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APPLICATION OF THE KINETIC THEORY OF PARTLY IONIZED GASES TO RADIATION TRANSFER AND IONIZATION IN HYDROGEN NEBULAE

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SUMMARY

The problem of ionization produced by a hot star embedded in a hydrogen nebula is studied from a basic standpoint. The necessary equation of radiative transfer is derived by introducing a distribution function in photon phase-space, and subjecting it to the procedures of classical kinetic theory. It is found that the distribution function can be separated into a singular part corresponding to "direct" radiation from the star, and a non-singular part corresponding to "diffuse" radiation. The concept of statistical equilibrium is applied to the electron population to provide an auxiliary equation, and the Stromgren equation derived as an approximation.

A qualitative discussion of the exact equations shows that for a wide range of physical parameters a well defined ionized region surrounds the star. Ionizing radiation undergoes little absorption within this region, but it is rapidly extinguished at the boundary. The radius of the ionized region and the form of the boundary are discussed, and their dependence on the various physical parameters interpreted. Numerical solutions neglecting the effect of diffuse radiation confirm the predictions for that case.

I hereby declare that this thesis contains no material which has been accepted for the award of any other degree or diploma in any University, and that to the best of my knowledge and belief, the thesis contains no material previously published or written by another person, except where due reference is made in the text.

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I. INTRODUCTION

Consider a hot star embedded in interstellar gas. Radiation in the Lyman continuum is absorbed and neutral hydrogen dissociated by the mechanism of photo-ionization. On the other hand dissociated electrons and protons continually recombine, and if recombination occurs directly to ground state then ionizing radiation is returned to the system. The intensity of high energy radiation thus depends on the distribution of electrons, protons and neutral hydrogen atoms throughout the system; however, the rate of dissociation of hydrogen atoms itself depends on the radiation field.

In this paper we study the manner in which ionizing radiation is transferred through the gas, and the consequent state of the gas, on the assumption that a steady state is attained. To render the problem tractable the interstellar gas is assumed to be pure hydrogen, and to be stationary and of uniform density. Two coupled equations in the radiation field and the degree of ionization can then be obtained, their solution providing a complete description of the system. A simpler approach has been given by Stromgren (1939), who modified the Saha equation with a "dilution factor". Stromgren's equation can be obtained from the exact equations, thus revealing its inherent approximations.

The equations governing the system are developed in Chapter II. The first expresses the radiation intensity at a point as a function of the degree of ionization at all points of the system. The derivation employs a distribution function in photon phase-space, which is subjected to the methods of classical kinetic theory. It is found that the radiation field can be split into two components, one corresponding to "direct" radiation from the star, the other to secondary "diffuse" radiation from recombinations. The second equation is obtained in Section 2.5 by applying the concept of statistical equilibrium to the electron population. The system of equations is finally expressed in dimensionless form in Section 2.7. Stromgren's approximation is obtained in Section 2.6.

Chapter III is devoted to a qualitative discussion of the equations; in Part A the diffuse radiation is neglected, but its effect is included in Part B. The method in Part A is to consider first a crude approximation which admits of an analytical solution; the discussion is then extended to a more refined approximation, and finally to the correct equations. In the first approximation ionization is confined to a fairly well defined region, with (dimensionless) radius R_o, and the solution depends only on R_o. R_o is defined in Eq. (3.11) and preceding equations, and is a complicated function of the physical parameters. The further discussion relates the complete system of equations to the simple solution. The second approximation is of a similar form to the Stromgren equation; it is therefore also possible to predict generally the radius of the ionized sphere in the Stromgren approximation. This is given by Eq. (3.34).

A numerical treatment is given in Chapter IV, with the aim of verifying certain predictions of Chapter III, Part A; diffuse radiation is neglected. The degree of ionization is determined as a function of radius for several sets of physical parameters, the radius in each case being expressed in units of R_o.

The values of all physical quantities used were taken from Kaye and Laby (1957).

II. DERIVATION OF THE EQUATIONS

2.1. Radiation Transfer; Introduction.

A distribution function $f(t,\underline{r},\underline{p})$ is introduced in the 6-D photon phase-space; p is the photon momentum defined by

$$p = \frac{h\nu}{c} \quad . \qquad \ldots\ldots (2.1)$$

If $d\underline{r}d\underline{p}$ is a volume element of phase-space at position $\underline{r}$, momentum $\underline{p}$, then $f(t,\underline{r},\underline{p})\, d\underline{r}d\underline{p}$ is the probability of finding a photon in this element at time t. The normalization is such that the integral of f over the whole of phase-space equals the total number of photons; thus $fd\underline{r}d\underline{p}$ is the expected number of photons in the element. More usually considered is the quantity I_ν : $I_\nu d\nu d\Omega$ is the intensity of radiation with frequency in the range $d\nu$ and direction in the solid angle $d\Omega$. From the definition of f,

$$I_\nu = \frac{h^4\nu^3}{c^2} f \quad . \qquad \ldots\ldots (2.2)$$

The methods of classical kinetic theory (e.g. Chapman and Cowling 1939) are used to develop an equation in f. A photon with momentum $\underline{p}$ has constant velocity $c\hat{\underline{p}}$, where c is the speed of light and $\hat{\underline{p}}$ a unit vector in the direction of $\underline{p}$. Hence a photon with position $\underline{r}$ and momentum $\underline{p}$ at time t is at position $\underline{r} + c\hat{\underline{p}}dt$ at time t + dt. If $d\underline{r}d\underline{p}$ is initially at $\underline{r},\underline{p}$ at time t, and moves in phase-space with the photons it contains, then the number of photons in the element at time $t + dt$ is $f(t + dt, \underline{r} + c\hat{\underline{p}}dt, \underline{p})d\underline{r}d\underline{p}$; and provided that no photons enter or leave the element during dt, this is equal to $f(t, \underline{r}, \underline{p})d\underline{r}d\underline{p}$.

In the nebula photons are being continually absorbed and emitted, scattered by free electrons or otherwise deflected; by analogy with kinetic theory all these processes are referred to as "collisions". The effect of collisions is to cause photons to enter or leave the element $d\underline{r}d\underline{p}$ during dt. If the net increase is denoted by $\frac{\partial_e f}{\partial t} d\underline{r}d\underline{p}dt$, then

$$f(t + dt, \underline{r} + c\hat{\underline{p}}dt, \underline{p})d\underline{r}d\underline{p} - f(t, \underline{r}, \underline{p})d\underline{r}d\underline{p} = \frac{\partial_e f}{\partial t} d\underline{r}d\underline{p}dt$$

i.e.

$$\frac{\partial f}{\partial t} + c\hat{\underline{p}}.\frac{\partial f}{\partial \underline{r}} = \frac{\partial_e f}{\partial t} \quad . \qquad \ldots\ldots (2.3)$$

As in kinetic theory, $\frac{\partial_e f}{\partial t}$ is called the "collision term". It is defined by the statement that $\frac{\partial_e f}{\partial t} d\underline{r}d\underline{p}dt$ is the net increase due to collisions in the number of photons in the element $d\underline{r}d\underline{p}$ during dt, evaluated at $\underline{r}, \underline{p}, t$.

2.2. The Collision Term.

For radiation in the Lyman continuum, the only important processes occuring in the nebula are absorption as a result of photoionization and emission during recombination of electrons and protons. Scattering by free electrons is unimportant because of the very small cross-section ϕ_0 for this process (according to Heitler 1944, p.37,

$\phi_o = 6.57 \times 10^{-25}$ cm^2.) The scattering of radiation along a path of length r may be represented by exp $(-n_1\phi_o r)$, where n_1 is the electron number density. Now in Chapter III it is shown that the hydrogen is ionized only within a limited spherical region. Substituting numerical values it can be shown in retrospect that the effect of electron scattering within this region is negligible for any physically reasonable values of the parameters.

It is assumed that the protons and hydrogen atoms are stationary, and that the electrons have a locally Maxwellian velocity distribution corresponding to a constant electron temperature T_e. Spitzer (1954) has established that the distribution is Maxwellian by showing that even if impurities are present elastic collisions occur far more often than either inelastic collisions, photo-ionization or recombinations. He has also shown that in a real nebula the temperature is substantially determined by the impurities (especially the O^+ ion) and not by the radiation field; typically T_e = 10,000 °K. The assumption of fixed heavy particles is considered in Appendix 1, where it is found that the approximation leads to negligible error.

As the star itself is also a source of radiation, we write

$$\frac{\partial ef}{\partial t} = \left(\frac{\partial ef}{\partial t}\right)_1 + \left(\frac{\partial ef}{\partial t}\right)_2 + \left(\frac{\partial ef}{\partial t}\right)_3 \qquad \ldots\ldots (2.4)$$

where the separate collision terms are to account for the effects of recombination, photo-electric absorption and stellar emission respectively.

(a) Recombinations: $\left(\frac{\partial ef}{\partial t}\right)_1$. The number densities of electrons, protons and neutral atoms are denoted by n_1, n_2, n_3 respectively, and the electronic mass and velocity by m and $\underline{u}$. Since the velocity distribution is Maxwellian, if

$$\psi = 4\pi n_1\left(\frac{m}{2\pi kT_e}\right)^{\frac{3}{2}} \exp\left(-\frac{mu^2}{2kT_e}\right)u^2 \qquad \text{......} \ (2.5)$$

then there are ψdu electrons per unit volume with speed in the range u, u + du. Radiation in the Lyman continuum is produced only by recombination directly to the ground state atom. If $\sigma_1(u)$ is the cross-section for electron and proton recombination to ground state, then for electronic speed in the range du the number of such recombinations in time dt in the volume element $d\underline{r}$ is β, where

$$\beta = n_2\psi u\, \sigma_1(u) du d\underline{r} dt \ . \qquad \text{......} \ (2.6)$$

Each recombination to ground state yields a photon with momentum p given by

$$cp = E + \frac{1}{2}mu^2 \qquad \text{......} \ (2.7)$$

where E is the ionization potential of the hydrogen atom. The photons are emitted isotropically, hence we can write

$$d\underline{p} = 4\pi p^2 dp \ ,$$

or from Eq. (2.7),

$$udu = \frac{c}{4\pi mp^2}\, d\underline{p} \ . \qquad \text{......} \ (2.8)$$

Then using Eqs. (2.5), (2.7) and (2.8), β can be expressed in the form

$$\beta = \gamma(\underline{r},\underline{p})d\underline{r}d\underline{p}dt \quad ,$$

and is the number of photons emitted in $d\underline{r}$ during dt with momentum in $d\underline{p}$. Thus by definition of the collision term,

$$\left(\frac{\partial_e f}{\partial t}\right)_1 = \gamma \quad .$$

Performing the various substitutions, finally

$$\left(\frac{\partial_e f}{\partial t}\right)_1 = 2cn_1n_2\left(\frac{1}{2\pi mkT_e}\right)^{\frac{3}{2}} \exp\left(-\frac{cp-E}{kT_e}\right)\frac{m(cp-E)}{p^2}\sigma_1 \quad \ldots\ldots (2.9)$$

where σ_1 is now considered to be a function of p by use of Eq. (2.7).

(b) Photo-ionization : $\left(\frac{\partial_e f}{\partial t}\right)_2$. It is assumed that all neutral hydrogen atoms are in the ground state; this is justified in Appendix 2. The cross-section for photo-ionization of the ground state atom is denoted by $\alpha(p)$. By definition of f, the number of photons per unit volume with momentum in $d\underline{p}$ is $fd\underline{p}$. Thus as the speed of each photon is c, the number of photons absorbed from the momentum range $d\underline{p}$ during dt in a volume element $d\underline{r}$ is

$$cn_3f\alpha d\underline{r}d\underline{p}dt.$$

Hence

$$\left(\frac{\partial_e f}{\partial t}\right)_2 = -cn_3\alpha f \quad . \qquad \ldots\ldots (2.10)$$

(c) Stellar Emission : $\left(\frac{\partial_e f}{\partial t}\right)_3$. The surface of the star is assumed to radiate as a black body at a temperature T_s. For the low gas densities occuring in nebulae, the radius of the star is many orders of magnitude smaller than the linear dimensions of the system, hence the star may be approximated by a point source. The collision term may then be written

$$\left(\frac{\partial ef}{\partial t}\right)_3 = cH(p)\ \delta(\underline{r})\ \delta\left(1 - \frac{\underline{p}.\underline{r}}{pr}\right) \quad . \qquad \ldots\ldots (2.11)$$

The factor $\delta(\underline{r})$ is to restrict emission to a point source at the origin, and the last factor is to satisfy the requirement that photons emitted from the star have $\underline{p}$ parallel to $\underline{r}$. It may be noted that

$$\int_{-\infty}^{0} \delta(x)dx = \frac{1}{2} \quad .$$

The factor $H(p)$ will subsequently be determined so as to give the emergent flux its correct quantity and quality.

2.3. The Relation between α and σ_1

The absorption coefficient α and the recombination cross-section σ_1 are cross-sections for converse processes, and a thermodynamic concept can be used to obtain a relationship between them; this has been done by Aller and Menzel (1945). With the formalism introduced here the derivation is straightforward, illustrating the advantages of this formalism in problems of photon transfer.

Consider the radiative transfer through a system in thermodynamic equilibrium at a temperature T. At any point of phase-space the terms $\left(\frac{\partial ef}{\partial t}\right)_1$ and $\left(\frac{\partial ef}{\partial t}\right)_2$ represent the rates of converse processes, hence the principle of detailed balancing can be applied, giving

$$\left(\frac{\partial ef}{\partial t}\right)_1 + \left(\frac{\partial ef}{\partial t}\right)_2 = 0 \quad . \qquad \ldots\ldots (2.12)$$

The recombination cross-section σ_1 is replaced by

$$\left[\sigma_1(p) + \epsilon(p)f\right] ,$$

since it is necessary in principle to consider stimulated recombinations. The appropriate form of f is that for a thermodynamic enclosure; using Eq. (2.2) this is

$$f = \frac{2}{h^3}\left[\exp\left(\frac{cp}{kT}\right) - 1\right]^{-1} . \qquad \text{...... (2.13)}$$

An independent equation in the number densities is required: this is given by the Saha equation (Saha 1920, Saha and Saha 1934), which is an expression for the degree of ionization in a thermodynamic enclosure. The equation is

$$\frac{n_1 n_2}{n_3} = \left(\frac{2\pi mkT}{h^2}\right)^{\frac{3}{2}} \exp\left(-\frac{E}{kT}\right)\frac{\omega_1\omega_2}{\omega_3} \qquad \text{...... (2.14)}$$

where ω_1, ω_2, ω_3 are statistical weights of the electron, the ion and the ground state atom respectively. The statistical weight has in each case the value $(2j + 1)$, thus $\omega_1 = 2$, and for the case of hydrogen $\omega_2 = 1$, $\omega_3 = 2$.

Eq. (2.12) then becomes

$$\frac{m(cp - E)}{p^2}\left[\sigma_1 \exp\left(\frac{cp}{kT}\right) - \left(\sigma_1 - \frac{2\epsilon}{h^3}\right)\right] = \alpha \exp\left(\frac{cp}{kT}\right) .$$

Since neither σ_1 nor ϵ depend on T, it is necessary that

$$\epsilon = \frac{1}{2} h^3 \sigma_1 , \qquad \text{...... (2.15)}$$

and
$$\frac{m(cp - E)}{p^2}\sigma_1 = \alpha . \qquad \text{......} \ (2.16)$$

These relations are true in any system. Stimulated recombinations are unimportant in most astrophysical applications because the radiation is extremely diluted. Even in thermodynamic equilibrium,

$$\epsilon f = \sigma_1 \left[\exp\left(\frac{cp}{kT}\right) - 1\right]^{-1} \quad ;$$

with T = 10,000 °K this gives ϵf six orders of magnitude smaller than σ_1 for radiation in the Lyman continuum.

Using Eq. (2.16) and neglecting the time dependence of f, the transfer equation (2.3) can now be written as

$$\hat{\underline{p}}\cdot\frac{\partial f}{\partial \underline{r}} = - n_3 \alpha f + H \ \delta(\underline{r})\, \delta\left(1 - \frac{\underline{p}\cdot\underline{r}}{pr}\right) + $$
$$+ 2n_1 n_2 \alpha \left(\frac{1}{2\pi mkT_e}\right)^{\frac{3}{2}} \exp\left(- \frac{cp - E}{kT_e}\right) . \qquad \text{......} \ (2.17)$$

This is linear in f, consequently f may be separated:

$$f = f_1 + f_2 \qquad \text{......} \ (2.18)$$

where
$$\hat{\underline{p}}\cdot\frac{\partial f_1}{\partial \underline{r}} = - n_3 \alpha f_1 + H\delta(\underline{r})\, \delta\left(1 - \frac{\underline{p}\cdot\underline{r}}{pr}\right) \qquad \text{......} \ (2.19)$$

and
$$\hat{\underline{p}}\cdot\frac{\partial f_2}{\partial \underline{r}} = - n_3 \alpha f_2 + 2n_1 n_2 \alpha \left(\frac{1}{2\pi mkT_e}\right)^{\frac{3}{2}} \exp\left(- \frac{cp - E}{kT_e}\right) . \qquad \text{......} \ (2.20)$$

The functions f_1 and f_2 respectively represent the "direct" radiation from the star and the "diffuse" radiation arising from recombinations.

2.4. Spherical Symmetry.

Spherical symmetry is assumed, thus n_1, n_2, n_3 are functions of r, and f depends only on p, r and θ, where θ is the angle between the radius vector and the forward direction of the momentum $\underline{p}$. It is convenient to introduce μ:

$$\mu = \cos\theta = \frac{\underline{p}\cdot\underline{r}}{pr} \quad . \qquad \ldots\ldots (2.21)$$

The equation in the direct radiation, Eq. (2.19), can then be solved in terms of n_3, giving

$$f_1 = \frac{H}{4\pi r^2} \exp\left(-\alpha\int_0^r n_3\, dr'\right)\delta(1-\mu). \qquad \ldots\ldots (2.22)$$

The solution may be verified by substituting it in Eq. (2.19) and integrating each side over any volume appropriate to the spherical coordinates, keeping $\underline{p}$ constant. The left hand side becomes a surface integral using the Gauss theorem, and the right hand side is evaluated as a volume integral, using the relations

$$n_3\alpha f_1 r^2 = -\frac{\partial}{\partial r}(r^2 f_1)$$

and
$$r^2\delta(\underline{r}) = \frac{1}{2\pi}\delta(r) \quad .$$

The δ-function in the solution implies that f_1 is zero except for $\mu = 1$, or $\theta = 0$, thus all photons travel directly outwards from the star until they are absorbed. Absorption is represented by the exponential term, and the inverse square decrease by the $\frac{1}{r^2}$ term.

To determine H(p), the energy density at the surface of the star is derived from Eq. (2.22) and compared with the expected value. If $E d\nu$ is the energy density in the frequency range $d\nu$, then

$$E = \frac{1}{c}\int I_\nu d\Omega$$

where the integration is over all solid angles. Using Eq. (2.2),

$$E = \frac{2\pi h^4\nu^3}{c^3}\int_{-1}^{1} f_1 d\mu$$

$$= \frac{h^4\nu^3}{4r^2c^3} H \exp\left(-\alpha\int_0^r n_3 dr'\right) \quad .$$

Thus if r_s is the stellar radius, the energy density at the surface of the star is

$$E_s = \frac{h^4\nu^3}{4r_s^2c^3} H \quad .$$

But the energy density at the surface of the star is one quarter of that appropriate to a thermodynamic enclosure at temperature T_s,

i.e. $$E_s = \frac{2\pi h\nu^3}{c^3}\left[\exp\left(\frac{h\nu}{kT_s}\right) - 1\right]^{-1}$$

$$\therefore \quad H = \frac{8\pi r_s^2}{h^3}\left[\exp\left(\frac{cp}{kT_s}\right) - 1\right]^{-1} \quad .$$

If T_s is not too large, then for radiation in the Lyman continuum $\exp\left(\frac{cp}{kT_s}\right) \gg 1$, hence

$$N = \frac{8\pi r_s^2}{h^3} \exp\left(-\frac{cp}{kT_s}\right) . \qquad \ldots\ldots (2.23)$$

Substituting in Eq. (2.22),

$$f_1 = \frac{2r_s^2}{h^3 r^2} \exp\left(-\alpha \int_0^r n_3 dr' - \frac{cp}{kT_s}\right) \delta(1-\mu) . \qquad \ldots\ldots (2.24)$$

In terms of p, r, μ, Eq. (2.20) for the diffuse radiation becomes

$$\mu \frac{\partial f_2}{\partial r} + \frac{(1-\mu^2)}{r} \frac{\partial f_2}{\partial \mu} = - n_3 \alpha f_2 + $$
$$+ 2 n_1 n_2 \alpha \left(\frac{1}{2\pi m k T_e}\right)^{\frac{3}{2}} \exp\left(-\frac{cp-E}{kT_e}\right). \qquad \ldots\ldots (2.25)$$

This has the form of the standard radiative transfer equation for spherical symmetry. In most astrophysical applications it is replaced by an approximate form (Eddington 1926, Rosseland 1936), and this approximation has been used in a study of ionization in hydrogen nebulae (Pottasch and Jeffries 1959). Some objections to the use of the approximate equation in this situation are raised and discussed in Appendix 3. Accordingly, the exact equation is retained in the following work.

2.5. Statistical Equilibrium.

Eqs. (2.24) and (2.25) describe radiative transfer through the nebula in terms of electron, proton and H-atom densities. To obtain an independent equation in n_1, n_2, n_3 the concept of statistical equilibrium is applied to the electron population. Electrons are produced by photoionization and lost through recombinations; since n_1 remains constant in time these processes must balance. Accordingly the number of electrons emitted per unit time and volume is equated to the number absorbed. It may be noted that detailed balance in phase-space does not apply. As in Section 2.2. photoelectric emission occurs only from ground state, but recombinations to all states must be considered. The total recombination cross-section upon all bound states is denoted by $\sigma(u)$.

The number of recombinations per unit time and volume for electrons with speed in the range u, u + du is

$$\zeta\, du = n_2 \psi \sigma(u)\, u du$$

where ψ is given by Eq. (2.5). Integrating over u, the rate of loss of electrons per unit volume is

$$V_1 = \int_0^\infty \zeta\, du \quad .$$

It is convenient to introduce p defined by

$$cp = E + \frac{1}{2} mu^2 \qquad \ldots\ldots (2.26)$$

as a parameter. However, the momentum of the emitted photon is p only if the recombination is to ground state. The cross-section σ is expressed in terms of σ_1 :

$$\sigma(p) = s(p)\,\sigma_1(p) \quad . \qquad \ldots\ldots (2.27)$$

Thus $s(p)$ is the ratio of the total cross-section to the partial cross-section σ_1. Performing the various substitutions and using Eq. (2.16),

$$V_1 = 8\pi c n_1 n_2 \left(\frac{1}{2\pi m k T_e}\right)^{\frac{3}{2}} \exp\left(\frac{E}{kT_e}\right) \times$$

$$\times \int_{\frac{E}{c}}^{\infty} \exp\left(-\frac{cp}{kT_e}\right) s(p)\alpha(p)p^2dp \quad . \qquad \ldots\ldots (2.28)$$

The number of electrons produced per unit time and volume by absorption of photons in the momentum range $d\underline{p}$ is

$$cn_3 f\alpha p^2 dp d\hat{\underline{p}} \quad ,$$

where $d\hat{\underline{p}}$ is an element of solid angle including the unit vector $\hat{\underline{p}}$. Thus the rate of electron production due to all radiation beyond the Lyman limit is

$$V_2 = cn_3 \int_{\frac{E}{c}}^{\infty} \left(\int f d\hat{\underline{p}}\right) \alpha\, p^2 dp \quad ,$$

where the inner integration is over all solid angles. As f has angular dependence on μ only,

$$V_2 = 2\pi c n_3 \int_{\frac{E}{c}}^{\infty} \left(\int_{-1}^{1} f d\mu\right) \alpha p^2 dp \quad . \qquad \ldots\ldots (2.29)$$

The assumption of statistical equilibrium then gives

$$V_1 = V_2 \qquad \ldots\ldots (2.30)$$

where V_1 and V_2 are given by Eqs. (2.28) and (2.29).

2.6. The Stromgren Equation.

The Stromgren equation (Stromgren 1939) can now be obtained as an approximation. The diffuse radiation is neglected, or f_2 taken as zero. Then from Eq. (2.24),

$$\int_{-1}^{1} f d\mu = \frac{r_s^2}{h^3 r^2} \exp\left(-\alpha\int_0^r n_3 dr' - \frac{cp}{kT_s}\right) .$$

The absorption coefficient α is assumed to be constant, and set equal to α_o, its value at the Lyman limit. Then if

$$\tau = \alpha_o \int_0^r n_3 dr' ,$$

$$\int_{-1}^{1} f d\mu = \frac{r_s^2}{h^3 r^2} e^{-\tau} \exp\left(-\frac{cp}{kT_s}\right) . \qquad \text{......} (2.31)$$

Thus

$$V_2 = 2\pi c\alpha_o n_3 \frac{r_s^2}{h^3 r^2} e^{-\tau} \int_{\frac{E}{c}}^{\infty} \exp\left(-\frac{cp}{kT_s}\right) p^2 dp$$

$$\approx 2\pi c\alpha_o n_3 \frac{r_s^2}{h^3 r^2} e^{-\tau} \exp\left(-\frac{E}{kT_s}\right) \frac{kT_s}{c} \left(\frac{E}{c}\right)^2 .$$

In evaluating the integral, p^2 has been replaced by its value at the lower limit of integration.

Eq. (2.28) giving the rate of electron recombinations is simplified by neglecting recombinations to excited states; thus $\sigma(p)$ is replaced by $\sigma_1(p)$, or equivalently $s(p)$ becomes one. Then

$$V_1 = 8\pi c \alpha_0 n_1 n_2 \left(\frac{1}{2\pi mkT_e}\right)^{\frac{3}{2}} \int_{\frac{E}{c}}^{\infty} \exp\left(-\frac{cp-E}{kT_e}\right) p^2 dp$$

$$\approx 8\pi c\, \alpha_0 n_1 n_2 \left(\frac{1}{2\pi mkT_e}\right)^{\frac{3}{2}} \frac{kT_e}{c} \left(\frac{E}{c}\right)^2 .$$

Since $V_1 = V_2$, we obtain

$$\frac{n_1 n_2}{n_3} = \left(\frac{2\pi mkT_s}{h^2}\right)^{\frac{3}{2}} \sqrt{\frac{T_e}{T_s}} \exp\left(-\frac{E}{kT_s}\right) \frac{r_s^2}{4r^2}\, e^{-\tau} \qquad \ldots\ldots (2.32)$$

which is the Stromgren equation, originally derived by modifying the Saha equation. The important approximations in the above derivation lie in neglecting the diffuse radiation, and in assuming all recombinations to occur to ground state. The effect of diffuse radiation is discussed in Chapter IIIB and is shown to be significant. Recombinations to excited states are of importance only for slow electrons, but the Maxwellian velocity distribution and the form of the recombination cross-section both ensure that in most cases the absorbed electron is in fact slow. (Eq. (2.16) shows that for small electron energies σ_1 varies inversely as the energy). The solution of the Stromgren equation is discussed in Section 3.5.

2.7. The Exact Equations.

We return now to the exact equations (2.24), (2.25) and (2.30). To simplify the system it is assumed that

$$n_2 + n_3 = n \qquad \text{...... (2.33)}$$

and that

$$n_1 = n_2 = xn \quad , \qquad \text{...... (2.34)}$$

where n is a constant. Then

$$n_3 = (1 - x)n \quad . \qquad \text{...... (2.35)}$$

The absorption cross-section $\alpha(p)$ is assumed to vary as p^{-3}. The exact dependence on p is of a complicated form (Heitler 1944, p 124), but a numerical analysis shows that the variation is reasonably represented by p^{-3}. From the same source, the cross-section α_0 at the threshold ($p = \frac{E}{c}$) is 6.28×10^{-18} cm^2.

It is convenient to write the equations in dimensionless form. Dimensionless forms of the radial distance, the momentum and the distribution function respectively are

$$R = n\alpha_0 r \qquad \text{...... (2.36)}$$

$$P = \frac{c}{E}p \qquad \text{...... (2.37)}$$

$$F = h^3 f \quad . \qquad \text{...... (2.38)}$$

Then $\alpha = \alpha_0 P^{-3}$ . ... (2.39)

The following quantities are also defined:

$$A = 2n \exp\left(\frac{E}{kT_e}\right)\left(\frac{h^2}{2\pi mkT_e}\right)^{\frac{3}{2}} \qquad \ldots\ldots (2.40)$$

$$I = 2\int_1^\infty \exp\left(-\frac{PE}{kT_e}\right) S(P)\,\frac{dP}{P} \qquad \ldots\ldots (2.41)$$

$$C = AI \qquad \ldots\ldots (2.42)$$

$$J(P) = \frac{A}{P^3}\exp\left(-\frac{PE}{kT_e}\right) \qquad \ldots\ldots (2.43)$$

$$B(R) = \int_0^R (1-X)\,dR' \qquad \ldots\ldots (2.44)$$

where $X(R) = x(r)$ $\qquad \ldots\ldots (2.45)$

and $S(P) = s(p)$. $\qquad \ldots\ldots (2.46)$

The determination of $S(P)$ and the evaluation of I in a particular case are discussed in Appendix 4.

With these definitions the equations become

$$F_1 = \frac{2R_s^2}{R^2}\exp\left(-\frac{PE}{kT_s} - \frac{B(R)}{P^3}\right)\delta(1-\mu) \qquad \ldots\ldots (2.47)$$

$$\mu\frac{\partial F_2}{\partial R} + \frac{(1-\mu^2)}{R}\,\frac{\partial F_2}{\partial \mu} = J(P)X^2 - \frac{1}{P^3}(1-X)F_2 \qquad \ldots\ldots (2.48)$$

$$C\,\frac{X^2}{1-X} = \int_1^\infty\left(\int_{-1}^1 F\,d\mu\right)\frac{dP}{P} \qquad \ldots\ldots (2.49)$$

where $F = F_1 + F_2$. $\qquad \ldots\ldots (2.50)$

These equations together with a suitable boundary condition on Eq. (2.48) form a closed system.

2.8. Diffuse Radiation.

Eq. (2.48), describing transfer of diffuse radiation, can be expressed in a more illuminating form with the introduction of new coordinates Z and Y to replace R and μ:

$$Z = R\mu = R \cos\theta \qquad \ldots\ldots (2.51)$$

$$Y = R\sqrt{1-\mu^2} = R \sin\theta \quad . \qquad \ldots\ldots (2.52)$$

If the momentum vector $\underline{P}$ at a point $\underline{R}$ defines a straight line through $\underline{R}$, then Y is the (dimensionless) distance of this line from the origin; Z is the distance along the line from the point of closest approach to $\underline{R}$. A photon which undergoes no "collisions" has Y constant along its trajectory, and travels in the direction of increasing Z. With these coordinates, X is a function of $\sqrt{Z^2 + Y^2}$, and Eq. (2.48) becomes

$$\frac{\partial F_2}{\partial Z} + \frac{1}{p^3}(1 - X) F_2 = JX^2 \quad . \qquad \ldots\ldots (2.53)$$

The boundary condition is that no radiation enters the system from infinity, thus in terms of Eq. (2.53),

$$F_2 = 0 \text{ at } Z = -\infty \; .$$

Using Eq. (2.53), F_2 may be expressed explicitly as an integral:

$$F_2 = \int_{-\infty}^{Z} J(P)X^2 \exp\left(-\frac{1}{P^3}\int_{Z''}^{Z}(1 - X)dZ'\right)dZ'' \quad , \ldots\ldots \quad (2.54)$$

where the integrations are carried out keeping Y constant. This equation is readily interpreted if the outer integral is written as a sum:

$$F_2\,(Z) = \sum_i JX_i^2\Delta Z_i \exp\left(-\frac{1}{P^3}\int_{Z_i}^{Z}(1-X)dZ'\right) \quad .$$

The factor $JX_i^2\Delta Z_i$ represents the emission in the Z direction from the line element ΔZ_i; each such contribution to the diffuse radiation field at Z is modified by absorption along the path from Z_i to Z.

III QUALITATIVE DISCUSSION

A. Direct Radiation

3.1. Introduction.

In this Chapter the equations derived in Chapter II are discussed qualitatively. In Part A the diffuse radiation arising from recombinations is neglected; the modifying effect of diffuse radiation is considered in Part B.

With F_2 taken as zero, Eqs. (2.47) and (2.49) give

$$C \frac{X^2}{1 - X} = \frac{R_s^2}{R^2} K(R) \qquad \text{...... (3.1)}$$

where $$K(R) = \int_1^\infty \exp\left(- GP - \frac{B(R)}{P^3}\right) \frac{dP}{P} \qquad \text{...... (3.2)}$$

and $$B(R) = \int_0^R (1-X) dR' \quad , \qquad \text{...... (3.3)}$$

$$G = \frac{E}{kT_s} \quad . \qquad \text{...... (3.4)}$$

A modified form of Eq. (3.2) leads to an equation of more tractable form: replacing the factor B/P^3 by B, Eq. (3.2) becomes

$$K = \int_1^\infty \exp\,(- B - GP) \frac{dP}{P} \quad .$$

Provided that T_s is not too large, exp (- GP) varies more rapidly than 1/P in the range of integration, and 1/P may be replaced by one, its value at the lower limit. (For $T_s = 40{,}000\ ^\circ K$, $G \approx 4$.) Thus

$$\int_1^\infty \exp(-GP)\frac{dP}{P} \approx G^{-1}\exp(-G) \qquad \ldots\ldots (3.5)$$

and

$$K \approx G^{-1}\exp(-G-B) \quad . \qquad \ldots\ldots (3.6)$$

If Eq. (3.6) is regarded as exact, Eq. (3.1) then becomes

$$\frac{1-X}{X^2} = aR^2\exp(B) \qquad \ldots\ldots (3.7)$$

where

$$a = \frac{C}{R_s^2}G\exp(G) \quad . \qquad \ldots\ldots (3.8)$$

Eq. (3.7) corresponds to an artificial system in which the absorption coefficient is independent of frequency; it will be shown in Section 3.4 that an understanding of this system is useful in considering the original system. Even Eq. (3.7) cannot be solved analytically, but it is closely related to an equation with an elementary solution:

$$1 - X = aR^2\exp(B) \quad . \qquad \ldots\ldots (3.9)$$

Accordingly, Eq. (3.9) is solved first and used to study the behaviour of Eq. (3.7); this in turn is used to discuss Eqs. (3.1) and (3.2).

3.2. Equation (3.9).

From Eq. (3.3),

$$\frac{dB}{dR} = 1 - X$$

where

$$B(0) = 0 \quad .$$

Thus Eq. (3.9) can be written

$$\frac{dB}{dR} = aR^2 \exp (B)$$

$$\therefore \qquad \int_0^B \exp (-B') \, dB' = \int_0^R aR'^2 \, dR'$$

$$\therefore \qquad B = \ln \left(\frac{1}{1 - \frac{1}{3}aR^3} \right) . \qquad \ldots\ldots (3.10)$$

For real B, Eq. (3.10) is valid only for $\frac{1}{3} aR^3 < 1$. Define R_o, y and b by

$$R_o = \left(\frac{3}{a}\right)^{\frac{1}{3}} \qquad \ldots\ldots (3.11)$$

$$y = \frac{R}{R_o} \qquad \ldots\ldots\ldots (3.12)$$

$$b = aR_o^2 = (9a)^{\frac{1}{3}} \quad . \qquad \ldots\ldots (3.13)$$

Then

$$B = \ln \left(\frac{1}{1 - y^3} \right) \text{ for } y < 1 \quad , \qquad \ldots\ldots (3.14)$$

and from Eq. (3.9),

$$1 - X = b\frac{y^2}{1 - y^3} \quad . \qquad \ldots\ldots (3.15)$$

Physically Eq. (3.9) may be interpreted as describing an artificial situation in which the rate of recombinations is independent of the electron density (whereas in the actual system the rate of recombinations varies as X^2.) Eq. (3.15) is then valid only for $X \geqslant 0$. If y_1 is the solution of

$$by_1^2 = 1 - y_1^3 \qquad \text{......} \; (3.16)$$

then Eq. (3.15) (and consequently also Eq. (3.14)) is valid for $y \leq y_1 < 1$; for $y > y_1$, X is zero.

From Eq. (3.15), X is unity for y zero, and falls monotonically to zero as y increases to y_1; however, the detailed behaviour depends on the value of a. In all cases with which we are concerned, a is a very small number. For example, with an electron temperature T_e = 10,000 °K, and a star of ten solar radii ($r_s = 7 \times 10^{11}$ cm),

$$a = 1.36 \times 10^{-11} \, G \exp(G). \frac{1}{n} \qquad (n \text{ in cm}^{-3}) \quad .$$

(Here we have used the value of I calculated in Appendix 4.) If attention is restricted to stellar temperatures T_s greater than 20,000 °K, and densities n greater than 100 atoms or ions per cm^3, then a is not greater than 2.9×10^{-9}. Consequently from Eq. (3.13), b is not greater than 3.0×10^{-3}.

The value of y_1 is then seen from Eq. (3.16) to be very close to unity (approximately $y_1 = 1 - \frac{1}{3}b$), hence

$$B(y_1) = \ln\left(\frac{1}{1 - y_1^3}\right)$$

$$\approx \ln\left(\frac{1}{b}\right) \quad .$$

For the limited range of parameters, $B(y_1)$ is not less than 5.8. On the other hand, at

$$y = (1 - \frac{1}{e})^{\frac{1}{3}} = 0.8582,$$

Eqs. (3.14) and (3.15) give

$$B = 1$$

and

$$1 - X = b\left[(1 - \frac{1}{e})^{\frac{2}{3}} \cdot e\right]$$

$$= 2.00\ b$$

which is less than 6×10^{-3} for parameters of interest.

Thus Eq. (3.9) has a solution with X close to unity until R is close to R_e, when it falls very rapidly to zero; the rapid decrease is associated with a rapid increase in B. On the other hand, if Eq. (3.9) is modified so that absorption is neglected (i.e. B taken as zero), Eq. (3.15) is replaced by

$$1 - X = by^2 \quad . \qquad \ldots\ldots (3.17)$$

The values of (1 - X) are similar in the two cases until y is fairly close to one, for example if $y = \frac{1}{2}$, Eq. (3.17) over-estimates (1 - X) by a factor of only 8/7. However, Eq. (3.17) gives X zero for $y = b^{-\frac{1}{2}}$, which is a large number (greater than 18 for the parameters considered), in contrast to Eq. (3.15) which gives X zero for some $y < 1$. The interpretation of the behaviour of the solution of Eq. (3.9) is then that for y small ($y \leqslant \frac{1}{2}$ say), X is determined entirely by the R^2 term, but as y approaches one the exponential term dominates completely, and reduces X to zero within a short distance. Physically, the inverse square decrease of intensity weakens the radiation initially, allowing recombinations to predominate. When the density of neutral hydrogen atoms is sufficient to cause substantial absorption, the absorption plays the dominant role in weakening the radiation.

The qualitative behaviour of the solution changes as a increases, and it is instructive to consider the solution for a large (physically unreal) value of a. Taking $a = \frac{1}{9}$, then from Eqs. (3.11) and (3.13), $R_o = 3$ and $b = 1$. From Eq. (3.16), y_1 is given by

$$y_1^2 = 1 - y_1^3$$

or

$$y_1 = 0.76 \quad .$$

Thus X falls to zero at $R = 0.76\ R_o$, where R_o is itself small. From Eq. (3.15)

$$1 - X = \frac{y^2}{1 - y^3} \quad \text{for } y < y_1 \quad ,$$

hence X falls smoothly to zero with no appearance of a sharp edge. The optical depth at $y = y_1$ is small:

$$B(y_1) = \ln\left(\frac{1}{1 - y_1^3}\right)$$

$$= 0.56 \quad .$$

On the other hand, with absorption entirely neglected,

$$1 - X = y^2 \quad ,$$

giving X zero at $y = 1$. The effect of absorption is thus to modify only slightly the decrease in X which results from inverse square weakening.

Large values of a (or small values of R_o) correspond to low stellar temperatures and low densities. It may therefore be inferred that as the stellar temperature and the density are decreased, the edge of the ionized region loses precision, and absorption becomes less important. Furthermore, the extent of the ionized region is reduced (in terms of the dimensionless coordinate R). The explanation is that with low values of T_s and n, the output of ionizing radiation from the star is very low, and the hydrogen atoms sparsely distributed. The radiation intensity is initially low, and decreases as $\frac{1}{R^2}$, allowing the degree of ionization to decrease rapidly. Any given photon thus reaches the edge of the ionized region before it has encountered many hydrogen atoms.

These remarks apply strictly only to Eq. (3.9), but it will be shown that Eq. (3.7) and the original equations are closely related. Thus for values of T_s and n of physical interest, a large ionized region with a rather sharp boundary surrounds the star. For lower temperatures and densities the region has a more diffuse boundary, and the ionized region is somewhat reduced in (dimensionless) extent.

3.3. Equation (3.7).

Using R_o, y and b defined by Eqs. (3.11), (3.12) and (3.13), Eqs. (3.3) and (3.7) can be written

$$B = \frac{3}{b} \int_o^y (1 - X) dy' \qquad \ldots\ldots (3.18)$$

$$\frac{1 - X}{X^2} = by^2 \exp (B) \quad . \qquad \ldots\ldots (3.19)$$

As y varies from o to ∞, the right hand side of Eq. (3.19) increases monotonically from o to ∞, hence from the form of the left hand side X decreases monotonically from 1 to o. Thus for any W in the range $o < W \leqslant 1$, there exists y_W satisfying

$$\frac{1-W}{W^2} = by_W^2 \exp \left(\frac{3}{b} \int_o^{y_W} (1 - X)\, dy' \right) , \qquad \ldots\ldots (3.20)$$

and $X \geqslant W$ if and only if $y \leqslant y_W$. From Eq. (3.19)

$$1 - X = bX^2y^2 \exp (B)$$

hence if $y \leqslant y_W$,

$$1 - X \geqslant bW^2y^2 \exp (B) \qquad \ldots\ldots (3.21)$$

i.e. $$\frac{b}{3} \frac{dB}{dy} \geqslant bW^2y^2 \exp (B).$$

Using the boundary condition $B = 0$ at $y = 0$, it follows that

$$B \geqslant \ln \left(\frac{1}{1 - W^2y^3} \right)$$

for $W^2y^3 < 1$ and $y < y_W$. For this range of y, Eq. (3.21) then gives

$$1 - X \geqslant W^2 b \frac{y^2}{1 - W^2 y^3}$$

$$\therefore \quad X \leqslant 1 - W^2 b \frac{y^2}{1 - W^2 y^3} \quad . \qquad \ldots\ldots (3.22)$$

Since the right hand side of Eq. (3.22) approaches $-\infty$ as y approaches $W^{-\frac{2}{3}}$ from below, it is concluded that y_W is less than $W^{-\frac{2}{3}}$: for if the contrary is supposed then for some $y < y_W$, X is zero, contradicting the assumption that $X > W > 0$ for all $y < y_W$. Thus if $y = W^{-\frac{2}{3}}$, then

$$y > y_W$$

$$\therefore \quad X(y) < W$$

$$\text{i.e.} \quad X(y) < y^{-\frac{3}{2}} \quad .$$

This inequality holds for any $y \geqslant 1$, since given y, the corresponding W may be defined by $y = W^{-\frac{2}{3}}$. Hence for $y \geqslant 1$

$$1 - X > 1 - y^{-\frac{3}{2}} \quad .$$

But from Eq. (3.18), since $(1 - X) \geqslant 0$,

$$B \geqslant \frac{3}{b} \int_1^y (1 - X)\, dy' \quad .$$

Hence

$$B > \frac{3}{b} \int_1^y (1 - y'^{-\frac{3}{2}})\, dy'$$

i.e. $$B > \frac{3}{b}\left(y + 2y^{-\frac{1}{2}} - 3\right) \quad .$$

Then using Eq. (3.19)

$$\frac{1 - X}{X^2} > by^2 \exp\left(\frac{3}{b}\left[y + 2y^{-\frac{1}{2}} - 3\right]\right) , \qquad \text{......} \ (3.23)$$

valid for $y \geqslant 1$.

As noted in Section 3.2, b is less than 3.0×10^{-3} for a restricted range of physical parameters. The factor $\frac{3}{b}$ occurs in the exponential of Eq. (3.23), hence the right hand side increases very rapidly as y increases from one. As an indication, for $y = 1.1$ and $b = 3.0 \times 10^{-3}$, the right hand side is a monotonically decreasing function of b, hence evaluating it with $y = 1.1$ and $b = 3.0 \times 10^{-3}$,

$$\frac{1 - X}{X^2} > 4.1$$

or $$X < 0.39 \quad ,$$

valid for all parameters in the restricted range. As y increases from 1.1, X falls very rapidly.

It is clear then that for physically interesting values of the parameters, the ionized region does not extend far beyond R_o. On the other hand, elementary considerations show that for any value of a, the function X(R) derived from Eq. (3.7) is everywhere greater than the solution of Eq. (3.9) discussed in Section 3.2. Thus as before, high values of T_s and n give a well-defined ionized region, with its boundary close to R_o. Low values of T_s and n give a less definite boundary, less definite even than in the previous case as X is everywhere larger. The only essential difference between the two cases is that with Eq. (3.7) X is everywhere greater than zero - the ionized region exhibits a tail. This is because as the degree of ionization falls, the electron and proton populations are diminished, and the recombination process occurs less often.

3.4. The Actual System.

In the preceding sections Eqs. (3.9) and (3.7) have been analysed, and conclusions drawn concerning the behaviour of the corresponding systems. It remains to indicate how the actual physical system is related.

Eq. (3.2) may be written as

$$K = \int_1^\infty \exp\,(-\,g(P))\,\frac{dP}{P} \qquad \ldots\ldots\ (3.24)$$

where $g(P) = GP + \dfrac{B}{P^3}$. $\qquad \ldots\ldots\ (3.25)$

For $P \geqslant 1$, $g\,(P) \leqslant GP + B$,

$\therefore \quad \exp\,(-\,g(P)) \geqslant \exp\,(-GP - B)$.

The exponential form on the right was that used to derive Eq. (3.7): hence for given B, the correct value of K, and thus also the correct value of X, are greater than those found from Eq. (3.7). (This is not strictly correct in that Eq. (3.5) is only approximate; for the present this complication is ignored). It can be inferred that for a given R the value of X is larger and that of B smaller than for the previous case.

The function g(P) has a single minimum at $P_o = (\frac{3B}{G})^{\frac{1}{4}}$, hence exp (- g(P)) has a maximum at P_o. If $B < \frac{1}{3}G$ then $P_o < 1$, and exp (- g(P)) decreases monotonically to zero as P increases from 1 to ∞. For B small compared to $\frac{1}{3}G$ it differs only slightly from exp (- GP - B), hence K is close to the value corresponding to Eq. (3.7), namely

$$K = G^{-1} \exp (- G - B) \quad , \qquad \text{......} (3.26)$$

and X is close to its value in the preceding section.

If $B > \frac{1}{3}G$ then $P_o > 1$, and exp (- g(P)) has its maximum inside the region of integration. It is to be expected then that K and X are considerably larger than the corresponding values calculated from Eq. (3.7). For B sufficiently large an approximate expression can be obtained for K. The function g(P) is expanded in a power series about P_o, retaining only second order terms. This gives

$$K \approx \int_1^\infty \exp \left(- GP_o - \frac{B}{P_o^3} - \frac{6B}{P_o^5} \left[P - P_o \right]^2 \right) \frac{dP}{P}$$

$$\approx \frac{1}{P_o} \exp (- GP_o - \frac{B}{P_o^3}) \int_1^\infty \exp \left(- \frac{6B}{P_o^5} \left[P - P_o \right]^2 \right) dP \quad .$$

For B sufficiently large, P_o is considerably greater than one, and it can be shown that the integrand becomes small at the lower limit. Replacing the lower limit by $-\infty$,

$$K \approx \sqrt{\frac{\pi P_o^3}{6B}} \exp\left(-GP_o - \frac{B}{P_o^3}\right)$$

$$\text{i.e.} \quad K \approx 1.093\,(G^3B)^{-\frac{1}{8}} \exp\left(-1.754\,(G^3B)^{\frac{1}{4}}\right) \quad . \qquad \ldots\ldots (3.27)$$

The approximations can be shown to improve as B increases, and for cases of interest Eq. (3.27) is reasonably accurate for $B > 10$.

We thus have expressions for K for B small and B large; with intermediate values of B, K may be expected to lie between the values given by Eqs. (3.26) and (3.27). In general the latter equation gives K decreasing far less rapidly with increasing B than does Eq. (3.26) since B occurs only to the power $\frac{1}{4}$. This is particularly so for small values of G, corresponding to large T_s. Nevertheless a fairly sharp edge may still be found provided that B increases sufficiently rapidly with R. Considering only densities $n > 10^2$ cm^{-3}, R_o is large, and it is largest for the smallest values of G, hence it is possible for B to increase very considerably within a relatively short distance. (E.g. with $T_s = 4 \times 10^4$, $G = 3.94$; if $n = 10^2$ then $R_o = 4.7 \times 10^3$.) We expect therefore a fairly well defined ionized region for all values of G; but if G is varied keeping constant R_o, the boundary should be most diffuse for small values of G.

Furthermore the edge of the ionized region may be not far beyond

R_o, at least for G not too small. The solution for X follows that of Eq. (3.7) until $B \sim \frac{1}{3}G$; if G is reasonably large the solutions may be similar out to a radius at which the boundary is effectively established. Thereafter X will decrease far more slowly with B than was the case previously, but as noted above, a large increase in B may be achieved in a relatively short distance. Any extension of the ionized region beyond R_o will be favoured by small values of G, or correspondingly, by large T_s.

To interpret physically the above behaviour, we observe that the radiation spectrum deviates increasingly from the black-body form as the effect of absorption increases. Radiation with frequency well above the Lyman limit is absorbed only slightly, hence penetrates further; and although such radiation is initially only a small proportion of the total radiation, ultimately it is more important than radiation near the threshold. For high stellar temperatures, the proportion of high frequency radiation relative to radiation near the threshold is large in the initial black-body spectrum, and the disturbing effect is favoured.

The conclusions are slightly modified if the approximation of Eq. (3.5) is considered. Strictly, the actual system should be compared to a system of radius R_o', where

$$R_o' = \left(\frac{3}{a'}\right)^{\frac{1}{3}} \qquad \ldots\ldots (3.28)$$

and $$a' = \frac{C}{R_s^{\,2}} \left(\int_1^{\infty} \exp\,(-\,GP)\, \frac{dP}{P} \right)^{-1} . \qquad \ldots\ldots (3.29)$$

Since $\frac{1}{P}$ is a decreasing function of P,

$$\int_1^\infty \exp(-GP)\frac{dP}{P} < G^{-1}\exp(-G)$$

$$\therefore \qquad a' > a$$

$$\therefore \qquad R_o' < R_o \quad .$$

To show that this effect is not important, we observe that near $P = 1$, $\frac{1}{P}$ behaves as $\exp(1 - P)$, and that for $P > 1$,

$$\frac{1}{P} > \exp(1 - P)$$

$$\therefore \quad \int_1^\infty \exp(-GP)\frac{dP}{P} > (G + 1)^{-1}\exp(-G)$$

$$\therefore \qquad a' < \frac{C}{R_s^2}(G + 1)\exp(G)$$

$$= \frac{G + 1}{G}a$$

$$\therefore \qquad R_o' > R_o\left(\frac{G}{G + 1}\right)^{\frac{1}{3}} \quad .$$

For $T_s < 50{,}000\ ^\circ K$, $\left(\frac{G}{G + 1}\right)^{\frac{1}{3}} > 0.91$.

Thus the discrepancy is fairly small, and R_o' is close to R_o. The approximation is worst for small G (i.e. large T_s), thus tending to offset the high temperature effect of increased penetration.

The final conclusion is that the effect of direct radiation alone is to give a fairly well defined spherical region in which ionization is substantially complete; outside this region the degree of ionization falls rapidly, and the ionizing radiation is extinguished. The radius of the sphere (in dimensionless form) is closely related to R_o, where R_o is given by Eq. (3.11). For high stellar temperatures the radius may be somewhat larger than R_o because of the small absorption cross-section for high frequency radiation.

3.5. The Stromgren Equation.

As has been shown in Section 2.6, the Stromgren approximation includes the effect of direct radiation only, and neglects variation of the absorption cross-section with frequency. It does in fact correspond very closely to Eq. (3.7).

Using the dimensionless quantities defined in Section 2.7, Eq. (2.32) becomes

$$C^{+} \frac{X^2}{1 - X} = \frac{R_s^2}{R^2} G^{-1} \exp(- G - B) \qquad \ldots\ldots (3.30)$$

where

$$C^{+} = AI^{+}$$

and

$$I^{+} = 2 \exp\left(- \frac{E}{kT_e}\right) \frac{kT_e}{E} \quad . \qquad \ldots\ldots (3.31)$$

The quantity I^+ is analogous to I; replacing I by I^+ corresponds to neglecting electron recombinations on excited states. Defining also

$$a^{+} = \frac{C^{+}}{R_s^2} B \exp(G),$$

Eq. (3.30) becomes

$$\frac{1 - X}{X^2} = a^+ R^2 \exp (B) \quad ,$$

which has the form of Eq. (3.7). Hence by the argument of Section 3.3. the ionized region has a sharp boundary at a radius R_o^+, where

$$R_o^+ = \left(\frac{3}{a^+}\right)^{\frac{1}{3}}$$

i.e. $$R_o^+ = \left(\frac{I}{I^+}\right)^{\frac{1}{3}} R_o \quad . \qquad \ldots\ldots (3.32)$$

In Appendix 4., I is evaluated for $T_e = 10,000\ ^oK$, giving

$$I = 2.5\ I^+$$

i.e. $$R_o^+ = 1.36\ R_o \qquad \ldots\ldots (3.33)$$

It is also suggested there that the ratio of I to I^+ is substantially independent of T_e, hence Eq. (3.33) should be reasonably valid for any T_e.

Thus the Stromgren equation leads to a sharply bounded ionized region of radius R_o^+, which is significantly larger than R_o. The discrepancy arises because recombinations to excited states have been neglected. The rate of electron recombinations is diminished by this assumption, giving a larger electron density and a smaller H-atom density, and allowing the ionizing radiation to penetrate further.

For astrophysical applications, it may be useful to have an explicit formula for the radius r_o^+ of the ionized sphere under the Stromgren approximation. Using the equation

$$n\alpha_0 r_o^+ = R_o^+ \quad ,$$

and the equations of this section, we find

$$r_o^+ = \left[\frac{3r_s^2}{4n^2\alpha_0}\left(\frac{2\pi mkT_s}{h^2}\right)^{\frac{3}{2}}\sqrt{\frac{T_e}{T_s}}\exp\left(-\frac{E}{kT_s}\right)\right]^{\frac{1}{3}} \quad . \quad \ldots\ldots (3.34)$$

This will probably be a better approximation than is indicated above, since diffuse radiation tends to increase the true radius (see Chapter IIIB).

B. Total Radiation

3.6. Introduction.

It has been shown in Part A that if diffuse radiation is neglected, the ionization is substantially confined to a spherical region surrounding the star. The region is rather sharply bounded, and has radius close to R_o. It is expected that the diffuse radiation will cause the ionized region to be somewhat extended, to a radius R_2 say. To gain some idea of the importance of diffuse radiation, a simplified model of the nebula is adopted: the gas is taken to be completely ionized within a sphere of radius R_2, and unionized outside. Thus

$$X = 1 \qquad \text{for } R \leqslant R_2$$

$$X = 0 \qquad \text{for } R > R_2 .$$

Eq. (2.49) is written as

$$\frac{X^2}{1 - X} = L_1 (R) + L_2 (R)$$

where
$$L_i (R) = \frac{1}{C} \int_1^\infty \left(\int_{-1}^1 F_i d\mu \right) \frac{dP}{P} \quad . \qquad \ldots\ldots (3.35)$$

The functions L_1 and L_2 describe the effect of direct and diffuse radiation respectively. They will be evaluated approximately using the above form of X, and the relative magnitudes at R_2 used to give some information on the importance of diffuse radiation.

3.7. Ratio of L_2 to L_1.

For $R \leqslant R_2$,

$$B\ (R) = \int_0^R (1 - X)\ dR' = 0 \quad .$$

Hence from Eq. (2.47),

$$\int_{-1}^{1} F_1 d\mu = \frac{R_s^2}{R^2} \exp\ (-\ GP)$$

$$\therefore \qquad L_1\ (R) = \frac{R_s^2}{CR^2} \int_1^\infty \exp\ (-\ GP)\ \frac{dP}{P}$$

$$\approx \frac{R_s^2}{CR^2}\ G^{-1} \exp\ (-\ G)$$

$$= \frac{1}{aR^2}$$

where a is defined by Eq. (3.8).

$$\therefore \qquad L_1\ (R_2) \approx \frac{1}{aR_2^2} \quad . \qquad \ldots\ldots (3.36)$$

In calculating L_2, the integral form of F_2 (Eq. (2.54)) is used:

$$F_2 = \int_{-\infty}^{Z} J(P) X^2 \exp\left(-\frac{1}{P^3}\int_{Z''}^{Z}(1 - X)\ dZ'\right) dZ''$$

where X is a function of Z and Y (defined by Eqs. (2.51) and (2.52)), and Y is constant in the integration. In terms of Z and Y,

$$X = 1 \text{ for } \sqrt{Z^2 + Y^2} \leqslant R_2$$

$$X = 0 \text{ for } \sqrt{Z^2 + Y^2} > R_2.$$

Thus for $|Z| \leqslant \sqrt{R_2^2 - Y^2}$,

$$F_2 = \int_{-\sqrt{R_2^2-Y^2}}^{Z} J \, dZ''$$

$$= J \left(Z + \sqrt{R_2^2 - Y^2} \right)$$

i.e. $$F_2 = J\left(R\mu + \sqrt{R_2^2 - R^2 + R^2\mu^2}\right) \qquad \text{......} \ (3.37)$$

$\therefore$ $$\int_{-1}^{1} F_2 d\mu = J(P)M(R) \qquad \text{......} \ (3.38)$$

where $$M(R) = R_2 + \frac{(R_2^2 - R^2)}{R} \ln\left(\sqrt{\frac{R_2 + R}{R_2 - R}}\right). \qquad \text{......} \ (3.39)$$

Eqs. (3.38) and (3.39) are valid for $R \leqslant R_2$.

Then from Eq. (3.35),

$$L_2(R) = \frac{1}{C} \int_1^{\infty} J(P)M(R)\frac{dP}{P}$$

and using Eq. (2.43),

$$L_2(R) = \frac{A}{C} M(R) \int_1^{\infty} \frac{1}{P^4} \exp\left(-\frac{PE}{kT_e}\right)dP.$$

With T_e = 10,000 °K, $\frac{E}{kT_e} \approx 16$, and for $P \geqslant 1$ the exponential term dominates the factor P^{-4}.

$\therefore$ $$L_2(R) \approx \frac{A}{C} M(R) \int_1^{\infty} \exp\left(-\frac{PE}{kT_e}\right)dP$$

$$= \frac{M(R)}{I} \frac{kT_e}{E} \exp\left(-\frac{E}{kT_e}\right) \quad ,$$

where we have used C = AI. The quantity $\bar{\bar{S}}$ is defined by

$$I = 2\bar{\bar{S}} \frac{kT_e}{E} \exp\left(- \frac{E}{kT_e} \right) \quad . \qquad \ldots\ldots (3.40)$$

Then $L_2(R) \approx \frac{M(R)}{2\bar{\bar{S}}}$

and $L_2(R_2) \approx \frac{1}{2\bar{\bar{S}}} \lim_{R \to R_2} \left[R_2 + \frac{(R_2^2 - R^2)}{R} \ln \left(\sqrt{\frac{R_2 + R}{R_2 - R}} \right) \right]$

i.e.
$$L_2(R_2) \approx \frac{1}{2\bar{\bar{S}}} R_2 \quad .$$

Hence finally

$$\frac{L_2(R_2)}{L_1(R_2)} \approx \frac{1}{2\bar{\bar{S}}} aR_2^3 \quad .$$

If we define $u = \frac{R_2}{R_o}$, then from Eq. (3.11),

$$\frac{L_2(R_2)}{L_1(R_2)} \approx \frac{3}{2\bar{\bar{S}}} u^3 \quad . \qquad \ldots\ldots (3.41)$$

In Appendix 4 it is shown that for T_e = 10,000 °K, $\bar{\bar{S}}$ = 2.5, and $\bar{\bar{S}}$ is substantially independent of T_e. Thus provided that u is fairly close to unity, the effects of diffuse and direct radiation are of comparable magnitude. A surprising feature of Eq. (3.41) is that the ratio is independent of all the physical parameters of the system. Since it was assumed that direct radiation alone would produce an ionized region of radius R_o, it may therefore be expected that the diffuse radiation will increase the radius to uR_o, where u is independent of the physical parameters. We note that the relative importance of diffuse radiation would be markedly increased if recombinations to excited states were neglected (i.e. $\bar{S} = 1$).

Rather drastic approximations have been introduced, and no reliance can be placed on the numerical quantities in Eq. (3.41); nevertheless it seems clear that the diffuse radiation will produce a significant effect, and that the magnitude of the effect will be to a large extent independent of the parameters of the system. However, the approach is too crude to give a worthwhile estimate of R_2.

With the limited range of physical parameters considered in Part A, the boundary of the ionized region should remain well defined when diffuse radiation is included, because the same physical conditions prevail (R_o is large, hence there can be considerable absorption in a relatively short distance). The situation is somewhat more complicated in that diffuse radiation is emitted everywhere, even at the boundary itself. However, the preceding analysis shows that the total effect of diffuse radiation at the edge of the sphere is not overwhelming, hence the contribution due to local emission at the boundary must be negligible. Furthermore the form of $J(P)$ (Eq. (2.43)) shows that the emitted diffuse radiation is strongly concentrated near the Lyman threshold, where the absorption coefficient is greatest.

IV. NUMERICAL SOLUTION NEGLECTING DIFFUSE RADIATION

4.1. Introduction.

The equations derived in Chapter II have been solved numerically, neglecting the effect of diffuse radiation. Eqs. (2.47) and (2.49) can then be written

$$\int_{-1}^{1} F d\mu = \frac{R_s^2}{R^2} \exp\left(- GP - \frac{B(R)}{P^3}\right) \qquad \ldots\ldots (4.1)$$

$$C \frac{X^2}{1 - X} = \int_{1}^{\infty}\left(\int_{-1}^{1} F d\mu\right)\frac{dP}{P} \qquad \ldots\ldots (4.2)$$

where

$$B(R) = \int_{0}^{R} (1 - X)\, dR' \quad , \qquad \ldots\ldots (4.3)$$

and

$$G = \frac{E}{kT_s} \quad . \qquad \ldots\ldots (4.4)$$

The solution for X was obtained by an iterative technique. To begin the process a functional form of X was postulated, and specified at evenly spaced values of R. The corresponding values of B were computed from Eq. (4.3), then those of ($\int_{-1}^{1} F d\mu$) from Eq. (4.1). Finally new values of X were obtained from Eq. (4.2), and used to commence a new iteration, the process continuing until convergence was established. The computing was carried out on the I.B.M. 7090 computer at the Weapons Research Establishment, Salisbury. The Fortran programme and computing details are given in Appendix 5.

4.2. Numerical Results.

From the discussion of Chapter IIIA, the solution for X is expected to depend only on the stellar temperature T_s and the parameter R_o, defined by

$$R_o = (\frac{3}{a})^{\frac{1}{3}}$$

where $a = \frac{C}{R_s^2} G \exp(G)$.

Accordingly the solution was obtained for several sets of physical parameters, chosen to reveal the dependence on T_s and R_o.

In each case the electron temperature and the stellar radius were taken as 10,000 °K and 7×10^{11} cm. The variable parameters, T_s and n, are given in Table 1, and the corresponding values of R_o are also listed (the value of I necessary in calculating R_o was taken from Appendix 4.). The solutions for the various cases are given in Table 2.

Table 1. Physical Parameters and R_o
($T_e = 10^4$ °K, $r_s = 7 \times 10^{11}$ cm. in each case)

	n (cm^{-3})	T_s (°K)	R_o
(i)	10^2	4×10^4	4.715×10^3
(ii)	10^3	4×10^4	1.016×10^4
(iii)	10^4	4×10^4	2.189×10^4
(iv)	10^2	2×10^4	1.005×10^3
(v)	1.033×10^4	2×10^4	4.715×10^3

Table 2(a). Degree of Ionization ($T_s = 4 \times 10^4$ °K).

$45 \cdot \frac{R}{R_o}$	$\frac{R}{R_o}$	X		
		(i)	(ii)	(iii)
27	0.6	.9997	.9998	.9999
36	0.8	.9992	.9996	.9998
45	1.00	.9976	.9989	.9995
49	1.089	.9939	.9972	.9987
51	1.133	.986	.994	.997
52	1.156	.975	.987	.994
53	1.178	.929	.958	.975
54	1.200	.555	.258	.076
55	1.222	.098	.013	.001

Table 2(b). Degree of Ionization ($T_s = 2 \times 10^4$ °K).

$45 \cdot \frac{R}{R_o}$	$\frac{R}{R_o}$	X	
		(iv)	(v)
27	0.6	.9986	.9997
36	0.8	.9965	.9993
45	1.000	.986	.9968
47	1.044	.973	.9936
48	1.067	.956	.989
49	1.089	.906	.959
50	1.111	.634	.075
51	1.133	.150	.001
52	1.156	.029	

4.3. Discussion of Results.

In each of the five cases it is found as predicted in Chapter IIIA that the ionized region is well defined, with radius somewhat greater than R_o. The manner in which the solution varies with R_o can be seen from either of the two series (i), (ii), (iii) or (iv), (v); it is evident that for constant T_s the boundary becomes increasingly sharp as R_o increases. The behaviour is in accord with prediction, and is due to the increasing importance of absorption compared to inverse square weakening. On the other hand cases (i) and (v) illustrate the effect of varying T_s keeping R_o constant. As expected, the higher temperature star produces a more extended region (in terms of R_o), and the boundary is more diffuse. Case (iv) is the extreme low density and low temperature case, corresponding to the smallest value of R_o (and thus the largest value of a) considered in Chapter IIIA. Here it is found that X falls off to zero relatively slowly, again in accord with prediction.

The radius (in units of R_o) is seen to depend almost entirely on T_s. In Table 3 the radius is given for each case, both in units of R_o and in centimetres (by use of Table 1 and Eq. (2.36)).

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Table 3. Radius of Ionized Region

	$\frac{R}{R_o}$	r (cm)
(i)	1.20	9.01×10^{18}
(ii)	1.20	1.94×10^{18}
(iii)	1.19	4.15×10^{17}
(iv)	1.11	1.78×10^{18}
(v)	1.10	8.00×10^{16}

4.4. The Optical Depth, B.

Table 4 lists the computed values of B for each of the five cases. It was predicted in Chapter IIIA that for $\frac{R}{R_o}$ small, B should have the same functional dependence on $\frac{R}{R_o}$ for all physical parameters (the functional form is given by Eq. (3.14)). This is seen to be true for constant T_s, but there is a small discrepancy between the values in Tables 4(a) and 4(b). However, a discrepancy of this nature is to be expected, since it was shown in Section 3.4. that expressing the radius in terms of R_o is not strictly correct. Rather should the radius be compared to the slightly smaller quantity R_o' (defined by Eq. (3.28)); and the difference between R_o and R_o' increases with increasing T_s.

Table 4(a). Optical Depth ($T_s = 4 \times 10^4$ °K)

$45 \cdot \frac{R}{R_o}$	$\frac{R}{R_o}$	B		
		(i)	(ii)	(iii)
27	0.6	0.275	0.275	0.276
36	0.8	0.751	0.751	0.752
45	1.000	2.03	2.03	2.04
49	1.089	3.62	3.63	3.65
51	1.133	5.52	5.55	5.61
52	1.156	7.47	7.59	7.73
53	1.178	12.0	13.0	14.2
54	1.200	33.8	77.1	175
55	1.222	108	314	779

Table 4(b). Optical Depth ($T_s = 2 \times 10^4$ °K)

$45 \cdot \frac{R}{R_o}$	$\frac{R}{R_o}$	B	
		(iv)	(v)
27	0.6	0.255	0.255
36	0.8	0.708	0.710
45	1.000	2.10	2.13
47	1.044	2.96	3.06
48	1.067	3.72	3.95
49	1.089	5.16	6.31
50	1.111	9.56	42.4
51	1.133	23.1	172
52	1.156	44.8	

Correlating Tables 2 and 4, one notes the large value of B achieved within the ionized region, particularly for the high temperature cases ((i), (ii), (iii)). For example, in case (i), B = 12.0 where X = .929. This effect can be understood from the discussion of Section 3.4. In brief, radiation at the Lyman limit is absorbed as exp (-B), but higher frequency radiation only as $\exp\left(-\frac{B}{P^3}\right)$, hence the latter penetrates further; and a high temperature star emits a bigger proportion of high frequency radiation. It is in fact for this reason that the high temperature cases (i), (ii), (iii) give a more extended ionized region.

V CONCLUSION

With diffuse radiation neglected, it has been shown that a hot star ionizes a limited region of a surrounding hydrogen nebula. In circumstances of physical interest the ionized region is well defined, the degree of ionization falling from close to unity to near zero within a relatively short distance. Within the ionized region radiation in the Lyman continuum is little absorbed, but as the boundary is approached it is extinguished extremely rapidly.

For low stellar temperatures and low gas densities the effect of absorption is less dominant, and consequently the boundary less clearly defined. A similar effect also occurs with a very hot star, since such a star emits much of its radiation with frequencies well beyond the Lyman limit, where the absorption cross-section is small. This part of the radiation penetrates further into the nebula, causing some loss of definition at the boundary.

It has also been shown that the effect of diffuse radiation is probably to extend the ionized region by a significant but not overwhelming amount, the relative increase being substantially independent of the physical parameters. Otherwise the conclusions remain unaltered. The true radius may be reasonably approximated by the Stromgren radius, as the chief errors in that approximation have opposing effects.

In conclusion, I wish to thank Professor H.S. Green for his encouragement, and for many helpful discussions. During the course of this work the writer was supported by the Colonial Sugar Refining Co. Ltd., Sydney.

Appendix 1

The Approximation of Fixed Heavy Particles

For either recombinations to the ground state hydrogen atom or photoionization from the ground state, the equations of conservation of energy and momentum are

$$E + \frac{1}{2} m_1 c_1^2 + \frac{1}{2} m_2 c_2^2 = cp + \frac{1}{2} m_3 c_3^2 \qquad \ldots\ldots (A1.1)$$

$$m_1 \underline{c}_1 + m_2 \underline{c}_2 = \underline{p} + m_3 \underline{c}_3 \qquad \ldots\ldots (A1.2)$$

where the suffices 1, 2, 3 refer to electrons, protons and hydrogen atoms respectively. It is convenient to introduce $\underline{G}$ and $\underline{g}$:

$$m_o \underline{G} = m_1 \underline{c}_1 + m_2 \underline{c}_2 \qquad \ldots\ldots (A1.3)$$

$$\underline{g} = \underline{c}_2 - \underline{c}_1 \qquad \ldots\ldots (A1.4)$$

where

$$m_o = m_1 + m_2 \quad . \qquad \ldots\ldots (A1.5)$$

Solving for $\underline{c}_1$ and $\underline{c}_2$,

$$\underline{c}_1 = \underline{G} - M_2\, \underline{g} \qquad \ldots\ldots (A1.6)$$

$$\underline{c}_2 = \underline{G} + M_1\, \underline{g} \quad . \qquad \ldots\ldots (A1.7)$$

Then Eqs. (A1.1) and (A1.2) become

$$E + \frac{1}{2}\frac{m_1m_2}{m_o} g^2 + \frac{1}{2} m_oG^2 = cp + \frac{1}{2} m_3c_3^{\ 2}$$

$$m_o\underline{G} = \underline{p} + m_3\underline{c}_3 \quad ,$$

or eliminating $\underline{c}_3$,

$$E + \frac{1}{2}\frac{m_1m_2}{m_o} g^2 - cp = -\frac{1}{2} m_o \left(1 - \frac{m_o}{m_3}\right) G^2 - \\ - \frac{m_o}{m_3} \underline{G}.\underline{p} + \frac{1}{2m_3} p^2 \quad . \qquad \ldots\ldots (A1.8)$$

Each particle species is assumed to have a Maxwellian velocity distribution corresponding to the temperature T. It will be shown that under reasonable physical conditions the terms on the right of Eq. (A1.8) are negligible for almost all reactions. Suppose f_1, f_2 are the velocity distribution functions of the electrons and protons respectively. The product f_1f_2 contains the factor

$$\exp\left(-\frac{1}{kT}\left[\frac{1}{2} m_1c_1^{\ 2} + \frac{1}{2} m_2c_2^{\ 2}\right]\right)$$

$$= \exp\left(-\frac{1}{kT}\left[\frac{1}{2}\frac{m_1m_2}{m_o} g^2 + \frac{1}{2} m_oG^2\right]\right) \quad .$$

Also, from Eqs. (A1.6) and (A1.7),

$$d\underline{c}_1 d\underline{c}_2 = d\underline{g}\ d\underline{G}$$

since the Jacobian is unity. Thus the distribution of the relative velocity g is Maxwellian, and the mean value of $\frac{1}{2}\frac{m_1m_2}{m_o} g^2$ is $\frac{3}{2}$ kT. Similarly the mean value of $\frac{1}{2} m_o G^2$ is $\frac{3}{2}$ kT. Now the factor

$$\left(1 - \frac{m_o}{m_3}\right) = \frac{m_3 - m_1 - m_2}{m_3}$$

is a very small number, the numerator being the mass equivalent of the binding energy E, and the denominator the atomic mass. For hydrogen its value is 1.45×10^{-8}. The factor $\frac{1}{2} m_o G^2 (1 - \frac{m_o}{m_3})$ is therefore negligible for probable values of G.

As temperature T = 10,000 °K, the root mean square value of G is 1.57×10^6 cm/sec, thus for G near its mean value, Gp is four orders of magnitude smaller than cp, and the term

$$\frac{m_o}{m_3} \underline{G} \cdot \underline{p} < Gp$$

can be neglected. The final term $\frac{p^2}{2m_3}$ is small because p is small: at the threshold,

$$\frac{p}{2m_3} = 217 << c$$

$$\therefore \quad \frac{p^2}{2m_3} << cp \quad .$$

Almost all reactions have p quite close to its threshold value, thus this factor too can be neglected.

The energy equation then becomes

$$E + \frac{1}{2} \; \frac{m_1 m_2}{m_o} g^2 = cp \quad , \qquad \ldots\ldots \text{(A1.9)}$$

and the energy of the emitted or absorbed photon depends only on the relative velocity g. On the other hand, if the heavy particles are assumed to be stationary, the equation is

$$E + \frac{1}{2} m_1 c_1^{\;2} = cp \quad . \qquad \ldots\ldots \text{(A1.10)}$$

For recombinations this has the same validity as Eq. (A1.9) since we have seen that the distribution of $\frac{1}{2}\frac{m_1 m_2}{m_0} g^2$ is the same as that of $\frac{1}{2} m_1 c_1^2$. But in any case the reduced electron mass differs negligibly from the true mass, and Eq. (A1.10) is a good approximation.

Appendix 2

The Ground State Approximation

It has been assumed in Chapter II that all neutral hydrogen atoms are in the ground state. The assumption is not strictly correct as atoms in excited states may result from recombinations, and may also be produced by absorption of line radiation or by collisions. However, the ground state approximation is justified if it can be established that ionization rarely occurs from excited states, and that the lifetime of any excited state before it cascades to ground state is short compared to the lifetime of the ground state. These conditions are shown to be satisfied except in the immediate neighbourhood of the star.

The lifetime of the ground state atom is governed by Lyman continuum and Lyman line radiation - continuum radiation ionizes the atom, line radiation excites it. In principle collisional excitation should also be considered, but with an electron temperature of 10,000 °K the electrons are insufficiently energetic to excite the ground state: the mean kinetic energy is $\sim$1eV, whereas excitation to the first excited state requires 10 eV. The effect of line radiation will be considered subsequently; for the present we consider only continuum radiation.

If N_1 is the number density of ground-state hydrogen atoms, then with the notation of Chapter II,

$$c\alpha N_1 f d\underline{p} d\underline{r} dt$$

photons are absorbed in $d\underline{r}$ during dt from the momentum range $d\underline{p}$. Each photon absorbed corresponds to ionization of an atom, hence the total number of ground-state atoms ionized in $d\underline{r}$ during dt is

$$2\pi \int_{\frac{E}{c}}^{\infty}\int_{-1}^{1} c\alpha N_1 f d\mu p^2 dp d\underline{r} dt.$$

Thus if τ_1^{-1} is the transition probability for photo-ionization from ground state,

$$\tau_1^{-1} = 2\pi c \int_{\frac{E}{c}}^{\infty}\int_{-1}^{1} f d\mu \alpha p^2 dp \quad .$$

If diffuse radiation is neglected (i.e. $f_2 = 0$), and the effect of absorption ignored, Eq. (2.24) gives

$$\int_{-1}^{1} f d\mu = \frac{r_s^2}{h^3 r^2} \exp\left(-\frac{cp}{kT_s}\right) \quad .$$

The absorption coefficient α can be written (Section 2.7.) as

$$\alpha = \alpha_0 \left(\frac{cp}{E}\right)^{-3} \quad .$$

$$\therefore \quad \tau_1^{-1} = \frac{2\pi}{h^3} c\, \alpha_0 \left(\frac{E}{c}\right)^3 \frac{r_s^2}{r^2} \int_{\frac{E}{c}}^{\infty} \exp\left(-\frac{cp}{kT_s}\right)\frac{dp}{p}$$

i.e. $\tau_1^{-1} \approx \frac{2\pi}{h^3} \alpha_0 \left(\frac{E}{a}\right)^2 \frac{r_s^2}{r^2} kT_s \exp\left(-\frac{E}{kT_s}\right)$ . ... (A2.1)

For $T_s = 40{,}000\ ^\circ K$,

$$\tau_1^{-1} \approx 7.7 \times 10^6 \frac{r_s^2}{r^2} \sec^{-1} . \qquad \text{...... (A2.2)}$$

It is necessary to show that the transition probability τ_1^{-1} is small compared to the probability for radiative decay from any excited state. It is sufficient to consider only the most long-lived excited state, the 2s state. Breit and Teller (1940) show that this state can decay only by emission of two photons - electric and magnetic dipole and electric quadrupole contributions are negligible. According to Spitzer and Greenstein (1951) the transition probability for 2-quantum emission is 8.23 $\sec^{-1}$. Near the star this probability is small compared to τ_1^{-1}, but for $r \geqslant 10^4\ r_s$,

$$\tau_1^{-1} << 8.23 \sec^{-1},$$

and the condition is satisfied. The discussion of Chapter III shows that $10^4 r_s$ is several orders of magnitude smaller than the radius of the ionized region, hence we conclude that the condition is satisfied except in the immediate neighbourhood of the star.

Photo-ionization from an excited state will be unimportant if the state undergoes radiative decay before a continuum photon can be absorbed. Here again the 2s state is critical, as any other excited state has a lifetime shorter by several orders of magnitude. The transition probability τ_{2s}^{-1} for photo-ionization from the 2s state can be calculated by the method used for τ_1^{-1}. The ionization potential of the 2s state is $\frac{1}{4}E$, hence this replaces E in Eq. (A2.1). Also α_0 is replaced by $2.34\,\alpha_0$, the absorption coefficient of the 2s state atom at the Balmer limit. This was calculated from the known cross-section for the converse process (Bates et al. 1939) using the method of Section 2.3. For T_s = 40,000 °K, it is then found that

$$\tau_{2s}^{-1} \approx 2.2. \text{x}\ 10^7 \frac{r_s^{\ 2}}{r^2}\ \text{sec}^{-1} \quad .$$

The condition

$$\tau_{2s}^{-1} << 8.23\ \text{sec}^{-1}$$

is thus satisfied for $r \geqslant 10^4 r_s$.

Finally the effect of line radiation is considered. Let N_i be the number density of atoms in the state i; then

$$\Sigma_i N_i = n_3$$

where n_3 is the total number density of neutral hydrogen atoms. As in Chapter II, n_1 and n_2 are the number densities of electrons and protons.

Consider a system in thermodynamic equilibrium at the temperature

T_e, with electron and proton number densities equal to the local values of n_1 and n_2. Let N_i^T be the corresponding population of the state i. Baker and Menzel (1938) have shown that diffusing Lyman line radiation in a nebula gives N_i comparable to N_i^T for all excited states. In particular, for $T_e = 10{,}000$ °K,

$$N_i < N_i^T \text{ for } i \geqslant 2 \qquad \ldots\ldots \text{(A2.3)}$$

whatever the assumed optical depth of the nebula in the Lyman lines.

In the case of thermodynamic equilibrium, the Boltzmann law gives

$$\frac{N_i^T}{N_1^T} = \frac{S_i}{S_1} \exp\left(-\frac{E - E_i}{kT_e}\right)$$

where E_i is the ionization potential and S_i the statistical weight of the state i. The ground state population N_1^T is related to n_1 and n_2 by the Saha equation (Eq. (2.14)):

$$\frac{n_1 n_2}{N_1^T} = \frac{\omega_1 \omega_2}{S_1} \left(\frac{2\pi mkT_e}{h^2}\right)^{\frac{3}{2}} \exp\left(-\frac{E}{kT_e}\right) \quad .$$

Thus for any i,

$$\frac{n_1 n_2}{N_i^T} = \frac{\omega_1 \omega_2}{S_i} \left(\frac{2\pi mkT_e}{h^2}\right)^{\frac{3}{2}} \exp\left(-\frac{E_i}{kT_e}\right) \quad . \qquad \ldots\ldots \text{(A2.4)}$$

On the other hand, in the nebula the ground-state population N_1 is given approximately by the Stromgren equation (Eq. (2.32)):

$$\frac{n_1 n_2}{N_1} = \frac{\omega_1 \omega_2}{S_1} \left(\frac{2\pi mkT_e}{h^2}\right)^{\frac{3}{2}} \frac{T_s}{T_e} \exp\left(-\frac{E}{kT_s}\right) \frac{r_s^2}{4r^2} \quad . \qquad \ldots\ldots \text{(A2.5)}$$

Here the general statistical weights (removed in Section 2.3.) have been restored to the equation, but the effect of absorption has been neglected. From Eqs. (A2.4) and (A2.5),

$$\frac{N_i^T}{N_1} = \frac{S_i}{S_1} \cdot \frac{T_s}{T_e} \exp\left(-\frac{E}{kT_s} + \frac{E_i}{kT_e}\right) \frac{r_s^2}{4r^2} \quad ,$$

and using the inequality (A2.3)

$$\frac{N_i}{N_1} \leqslant \frac{S_i}{S_1} \frac{T_s}{T_e} \exp\left(-\frac{E}{kT_s} + \frac{E_i}{kT_e}\right) \frac{r_s^2}{4r^2} \text{ for } i \geqslant 2 \quad \ldots\ldots\ldots \text{(A2.6)}$$

The most highly populated state is the first (i = 2). If the typical temperatures $T_e = 10,000\ ^{\circ}K$, $T_s = 40,000\ ^{\circ}K$ are assumed, then since

$$E_2 = \frac{1}{4} E,$$

$$\therefore \quad \frac{N_2}{N_1} \leqslant \frac{S_2}{S_1} \frac{r_s^2}{r^2} \quad .$$

Close to the star the result is not definitive, but for large values of r it follows that very few atoms are in excited states; certainly this is true for $r \geqslant 10^4 r_s$.

The ground state approximation is therefore valid outside the immediate neighbourhood of the star. Close to the star the assumption may not be true, and a substantial proportion of neutral atoms may be in excited states (particularly the 2s state). However, the discussion of Chapter III shows that in this region the hydrogen is almost completely ionized, hence the deviation near the star is unimportant.

Appendix 3

The Approximate Transfer Equation

With spherical symmetry, the transfer equation has the form (Eq. (2.48)):

$$\mu\frac{\partial F_2}{\partial R} + \frac{(1-\mu^2)}{R}\frac{\partial F_2}{\partial \mu} = J(P)X^2 - \frac{1}{P^3}(1-X)\,F_2\,. \qquad \ldots\ldots (A3.1)$$

In astrophysical applications it is customary to use an approximate equation:

$$\frac{1}{3}\frac{d^2}{d\eta^2}\Phi = \Phi - JP^3\frac{X^2}{1-X} \qquad \ldots\ldots (A3.2)$$

where

$$\Phi = \frac{1}{2}\int_{-1}^{1} F_2 d\mu \qquad \ldots\ldots (A3.3)$$

and

$$\eta = \frac{1}{P^3}\int_0^R (1-X)\,dR' \,. \qquad \ldots\ldots (A3.4)$$

The approximate equation has been applied to the case of a low density nebula (Pottasch and Jeffries 1959); we intend to show that Eq. (A3.2) is suspect in this situation. The two approximations inherent in Eq. (A3.2) have been justified by Eddington (1926) and Rosseland (1936) for the case of stellar interiors.

In the standard derivation, the term

$$\frac{(1-\mu^2)}{R}\frac{\partial F_2}{\partial \mu}$$

is dropped from Eq. (A3.1), the justification being that the curvature of the system is negligible: according to Rosseland, "In most problems of stellar atmospheres it is legitimate to assume the atmosphere to be stratified in plane parallel layers". The assumption is reasonable for stellar interiors, since in that case the absorption coefficient per unit length is large, and emission and absorption of radiation is essentially a local phenomenon. The situation in a nebula is quite different, as absorption within the ionized region is small (see Chapter III), allowing a photon to travel a distance comparable with the dimensions of the system. The effect of curvature is therefore important.

With that term neglected, Eqs. (A3.1) and (A3.4) give

$$\mu \frac{\partial F_2}{\partial \eta} = JP^3 \; \frac{X^2}{1 - X} - F_2 \quad . \qquad \ldots\ldots \text{(A3.5)}$$

Integrating with respect to μ,

$$\frac{d\Psi}{d\eta} = JP^3 \frac{X^2}{1 - X} - \Phi$$

where

$$\Psi = \frac{1}{2} \int_{-1}^{1} F_2 \mu \, d\mu \quad .$$

A second equation is obtained from Eq. (A3.5) by multiplying through by μ before integrating:

$$\frac{1}{2} \; \frac{d}{d\eta} \int_{-1}^{1} F_2 \mu^2 d\mu = - \Psi \quad .$$

Then eliminating Ψ,

$$\frac{1}{2} \; \frac{d^2}{d\eta^2} \int_{-1}^{1} F_2 \mu^2 d\mu = \Phi - JP^3 \frac{X^2}{1 - X} \quad .$$

An approximation due to Eddington is then used:

$$\int_{-1}^{1} F_2 \mu^2 \, d\mu \approx \frac{1}{3} \int_{-1}^{1} F_2 d\mu \quad , \qquad \ldots\ldots \text{(A3.6)}$$

giving Eq. (A3.2).

The two sides of the Eddington approximation can be evaluated using the simple model of the nebula introduced in Chapter IIIB. Within the ionized sphere of radius R_2, F_2 is given by Eq. (3.37):

$$F_2 = J(R\mu + \sqrt{R_2^2 - R^2 + R^2\mu^2})$$

$$\therefore \quad \int_{-1}^{1} F_2 d\mu = J\left[R_2 + \frac{(R_2^2 - R^2)}{R} \ln\left(\sqrt{\frac{R_2 + R}{R_2 - R}}\right)\right] \qquad \ldots\ldots \text{(A3.7)}$$

$$\text{and} \quad \int_{-1}^{1} F_2 \mu^2 d\mu = \frac{1}{4} J\left[\frac{R_2(R_2^2 + R^2)}{R^2} - \frac{(R_2^2 - R^2)^2}{R^3} \ln\left(\sqrt{\frac{R_2 + R}{R_2 - R}}\right)\right] . \qquad \ldots\ldots \text{(A3.8)}$$

At $R = 0$, F_2 is independent of μ, and Eq. (A3.6) is exact:

$$\frac{1}{3} \int_{-1}^{1} F_2 \, d\mu = \int_{-1}^{1} F_2 \mu^2 d\mu = \frac{2}{3} JR_2 \quad .$$

However, at $R = R_2$,

$$\int_{-1}^{1} F_2 d\mu = JR_2$$

$$\text{and} \quad \int_{-1}^{1} F_2 \mu^2 d\mu = \frac{1}{2} JR_2$$

$$\therefore \quad \int_{-1}^{1} F_2 \mu^2 d\mu = \frac{1}{2} \int_{-1}^{1} F_2 d\mu \quad .$$

Thus at the boundary of the nebula the Eddington approximation is considerably in error. The rather extreme model adopted probably emphasises the discrepancy, but according to the discussion of

Chapter III the true situation may be rather similar.

It is concluded that the approximate transfer equation is of doubtful validity in this application.

Appendix 4.

Recombinations to All Bound States

In Section 2.7 the quantity I was defined:

$$I = 2 \int_1^{\infty} \exp \left(-\frac{PE}{kT_e}\right) S(P) \frac{dP}{P} \qquad \ldots\ldots \text{(A4.1)}$$

where S(P) is the ratio of the total recombination cross-section σ to the recombination cross-section to ground state, σ_1. The parameter P is defined by

$$P = \frac{cp}{E} = 1 + \frac{\frac{1}{2} mu^2}{E} \quad .$$

If recombinations on excited states are neglected, the right hand side of Eq. (A4.1) becomes

$$2 \int_1^{\infty} \exp \left(-\frac{PE}{kT_e}\right) \frac{dP}{P}$$

$$\approx 2 \frac{kT_e}{E} \exp \left(-\frac{E}{kT_e}\right) \quad .$$

For $T_e = 10^4$ °K, $\frac{E}{kT_e} \approx 16$, and the approximation is better than 6%. The effect of recombinations on excited states can be expressed in terms of the quantity $\bar{S}$, defined by

$$I = 2\bar{S} \frac{kT_e}{E} \exp \left(-\frac{E}{kT_e}\right) \quad . \qquad \ldots\ldots \text{(A4.2)}$$

I and $\bar{S}$ have been calculated numerically for an electron temperature of 10,000 °K, this temperature having been suggested by Spitzer (1954) as characteristic of emission nebulae. Here the method and the results are indicated.

It is convenient to introduce a new parameter Q:

$$Q = P - 1 = \frac{\frac{1}{2} mu^2}{E}$$

Defining $\qquad V(Q) = S(P) \quad ,$

then V(Q) is the ratio of the cross-sections, where the cross-sections are functions of the electron kinetic energy in units of E. In terms of Q,

$$I = 2 \int_0^\infty \exp\left(-\frac{(Q+1)E}{kT_e}\right) \frac{V(Q)}{(Q+1)}\, dQ \quad . \qquad \ldots\ldots (A4.3)$$

Before I can be calculated it is necessary to determine V(Q). The cross-section σ_1 presents no problem, since it is related to the absorption coefficient α by Eq. (2.16). However, the published information on the total recombination cross-section σ is limited. Bates, Buckingham, Massey and Unwin (1939) have calculated σ for four very low values of the electron kinetic energy, their method being to sum the partial cross-sections to each bound state. Their results are used in deriving Table 5.

Table 5. The Function V(Q).

$Q = \frac{\frac{1}{2}mu^2}{E}$	V(Q)
2.50×10^{-3}	4.06
5.07×10^{-3}	3.63
9.56×10^{-3}	3.23
2.06×10^{-2}	2.84

The only other available information comes from Wessel (1930). Wessel has shown that

$$V(Q) = \sum_{n=1}^{\infty} \frac{1}{n^3} = 1.197 \qquad \text{for } Q \gg 1 ,$$

$$V(1) = 1.290 ,$$

and $V(0.25) > 1.512$.

The results of Table 5 were plotted on a small scale graph, and a smooth curve drawn through the points. A plausible value of V(0) was found to be 4.76. From this graph it was also possible to estimate the gradients at $Q = 0$ and at $Q = 2 \times 10^{-2}$:

$$V'(0) \approx -70$$

$$V'(2 \times 10^{-2}) \approx -25 .$$

A large scale graph was then plotted (with domain (0,1)), using the latter value of the gradient and the information from Wessel. In both cases the graph appeared reasonably well determined by the requirement that the curve be smoothly varying, the uncertainty being greatest for the second graph.

The integral in Eq. (A4.3) was then evaluated using Simpson's rule, considering only the domain $(0, \frac{1}{2})$; for $Q > \frac{1}{2}$ the integrand is negligibly small. Using 50 points for the integration, it was found that

$$I = 4.43 \times 10^{-8} \quad .$$

More than half the contribution to the integral comes from $Q \lesssim 4 \times 10^{-2}$, in which range $V(Q)$ is expected to be reasonably accurate.

From Eq. (A4.2), it follows that

$$\bar{S} = 2.50 \quad .$$

The effect of recombinations is thus quite considerable. The calculation has been carried out for only one value of T_e, but it is suggested that $\bar{S}$ will be close to 2.5 over a considerable range of temperatures. For the quantity $\bar{S}$ is effectively a weighted average of the function $V(Q)$. In the immediate neighbourhood of the origin $V(Q)$ varies extremely rapidly, and there the weight function $\exp\left(-\frac{PE}{kT_e}\right)$ is relatively unimportant except for impossibly high temperatures. On the other hand $V(Q)$ flattens out very rapidly, and by the time the weight function assumes importance (when $V'(Q) \sim \frac{E}{kT_e}$), $V(Q)$ is already considerably reduced from its maximum, and thereafter $V(Q)$ changes only slowly.

Appendix 5.

Computing Details

1. The Fortran Programme

```
       DIMENSIONR(100),COMPX(100),Z(100),B(100),X(100)
    1  FORMAT(I3,I2,E7.4)
    2  FORMAT(6E10.6)
    5  FORMAT(5E10.6)
    6  FORMAT(2I2)
    7  FORMAT(1Hb,7HDELTAR=1PE9.3,4HbbA=1PE10.4,5HbbDS=1PE10.4)
    8  FORMAT(1Hb,9HI,bbbR(I))
    9  FORMAT(1Hb,4(I5,1PE12.4))
   10  FORMAT(1Hb,17HI,bbbX(I),bbbB(I))
   11  FORMAT(1Hb,3HIT=I2)
   12  FORMAT(1Hb,2(I8,1PE15.6,1PE12.3))
   16  READINPUTTAPE2,1,MAXI,LIM,DELTAR
       LIMX=LIM+1
       READINPUTTAPE2,2,(X(I),I=LIMX,MAXI)
       READINPUTTAPE2,5,A,C,D,DS,RS
       READINPUTTAPE2,6,NIT,L
       PRINT7,DELTAR,A,DS
   23  R(1)=DELTAR
       I=1
   26  I=I+1
       R(I)=R(I-1)+DELTAR
       IF(I-MAXI)26,27,27
   27  PRINT8
       PRINT9,(I,R(I),I=LIMX,MAXI)
       I=0
   28  I=I+1
       X(I)=1.0
       COMPX(I=0.0
       IF(I-LIM)28,29,29
   29  PRINT10
       IT=0
   30  IT=IT+1
       KB=LIM
       I=LIM
   32  I=I+1
       COMPX(I)=1.0-X(I)
       KB=2*LIM-1-KB
       HB=2.0
       SUMB=0.0
       KBB=KB
   33  KBB=KBB+1
       HB=6.0-HB
       SUMB=SUMB+COMPX(KBB)*HB
```

```
 34   IF(KBB+1-I)33,35,35
 35   SUMB=SUMB+COMPX(I)
      B(I)=SUMB*DELTAR/3.0
 37   IF(I-MAXI)32,38,38
 38   IF((IT-1)-((IT-1)/L)*L)40,39,40
 39   ITL=IT-1
      PRINT11,ITL
      PRINT12,(I,X(I),B(I),I=LIMX,MAXI)
 40   I=LIM
 42   I=I+1
      SUM=0.0
      T=0.0
      H=2.0
 47   T=T+0.05
      H=6.0-H
      E=EXPF(-DS/T-B(I)*T*T*T)/T
      SUM=SUM+E*H
 48   IF(T-0.93)47,49,49
 49   SUM=SUM+EXPF(-DS-B(I))
      RHS=(RS/R(I))**2*0.05/3.0*SUM
160   RAT=RHS/C
      IF(RAT-100.0)300,301,301
300   Z(I)=-0.5*RAT+SQRTF(0.25*RAT*RAT+RAT)
      GOTO162
301   Z(I)=1.0-(1.0/RAT)+2.0*(1.0/RAT)**2-5.0*(1.0/RAT)**3
162   IF(I-MAXI)42,165,165
165   K=LIM
170   K=K+1
      X(K)=Z(K)
      IF(K-MAXI)170,175,175
175   IF(IT-NIT)30,180,180
180   KB=LIM
      I=LIM
185   I=I+1
      KB=2*LIM-1-KB
      HB=2.0
      SUMB=0.0
      KBB=KB
187   KBB=KBB+1
      HB=6.0-HB
      SUMB=SUMB+(1.0-X(KBB))*HB
188   IF(KBB+1-I)187,189,189
189   SUMB=SUMB+(1.0-X(I))
      B(I)=DELTAR/3.0*SUMB
      IF(I-MAXI)185,190,190
190   PRINT11,NIT
      PRINT12,(I,X(I),B(I),I=LIMX,MAXI)
      GOTO16
      END(0,0,1,1,0,0,0,0,0,0,0,0,0,0,0)
```

2. Explanation of the Programme.

The general purpose and method of the programme are outlined in Section 4.1. The input data is as follows:

MAXI: The total number of points at which X is specified

LIM: In all iterations X(I) is taken as one for I $\leqslant$ LIM.

DELTAR: The spacing between the radial points (thus (MAXI) x (DELTAR) is the total dimensionless radius of the system).

X(I): The initially postulated form of X.

A,C,D,DS,RS: Physical Data. A and C are as defined in the text; $D = \frac{E}{kT_e}$; $DS = \frac{E}{kT_s}$; $RS = R_s$.

NIT: The number of iterations to be performed.

L: The iterated solutions are printed after every L iterations; e.g. if NIT = 8 and L = 3 then the results of the third and sixth iterations are printed in addition to the input form and the result of the eighth iteration. Print-outs include the corresponding values of the optical depth B(I).

At the beginning of each iteration the values of B(I) are computed using Simpson's rule; and if IT (the iterative number) is suitably related to L the values of X(I) and B(I) are printed. For each radial point the value of

$$(\mathrm{RHS}) = \frac{R_s^2}{R^2}\int_1^\infty \exp\left(-\frac{E}{kT_s}P - \frac{B}{P^3}\right)\frac{dP}{P}$$

is then computed (statement 42), again using Simpson's rule. (In the actual calculation $\frac{1}{T}$ is substituted for P). The value of X to be used in the next iteration is computed (statement 160), and stored as Z(I) until the iteration is complete; the formal solution of

$$\frac{X^2}{1 - X} = (RHS/C) = (RAT)$$

is used for small values of (RAT), but for large values a series expansion is employed. When NIT iterations have been performed, exit occurs at statement 180. Optical depths appropriate to the output form of X are computed as before, and the values of X(I) and B(I) printed.

The listed programme is a truncated form of the one actually used, as the programme was originally designed to include optionally the effect of diffuse radiation. That part was unsuccessful, essentially because of the large extent of the ionized region (in terms of R); with this circumstance it is difficult to solve Eq. (2.53) numerically. A degree of clumsiness in the listed programme is due to its intended dual purpose.

3. Operational Details.

Preliminary runs showed that if the initially postulated form of X represents a too small ionized region then the iterative process converges rather slowly - in fact unless the points are reasonably closely spaced, iterations only smooth the input data. On the other hand convergence is quite rapid if the input form of X is everywhere too large. Comparison of the two cases with fine spacing and a large number of iterations established that the result is independent of the input form of X. The spacing of points was varied to prove that the results are also independent of the spacing.

The final results quoted in Chapter IV were obtained with MAXI = 63, LIM = 3 and DELTAR chosen so that

$$45(\text{DELTAR}) = R_o \quad .$$

In the input data X(I) was taken as one for all values of I, and in each case 36 iterations were performed.

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