colon\ H_1(T) = H^1(T^\vee) \to H^2(G/T)$, viewed in $\operatorname{Hom}(H_1(T), H^2(G/T)) \cong H^2(G/T) \otimes H^1(T)$, which makes sense since this element is easily seen to be killed by d_2. Similarly for $h^\vee \in H^3(G^\vee)$. Then (G, h) together with the map $G/T \to \widetilde{R}$ defined by the homotopy commutative diagram of fibrations

$$\begin{tikzcd}
T \arrow[r, hook] \arrow[d, equal] & G \arrow[r, "p"] \arrow[d, "{(h,\phi\circ c(p^\vee))}"'] & G/T \arrow[d, dotted] \\
T \arrow[r] & K(\mathbb{Z},3) \times K(\mathbb{Z}^n,2) \arrow[r] & \widetilde{R}
\end{tikzcd}$$

(the dashed arrow is a homotopy push-out) is a set of T-duality data, and the dual data corresponds to $(G^\vee, h^\vee)$ viewed also as a bundle over G/T via ϕ.

Proof. The one thing to check is that $\phi \circ c(p^\vee)$ is killed by d_2 and thus gives a genuine class in $H^3(G)$. This involves showing that the map is W-invariant. But $S^2(H^1(T) \otimes_{\mathbb{Z}} \mathbb{Q})$ contains a unique W-invariant element up to scale, namely the Killing form, so we just need to check that $\psi^{-1} \circ \phi \circ c(p^\vee)\colon\ H_1(T) \to H^1(\widetilde{T})$ is a multiple of this form, which is immediate from the formula.

As explained in [18], it is then almost a tautology that the T-dual of this data is the data for $(G^\vee, h^\vee)$, since G and $G^\vee$ play completely symmetrical roles here. □

Corollary 7. *In the setting of Theorems 4 and 6, one gets an isomorphism of twisted K-groups*

$$K^\bullet(G, h) \cong K^{\bullet+n}(G^\vee, h^\vee).$$

Proof. This follows immediately from [19] or [51], applied to the T-duality. □

2.3. **Twisted K-theory for Langlands dual pairs.** The results of Section 2.2, especially Corollary 7, provide new examples of isomorphisms between twisted K-groups on different compact simple Lie groups. This provides additional motivation for understanding the computation of such twisted K-groups and computing exactly what one gets in the case of the T-duality pairs coming from Langlands duality. We start with a few very explicit examples.

Example 8 (SU(2) and SO(3))**.** We begin with the simplest case of $G = \mathrm{SU}(2)$, $G^\vee = \mathrm{SO}(3)$, the only dual pair in rank 1 (though this pair has other aliases, e.g., $G = \mathrm{Sp}(1)$ and $G^\vee = \mathrm{PSp}(1)$, or $G = \mathrm{Spin}(3)$ and $G^\vee = \mathrm{SO}(3)$). In this case $G/T = S^2$, G is topologically S^3, p is the Hopf bundle $S^3 \to S^2$, and $G^\vee$ is topologically $\mathbb{RP}^3$. It is easy to see that the Chern class $c(p) = 1$ (for the usual identification of $H^2(S^2)$ with $\mathbb{Z}$) and $c(p^\vee) = 2$. Since T-duality for circle bundles (see for example [13, 50, 60]) requires $p_!(h) = c(p^\vee)$ and $(p^\vee)_! = c(p)$, we easily compute from the Gysin sequences that $h = 2$ and $h^\vee = 1$, again for the usual identifications of the H^3 groups with $\mathbb{Z}$.

Let us now check Corollary 7 in this example. We can compute the twisted K-theory via the Atiyah-Hirzebruch spectral sequence (AHSS) for twisted K-theory, first discussed in [58, 59] and further explained in [4, 5]. Since G and $G^\vee$ are 3-dimensional, the only differential is d_3, and $\mathrm{Sq}^3 = 0$ for both G and $G^\vee$. So (for any choice of h and $h^\vee$), the twisted K-groups are computed from the spectral sequence shown in Figure 2. (By Bott periodicity, the differentials repeat vertically with period 2.)

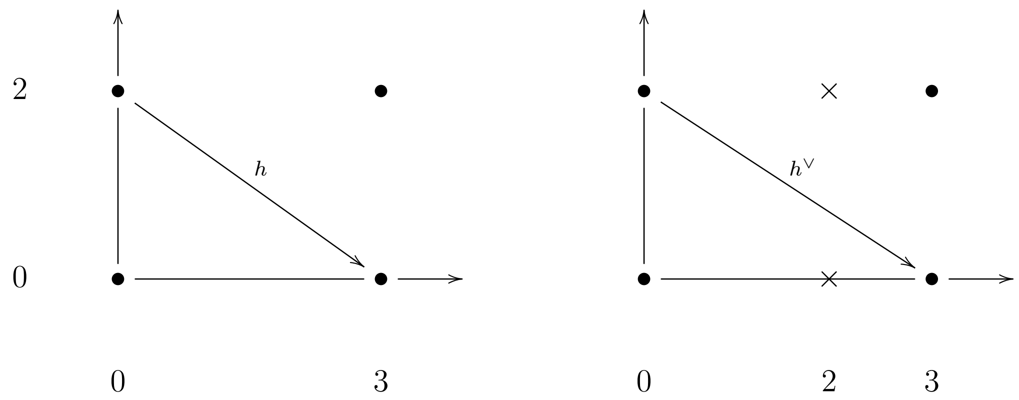

FIGURE 2. The AHSS for twisted K-theory of SU(2) (left) and SO(3) (right). The symbol $\bullet$ denotes a copy of $\mathbb{Z}$; $\times$ denotes a copy of $\mathbb{Z}/2$.

From the figure, it is apparent that (assuming h and $h^\vee$ are both non-zero) we get $K^\bullet(G, h) = \mathbb{Z}/h$ in odd degree, 0 in even degree. Similarly, we get $K^\bullet(G^\vee, h^\vee) = \mathbb{Z}/h^\vee$ in odd degree, $\mathbb{Z}/2$ in even degree. (This calculation also appears in [15, §3.2].) So with the choice $h = 2$ and $h^\vee = 1$, we get $\mathbb{Z}/2$ in odd degree for G, $\mathbb{Z}/2$ in even degree for $G^\vee$, as is consistent with Corollary 7. Note, incidentally, that no other positive choices for h and $h^\vee$ would work here. ☐

Example 9 (SU(3) and PSU(3))**.** Now consider the case of the Langlands dual pair $G = \mathrm{SU}(3)$ and $G^\vee = \mathrm{PSU}(3)$. We want to compute the appropriate values of h and $h^\vee$ for this case. Recall from Theorem 1 that if $\pi\colon G \to G^\vee$ is the 3-to-1 covering map, then $\pi^*\colon H^3(G^\vee) \to H^3(G)$ is an isomorphism in this case.

Since G and $G^\vee$ are bundles over G/T, we need the cohomology of the flag variety G/T. This is a classical calculation of Borel [9, Proposition 29.2]:

$$H^\bullet(G/T) \cong S^\bullet(H^1(T))/S^\bullet(H^1(T))^W,$$

which in our case reduces to

$$\begin{aligned}(13)\quad \mathbb{Z}[x_1,x_2,x_3]/(x_1+x_2+x_3,\ x_1x_2+x_1x_3+x_2x_3,\ x_1x_2x_3)\\ = \mathbb{Z}[x_1,x_2]/(x_1^2+x_2^2+x_1x_2, x_1x_2^2+x_1^2x_2).\end{aligned}$$

Here x_1, x_2, and x_3 represent a basis for the weight lattice of the maximal torus of $\mathrm{U}(3)$: $x_j(\mathrm{diag}(z_1,z_2,z_3)) = z_j$, and the Weyl group invariants are generated by the elementary symmetric functions. Imposing the relation $x_1+x_2+x_3=0$ is equivalent to restricting to $\mathrm{SU}(3)$. Inside the weight lattice $H^1(T)=\mathbb{Z}x_1+\mathbb{Z}x_2$ we have the root lattice $H^1(T^\vee)$, spanned by the simple roots $\alpha = x_1-x_2$ and $x_2-x_3 = x_1+2x_2 = \alpha+3x_2$. So the index of $H^1(T^\vee)$ in $H^1(T)$ is

$$\det\begin{pmatrix}1 & 1\\ -1 & 2\end{pmatrix} = 3,$$

as it should be (since 3 is the degree of the covering $G\to G^\vee$).

We would now like to compute h and $h^\vee$ explicitly, as multiples of the generators of $H^3(G)$ and $H^3(G^\vee)$. For this we can split the torus bundles $G\to G/T$ and $G^\vee \to G/T$ into circle bundles and use [51, Theorem 5.6]. There are many ways to do this, corresponding to different choices of basis in the weight lattice $H^1(T)$, but it is convenient to split the bundles as shown:

$$\begin{tikzcd} & G \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X_1 \arrow[dr, "p_1"'] & & X_2 \arrow[dl, "p_2"] \\ & G/T & \end{tikzcd} \qquad \begin{tikzcd} & G^\vee \arrow[dl, "\pi_1^\vee"'] \arrow[dr, "\pi_2^\vee"] & \\ X_1^\vee \arrow[dr, "p_1^\vee"'] & & X_2^\vee \arrow[dl, "p_2^\vee"] \\ & G^\vee/T^\vee = G/T & \end{tikzcd}$$

with $c(p_1) = x_1-x_2$, $c(p_2)=x_2$, $c(p_1^\vee) = x_1-x_2$, $c(p_2^\vee)=3x_2$. Note that in this case $X_1 = X_1^\vee$, since these are both circle bundles over G/T with the same Chern class x_1-x_2. [51, Theorem 5.6] implies (with our current notation) that

(14)
$$(\pi_2)_!(h) = p_2^*(c(p_1^\vee)) = p_2^*(x_1-x_2), \quad (\pi_2^\vee)_!(h^\vee) = (p_2^\vee)^*(c(p_1)) = (p_2^\vee)^*(x_1-x_2).$$

A simple calculation with the Serre spectral sequence (or the Gysin sequence, which is basically the same thing) shows that the cohomology ring of X_2 is torsion-free, and is an exterior algebra on generators in dimensions 2 and 5. Furthermore, $p_2^*(x_2)=0$ and $p_2^*(x_1)$ generates $H^2(X_2)$. So from the first equality in (14), $(\pi_2)_!(h)$ generates $H^2(X_2)$. It follows that h has to generate $H^3(G)$, i.e., $h=1$ (with our usual identifications). Next we look at $X_2^\vee$. This time, since $c(p_2^\vee)=3x_2$,

$$H^2(X_2^\vee) = \mathbb{Z}\left((p_2^\vee)^*(x_1)\right) \oplus (\mathbb{Z}/3)\left((p_2^\vee)^*(x_2)\right).$$

From (14), $(\pi_2^\vee)_!(h^\vee)$ is still primitive, so we also have $h^\vee = 1$. Note that $\pi^*(h^\vee) = h$ for the covering $\pi\colon G\to G^\vee$.

It remains to compute the twisted K-theory for (G,h) and for $(G^\vee, h^\vee)$ and to check the result of Corollary 7. For G this is easy — either we can appeal to [14] or [23], or else we can compute directly from the AHSS, since $H^*(G) = \bigwedge(x_3, x_5)$ and $h = x_3$, $\mathrm{Sq}^3\, x_j = 0$, so that d_3 is multiplication by x_3, which sends x_3 to 0 and x_5 to x_3x_5. So $\ker d_3 = \mathbb{Z}x_3 + \mathbb{Z}x_3x_5 = \operatorname{image} d_3$ and $E_4 = 0$. Thus $K^\bullet(G,h) = 0$ (in both even and odd degree). In the case of $G^\vee$, the cohomology is a bit more complicated because (via (1)) there is also a 3-torsion generator y in degree 2 with $y^2 \neq 0$, $y^3 = 0$. The 5-dimensional torsion-free generator x_5 reduces mod 3 to x_1y^2. Rationally, things are just as for G, so it suffices to see what happens mod 3. The class $h^\vee$ mod 3 is x_3, in the notation of (1). A basis for $H^\bullet(G^\vee, \mathbb{F}_3)$ is given by 1, x_1, y, x_1y, x_3, y^2, x_1x_3, x_3y, x_1y^2, x_1x_3y, x_3y^2, and $x_1x_3y^2$. Multiplication by x_3 kills the 6 of these monomials containing an x_3 factor, and sends the other 6 monomials to these 6. So once again we see that $\ker d_3 = \operatorname{image} d_3$ and the twisted K-theory $K^\bullet(G^\vee, h^\vee) = 0$ (in both even and odd degree). This is consistent with Corollary 7.

Example 10 (Sp(2) = Spin(5) and PSp(2) = SO(5))**.** We can do a similar calculation for the Langlands dual pair $G = \mathrm{Sp}(2)$ and $G^\vee = \mathrm{PSp}(2)$, and compute the appropriate values of h and $h^\vee$ for this case. Recall from Theorem 1 that if $\pi\colon G \to G^\vee$ is the 2-to-1 covering map, then $\pi^*\colon H^3(G^\vee) \to H^3(G)$ is an isomorphism in this case.

We can take the weight lattice $H^1(T)$ to be $\mathbb{Z}x_1 \oplus \mathbb{Z}x_2$, with the root lattice $H^1(T^\vee)$ spanned by the simple roots $x_1 - x_2$ and $2x_2$. The Weyl group W is the dihedral group of order 8 generated by the reflection $x_1 \mapsto x_1, x_2 \mapsto -x_2$ and by the rotation $x_1 \mapsto x_2, x_2 \mapsto -x_1$. The ring of W-invariants in $\mathbb{Z}[x_1, x_2]$ is generated by $x_1^2 + x_2^2$ and by $x_1^2x_2^2$, and $H^\bullet(G/T) \cong \mathbb{Z}[x_1, x_2]/(x_1^2 + x_2^2, x_1^2x_2^2)$, here with x_j of degree 2 in the cohomology ring.

In order to apply [51, Theorem 5.6], we again split the torus bundles $G \to G/T$ and $G^\vee \to G/T$ as:

$$\begin{tikzcd}
 & G \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
X_1 \arrow[dr, "p_1"'] & & X_2 \arrow[dl, "p_2"] \\
 & G/T &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & G^\vee \arrow[dl, "\pi_1^\vee"'] \arrow[dr, "\pi_2^\vee"] & \\
X_1^\vee \arrow[dr, "p_1^\vee"'] & & X_2^\vee \arrow[dl, "p_2^\vee"] \\
 & G^\vee/T^\vee = G/T &
\end{tikzcd}$$

with $c(p_1) = x_1 - x_2$, $c(p_2) = x_2$, $c(p_1^\vee) = x_1 - x_2$, $c(p_2^\vee) = 2x_2$. Note that $X_1 = X_1^\vee$, since these are both circle bundles over G/T with the same Chern class $x_1 - x_2$. As in Example 9, the cohomology ring of X_2 is torsion-free, and is an exterior algebra on generators in dimensions 2 and 7. Furthermore, $p_2^*(x_2) = 0$ and $p_2^*(x_1)$ generates $H^2(X_2)$. So from the first equality in (14), $(\pi_2)_!(h)$ generates $H^2(X_2)$. It follows that h has to generate $H^3(G)$, i.e., $h = 1$ (with our usual identifications).

Next we look at $X_2^\vee$. This time, since $c(p_2^\vee) = 2x_2$,

$$H^2(X_2^\vee) = \mathbb{Z}\left((p_2^\vee)^*(x_1)\right) \oplus (\mathbb{Z}/2)\left((p_2^\vee)^*(x_2)\right).$$

From (14), $(\pi_2^\vee)_!(h^\vee)$ is still primitive, so we also have $h^\vee = 1$. Note that $\pi^*(h^\vee) = h$ for the covering $\pi\colon G \to G^\vee$.

As in the last example, one can check that the twisted K-theory vanishes on both sides of the duality. (A quick way of proving this will be given in the following Section 2.4.)

The phenomena that showed up in Examples 8, 9, and 10 can all be explained by the following theorem, which also explains why we were interested in Theorem 1.

Theorem 11. *Let G be a compact simply connected simple Lie group, not of type B_n or C_n with $n \geq 3$, so that $G^\vee$, the corresponding Langlands dual, is of adjoint type, and there is a covering map $\pi\colon G \to G^\vee$. Then for the Langlands dual pair G and $G^\vee$, with h and $h^\vee$ the classes of Theorem 6, $h^\vee$ is the generator of $H^3(G^\vee)$ specified by the usual orientation of an embedded rank-1 subgroup, i.e., it is 1 when we identify $H^3(G^\vee)$ with $\mathbb{Z}$, and $h = \pi^*(h^\vee)$ as computed in Theorems 1 and 3.*[7] *Furthermore, the twisted K-theory groups $K^\bullet(G, h)$ and $K^\bullet(G^\vee, h^\vee)$ vanish unless $G = \mathrm{SU}(2)$ and $G^\vee = \mathrm{SO}(3)$.*

Proof. The key here is to observe that the bundle $p\colon G \to G/T$ and the bundle $p^\vee\colon G^\vee \to G^\vee/T^\vee$ are related by the covering map $\pi : G \to G^\vee$. Thus if E is the Poincaré bundle (the pull-back of p and $p^\vee$), we have a commuting diagram

(15)
$$\begin{tikzcd}
 & E \arrow[dl, "p^*(p^\vee)"'] \arrow[dr, "(p^\vee)^*(p)"] & \\
G \arrow[rr, "\pi"] \arrow[dr, "p"'] & & G^\vee \arrow[dl, "p^\vee"] \\
 & G/T &
\end{tikzcd}.$$

First, look at the classes h and $h^\vee$. They both pull back to the same thing in $H^3(E)$, so

$$(p^*(p^\vee))^* h = ((p^\vee)^*(p))^*(h^\vee).$$

By commutativity of (15), $(p^\vee)^*(p) = \pi \circ p^*(p^\vee)$, so that means that

$$\left(p^*(p^\vee)\right)^* (h - \pi^* h^\vee) = 0,$$

or $h - \pi^* h^\vee = 0$ modulo the kernel of $(p^*(p^\vee))^*$. But

$$c(p^*(p^\vee)) = p^*(c(p^\vee)) = p^*(c(p) \circ \pi).$$

[7] In the exceptional case where G is of type D_n with $n \equiv 0 \pmod 4$, $h^\vee$ is still the generator of an infinite cyclic summand in $H^3(G^\vee)$.

However, from the Serre spectral sequence of the T-bundle p, p^* kills the image of $c(p)\colon H^1(T) \to H^2(G/T)$. So the Chern class of $p^*(p^\vee)$ vanishes, i.e., this is a trivial bundle, and hence the pull-back map on cohomology is faithful. Hence $h = \pi^*(h^\vee)$.

Next, we show that $h^\vee$ has to be primitive, i.e., has to be ± 1 as a multiple of the generator of $H^3(G^\vee)$. (The multiple is positive because of the normalization in Theorem 6, so it's therefore exactly 1.) This follows from the simple connectivity of G together with the following Lemma 12. Indeed, suppose $h^\vee = k\alpha$ with $\alpha \in H^3(G^\vee)$, $k > 1$. Then by Lemma 12, the classifying map for the T-duality data for $(G^\vee \to G/T, h^\vee)$ factors as $f_k \circ (\text{something})\colon G/T \to \widetilde{R}$, and so by the Lemma the classifying map $c(p)\colon G/T \to BT$ for the T-dual is divisible by k. But this would imply that G has a k-fold covering, contradicting the assumption that G is 1-connected.

Finally, putting what we have done so far together with Theorems 1 and 3, we see that $h = 1$ or 2. By the results of [14, 23], that implies that $K^\bullet(G, h) = 0$ unless $G = \mathrm{SU}(2)$. (In all other cases, we get a direct sum of copies of $\mathbb{Z}/h'$, where $h' = \frac{h}{\gcd(h,m)}$ and m is divisible by 2, so that $h' = 1$.) By Corollary 7, we also get $K^\bullet(G^\vee, h^\vee) = 0$. ☐

Lemma 12. *Let $k > 1$ be a positive integer and let $\widetilde{R}$ be the classifying space for T-duality data for $\mathbb{T}^n$-bundles over simply connected spaces, as defined in* [51]*; recall that this is a fibration as diagrammed in* (12)*. Then, up to homotopy, $\widetilde{R}$ has a unique endomorphism f_k which induces multiplication by k on $\pi_3(\widetilde{R}) \cong \mathbb{Z}$ and which on $\pi_2(\widetilde{R}) \cong \mathbb{Z}^n \times \mathbb{Z}^n$ is the identity on the first factor and multiplication by k on the second factor. The effect of f_k on T-duality data $(E \xrightarrow{p} Z, h)$ is to keep the bundle p the same and to multiply the H-flux h by k. On the T-dual data, f_k multiplies the characteristic class $c(p^\vee)$ by k.*

Proof. This is a simple exercise in obstruction theory, based on the observation that the given maps on homotopy groups are compatible with the k-invariant of the bundle (12). ☐

Theorem 11 in a sense is disappointing, in that we get no nontrivial isomorphisms of twisted K-groups, but it completely settles the nature of the T-dualities coming from Langlands duality.

2.4. **Twisted K-theory of compact simple Lie groups.** We now move on to the question of how to compute the twisted K-groups $K^*(G, h)$ more generally, where G is a connected compact simple Lie group and $h \in H^3(G) \cong \mathbb{Z}$. (Strictly speaking, a twist for K-theory is not exactly the same as a class in H^3, but the difference won't matter for our purposes since we just want to compute the groups as abstract groups.) Methods for computing $K^*(G, h)$ were discussed in [48, 53] in a few cases, and the calculation for simply connected G was done completely in [14, 23]. However, the methods used there do not work when G is not simply connected, at least not without a lot of additional work, so other techniques are needed. We illustrate another method using the Segal spectral sequence (from [61, Proposition 5.2]). We begin with an abstract result, which could be made even more general, though the case given is sufficient for our purposes

Theorem 13. *Let $F \xrightarrow{\iota} E \to B$ be a fiber bundle, say of compact metrizable spaces with finite homotopy type, and let $h \in H^3(E)$. Then there is a spectral sequence $H^p(B, K^q(F, \iota^* h)) \Rightarrow K^\bullet(E, h)$.*

Proof. In the absence of the twist, this is precisely the spectral sequence of [61, Proposition 5.2] in the case where the cohomology theory used is complex K-theory, and if $E = B$ and $F = \mathrm{pt}$, it reduces to the usual AHSS. Similarly, if $E = B$ and $F = \mathrm{pt}$, but $h \neq 0$, this is the AHSS for twisted K-theory. To get the general case, we can assume (by homotopy invariance) that B is a finite CW-complex, and filter B by its skeleta. This induces a filtration of $K^\bullet(E, h)$ for which this is the induced spectral sequence (by Segal's proof).

There is also another way to see this in terms of continuous-trace algebras. $K^\bullet(E, h)$ is the K-theory of a continuous-trace C^*-algebra A with spectrum B and Dixmier-Douady class h. Using the bundle projection, we can write A as the algebra of sections of a bundle of C^*-algebras over B, where the fiber algebra over a point $b \in B$ is a continuous-trace algebra with spectrum $E_b \cong F$ and the Dixmier-Douady class is the pull-back of h to this fiber. Projections in the fiber algebras locally extend to a neighborhood of the fiber, and the differentials of the spectral sequence measure the obstructions to extending them over the inverse images of bigger and bigger skeleta of B. □

Next let us try to explain the somewhat puzzling results of [14] and [23], which, to pick just the simplest case, say that $K^\bullet(\mathrm{SU}(n+1), h)$ is isomorphic to $\mathbb{Z}/h'$ tensored with an exterior algebra on $n-1$ generators, where

$$h' = \frac{h}{\gcd(h, \operatorname{lcm}(1, 2, \cdots, n))}. \tag{16}$$

The method of proof of this result was quite indirect — [14] used the Hodgkin Künneth spectral sequence in equivariant K-theory together with the calculations of Freed-Hopkins-Teleman [27, 28][8], while [23] used a Rothenberg-Steenrod spectral sequence and K-theory of loop spaces.

Theorem 13 suggests a potentially much simpler method for doing the calculations inductively, which we will consider here for $\mathrm{SU}(n+1)$ and $\mathrm{PSU}(n+1)$, though it could be extended to other classical groups as well. Let $G = \mathrm{PSU}(n+1)$ and $\widetilde{G} = \mathrm{SU}(n+1)$. The transitive action of $\widetilde{G} = \mathrm{SU}(n+1)$ on the unit sphere S^{2n+1} in $\mathbb{C}^{n+1}$ descends to a transitive action of G on $L^{2n+1}(n+1)$, the lens space

[8]There is indirect physics input here since Freed-Hopkins-Teleman showed that the *equivariant* twisted K-theory is the same as the Verlinde ring of the associated WZW model.

obtained by dividing S^{2n+1} by the action of μ_{n+1}, the $(n+1)$-th roots of unity. We thus get a commuting diagram of fibrations

$$\begin{tikzcd} \mathrm{SU}(n) \arrow[r] \arrow[d, equal] & \mathrm{SU}(n+1) \arrow[r] \arrow[d, "\pi"] & S^{2n+1} \arrow[d] \\ \mathrm{SU}(n) \arrow[r] & \mathrm{PSU}(n+1) \arrow[r] & L^{2n+1}(n+1). \end{tikzcd} \tag{17}$$

The $\mathrm{SU}(n)$ at the bottom is really the image in $\mathrm{PSU}(n+1)$ of matrices of the form $\begin{pmatrix} \zeta & 0 \\ 0 & \zeta A \end{pmatrix}$, $A \in \mathrm{SU}(n)$ and $\zeta \in \mu_{n+1}$, but this can be identified with $\mathrm{SU}(n)$. The diagram (17) gives rise to a morphism of Serre spectral sequences

$$\begin{tikzcd} H^p(S^{2n+1}, H^q(\mathrm{SU}(n))) \arrow[r, Rightarrow] & H^{p+q}(\mathrm{SU}(n+1)) \\ H^p(L^{2n+1}(n+1), H^q(\mathrm{SU}(n))) \arrow[r, Rightarrow] \arrow[u, "\pi^*"] & H^{p+q}(\mathrm{PSU}(n+1)). \arrow[u, "\pi^*"] \end{tikzcd}$$

Here $H^\bullet(\mathrm{SU}(n))$ is an exterior algebra on generators $x_3, x_5, \cdots, x_{2n-1}$ (with the subscript indicating the degree), so the first sequence collapses just for dimensional reasons. The second sequence looks as if it could have a nontrivial differential $d_4\colon\ x_3 \mapsto cy^2$, where $y \in H^2(L^{2n+1}(n+1)) \cong \mathbb{Z}/(n+1)$ is the usual generator. However, the calculations of $H^\bullet(\mathrm{PSU}(n+1), \mathbb{F}_p)$ by Borel in [10, Théorème 11.4] and by Baum and Browder in [6] (which already appeared in the proof of Theorem 1) show that in many cases (e.g., $n+1$ odd), $d_4(x_3) = 0$ in the spectral sequence for the lower fibration in (17). Then the restriction maps $H^3(G) \to H^3(\mathrm{SU}(n))$ and $H^3(\widetilde{G}) \to H^3(\mathrm{SU}(n))$ are both isomorphisms.

Now apply Theorem 13 to the fibrations of (17). We get spectral sequences

$$H^p(S^{2n+1}, K^q(\mathrm{SU}(n), h)) \Rightarrow K^{p+q}(\mathrm{SU}(n+1), h)$$

and (say when $n+1$ is odd, so we can still identify $H^3(\mathrm{PSU}(n+1))$ with $H^3(\mathrm{SU}(n))$)

$$H^p(L^{2n+1}(n+1), K^q(\mathrm{SU}(n), h)) \Rightarrow K^{p+q}(\mathrm{PSU}(n+1), h).$$

Assuming we've inductively proved (16) for smaller values of n, this now gives information about $K^\bullet(\mathrm{SU}(n+1), h))$ or $K^\bullet(\mathrm{PSU}(n+1), h))$. We will show how this can be used to get information about these with relatively elementary methods, not involving the techniques used by Braun and Douglas.

For example, while the following is weaker than the results of [53, 48] (for $\mathrm{SU}(3)$) and of [14, 23] (for $\mathrm{SU}(n+1)$ in general), it does give some highly nontrivial information, and the proof is relatively elementary.

Theorem 14. *Suppose $h \in \mathbb{Z} \cong H^3(\mathrm{SU}(n+1))$ is relatively prime to $n!$. Then $K^\bullet(\mathrm{SU}(n+1), h)$ is $\mathbb{Z}/h$ tensored with an exterior algebra on $n-1$ odd generators.*

Proof. We proceed by induction on n. The case $n = 1$ is covered by Example 8, where we computed that $K^\bullet(\mathrm{SU}(2), h) \cong \mathbb{Z}/h$ in odd degree. So assume that $n > 1$ and that the result holds for smaller values of n, and use the Segal spectral sequence (Theorem 13) associated to the fibration

$$\mathrm{SU}(n) \to \mathrm{SU}(n+1) \to S^{2n+1}. \tag{18}$$

Since $n > 1$, it is clear from the fibration that the restriction map $H^3(\mathrm{SU}(n+1)) \to H^3(\mathrm{SU}(n))$ is an isomorphism. Also, if h is relatively prime to $n!$, then it is certainly relatively prime to $(n-1)!$. So we get a Segal spectral sequence

$$H^p(S^{2n+1}, K^q(\mathrm{SU}(n), h)) \Rightarrow K^{p+q}(\mathrm{SU}(n+1), h),$$

and by inductive hypothesis, $K^q(\mathrm{SU}(n), h) \cong (\mathbb{Z}/h) \otimes \bigwedge(y_1, \cdots, y_{n-2})$, with the $\mathbb{Z}/h$ and the y's all in odd degree. We just need to show that the spectral sequence collapses. There is only one possible differential, d_{2n+1}, related to the homotopical nontriviality of the fibration. However, the long exact homotopy sequence of the fibration includes the following:

$$\pi_{2n+1}(\mathrm{SU}(n+1)) \to \pi_{2n+1}(S^{2n+1}) \to \pi_{2n}(\mathrm{SU}(n)) \to \pi_{2n}(\mathrm{SU}(n+1)). \tag{19}$$

By [11, §8, Theorem 5], $\pi_{2n+1}(\mathrm{SU}(n+1)) \cong \mathbb{Z}$, $\pi_{2n}(\mathrm{SU}(n+1)) = 0$, and $\pi_{2n}(\mathrm{SU}(n)) \cong \mathbb{Z}/n!$. So substituting back into (19) we get

$$\cdots \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n! \to 0.$$

That means that after inverting $n!$ (but not without doing this), the map

$$\pi_{2n+1}(\mathrm{SU}(n+1)) \to \pi_{2n+1}(S^{2n+1})$$

splits, i.e., the fibration (18) splits. (A splitting of the generator of $\pi_{2n+1}(S^{2n+1})$ would be a map $S^{2n+1} \to \mathrm{SU}(n+1)$ splitting the fibration.) Or equivalently, the class of the bundle (18) is given by a class in

$$[S^{2n+1}, B\,\mathrm{SU}(n)] = \pi_{2n+1}(B\,\mathrm{SU}(n)) \cong \pi_{2n}(\mathrm{SU}(n)) \cong \mathbb{Z}/n!.$$

So when $\gcd(h, n!) = 1$, d_{2n+1} vanishes and the conclusion follows. □

Remark 15. Note that Theorem 14, and especially the appeal to Bott's theorem [11, §8, Theorem 5], "explains" the strange denominator in formula (16), at least to some extent. We can also handle the symplectic groups the same way.

Theorem 16. *Suppose $h \in \mathbb{Z} \cong H^3(\mathrm{Sp}(n))$ is relatively prime to $(2n+1)!$. Then $K^\bullet(\mathrm{Sp}(n), h))$ is $\mathbb{Z}/h$ tensored with an exterior algebra on $n-1$ odd generators.*

Proof. This proceeds exactly the same way as Theorem 14, using the homotopy sequence of the fibration

$$\mathrm{Sp}(n-1) \to \mathrm{Sp}(n) \to S^{4n-1}.$$

By Bott periodicity and stability, $\pi_{4n-1}(\mathrm{Sp}(n)) \cong \mathbb{Z}$ and $\pi_{4n-2}(\mathrm{Sp}(n)) = 0$. The group $\pi_{4n-2}(\mathrm{Sp}(n-1))$ is not in the stable range since $4n-2 = 4(n-1)+2$, but it is computed in [37] and [52], and turns out to be $\mathbb{Z}/(2n+1)!$ or $\mathbb{Z}/(2(2n+1)!)$, depending on whether n is odd or even. Thus we get the long exact homotopy sequence

$$(\mathbb{Z} \cong \pi_{4n-1}(\mathrm{Sp}(n))) \to \pi_{4n-1}(S^{4n-1}) \to (\mathbb{Z}/(2n+1)! \text{ or } \mathbb{Z}/(2(2n+1)!)) \to 0,$$

and so the fibration splits after inverting primes $\leq 2n+1$. The rest of the proof is as before. $\square$

Results for other groups can be proved with similar methods. For example, we have:

Theorem 17. *Suppose $h \in \mathbb{Z} \cong H^3(G_2)$ is relatively prime to* 2, 3, *and* 5. *Then $K^\bullet(G_2, h)$ is $\mathbb{Z}/h$ in both even and odd degree.*

Proof. This proceeds similarly, using the homotopy sequence of the fibration [10, Lemme 17.1]

$$\mathrm{SU}(2) \to G_2 \to V_{7,2}.$$

The Stiefel manifold $V_{7,2}$ is 11-dimensional and has only one nontrivial homology group below the top dimension, namely a $\mathbb{Z}/2$ in dimension 5. The restriction map $H^3(G_2) \to H^3(\mathrm{SU}(2))$ is an isomorphism, so we can apply the Segal spectral sequence and we get a spectral sequence

$$E_2^{p,q} = H^p(V_{7,2}, K^q(\mathrm{SU}(2), h)) \Rightarrow K^\bullet(G_2, h).$$

Here $K^q(\mathrm{SU}(2), h) \cong \mathbb{Z}/h$ for q odd and is 0 for q even. If h is odd, $E_2^{p,q}$ is non-zero only for $p = 11$ and q odd, and d_{11} is the only possible differential. However, we have the long exact homotopy sequence

$$\pi_{11}(G_2) \to \pi_{11}(V_{7,2}) \to \pi_{10}(\mathrm{SU}(2)) \to \pi_{10}(G_2),$$

and since $\pi_{10}(\mathrm{SU}(2)) \cong \mathbb{Z}/15$, the map $\pi_{11}(G_2) \to \pi_{11}(V_{7,2})$ is surjective after inverting 3 and 5. Furthermore, after inverting 2, $V_{7,2}$ becomes homotopy equivalent to S^{11} by the Hurewicz Theorem modulo the Serre class of 2-primary torsion groups. So when h is prime to 2, 3, and 5, the spectral sequence is just as for $S^3 \times S^{11}$, must collapse, and gives the result. $\square$

Similar techniques can also be used to compute twisted K-theory in some cases for non-simply connected groups. Here is a representative example:

Theorem 18. *Let $h \in \mathbb{Z}$, identified with $H^3(\mathrm{PSU}(3))$. Then if* $\gcd(h, 3) = 1$,

$$K^\bullet(\mathrm{PSU}(3), h) \cong K^\bullet(\mathrm{SU}(3), h) \cong \begin{cases} \mathbb{Z}/h \otimes \bigwedge(x_1), & h \text{ odd}, \\ \mathbb{Z}/(h/2) \otimes \bigwedge(x_1), & h \text{ even}. \end{cases}$$

On the other hand, if $h = 3$, then both $K^{even}(\mathrm{PSU}(3), h)$ and $K^{odd}(\mathrm{PSU}(3), h)$ are finite groups of order 27.

Proof. Since we've seen that the restriction map $H^3(\mathrm{PSU}(3)) \to H^3(\mathrm{SU}(2))$ (coming from (17)) is an isomorphism, we get from (17) a Segal spectral sequence

$$H^p(L^5(3), K^q(\mathrm{SU}(2), h)) \Rightarrow K^{p+q}(\mathrm{PSU}(3), h). \tag{20}$$

Here we know that $K^q(\mathrm{SU}(2), h)$ is $\mathbb{Z}/h$, concentrated in odd degree. There are now various cases. If h is prime to 3, then the lens space $L^5(3)$ looks like S^5 mod h, and the spectral sequence becomes the same as for $K^\bullet(\mathrm{SU}(3), h)$, for which we know the answer by [53, 48, 14, 23].

So consider the case where $h = 3$. In this case, $H^\bullet(L^5(3), \mathbb{F}_3) \cong \mathbb{F}_3[x_1, y_2]/(x_1^2, y_2^3)$ with $\beta x_1 = y_2$. So the E_2 stage of (20) gives groups of order 27 in both even and odd degree. So it will suffice to show that the spectral sequence collapses at E_2. Since $E_2^{p,q}$ is only non-zero for q odd and for $p \le 5$, the only possible differentials are d_3 and d_5. We consider them one at at time. Both differentials would vanish if PSU(3) looked like $S^3 \times L^5(3)$, and have to do with nontriviality of the fibration in (17). So how nontrivial is it? The homotopy groups of PSU(3) are given by $\pi_j(\mathrm{PSU}(3)) \cong \mathbb{Z}/3, j = 1;\ 0, j = 2;\ \mathbb{Z}, j = 3;\ 0, j = 4$; and $\mathbb{Z}, j = 5$. Since PSU(3) is a topological group, it is certainly simple and has a Postnikov system even though it isn't simply connected. The first nontrivial stage of this Postnikov system is a fibration $K(\mathbb{Z}, 3) \to X_1 \to K(\mathbb{Z}/3, 1)$. The next stage is a fibration $K(\mathbb{Z}, 5) \to X_2 \to X_1$. The k-invariant of X_1 lies in $H^4(K(\mathbb{Z}/3, 1)) \cong \mathbb{Z}/3$, and corresponds to the transgression $d_4\colon H^3(K(\mathbb{Z}, 3)) \to H^4(K(\mathbb{Z}/3, 1))$ in the Serre spectral sequence for the fibration defining X_1. But $H^\bullet(K(\mathbb{Z}, 3), \mathbb{F}_3) \cong \mathbb{F}_3[x_1, y_2]/(x_1^2)$ with $\beta x_1 = y_2$, which agrees with the cohomology of $L^5(3)$ up to degree 5, while by (1), the element corresponding to y_2^2 is nonzero in $H^4(\mathrm{PSU}(3), \mathbb{F}_3)$. Thus d_4 has to vanish and $X_1 \simeq K(\mathbb{Z}, 3) \times K(\mathbb{Z}/3, 1)$. This means PSU(3) closely approximates $K(\mathbb{Z}, 3) \times K(\mathbb{Z}/3, 1)$ through dimension 4. Hence as far as d_3 in the Segal spectral sequence is concerned, we might as well have $\mathrm{PSU}(3) \simeq S^3 \times L^5(3)$, which causes d_3 to vanish. Vanishing of d_5 then comes from comparison of spectral sequences; the diagram (17) plus the fact that there are no differentials before d_5 implies that d_5 for $K^\bullet(\mathrm{PSU}(3), h)$ must agree with d_5 for $K^\bullet(\mathrm{SU}(3), h)$, which we know must vanish if h is odd. So the result follows. ☐

Remark 19. Note that even when the Segal spectral sequence degenerates, there is still an extension problem in going from E_∞ to the actual twisted K-groups. In the situation of Theorem 18, one can solve the extension problem by noting that the method of proof really showed something stronger, namely that

$$K^\bullet(\mathrm{PSU}(3), 3) \cong K^\bullet(\mathrm{SU}(2) \times L^5(3), 3) \cong K^{\bullet+1}(L^5(3), \mathbb{F}_3).$$

This turns out to be elementary abelian, for two reasons. First, the universal coefficient theorem for K-theory with $\mathbb{F}_p$ coefficients splits, unlike what happens for KO-theory with $\mathbb{F}_2$ coefficients. (See [2, §2] for an explanation.) And secondly, the complex K-theory $\widetilde{K}^0(L^5(3))$ turns out to be elementary abelian, unlike what happens for certain other lens spaces. For an explanation of this, see [41] and especially the *Mathematical Reviews* review of this paper by Hirzebruch.

The final result that the twisted K-groups in both even and odd degree are $(\mathbb{Z}/3)^3$ was obtained previously in [32, (2.20)] from the physics perspective of D-brane charges in WZW theories for PSU(3). More complicated calculations for PSU(9), where the result is not so simple, appear in [33].

Note that exactly the same method proves:

Theorem 20. *Let* $G = \mathrm{PSU}(n+1)$ *or* $\mathrm{PSp}(n)$, $n \geq 2$, *with universal cover* $\widetilde{G} = \mathrm{SU}(n+1)$ *or* $\mathrm{Sp}(n)$. *Suppose* $\gcd(h, n+1) = 1$ (*in the case of* $\mathrm{PSU}(n+1)$) *or* h *is odd* (*in the case of* $\mathrm{PSp}(n)$). *Then* $\pi^* \colon K^\bullet(G, h) \to K^\bullet(\widetilde{G}, \pi^* h)$ *is an isomorphism of twisted K-theory groups.*

Proof. We indicate the details for $\mathrm{PSp}(n)$; the other case is analogous. Consider the commuting diagram of fibrations (4), which gives a morphism of spectral sequences

$$\begin{tikzcd}
H^p(\mathbb{RP}^{4n-1}, K^q(\mathrm{Sp}(n-1), \iota^* h)) \arrow[r, Rightarrow] \arrow[d, "\pi^*"'] & K^{p+q}(\mathrm{PSp}(n), h) \arrow[d, "\pi^*"'] \\
H^p(S^{4n-1}, K^q(\mathrm{Sp}(n-1), \iota^* \pi^* h)) \arrow[r, Rightarrow] & K^{p+q}(\mathrm{Sp}(n), \pi^* h),
\end{tikzcd}$$

with ι^* the pull-back to the fiber. Note that by Theorem 1, $\pi^* h = h$ or $2h$. But in either event, by the results of [14, 23], since h is odd, $K^\bullet(\mathrm{Sp}(n-1), \pi^* h)$ is $\mathbb{Z}/h' \otimes \bigwedge(x_1, \cdots x_{n-2})$, with x_j of odd degree, and with h' *odd*. (Furthermore, the value of h' is independent of whether or not h is multiplied by 2.) So

$$\pi^* \colon H^p(\mathbb{RP}^{4n-1}, K^q(\mathrm{Sp}(n-1), \iota^* h)) \to H^p(S^{4n-1}, K^q(\mathrm{Sp}(n-1), \iota^* \pi^* h))$$

is an isomorphism for all p and q. The result now follows by the comparison theorem for spectral sequences. In the case of $\mathrm{PSU}(n+1)$, we use the diagram (17); note that $\pi^* = 1$ if n is even and is 2 if n is odd. But in that case h, being relatively prime to $n+1$, is odd. In all cases, the order of the torsion in $K^\bullet(\mathrm{SU}(n), \iota^* h)$ is prime to the order of the torsion in the cohomology of the lens space $L^{2n+1} = S^{2n+1}/\mu_{n+1}$. So everything works the same way. □

2.5. **Level-rank dualities.** To conclude this section, we finally mention level-rank dualities, which are certain mysterious relations between WZW theories that involve swapping a parameter for the group with one for the Virasoro level. (See, e.g., [55, 54, 29, 30, 56, 57].) This is quite a deep subject and does seem to be connected with twisted K-theory calculations, though not in a totally straightforward way. In [14, §6] Braun calls this "level-rank nonduality," but this seems to be too harsh, as even in Braun's "nonduality" examples such as B_2 at level 1 and E_8 at level 2 (with only a few exceptions, such as G_2 at level 1 [14, §4], where the twisted K-theory vanishes), the *order of the torsion* in the associated twisted

K-groups is the same, as it is this order that gives the dimension of the associated fusion rings, which are equal for the dual theories.

A few examples will illustrate this point. The most basic case of level-rank dualities is between WZW theories on $\mathrm{SU}(n)$ at level k and $\mathrm{SU}(k)$ at level n [54]. (This has evolved into the "strange duality" conjecture proved in [49, 7, 8].) While these theories might at first seem to have nothing to do with one another, there is one immediate connection. The associated twisting class in both cases is $n + k$ (level plus dual Coxeter number), and if this is relatively prime to both $(n - 1)!$ and $(k-1)!$ (for example, in the case of $\mathrm{SU}(3)_4$ and $\mathrm{SU}(4)_3$), it follows immediately from Theorem 14 that the order of the torsion in the two twisted K-groups is the same. Thus we have a very quick way of verifying this in certain cases. In some cases the duality descends to the adjoint group (cf. [29, (9.3)]); the order of the torsion is also the same for $\mathrm{PSU}(3)_4$ and $\mathrm{PSU}(4)_3$, since the twisting is by $3+4 = 7$ in both cases and we can apply Theorem 20.

3. Orientifold dualities for Lie groups with involution

This section of the paper was motivated by the observation (see for example [34, 21, 22]) that there are interesting examples of T-dualities between orientifold string theories on Calabi-Yau manifolds with holomorphic and anti-holomorphic involutions (in types IIB and IIA, respectively). Such dualities of orientifold theories are expected to result in isomorphisms of (possibly twisted) KR-groups, with a degree shift that depends on the number of circles in which on dualizes. If we interpret the term "Calabi-Yau manifold" in the broadest sense, as a complex manifold, not necessarily compact or simply connected, with vanishing first Chern class, then complex Lie groups provide plenty of examples, since their holomorphic tangent bundles are parallelizable (and thus certainly have vanishing Chern classes). Furthermore, semi-simple Lie groups are the natural setting for WZW models in string theory and conformal field theory, so they provide an obvious case of interest. We further specialize to the case where the orientifold involution is a group automorphism. While this is not the only possibility, and other literature on WZW orientifolds (for example [39, 16, 38, 17, 35]) deals with the case of group anti-automorphisms (or twists thereof), the automorphism case certainly seems like a natural case to consider, and it could be that orientifold theories on a group G with a group involution ι and $G^\iota = H$ are related to "coset theories" on G/H.

The T-dualities we exhibit here are motivated by examples of "group duality" (such as Matsuki duality) that occur in representation theory. We begin with a proposition which is basically well-known.

Proposition 21. *Let G be a connected semi-simple complex Lie group, and let G_0 be a real form of G (so that $G_0 = G^\iota$ for some anti-holomorphic involution ι of G). Let θ be the Cartan involution of G_0, with fixed-point subgroup $K_0 = G_0^\theta$. Then θ can be chosen to commute with ι and to extend to a holomorphic involution of G (which we will again denote by θ) commuting with ι.*

Proof. As we mentioned, this is standard; see for example [25]. □

Theorem 22. *Let G, G_0, K_0, ι, and θ be as in Proposition 21. Then there is a natural isomorphism $KR^\bullet(G,\iota) \cong KR^{\bullet+2(m-n)}(G,\theta)$, where $m = \dim G_0 - \dim K_0$, $n = \dim K_0$. The isomorphism can be viewed as coming from a T-duality of orientifold theories for the two involutions.*

Proof. Let $\mathfrak{g}$ and $\mathfrak{g}_0$ be the Lie algebras of G and G_0, respectively, and let $\mathfrak{k}_0$ be the Lie algebra of K_0, $\mathfrak{k}$ its complexification (the Lie algebra of the complex subgroup K of G given by $K = G^\theta$). Let $\tau = \theta\iota$; since θ and ι commute, this is also an involution, and it's anti-holomorphic since θ is holomorphic and ι is anti-holomorphic. Thus $U = G^\tau$ is another real form of G. Let $\mathfrak{p}$ be the -1-eigenspace of θ; note that the the -1-eigenspace of ι is $i\mathfrak{g}_0$. So we have the decomposition

$$\mathfrak{g} = \mathfrak{g}_0 + i\mathfrak{g}_0 = \mathfrak{k}_0 + i\mathfrak{k}_0 + \mathfrak{p}_0 + i\mathfrak{p}_0, \tag{21}$$

where $\mathfrak{p}_0 = \mathfrak{p} \cap \mathfrak{g}_0$. The Cartan decomposition of G_0 is $G_0 = K_0 \exp(\mathfrak{p}_0)$. The involution τ acts by $+1$ on $\mathfrak{k}_0$ and on $i\mathfrak{p}_0$, and by -1 on $i\mathfrak{k}_0$ and on $\mathfrak{p}_0$. The Killing form is negative definite on $\mathfrak{k}_0$ and positive definite on $\mathfrak{p}_0$, hence is negative definite on $\mathfrak{u} = \mathfrak{k}_0 + i\mathfrak{p}_0$, the Lie algebra of U. So U is compact; it is the compact real form of G (which is unique up to inner automorphisms). So the Cartan decomposition of G is $G = U \exp(i\mathfrak{k}_0 + \mathfrak{p}_0)$.

With all these preliminaries, we can now understand the involutions θ and ι. Topologically, $G = U \times (i\mathfrak{k}_0) \times \mathfrak{p}_0$, where $i\mathfrak{k}_0$ and $\mathfrak{p}_0$ are real vector spaces of dimensions $\dim K_0$ and $\dim G_0 - \dim K_0$, respectively. The involutions ι and θ both send U to itself with fixed-point subgroup K_0. But ι is -1 on $i\mathfrak{k}_0$ and $+1$ on $\mathfrak{p}_0$, whereas θ is $+1$ on $i\mathfrak{k}_0$ and -1 on $\mathfrak{p}_0$. So as Real spaces in the sense of Atiyah [3], but with the opposite indexing convention as in [47], $(G,\theta) = (U,\theta) \times \mathbb{R}^{n,m}$, where $n = \dim K_0$ and $m = \dim G_0 - \dim K_0$, while $(G,\iota) = (U,\theta) \times \mathbb{R}^{m,n}$. So

$$KR^\bullet(G,\theta) \cong KR^{\bullet-m+n}(U,\theta), \quad KR^\bullet(G,\iota) \cong KR^{\bullet-n+m}(U,\theta).$$

The result follows. □

Remark 23. Theorem 22 may seem too weak since it does not incorporate the twist in KR-theory given by the B-field and H-flux (explained in [21, §4 and §5]). This may be remedied by observing that the twist is always almost trivial on the Euclidean space factors $\mathfrak{k}_0$ and $\mathfrak{p}_0$, since they are contractible. (The Dixmier-Douady class must be trivial, and so the only way of twisting is by changing the signs of the O-planes, which simply shifts KR in degree by 4.) Then as long as one uses the same twist on (U,θ) and (U,ι), Theorem 22 continues to hold even in the presence of a topologically nontrivial B-field.

Another comment is that it might seem surprising at first that the degree shift in Theorem 22 is always even, unlike the situation for torus orientifolds in [34, 21, 22], where switching between type IIA and type IIB via a single T-duality involves a shift of 1. This can be explained by the fact that here we are dealing with a slightly different notion of T-duality — it is still "target space" duality, but we no longer have torus bundles. The simplest case of this new kind of T-duality is the duality given by the Fourier transform, between $\mathbb{R}^{1,0}$ and $\mathbb{R}^{0,1}$. (The switch in the involution comes from the fact that the Fourier transform of a real-valued function satisfies $\overline{f(x)} = f(-x)$, not $\overline{f(x)} = f(x)$.) For such T-dualities of real or complex vector spaces, the degree shift in KR-theory is always even.

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