Graphs With Transitive Automorphism Groups

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Summary

Graph theory to date tends to be mostly of a combinatorial or topological nature. More algebraic aspects of graph theory have been studied, however they deal almost exclusively with the problem of determining graphs with given combinatorial properties whose automorphism group is isomorphic to a given abstract group. A bibliography to the literature on this subject is given in [11] p.263.

This thesis discusses a different algebraic side of graph theory, namely the theory of graphs with transitive automorphism groups. A large class of such graphs is given by the Cayley graphs, and the problem immediately arises as to how the class $\mathcal{T}$ of graphs with transitive automorphism groups is related to the class $\mathcal{L}$ of graphs which are isomorphic to Cayley graphs. This problem is discussed in the first four chapters of the thesis.

In chapter I the basic definitions and terminology are given, and it is shown by examples that $\mathcal{T}$ properly contains $\mathcal{L}$.

In chapter II it is shown that $\mathcal{T}$ and $\mathcal{L}$ are both closed under cartesian products, but not under the reverse operation of factorising a graph with respect to cartesian products. It is also shown that to each simple graph G in $\mathcal{T}$, there exists a complete graph whose cartesian product with G is in $\mathcal{L}$.

Two natural generalizations of Cayley graphs are discussed in chapter III. The first of these is shown to give arbitrary simple graphs in $\mathcal{T}$, generalizing a theorem of Sabidussi which

characterises the graphs in $\mathcal{L}$ by means of their automorphism groups. This is used to deduce a theorem on homomorphisms, which states in Reidemeister's language that any simple graph in γ may be covered by a graph in $\mathcal{L}$. Similar results hold for the second generalization of Cayley graphs, and are used to deduce a further characterisation of the graphs in $\mathcal{L}$.

The problem of finding usable sufficient conditions for a graph to be in $\mathcal{L}$ is discussed in §7 using the results of chapter III, and in §10 using Petersen's alternating path method. It is for example shown that if a regular graph of degree 2 with p^2 vertices (p prime) is in γ, then it is already in $\mathcal{L}$. This is deduced from a rather stronger result involving the alternate composition graph of a graph.

Some further applications of the alternating path method are also considered in §9 and §10, and the strong practical applications of this method are demonstrated in §11 in the construction of an infinite set of regular graphs of degree 2 which are in $\mathcal{L}$ but not in γ.

In chapter V Hamiltonian arcs in Cayley graphs are discussed. It is shown for instance that a connected Cayley graph of a finite abelian group always has a Hamiltonian arc, and the problem of existence and classification of Hamiltonian arcs in Cayley graphs is solved or partially solved in a number of other special cases.

D e c l a r a t i o n

I hereby declare that this thesis contains no material which has been accepted for the award of any other degree or diploma in any University and that, to the best of my knowledge and belief, this thesis contains no material previously published or written by another person, except when due reference is made in the text of this thesis.

Signed

Bonn, 15th March 1967.

Notes on Terminology

The basic graph theoretical terminology follows as far as possible Oystein Ore's "Theory of Graphs" (A.M.S. Colloquium publications vol.38).

The terminology for permutation groups is that of H. Wielandt "Finite Permutation Groups" (Academic press 1964). This terminology differs from the classical terminology in a few instances. In particular "regular" is used instead of "regular transitive" to describe a transitive permutation group whose stabilizer subgroups are trivial, and the term "block" is used for "set of imprimitivity".

In the first three chapters the arithmetic used is cardinal arithmetic, though it is often restricted to the usual finite arithmetic.

CHAPTER I : Introduction

§1. Basic Definitions

Intuitively a graph consists of a configuration of points with lines joining them, and each line may or may not have a direction assigned to it. We call the lines directed or undirected edges of the graph according to whether they have an assigned direction or not. In order to avoid the inconvenience of having to consider directed and undirected edges simultaniously, we will consider an undirected edge to be a pair of oppositely oriented directed edges.

A graph may have several edges from one given vertex to another. For the present it is inconvenient to distinguish these edges, so we shall characterise an edge simply by its initial and terminal vertices and its "multiplicity" - the number of times it occurs in the graph. These considerations motivate the following formal definitions.

A graph G is a triple (V,E,ρ) consisting of a vertex set V; an edge set E of ordered pairs of vertices; and a multiplicity function ρ, which maps $V \times V$ into the class of all cardinal numbers and has the property that $\rho(a,b) \neq 0$ if and only if $(a,b) \in E$.

If $(a,b) \in E$, we say G has an edge from a to b of multiplicity $\rho(a,b)$, and we call a and b respectively the initial and terminal vertices of the edge. If $\rho(a,b) = 1$ we call (a,b) a simple edge, otherwise a multiple edge. An edge of the form (a,a) is called a loop at a.

A graph is already fully described by its vertex set and multiplicity function. Further a simple graph - that is a graph with no multiple edges - is fully described by its vertex and edge sets, and will therefore often be described only by the pair (V,E).

If $G = (V,E,\rho)$ and $G' = (V',E',\rho')$ are graphs with $V' \subseteq V$, $E' \subseteq E$, and $\rho'(a,b) \leq \rho(a,b)$ for all $(a,b) \in V' \times V'$, then we call G′ a subgraph of G. If $\rho'(a,b) = \rho(a,b)$ for all $(a,b) \in V' \times V'$, we call G′ the full subgraph of G on the set V′.

The number of outgoing edges at a vertex a of G is called the local out-degree at a, denoted by $\rho(a)$. The local in-degree $\rho^*(a)$ at a is similarly defined. Clearly

(1) $$\rho(a) = \sum_{b \in V} \rho(a,b) \quad ; \quad \rho^*(a) = \sum_{b \in V} \rho(b,a).$$

We say G is out-regular of degree n (n a finite or infinite cardinal number) if $\rho(a) = n$ for all $a \in V$. Similarly G is in-regular of degree n if $\rho^*(a) = n$ for all $a \in V$. G is half-regular if it is both in- and out-regular, and is regular of degree n if it is both in- and out-regular of degree n.

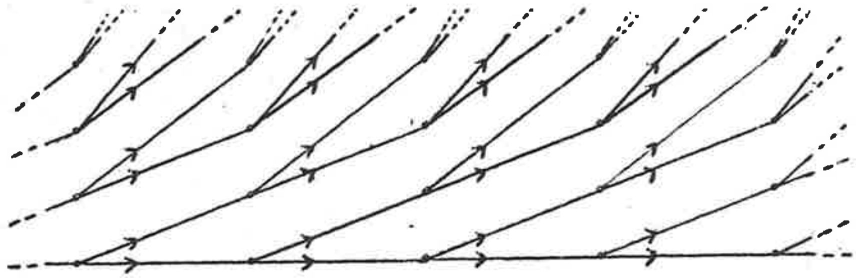
We say G is finite if the number $|V|$ of vertices and the number $\sum_{a \in V} \rho(a)$ $(= \sum_{a \in V} \rho^*(a))$ of edges are both finite.

The following lemma is trivial:

<u>Lemma 1.1.</u> A finite half-regular graph is regular.

That the finiteness condition is necessary is shown by the

<u>Example 1.1.</u> $G = (V,E)$ is the simple graph with vertex set $V = \{(m,n): m \text{ and } n \text{ integers}, n \geqslant 0\}$, and edges $((m,n),(m+1,2n))$ and $((m,n),(m+1,2n+1))$ for each (m,n) in V.



This graph is in-regular of degree 1 and out-regular of degree 2, hence half-regular but not regular.

The <u>converse graph</u> G^* of a graph $G = (V,E,\rho)$ is the graph with vertex set V and multiplicity funtion ρ^* defined by

(2) $$\rho^*(a,b) = \rho(b,a) \quad , \quad ((a,b) \in V \times V).$$

It is obtained by reversing the edges of G.

If $G_i = (V,E_i,\rho_i)$, $i \in I$, is a set of graphs on the vertex set V, their <u>edge direct sum</u> is the graph $G = (V,E,\rho)$ with

(3) $$\rho(a,b) = \sum_{i \in I} \rho_i(a,b)$$

for all $(a,b) \in V \times V$. It is denoted by $\sum_{i \in I} G_i$, or if I is finite, say $I = \{1,2, \ldots ,n\}$, also by $G_1+G_2+ \ldots +G_n$.

The <u>undirected graph</u> G_u of a graph G is the edge direct sum

$G+G^*$ of G and its converse graph G^*.

We say that the vertex a of G is path-connected to the vertex b of G if there exists a sequence $a=a_0,a_1, \dots ,a_s=b$ of vertices of G such that (a_{i-1},a_i) is an edge of G for each $i = 1,2, \dots ,s$. The vertices a and b are said to be connected in G if they are path-connected in G_u.

The relation of connectedness is clearly an equivalence relation on the vertex set. The full subgraphs of G on the equivalence classes of this relation are called the connected components of G, or simply the components of G. G is connected if it has only one component.

The following lemma is standard, so we ommit the proof.

Lemma 1.2. Path-connectedness and connectedness are equivalent concepts in finite regular graphs.

That this lemma does not hold in general for infinite regular graphs is shown by the graph whose vertices are the integers and whose edges are all integer pairs of the form (i,i+1). For instance the vertex 1 is connected but not path-connected to the vertex 0 in this graph.

§2. Homomorphisms, Symmetric and Group Graphs

Let $G = (V,E,\rho)$ and $G_1 = (V_1,E_1,\rho_1)$ be graphs, and let φ be a mapping of V onto V_1. We say φ is a homomorphism of G onto

G_1 if for all $(a,b) \in V_1 \times V_1$

(1) $$\rho_1(a,b) = \sum \rho(c,d) ,$$

the summation being over all $(c,d) \in V \times V$ for which $(c\varphi, d\varphi) = (a,b)$.

A one-one homomorphism is called an <u>isomorphism</u>. In this case (1) reduces to the condition:

(2) $$\rho_1(c\varphi, d\varphi) = \rho(c,d) \quad , \quad \text{for all } (c,d) \in V \times V.$$

An <u>automorphism</u> of the graph G is an isomorphism of G onto itself. The set of all automorphisms of G forms a group under composition, called the <u>automorphism group</u> of G and denoted by $\Gamma(G)$. We consider an automorphism to be a permutation of the vertex set, so $\Gamma(G)$ is a permutation group.

If $\Gamma(G)$ is a transitive group we say that G is a <u>symmetric graph</u>. Since an automorphism must map a vertex onto a vertex of the same local in- and out-degrees, a symmetric graph is half-regular, and hence if finite it is regular. Example 1.1 gives an infinite symmetric graph which is not regular. The symmetry of this graph follows from the fact that the following two permutations of the vertex set are automorphisms of the graph and generate a transitive permutation group: (The arrow means "is mapped onto".)

$$\alpha : (m,n) \mapsto (m+1,n) \qquad \text{for all } (m,n) \in V;$$

$$\beta : (m,n) \mapsto \begin{cases} (m,n) & \text{if } m \leq 0 \text{ or } n \geq 2^m, \\ (m,n+2^{m-1}) & \text{if } 0 \leq n < 2^{m-1} \text{ and } m \geq 1, \\ (m,n-2^{m-1}) & \text{if } 2^{m-1} \leq n < 2^m \text{ and } m \geq 1. \end{cases}$$

The verification that α and β are automorphisms is easy. To show that they generate a transitive group, we consider an arbitrary vertex (m,n) of the graph. If the binary expansion of n is $n = \varepsilon_0 2^0 + \varepsilon_1 2^1 + \ldots + \varepsilon_s 2^s$, where each ε_i is either 0 or 1, then one readily verifies that $\alpha^{s+1-m}\beta^{\varepsilon_s}\alpha^{-1}\beta^{\varepsilon_{s-1}}\alpha^{-1} \ldots \alpha^{-1}\beta^{\varepsilon_0}\alpha^{-1}$ maps (0,0) onto (m,n), so our statement is proved. We remark that α and β do not generate the full automorphism group, for one can show that the full automorphism group is uncountable and hence it cannot be finitely generated.

We now define a spcial type of symmetric graph. Let H be a group and S a subset of H; the <u>Cayley graph</u> [H,S] of H with respect to S is the simple graph with vertex set H, and edge set

(3) $$E = \{(a,as) : a \in H, s \in S\} .$$

A Cayley graph generally has an associated "colouring" of the edges - that is, a mapping of E into some set of "colours" - however for our purposes this is redundant.

The graph G is a <u>group graph</u> of the group H if it is isomorphic to [H,S] for some subset S of H. Since Cayley graphs of non-isomorphic groups can be isomorphic, a graph can be a group graph of several different groups.

The Cayley graph [H,S] is clearly regular of degree the cardinality of S, so group graphs are regular.

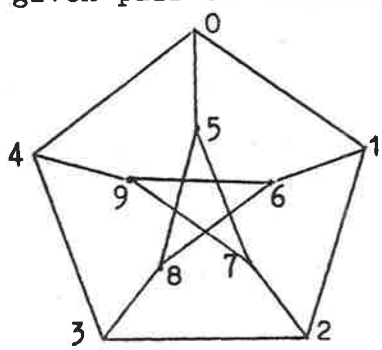
The following characterisation of group graphs is due to G. Sabidussi [16]. He only proves it for undirected graphs, but his proof holds without change for directed graphs.

Theorem 2.1. G. Sabidussi. The simple graph G is a group graph of the group H if and only if $\Gamma(G)$ contains a regular subgroup isomorphic to H.

Since we shall prove a rather more general theorem in §5, we do not prove this theorem here.

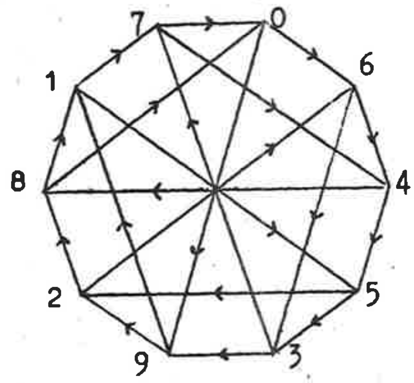
Example 1.1 gives a simple symmetric graph which is not a group graph; indeed this graph is not even regular. Finite simple symmetric graphs which are not group graphs appear to be rare, at least among small graphs. We give two examples.

Example 2.1. The Petersen graph. This graph is undirected of degree 3. For convenience we draw the pair of oppositely oriented edges connecting a given pair of vertices as one line.



R. Frucht [8] showed that the automorphism group of this graph is isomorphic to the symmetric group of degree 5 and is generated by the permutations (01234)(56789) and (26)(39)(78). If the graph were a group graph, then by theorem 2.1 its automorphism group would contain a regular subgroup. This subgroup would have order 10, and its elements of order 2 would be fixpoint free. But one readily verifies that every automorphism of order 2 leaves 4 vertices fixed. Hence the Petersen graph is not a group graph.

Example 2.2.



This graph has degree 2. In §11 we shall show that its automorphism group is transitive of order 20, and is generated by (01234)(56789) and (05)(1748)(2936). Every automorphism of order 2 leaves 2 vertices fixed, so as above, this graph is not a group graph.

To close this chapter we consider briefly the trivial cases of symmetric graphs.

If V is any set, we denote by $\Delta(V)$ the diagonal of $V \times V$; that is the set of pairs of the form (a,a) with $a \in V$. The simple graphs $(V, V\times V)$, $(V, V\times V - \Delta(V))$, $(V, \emptyset)$, and $(V, \Delta(V))$ are called respectively the complete graph, the complete graph without loops, the trivial graph, and the trivial graph with loops or identity graph on the vertex set V. They each have the full symmetric group on V as automorphism group.

Conversely we have as an immediate consequence of the definition of "doubly transitive":

Theorem 2.2. If the graph $G = (V,E,\rho)$ has doubly transitive automorphism group, then it is the edge direct sum of complete graphs without loops and identity graphs on V, and $\Gamma(G)$ is the full symmetric group on V.

CHAPTER II : Cartesian Products of Graphs

§3. Cartesian Products of Graphs

We define a cartesian product of graphs in the natural way by taking the cartesian product of the vertex sets and then defining the structure componentwise. More formally:

Let $G_1 = (V_1,E_1,\rho_1)$ and $G_2 = (V_2,E_2,\rho_2)$ be two graphs. Their cartesian product is the graph $G_1 \times G_2 = (V_1 \times V_2, E, \rho)$ where ρ is defined by

(1) $$\rho((a,c),(b,d)) = \rho_1(a,b)\rho_2(c,d)$$

for all $((a,c),(b,d)) \in (V_1 \times V_2) \times (V_1 \times V_2)$. Note that $((a,c),(b,d)) \in E$ if and only if $(a,b) \in E_1$ and $(c,d) \in E_2$. Hence under the canonical identification of $(V_1 \times V_2) \times (V_1 \times V_2)$ with $(V_1 \times V_1) \times (V_2 \times V_2)$, E is just $E_1 \times E_2$.

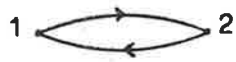
We shall restrict ourselves to cartesian products of pairs of graphs, however it is clear that cartesian products of arbitrary sets of graphs may be similarly defined, and the following discussion can be correspondingly generalized.

An immediate consequence of the definition is that a cartesian product of non-trivial graphs is simple if and only if the factors are simple. One verifies easily that the same holds for regularity and half-regularity of graphs, if one restricts the graphs considered to be locally finite (that is all local degrees are finite); for the local in-degree at a vertex (a,c) of $G_1 \times G_2$

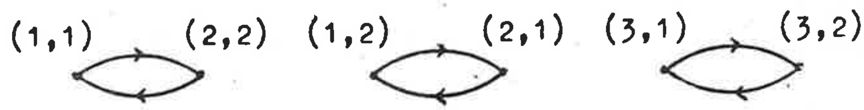
is just the product of the local in-degrees of a and c in G_1 and G_2 respectively, and similarly for local out-degrees.

The corresponding statement does not hold for group graphs or symmetric graphs. For instance the graph

is not symmetric, even though its cartesian product with the graph



is the graph



which is a group graph.

However although the symmetry of a cartesian product of graphs does not imply the symmetry of the factors, the reverse implication does hold.

<u>Theorem 3.1.</u> (i). If G_1 and G_2 are group graphs of the groups H and K respectively, then $G_1 \times G_2$ is a group graph of $H \times K$.

(ii). If G_1 and G_2 are symmetric graphs, then so is $G_1 \times G_2$.

<u>Proof.</u> (i). Without loss of generality $G_1 = [H,S]$ and $G_2 = [K,T]$ where S and T are subsets of H and K respectively. Let $G = [H \times K, S \times T]$. Since multiplication in $H \times K$ is defined componentwise and the edges of $G_1 \times G_2$ are defined componentwise, G and $G_1 \times G_2$ have the same edge sets. Since they are both simple graphs, they are equal.

(ii). Let $G_1 = (V_1, E_1, \rho_1)$ and $G_2 = (V_2, E_2, \rho_2)$ be symmetric graphs with cartesian product $G_1 \times G_2 = (V_1 \times V_2, E, \rho)$. Let $\Gamma = \Gamma(G_1) \times \Gamma(G_2)$

be considered as a permutation group on $V_1 \times V_2$ in the natural way; that is $(a,b)(\pi,\sigma) = (a\pi, b\sigma)$ for all $(a,b) \in V_1 \times V_2$ and $(\pi,\sigma) \in \Gamma$. By transitivity of $\Gamma(G_1)$ and $\Gamma(G_2)$, we can find for arbitrary (a,c) and (b,d) in $V_1 \times V_2$, $\pi \in \Gamma(G_1)$ and $\sigma \in \Gamma(G_2)$ with $a\pi = b$ and $c\sigma = d$. Then $(a,c)(\pi,\sigma) = (b,d)$, so Γ is transitive.

Now for any $((a,c),(b,d)) \in (V_1 \times V_2) \times (V_1 \times V_2)$ and any $(\pi,\sigma) \in \Gamma$ we have $\rho((a,c)(\pi,\sigma),(b,d)(\pi,\sigma)) = \rho((a\pi,c\sigma),(b\pi,d\sigma)) = \rho_1(a\pi,b\sigma)\rho_2(c\pi,d\sigma) = \rho_1(a,b)\rho_2(c,d) = \rho((a,c),(b,d))$, so (π,σ) is an automorphism of $G_1 \times G_2$, so $\Gamma \subseteq \Gamma(G_1 \times G_2)$. Hence $\Gamma(G_1 \times G_2)$ is transitive. Q.E.D.

Theorem 3.2. If $G = (V,E)$ is a simple symmetric graph and H is any transitive subgroup of $\Gamma(G)$, then there exists a complete graph C such that $C \times G$ is a group graph of H.

Proof. Let a be any vertex of G and H_a the stabilizer subgroup of a in H. The Cayley graph $C = [H_a, H_a]$ is clearly the complete graph with vertex set H_a. Let S be the subset of H defined by

(2) $$S = \{\pi \in H : (a\pi, a) \in E\}.$$

We shall show that $C \times G$ is isomorphic to $[H,S]$.

Let R be a set of left coset representatives of H_a in H. Then each element of H has a unique representation in the form $\tau\sigma$ with $\tau \in R$ and $\sigma \in H_a$. Further R^{-1} is a set of right coset representatives of H_a in H, so each vertex of G has a unique representation in the form $a\tau^{-1}$ with $\tau \in R$. Hence the following mapping ψ of the vertex set H of $[H,S]$ onto the vertex set $H_a \times V$ of $C \times G$ is well defined and one-one.

(3) $\psi: \tau\sigma \longmapsto (\sigma, a\tau^{-1})$, $(\tau \in R, \sigma \in H_a)$.

Call the edge sets of $[H,S]$ and $C \times G$ E_1 and E_2 respectively. Then for any $\tau_1, \tau_2 \in R$ and $\sigma_1, \sigma_2 \in H_a$ we have $(\tau_1\sigma_1, \tau_2\sigma_2) \in E_1 \Leftrightarrow (\tau_1\sigma_1)^{-1}\tau_2\sigma_2 \in S \Leftrightarrow (a\sigma_1^{-1}\tau_1^{-1}\tau_2\sigma_2, a) \in E \Leftrightarrow (a\sigma_1^{-1}\tau_1^{-1}, a\sigma_2^{-1}\tau_2^{-1}) \in E$. But σ_1^{-1} and σ_2^{-1} are in H_a, so $a\sigma_1^{-1} = a\sigma_2^{-1} = a$. Hence $(a\sigma_1^{-1}\tau_1^{-1}, a\sigma_2^{-1}\tau_2^{-1}) \in E \Leftrightarrow (a\tau_1^{-1}, a\tau_2^{-1}) \in E$. But C is complete, so $(a\tau_1^{-1}, a\tau_2^{-1}) \in E \Leftrightarrow ((\sigma_1, a\tau_1^{-1}), (\sigma_2, a\tau_2^{-1})) \in E_2$. We have thus shown that $(\tau_1\sigma_1, \tau_2\sigma_2) \in E_1 \Leftrightarrow ((\tau_1\sigma_1)\psi, (\tau_2\sigma_2)\psi) \in E_2$, so since $[H,S]$ and $C \times G$ are both simple graphs, ψ is an isomorphism of $[H,S]$ onto $C \times G$.

Q.E.D.

Theorem 3.3. (i). The graph $G = (V,E)$ is a group graph if and only if its components are mutually isomorphic group graphs.

(ii). The graph $G = (V,E,\rho)$ is symmetric if and only if its components are mutually isomorphic symmetric graphs.

Proof. "If". Let G have n components, all isomorphic to G_1. Then G is isomorphic to $G_1 \times T$, where T is the n-vertex trivial graph with loops. Since T is certainly a group graph, the "if" follows in both cases from theorem 3.1.

"Only if". (i). If G is a group graph isomorphic to the Cayley graph $[H,S]$, then its components are all isomorphic to $[K,S]$ where K is the subgroup of H generated by S.

(ii). We first remark that an automorphism of a graph maps paths into paths, so it preserves the relation of connectedness and hence just permutes the components of the graph among themselves.

If G_1 and G_2 are any two components of the symmetric graph G, we choose vertices a and b of G_1 and G_2 respectively. Since $\Gamma(G)$ is transitive there is a $\pi \in \Gamma(G)$ with $a\pi = b$. This π must map G_1 isomorphically onto G_2, so the components of G are mutually isomorphic. In the case $G_1 = G_2$, π induces an automorphism of G_1 which maps a onto b, so since a and b can be chosen arbitrarily in G_1, G_1 is symmetric. Q.E.D.

CHAPTER III : Generalized Cayley Graphs

§4. Introduction

Given a group H and a subset S of H one can define generalizations of the Cayley graph [H,S] by taking as vertices the left or right cosets of some subgroup U in H instead of the elements of H.

The first of these two generalizations gives arbitrary simple symmetric graphs, as we show in §5. It has apparently not been studied before.

The second generalization gives arbitrary connected regular graphs of countable degree with a weak restriction on multiple edges, as was shown by Reidemeister [15]. We state his results in §6. Reidemeister only considered undirected graphs of finite degree, but the relevant parts of his work carry over with no change to the more general case stated here.

§5. Symmetric Generalized Cayley Graphs

Let H be a group, U a subgroup, and S a subset of H. The graph [H,U,S] is defined to be the simple graph (V',E') with

(1) $$V' = \{xU : x \in H\} ,$$

(2) $$E' = \{(xU,xsU) : x \in H, s \in S\}.$$

If U is the trivial subgroup then [H,U,S] is just the Cayley graph [H,S].

The following is a generalization of theorem 2.1.

<u>Theorem 5.1.</u> The graph $[H,U,S]$ contains in its automorphism group the transitive representation of H as a permutation group on the left cosets of U, and is hence symmetric.

Conversely if G is a simple symmetric graph then G is isomorphic to a graph of the form $[H,U,S]$. H can be chosen as any transitive subgroup of $\Gamma(G)$, and U as a stabilizer subgroup of H.

<u>Proof.</u> Let $[H,U,S]$ have vertex and edge sets V' and E' as defined in (1) and (2). The element of the representation of H as a permutation group on the left cosets of U which corresponds to the element $h \in H$ is the permutation which maps the coset xU onto the coset $h^{-1}xU$ for each $x \in H$. This clearly maps E' onto itself, so the first part of the theorem is proved.

Now let $G = (V,E)$ be any simple symmetric graph. Let H be any transitive subgroup of $\Gamma(G)$, a any vertex of G, and U the stabilizer subgroup H_a of a in H. Define

(3) $$S = \{\pi \in H : (a\pi, a) \in E\}.$$

Denote the graph $[H,U,S]$ by G', with vertex and edge sets V' and E'. Let φ be the mapping of V' onto V defined by

(4) $$\varphi : \pi U \longmapsto a\pi^{-1} \quad , \quad (\pi \in H).$$

φ is well defined and one-one since $U = H_a$, so $\pi_1 U = \pi_2 U \iff \pi_1^{-1}\pi_2 \in U \iff a\pi_1^{-1} = a\pi_2^{-1}$. It is defined on the whole of V' and is "onto" since H is transitive. We show it is an isomorphism of G' onto G.

If $(\pi_1 U, \pi_2 U)$ is an arbitrary element of $V' \times V'$, then $(\pi_1 U, \pi_2 U) \in E' \Longleftrightarrow \pi_2 U = \pi_1 \lambda \pi U$ for some $\pi \in S$ and $\lambda \in U \Longleftrightarrow \pi_2 \delta = \pi_1 \lambda \pi$ for some $\pi \in S$ and $\lambda, \delta \in U \Longleftrightarrow \lambda^{-1} \pi_1^{-1} \pi_2 \delta \in S$ for some $\lambda, \delta \in U \Longleftrightarrow (a\lambda^{-1}\pi_1^{-1}\pi_2\delta, a) \in E \Longleftrightarrow (a\lambda^{-1}\pi_1^{-1}, a\delta^{-1}\pi_2^{-1}) \in E$. But $\lambda^{-1}, \delta^{-1} \in U = H_a$, so $a\lambda^{-1} = a\delta^{-1} = a$, so $(a\lambda^{-1}\pi_1^{-1}, a\delta^{-1}\pi_2^{-1}) = (a\pi_1^{-1}, a\pi_2^{-1})$ which is just the image of $(\pi_1 U, \pi_2 U)$ under φ. Since G and G′ are both simple graphs, φ is an isomorphism. Q.E.D.

We note that an edge of [H,U,S] generally has many representations in the form (xU, xsU) with $x \in H$ and $s \in S$. We may choose a single representation of each edge by the following lemma.

<u>Lemma 5.2.</u> Let H be a group, U a subgroup, and S a subset of H. If T is a set of representatives of the left cosets of U that occur in USU, then

(i). [H,U,S] = [H,U,T] ;

(ii). If R is any set of left coset representatives of U in H, then each edge of [H,U,T] has a unique representation in the form (rU, rtU) with $r \in R$ and $t \in T$.

<u>Proof.</u> We first note three properties of T:

(5) If s and t are distinct elements of T then $sU \neq tU$;

(6) $TU = USU$;

(7) $UT \subseteq TU$.

(5) and (6) are just a restatement of the definition of T. (7) holds since $T \subseteq USU$, so $UT \subseteq UUSU = USU = TU$.

(i). Since $T \subseteq USU$, any $t \in T$ is expressible as $t = usv$ with $u, v \in U$

and $s \in S$. Hence any edge (xU,xtU) of $[H,U,T]$ is expressible as $(xU,xtU) = (xU,xusvU) = (xuU,xusU)$, and is hence an edge of $[H,U,S]$. Conversely any $s \in S$ is certainly in $USU = TU$, so it is expressible as $s=tu$ with $t \in T$ and $u \in U$. Thus any edge (xU,xsU) of $[H,U,S]$ is expressible as $(xU,xsU) = (xU,xtuU) = (xU,xtU)$, and is hence an edge of $[H,U,T]$. Hence (i) is proved.

(ii): Let R be any set of left coset representatives of U in H. Let (xU,xtU) with $x \in H$ and $t \in T$ be any edge of $[H,U,T]$. $RU = H$, so $x=ru$ for some $r \in R$ and $u \in U$. By (7) $ut=t'v$ for some $t' \in T$ and $v \in U$. Hence $(xU,xtU) = (ruU,rutU) = (rU,rt'vU) = (rU,rt'U)$. It remains only to show that this representation is unique.

Indeed if $(rU,rtU) = (r'U,r't'U)$ with $r,r' \in R$ and $t,t' \in T$, then from $rU=r'U$ follows $r=r'$, since R is a set of left coset representatives of U in H. Hence $tU=t'U$, so $t=t'$ by (5). Q.E.D.

An immediate corollary of the preceding lemma is:

<u>Corollary 5.3.</u> If H is a group, U a subgroup, and S and S′ subsets of H, then $[H,U,S] = [H,U,S']$ if $USU = US'U$.

We remark without proof that the converse also holds.

If G is a graph and n any cardinal number, we denote by nG the edge direct sum of n copies of G.

<u>Theorem 5.4.</u> If G is a simple symmetric graph and H is a transitive subgroup of $\Gamma(G)$ such that the stabilizer subgroups of H have order n, then nG is a homomorphic image of $[H,T]$ for a suitable subset T of H.

Proof. By theorem 5.1 and lemma 5.2 we may assume $G = [H,U,T]$, where T is as in lemma 5.2 (ii).

Let Ψ be the mapping of $[H,T]$ onto G defined by

(8) $$\Psi: h \longmapsto hU \quad , \quad (h \in H).$$

Let R be any set of left coset representatives of U in H and u be any element of U. Then Ru is also a set of left coset representatives of U in H. Define $E_u = \{(x,xt): x \in Ru, t \in T\}$. By lemma 5.2 (ii), E_u is mapped one-one onto the edge set of G by Ψ. But the n sets Ru, $u \in U$, partition H, so the n sets E_u, $u \in U$, partition the edge set of $[H,T]$. Hence Ψ maps the edge set of $[H,T]$ n-fold onto the edge set of $[H,U,T]$, so it is a homomorphism of $[H,T]$ onto $n[H,U,T] = nG$. Q.E.D.

§6. Regular Generalized Cayley Graphs

If V is a set, a permutation graph on V is a regular graph of degree 1 with vertex set V. If π is any permutation of V, we define a corresponding permutation graph $P_\pi = (V,E_\pi)$ on V by defining

(1) $$E_\pi = \{(a,a\pi): a \in V\}.$$

This defines a one-one correspondence between the permutations of V and the permutation graphs on V.

A permutation subgraph of a graph G is a subgraph of G which is a permutation graph on the full vertex set of G.

We now define the second type of generalized Cayley graph.

Let H be a group, U a subgroup, and S a subset of H. We choose a set R of right coset representatives of U in H and define the graph $\langle H,U,S\rangle$ to be the graph with vertex and edge sets

(2) $$V' = \{Ur : r \in R\} ;$$

(3) $$E' = \{(Ur,Urs) : r \in R, s \in S\}.$$

If m is the number of representations an edge has in the form (Ur,Urs) with $r \in R$ and $s \in S$, we give this edge multiplicity m.

Clearly $\langle H,U,S\rangle$ is independant of the chosen set R of right coset representatives, and it is regular of degree $|S|$. If U is the trivial subgroup then $\langle H,U,S\rangle = [H,S]$.

Translating the results of Reidemeister [15] ch. 4 §17 to the language used here gives:

Theorem 6.1. Let G be a connected regular graph expressible as edge direct sum of a set $P_\pi, \pi \in S$, of distinct permutation subgraphs. Then the permutation group H generated by S is transitive, and G is isomorphic to $\langle H,U,S\rangle$ where U is any stabilizer subgroup of H.

Analogously to theorem 5.4, or alternatively as a corollary of the discussion in [15] ch. 4 §19, one obtains:

Theorem 6.2. Under the conditions of theorem 6.1, nG is a homomorphic image of [H,S], where n is the order of any stabilizer subgroup H_a of H.

The following lemma shows that the conditions of theorems

6.1 and 6.2 are by no means very restrictive.

Lemma 6.3 (O. Ore [11] p.160). A connected regular graph of at most countable degree is expressible as the edge direct sum of permutation subgraphs.

These permutation subgraphs need not be distinct. The condition of theorems 6.1 and 6.2 that the P_π be distinct is a (rather weak) restriction on the multiple edges of G.

In Reidemeister's language theorem 6.2 states that under the given conditions G has an n-fold covering by [H,S]. A "covering" is basicly a homomorphism in our sense, with the added condition that if the homomorphism maps the vertex a of the one graph onto the vertex b of the other, then every edge at b should be the image of some edge at a. This condition is clearly satisfied by the homomorphism of theorem 5.4, so theorem 5.4 can also be interpreted as a theorem on coverings.

Theorem 6.4. The connected graph G is a group graph if and only if it is the edge direct sum of distinct permutation subgraphs P_π, $\pi \in S$, where S generates a regular permutation group. In fact G is isomorphic to [H,S], where H is the group generated by S.

Proof. The "if" is a direct corollary of theorem 6.1.

"Only if": Suppose G is a connected group graph. Without loss of generality G = [K,T] where K is a group and T a subset of K. Let H and S be the images of K and T under the natural isomorphism of

K onto its right regular representation as a permutation group on the elements of K. G is then the edge direct sum of the permutation graphs P_π, $\pi \in S$. Since G is connected, S generates a transitive group L by theorem 6.1. L is a subgroup of H; but H, as a regular group, has only itself as transitive subgroup, so S generates H. Q.E.D.

§7. Applications

We now apply the characterisations of group graphs given by theorems 2.1 and 6.4 to the problem of finding usable sufficient conditions for a graph to be a group graph.

Theorem 7.1. If G is a simple graph satisfying one of the following conditions, then it is a group graph.

(i). G is symmetric and has a prime number of vertices.

(ii). G is symmetric and regular of degree 1.

(iii). G is symmetric and regular of degree 2 and has both directed and undirected edges.

(iv). G is symmetric or connected and is expressible as the edge direct sum of permutation subgraphs P_π, $\pi \in S$, such that any two elements of S commute.

Proof. We may assume in each case that G is connected. For if G is not connected, it suffices to consider the components of G by theorem 3.3.

(i): Let G be simple and symmetric with p vertices, where p is a prime number. $\Gamma(G)$ is transitive of degree p, so its order is divisible by p, so it has an element of order p by Cauchy's theorem. This element must be a p-cycle, so it generates a regular cyclic subgroup of $\Gamma(G)$, so G is a group graph by theorem 2.1.

(ii): If G is connected and regular of degree 1, then it is a permutation graph P_π corresponding to a cyclic permutation π. π generates a regular group, so G is a group graph by theorem 6.4.

(iii): Let G be connected, symmetric, and regular of degree 2, and have both directed and undirected edges. Since $\Gamma(G)$ is transitive, it suffices to show that any automorphism which fixes a vertex of G is trivial, for then $\Gamma(G)$ is regular, so theorem 2.1 gives the desired conclusion. At each vertex G has one undirected edge, one incoming directed edge, and one outgoing directed edge; for otherwise G has only directed or only undirected edges at some vertex, and by symmetry this would hold at every vertex, contradicting the assumption that G have both directed and undirected edges. If π is an automorphism which leaves the vertex a fixed, it must permute the edges at a. But the edges at a are all of different "types", so π leaves them fixed, and hence leaves any vertex adjacent to a fixed. Since G is connected, repeating the argument shows that π leaves all vertices fixed and is thus trivial.

(iv): Let G be a connected graph expressible as the edge direct sum of the permutation subgraphs $P_\pi, \pi \in S$, where any two elements of S commute. The group generated by S is transitive by theorem

6.1. But it is abelian, and an abelian transitive group is regular. Hence G is a group graph by theorem 6.4. Q.E.D.

We remark that one can say rather more in cases (i), (ii), and (iv) of the above theorem. In fact we showed in the proof that G is a group graph of a cyclic group in case (i). One can easily show that the same holds in case (ii), if G has a finite number of components. In general one can only say that G is a group graph of an abelian group in cases (ii) and (iv). This needs a simple extension of part (i) of theorem 3.3 or can alternatively be proved directly with no difficulty.

CHAPTER IV : Regular Graphs of Degree 2; The Alternating Path Method

§8. Introduction

In this chapter we shall discuss the "alternating path method", applying it in §9 to the investigation of permutation and related subgraphs of a finite regular graph of degree 2 and in §10 to the consideration of the automorphism groups of finite regular symmetric graphs of degree 2. In §11 we apply the results of §9 and §10 to the construction of an infinite set of symmetric graphs which are not group graphs.

The alternating path method was introduced by Petersen [12], and has since become a standard tool in the investigation of subgraphs of bipartite (c.f. [11] p.106) and directed graphs. The results of §9 are standard, though not in the precise form given here. They are for instance contained in essense in O. Ore's discussion of the matching theorems ([11] ch.7) if one translates the language of bipartite graphs to that of directed graphs (c.f. [11] p.159). The application of the alternating path method to automorphism groups appears to be new.

Much of the following can be generalized to infinite graphs, however for simplicity in presentation we do not do so, and only indicate the generalizations where they are of interest.

§9. Alternating Paths

Throughout this section we assume that the multiple edges of a graph $G = (V,E,\rho)$ are distinguishable, and distinguish them by subscripts $(a,b)_1,(a,b)_2, \ldots ,(a,b)_m$, where m is the multiplicity of (a,b). Hence when we say that two edges (a,b) and (c,d) are distinct, we mean that either $a \neq c$, or $b \neq d$, or they are a pair of the form $(a,b)_i$, $(a,b)_j$ with $i \neq j$. The edges of the converse graph G* are distinguished correspondingly and are furthermore assumed to be distinct from the edges of G.

The alternate composition graph $\overline{G}$ of G is the graph with vertex set V, and an edge (a,b) for each pair of distinct edges (a,c) and (b,c) of G. $\overline{G^*}$ is the alternate composition graph of the converse graph G* of G. This definition differs from that of O. Ore ([11] p.158) in that he does not require that (a,c) and (b,c) be distinct, so for each $a \in V$ $\rho(a)$ loops are added to $\overline{G}$ at the vertex a.

Lemma 9.1. (i). $\overline{G}$ and $\overline{G^*}$ are undirected.

(ii). $\Gamma(G) \subseteq \Gamma(\overline{G})$; $\Gamma(G) \subseteq \Gamma(\overline{G^*})$.

(iii). If G is regular of degree n then $\overline{G}$ and $\overline{G^*}$ are regular of degree n(n-1).

Proof. Since $\Gamma(G) = \Gamma(G^*)$ and G* is regular of degree n if G is, it suffices to prove the statements for $\overline{G}$ only.

(i) is trivial from the definition.

(ii). An automorphism π of G maps pairs of edges of the form (a,c), (b,c) into similar pairs, so it maps edges of $\overline{G}$ into edges of $\overline{G}$. The same holds for π^{-1} since π^{-1} is also an automorphism of G. Hence π is an automorphism of $\overline{G}$, so $\Gamma(G) \subseteq \Gamma(\overline{G})$.

(iii). If a is any vertex of the regular graph G of degree n, then to each of the n edges of G of the form (a,c) there are n-1 edges distinct from it of the form (b,c). a hence has n(n-1) out-going edges in $\overline{G}$, so since $\overline{G}$ is undirected, a also has n(n-1) in-coming edges in $\overline{G}$. But a was an arbitrary vertex, so $\overline{G}$ is regular of degree n(n-1).

Q.E.D.

We now assume that G is a finite regular graph of degree 2. Since we are distinguishing edges, G is characterised by its vertex and edge sets V and E alone. $\overline{G}$ and $\overline{G^*}$ are undirected and regular of degree 2, so they each consist of disjoint unions of undirected cyclic graphs.

Let

(1) $$A = (a_0,a_1)(a_1,a_2) \ldots (a_{r-1},a_r)$$

be a path in the undirected graph G_u, whose edges belong alternately to G and G*. The edges $(a_0,a_1),(a_2,a_1),(a_2,a_3),(a_4,a_3), \ldots$ then all belong to G or all belong to G*. We make the requirement that they be distinct, and call A an alternating path of G. If the above edges all belong to G we call A an α-path and denote the above set of edges by E(A). Otherwise we call A an α^*-path and denote the set of edges $(a_1,a_0),(a_1,a_2),(a_3,a_2),(a_3,a_4), \ldots$ of G by E(A). E(A) is called the edge set of A. The set of initial (terminal) vertices of edges in $E(A)$ is called the initial (terminal) vertex set of A, denoted by $V(A)$ ($V^*(A)$).

We call the alternating path of (1) cyclic if its legnth r is even and $a_0 = a_r$. We then call the corresponding subgraph $(V(A) \cup V^*(A), E(A))$ of G an alternating circuit of G, denoting it often by the same letter A.

An alternating circuit has several representations by alternating paths; for instance if

(2) $$C = (a_0,a_1)(a_1,a_2) \dots (a_{2s-1},a_0)$$

is a cyclic α-path, then

(3) $$C' = (a_1,a_0)(a_0,a_{2s-1})(a_{2s-1},a_{2s-2}) \dots (a_2,a_1)$$

is a cyclic α^*-path representing the same alternating circuit of G.

Let G have precisely n alternating cicuits

(4) $$A_1, A_2, \dots, A_n,$$

and define

(5) $$V_i = V(A_i) \; ; \; V_i^* = V^*(A_i) \; ; \; E_i = E(A_i) \; ; \; (i=1,2, \dots ,n).$$

Lemma 9.2. $\overline{G}$ and $\overline{G^*}$ each have precisely n components and these may be so indexed that the component $\overline{G}_i$ of $\overline{G}$ has vertex set V_i and the component $\overline{G^*}_i$ of $\overline{G^*}$ has vertex set V_i^* for each $i=1,2, \dots n$.

Proof. Let $\overline{G}_1$ be any component of $\overline{G}$. $\overline{G}_1$ is an undirected cycle, so we may write its edges in a sequence

(6) $$[a_0,a_2],[a_2,a_4], \dots ,[a_{2s-2},a_0],$$

where the square bracket is used to denote undirected (that is pairs of oppositely oriented) edges. By definition of $\overline{G}$ we can find a vertex a_{2i-1} for each i=1,2, ... ,s such that (a_{2i-2},a_{2i-1}) and (a_{2i},a_{2i-1}) are edges of G (indices modulo 2s). $(a_0,a_1)(a_1,a_2)$ ... (a_{2s-1},a_0) is then a cyclic α-path of G which defines a unique alternating circuit whose initial vertex set is the vertex set $\{a_0,a_2, \dots ,a_{2s-2}\}$ of $\overline{G}_1$. The uniqueness is clear since G has only 2 outgoing edges at each vertex.

Conversely if an alternating circuit is given we may represent it by the α-path of (2) say. It is then clearly obtained by the above argument from the component of $\overline{G}$ whose edges are as in (6).

The above argument hence defines a one-one correspondence with the desired property between the sets V_i and the components of $\overline{G}$. The statement for $\overline{G^*}$ follows similarly. Q.E.D.

<u>Lemma 9.3.</u> (i). The V_i $(1 \leq i \leq n)$ partition V.
(ii). The V_i^* $(1 \leq i \leq n)$ partition V.
(iii). The E_i $(1 \leq i \leq n)$ partition E.

<u>Proof.</u> (i) and (ii) are consequences of lemma 9.2, since the vertex sets of the components of a graph partition the vertex set of the graph.

(iii). Each edge (a,b) of G occurs in one of the E_i, since $a \in V_i$ for some i and both outgoing edges of G at a are in the corresponding E_i. (a,b) cannot be in both E_i and E_j $(i \neq j)$ as this would imply that a is in both of the disjoint sets V_i and V_j. Q.E.D.

Now let the alternating circuit A_i be represented by the cyclic α-path of (2). We split the set $E_i = \{(a_0,a_1),(a_2,a_1),(a_2,a_3),(a_4,a_3), \dots ,(a_0,a_{2s-1})\}$ into two disjoint sets

$$E_i^1 = \{(a_0,a_1),(a_2,a_3), \dots ,(a_{2s-2},a_{2s-1})\} \tag{7}$$
$$E_i^2 = \{(a_2,a_1),(a_4,a_3), \dots ,(a_0,a_{2s-1})\}$$

which are uniquely defined up to order, but may be exchanged by taking a different path representing A_i. The E_i^j $(1 \leq i \leq n, j=1,2)$ partition E since the E_i do. Furthermore for each i the initial and terminal vertices of the edges in E_i^1 run once through V_i and V_i^* respectively, and the same holds for E_i^2.

<u>Theorem 9.4.</u> If $G = (V,E,\rho)$ is a finite regular graph of degree 2, then the subgraph $P = (V,E_P)$ is a permutation subgraph of G if and only if for each i=1,2, ... ,n E_P contains one of the sets E_i^1 and E_i^2 and is disjoint from the other.

<u>Proof.</u> Let E_P have the given property. If a is any vertex of G then $a \in V_i$ for some i with $1 \leq i \leq n$. Each of E_i^1 and E_i^2 contains precisely one outgoing edge at a, so since E_P contains one of these sets and is disjoint from the other, E_P contains precisely one outgoing edge at a. Since a was arbitrary, P is out-regular of degree 1. Similarly P is in-regular of degree 1, so it is a permutation subgraph of G.

Conversely let P be a permutation subgraph of G and let (a_0,a_1) be any edge of P. Let the alternating circuit A_i whose edge set contains (a_0,a_1) be represented by the α-path of (2). Then $(a_2,a_1) \notin E_P$ as $(a_0,a_1) \in E_P$ and E_P contains only one incoming edge at a_1. Hence $(a_2,a_3) \in E_P$ as E_P must contain an outgoing edge at a_2. Hence $(a_4,a_3) \notin E_P$ as E_P contains only one incoming edge at a_3. Continuing the argument shows that E_P contains one of the sets E_i^1 and E_i^2 and is disjoint from the other. Since P has an edge from every vertex of G, and hence certainly from every V_i, this must hold for every $i=1,2, \ldots ,n$, so E_P has the stated form.

Q.E.D.

Now let $B = (V,E_B,\rho_B)$ be a subgraph of G with the property: There exist distinct vertices a and b of G, called respectively the <u>initial</u> and <u>terminal</u> vertex of B, such that

(8)
$$\rho_B(x) = 1 \text{ for all } x \in V-\{b\}, \rho_B(b) = 0 ;$$
$$\rho_B^*(x) = 1 \text{ for all } x \in V-\{a\}, \rho_B^*(a) = 0 .$$

We call such a subgraph a <u>broken permutation subgraph of G</u>. It is the disjoint union of a (possibly empty) set of directed cycles together with one directed arc. This arc has initial and terminal vertices a and b respectively.

<u>Theorem 9.5.</u> If $G = (V,E,\rho)$ is a finite regular graph of degree 2, then the subgraph $B = (V,E_B)$ is a broken permutation subgraph with initial and terminal vertices a and b if and only if the following conditions are satisfied:

(i). There is an alternating circuit A of G representable by a cyclic α^*-path

(9) $$(a_0,a_1)(a_1,a_2)\ \ldots\ (a_{2t-2},a_{2t-1})(a_{2t-1},a_{2t})\ \ldots\ \ldots\ (a_{2s-2},a_{2s-1})(a_{2s-1},a_0)$$

with $a = a_0 \neq a_{2t-1} = b$ for some $1 \leq t \leq s$.

(ii). The edges of E_B that are in $E(A)$ are precisely

(10) $$\begin{aligned} &(a_{2j-1},a_{2j}) \quad \text{for } 1 \leq j \leq t-1; \\ &(a_{2j-1},a_{2j-2}) \text{ for } t+1 \leq j \leq s. \end{aligned}$$

(iii). For any alternating circuit $A_i \neq A$, E_B contains one of the sets E_i^1 and E_i^2 and is disjoint from the other.

Proof. One readily verifies that if E_B satisfies (i), (ii), and (iii), then B is a broken permutation subgraph of G with initial vertex a and terminal vertex b.

If B is a broken permutation subgraph of G with initial vertex a and terminal vertex b we choose j such that $a \in V_j^*$ and put $A = A_j$. We may then represent A by an α^*-path as in (9) with $a = a_0$. Suppose $b \neq a_{2t-1}$ for each $t=1,2, \ldots ,s$. Since B has no incoming edges at $a = a_0$, $(a_1,a_0) \notin E_B$. Since B must have an outgoing edge at a_1, $(a_1,a_2) \in E_B$. Since B has only one incoming edge at a_2, $(a_3,a_2) \notin E_B$. Continuing the argument yields finally that $(a_{2s-1},a_0) \in E_B$; but this is a contradiction as B has no incoming edges at $a = a_0$, so our supposition was false and $b = a_{2t-1}$ for some t with $1 \leq t \leq s$.

The same method of considering edges sequentially around the alternating circuits of G now yields (ii) and (iii). Q.E.D.

§10. Finite Regular Symmetric Graphs of Degree 2

Throughout this section let $G = (V,E,\rho)$ be a finite symmetric graph of degree 2 with m vertices and n alternating circuits.

We index the alternating circuits, their initial and terminal vertex sets, and the components of $\overline{G}$ and $\overline{G^*}$ as in lemma 9.2 and in the comments immediately preceding that lemma.

Any automorphism π of G certainly maps alternating circuits onto alternating circuits, so π permutes the A_i $(1 \leq i \leq n)$ among themselves. Hence π permutes the V_i among themselves and the V_i^* among themselves in such a way that it maps V_i^* onto V_j^* whenever it maps V_i onto V_j. Hence π permutes the nonempty sets of the form $V_i \cap V_i^*$ $(1 \leq i \leq n)$ among themselves and π permutes the nonempty sets of the form $V_i \cap V_j^*$ $(1 \leq i,j \leq n,\ i \neq j)$ among themselves.

<u>Lemma 10.1.</u> (i). $\Gamma(G)$ acts transitively on the V_i and on the V_i^*. The V_i and V_i^* are all of equal size.
(ii). $\Gamma(G)$ acts transitively on the nonempty sets of the form $V_i \cap V_j^*$ $(1 \leq i,j \leq n,\ i \neq j)$. In particular they are all of equal size.
(iii). Either $V_i = V_i^*$ for each i or $V_i \cap V_i^* = \emptyset$ for each i $(1 \leq i \leq n)$.
<u>Proof.</u> (i): For any V_s and V_t choose $a \in V_s$ and $b \in V_t$. By symmetry of G there is a $\pi \in \Gamma(G)$ with $a\pi = b$. $V_s\pi$ is one of the sets V_i, and is not disjoint from V_t since $b = a\pi \in V_s\pi$. Hence $V_s\pi = V_t$, so since V_s and V_t were arbitrary, $\Gamma(G)$ acts transitively on the V_i. Similarly $\Gamma(G)$ acts transitively on the V_i^*.

In particular the n sets V_i are all of equal size. But they partition V which has m elements, so $|V_i| = m/n$ for each i. Similarly $|V_i^*| = m/n$ for each i, so $|V_i| = |V_j^*|$ for all i and j with $1 \leqslant i,j \leqslant n$.

(ii) and (iii): Let s,t,k,l be arbitrary with $1 \leqslant s,t,k,l \leqslant n$ and $V_s \cap V_t^* \neq \emptyset$ and $V_k \cap V_l^* \neq \emptyset$. Choose $a \in V_s \cap V_t^*$ and $b \in V_k \cap V_l^*$. If π is an automorphism with $a\pi = b$, then by the argument of (i), $V_s\pi = V_k$ and $V_t^*\pi = V_l^*$, whence $(V_s \cap V_t^*)\pi = (V_k \cap V_l^*)$. Hence $\Gamma(G)$ acts transitively on the nonempty sets of the form $V_i \cap V_j^*$ $(1 \leqslant i,j \leqslant n)$. But any automorphism permutes the $V_i \cap V_i^*$ among themselves and the $V_i \cap V_j^*$ $(i \neq j)$ among themselves. Hence either all the $V_i \cap V_i^*$ are empty, or all the $V_i \cap V_j^*$ $(i \neq j)$ are empty. The latter implies that $V_i = V_i \cap V_i^*$, whence $V_i = V_i^*$ for each i. Q.E.D.

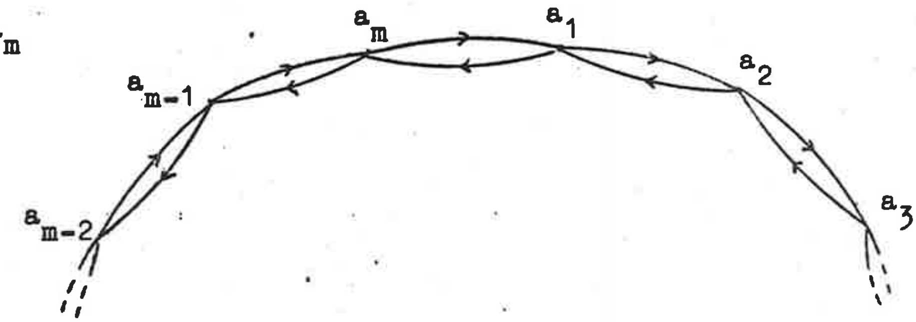
<u>Lemma 10.2.</u> If $1 \leqslant i \leqslant n$ and $|V_i| \neq 2$ then the action of an automorphism $\pi \in \Gamma(G)$ on the set V_i^* is determined by its action on V_i.

<u>Proof.</u> Note that the definition of V_i and V_i^* from the alternating circuit A_i implies that V_i^* is just the set of terminal vertices of edges whose initial vertices are in V_i.

Suppose $|V_i| \neq 2$ and suppose the action of π on V_i is known. If $|V_i| = 1$ then $|V_i^*| = 1$ and the lemma is trivial. If $|V_i| \geqslant 3$ then to any $c \in V_i^*$ there is a pair a,b of vertices in V_i such c is the unique vertex of V_i^* for which (a,c) and (b,c) are in E. But $a\pi$ and $b\pi$ are known, and $c\pi$ is the unique vertex with $(a\pi, c\pi)$ and $(b\pi, c\pi)$ in E, so the action of π on c is determined. Q.E.D.

We shall need the following simple lemma:

Lemma 10.3. The undirected cyclic graph with m vertices $a_1, a_2, \ldots, a_m$



has transitive automorphism group Γ generated by the permutations $\alpha = (a_1\ a_2\ \ldots\ a_m)$ and $\beta = (a_1\ a_{2k})(a_2\ a_{2k-1}) \ldots (a_k\ a_{k+1})$ where $k = [m/2]$.

If m is odd, the only regular subgroup of Γ is the cyclic subgroup generated by α; if m is even there is also the regular dihedral subgroup generated by α^2 and β.

Γ is primitive if and only if m is prime.

Proof. The statements of the lemma are all easily verified. That m prime implies Γ primitive is given by theorem 8.3 of Wielandt [17]. If m is not prime, then the set $\{a_p, a_{2p}, \ldots, a_{qp}\}$ is a nontrivial block of Γ for any nontrivial factorisation $m = pq$ of m. Q.E.D.

The final statement of lemma 10.3 holds for much more general graphs.

Theorem 10.4. If the finite symmetric graph of degree 2 has m vertices and if $\Gamma(G)$ is not doubly transitive, then $\Gamma(G)$ is primitive if and only if m is prime.

Remarks. The condition that $\Gamma(G)$ be not doubly transitive only eliminates trivial cases (c.f. theorem 2.2). If G is infinite symmetric with local degrees not exceeding 2 then a similar proof gives that $\Gamma(G)$ is always imprimitive.

Proof of 10.4. If m is prime then $\Gamma(G)$ as a transitive group of prime degree is primitive ([17] thm.8.3).

Conversely suppose $\Gamma(G)$ is primitive. Then $\Gamma(\overline{G})$ is primitive as it contains $\Gamma(G)$ by lemma 9.1. If the V_i each had k elements with $1 < k < m$, then they would be nontrivial blocks for $\Gamma(\overline{G})$. Hence either $|V_i| = 1$ for each i, or $\overline{G}$ is connected.

If $|V_i| = 1$ for each i then G has only double edges. G is not disconnected, for if it were, the vertex sets of its components would be nontrivial blocks of $\Gamma(G)$. Hence G is the edge direct sum of two copies of a directed cyclic graph, so its automorphism group is cyclic generated by $(a_1\ a_2\ \dots\ a_m)$ say. m is prime since otherwise $\{a_p, a_{2p}, \dots, a_{qp}\}$ would be a block of $\Gamma(G)$ for any nontrivial factorisation $m = pq$ of m.

If $\overline{G}$ is connected, then it is an undirected cycle of legnth m, so by lemma 10.3 the primitivity of $\Gamma(\overline{G})$ implies again that m is prime.

Q.E.D.

In the remainder of this section we discuss conditions for a finite symmetric graph of degree 2 to be a group graph.

Theorem 10.5. Let G be a finite simple symmetric graph of degree 2 with m vertices, m an odd number. Let each of the n components of $\overline{G}$ have precisely k vertices. If no nontrivial factor of k is less than n then G is a group graph. If G is furthermore connected then $\Gamma(G)$ is regular or contains a regular subgroup of index 2.

We use the following lemma:

<u>Lemma 10.6.</u> Let the finite simple symmetric graph G of degree 2 satisfy the conditions:

(i). G has an odd number m of vertices;

(ii). Each V_s^* is equal to some V_t $(1 \leq s,t \leq n)$.

Then G is a group graph. If G is connected, then $\Gamma(G)$ is regular or has a regular subgroup of index 2.

<u>Proof.</u> The lemma is trivial for m = 3. We use induction on m. If G is disconnected and satisfies (i) and (ii), then its components certainly satisfy (i) and (ii) and have a smaller number of vertices than G, so they are group graphs by induction hypothesis. G is then a group graph by theorem 3.3.

We may hence assume G to be connected. By lemma 1.2 any two vertices of G are path connected. We remark also that $|V_i|$ is a divisor of m and hence not 2 as m is odd, so lemma 10.2 is applicable.

By (ii) we may define a sequence of sets $V_{t_0}=V_1, V_{t_1}, V_{t_2}, \ldots$ with $V_{t_{i+1}} = V_{t_i}^*$ for each i=0,1,2, ... $V_{t_1} = V_1^*$ is the set of vertices of G which can be reached by an edge from V_1. $V_{t_2} = V_{t_1}^*$ is the set of vertices of G which can be reached by an edge from V_{t_1}, and hence by a directed path of legnth 2 from V_1. In general

V_{t_i} is the set of vertices that can be reached by a directed path of legnth i from V_1, so as any two vertices of G are path connected, every vertex of G occurs in some V_{t_i} $(i \geqslant 0)$.

If π is any automorphism of G whose action is given on V_1, then by repeated application of lemma 10.2 its action is determined on $V_{t_1} = V_1^*$, $V_{t_2} = V_{t_1}^*$, ... Hence its action is determined on the whole of V.

Let $a \in V_1$ and let π be any automorphism of G which leaves a fixed. Then π must map V_1 onto itself, so its restriction to V_1 gives an automorphism of $\overline{G}_1$. But the automorphism group of $\overline{G}_1$ is by lemma 10.3 transitive of order twice its degree, so its a-stabilizer subgroup has order 2. Since π is already uniquely defined by its action on V_1, there are at most two automorphisms of G which leave a fixed, so the a-stabilizer subgroup Γ_a of $\Gamma = \Gamma(G)$ has order 1 or 2.

If $|\Gamma_a| = 1$ then Γ, as a transitive group with trivial stabilizer subgroup, is regular, so G is a group graph by theorem 2.1.

If $|\Gamma_a| = 2$ then Γ has order 2m. Since m is odd, Γ has a normal subgroup H say of order m ([17] thm.4.6). The stabilizer subgroups of Γ have order 2, so any nontrivial automorphism of G which fixes a vertex must have order 2. Hence no nontrivial element of H fixes a vertex, for H has odd order. Hence H, as a group of order equal to its degree and with trivial stabilizer subgroups, is regular, so G is a group graph by theorem 2.1. Q.E.D.

Proof of theorem 10.5. Suppose G satisfies the conditions of theorem 10.5. It suffices to prove that G satisfies the conditions of lemma 10.6. Condition (i) is satisfied by assumption, so we need only verify (ii).

Let $1 \leq s \leq n$. We must find a t with $1 \leq t \leq n$ and $V_s^* = V_t$. If $V_s = V_s^*$ we have finished, so by lemma 10.1(iii) we may assume $V_s \cap V_s^* = \emptyset$.

Let p be the number of nonempty sets of the form $V_i \cap V_s^*$ $(1 \leq i \leq n)$. Since $V_s \cap V_s^* = \emptyset$, $p < n$, so by assumption p cannot be a nontrivial divisor of k. But p divides k, for the p nonempty sets of the form $V_i \cap V_s^*$ $(1 \leq i \leq n)$ partition V_s^* which has k elements, and they have equal size by lemma 10.1(ii). Hence $p = 1$, so there is precisely one t with $1 \leq t \leq n$ and $V_t \cap V_s^* \neq \emptyset$. It follows that $V_t \cap V_s^* = V_s^*$, so $V_t = V_s^*$ as they both have equal size. Q.E.D.

Corollary 10.7. If G is a simple symmetric graph of degree 2 with p^2 vertices where p is prime, then G is a group graph.

Proof. For $p = 2$ one verifies the statement by constructing the possibilities. Hence assume $p \geq 3$. Since the sets V_i $(1 \leq i \leq n)$ have equal size and partition V, n must be a divisor of p^2. If $n = p^2$ then $|V_i| = 1$ for each i, so G has only double edges, contradicting assumption. Hence $n = 1$ or p and the conditions of theorem 10.5 are satisfied. Q.E.D.

Theorem 10.8. If G is a finite symmetric graph of degree 2 with connected alternate composition graph $\overline{G}$, then G is a group graph of a cyclic group.

Remark. The finiteness condition is not necessary; a similar proof holds in the infinite case.

Proof of 10.8. Let G have m vertices. Since $\overline{G}$ is connected, it is an undirected cyclic graph with m vertices, so $\Gamma(\overline{G})$ is as in lemma 10.3. If $\Gamma(G) = \Gamma(\overline{G})$ then $\Gamma(G)$ contains a regular cyclic subgroup by lemma 10.3, so theorem 2.1 gives the desired result. We may hence assume $\Gamma(G) \subset \Gamma(\overline{G})$. The order of $\Gamma(G)$ is then a factor of $|\Gamma(\overline{G})| = 2m$, and it is a multiple of its degree m, so $|\Gamma(G)| = m$. Hence $\Gamma(G)$ is regular, as it is transitive of order equal to its degree. If $\Gamma(G)$ is cyclic we have finished. But by lemma 10.3 the only other possibility is that $\Gamma(G)$ is regular dihedral. This cannot occur, as then G would be a group graph of degree 2 of a dihedral group, and one verifies easily that $\overline{G}$ would then be disconnected with either 2 or m/2 components. Q.E.D.

The alternating path method has useful practical applications to the calculation of automorphism groups of regular graphs of degree 2 and to the construction of graphs with given properties.

For instance if one uses theorem 9.4 to calculate the permutation subgraphs of a given graph, then using the fact that automorphisms of the graph must permute the permutation subgraphs among themselves reduces the calculation of the automorphism group. We use this method in the next section.

Another application is in the search for finite symmetric graphs of degree 2 which are not group graphs. One can show for instance that the graph of example 2.2 is the only symmetric graph of degree 2 with less than 12 vertices which is not a group graph. However even with the methods of this chapter this involves a tedious consideration of numerous cases.

By a similar argument to the proof of theorem 10.5 one can show that theorem 10.5 still holds if m is even of the form 2p, where p is a prime number congruent to -1 modulo 4. As a final application of the methods of this chapter we construct examples which show that this is not so if $p \equiv +1$ modulo 4.

§11. A Set of Symmetric Graphs which are not Group Graphs

Let p be a prime number congruent to +1 modulo 4. If x is an integer we denote by $\underline{x}$ the unique number with $0 \leq \underline{x} \leq p-1$ and $\underline{x} \equiv x$ modulo p.

Let b be a primitive root of p and $a = \underline{b^{(p-1)/4}}$, that is a is a primitive 4th root of unity modulo p. then in particular

(1) $$\underline{a^2} = p-1.$$

Let G be the simple graph with vertex set $V = \{0,1, \dots ,2p-1\}$ and edge set $E = E_1^1 \cup E_1^2 \cup E_2^1 \cup E_2^2$, where

(2)
$$\begin{aligned} E_1^1 &= \{(i, p+\underline{i+1}) : 0 \leq i \leq p-1\}, \\ E_1^2 &= \{(i, p+\underline{i-1}) : 0 \leq i \leq p-1\}, \\ E_2^1 &= \{(p+i, \underline{i-a}) : 0 \leq i \leq p-1\}, \\ E_2^2 &= \{(p+i, \underline{i+a}) : 0 \leq i \leq p-1\}. \end{aligned}$$

Using (1), one readily verifies that the permutations

(3) $$\alpha = (012 \ldots p-1)(p\ p+1 \ldots 2p-1),$$

(4) $$\beta = (0p)(1\ p+a\ \underline{-1}\ p+\underline{-a}) \ldots (i\ p+\underline{ia}\ \underline{-i}\ p+\underline{-ia}) \ldots$$
$$\ldots ((p-1)/2\ p+\underline{(p-1)a/2}\ \underline{-(p-1)/2}\ p+\underline{-(p-1)a/2})$$

are automorphisms of G. They clearly generate a transitive group, so G is symmetric. We now show that G is not a group graph.

The subgraph $A_1 = (V, E_1^1 \cup E_1^2)$ is clearly composed of complete alternating circuits of G. Using the fact that the initial vertex sets of the alternating circuits of G must all have 2, p, or 2p elements one sees that A_1 is in fact itself an alternating circuit of G. Similarly $A_2 = (V, E_2^1 \cup E_2^2)$ is an alternating circuit of G, and since A_1 and A_2 together involve all the edges of G, they are the only alternating circuits of G. Hence the sets E_i^j $(i,j = 1,2)$ are the sets defined in (7) of §9, so by theorem 9.4 the permutation subgraphs of G are just $P_{ij} = (V, E_1^i \cup E_2^j)$ $(i,j = 1,2)$. We denote the permutation of V which corresponds to P_{ij} by π_{ij} (c.f. §6). Now $(0,p+1),(p+1,\underline{1-a}),(\underline{1-a},p+\underline{2-a}),(p+\underline{2-a},\underline{2-2a}), \ldots$ are the edges of P_{11} so

(5) $$\pi_{11} = (0\ p+1\ \underline{1-a}\ p+\underline{2-a}\ \underline{2-2a} \ldots p+\underline{i-(i-1)a}\ \underline{i-ia} \ldots$$
$$\ldots p+\underline{p-(p-1)a}).$$

Similarly

(6) $$\pi_{22} = (0\ p+\underline{-1}\ \underline{-1+a}\ p+\underline{-2+a}\ \underline{-2+2a} \ldots p+\underline{-p+(p-1)a}),$$

(7) $$\pi_{12} = (0\ p+1\ \underline{1+a}\ p+\underline{2+a}\ \underline{2+2a} \ldots p+\underline{p+(p-1)a}),$$

(8) $$\pi_{21} = (0\ p+\underline{-1}\ \underline{-1-a}\ p+\underline{-2-a}\ \underline{-2-2a} \ldots p+\underline{-p-(p-1)a}).$$

Hence the permutation subgraphs of G are all cyclic.

G can be expressed as the edge direct sum of permutation subgraphs in precisely two ways: $G = P_{11} + P_{22} = P_{12} + P_{21}$. Since π_{22} is not a power of π_{11} and π_{21} is not a power of π_{12}, neither π_{11} and π_{22} nor π_{12} and π_{21} generate a regular permutation group, so by theorem 6.4 G is not a group graph.

We now show that $\Gamma(G)$ has order 4p and is generated by α and β. Since the group generated by α and β contains α and β whose orders are p and 4, it has order at least 4p, so since it is contained in $\Gamma(G)$, it suffices to prove the first statement. To do this it suffices to show that a stabilizer subgroup of $\Gamma(G)$ has order 2.

Let π be an element of $\Gamma(G)$ which leaves 0 fixed. π permutes the permutation subgraphs of G, so in particular it maps P_{11} onto P_{11}, P_{12}, P_{21}, or P_{22}. If it maps P_{11} onto P_{11} then it is the trivial automorphism. If it maps P_{11} onto P_{12} then it maps $p+\underline{-1} = p+\underline{-a-(-a-1)a}$ onto $p+\underline{-a+(-a-1)a} = p+\underline{1-2a}$, so it maps the edge $(0,p+\underline{-1})$ onto $(0,p+\underline{1-2a})$ which is not an edge. This is hence impossible. Similarly π cannot map P_{11} onto P_{21}, but the permutation which maps P_{11} onto P_{22} does give an automorphism. Hence there are just 2 automorphisms of G which leave 0 fixed, which is what we wished to prove.

We remark that for $p = 5$ and $a = 2$, G is just the graph of example 2.2 and α and β are the two permutations given there.

One can show that if p is a prime number congruent to 1 modulo 4, and if G' is a symmetric graph of degree 2 with 2p vertices which is not a group graph, then if $|\Gamma(G')| = 4p$ or if $\overline{G'}$ does not have p components, G′ is isomorphic to the graph G constructed above. I do not know whether the last condition is necessary. The proof of this statement needs other tools to those we have developed here, so we ommit it.

CHAPTER V : Maximal Schreier Words in Finite Groups

§12. Introduction

If G is a directed graph, a Hamiltonian arc in G is a (possibly infinite) path in G, which starts from some vertex and passes precisely once through every other vertex of G. A Hamiltonian circuit of G is a cyclic path in G (that is a path whose first and last vertices are equal, or in the infinite case a two way infinite path) which passes precisely once through each vertex of G.

A Hamiltonian circuit is just a cyclic permutation subgraph, and in the finite case a Hamiltonian arc is just a connected broken permutation subgraph (c.f. §9).

In this chapter we consider Hamiltonian arcs and circuits in finite group graphs; or more precisely, in finite Cayley graphs.

A Hamiltonian arc can clearly only exist if the graph is connected; in the Cayley graph [H,S] this just means that S must be a generating set for H. We note also that in a symmetric graph, and hence certainly in a Cayley graph, it suffices to consider Hamiltonian arcs which start from some fixed base point, for any Hamiltonian arc may be mapped by a suitable graph automorphism to start from this base point.

Now let H be a finite group, S a generating set of H, and suppose we have a Hamiltonian arc B starting from the identity element e of H in the Cayley graph $[H,S]$. If $|H| = h$ then B must have legnth h-1, so it has the form

$$\text{(1)} \qquad B = (e,s_1)(s_1,s_1s_2)(s_1s_2,s_1s_2s_3) \cdots \cdots (s_1s_2 \ldots s_{h-2}, s_1s_2 \ldots s_{h-2}s_{h-1})$$

where $s_1, s_2, \ldots, s_{h-1}$ are (not necessarily distinct) elements of S, and $e, s_1, s_1s_2, \ldots, s_1s_2 \ldots s_{h-1}$ are distinct elements of H. Hence B defines a word $s_1s_2 \ldots s_{h-1}$ in the free monoid F(S) generated by S such that the initial segments $e, s_1, s_1s_2, \ldots$ $s_1s_2 \ldots s_{h-1}$ of legnths $0,1,2, \ldots, h-1$ respectively have distinct values in H. Conversely it is clear that any such word defines a Hamiltonian arc (1) in $[H,S]$. We call such a word of legnth h-1 in F(S), whose initial segments have distinct values in H, a <u>maximal Schreier word of H</u> with respect to S or in the elements of S.

We are using the symbol $w = s_1s_2 \ldots s_{h-1}$ simultaniously as an element of the free monoid F(S), that is a word for which initial segments can be defined, and as an element of H for which initial segments can certainly not be defined. Although this is formally incorrect, it is rather more convenient than using a different symbol for the multiplication in F(S). Where it could lead to confusion we distinguish the two concepts by writing $w \in F(S)$ or $w \in H$, or by saying "the word w" or "the value of w".

If w is an arbitrary word of legnth n, we denote the value of its initial segment of legnth i $(0 \leq i \leq n)$ by w_i. In particular $w_0 = e$ and w_n is the value of the word w.

In the following we shall use the abbreviation ms-word for maximal Schreier word. The ms-word $w = s_1 s_2 \dots s_{h-1} \in F(S)$ is called precyclic if $w_{h-1} s_h = e$ for some $s_h \in S$, and in this case we call the word $ws_h = s_1 s_2 \dots s_{h-1} s_h \in F(S)$ a cyclic ms-word. There is clearly a one-one correspondence between the precyclic and cyclic ms-words of H with respect to S, for one may obtain the one from the other in a unique fashion by adding or dropping the final letter. The cyclic ms-words of H with respect to S just correspond to the Hamiltonian circuits in [H,S].

If the group H has a ms-word with respect to the generating set S, we shall call S a Hamiltonian generating set.

We may now restate the aim of this chapter as a discussion of ms-words, cyclic ms-words, and Hamiltonian generating sets in finite groups. In particular we shall consider the problems of existence and classification for ms-words and cyclic ms-words, and the problem of how small a Hamiltonian generating set can be for a given group.

Notation. The identity element of an abstract group will always be denoted by e. The notation $H = \langle x_1, \dots, x_n : r_1, \dots, r_m \rangle$ means that the group is given by the generators $x_1, \dots, x_n$ and relations $r_1, \dots, r_m$. If a,b, ... are elements or subsets of the group H, then the notation gp{a,b, ...} is used for the subgroup of H generated by a,b, ... Finally S^n and A^n mean respectively the symmetric group and the alternating group on the set {1,2, ... ,n}, and C_n denotes the cyclic group of order n.

§13. Basic Properties of Maximal Schreier Words

We shall use the following lemma implicitely in much of the following discussion.

Lemma 13.1. If H is a finite group and S is a generating set of H then

(i). The word $w = s_1s_2 \ldots s_{h-1} \in F(S)$ is a ms-word of H if and only if no nontrivial segment has value e, and $h = |H|$.

(ii). The word $v = s_1s_2 \ldots s_h \in F(S)$ is a cyclic ms-word of H if and only if no nontrivial proper segment of v has value e, and $h = |H|$.

(By nontrivial segment of a word $x_1x_2 \ldots x_n$ we mean a subword of the form $x_ix_{i+1} \ldots x_j$ where $1 \leq i \leq j \leq n$. It is a proper segment if $i \neq 1$ or $j \neq n$.)

Proof. (i). Let $1 \leq i \leq j \leq h-1$. Then $w_{i-1} = w_j \Longleftrightarrow s_is_{i+1} \ldots s_j = w_{i-1}^{-1}w_j = e$. Hence the initial segments of w have distinct values if and only if no nontrivial segment of w has value e. They run through the elements of H if and only if they are distinct and there are $|H|$ of them, that is $|H| = h$.

(ii). Suppose $h = |H|$ and no nontrivial proper segment of v has value e. Then in particular no nontrivial segment of $s_1s_2 \ldots s_{h-1}$ has value e, so this is a ms-word. If $v_h \neq e$, we would have $v_h = v_i$ for some i with $1 \leq i \leq h-1$, and the nontrivial proper segment $s_{i+1}s_{i+2} \ldots s_h$ would have value $v_i^{-1}v_h = e$. Hence $v_h = e$ so v is a cyclic ms-word.

Conversely if v is a cyclic ms-word then $s_1 s_2 \ldots s_{h-1}$ is a ms-word, so no nontrivial segment of it has value e. The only other nontrivial proper segments of v are of the form $s_{i+1} s_{i+2} \ldots s_h$ $(1 \leq i \leq h-1)$, and if this had value e we would have $e = v_h = v_i s_{i+1} s_{i+2} \ldots s_h = v_i$, contradicting the fact that $s_1 s_2 \ldots s_{h-1}$ is a ms-word. Hence no nontrivial proper segment of v has value e. Q.E.D.

The following lemma states that if we know the ms-words for some generating set S of H, then we know them for any set related to S by automorphisms and antiautomorphisms of H. Thus for instance the 108 distinct pairs of generators of S^4 fall into five classes under the action of automorphisms, so to find all ms-words of S^4 in a pair of generators one need only consider 5 pairs.

Lemma 13.2. If H is a finite group, S a generating subset, and φ an automorphism of H, then the following statements are equivalent:

(i). $w = s_1 s_2 \ldots s_n$ is a ms-word (cyclic ms-word) of H with respect to S.

(ii). $w\varphi = s_1\varphi s_2\varphi \ldots s_n\varphi$ is a ms-word (cyclic ms-word) of H with respect to $S\varphi$.

(iii). $w^{-1} = s_n^{-1} s_{n-1}^{-1} \ldots s_1^{-1}$ is a ms-word (cyclic ms-word) of H with respect to S^{-1}.

Proof. The proof is trivial using lemma 13.1. Q.E.D.

A further useful method in the classification of (cyclic) ms--words is given by the operations of cycling and reversing a given word. If $w = s_1 s_2 \ldots s_n$ is any word in $F(S)$, then the cycled words of w are the words $w^{(i)} = s_{i+1}s_{i+2} \ldots s_n s_1 s_2 \ldots s_{i-1}s_i$ $(i = 0,1, \ldots ,n-1)$, and the reverse word of w is the word $w^R = s_n s_{n-1} \ldots s_2 s_1$.

Theorem 13.3. If $v = s_1 s_2 \ldots s_h$ is a cyclic ms-word of H with respect to S, then so are the cycled words of v.

Proof. A nontrivial proper segment of $v^{(i)}$ must either be a proper segment of v, and hence not have value e, or have the form $s_j s_{j+1} \ldots s_h s_1 s_2 \ldots s_k$ with $k \leq i < j$ and $k+1 < j$. But this has value $(s_{k+1} \ldots s_{j-1})^{-1}$ since $s_1 s_2 \ldots s_h$ has value e, and hence cannot be e as then $s_{k+1} \ldots s_{j-1}$ would be a nontrivial proper segment of v with value e. Q.E.D.

We now discuss a sufficient condition for the reverse word of a (cyclic) ms-word to be a (cyclic) ms-word.

If H is a group generated by the subset S, we call H reversible over S if there exists an automorphism of H which maps each element of S onto its inverse. If such an automorphism exists then it is unique, for an automorphism is defined by its action on a generating set.

If H is given by the set S of generators and a set of defining relations, then H is clearly reversible over S if and only if the relations obtained from the given relations by replacing each letter by its inverse are again relations of H. For instance the metacyclic group of order 21:

(1) $H = \langle s,t \colon s^7 = t^3 = e,\ ts = s^2t \rangle$

is not reversible over $S = \{s,t\}$ since $t^{-1}s^{-1} = (s^{-1})^2t^{-1}$ does not hold in H.

It is well known that a group is abelian if and only if the inversion mapping is an automorphism of the group. Hence:

Lemma 13.4. The group H is reversible over every set of generators if and only if H is abelian.

By simply checking all possible cases one may show:

Lemma 13.5. If H has order less than 16, or is dihedral, or is the symmetric group S^4 or the alternating group A^5, then H is reversible over any pair of generators.

This list may easily be extended. It shows however that reversibility is surprizingly common among small groups with very small generating sets, so the following theorem has practical value as well as academic interest.

Theorem 13.6. If the group H is reversible over the generating set S, and $w = s_1s_2 \ldots s_n$ is a ms-word (cyclic ms-word) of H with respect to S, then so is the reverse word $w^R = s_ns_{n-1} \ldots s_1$.

Proof. Let φ be the automorphism of H which maps each element of S onto its inverse. By lemma 13.2(ii) $w\varphi = s_1^{-1}s_2^{-1} \ldots s_n^{-1}$ is a ms-word (cyclic ms-word) of H with respect to $S\varphi = S^{-1}$, so by lemma 13.2(iii) $(w\varphi)^{-1} = s_ns_{n-1} \ldots s_1$ is a ms-word (cyclic ms-word) of H with respect to $(S^{-1})^{-1} = S$. Q.E.D.

An example of a ms-word whose reverse word is not a ms-word is given by taking H to be the group of (1) above, $S = \{s,t\}$, and

(2) $$w = s^5ts^3ts^4t^2s^2ts \in F(S).$$

This word has legnth 20, and the values of its initial segments are in increasing order of legnth $e,s,s^2,s^3,s^4,s^5,s^5t,t,s^2t,s^4t,s^4t^2,st^2,s^5t^2,s^2t^2,s^6t^2,s^6,s^6t,st,s^3t,s^3t^2,t^2$. Since each element of H has a unique expression in the form s^it^j with $0 \leqslant i \leqslant 6$ and $0 \leqslant j \leqslant 2$, these elements are distinct and run through H, so w is a ms-word; in fact w is even precyclic as $w_{20}t = t^2t = e$. The reverse word $w^R = sts^2t^2s^4ts^3ts^5$ has a segment $sts^2t^2s^2$ with value e, so it is not a ms-word.

Observe that $(s^6t)^3 \in F(S)$ is a cyclic ms-word in the above group, and so is its reverse word $(ts^6)^3$, despite the fact that H is not reversible over S. Hence theorem 13.6 does not give a necessary condition for the reverse of a ms-word to be a ms-word.

§14. <u>A Bound on the Size of a Smallest Hamiltonian Generating Set</u>

<u>Theorem 14.1.</u> If H is a finite soluble group which has a subnormal series of legnth k with cyclic factors, then H has a Hamiltonian generating set with k elements.

We shall prove this by means of the following general theorem:

<u>Theorem 14.2.</u> Let the finite group H have generating set $S = \{s_1,s_2, \ldots ,s_k\}$ and let $H_0 = gp\{e\}$, $H_1 = gp\{s_1\}$, $H_2 = gp\{s_1,s_2\}$, $\ldots ,H_{i+1} = gp\{H_i,s_{i+1}\}, \ldots , H_k = H$. Let $j(i) = |H_i:H_{i-1}|$ for

each $i = 1,2, \ldots ,k$, and define a sequence of words in $F(S)$ by

(1)
$$w(0) = e \quad ; \quad w(1) = s_1^{j(1)-1} \quad ;$$
$$w(i+1) = (w(i)s_{i+1})^{j(i+1)-1}w(i) \quad , \quad (i = 1,2, \ldots ,k-1).$$

Let $v(i)$ denote the value of the word $w(i)s_{i+1}$ and let $V_i = gp\{v(i)\}$ $(i = 0,1, \ldots ,k-1)$. Then $w(i)$ is a ms-word of H_i with respect to $\{s_1,s_2, \ldots ,s_i\}$ for each $i = 1,2, \ldots ,k$ if and only if

(2) $\quad V_iH_i = H_{i+1}$ for each $i = 1,2, \ldots ,k-1$.

Remark. Since V_i and H_i certainly generate H_{i+1}, (2) can be replaced by either of the conditions:

(2)′ $\quad V_iH_i = H_iV_i$ for each $i = 1,2, \ldots ,k-1$.

(2)′′ $\quad V_iH_i$ is a group for each $i = 1,2, \ldots ,k-1$.

Proof. Suppose $w(i)$ is a ms-word of H_i for each i. Then every element of H_{i+1} $(0 \leq i \leq k-1)$ is the value of some initial segment of $w(i+1)$. But each initial segment of $w(i+1)$ has the form $(w(i)s_{i+1})^l u$ with $0 \leq l \leq j(i+1)-1$ and u an initial segment of $w(i)$; and this has value $v(i)^l x$ where x is the value of u. But $v(i)^l x \in V_iH_i$, so $H_{i+1} \subseteq V_iH_i$. Since both V_i and H_i are in H_{i+1}, $H_{i+1} = V_iH_i$.

Suppose conversely that $V_iH_i = H_{i+1}$ for each $i = 1,2, \ldots ,k-1$. $w(1)$ is certainly a ms-word of H_1 with respect to $\{s_1\}$. We assume that $w(i)$ is a ms-word of H_i with respect to $\{s_1,s_2, \ldots ,s_i\}$ and deduce the corresponding statement for $i+1$, proving the theorem by induction.

We have already shown that the values of the initial segments of $w(i+1)$ run through the elements of the form $v(i)^l x$ with

$0 \leqslant l \leqslant j(i+1)-1$ and x the value of an initial segment of $w(i)$, and by assumption $w(i)$ is a ms-word of H_i, so x runs through H_i. Thus it suffices to show that each element of H_{i+1} has a unique expression in the form $v(i)^l x$ with $0 \leqslant l \leqslant j(i+1)-1$ and $x \in H_i$.

Let j be the least positive number such that $v(i)^j \in H_i$. Since $V_i H_i = H_{i+1}$ every element of H_{i+1} is expressible in the form $v(i)^l x$ with $x \in H_i$, and since we may absorb powers of $v(i)^j$ into x, we may assume that $0 \leqslant l \leqslant j-1$. It remains to show that this representation is unique, for then it also follows that $|H_{i+1}| = j|H_i|$, whence $j = j(i+1)$ and the proof is complete.

Suppose $v(i)^r x = v(i)^s y$ with $0 \leqslant r,s \leqslant j-1$ and $x,y \in H_i$. Without loss of generality $r \leqslant s$. Then $v(i)^{s-r} = xy^{-1} \in H_i$ and $0 \leqslant s-r \leqslant j-1$, so $r = s$ by minimality of j, whence also $x = y$. Q.E.D.

Proof of theorem 14.1. Let

(3) $$\{e\} = H_0 \subset H_1 \subset H_2 \subset \dots \subset H_{k-1} \subset H_k = H$$

be a subnormal series for H with cyclic factors. For each $i = 1,2, \dots ,k$ let s_i be a generator of H_i modulo H_{i-1}. If the V_i are defined as in theorem 14.2, then since H_i is normal in H_{i+1} for each $i = 1,2, \dots ,k-1$, $V_i H_i$ is a group for each $i = 1,2, \dots ,k-1$. Hence (2)'' is satisfied, so by theorem 14.2 the s_i ($i = 1,2, \dots k$) form a Hamiltonian generating set for H. Q.E.D.

Corollary 14.3. If the finite soluble group H has order $p_1 p_2 \dots p_k$, where the p_i are (not necessarily distinct) prime numbers, then H has a Hamiltonian generating set with k elements.

Indeed we may take any composition series of H as the subnormal series of theorem 14.1.

One may considerably weaken the solubility condition in theorem 14.1 and corollary 14.3. In fact it is sufficient that H have a series (3) of subgroups such that for each $i = 0,1, \dots ,$ $k-1$, H_{i+1} has a cyclic subgroup V_i with $H_{i+1} = V_i H_i$. For then we may take a generator $v(i)$ of each V_i and define inductively $s_1 = v(1)$, $x(1) = v(1)^{j(1)-1}$; $s_{i+1} = x(i)^{-1}v(i)$, $x(i+1) =$ $v(i)^{j(i+1)-1}x(i)$, $(i = 1,2, \dots ,k-2)$; $s_k = x(k-1)^{-1}v(k-1)$; where $j(i) = |H_i : H_{i-1}|$ for $i = 1,2, \dots ,k$. It is then easily seen that the s_i and $v(i)$ are as in theorem 14.2 and the $x(i)$ are just the values of the words $w(i)$ of theorem 14.2. It follows that the set $\{s_1, s_2, \dots ,s_k\}$ is a Hamiltonian generating set of H.

The generalized solubility condition is satisfied for instance by the symmetric group S^{k+1} of degree $k+1$, taking $H_i = S^{i+1}$ $(0 \leqslant i \leqslant k)$ and $V_i = gp\{(123 \dots i+2)\}$ $(0 \leqslant i \leqslant k-1)$. However one verifies easily that it is not satisfied for instance by the alternating group A^6.

If H is the elementary abelian group of order p^n (p prime), then H has no generating set with less than n elements, so the bound of corollary 14.3 is attained. However for the symmetric group S^{k+1} the above discussion only shows that a Hamiltonian generating set with k elements exists, and we shall see in §17 that there exists one with only 3 elements. Thus the problem of finding the minimal size of Hamiltonian generating sets remains far from solved.

It is likely that the minimal size of a Hamiltonian generating set can strictly exceed the minimal size of an arbitrary generating set but I know of no example to prove this. It would appear that small relatively free groups (cf. Hanna Neumann "Varieties of Groups" Ergebnisse der Math. v.37 1967 p.9) would be likely to provide an example, as they have only one automorphism class of smallest generating sets. However the only non-abelian (cf. §15 for abelian case) groups of this type which are small enough to permit direct calculation of ms-words turn out to have ms-words in the relevant pair of generators. They are the groups

(4) $H = \langle s, t : s^3 = t^3 = e,\ [t, s^{-1}] = [t^{-1}, s] = [s, t] \rangle$,

(the Burnside group of exponent 3 and rank 2) and

(5) $H = \langle s, t : s^4 = t^4 = [s, t]^2 = e,\ [t, s^{-1}] = [t^{-1}, s] = [s, t] \rangle$,

(the relatively free class 2 nilpotent group of exponent 4 with rank 2), and they have cyclic ms-words $(s^2ts^2tst^2)^3$ and $(ts^3t^3s^3ts^2ts^2)^2$ respectively.

§15. Maximal Schreier Words in Abelian Groups

If H is an abelian group and $S = \{s_1, s_2, \ldots, s_k\}$ is an arbitrary generating set, and if the subgroups H_i and V_i are defined as in theorem 14.2, then condition (2)′ of §14 is certainly satisfied, so S is a Hamiltonian generating set of H. Hence:

Theorem 15.1. Every generating set of a finite abelian group is

Hamiltonian.

Theorem 14.2 gives a ms-word of H in S for each ordering of the set S, however one does not in general obtain all ms-words of H in this way. For instance if H is the abelian group

(1) $$H = \langle s,t : s^4 = (st)^2 = [s,t] = e\rangle$$

of order 8, then the two ways of ordering the generating set $\{s,t\}$ give by theorem 14.2 the two ms-words s^3ts^3 and t^3st^3. Both are precyclic, and if one extends either of them to a cyclic ms-word and cycles it one obtains new ms-words which are not given by the construction in theorem 14.2.

Although the finite abelian group H always has a ms-word in a given set of generators, it need not have a cyclic ms-word in these generators. For instance if we consider the cyclic group

(2) $$H = \langle r : r^6 = e\rangle$$

of order 6, and put $s = r^3$ and $t = r^2$, then the only ms-words of H with respect to s and t are t^2st^2 and ststs, neither of which is precyclic.

§16. Maximal Schreier Words in Two Generators - the Coset Method

In this section we derive a condition which greatly restricts the possible forms that a ms-word on two generators can take, and hence has great practical value for the actual calculation of ms--words. This condition results from the direct translation of the alternating path method (§9) to group graphs. Though the ideas of this section are not new - they have been indicated or implicitely applied by Fletcher [7], Dickinson [6], and Rapaport [14] for cyclic ms-words, and by Rankin [13] for general permutation subgraphs of Cayley graphs of degree 2 - they have not been stated in an explicit form for general 2 generator ms-words, and the fact that they are nothing but the alternating path method seems to have escaped notice. Rankin, Fletcher, and Dickinson (loc. cit.) describe an application to campanology.

Let H be a finite group generated by the set $S = \{s,t\}$. Then $[H,S]$ is regular of degree 2, so the methods of §9 are applicable. We first consider the form of the alternating paths of $[H,S]$.

Let the order of st^{-1} be m. Then for any x H the path

(1) $$A_x = (x,xs)(xs,xst^{-1})(xst^{-1},xst^{-1}s)\ \dots$$
$$\dots\ (x(st^{-1})^{m-1},x(st^{-1})^{m-1}s)(x(st^{-1})^{m-1}s,x)$$

is a cyclic alternating α-path of $[H,S]$. We denote the corresponding alternating circuit of $[H,S]$ also by A_x.

$V(A_x)$ is the set of elements $x(st^{-1})^i$, $i = 0,1, \dots ,m-1$; that is $V(A_x) = xC$ where C is the cyclic subgroup $gp\{st^{-1}\}$ of H.

Since x was arbitrary, and the alternating circuit containing a given vertex in its initial vertex set is unique, it follows that every alternating circuit has the form A_x for some $x \in H$.

The partition (§9,(7)) of the edge set $E(A_x)$ may be taken as

(2) $$E^1(A_x) = \{(y,ys):y \in xC\}\ ;\ E^2(A_x) = \{(y,yt):y \in xC\}.$$

Now if $w = s_1s_2 \dots s_{h-1}$ is any ms-word of H in s and t, we define its <u>word function</u> f_w to be the mapping of $H-\{w_{h-1}\}$ to S defined by

(3) $f_w(w_i) = s_{i+1}$, $(i = 0,1, \ldots ,h-2)$.

The ms-word w is uniquely defined by its word function, for the corresponding Hamiltonian arc of [H,S] is fully defined by the fact that its edges are just the pairs $(x,xf_w(x))$ with $x \neq w_{h-1}$.

Noting that a Hamiltonian arc is just a special type of broken permutation subgraph, theorem 9.5 gives:

Theorem 16.1. If $w = s_1 s_2 \ldots s_{h-1}$ is a ms-word of H in the elements s and t, f_w its word function, and $C = gp\{st^{-1}\}$ has order m, then

(i). $w_{h-1} = t^{-1}(st^{-1})^k$ for some k with $0 \leq k \leq m-1$;

(ii). $f_w(t^{-1}(st^{-1})^j) = s$ if $0 \leq j < k$,

$= t$ if $k < j \leq m-1$,

undefined if $k = j$;

(iii). If $xC \neq t^{-1}C$ then f_w is defined and constant on xC.

Proof. Let B be the Hamiltonian arc corresponding to w. Then B is a broken permutation subgraph of [H,S] with initial vertex e and terminal vertex w_{h-1}. The α^*-path of theorem 9.5 must have the form either

(4) $$(e,t^{-1})(t^{-1},t^{-1}s)(t^{-1}s,t^{-1}st^{-1}) \ldots$$
$$\ldots (t^{-1}(st^{-1})^{k-1}s,t^{-1}(st^{-1})^k)(t^{-1}(st^{-1})^k,t^{-1}(st^{-1})^k s) \ldots$$
$$\ldots (t^{-1}(st^{-1})^{m-1},e)$$

with $w_{h-1} = t^{-1}(st^{-1})^k$; or the same with s and t exchanged throughout.

In the second case we may use the fact that $(ts^{-1})^i = (st^{-1})^{-i}$

to replace powers of ts^{-1} by powers of st^{-1}, and then on reversing the representation of B we get a representation as in (4).

(i) now follows immediately from theorem 9.5; and using the fact that the edges of B are just the pairs $(x, xf_w(x))$ $(x \neq w_{h-1})$ and using (2), (ii) and (iii) are direct translations of the corresponding statements of theorem 9.5. Q.E.D.

<u>Corollary 16.2.</u> If the number of times s occurs in w is l, then $w_{h-1} = t^{-1}(st^{-1})^l$.

<u>Proof.</u> l is the number of elements of $H-\{w_{h-1}\}$ for which $f_w(x) = s$. By (ii) $f_w(x)$ equals s for precisely k elements of the coset $t^{-1}C$, and by (iii) $f_w(x)$ equals for either 0 or m elements of each other coset of C. Hence $l \equiv k$ modulo m, so $(st^{-1})^l = (st^{-1})^k$, as st^{-1} has order m. The corollary now follows from (i). Q.E.D.

<u>Corollary 16.3.</u> If st^{-1} has order 2 then every ms-word of H in s and t is precyclic.

Indeed w_{h-1} can then only take one of the two values t^{-1} or $t^{-1}(st^{-1}) = s^{-1}(st^{-1})^2 = s^{-1}$, by theorem 16.1(i).

Theorem 16.1 also gives a bound on the total number of ms-words of H in s and t. Indeed if C has index n in H then f_w can take one of 2 possible values on each of the n-1 left cosets of C other than $t^{-1}C$, and on $t^{-1}C$ f_w is determined by k which can take m possible values. Hence H has at most $2^{n-1}m$ ms-words in s and t. In fact $2^{n-1}m$ is just the number of broken permutation subgraphs of $[H,\{s,t\}]$ with initial vertex e.

§17. Some Special Cases

In this section we discuss some cases where the calculation of small Hamiltonian generating sets and ms-words is rather easier than in the general case.

1). Symmetric groups.

The symmetric group S^2 of degree 2 is cyclic, so it has a one element Hamiltonian generating set.

S^3 and S^4 have ms-words in every pair of generators. For S^4 these are listed in Appendix 1; for S^3 they are given by theorem 17.2.

For general S^n $(n \geqslant 3)$ we have the following theorem:

Theorem 17.1. (Rapaport [14]) If $n \geqslant 3$ then the symmetric group S^n has a cyclic ms-word in the three generators $r = (12)$, $s = (12)(34) \ldots (2l-1\ 2l)$, $t = (23)(45) \ldots (2m\ 2m+1)$, where $l = [n/2]$ and $m = [(n-1)/2]$.

Hence a symmetric group always has a three element Hamiltonian generating set. It seems likely that this can be reduced to 2.

2). Alternating groups.

The alternating group A^3 is cyclic, so it has a one element Hamiltonian generating set.

A^4 has ms-words in every pair of generators; they are listed in Appendix 1.

Direct calculation gives that for $s = (12345)$ and $t = (321)$

$(s^4t)^2s^2t^2(s^4t)^2(s^2t)^2s^4tst^2sts^3tsts^4tst^2sts^3$ is a (non-precyclic) ms-word of A^5. We shall prove in §18 that A^5 has no ms-word in the generators (12)(34) and (135).

I have no results for A^n, $n > 5$, other than the result that for $n \geqslant 9$ A^n has a non-Hamiltonian generator pair (corollary 18.2). Direct calculation can only give very special results and is in any case too time consuming, even for an electronic computer, for $n \geqslant 6$. This holds even though for $n = 5$ hand calculation is still feasible, and was used to find the above ms-word.

3). Dihedral groups.

If H is the dihedral group of order 2n:

(1) $$H = \langle r,s : r^2 = s^2 = (rs)^n = e \rangle ,$$

then the only generator pairs of H are $\{r,s\}$ and $\{s,rs\}$ and images of these under automorphisms. Hence by lemma 13.2 it suffices to consider the pairs $\{r,s\}$ and $\{s,t\}$, where $t = rs$.

Theorem 17.2. $(rs)^n$ is a cyclic ms-word of H. Any ms-word of H in r and s is precyclic and the corresponding cyclic word may be obtained by cycling $(rs)^n$.

$(t^{n-1}s)^2$ is a cyclic ms-word of H. Any ms-word of H in s and t is precyclic and the corresponding cyclic ms-word may be obtained by cycling $(t^{n-1}s)^2$.

Proof. One verifies easily that $(rs)^n$ is a cyclic ms-word of H. Any ms-word of H in r and s must have the form rsrs.... or srsr... since r and s both have order 2. The first part of the theorem now follows immediately.

One verifies easily that $(t^{n-1}s)^2$ is a cyclic ms-word of H. Let $w \in F(S)$ be any ms-word of H in s and t and let k be the number of times s occurs in w. We distinguish 3 cases:

$k \geq 3$. Then w has a segment st^ist^js for some $i,j \geq 0$. If $i \geq j$ then t^jst^js is a segment of w with value e. If $i \leq j$ then st^ist^i is a segment of w with value e. This case can hence not occur.

$k = 2$. Then $w = t^ist^jst^l$ for some $i,j,l \geq 0$. Since w has legnth 2n-1 we must have

(2) $\quad i+j+l = 2n-3.$

Clearly $j \leq n-1$. Suppose $j \leq n-2$. Then if $i \geq j$, w has a segment t^jst^js with value e, and if $i \leq j$ then $l = 2n-3-i-j \geq 2n-3-i-(n-1) = n-2-i \geq j-i$, so w has a segment $t^ist^jst^{j-i}$ with value e. Hence $j \leq n-2$ gives a contradiction so $j = n-1$. (2) now gives that $w = t^ist^{n-1}st^{n-2-i}$ where $0 \leq i \leq n-2$. The word $wt = t^ist^{n-1}st^{n-1-i}$ is obtained by cycling $(t^{n-1}s)^2$, so w is a ms-word with the desired property.

$k = 1$. Then $w = t^ist^j$ for some $i,j \geq 0$. Consideration of the legnth of w gives $i = j = n-1$, so $w = t^{n-1}st^{n-1}$. Hence $ws = (t^{n-1}s)^2$, so w has the stated property. Q.E.D.

4). Groups expressible as the product of two cyclic subgroups.

Theorem 17.3. If the element t of H has order m, then H has a ms-word of the form $(t^{m-1}s)^{l-1}t^{m-1}$ if and only if

(3) $\quad H = BC$ where $B = gp\{t\}$ and $C = gp\{st^{-1}\}$.

Proof. Putting $s_1 = t$, $s_2 = s$, $H_1 = B$, and $V_1 = gp\{t^{-1}s\}$ in theorem 14.2 with $k = 2$ gives that H has a ms-word of the form $(t^{m-1}s)^{l-1}t^{m-1}$ if and only if $H = BV_1$. But $st^{-1} = t(t^{-1}s)t^{-1}$, so $C = tV_1t^{-1}$. Hence $H = BC \Longleftrightarrow H = BtV_1t^{-1} \Longleftrightarrow Ht = BtV_1$. But $Bt = B$ and $Ht = H$, so $H = BC \Longleftrightarrow H = BV_1$. Q.E.D.

Since B and C of theorem 17.3 certainly generate H, $H = BC$ if and only if $BC = CB$. Hence in particular:

Corollary 17.4. If B or C of theorem 17.3 is normal in H, then (3) is satisfied, so $(t^{m-1}s)^{l-1}t^{m-1}$ is a ms-word of H for suitable l.

Rankin [13]pp.21-23 discusses the existence of cyclic ms-words in the case that C is a normal subgroup of H. In fact he gives necessary and sufficient conditions for the Cayley graph $[H,\{s,t\}]$ to have a permutation subgraph with any given number of components.

We shall need the following lemma in the next section.

Lemma 17.5. Let H be generated by s and t and let C be the subgroup $C = gp\{st^{-1}\}$. Then

(i). If $s \in C$ or $t \in C$ then $H = C$.

(ii). If t has prime order p and $|H:C| < p$ then $H = C$.

(iii). If t has prime order p and $|H:C| = p$ then $(t^{p-1}s)^l$ is a cyclic ms-word of H, where $l = |C|$.

Proof. (i). If $s \in C$ then $t = (st^{-1})^{-1}s \in C$ as $st^{-1} \in C$. But s and t generate H so $H = C$. Similarly $t \in C$ implies $H = C$.

(ii) and (iii). The least positive power of t which is in C must be a divisor of the order p of t, so it is either 1 or p. If it is 1 then by (i) H = C. If it is p then the cosets $C, tC, \dots, t^{p-1}C$ are distinct so C has index at least p in H, proving (ii). If C has index precisely p in H, then these cosets cover H, so H = BC where B = gp{t}. Hence by theorem 17.3 $(t^{p-1}s)^{l-1}t^{p-1}$ is a ms-word of H. l is here the order of $t^{-1}s = t^{p-1}s$, so $(t^{p-1}s)^l$ is a cyclic ms-word of H. Q.E.D.

§18. Maximal Schreier Words in two Generators of Orders 2 and 3

In this section we consider ms-words of a group H in the generators s and t, where s and t have orders 2 and 3 respectively.

If H is abelian then it is cyclic of order 6, and direct calculation gives that t^2st^2 and ststs are the only ms-words of H in s and t.

This completes the discussion of the abelian case, so in the following we need only consider non-abelian H.

Theorem 18.1. Let H be a finite non-abelian group generated by the elements s and t of orders 2 and 3 respectively, and let $C = gp\{st^{-1}\}$ have order m and index n in H. Then

(i). $n \geqslant 3$;

(ii). If n = 3 then $(st^2)^m$ is a cyclic ms-word of H. Any ms-word of H is precyclic and the corresponding cyclic ms-word may be obtained by cycling $(st^2)^m$.

(iii). If $n > 3$ and if H has a ms-word w in s and t, then w is not precyclic and one of the following cases holds:

a). $n = 4$ and $w = (st^2)^{m-1}st(st^2)^{m/3}$ or $w = (t^2s)^{m/3}ts(t^2s)^{m-1}$;

b). $n = 5$ and $w = (st^2)^{m-1}st(st^2)^{2m/3}$ or $w = (t^2s)^{2m/3}ts(t^2s)^{m-1}$;

c). $n = 6$ and $w = (st^2)^{m-1}stst(st^2)^{m-1}s$;

d). $n = 7$ and $w = (st^2)^{m-1}st(st^2)^{m/3}st(st^2)^{m-1}s$;

e). $n = 8$ and $w = (st^2)^{m-1}st(st^2)^{2m/3}st(st^2)^{m-1}s$;

In particular if $n \geqslant 9$ then H has no ms-word in s and t.

<u>Proof.</u> We write $|H| = h$ and $gp\{t\} = B$. Clearly $h = mn$, for m and n are respectively order and index of the subgroup C of H. Further h is divisible by 3, for H contains the element t of order 3. Finally $m > 1$, for $m = 1$ would imply $st^{-1} = e$, so $s = t$, but s and t have unequal orders.

If $n < 3$ then by lemma 17.5(ii) $H = C$, so H is abelian, contrary to assumption. Hence $n \geqslant 3$, proving (i).

If $n = 3$ then by lemma 17.5(iii) $(st^2)^m$ is a cyclic ms-word of H, proving the first statement of (ii).

Let $n \geqslant 3$ and w be a ms-word of H in s and t. Since $t^3 = s^2 = e$, w must have the form

$$(1) \qquad w = t^{i_0}st^{i_1}s \ldots st^{i_{p-1}}st^{i_p} \in F(S),$$

with $0 \leqslant i_j \leqslant 2$ for $j = 0,p$, and $1 \leqslant i_j \leqslant 2$ for $j = 1,2, \ldots ,p-1$.

We write $k_j = j+\sum_{l=0}^{j-1} i_l$,$(1 \leqslant j \leqslant p)$. Then the value of the initial segment of legnth k_j of w is just

$$(2) \qquad w_{k_j} = t^{i_0}st^{i_1}s \ldots st^{i_{j-1}}s \quad H ,(1 \leqslant j \leqslant p).$$

We consider a number of cases:

<u>Case 1.</u> $i_j = 2$ for all j with $1 \leqslant j \leqslant p-1$.

Then $w = t^{i_0}(st^2)^{p-1}st^{i_p}$, so w has legnth $3(p-1)+1+i_0+i_p$. But w has legnth $h-1$ and h is divisible by 3, so $i_0+i_p \equiv 1$ modulo 3. Since $0 \leqslant i_o, i_p < 3$, we have the three possibilities: $(i_0, i_p) =$ $(1,0)$ or $(0,1)$ or $(2,2)$. Suppose $(i_0, i_p) = (1,0)$; then $w =$ $t(st^2)^{p-1}s$, and equating the legnth of this with $h-1$ gives $h = 3p$. Now $m = p$, for if m were greater than p, $n = h/m$ would be less than 3, contradicting part (i), and if m were less than p, $(st^2)^m$ would be a nontrivial segment of w with value e. Hence $n = 3$ and $w = t(st^2)^{m-1}s$. $wt = t(st^2)^{m-1}st$ can be obtained by cycling the cyclic ms-word $(st^2)^{m-1}$, so w is a precyclic ms-word. Similarly the cases $(i_0, i_p) = (0,1)$ or $(2,2)$ also lead to the conclusion that $n = 3$ and w is derived from the cyclic ms-word $(st^2)^m$.

<u>Case 2.</u> $i_j = 1$ for at least one j with $1 \leqslant j \leqslant p-1$.

To complete the proof we must show that this case leads to the conclusions $n > 3$, w is not precyclic, and w is of one of the forms given in (iii) a) - e). We first show:

(A). If $1 \leqslant j \leqslant p-1$ and $i_j = 1$, then either $w_{k_j}t^2 = e$ and $i_0 = 0$ or $w_{k_j}t^2 = w_{h-1}$ and $i_p = 0$.

Indeed, suppose $1 \leqslant j \leqslant p-1$ and $i_j = 1$. If $w_{k_j}t^2 = w_l$ with $k_1 \leqslant l < k_p$, we can find a q with $1 \leqslant q \leqslant p-1$ such that $k_q \leqslant l < k_{q+1}$. Since $1 \leqslant q \leqslant p-1$, $i_q \geqslant 1$, so $w_{k_q+1} = w_{k_q}t$. It follows that w_{k_q}, w_{k_q+1}, w_{k_j}, and w_{k_j+1} are four elements of the left coset $w_{k_j}B$ of B. They are hence not distinct, as $B = 3$, so we must have $q = j$.

But this implies that $w_{k_j}t^2 = w_{k_j}$ or w_{k_j+1}, which is clearly impossible. Hence $w_{k_j}t^2 = w_l$ with $0 \leq l < k_1$ or $k_p \leq l \leq h-1$. If $0 \leq l < k_1$ then $w_{k_j}, w_{k_j+1}, w_0, \ldots, w_{k_1-1}$ are k_1+2 distinct elements of B, so $k_1 \leq 1$. $k_1 = 0$ is impossible as $k_1 = 1+i_0$, so $k_1 = 1$, whence $i_0 = 0$ and $l = 0$. Similarly $k_p \leq l \leq h-1$ implies $i_p = 0$ and $l = h-1$. (A) is hence proved.

It follows that the condition of Case 2 implies that w is not precyclic, for we know that there exists a j with $1 \leq j \leq p-1$ such that $w_{k_j}t^2 = e$ or w_{h-1}. If $w_{k_j}t^2 = w_{h-1}$ then $i_p = 0$, so the word w ends with an s. Hence $w_{h-1}s = w_{h-2} \neq e$ and $w_{h-1}t = w_{k_j}t^2t = w_{k_j} \neq e$. If $w_{k_j}t^2 = e$ then $i_0 = 0$, so $w_1 = s$. Hence $w_{h-1} \neq s$, so $w_{h-1}s \neq s^2 = e$; also $w_{h-1}t \neq w_{k_j}t^2 = e$.

It remains to show that w is of one of the types given in (iii) and that n takes the corresponding value. (A) implies that $i_j = 1$ can occur for at most two values of j with $1 \leq j \leq p-1$. Thus two possibilities must be considered:

<u>Case 2a.</u> $i_j = i_l = 1$ with $1 \leq j,l \leq p-1$ and $j \neq l$.

Without loss of generality we may assume by (A) that $w_{k_j}t^2 = e$ and $w_{k_l}t^2 = w_{h-1}$. We also have that $i_0 = i_p = 0$ so one of the following cases holds:

(3) $\qquad w = (st^2)^{l-1}st(st^2)^{j-l-1}st(st^2)^{p-j-1}s \quad ,(l < j);$

(4) $\qquad w = (st^2)^{j-1}st(st^2)^{l-j-1}st(st^2)^{p-l-1}s \quad ,(j < l).$

In (3) $(st^2)^{l-1}st(st^2)^{j-l-1}st^2 = w_{k_j}t^2 = e$, so $(st^2)^{l-1}st =$

$(st^2)^{1-j}$, so $st = (st^2)^{1-j}$. Hence $t = t^4 = t^2sst^2 = (st)^{-1}st^2 = (st^2)^{j-1}st^2 = (st^2)^j \in C$, so by lemma 17.5(i) $H = C$. This is contrary to the assumption that H be non-abelian, so (3) cannot occur.

In (4) $(st^2)^j = (st^2)^{j-1}st^2 = w_{k_j}t^2 = e$, so m divides j. $m < j$ would imply that $(st^2)^m$ is a nontrivial segment of w with value e, so $m = j$. Also $(st^2)^{p-1} = (w_{k_1}t)^{-1}w_{h-1}t^2 = (w_{h-1}t^{-1})^{-1}w_{h-1}t^2 = t^3 = e$, so $m = p-1$. Hence writing $j-1-1 = q$, (4) becomes $w = (st^2)^{m-1}st(st^2)^q st(st^2)^{m-1}s$. w hence has legnth $6(m-1)+3q+5$, and equating this with $h-1 = mn-1$ gives

(5) $$mn = 6m+3q.$$

But clearly $0 \leq q < m$, so $6m \leq mn < 9m$, so $n = 6$, 7, or 8. Inserting these values back in (5) gives $q = 0$, $m/3$, or $2m/3$ respectively. This just gives cases (iii) c), d), and e) of the theorem.

Case 2b. $i_j = 1$ for precisely one value of j with $1 \leq j \leq p-1$.

By (A) this case gives the two possibilities $w_{k_j}t^2 = e$ and $i_0 = 0$ or $w_{k_j}t^2 = w_{h-1}$ and $i_p = 0$. These give respectively the two possibilities:

(6) $$w = (st^2)^{j-1}st(st^2)^{p-j-1}st^{i_p} ;$$

(7) $$w = t^{i_0}(st^2)^{j-1}st(st^2)^{p-j-1}s .$$

Comparing the legnths in each case with $h-1$, and noting that h is divisible by 3, gives $i_p = 2$ and $i_0 = 2$ respectively.

In (6) $(st^2)^j = (st^2)^{j-1}st^2 = w_{k_j}t^2 = e$, so $j = m$. On writing $p-j = q$, (6) becomes $w = (st^2)^{m-1}st(st^2)^q$. This has legnth $3(m-1)+2+3q$, so equating with $mn-1$ gives

(8) $mn = 3m+3q$.

But $0<q$ as $j<p$; and certainly $q<m$. Hence $3m<mn<6m$, so $n = 4$ or 5. Inserting back into (8) gives $q = m/3$ and $2m/3$ respectively, which gives the first possibilities in cases (iii) a) and b) of the theorem. Similar considerations show that (7) leads to the second possibility in each of these cases (alternatively one may use the fact that a word of type (7) is just the reverse of one of type (6)), so the theorem is proved. Q.E.D.

Examples.

18.1. $H = S^3$ (the symmetric group of degree 3) of order $h = 6$; $s = (12)$, $t = (123)$. Then $st^{-1} = (23)$, so $m = 2$ and $n = 3$. Hence by theorem 18.1(ii), $(st^2)^2$ is a cyclic ms-word, and the words st^2st, t^2st^2, and tst^2s obtainable from $(st^2)^2$ give all ms-words of S^3 in s and t.

18.2. $H = C_3 wr C_2$ (the wreath product of a cyclic group of order 3 by a cyclic group of order 2) of order $h = 18$. This group may be presented by $H = \langle s,t : s^2 = t^3 = [s^{-1}ts,t] = e\rangle$. One verifies easily that st^{-1} has order $m = 6$; hence $n = 3$, and we again have the cyclic case, every ms-word of H in s and t being derivable from the cyclic ms-word $(st^2)^6$.

Using standard group theoretical methods one can show that the above two examples are the only examples (up to isomorphism) with $n = 3$, so they are the only cases (up to isomorphism) of groups having cyclic ms-words in two generators of orders 2 and 3.

<u>18.3.</u> $H = A^4$ of order $h = 12$; $s = (12)(34)$, $t = (123)$. Then $st^{-1} = (234)$, so $m = 3$ and $n = 4$. By theorem 18.1(iii) a) a ms-word of H in s and t must have the form $(st^2)^2stst^2$ or $t^2sts(t^2s)^2$. One verifies easily that both of these are in fact ms-words.

<u>18.4.</u> $H = S^4$ of order $h = 24$; $s = (12)$, $t = (134)$. Then $st^{-1} = (1234)$, so $m = 4$ and $n = 6$. By theorem 18.1(iii) c), a ms-word of H in s and t must have the form $(st^2)^3stst(st^2)^3s$. This is a ms-word.

<u>18.5.</u> $H = A^5$ of order $h = 60$; $s = (12)(34)$, $t = (135)$. Then $st^{-1} = (12534)$, so $m = 5$ and $n = 12$. Since $n \geqslant 9$, A^5 has no ms-word in s and t. (However it has a ms-word in a different generator pair; cf. §17.)

<u>18.6.</u> $H = C_2 wr C_3$ (the wreath of a cyclic group of order 2 by one of order 3. This is isomorphic to $A^4 \times C_2$). The order is $h = 24$; the group may be presented by $H = \langle s,t : s^2 = t^3 = [t^{-1}st, s] = e\rangle$. In this case st^{-1} has order $m = 6$, so $n = 4$. A ms-word of H in s and t must hence have the form $(st^2)^5st(st^2)^2$ or $(t^2s)^2ts(t^2s)^5$. But these have segments st^2stst^2st and tst^2stst^2s respectively with value e, so they are not ms-words. Hence $C_2 wr C_3$ has no ms-word in s and t. (It has however a ms-word $pqp^2qp^2q^2p^5q^2p^2qp^2qp$ in the generators $p = st$ and $q = t^2$).

In the following a "(2,3) generator pair" will mean a pair $\{s,t\}$ of generators where s and t have orders 2 and 3 respectively. m and n will denote, as usual, the order and index of the subgroup

$C = gp\{st^{-1}\}$ of the group considered.

Burnside ([2] §§296 and 301) classifies the (2,3) generated groups with m = 2,3,4,5,6. For m = 2,3,4,5 these groups are as in examples 18.1, 18.3, 18.4, 18.5 respectively. For m = 6 there are already an infinite number of such groups; only a finite number of them have $n \leq 8$, so the (2,3) generator pair is non-Hamiltonian for almost all of them. For $m \geq 7$ these groups do not appear to have been classified.

G. A. Miller [10] has shown that every nontrivial symmetric or alternating group can be (2,3) generated, with the following exeptions: S^2, S^5, S^6, S^8, A^3, A^6, A^7, A^8. For $k \geq 5$, A^k and S^k have no cyclic subgroup of index less than 10, so $n \geq 10$, so any (2,3) generator pair is non-Hamiltonian. For S^3, A^4, and S^4, the only (2,3) generator pairs are those given in examples 18.1, 18.3, and 18.4 (up to automorphism). Hence

Corollary 18.2. S^k has a (2,3) generator pair if and only if k = 3, 4, 7, or $k \geq 9$. This pair is Hamiltonian if and only if k = 3 or 4.

A^k has a (2,3) generator pair if and only if k = 4,5 or $k \geq 9$. This pair is Hamiltonian if and only if k = 4.

One may obtain analogous results for other classes of groups. For instance if p is an odd prime then the fractional linear group LF(2,p) has a (2,3) generator pair with m = p and $n = (p^2-1)/2$ (Coxeter and Moser [5] p.94). This generator pair is non-Hamiltonian for $p > 3$, since $n = (p^2-1)/2$ is then greater than 8.

The last paragraph and the second part of corollary 18.2 both give infinite sets of simple groups with non-Hamiltonian (2,3) generator pairs. In fact the number of simple groups with Hamiltonian (2,3) generator pairs is finite; for such a group has a subgroup C of index $n \leqslant 8$ by theorem 18.1, and since the permutation representation on the left cosets of C must be faithful of degree n, the group has order at most 8!. I have been unable to prove or disprove the corresponding statement for arbitrary groups.

To close this chapter we make the following two comments:

For abelian groups every generating set is Hamiltonian, but we have seen that almost all simple groups have non-Hamiltonian generating sets. It therefore seems probable that the existence of Hamiltonian generating sets and ms-words is fairly closely connected with the commutatorial structure of the group. This is verified to some extent by theorem 14.1, however I have been unable to find any strong connections.

The only non-Hamiltonian generating sets that I know of are all (2,3) generating pairs. It is extremely unlikely that this is in fact necessary for a generating set to be non-Hamiltonian. However very rough plausibility considerations imply that the larger the order of the generators, the greater the probability that they form a Hamiltonian generating set. It is hence possible that there may exist sufficient conditions of the type "a given function of the orders of the generators is greater than a given bound" for a generating set to be Hamiltonian, or results of a similar type may hold. I have however been unable to obtain any results in this direction.

APPENDIX 1 : Tables of Maximal Schreier Words

If S is a generating set of the group H, we call S minimal if no proper subset of S generates H.

The following tables give all ms-words in a minimal generator pair for the following groups: all groups of order less than 12 which have a minimal generator pair, A^4, and S^4. The only reason for not considering generator pairs which are not minimal is that there are too many such cases; - not because they lack interest.

The dihedral groups are dealt with by theorem 17.2, and are hence ommitted from these tables.

For each group we consider one generator pair out of each automorphism class of minimal generator pairs. For each group and generator pair we list a set of cyclic ms-words from which all other cyclic ms-words are obtainable by cycling and reversal, and a set of non-precyclic ms-words from which all other non-precyclic ms-words are obtainable by reversal (cf. lemma 13.5 and theorem 13.6).

$C_6 = \langle r : r^6 = e \rangle$.

$s = r^3$, $t = r^2$.

Cyclic : None

Non-precyclic : t^2st^2 , ststs .

$C_2 \times C_4 = \langle p,q : p^2 = q^4 = [p,q] = e \rangle$.

$s = p$, $t = q$.

Cyclic : $(st)^4$.

Non-precyclic : t^3st^3.

$s = pq$, $t = q$.

Cyclic : $(s^3t)^2$, $(t^3s)^2$.

Non-precyclic : None.

$Q = \langle p,q : p^4 = q^4 = pqp^{-1}q = e \rangle$ (Quaternion group).

$s = p$, $t = q$.

Cyclic : $(st)^4$.

Non-precyclic : s^3ts^3, t^3st^3.

$C_3 \times C_3 = \langle p,q : p^3 = q^3 = [p,q] = e \rangle$.

$s = p$, $t = q$.

Cyclic : $(s^2t)^3$, $(t^2s)^3$.

Non-precyclic : None.

$C_{10} = \langle r : r^{10} = e \rangle$.

$s = r^5$, $t = r^2$.

Cyclic : None.

Non-precyclic : $(st)^4s$, t^4st^4, $t^2ststst^2$.

A^4 (Alternating group on $\{1,2,3,4\}$).

$s = (12)(34)$, $t = (123)$.

Cyclic : None.

Non-precyclic : $(st^2)^2stst^2$.

$s = (123)$, $t = (234)$.

Cyclic : None.

Non-precyclic : $s^2t^2s^2ts^2ts$, $t^2s^2t^2st^2st$.

$s = (123)$, $t = (243)$.

Cyclic : $(s^2t^2st)^2$, $(t^2s^2ts)^2$.

Non-precyclic : None.

S^4 (Symmetric group on $\{1,2,3,4\}$).

$s = (1234)$, $t = (123)$.

Cyclic : $(s^3ts^2tstst^2)^2$, $(s^3ts^2t^2s^2t^2)^2$, $(s^3tsts^2t^2st)^2$, $(s^3tst^2s^3t^2)^2$, $s^3ts^2ts(t^2s^3)^2(ts)^2t^2$, $s^3ts^2tst^2s^2ts^3t^2s^2tst^2$, $s^3ts^2t^2s^3t^2sts^3(ts)^2t^2$, $s^3ts^2t^2st(s^3tst^2)^2$, $s^3ts^2t^2st(s^3t^2)^2s^2t^2$, $s^3tsts^3t^2ststs^3tstst^2$, $s^3ts(t^2s^3)^3t^2st$.

Non-precyclic : None.

$s = (12)$, $t = (134)$.

Cyclic : None.

Non-precyclic : $(st^2)^3stst(st^2)^3s$.

$s = (12)$, $t = (1234)$.

Cyclic : None.

Non-precyclic : $st^3st(st^3)^2stst^2st^3$, $(st^3)^2st^2st(st^3)^2st$, $tst(st^3)^2(stst^3)^2$.

$s = (123)$, $t = (1432)$.

Cyclic : None.

Non-precyclic : $s^2t^3(s^2t^2)^2st^3(st^2)^2$, $st(s^2t)^2s^2t^3(s^2t)^3s$, $ts^2t^3s^2ts^2t^3st^2st^3st$, $t^3s^2(t^3s)^3t^2st^3$.

$s = (1234)$, $t = (1324)$.

Cyclic : None.

Non-precyclic : $s^3t^2sts^2t^2s^3t^3s^3t^2s$, $(s^3t^3)^2s^3t^2s^2(ts)^2$, $s^3t^3s^2tsts^2t^2s^3t^3s^2$, $s^2ts^3ts^3t^2st(s^3t)^2s$, $s^2tst^2s^3t^2sts^3t^3s^2ts$, $s^2t(st^3)^2sts^2t^2stst^2s$, $s^2t^3s^2t(st^3)^2s^2tst^2s$, $sts^3tst^2s^3t^2stst^2s^2ts$, $(st)^2s^2t^2(stst^3)^2sts$, $(s^3tst)^2(s^3t)^2sts$, $s^3tst^3s^3t^2s^2tsts^2t^2s$, $s^3t^2st(s^3t)^2sts^3ts^2$, and all words obtained from these twelve words by exchanging s and t.

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