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Chirality of quark modes

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A model for the QCD vacuum based on a domainlike structured background gluon field with a definite duality attributed to the domains has been shown elsewhere to give confinement of static quarks, a reasonable value for the topological susceptibility, and indications that chiral symmetry is spontaneously broken. In this paper, we study in detail the eigenvalue problem for the Dirac operator in such a gluon mean field. A study of the local chirality parameter shows that the lowest nonzero eigenmodes possess a definite mean chirality correlated with the duality of a given domain. A probability distribution of the local chirality qualitatively reproduces histograms seen in lattice simulations.

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I. INTRODUCTION

In a previous paper [1], we formulated a model which characterizes the QCD vacuum by a ‘‘lumpy’’ distribution of field strength and topological charge density. For lack of a better name, we shall refer to the model as ‘‘the domain model.’’ The formulation is given concretely in terms of a partition function which describes a statistical ensemble of domains, each of which is characterized by a set of internal parameters associated with the mean background gluon field, and the internal dynamics are represented by fluctuation fields. Correlation functions in this model can be calculated by taking the mean field into account explicitly and decomposing over the fluctuations. We briefly review the details and assumptions behind the model in the next section but state here unambiguously that the ‘‘domains’’ in question are assumed to be purely quantum in nature. They are not semiclassical solutions of Yang-Mills theory and are not argued to exist as topologically stable classical configurations, rather they seek to characterize the average bulk properties of the ensemble of fields that determine the gluonic vacuum. In particular, it is not assumed that the topological charge associated with a domain should be an integer. The rationale of such an extremely simplified construction can be understood as an attempt to implicitly incorporate effects of the presence of singular pure gauge configurations in the QCD Euclidean functional integral into a practical calculational scheme with a mean-field description of the QCD ground state. A self-consistent mean-field approach requires nonperturbative calculation of the free energy as a functional of the mean field whose minima should determine its form, but this is beyond the reach of analytical methods. Nevertheless, there is an accumulation of semiqualitative arguments [1] in favor of the ansatz for the mean field we have chosen.

In the gluonic sector, the model depends on two parameters, a mean-field strength per domain B and a mean size for domains R, which is sufficient for an adequate description of the pure glue characteristics of the QCD vacuum—the gluon condensate, topological susceptibility, and string tension. The Wilson loop in such a gluonic background was found to exhibit an area law dependence for large loops. Thus a confinement of static fundamental charges is captured by the model; some dynamical gluon degrees of freedom turn out also to be nonpropagating. The absolute value of the underlying average topological charge per domain was determined to be approximately $q=0.15$ and the density of domains to be as high as 42 fm^{-4}. Although tentative signals of spontaneous chiral symmetry breaking were also obtained in [1], a more rigorous consideration of the fermionic spectrum and eigenmodes as well as the calculation of the quark determinant is required, which was missed in [1].

In this paper, we solve the eigenvalue problem for the Dirac operator for the gluonic background and boundary conditions adopted in the model and examine the chirality properties of the eigenmodes. This is a necessary step for checking the status of chiral symmetry breaking in the domain model. But in view of recent lattice results, this problem is valuable also in its own right.

There are strong hints in lattice Monte Carlo simulations at intermediate-range structures in individual gluon configurations once fluctuations are filtered out by some means. For example, cooling or relaxation algorithms are well established now [2], and can reveal instantoniclike structures after several sweeps of a given lattice configuration. However, as these algorithms are designed precisely to locally minimize the action, it is natural they should bring objects with integer topological charge into relief. Alternately, and more relevant to the present work, low-lying and zero modes of the massless Dirac operator can be used as a probe of long-range gluonic structures [3], although only recently did this become more reliable with lattice fermions with good chirality properties. For example, exact index theorems are found to be satisfied on the lattice [4,5] and the zero modes are seen to correlate precisely with instantonic structures in the raw lat-

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tice configuration, in the absence of cooling [6]. However, the exact zero modes of any finite volume simulation cannot be those relevant to spontaneous chiral symmetry breaking, rather the discrete spectrum of low-lying non-zero modes should, in the infinite volume limit, go over to a continuous band at zero, saturating the Banks-Casher relationship [7]. Such modes are sometimes called ''pseudo-zero-modes.'' Low-lying nonzero modes with strong signs of chirality in regions of high action and topological charge densities would be a tool for identification of the properties of gluonic configurations relevant to chiral symmetry breaking. Indeed, after an initial negative result [8], recent results have emerged showing precisely this: low-lying nonzero modes of the overlap Dirac operator which seem to exhibit strong chirality, as measured by the local parameter X introduced in [8] and defined by

$$\tan\left(\frac{\pi}{4}[1-X(x)]\right)=\frac{|\psi_-(x)|}{|\psi_+(x)|}, \qquad (1)$$

in regions where the probability density $\psi^\dagger(x)\psi(x)$ of these modes is maximal [6,9–11]. The verification of the instantonic nature of these objects and their relevance to spontaneous chiral symmetry breaking in the infinite volume are still being argued in the literature (see, for example, the recent studies of [11,12] and [13]).

An unbiased summary of the totality of available lattice results can be formulated as follows: that they support the importance of gluon configurations producing regions of approximately ''locked'' chromoelectric/magnetic fields for chiral symmetry breaking but do not yet confirm or rule out a specifically instantonic nature for these configurations [14]. A potential test which might clarify this would be a comparison of hadronic correlation functions between vacuum models and lattice simulations. Such results are already available for instanton-based models [15]. A search for complementary scenarios for the vacuum consistent with lattice results and incorporating confinement (missed in the instanton models) is evidently timely. This paper is a step in that direction for the domain scenario.

The core of this paper is the Dirac eigenvalue/function problem for a spherical four-dimensional Euclidean region of radius R with baglike boundary conditions on the fermions and in the presence of a covariantly constant (anti-) self-dual gauge field

$$\not{D}\psi(x)=\lambda\psi(x),$$

$$i\not{\eta}(x)e^{i\alpha\gamma_5}\psi(x)=\psi(x), \quad x^2=R^2. \qquad (2)$$

Here $\eta_\mu(x)=x_\mu/|x|$, D_μ is the covariant derivative in the fundamental representation,

$$D_\mu=\partial_\mu-i\hat{B}_\mu=\partial_\mu+\frac{i}{2}\hat{n}B_{\mu\nu}x_\nu,$$

$$\hat{n}=t^a n^a, \quad \tilde{B}_{\mu\nu}=\pm B_{\mu\nu},$$

where the (anti-)self-dual tensor $B_{\mu\nu}$ is constant, and the Euclidean γ matrices are in an anti-Hermitean representation.

The outcomes of this study are the peculiar chiral properties of eigenspinors $\psi(x)$: there are no zero modes and none of the modes is chiral but at the center of a domain the local chirality parameter $X(x)$ is found to be

$$X(0)=\pm 1$$

for all modes with zero orbital momentum. The sign of chirality and the duality of the tensor $B_{\mu\nu}$ are locked: $(+1)-1$ for an (anti-)self-dual field. Simultaneously the normal density for these modes is maximal at the center. At the boundary the local chirality X is equal to zero for all modes. The chirality of the lowest mode is a monotonic function inside the region while for the higher radial excitations the chirality alternates. The detailed form of $X(x)$ changes with the variation of an arbitrary angle α in the boundary condition. This angle is treated as a random variable. Calculating chiralities averaged over a small central region for the various lowest modes, and operating in the whole ensemble of domains, we end up with a histogram which represents the probability of finding a given smeared chirality among the set of lowest modes. The histogram qualitatively reproduces the lattice results for the chirality of low-lying Dirac modes such as those of [6] and others.

After reviewing the domain model in the next section, we present details of the solution of the above-formulated problem and then study the chirality properties of the eigenmodes. We conclude with a discussion and future prospects. Technical details of calculations and conventions for this paper are relegated to the Appendixes.

II. REVIEW OF THE MODEL

It has been suggested [16] that the restrictive influence of pure gauge singularities (present in instanton, monopole, and vortex configurations) on surrounding quantum fluctuations may be used for an approximate treatment of QCD dynamics. Due to the complex structure of the manifold of gauge orbits in QCD, singular gauge fields may be unavoidable in the course of the elimination of redundant variables. Obstructions such as the Gribov problem and condensation of monopoles are two examples of this potentially more general statement. This has also long been advocated by van Baal [17] in his studies of the fundamental domain in small volume studies on the torus and sphere. In particular, the proposal has been made that ''domain formation'' at larger volumes can be driven essentially by the nontrivial topology of the gauge field manifold. Moreover, it is stressed in [17] that the full set of singular fields, instantons, monopoles, and vortices must play a role in this. One can also add to this hierarchy domain-wall singularities [18] which are not topologically stable on their own but can be part of a complicated object: a domain wall can start and end on a lower-dimensional topologically nontrivial singularity of lower dimension, namely a vortex, and in this sense should not be neglected also.

An arbitrary gauge field configuration $\mathcal{A}$ containing a pure gauge singularity can be represented in the vicinity of the singularity as

$$\mathcal{A}_\mu = S_\mu + Q_\mu$$

with S_μ a pure gauge singular field. If we now substitute this into the Yang-Mills Lagrangian, we will see that the requirement of finiteness of the action density imposes specific conditions on the behavior of Q in the vicinity of the singularity in S. The model we consider focuses on domain-wall singular hypersurfaces which are the most restrictive for Q; an inclusion of lower-dimensional singularities is a complicated task beyond the scope of the present work. In the case of a domain wall, the constraining influence of S on gluon fluctuations Q and quark fields ψ is expressed via the boundary conditions

$$[Q,S]=0, \tag{3}$$

$$\overline{\psi}(x)\,\not{n}\,(x)\psi(x)=0 \tag{4}$$

for x being on the singular hypersurface of the pure gauge field S. These conditions ensure a nonvanishing weight for such fields in the functional integral.

Domain-wall singular pure gauge configurations are topologically trivial. This implies that the field S can be characterized by a definite color direction n^a and the matrix $n^a t^a$ can always be tuned to belong to the Cartan subalgebra of $SU_c(3)$. The off-diagonal (or, equivalently, orthogonal to n^a) components of the fluctuations Q must then satisfy Dirichlet boundary conditions, while those fluctuations longitudinal to n^a are not restricted at the domain wall. A typical configuration of this type looks like a system of domains which are coupled in a sense that fluctuations inside neighboring domains interact with each other via exchange by the gluon modes longitudinal to the color direction of the domain boundaries. It should be stressed that unavoidably there are obstructions of color direction at the domain-wall junctions where lower-dimensional topologically nontrivial singularities are situated.

To be specific and to deal with an analytically tractable model, we introduce several drastic simplifications: we disengage ourselves from the obstructions in the color direction and substitute the coupling between domains by the presence of a mean field. Inside and on the boundary of the domain, the field is taken to be covariantly constant (anti-)self-dual such that the strength over the whole Euclidean space reads

$$F^a_{\mu\nu}(x)=\sum_{j=1}^{N} n^{(j)a}B^{(j)}_{\mu\nu}\theta(1-(x-z_j)^2/R^2),$$

$$B^{(j)}_{\mu\nu}B^{(j)}_{\mu\rho}=B^2\delta_{\nu\rho}. \tag{5}$$

The individual color and space orientations in each domain are random. In particular, effective action arguments were used in [1] to constrain the form of $n^{(j)a}$ such that the matrix $\hat{n}^{(j)}=t^3\cos\xi_j+t^8\sin\xi_j$ with angles $\xi_j\in\{\pi/6(2k+1),k=0,\ldots,5\}$ corresponding to the discrete Weyl subgroup. Domains are taken to be hyperspherical with a mean radius R and centered at random points z_j. For a detailed motivation of these steps, we refer the reader to [1].

In this way, consideration is reduced to a model with essentially two free parameters: the mean-field strength B and the mean domain radius R. The partition function for this simplified system can be written down as

$$\mathcal{Z}=\mathcal{N}\lim_{V,N\to\infty}\prod_{i=1}^{N}\int_\Sigma d\sigma_i\int_{\mathcal{F}^i_\psi}\mathcal{D}\psi^{(i)}\mathcal{D}\overline{\psi}^{(i)}$$
$$\times\int_{\mathcal{F}^i_Q}\mathcal{D}Q^i\,\delta[D(\breve{\mathcal{B}}^{(i)})Q^{(i)}]\Delta_{\rm FP}[\breve{\mathcal{B}}^{(i)},Q^{(i)}]$$
$$\times e^{-S^{\rm QCD}_{V_i}[Q^{(i)}+\mathcal{B}^{(i)},\psi^{(i)},\overline{\psi}^{(i)}]}, \tag{6}$$

where the functional spaces of integration $\mathcal{F}^i_Q$ and $\mathcal{F}^i_\psi$ are specified by the boundary conditions $(x-z_i)^2=R^2$,

$$\breve{n}_i Q^{(i)}(x)=0, \tag{7}$$

$$i\,\not{\eta}_i(x)e^{i\alpha_i\gamma_5}\psi^{(i)}(x)=\psi^{(i)}(x), \tag{8}$$

$$\overline{\psi}^{(i)}e^{i\alpha_i\gamma_5}i\,\not{\eta}_i(x)=-\overline{\psi}^{(i)}(x). \tag{9}$$

Here $\breve{n}_i=n^a_i t^a$ with the color generators t^a in the adjoint representation. The conditions Eqs. (8) and (9) represent specific (though not unique) choices for the implementation of Eq. (4) which manifest the explicit breaking of chiral symmetry by the boundary condition, as occurs, for example, in bag models for the nucleon. The thermodynamic limit assumes $V,N\to\infty$ but with the density $v^{-1}=N/V$ taken fixed and finite. The partition function is formulated in a background field gauge with respect to the domain mean field. The measure of integration over parameters characterizing domains is

$$\int_\Sigma d\sigma_i\cdots=\frac{1}{48\pi^2}\int_V\frac{d^4z_i}{V}\sum_{\nu_i=-\infty}^{\infty}\int_{(2\nu_i-1)\pi}^{(2\nu_i+1)\pi}d\alpha_i$$
$$\times\int_0^{2\pi}d\varphi_i\int_0^{\pi}d\theta_i\sin\theta_i\int_0^{2\pi}d\xi_i$$
$$\times\sum_{l=0,1,2}^{3,4,5}\delta\left(\xi_i-\frac{(2l+1)\pi}{6}\right)$$
$$\times\int_0^{\pi}d\omega_i\sum_{k=0,1}\delta(\omega_i-\pi k)\cdots, \tag{10}$$

where (θ_i,φ_i) are the spherical angles of the chromomagnetic field, ω_i is the angle between chromoelectric and chromomagnetic fields, and ξ_i is an angle parametrizing the color orientation. It should be noted that because of the axial anomaly and that nothing *a priori* constrains the topological charge per domain to be integral, the fermion determinant is a single-valued function of α_i only if an appropriate Riemann surface is constructed. Here ν_i enumerates the Riemann sheets to be taken into account.

This partition function describes a statistical system of density v^{-1} composed of extended domainlike structures, each of which is characterized by a set of internal parameters and whose internal dynamics are represented by the fluctuation fields. It respects all the symmetries of the QCD Lagrangian, since the statistical ensemble is invariant under space-time and color gauge symmetries. For the same reason, if the quarks are massless, then the chiral invariance is respected.

Field eigenmodes satisfying the above boundary conditions in the presence of an (anti-)self-dual gluon field and corresponding Green functions can be calculated explicitly. For gluons, this was shown in [1]. For quarks, this will be shown in this paper. On this basis one can compute any correlation function taking the mean field into account exactly and decomposing the integrand over fluctuations. In particular, correlation functions of the mean field itself have a finite radius R, which is more or less obvious and is discussed in [1] in detail.

Within this framework the gluon condensate to lowest order in fluctuations is immediately obtained in the form

$$g^2\langle F^a_{\mu\nu}(x)F^a_{\mu\nu}(x)\rangle=4B^2, \tag{11}$$

and the topological susceptibility reads

$$\chi=\int d^4x\langle Q(x)Q(0)\rangle=\frac{B^4R^4}{128\pi^2}.$$

Less trivial is the manifestation of an area law for static quarks. Computation of the Wilson loop for a circular contour of a large radius $L\gg R$ gives a string tension $\sigma=Bf(\pi BR^2)$ with the function

$$f(z)=\frac{2}{3z}\left(3-\frac{\sqrt{3}}{2z}\int_0^{2z/\sqrt{3}}\frac{dx}{x}\sin x-\frac{2\sqrt{3}}{z}\int_0^{z/\sqrt{3}}\frac{dx}{x}\sin x\right).$$

Estimations of the values of these quantities are known from lattice calculation or phenomenological approaches and can be used to fit B and R. As described in [1], these parameters are fixed to be

$$\sqrt{B}=947\ \text{MeV},\quad R=(760\ \text{MeV})^{-1}=0.26\ \text{fm} \tag{12}$$

with the average absolute value of topological charge per domain turning out to be $q\approx0.15$ and the density of domains $v^{-1}=42\ \text{fm}^{-4}$. The topological susceptibility then turns out to be $\chi\approx(197\ \text{MeV})^4$, comparable to the Witten-Veneziano value [19].

III. SPECTRUM OF THE DIRAC OPERATOR IN A DOMAIN

We have mentioned already that the boundary conditions on fermions violate chiral symmetry explicitly, which can only be restored by a random assignment of values of α over the complete ensemble of domains in Euclidean space.

In this section, we address the eigenvalue problem for the massless Dirac operator as it is stated in Eqs. (2). Here we give the scheme for solving the problem, with technical details given in the Appendixes. The Dirac matrices in Euclidean space are chosen to be anti-Hermitean and taken in the chiral representation.

For boundary conditions on a hypersphere and covariantly constant background field of definite duality, it is natural to use hyperspherical coordinates (r,Ω), given in detail in Appendix A. In such coordinates, rather than work with the covariant derivative itself, it is more convenient to introduce the operator $\not{x}\not{D}$ which can be easily expanded into intrinsic and orbital angular momentum generators. Any spinor can be represented in the form

$$\psi=i\not{x}\chi+\varphi,\quad \bar{\psi}=i\bar{\chi}\not{x}+\bar{\varphi}, \tag{13}$$

where φ and χ have the same chirality. This is simply a decomposition into a sum of chiral components. The eigenvalue equation (2) can be rewritten then identically as

$$\chi=-\frac{1}{i\lambda}\not{x}\not{D}\varphi,\quad \not{D}^2\varphi=\lambda^2\varphi. \tag{14}$$

In these terms the boundary conditions take the form

$$\chi=-e^{\mp i\alpha}\varphi,\quad \bar{\chi}=\bar{\varphi}e^{\mp i\alpha},\quad x^2=R^2, \tag{15}$$

where upper (lower) signs correspond to φ and χ with chirality ∓1.

A solution of Eqs. (14) is achieved by separating the angular and radial coordinates. To do this one has to represent, respectively, $\not{D}^2$ and $\not{x}\not{D}$ in terms of momentum generators and projectors onto the various spin and color polarization subspaces. In four-dimensional Euclidean space, the angular momentum operators can be represented as

$$\mathbf{K}_{1,2}=\frac{1}{2}(\mathbf{L}\pm\mathbf{M})$$

with $\mathbf{L}$ the usual three-dimensional angular momentum operator and $\mathbf{M}$ the Euclidean version of the boost operator. These correspond to the decomposition of the four-dimensional rotational group $SO(4)$ into a product of two $SO(3)$ groups. They lead to Casimir operators and eigenvalues

$$\mathbf{K}_1^2=\mathbf{K}_2^2\to\frac{k}{2}\left(\frac{k}{2}+1\right),\quad k=0,1,\dots,\infty,$$

$$K^z_{1,2}\to m_{1,2},\quad m_{1,2}=-k/2,-k/2+1,\dots,k/2-1,k/2,$$

and the correponding angular eigenfunctions $C_{km_1m_2}(\Omega)$, given explicitly in Appendix A, are labeled by orbital momentum k and two azimuthal numbers m_1 and m_2. Eigenstates are also characterized by the color-spin polarization related to the projectors

$$O_\pm=N_+\Sigma_\pm+N_-\Sigma_\mp \tag{16}$$

with

$$N_{\pm}=\frac{1}{2}(1\pm\hat{n}/|\hat{n}|),\quad \Sigma_{\pm}=\frac{1}{2}(1\pm\mathbf{\Sigma B}/B)$$

being, respectively, the separate projectors for color and spin polarizations. Below we denote the polarization with respect to O by $\kappa=\pm$.

It is shown in Appendix B that if the background field is (anti-)self-dual, the boundary condition can only be implemented if spinors φ and χ are (right-) left-handed. Also the presence of the homogeneous background field reduces the spherical symmetry of the problem down to an axial symmetry. In the representation implemented here, this manifests itself as a restriction on the values of one of the azimuthal quantum numbers, namely $m_2=\pm k/2$ for the self-dual case and $m_1=\pm k/2$ for the anti-self-dual one. The sign in front of $k/2$ is correlated with the spin polarization of the state as seen in the explicit expressions for the eigenspinors below.

Thus for the self-dual case, $\gamma_5\varphi=-\varphi$, $\gamma_5\chi=-\chi$ so that the eigenspinors in the self-dual field can be labeled as $\psi^{-\kappa}_{km_1}(x)$, while in the anti-self-dual field they are $\psi^{+\kappa}_{km_2}(x)$. With details in Appendix B, we simply write down here the result for the self-dual case,

$$\psi^{-\kappa}_{km_1}=i\not{\eta}\,\chi^{-\kappa}_{km_1}+\varphi^{-\kappa}_{km_1},$$

$$\chi^{-+}_{km_1}=-(i\Lambda)^{-1}z^{(k+1)/2}e^{-z/2}\left[M(k+2-\Lambda^2,k+2,z)-\frac{k+2-\Lambda^2}{k+2}M(k+3-\Lambda^2,k+3,z)\right]\begin{pmatrix}0\\0\\N_-\mathcal{C}_{km_1(k/2)}(\Omega)\\N_+\mathcal{C}_{km_1-(k/2)}(\Omega)\end{pmatrix},\tag{17}$$

$$\varphi^{-+}_{km_1}=z^{k/2}e^{-z/2}M(k+2-\Lambda^2,k+2,z)\begin{pmatrix}0\\0\\N_-\mathcal{C}_{km_1(k/2)}(\Omega)\\N_+\mathcal{C}_{km_1-(k/2)}(\Omega)\end{pmatrix},\tag{18}$$

$$\chi^{--}_{km_1}=z^{(k+1)/2}e^{-z/2}\frac{i\Lambda}{k+2}M(1-\Lambda^2,k+3,z)\begin{pmatrix}0\\0\\N_+\mathcal{C}_{km_1(k/2)}(\Omega)\\N_-\mathcal{C}_{km_1-(k/2)}(\Omega)\end{pmatrix},$$

$$\varphi^{--}_{km_1}=z^{k/2}e^{-z/2}M(-\Lambda^2,k+2,z)\begin{pmatrix}0\\0\\N_+\mathcal{C}_{km_1(k/2)}(\Omega)\\N_-\mathcal{C}_{km_1-(k/2)}(\Omega)\end{pmatrix},\tag{19}$$

where $M(a,b,x)$ is the confluent hypergeometric function and

$$z=\hat{B}r^2/2,\quad \Lambda=\lambda/\sqrt{2\hat{B}},\quad \hat{B}=|\hat{n}|B.$$

The projectors $N_{\pm}$ act on the color vectors which are implicit in the above equations. The eigenfunctions $\psi^{+\kappa}_{km_2}$ for the anti-self-dual case are obtained by the change $m_1\to m_2$ and the shift of nonzero elements of the angular part to the first two positions of the spinor.

The eigenvalues are determined by the boundary condition at $z=z_0=\hat{B}R^2/2$, which for Λ^{-+}_k takes the form

$$e^{-i\alpha}M(k+2-\Lambda^2,k+2,z_0)-\frac{\sqrt{z_0}}{i\Lambda}\left[M(k+2-\Lambda^2,k+2,z_0)-\frac{k+2-\Lambda^2}{k+2}M(k+3-\Lambda^2,k+3,z_0)\right]=0,\tag{20}$$

and for Λ^{--}_k,

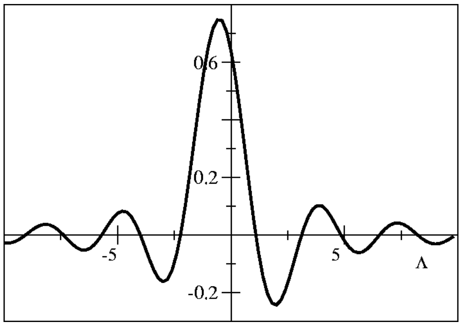

FIG. 1. Graphical representation of the left-hand side of Eq. (21) for $k=0$, a self-dual domain, and $\alpha=\pi/2$. Zeros of the function are the radial eigenvalues Λ_{0n}^{--}.

$$e^{-i\alpha}M(-\Lambda^2,k+2,z_0)+\frac{i\Lambda\sqrt{z_0}}{k+2}M(1-\Lambda^2,k+3,z_0)=0. \tag{21}$$

The equations for the eigenvalues in an anti-self-dual domain are the same as above but with $\alpha\to-\alpha$ as follows from Eqs. (15). The eigenvalues can be calculated numerically. They form a discrete set. Zero modes are absent, which is to be expected for these types of boundary conditions [21]. A graphical solution of Eq. (21) at $\alpha=\pi/2$ is presented in Fig. 1 to illustrate the structure of the spectrum. In general, the eigenvalues are complex. The spectrum is real for $\alpha=\pm\pi/2$, which is the only value for which the boundary condition Eq. (7) imposed on $\overline{\psi}_{km_{1,2}}^{\pm\kappa}$ is Hermitean conjugated to the condition for $\psi_{km_{1,2}}^{\pm\kappa}$ and the general fermion field $\overline{\psi}$ can be decomposed in terms of the basis of conjugate eigenfunctions $\psi_{km_{1,2}}^{\pm\kappa\dagger}$. For other values of α, a biorthogonal basis should be introduced. In particular, at $\alpha=0$ eigenvalues are complex and come in complex conjugated pairs. The partition function is nevertheless real since if $\lambda_{\rm sd}(\alpha)$ is an eigenvalue for the self-dual case, then for the anti-self-dual domain there is an eigenvalue $\lambda_{\rm asd}(\alpha)$ such that

$$\lambda_{\rm asd}(\alpha)=-\lambda^*_{\rm sd}(\alpha). \tag{22}$$

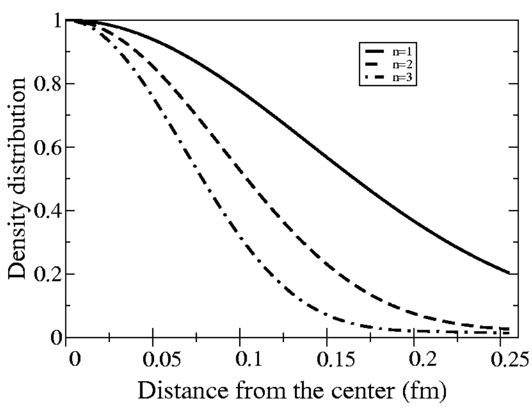

FIG. 2. Normal density distribution for modes with $k=0$ and $n=1,2,3$.

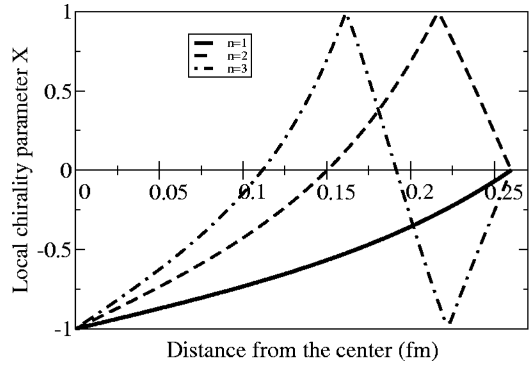

FIG. 3. Chirality parameter for the three lowest radial modes ψ_{00}^{--}, self-dual domain, $\alpha=\pi/2$.

We stress that the definition of λ here does not include an imaginary unity in front of the Dirac operator.

As seen from Fig. 1, in contradistinction to the eigenvalue problem in infinite volume on the space of square integrable functions, the spectrum is not symmetric under reflections $\lambda\to-\lambda$. This comes from the fact that γ_5 does not commute with the boundary condition so that $\gamma_5\psi$ is not an eigenfunction if ψ is an eigenfunction. An asymmetry of the spectrum is typical for the Dirac operator in odd-dimensional spaces (see [20] and references therein) and has important consequences there for the effective action. In our case, the unusual boundary conditions are responsible for the asymmetry in four-dimensional Euclidean space [21].

The most interesting feature of the fermionic eigenmodes becomes manifest if one considers the local chirality $X(x)$ of the lowest eigenmodes as defined by Eq. (1).

IV. CHIRALITY OF LOW-LYING MODES

It is obvious that none of the solutions are eigenstates of γ_5. However, at the domain center $x^2=0$ [or $(x-z_j)^2=0$ in general] all the purely radial modes with $k=0$ have a maximum in the probability density, they are chiral, and the sign of their chirality is determined by the duality of the mean field in a domain which is illustrated in Figs. 2 and 3. The probability density naturally vanishes at the domain center for the modes with $k>0$, as is seen in Fig. 4. To demonstrate

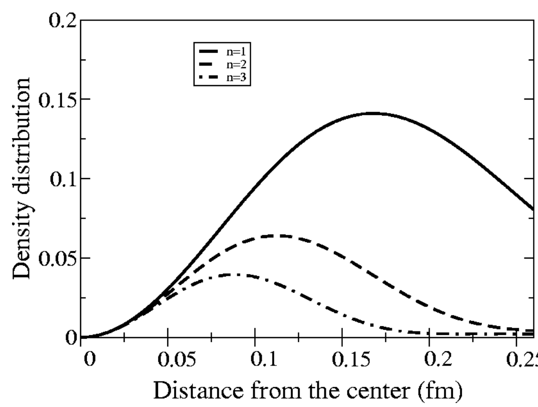

FIG. 4. Plot of the radial dependence of the normal density distribution for the modes with $k=1$ and $n=1,3,5$.

this analytically, let us turn to the local chirality parameter given in the Introduction, which we rewrite here in a more detailed form,

$$\tan\left(\frac{\pi}{4}[1-X(x)]\right)=\sqrt{\frac{\psi^{\dagger}(x)(1-\gamma_5)\psi(x)}{\psi^{\dagger}(x)(1+\gamma_5)\psi(x)}},$$

$$-1\leqslant X(x)\leqslant 1,$$

which takes the extremal values $X=\pm 1$ at positions x where $\psi(x)$ is purely right (left) -handed. Because φ and χ have the same chirality, the representation Eq. (17) immediately gives, for the self-dual domain,

$$\tan\left(\frac{\pi}{4}[1-X_{\rm sd}^{-\kappa}(x)]\right)=\frac{|\varphi_{00}^{-\kappa}(x)|}{|\chi_{00}^{-\kappa}(x)|},$$

while for the anti-self-dual case the local chirality reads

$$\tan\left(\frac{\pi}{4}[1-X_{\rm asd}^{+\kappa}(x)]\right)=\frac{|\chi_{00}^{+\kappa}(x)|}{|\varphi_{00}^{+\kappa}(x)|}.$$

Moreover, due to the relation Eq. (22),

$$|\varphi_{00}^{+\kappa}(x)|=|\varphi_{00}^{-\kappa}(x)|,\quad |\chi_{00}^{+\kappa}(x)|=|\chi_{00}^{-\kappa}(x)|.$$

Representations Eqs. (18) and (19) show that

$$0<\lim_{x^2\to 0}|\varphi_{00}^{+\kappa}(x)|<\infty,\quad \lim_{x^2\to 0}|\chi_{00}^{+\kappa}(x)|=0,$$

which finally results in

$$X_{\rm sd}^{-\kappa}(0)=-1,\quad X_{\rm asd}^{+\kappa}(0)=1.$$

The local chirality parameter X as a function of distance from the domain center for the lowest few modes is plotted in Fig. 3. There is a peak in X at the domain center. Away from the center X decreases due to a competition of left and right components of the eigenmodes as the χ component becomes nonvanishing. As is seen from Fig. 3, the chirality of the lowest mode $(n=1)$ monotonically decreases with distance from the center. The chirality parameter for the excited modes alternates between extremal values, the number of alternations is correlated with the radial number n, and the half-width decreases with growing n. The chirality parameter X is zero at the boundary for all modes. Qualitatively this picture does not depend on the angle α. The "width" of the peaks at half-maximum for the lowest $(n=0)$ radial modes varies for different values α and is of the order of $0.12-0.14$ fm if the values of B and R are fixed from the gluon condensate and the string tension, consistent with the lattice observations of [11].

We can now study the chirality characteristics of the ensemble of fermion fields entering the partition function Eq. (6) with all values of α treated with equal probability consistent with an explicit chiral symmetry. On the lattice [6,8], peaks in X or $\psi^{\dagger}\psi$ would only be localizeable within a size corresponding to the lattice spacing. To take this into account, we average $X(x)$ over a small neighborhood of the domain center. Thus we compute the probability to find a given value of $\overline{X}$, the smeared X, among the chiralities for the lowest modes. The result given in Fig. 5 was obtained for three sets of modes: with $n\leqslant 2,k=0$ (solid line), $n\leqslant 4,k=0$ (dashed line), and $n\leqslant 6,k=0$ (dot-dashed line), and all possible values of α and spin-color polarizations. The solid line, formed from the lowest modes, evidently indicates two narrow peaks with $\overline{X}\approx\pm 0.87$. This double peaking is not unexpected in view of the above-discussed chirality properties. Including higher modes broadens the peaks and shifts their maxima. This feature as well as the above-mentioned values for the half-width and the density of domains is in qualitative and quantitative agreement with recent lattice results [6,9–11]. It should be stressed that orbital excitations $(k>0)$ are not included in the histograms because the probability density for orbital modes vanishes at the center. However, there are maxima in the probability density for these modes in peripheral regions of the domain. The local chirality X is significantly smaller in peak value than those for the radial modes at the center. Inclusion of orbital modes will broaden the peaks more and build up the central plateau.

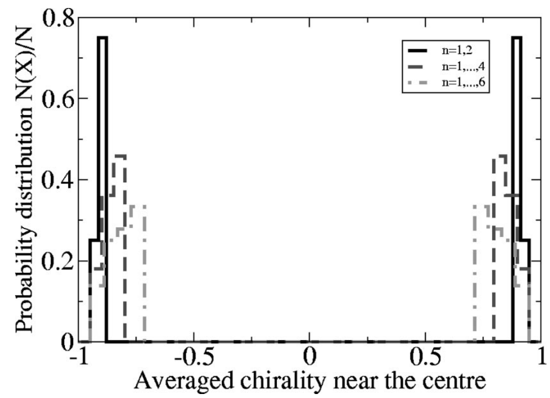

FIG. 5. Histogram of chirality parameter $\overline{X}$ averaged over the central region with radius 0.025 fm. Plots given in solid, dashed, and dot-dashed lines incorporate all modes with $n\leqslant 2$, $n\leqslant 4$, and $n\leqslant 6$, respectively.

V. DISCUSSION AND CONCLUSIONS

The statement that signals for spontaneous chiral symmetry breaking should be identifiable in the specific chirality properties of fermionic eigenmodes for some "dominant" gluonic background field is generally accepted. Such signals are now being seen on the lattice, but nevertheless there are not very many analytically explicit examples of this relationship available. Instanton-motivated models are certainly the most advanced example of this kind.

We have studied the spectrum of quark modes in a domainlike structured gluon background field. Such a background is argued to characterize the bulk average properties of the vacuum in the presence of strong intermediate range fluctuations and is not the result of a semiclassical approximation. The spectrum exhibits definite chirality properties. In particular, there are no zero modes because of the conditions which fermion fields must satisfy on the boundaries of domains. Nonetheless, at the center of domains all radial modes are purely chiral and the sign of their chirality depends on whether the underlying gluon field is self-dual or anti-self-dual. Moreover, the sign of chirality at the center persists over the whole domain for the lowest modes. Studying the local chirality parameter X in a chirally symmetric ensemble of domains, we obtain qualitatively similar results to those seen in lattice calculations. We stress that this comparison with lattice results takes place at the level of an ensemble of configurations not on a configuration-by-configuration basis.

Insofar as these lattice results for chirality are argued as supporting the evidence for spontaneous chiral symmetry breaking, the same can be said of the domain model. We note the absence of any explicit zero modes in achieving this. Namely, the range of configurations needed to produce the types of effects seen on the lattice is not restricted to instantonlike fields. It suffices that a given gluon background admit strongly chiral low-lying nonzero modes. In this respect, the more significant property of the gluon background is the ''locking'' of chromoelectric and chromomagnetic fields into self-dual or anti-self-dual fields in relatively large but finite regions of space restricted by the hypersurfaces on which pure gauge singularities are assumed to be situated. It should be stressed that in the thermodynamic limit, the number of domains is growing but their sizes stay fixed around some finite mean value.

The solutions obtained in this paper provide a basis for computation of chiral condensate $\langle\bar\psi\psi\rangle$, in particular in the presence of an explicit CP-violating θ term. This work is in progress.

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APPENDIX A: CONVENTIONS

We use a chiral representation for the anti-Hermitian Dirac matrices in four Euclidean space,

$$\{\gamma_\mu,\gamma_\nu\}=-2\delta_{\mu\nu},\quad \gamma_\mu^+=-\gamma_\mu,$$

$$\gamma_i=\begin{pmatrix}0 & \sigma_i\\ -\sigma_i & 0\end{pmatrix},\quad \gamma_4=i\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},$$

$$\gamma_5=\gamma_1\gamma_2\gamma_3\gamma_4=\mathrm{diag}(1,1,-1,-1).$$

The background field is specified as

$$\hat B_\mu(x)=-\frac{1}{2}\hat n B_{\mu\nu}x_\nu$$

$$B_{ij}=\varepsilon_{ijk}B_k,\quad B_i=\frac{1}{2}\varepsilon_{ijk}B_{jk},\quad E_i=B_{i4}=\pm B_i,$$

$$B_{\mu\nu}B_{\mu\rho}=\delta_{\nu\rho}B^2,\quad \tilde B_{\mu\nu}=\frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}B_{\alpha\beta}=\pm B_{\mu\nu}.$$

In addition, the following conventions and relations have been used:

$$\sigma_{\mu\nu}=\frac{1}{2i}[\gamma_\mu,\gamma_\nu],\quad \Sigma_i=\frac{1}{2}\varepsilon_{ijk}\sigma_{jk},$$

$$\sigma_{ij}B_{ij}=2\Sigma_i B_i,\quad \Sigma_\pm=\frac{1}{2}\left(1\pm\frac{\Sigma_i B_i}{B}\right),$$

$$\sigma_{i4}=-\frac{1}{2}\gamma_5\varepsilon_{i4\mu\nu}\sigma_{\mu\nu}=-\frac{1}{2}\gamma_5\varepsilon_{ijk}\sigma_{jk}=-\gamma_5\Sigma_i,$$

$$\gamma_5\sigma_{\alpha\beta}=-\tilde\sigma_{\alpha\beta},\quad \mathrm{Tr}\,\gamma_\mu\gamma_\nu\gamma_\alpha\gamma_\beta\gamma_5=4\varepsilon_{\mu\nu\alpha\beta},$$

and in particular

$$\begin{aligned}\sigma_{\alpha\beta}B_{\alpha\beta}&=\sigma_{ij}B_{ij}+2\sigma_{i4}B_{i4}=2\Sigma_iB_i\mp 2B_i\gamma_5\Sigma_i=4P_\mp\Sigma_iB_i\\ &=4BP_\mp(\Sigma_+-\Sigma_-).\end{aligned}$$

We use the following hyperspherical coordinate system in R^4:

$$x_1=r\sin\eta\cos\phi,\quad x_2=r\sin\eta\sin\phi,$$

$$x_3=r\cos\eta\cos\chi,\quad x_4=r\cos\eta\sin\chi,\tag{A1}$$

and define the angular momentum operators as follows:

$$L_i=-i\epsilon_{ijk}x_j\partial_k,$$

$$M_i=-i(x_4\partial_i-x_i\partial_4),$$

respectively, for spatial rotations and Euclidean ''boosts.'' As mentioned, it is more convenient to work in the basis

$$\mathbf{K}_{1,2}=(\mathbf{L}\pm\mathbf{M})/2,$$

which generates the following Lie algebra:

$$[K_1^i,K_1^j]=i\epsilon^{ijk}K_1^k,\quad [K_2^i,K_2^j]=i\epsilon^{ijk}K_2^k,\quad [K_1^i,K_2^j]=0.$$

Thus the ladder operators

$$K_{1,2}^\pm=(K_{1,2}^1\pm iK_{1,2}^2)$$

satisfy the algebra

$$[K_{1,2}^3,K_{1,2}^\pm]=\pm K_{1,2}^\pm$$

and correspond to raising and lowering operators of m_1,m_2.

The angular eigenfunctions corresponding to the $\mathbf{K}_{1,2}$ generators are

$$C_{km_1m_2}(\eta,\phi,\chi) = (-1)^{|m_1+m_2|}(2\pi)^{-1}\Theta_k^{m_1-m_2,m_1+m_2}(\eta)\times\exp i[(m_1-m_2)\chi+(m_1+m_2)\phi],$$

$$\Theta_k^{k-r-s,s-r}(\eta) = \sqrt{2(k+1)(k-r)!(k-s)!r!s!}\times\sum_{n=0}^{r}\frac{(-1)^{r-n}\cos^{k-r-s+2n}\eta\sin^{r+s-2n}\eta}{(k-r-s+n)!n!(r-n)!(s-n)!},$$

$$s=(k+m_2)/2,\quad r=(k-m_1)/2,$$

where k,m_1,m_2 are, respectively, the orbital angular momentum and the two azimuthal quantum numbers, relevant for an $O(4)=O(3)\times O(3)$ symmetry. They take the following values:

$$k=0,1,2,\ldots,\quad m_1,m_2=-\frac{k}{2},\ldots,\frac{k}{2}.$$

APPENDIX B: DIRAC EIGENVALUE PROBLEM IN A DOMAIN

Here we give further details of the solution of Eq. (2). Using the notation given in the main body of the text, we can decompose the field φ over a set of chiral and color-spin projectors,

$$\varphi=P_{\pm}\Phi_0+P_{\mp}O_{+}\Phi_{+1}+P_{\mp}O_{-}\Phi_{-1},$$

where (lower) upper signs correspond to the (anti-)self-dual field background field, and fields Φ_ζ must satisfy the second-order equation

$$(-D^2+2\zeta\hat{B}-\lambda^2)P_{\mp}O_\zeta\Phi_\zeta=0. \tag{B1}$$

We remind the reader that implicitly Φ_ζ^α is the color vector in the fundamental representation.

If we were solving the problem for square-integrable eigenfunctions in infinite volume, then all three components Φ_ζ would enter the final set of eigenfunctions. Moreover, as is seen from Eq. (B1), the equation for the component Φ_{-1} would produce zero modes with chirality ∓ 1. The spectrum in this case would be discrete and all nonzero eigenvalues of the Dirac operator come in pairs: if ψ is an eigenfunction with eigenvalue λ, then $\gamma_5\psi$ is an eigenfunction with eigenvalue $-\lambda$—quite a standard state of affairs.

However, the baglike boundary conditions Eq. (8) we must satisfy in the present case change the structure of eigenfunctions and eigenvalues drastically. First of all, because of the identities

$$\gamma_\mu B_{\mu\rho}x_\rho P_{\mp}\Sigma_{+}=iB\not{x}P_{\mp}\Sigma_{+},$$

$$\gamma_\mu B_{\mu\rho}x_\rho P_{\mp}\Sigma_{-}=-iB\not{x}P_{\mp}\Sigma_{-},$$

and

$$\gamma_\mu B_{\mu\rho}x_\rho P_{\pm}\Sigma_{+}=\left\{\frac{2i}{x^2}B[x_i^2-(B_ix_i)^2/B^2]-iB\right\}\not{x}P_{\pm}\Sigma_{+},$$

$$\gamma_\mu B_{\mu\rho}x_\rho P_{\pm}\Sigma_{-}=-\left\{\frac{2i}{x^2}B[x_i^2-(B_ix_i)^2/B^2]-iB\right\}\not{x}P_{\pm}\Sigma_{-},$$

which can be straightforwardly derived by expanding both sides over a complete set of Dirac matrices, the baglike boundary condition can be satisfied only for the trivial solution $\Phi_0(x)\equiv 0$. The significance of this observation is that for (anti-)self-dual domains, the boundary condition can only be implemented on eigenspinors $\psi=i\not{n}\chi+\phi$ for (positive) negative chirality φ and χ. The function ψ in turn is not an eigenspinor of γ_5, which is natural because the boundary condition violates chiral symmetry. Furthermore, zero modes are removed from the spectrum because they must be chiral, but this is forbidden by boundary conditions. And, finally, if ψ is an eigenfunction with eigenvalue λ, then $\gamma_5\psi$ is not an eigenfunction anymore, and there is no eigenvalue $-\lambda$ in the spectrum.

In order to find equations for components Φ_ζ^α of the corresponding spinors, we use that

$$\varphi_{-\pm}=P_{-}O_{\pm}\Phi_{\pm 1}=(0,0,N_{\mp}\Phi_{\pm 1}^3,N_{\pm}\Phi_{\pm 1}^4)^{\mathrm{T}},$$

$$\varphi_{+\pm}=P_{+}O_{\pm}\Phi_{\pm 1}=(N_{\mp}\Phi_{\pm 1}^1,N_{\pm}\Phi_{\pm 1}^2,0,0)^{\mathrm{T}}. \tag{B2}$$

In hyperspherical coordinates Eqs. (A1), the equations for the spinor components read (here and below we write down equations for the self-dual case only)

$$\left\{-\left[\frac{1}{r^3}\partial_r r^3\partial_r-\frac{4}{r^2}\mathbf{K}_1^2+2\frac{\hat{n}}{|\hat{n}|}\hat{B}K_{2z}-\frac{1}{4}\hat{B}^2r^2\right]+2\hat{B}-\lambda^2\right\}\times N_{-}\Phi_{+1}^3=0,$$

$$\left\{-\left[\frac{1}{r^3}\partial_r r^3\partial_r-\frac{4}{r^2}\mathbf{K}_1^2+2\frac{\hat{n}}{|\hat{n}|}\hat{B}K_{2z}-\frac{1}{4}\hat{B}^2r^2\right]+2\hat{B}-\lambda^2\right\}\times N_{+}\Phi_{+1}^4=0,$$

$$\left\{-\left[\frac{1}{r^3}\partial_r r^3\partial_r-\frac{4}{r^2}\mathbf{K}_1^2+2\frac{\hat{n}}{|\hat{n}|}\hat{B}K_{2z}-\frac{1}{4}\hat{B}^2r^2\right]-2\hat{B}-\lambda^2\right\}\times N_{+}\Phi_{-1}^3=0,$$

$$\left\{-\left[\frac{1}{r^3}\partial_r r^3\partial_r-\frac{4}{r^2}\mathbf{K}_1^2+2\frac{\hat{n}}{|\hat{n}|}\hat{B}K_{2z}-\frac{1}{4}\hat{B}^2r^2\right]-2\hat{B}-\lambda^2\right\}\times N_{-}\Phi_{-1}^4=0. \tag{B3}$$

The anti-self-dual case is reconstructed by the change $K_{2z}\to K_{1z}$ and $\Phi_\zeta^3\to\Phi_\zeta^1$, $\Phi_\zeta^4\to\Phi_\zeta^2$. In [1], we derived the general solution for equations of this type. The requirement of regularity at the origin then gives

$$\Phi_{+1}^{3,km_1m_2}=N_{-}z^{k/2}e^{-z/2}M\left(\frac{k}{2}+m_2-\Lambda^2+2,k+2,z\right)\times\mathcal{C}_{km_1m_2}(\varphi,\chi,\eta),$$

$$\Phi_{+1}^{4,km_1m_2}=N_+z^{k/2}e^{-z/2}M\left(\frac{k}{2}-m_2-\Lambda^2+2,k+2,z\right)\times\mathcal{C}_{km_1m_2}(\varphi,\chi,\eta),$$

$$\Phi_{-1}^{3,km_1m_2}=N_+z^{k/2}e^{-z/2}M\left(\frac{k}{2}-m_2-\Lambda^2,k+2,z\right)\times\mathcal{C}_{km_1m_2}(\varphi,\chi,\eta),$$

$$\Phi_{-1}^{4,km_1m_2}=N_-z^{k/2}e^{-z/2}M\left(\frac{k}{2}+m_2-\Lambda^2,k+2,z\right)\times\mathcal{C}_{km_1m_2}(\varphi,\chi,\eta),$$

where $\Lambda=\lambda/\sqrt{2\hat{B}}$, $z=\hat{B}r^2/2$. Thus the two independent mutually orthogonal solutions are

$$\varphi^{-+}=\begin{pmatrix}0\\0\\N_-z^{k/2}e^{-z/2}M\left(\frac{k}{2}+m_2-\Lambda^2+2,k+2,z\right)\mathcal{C}_{km_1m_2}\\N_+z^{k'/2}e^{-z/2}M\left(\frac{k'}{2}-m_2'-\Lambda'^2+2,k'+2,z\right)\mathcal{C}_{k'm_1'm_2'}\end{pmatrix},\tag{B4}$$

$$\varphi^{--}=\begin{pmatrix}0\\0\\N_+z^{k/2}e^{-z/2}M\left(\frac{k}{2}-m_2-\Lambda^2,k+2,z\right)\mathcal{C}_{km_1m_2}\\N_-z^{k'/2}e^{-z/2}M\left(\frac{k'}{2}+m_2'-\Lambda'^2,k'+2,z\right)\mathcal{C}_{k'm_1'm_2'}\end{pmatrix},\tag{B5}$$

where a "prime" indicates that angular quantum numbers and eigenvalues in the third line need not coincide with those in the fourth in order that these spinors be eigenmodes of Eq. (14).

To obtain an explicit representation for χ, we use the identity

$$-\not{n}(x)\not{D}=\partial_r+2R^{-1}(\mathbf{\Sigma}\cdot\mathbf{K}_1P_++\mathbf{\Sigma}\cdot\mathbf{K}_2P_-)-i\not{n}\frac{\hat{n}}{2}\gamma_\mu B_{\mu\nu}x_\nu,\tag{B6}$$

where the action of the last term on $P_\mp O_\zeta\Phi^\zeta$ can be determined via the identity

$$\frac{\hat{n}}{2}\gamma_\mu B_{\mu\nu}x_\nu P_\mp O_\zeta=i\zeta\not{n}\frac{\hat{B}R}{2}P_\mp O_\zeta,$$

and the action of the $\mathbf{\Sigma}\cdot\mathbf{K}$ terms via

$$\begin{aligned}\mathbf{\Sigma}\cdot\mathbf{K}_{1,2}\sum_\zeta O_\zeta\Phi_\zeta&=(\Sigma_3K^z_{1,2}+\Sigma^{(+)}K^-_{1,2}+\Sigma^{(-)}K^+_{1,2})\sum_\zeta O_\zeta\Phi_\zeta\\&=\frac{\hat{n}}{|\hat{n}|}K^z_{1,2}(O_+\Phi_{+1}-O_-\Phi_{-1})\\&\quad+N_+\Sigma^{(+)}K^-_{1,2}\Phi_{+1}+N_-\Sigma^{(+)}K^-_{1,2}\Phi_{-1}\\&\quad+N_-\Sigma^{(-)}K^+_{1,2}\Phi_{+1}+N_+\Sigma^{(-)}K^+_{1,2}\Phi_{-1}.\end{aligned}\tag{B7}$$

As well as the ladder operators $K^\pm$, we also have analogous operators for the spin,

$$\Sigma^{(\pm)}=\frac{1}{2}(\Sigma_1\pm i\Sigma_2).$$

For $B_i=B\delta_{i3}$, the following identities are also useful for implementing the above:

$$\Sigma_3O_\pm=\pm\frac{\hat{n}}{|\hat{n}|}O_\pm,\quad\Sigma^{(+)}O_\pm=N_\pm\Sigma^{(+)},\quad\Sigma^{(-)}O_\pm=N_\mp\Sigma^{(-)},$$

and

$$\begin{aligned}P_-\Sigma^{(+)}\Psi&=(0,0,-\Psi^4,0)^{\mathrm{T}},\\P_-\Sigma^{(-)}\Psi&=(0,0,0,-\Psi^3)^{\mathrm{T}},\\P_+\Sigma^{(+)}\Psi&=(-\Psi^2,0,0,0)^{\mathrm{T}},\\P_+\Sigma^{(-)}\Psi&=(0,-\Psi^1,0,0)^{\mathrm{T}}.\end{aligned}\tag{B8}$$

We thus get for the self-dual case

$$\chi^{-+}=\begin{pmatrix} 0\\ 0\\ \frac{1}{i\lambda}\left[\partial_r-\frac{\hat{B}r}{2}-\frac{2}{r}K_2^3\right]N_-z^{k/2}e^{-z/2}M\left(\frac{k}{2}+m_2-\Lambda^2+2,k+2,z\right)\mathcal{C}_{km_1m_2}\\ \frac{1}{i\lambda'}\left[\partial_r-\frac{\hat{B}r}{2}+\frac{2}{r}K_2^3\right]N_+z^{k'/2}e^{-z/2}M\left(\frac{k'}{2}-m_2'-\Lambda'^2+2,k'+2,z\right)\mathcal{C}_{k'm_1'm_2'}\end{pmatrix}$$
$$-\frac{2}{r}\begin{pmatrix} 0\\ 0\\ \frac{1}{i\lambda'}K_2^-N_+z^{k'/2}e^{-z/2}M\left(\frac{k'}{2}-m_2'-\Lambda'^2+2,k'+2,z\right)\mathcal{C}_{k'm_1'm_2'}\\ \frac{1}{i\lambda}K_2^+N_-z^{k/2}e^{-z/2}M\left(\frac{k}{2}+m_2-\Lambda^2+2,k+2,z\right)\mathcal{C}_{km_1m_2}\end{pmatrix}, \tag{B9}$$

and

$$\chi^{--}=\begin{pmatrix} 0\\ 0\\ \frac{1}{i\lambda}\left[\partial_r+\frac{\hat{B}r}{2}-\frac{2}{r}K_2^3\right]N_+z^{k/2}e^{-z/2}M\left(\frac{k}{2}-m_2-\Lambda^2,k+2,z\right)\mathcal{C}_{km_1m_2}\\ \frac{1}{i\lambda'}\left[\partial_r+\frac{\hat{B}r}{2}+\frac{2}{r}K_2^3\right]N_-z^{k'/2}e^{-z/2}M\left(\frac{k'}{2}+m_2'-\Lambda'^2,k'+2,z\right)\mathcal{C}_{k'm_1'm_2'}\end{pmatrix}$$
$$-\frac{2}{r}\begin{pmatrix} 0\\ 0\\ \frac{1}{i\lambda'}K_2^-N_-z^{k'/2}e^{-z/2}M\left(\frac{k'}{2}+m_2'-\Lambda'^2,k'+2,z\right)\mathcal{C}_{k'm_1'm_2'}\\ \frac{1}{i\lambda}K_2^+N_+z^{k/2}e^{-z/2}M\left(\frac{k}{2}-m_2-\Lambda^2,k+2,z\right)\mathcal{C}_{km_1m_2}\end{pmatrix}. \tag{B10}$$

By inspection, the boundary condition $\chi=-e^{-i\alpha}\varphi$ can only be fulfilled if terms with raising/lowering operators of the azimuthal quantum numbers vanish since these terms contain the projectors $N_\pm$ while the rest of the terms entering the boundary condition contain $N_\mp$ [see Eqs. (B4) and (B5)]. In particular, $m_2'=-k/2$, $m_2=k/2$. Finally, evaluating the derivatives of the confluent hypergeometric functions with the help of relation

$$M_z'(a,b,z)=\frac{a}{b}M(a+1,b+1,z)$$

leads to the solutions given in the main body of the paper.

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