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EXOTIC TWISTED EQUIVARIANT K-THEORY

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Abstract. In this paper we introduce *exotic twisted* $\mathbb{T}$*-equivariant K-theory* of loop space LZ depending on the (typically non-flat) holonomy line bundle $\mathcal{L}^B$ on LZ induced from a gerbe on Z. We also define *exotic twisted* $\mathbb{T}$*-equivariant Chern character* that maps the exotic twisted $\mathbb{T}$-equivariant K-theory of LZ into the exotic twisted $\mathbb{T}$-equivariant cohomology as defined earlier in [9], and which localises to twisted cohomology of Z.

Contents

Introduction

In [9], we introduced the *exotic twisted* $\mathbb{T}$*-equivariant cohomology* for the loop space LZ of a smooth manifold Z via the invariant differential forms on LZ with coefficients in the (typically non-flat) holonomy line bundle $\mathcal{L}^B$ of a gerbe, with differential an equivariantly flat superconnection $\nabla^{\mathcal{L}^B} - \iota_K + \bar{H}$ in the sense of [13], where K is the rotation vector field and $\bar{H}$ is a degree 3 circle-invariant form on LZ that is completely determined by H, the curvature of the gerbe.

This exotic twisted $\mathbb{T}$-equivariant cohomology theory has two applications.

First we introduced in [9] the twisted Bismut-Chern character form, generalising [2], which is a loop space refinement of the twisted Chern character form in [4] and represents classes in the completed periodic exotic twisted $\mathbb{T}$-equivariant cohomology $h^{\bullet}_{\mathbb{T}}(LZ, \nabla^{\mathcal{L}^B} : \bar{H})$ of LZ.

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More precisely, we define these in such a way that the following diagram commutes,

$$
\begin{tikzcd}
K^\bullet(Z,H) \arrow[rr, "BCh_H"] \arrow[dr, "Ch_H"'] & & h^\bullet_{\mathbb{T}}(LZ, \nabla^{\mathcal{L}^B} : \bar{H}) \arrow[dl, "res"] \\
& H^\bullet(\Omega(Z)[[u,u^{-1}]], d + u^{-1}H) &
\end{tikzcd}
\tag{0.1}
$$

where res is the localisation map, degree$(u) = 2$.

Secondly, in [9] we establish a localisation theorem (about the map res) for the completed periodic exotic twisted $\mathbb{T}$-equivariant cohomology for loop spaces and apply it to establish T-duality in a background flux in type II String Theory from a loop space perspective. Continuing along this clue, we recently used in [10] the exotic twisted $\mathbb{T}$-equivariant cohomology to enhance T-duality on twisted differential forms on circle bundles, where we also showed the exchange of winding and momentum for the first time in the model of [5, 6]. For an alternate approach to T-duality on loop space using twisted chiral de Rham cohomology instead, see [12].

In this paper, we introduce *exotic twisted* $\mathbb{T}$*-equivariant K-theory*, $K^0_{\mathbb{T}}(LZ, \nabla^{\mathcal{L}^B} : \mathcal{G})$, for the loop space LZ, where $\mathcal{G}$ is the weak $\mathbb{T}$-invariant gerbe on LZ whose Dixmier-Douady class is $\bar{H}$. We also define an *exotic twisted* $\mathbb{T}$*-equivariant Chern character*,

$$
Ch_{\nabla^{\mathcal{L}^B}:\mathcal{G}} : K^0_{\mathbb{T}}(LZ, \nabla^{\mathcal{L}^B} : \mathcal{G}) \longrightarrow h^{even}_{\mathbb{T}}(LZ, \nabla^{\mathcal{L}^B} : \bar{H})
$$

that make the below diagram commutative :

$$
\begin{tikzcd}
& K^0_{\mathbb{T}}(LZ, \nabla^{\mathcal{L}^B} : \mathcal{G}) \arrow[dl, "res"'] \arrow[dr, "Ch_{\nabla^{\mathcal{L}^B}:\mathcal{G}}"] & \\
K^\bullet(Z,H) \arrow[dr, "Ch_H"'] & & h^\bullet_{\mathbb{T}}(LZ, \nabla^{\mathcal{L}^B} : \bar{H}) \arrow[dl, "res"] \\
& H^\bullet(\Omega(Z)[[u,u^{-1}]], d + u^{-1}H) &
\end{tikzcd}
\tag{0.2}
$$

It follows that the exotic twisted $\mathbb{T}$-equivariant K-theory is the correct version of K-theory that corresponds via a Chern character map to the exotic twisted $\mathbb{T}$-equivariant cohomology as defined in [9]. However we would like to point out that the map BCh_H in figure (0.1) does not make the upper triangle of figure (0.2) commutative.

Our construction of the exotic twisted $\mathbb{T}$-equivariant K-theory can be done on more general spaces rather than loop spaces, namely on the good $\mathbb{T}$-manifolds, which apply to the circle bundles in the T-duality setting. Actually this paper lays the foundation for work in progress, [11], where we will use the exotic twisted $\mathbb{T}$-equivariant K-theory on LZ to enhance T-duality on objects in (twisted) K-theory on circle bundles, similarly in spirit to what we did in [10].

The plan of this paper is as follows.

In Section 1, we introduce the concept of *weak* $\mathbb{T}$*-invariant gerbes* and study the coupling of them to $\mathbb{T}$-equivariant line bundles on possibly infinite dimensional good $\mathbb{T}$-manifolds. A pair of coupled weak $\mathbb{T}$-invariant gerbe and $\mathbb{T}$-equivariant line bundles will be the initially input data for an exotic twisted $\mathbb{T}$-equivariant K-theory (see Section 3).

In Section 2, we establish the correspondence of the exotic twisted $\mathbb{T}$-equivariant cohomology about differential forms on M with coefficients in a line bundle ξ to certain cohomology theory about differential forms on $S\xi$, the circle bundle of ξ (see Theorem 2.3). Such a passage from M to $S\xi$ is needed to be established because when we attempt to develop the exotic twisted $\mathbb{T}$-equivariant K-theory, we realize that it is difficult to be done on M itself, instead one needs to pass to the circle bundle of ξ. This space has more room to develop the correct K-theory, who possesses a Chern character landing into the exotic twisted $\mathbb{T}$-equivariant cohomology.

In Section 3, we introduce exotic twisted $\mathbb{T}$-equivariant K-theory for possibly infinite dimensional $\mathbb{T}$-manifolds, and the exotic twisted $\mathbb{T}$-equivariant Chern character that lands into exotic twisted $\mathbb{T}$-equivariant cohomology. We also establish the transgression formulae in this context, using a new version of Chern-Simons forms.

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1. Coupling of $\mathbb{T}$-equivariant line bundles and weak $\mathbb{T}$-invariant gerbes

Let M be a (possibly infinite dimensional) $\mathbb{T}$-manifold. We call M a **good $\mathbb{T}$-manifold** if M has an open cover $\{U_\alpha\}$ such that all finite intersections $U_{\alpha_0\alpha_1\cdots\alpha_p} = U_{\alpha_0}\cap U_{\alpha_1}\cdots U_{\alpha_p}$ are $\mathbb{T}$-invariant. Let K be the Killing vector field of the $\mathbb{T}$-action.

Definition 1.1. *The system $(\{U_\alpha\}, H, B_\alpha, A_{\alpha\beta})$ is called a gerbe on M, if*

$$H\in\Omega^3(M),\ \ B_\alpha\in\Omega^2(U_\alpha),\ \ \ A_{\alpha\beta}\in\Omega^1(U_{\alpha\beta}),$$

such that $\frac{1}{2\pi i}H$ has integral period,

$$\begin{aligned} H &= dB_\alpha \ \text{ on } U_\alpha,\\ B_\alpha - B_\beta &= dA_{\alpha\beta} \ \text{ on } U_{\alpha\beta}, \end{aligned} \tag{1.1}$$

and there exist $C_{\alpha\beta\gamma}\in C^\infty(U_{\alpha\beta\gamma}, U(1))$ such that

$$A_{\alpha\beta}+A_{\beta\gamma}-A_{\alpha\gamma} = d\ln C_{\alpha\beta\gamma}.$$

It is easy to see that different choices of $C_{\alpha\beta\gamma}$ differ by a $U(1)$-valued constant scalar on each connected component of $U_{\alpha\beta\gamma}$.

Remark 1.2. *Our definition of a gerbe here is slightly more general than the gerbe in the usual sense. We don't require $C_{\beta\gamma\delta}C^{-1}_{\alpha\gamma\delta}C_{\alpha\beta\delta}C^{-1}_{\alpha\beta\gamma}=1$ on each nonempty intersection $U_\alpha\cap U_\beta\cap U_\gamma\cap U_\delta$.*

Definition 1.3. *A gerbe* $(\{U_\alpha\}, H, B_\alpha, A_{\alpha\beta})$ *is called a* **weak $\mathbb{T}$-invariant gerbe** *on* M *if*
(i) $H, B_\alpha, A_{\alpha\beta}$ *are all* $\mathbb{T}$*-invariant;*
(ii) $\iota_K A_{\alpha\beta} + \iota_K A_{\alpha\beta} - \iota_K A_{\alpha\gamma}$ *takes values in* $2\pi i \cdot \mathbb{Z}$ *on each connected component of* $U_{\alpha\beta\gamma}$.

Remark 1.4. *The second condition is equivalent to*

$$L_K C_{\alpha\beta\gamma} = 2\pi i n C_{\alpha\beta\gamma}$$

for some $n \in \mathbb{Z}$ *on each connected component of* $U_{\alpha\beta\gamma}$. *Actually we have*

$$\iota_K A_{\alpha\beta} + \iota_K A_{\alpha\beta} - \iota_K A_{\alpha\gamma} = \iota_K \left(C^{-1}_{\alpha\beta\gamma} dC_{\alpha\beta\gamma} \right) = C^{-1}_{\alpha\beta\gamma} \iota_K dC_{\alpha\beta\gamma} = C^{-1}_{\alpha\beta\gamma} L_K C_{\alpha\beta\gamma}.$$

If all the n *is equal to* 0*, i.e.* $C_{\alpha\beta\gamma}$*'s are* $\mathbb{T}$*-invariant, we call it a* **$\mathbb{T}$-invariant gerbe.**

Let ξ be a $\mathbb{T}$-equivariant complex line bundle over M equipped with a $\mathbb{T}$-invariant connection ∇^ξ.

Definition 1.5. *The* $\mathbb{T}$*-equivariant line bundle* (ξ, ∇^ξ) *and the weak* $\mathbb{T}$*-invariant gerbe* $(\{U_\alpha\}, H, B_\alpha, A_{\alpha\beta})$ *are called* **coupled on** M *if under some* $\mathbb{T}$*-invariant local basis* $\{s_\alpha\}$,
(i) $-\iota_K B_\alpha$ *is the connection 1-form of* ∇^ξ *on* U_α *for each* α*;*
(ii) $e^{-\iota_K A_{\alpha\beta}}$ *is the transition function of* ξ *on* $U_{\alpha\beta}$ *for each* α, β.

Lemma 1.6. *If the* $\mathbb{T}$*-equivariant line bundle* (ξ, ∇^ξ) *and the weak* $\mathbb{T}$*-invariant gerbe* $(\{U_\alpha\}, H, B_\alpha, A_{\alpha\beta})$ *are coupled on* M*, then the equivariant super connection* $\nabla^\xi - u\iota_K + u^{-1}H$ *on* ξ *is equivariantly flat, i.e.*

$$(\nabla^\xi - u\iota_K + u^{-1}H)^2 + uL^\xi_K = 0. \tag{1.2}$$

Proof. The proof is similar to the proof of Lemma 1 in [9].

□

We provide some examples of coupled $\mathbb{T}$-equivariant line bundles and weak $\mathbb{T}$-invariant gerbes.

Example 1. Let M be a smooth manifold. Let $\{U_\alpha\}$ be a *Brylinski open cover* of M, i.e. $\{U_\alpha\}$ is a maximal open cover of M with the property that $H^i(U_{\alpha_I}) = 0$ for $i = 2, 3$ where $U_{\alpha_I} = \bigcap_{i \in I} U_{\alpha_i}$, $|I| < \infty$. Then the free loop space LM is good $\mathbb{T}$-manifold with the open cover $\{LU_\alpha\}$, where the $\mathbb{T}$-action is the loop rotating action.

Let τ be the transgression

$$\tau : \Omega^\bullet(U_{\alpha_I}) \longrightarrow \Omega^{\bullet-1}(LU_{\alpha_I}) \tag{1.3}$$

is the transgression map defined as

$$\tau(\xi_I) = \int_{\mathbb{T}} ev^*(\xi_I), \qquad \xi_I \in \Omega^\bullet(U_{\alpha_I}). \tag{1.4}$$

Here ev is the evaluation map

$$ev : \mathbb{T} \times LM \to M : (t, \gamma) \to \gamma(t). \tag{1.5}$$

Let $\omega \in \Omega^i(M)$. Define $\hat{\omega}_s \in \Omega^i(LM)$ for $s \in [0,1]$ by

$$\hat{\omega}_s(X_1, \ldots, X_i)(\gamma) = \omega(X_1\big|_{\gamma(s)}, \ldots, X_i\big|_{\gamma(s)}) \tag{1.6}$$

for $\gamma \in LM$ and $X_1, \ldots, X_i$ are vector fields on LM defined near γ. Then one checks that $d\hat{\omega}_s = \widehat{d\omega}_s$. The i-form

$$\bar{\omega} = \int_0^1 \hat{\omega}_s ds \in \Omega^i(LM) \tag{1.7}$$

is $\mathbb{T}$-invariant, that is, $L_K(\bar{\omega}) = 0$. Moreover $\tau(\omega) = \iota_K \bar{\omega}$. Here K is the vector field on LM generating rotation of loops.

Let $(\{U_\alpha\}, H, B_\alpha, A_{\alpha\beta})$ be a gerbe on M. Associated to this gerbe, there exists a pair of coupled $\mathbb{T}$-equivariant line bundle and weak $\mathbb{T}$-invariant gerbe on LM.

The holonomy of this gerbe is a $\mathbb{T}$-equivariant line bundle $\mathcal{L}^B \to LM$ over the loop space LM. $\mathcal{L}^B$ has Brylinski local sections $\{\sigma_\alpha\}$ with respect to $\{LU_\alpha\}$ such that the transition functions are $\{e^{-\int_0^1 \iota_K A_{\alpha\beta}} = e^{-\tau(A_{\alpha\beta})}\}$, i.e. $\sigma_\alpha = e^{-\int_0^1 \iota_K A_{\alpha\beta}} \sigma_\beta$. The Brylinski sections are $\mathbb{T}$-invariant. $\mathcal{L}^B$ comes with a natural connection, whose definition with respect to the basis $\{\sigma_\alpha\}$ is

$$\nabla^{\mathcal{L}^B} = d - \iota_K \bar{B}_\alpha = d - \tau(B_\alpha). \tag{1.8}$$

For more details, cf. [8].

On the other hand, averaging the gerbe $(\{U_\alpha\}, H, B_\alpha, A_{\alpha\beta})$ gives rise to a gerbe

$$(\{LU_\alpha\}, \bar{H}, \bar{B}_\alpha, \bar{A}_{\alpha\beta})$$

on LM. First it is not hard to see that $\frac{1}{2\pi i}\bar{H}$ still has integral period. It is evident that

$$\begin{aligned} \bar{H} &= d\bar{B}_\alpha \quad \text{on } LU_\alpha, \\ \bar{B}_\alpha - \bar{B}_\beta &= d\bar{A}_{\alpha\beta} \quad \text{on } LU_{\alpha\beta}. \end{aligned} \tag{1.9}$$

If on $U_{\alpha\beta\gamma}$,

$$A_{\alpha\beta} + A_{\beta\gamma} - A_{\alpha\gamma} = d\ln C_{\alpha\beta\gamma}, \tag{1.10}$$

then

$$\iota_K \bar{A}_{\alpha\beta} + \iota_K \bar{A}_{\beta\gamma} - \iota_K \bar{A}_{\alpha\gamma} = \tau(d\ln C_{\alpha\beta\gamma}) \in 2\pi i \mathbb{Z} \tag{1.11}$$

on each connected component of $LU_{\alpha\beta\gamma}$. By (1.11), if x_0 is a fixed loop in $U_{\alpha\beta\gamma}$ and x is any loop in $U_{\alpha\beta\gamma}$, then

$$e^{\int_{x_0}^x (\bar{A}_{\alpha\beta} + \bar{A}_{\beta\gamma} - \bar{A}_{\alpha\gamma})} \tag{1.12}$$

does not depend on the choice of paths from x_0 to x in $LU_{\alpha\beta\gamma}$. By (1.10), it is not hard to see that $\int_{x_0}^x (\bar{A}_{\alpha\beta} + \bar{A}_{\beta\gamma} - \bar{A}_{\alpha\gamma})$ is pure imaginary. Then we further have

$$\bar{A}_{\alpha\beta} + \bar{A}_{\beta\gamma} - \bar{A}_{\alpha\gamma} = d\ln e^{\int_{x_0}^x (\bar{A}_{\alpha\beta} + \bar{A}_{\beta\gamma} - \bar{A}_{\alpha\gamma})}, \tag{1.13}$$

where $e^{\int_{x_0}^x (\bar{A}_{\alpha\beta} + \bar{A}_{\beta\gamma} - \bar{A}_{\alpha\gamma})}$ is an $U(1)$-valued function on $LU_{\alpha\beta\gamma}$. Therefore $(\{LU_\alpha\}, \bar{H}, \bar{B}_\alpha, \bar{A}_{\alpha\beta})$ is a gerbe on LM.

It is obvious that $\bar{H}, \bar{B}_\alpha, \bar{A}_{\alpha\beta}$ are all $\mathbb{T}$-invariant. Combining (1.11), we see that the gerbe $(\{LU_\alpha\}, \bar{H}, \bar{B}_\alpha, \bar{A}_{\alpha\beta})$ is a weak $\mathbb{T}$-invariant gerbe on LM.

As under the Brylinski sections, the local connection 1-form of $\left(\mathcal{L}^B, \nabla^{\mathcal{L}^B}\right)$ is

$$-\tau(B_\alpha) = -\iota_K \bar{B}_\alpha,$$

and the transition function of $\mathcal{L}^B$ is

$$e^{-\int_0^1 \iota_K A_{\alpha\beta}} = e^{-\iota_K \bar{A}_{\alpha\beta}},$$

we see that $\left(\mathcal{L}^B, \nabla^{\mathcal{L}^B}\right)$ and $(\{LU_\alpha\}, \bar{H}, \bar{B}_\alpha, \bar{A}_{\alpha\beta})$ are coupled on LM.

Example 2. In [5, 6], T-duality in a background flux has the following settings. There is a principal circle bundle $\mathbb{T} \to Z \xrightarrow{\pi} X$ with a $\mathbb{T}$-invariant connection Θ and a background $\mathbb{T}$-invariant flux H, which is a $\mathbb{T}$-invariant closed 3-form on Z. Let $\{U_\alpha\}$ be a good cover of X. The cover $\{\pi^{-1}(U_\alpha)\}$ makes Z a good $\mathbb{T}$-manifold.

The T-dual circle bundle $\hat{\mathbb{T}} \to \hat{Z} \xrightarrow{\hat{\pi}} X$ with a $\mathbb{T}$-invariant connection $\hat{\Theta}$ and a background $\hat{\mathbb{T}}$-invariant flux $\hat{H}$. The cover $\{\hat{\pi}^{-1}(U_\alpha)\}$ makes $\hat{Z}$ a good $\mathbb{T}$-manifold.

Denote $v, \hat{v}$ the Killing vector field on $Z, \hat{Z}$ respectively. The gerbe $(\{\pi^{-1}(U_\alpha)\}, H, B_\alpha, A_{\alpha\beta})$ on Z and the gerbe $(\{\hat{\pi}^{-1}(U_\alpha)\}, \hat{H}, \hat{B}_\alpha, \hat{A}_{\alpha\beta})$ be the gerbe on $\hat{Z}$ satisfy the following relations

$$e^{-\iota_v A_{\alpha\beta}} = \hat{g}_{\alpha\beta},\ -\iota_v B_\alpha = \hat{\eta}_\alpha,\ \iota_v H = F^{\hat{\Theta}} \tag{1.14}$$

and

$$e^{-\iota_v \hat{A}_{\alpha\beta}} = g_{\alpha\beta},\ -\iota_{\hat{v}} \hat{B}_\alpha = \eta_\alpha,\ \iota_{\hat{v}} \hat{H} = F^{\Theta}, \tag{1.15}$$

where $\hat{g}_{\alpha\beta}$ is the transition functions of the bundle $\hat{Z}$, $\hat{\eta}_\alpha$ is the local connection 1-form of $\hat{\Theta}$ on U_α, $F^{\hat{\Theta}}$ is the curvature 2-form of $\hat{\Theta}$ on X and the similar meaning for the notations without hats on the dual side.

In the setting, $B_\alpha, A_{\alpha\beta}$ are all chosen to be $\mathbb{T}$-invariant. Moreover as $e^{-\iota_v A_{\alpha\beta}} = \hat{g}_{\alpha\beta}$, we conclude that $\iota_v A_{\alpha\beta} + \iota_v A_{\alpha\beta} - \iota_v A_{\alpha\gamma}$ takes values in $2\pi i \cdot \mathbb{Z}$ on each $U_{\alpha\beta\gamma}$. Therefore $(\{\pi^{-1}(U_\alpha)\}, H, B_\alpha, A_{\alpha\beta})$ is a weak $\mathbb{T}$-invariant gerbe on Z. Similarly $(\{\hat{\pi}^{-1}(U_\alpha)\}, \hat{H}, \hat{B}_\alpha, \hat{A}_{\alpha\beta})$ is a weak $\hat{\mathbb{T}}$-invariant gerbe on $\hat{Z}$.

$(\hat{Z}, \hat{\Theta})$ and the standard representation of circle on complex plane give rise to a complex line bundle with connection $(\hat{\xi}, \nabla^{\hat{\xi}})$ on X. Dually, there is a similar (ξ, ∇^{ξ}) on X coming from (Z, Θ). As

$$e^{-\iota_v A_{\alpha\beta}} = \hat{g}_{\alpha\beta}, -\iota_v B_\alpha = \hat{\eta}_\alpha,$$

the $\mathbb{T}$-equivariant line bundle $(\pi^*\hat{\xi}, \pi^*\nabla^{\hat{\xi}})$ and the weal $\mathbb{T}$-invariant gerbe $(\{\pi^{-1}(U_\alpha)\}, H, B_\alpha, A_{\alpha\beta})$ are coupled on Z. Dually, the $\hat{\mathbb{T}}$-equivariant line bundle $(\hat{\pi}^*\xi, \hat{\pi}^*\nabla^{\xi})$ and the $\hat{\mathbb{T}}$-invariant gerbe $(\{\hat{\pi}^{-1}(U_\alpha)\}, \hat{H}, \hat{B}_\alpha, \hat{A}_{\alpha\beta})$ are coupled on $\hat{Z}$.

2. Exotic twisted equivariant cohomology and $U(1)$-bundles

Let M be a good $\mathbb{T}$-manifold, i.e. M has an open cover $\{U_\alpha\}$ such that all finite intersections $U_{\alpha_0\alpha_1\cdots\alpha_p} = U_{\alpha_0} \cap U_{\alpha_1} \cdots U_{\alpha_p}$ are $\mathbb{T}$-invariant.

Let K be the Killing vector field of the $\mathbb{T}$-action. Denote by L_K^ξ, ι_K the Lie derivative and contraction along the direction K respectively.

Let $\xi \to M$ be a $\mathbb{T}$-equivariant Hermitian line bundle over M equipped with a $\mathbb{T}$-invariant Hermitian connection ∇^ξ. Let $H \in \Omega^3_{cl}(M)$ be a $\mathbb{T}$-invariant closed 3-form (see [3] as a general reference for differential forms) such that the equivariant super connection $\nabla^\xi - u\iota_K + u^{-1}H$ is equivariantly flat, i.e.

$$(\nabla^\xi - u\iota_K + u^{-1}H)^2 + uL_K^\xi = 0, \tag{2.1}$$

where u is a degree 2 indeterminate. For relevant references to equivariant differential forms, see [13, 1].

In the previous section, we have seen examples that satisfy these settings.

Let $\pi : S\xi \to M$ be the principal $U(1)$-bundle of ξ. Let v be the vertical tangent vector field on $S\xi$, i.e. the Killing vector field of the $U(1)$-action.

It is clear that $S\xi$ also admits the induced $\mathbb{T}$-action. As the action of $\mathbb{T}$ on the fibers of ξ is linear, i.e. $g(\lambda \cdot v) = \lambda \cdot g(v), \forall g \in \mathbb{T}, \lambda \in U(1)$, one deduces that the $\mathbb{T}$-action and the $U(1)$-action commute. Therefore we have

$$[K, v] = 0. \tag{2.2}$$

The condition $(\nabla^\xi - u\iota_K + u^{-1}H)^2 + uL_K^\xi = 0$ is equivalent to the following three equalities,

$$\begin{cases} \mu_K^\xi = L_K^\xi - [\nabla^\xi, \iota_K] = L_K^\xi - \nabla_K^\xi = 0 \\ (\nabla^\xi)^2 - \iota_K H = 0 \\ dH = 0 \end{cases} \tag{2.3}$$

Let Θ be the connection 1-form on $S\xi$ for (ξ, ∇^ξ).

Lemma 2.1.

$$\iota_K\Theta = 0, \ \ L_K\Theta = 0 \tag{2.4}$$

and

$$d\Theta = \iota_K \pi^* H. \tag{2.5}$$

Proof. Let $\{U_\alpha\}$ be a $\mathbb{T}$-cover of M. Choose a $\mathbb{T}$-invariant local basis s_α of ξ on U_α. Let η_α be the connection 1-form corresponding to s_α. By the first relation in (2.3), we have

$$0 = \mu_K^\xi(s_\alpha) = (L_K^\xi - [\nabla^\xi, \iota_K])(s_\alpha) = (\iota_K\eta_\alpha) \otimes s_\alpha,$$

and therefore we have

$$\iota_K \eta_\alpha = 0. \tag{2.6}$$

As s_α is $\mathbb{T}$-invariant, we get a local $\mathbb{T}$-equivariant diffeomorphism $\phi_\alpha : U_\alpha \times S^1 \to \pi^{-1}(U_\alpha)$ such that on the left hand side, $\mathbb{T}$ only acts on U_α. Then as $\phi_\alpha^*(\Theta)|_{U_\alpha \times S^1} = \eta_\alpha + d\theta$, we deduce that

$$\iota_K \Theta = 0,\ \ L_K \Theta = 0.$$

By the second relation in (2.3), we get

$$d\Theta + \frac{1}{2}\Theta^2 - \iota_K \pi^* H = 0$$

or

$$d\Theta = \iota_K \pi^* H.$$

□

Consider the $C^\infty(M)$-module

$$\widetilde{\Omega}^*(S\xi) := \{\omega \in \Omega^*(S\xi) | \iota_v \omega = 0, L_v \omega = -\omega\}. \tag{2.7}$$

Theorem 2.2.

$$\left(\widetilde{\Omega}^*(S\xi)^{\mathbb{T}}[[u, u^{-1}]], d - \iota_V - u\iota_K + \Theta + u^{-1}\pi^* H\right)$$

is a chain complex.

Proof. We need to show that:
(i) if $\omega \in \widetilde{\Omega}^*(S\xi)^{\mathbb{T}}$, then

$$(d - \iota_v - u\iota_K + \Theta + u^{-1}\pi^* H)\omega \in \widetilde{\Omega}^*(S\xi)^{\mathbb{T}};$$

(ii)

$$(d - \iota_v - u\iota_K + \Theta + u^{-1}\pi^* H)^2 + uL_K = 0.$$

(i) holds as we have following three equalities,

$$\begin{aligned} &[d - \iota_v - u\iota_K + \Theta + u^{-1}\pi^* H, \iota_v] \\ =&L_v - [\iota_v, \iota_v] - u\iota_{[K,v]} + \iota_v \Theta + u^{-1}\iota_v(\pi^* H) \\ =&L_v + \iota_v \Theta \\ =&0 \text{ on } \widetilde{\Omega}^*(S\xi); \end{aligned} \tag{2.8}$$

$$\begin{aligned} &[d - \iota_v - u\iota_K + \Theta + u^{-1}\pi^* H, L_v] \\ =&[d, L_v] - \iota_{[v,v]} - u\iota_{[K,v]} + L_v \Theta + u^{-1} L_v(\pi^* H) \\ =&0; \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} &[d - \iota_v - u\iota_K + \Theta + u^{-1}\pi^* H, L_K] \\ =&[d, L_K] - \iota_{[v,K]} - u\iota_{[K,K]} + L_K \Theta + u^{-1} L_K(\pi^* H) \\ =&0. \end{aligned} \tag{2.10}$$

To show (ii), we have

$$
\begin{aligned}
&(d-\iota_v-u\iota_K+\Theta+u^{-1}\pi^*H)^2\\
=&(d-\iota_v-u\iota_K)^2+(d-\iota_v-u\iota_K)(\Theta+u^{-1}\pi^*H)+(\Theta+u^{-1}\pi^*H)^2\\
=&-L_v-uL_K+d\Theta-\iota_v\Theta-\pi^*\iota_K H\\
=&(-L_v-\iota_v\Theta)+(d\Theta-\iota_K\pi^*H)-uL_K\\
=&-uL_K \text{ on } \widetilde{\Omega}^*(S\xi).
\end{aligned} \tag{2.11}
$$

□

Let $\pi^*\xi$ be the pull back bundle of ξ on $S\xi$. Clearly this is a trivial bundle which has a canonical global nowhere vanishing section

$$\gamma:(x,y)\to y,\quad x\in M, y\in\pi^{-1}(x).$$

Consider the map

$$f:\Omega^*(M,\xi)\to\Omega^*(S\xi),\quad \omega\mapsto\gamma^{-1}\cdot\pi^*\omega. \tag{2.12}$$

It is not hard to see that

$$\mathrm{Im}(f)=\widetilde{\Omega}^*(S\xi),\ker(f)=\{0\}.$$

We therefore get an isomorphism of $C^\infty(M)$-modules:

$$f:\Omega^*(M,\xi)\to\widetilde{\Omega}^*(S\xi). \tag{2.13}$$

Since γ is a $\mathbb{T}$-invariant global section of $\pi^*\xi$, we see that f sends $\mathbb{T}$-invariant invariant parts to $\mathbb{T}$-invariant invariant parts. Hence we get an isomorphism of $C^\infty(M)$-modules, which we still denote by f:

$$f:\Omega^*(M,\xi)^{\mathbb{T}}\to\widetilde{\Omega}^*(S\xi)^{\mathbb{T}}. \tag{2.14}$$

Theorem 2.3.
(2.15)

$$f:\left(\Omega^*(M,\xi)^{\mathbb{T}}[[u,u^{-1}]],\nabla^\xi-u\iota_K+u^{-1}H\right)\to\left(\widetilde{\Omega}^*(S\xi)^{\mathbb{T}}[[u,u^{-1}]],d-\iota_v-u\iota_K+\Theta+u^{-1}\pi^*H\right)$$

is a chain map and therefore induces an isomorphism on cohomology

$$f_*:\ h^*_{\mathbb{T}}(M,\nabla^\xi:H)\to H^*\left(\widetilde{\Omega}^*(S\xi)^{\mathbb{T}}[[u,u^{-1}]],d-\iota_v-u\iota_K+\Theta+u^{-1}\pi^*H\right), \tag{2.16}$$

where $h^*_{\mathbb{T}}(M,\nabla^\xi:H)$ *is the* **completed periodic exotic twisted $\mathbb{T}$-equivariant cohomology** [9].

Proof. Let $\omega\in\Omega^*(M,\xi)^{\mathbb{T}}[[u,u^{-1}]]$. We have

$$
\begin{aligned}
&(d-\iota_v-u\iota_K+\Theta+u^{-1}\pi^*H)(f(\omega))\\
=&(d-\iota_v-u\iota_K+\Theta+u^{-1}\pi^*H)(\gamma^{-1}\cdot\pi^*\omega)\\
=&(d-u\iota_K+\Theta)(\gamma^{-1}\cdot\pi^*\omega)+u^{-1}\pi^*H(\gamma^{-1}\cdot\pi^*\omega).
\end{aligned} \tag{2.17}
$$

Let $\{U_\alpha\}$ be an $\mathbb{T}$-cover of M. Let s_α be $\mathbb{T}$-invariant local basis of the ξ on U_α. Suppose $\omega|_{U\alpha}=\omega_\alpha\otimes s_\alpha$.

Then

$$d(\gamma^{-1}\cdot\pi^*\omega)$$
$$=d(\pi^*\omega_\alpha\cdot(\gamma^{-1}\cdot\pi^*s_\alpha)) \tag{2.18}$$
$$=\pi^*(d\omega_\alpha)(\gamma^{-1}\cdot\pi^*s_\alpha)-\pi^*(\omega_\alpha)d(\gamma^{-1}\cdot\pi^*s_\alpha).$$

Therefore locally, we have

$$d(\gamma^{-1}\cdot\pi^*\omega)+\Theta(\gamma^{-1}\cdot\pi^*\omega)$$
$$=\pi^*(d\omega_\alpha)(\gamma^{-1}\cdot\pi^*s_\alpha)-\pi^*(\omega_\alpha)d(\gamma^{-1}\cdot\pi^*s_\alpha)+\Theta(\pi^*\omega_\alpha)(\gamma^{-1}\cdot\pi^*s_\alpha) \tag{2.19}$$
$$=\pi^*(d\omega_\alpha)(\gamma^{-1}\cdot\pi^*s_\alpha)-\pi^*(\omega_\alpha)(\gamma^{-1}\cdot\pi^*\omega)[\Theta-(\gamma^{-1}\cdot\pi^*\omega)^{-1}d(\gamma^{-1}\cdot\pi^*s_\alpha)]$$
$$=[\pi^*(d\omega_\alpha)-\pi^*(\omega_\alpha)\eta_\alpha](\gamma^{-1}\cdot\pi^*s_\alpha),$$

where $\eta_\alpha=\Theta-(\gamma^{-1}\cdot\pi^*\omega)^{-1}d(\gamma^{-1}\cdot\pi^*s_\alpha)$ is connection one form for the basis s_α of the connection ∇^ξ on U_α.

Moreover, we have

$$\iota_K(\gamma^{-1}\cdot\pi^*\omega)$$
$$=\iota_K(\pi^*(\omega_\alpha)(\gamma^{-1}\cdot\pi^*s_\alpha)) \tag{2.20}$$
$$=\iota_K(\pi^*(\omega_\alpha))(\gamma^{-1}\cdot\pi^*s_\alpha).$$

Therefore,

$$[d(\gamma^{-1}\cdot\pi^*\omega)+\Theta(\gamma^{-1}\cdot\pi^*\omega)+\iota_K(\gamma^{-1}\cdot\pi^*\omega)]|_{U_\alpha}$$
$$=\pi^*(d\omega_\alpha+\omega_\alpha\eta_\alpha-u\iota_K\omega_\alpha)(\gamma^{-1}\cdot\pi^*s_\alpha)$$
$$=\gamma^{-1}\pi^*[(d\omega_\alpha+\omega_\alpha\eta_\alpha-u\iota_K\omega_\alpha)\otimes s_\alpha] \tag{2.21}$$
$$=\gamma^{-1}\pi^*[(\nabla^\xi-u\iota_K)\omega]|_{U_\alpha}$$
$$=f((\nabla^\xi-u\iota_K)\omega)|_{U_\alpha}.$$

And so we have

$$(d-\iota_v-u\iota_K+\Theta+u^{-1}\pi^*H)(f(\omega))=f((\nabla^\xi-u\iota_K+u^{-1}H)\omega). \tag{2.22}$$

□

3. Exotic twisted equivariant K-theory and the Chern character

3.1. **Gerbe modules and twisted K-theories.** A geometric realization of the gerbe $\mathcal{G}=(\{U_\alpha\},H,B_\alpha,A_{\alpha\beta})$ is $\{(L_{\alpha\beta},\nabla^L_{\alpha\beta})\}$, a collection of trivial line bundles $L_{\alpha\beta}\to U_{\alpha\beta}$ such that there are isomorphisms $L_{\alpha\beta}\otimes L_{\beta\gamma}\cong L_{\alpha\gamma}$ on $U_{\alpha\beta\gamma}$ and collection of connections $\{\nabla^L_{\alpha\beta}\}$ such that $\nabla^L_{\alpha\beta}=d+A_{\alpha\beta}$. Note that as here we are using slightly more general version of gerbe (see Definition 1.1 and Remark 1.2), isomorphisms $L_{\alpha\beta}\otimes L_{\beta\gamma}\cong L_{\alpha\gamma}$ are not uniquely fixed, but may differ by a multiplication by a constant $U(1)$-valued scalar. Then we have

$$(\nabla^L_{\alpha\beta})^2=F^L_{\alpha\beta}=B_\beta-B_\alpha. \tag{3.1}$$

Let $E = \{E_\alpha\}$ be a collection of (infinite dimensional) Hilbert bundles $E_\alpha \to U_\alpha$ whose structure group is reduced to $U_{\mathfrak{J}}$, which are unitary operators on the model Hilbert space $\mathfrak{H}$ of the form identity + trace class operator. Here $\mathfrak{J}$ denotes the Lie algebra of trace class operators on $\mathfrak{H}$. In addition, assume that on the overlaps $U_{\alpha\beta}$ that there are isomorphisms

$$\phi_{\alpha\beta} : L_{\alpha\beta} \otimes E_\beta \cong E_\alpha, \tag{3.2}$$

which are consistently defined on triple overlaps because of the gerbe property. Then $\{E_\alpha\}$ is said to be a *gerbe module* for the gerbe $\{L_{\alpha\beta}\}$. A *gerbe module connection* ∇^E is a collection of connections $\{\nabla^E_\alpha\}$ is of the form $\nabla^E_\alpha = d + A^E_\alpha$ where $A^E_\alpha \in \Omega^1(U_\alpha) \otimes \mathfrak{J}$ whose curvature F^E_α on the overlaps $U_{\alpha\beta}$ satisfies

$$\phi^{-1}_{\alpha\beta}(F^{E_\alpha})\phi_{\alpha\beta} = F^{L_{\alpha\beta}} I + F^{E_\beta} \tag{3.3}$$

Using equation (3.1), this becomes

$$\phi^{-1}_{\alpha\beta}(B_\alpha I + F^E_\alpha)\phi_{\alpha\beta} = B_\beta I + F^E_\beta. \tag{3.4}$$

It follows that $\exp(B)\,\mathrm{Tr}\left(\exp(F^E) - I\right)$ is a globally well defined differential form on Z of even degree. Notice that $\mathrm{Tr}(I) = \infty$ which is why we need to consider the subtraction.

Let $E = \{E_\alpha\}$ and $E' = \{E'_\alpha\}$ be a gerbe modules for the gerbe $\{L_{\alpha\beta}\}$. Then an element of twisted K-theory $K^0(Z, \mathcal{G})$ is represented by the pair (E, E'), see [4]. Two such pairs (E, E') and (G, G') are equivalent if $E \oplus G' \oplus K \cong E' \oplus G \oplus K$ as gerbe modules for some gerbe module K for the gerbe $\{L_{\alpha\beta}\}$. We can assume without loss of generality that these gerbe modules E, E' are modeled on the same Hilbert space $\mathfrak{H}$, after a choice of isomorphism if necessary.

Suppose that $\nabla^E, \nabla^{E'}$ are gerbe module connections on the gerbe modules E, E' respectively. Then we can define the *twisted Chern character* as

$$Ch_H : K^0(Z, \mathcal{G}) \to H^{even}(Z, H)$$
$$Ch_H(E, E') = \exp(-B)\,\mathrm{Tr}\left(\exp(-F^E) - \exp(-F^{E'})\right)$$

That this is a well defined homomorphism is explained in [4, 14]. To define the twisted Chern character landing in $(\Omega^\bullet(Z)[[u, u^{-1}]])_{(d+u^{-1}H)-cl}$, simply replace the above formula by

$$Ch_H(E, E') = \exp(-u^{-1}B)\,\mathrm{Tr}\left(\exp(-u^{-1}F^E) - \exp(-u^{-1}F^{E'})\right).$$

The above theory can be extended to equivariant setting with a compact group action on all the data [14].

3.2. **Exotic twisted equivariant K-theory.** Let M be a good $\mathbb{T}$-manifold with an $\mathbb{T}$-invariant cover $\{U_\alpha\}$.

Let $\xi \to M$ be a $\mathbb{T}$-equivariant Hermitian line bundle over M equipped with a $\mathbb{T}$-invariant Hermitian connection ∇^ξ. Let $\pi : S\xi \to M$ be the principal $U(1)$-bundle of ξ.

Let $\mathcal{G} = (\{U_\alpha\}, H, B_\alpha, A_{\alpha\beta})$ be a weak $\mathbb{T}$-invariant gerbe on M.

Assume that (ξ, ∇^ξ) and $(\{U_\alpha\}, H, B_\alpha, A_{\alpha\beta})$ are coupled on M.

Associated to these data $((\xi,\nabla^\xi);\mathcal{G})$, we will introduce a version of twisted K-theory and twisted Chern character in this section.

It is clear that the open cover $\{\pi^{-1}(U_\alpha)\}$ makes $S\xi$ a good $(\mathbb{T}\times U(1))$-manifold. Here to distinguish the two circle actions, we denote by $\mathbb{T}$ the circle acting on the base M and by $U(1)$ the circle acting on the fibers.

Denote $\mathcal{G}^\xi := (\{\pi^{-1}(U_\alpha)\}, \pi^*H, \pi^*B_\alpha, \pi^*A_{\alpha\beta})$, which is a $(\mathbb{T}\times U(1))$-invariant gerbe on $S\xi$. Let $\{(\hat{L}_{\alpha\beta}, \nabla^{\hat{L}_{\alpha\beta}} = d+\pi^*A_{\alpha\beta})\}$ be the system of $(\mathbb{T}\times U(1))$-line bundles with $(\mathbb{T}\times U(1))$-invariant connections on $U_{\alpha\beta}\times U(1)$ be the geometrization of the gerbe $\mathcal{G}^\xi$.

Let v be the vertical tangent vector field on $S\xi$, i.e. the Killing vector field of the $U(1)$-action. Let K be the Killing vector field of the $\mathbb{T}$-action. Let u be a degree 2 indeterminate.

Definition 3.1. *$E=\{E_\alpha,\nabla^{E_\alpha}\}$ is called a $(\mathbb{T}\times U(1))$-equivariant* **gerbe module with horizontal connection** *for the gerbe $\{\hat{L}_{\alpha\beta}\}$ if*
(a) the $(\mathbb{T}\times U(1))$-invariant connections ∇^{E_α}'s vanish on the vertical direction, i.e. $\nabla^{E_\alpha}_v\equiv 0$;
(b) there are $(\mathbb{T}\times U(1))$-equivariant isomorphisms

$$\phi_{\alpha\beta}:\hat{L}_{\alpha\beta}\otimes E_\beta\cong E_\alpha,$$

which respect the connections and are consistently defined on triple overlaps because of the gerbe property.

Let (E,E') and (G,G') be two pairs of $(\mathbb{T}\times U(1))$-equivariant gerbe modules with horizontal connections for the gerbe $\{\hat{L}_{\alpha\beta}\}$. We say they are equivalent, denoted by

$$(E,E')\sim(G,G')$$

if there exists some K, a $(\mathbb{T}\times U(1))$-equivariant gerbe modules with horizontal connection, such that

$$E\oplus G'\oplus K\cong E'\oplus G\oplus K$$

as $(\mathbb{T}\times U(1))$-equivariant gerbe modules with horizontal connections. Clearly this is an equivalence relation. As usual, we define

$$\hat{K}^0_{\mathbb{T}}(M,\nabla^\xi:\mathcal{G}) := \{(E,\nabla^E,E',\nabla^{E'})\}/\{\sim\}. \tag{3.5}$$

If the horizontal gerbe module connections are forgotten, one defines the **exotic twisted $\mathbb{T}$-equivariant K-theory** of the coupled pair $((\xi,\nabla^\xi),\mathcal{G})$, denoted as $K^0_{\mathbb{T}}(M,\nabla^\xi:\mathcal{G})$, by

$$K^0_{\mathbb{T}}(M,\nabla^\xi:\mathcal{G}) := \{(E,E')\}/\{\sim\}. \tag{3.6}$$

3.3. **Exotic twisted equivariant Chern Character.** Let $E=\{E_\alpha,\nabla^{E_\alpha}\}$ be a $(\mathbb{T}\times U(1))$-equivariant gerbe module with horizontal connection for the gerbe $\{\hat{L}_{\alpha\beta}\}$. For the equivariant curvatures along the direction $v+uK$, we have

$$\phi^{-1}_{\alpha\beta}(F^{E_\alpha}+\mu^{E_\alpha}_{v+uK})\phi_{\alpha\beta} = (F^{\hat{L}_{\alpha\beta}}+\mu^{\hat{L}_{\alpha\beta}}_{v+uK})I + (F^{E_\beta}+\mu^{E_\beta}_{v+uK}), \tag{3.7}$$

where μ stands for the moment. However

$$F^{\hat{L}_{\alpha\beta}} = \pi^*B_\beta-\pi^*B_\alpha, \tag{3.8}$$

$$(3.9)\qquad \mu_{v+uK}^{\hat{L}_{\alpha\beta}} = (\iota_v + u\iota_K)\pi^* A_{\alpha\beta} = u\iota_K\pi^* A_{\alpha\beta} = 2\pi i u\theta_\beta - 2\pi i u\theta_\alpha,$$

where θ_α (resp. θ_β) is the vertical coordinates of $\pi^{-1}(U_\alpha)$ (resp. $\pi^{-1}(U_\beta)$). So we have

$$(3.10)\qquad \phi_{\alpha\beta}^{-1}(F^{E_\alpha} + \mu_{v+uK}^{E_\alpha} + \pi^* B_\alpha + 2\pi i u\theta_\alpha)\phi_{\alpha\beta} = F^{E_\beta} + \mu_{v+uK}^{E_\beta} + \pi^* B_\beta + 2\pi i u\theta_\beta.$$

Let $E' = \{E'_\alpha\}$ be another $(\mathbb{T}\times U(1))$-equivariant gerbe module for the gerbe $\{\hat{L}_{\alpha\beta}\}$. Then $\exp(-u^{-1}\pi^* B_\alpha - 2\pi i\theta_\alpha)\,\mathrm{Tr}\left(\exp(-u^{-1}(F^{E_\alpha} + \mu_{v+uK}^{E_\alpha})) - \exp(-u^{-1}(F^{E'_\alpha} + \mu_{v+uK}^{E'_\alpha}))\right)$ can be glued together as a global differential form in $\Omega^*(S\xi)[[u,u^{-1}]]$. Simply denote this form by

$$(3.11)\qquad ch_{\nabla^\xi:\mathcal{G}}(\nabla^E,\nabla^{E'}) = \exp(-u^{-1}\pi^* B - 2\pi i\theta)\,\mathrm{Tr}\left(-\exp(u^{-1}(F^E + \mu_{v+uK}^E)) - \exp(-u^{-1}(F^{E'} + \mu_{v+uK}^{E'}))\right).$$

Theorem 3.2. *(i) The following equalities hold,*

$$(3.12)\qquad \iota_v ch_{\nabla^\xi:\mathcal{G}}(\nabla^E,\nabla^{E'}) = 0,\quad L_v ch_{\nabla^\xi:\mathcal{G}}(\nabla^E,\nabla^{E'}) = -ch_{\nabla^\xi:\mathcal{G}}(\nabla^E,\nabla^{E'}),$$

$$(3.13)\qquad (d - \iota_v - u\iota_K + \Theta + u^{-1}\pi^* H) ch_{\nabla^\xi:\mathcal{G}}(\nabla^E,\nabla^{E'}) = 0.$$

(ii) If $(\nabla_0^E,\nabla_0^{E'}),(\nabla_1^E,\nabla_1^{E'})$ are two horizontal gerbe module connections, then there exists $cs(\nabla_0^E,\nabla_0^{E'};\nabla_1^E,\nabla_1^{E'}) \in \widetilde{\Omega}^*(S\xi)[[u,u^{-1}]]$ such that

$$(3.14)\qquad ch_{\nabla^\xi:\mathcal{G}}(\nabla_1^E,\nabla_1^{E'}) - ch_{\nabla^\xi:\mathcal{G}}(\nabla_0^E,\nabla_0^{E'}) = (d - \iota_v - u\iota_K + \Theta + u^{-1}\pi^* H) cs(\nabla_0^E,\nabla_0^{E'};\nabla_1^E,\nabla_1^{E'}).$$

Proof. (i) Consider the local expression

$$ch_{\nabla^\xi:\mathcal{G}}(\nabla^E,\nabla^{E'})|_{\pi^{-1}(U_\alpha)}$$
$$= \exp(-u^{-1}\pi^* B_\alpha - 2\pi i\theta_\alpha)\,\mathrm{Tr}\left(\exp(-u^{-1}(F^{E_\alpha} + \mu_{v+uK}^{E_\alpha})) - \exp(-u^{-1}(F^{E'_\alpha} + \mu_{v+uK}^{E'_\alpha}))\right).$$

Obviously, $\iota_v\pi^* B_\alpha = 0$. On the other hand, as ∇^{E_α} is horizontal connection, we have $\nabla_v^{E_\alpha} = 0$, but this equivalent to

$$[\nabla^{E_\alpha}, \iota_v] = L_v.$$

Therefore

$$\iota_v(F^{E_\alpha}) = [\iota_v, (\nabla^{E_\alpha})^2] = (L_v - \nabla^{E_\alpha}\iota_v)\nabla^{E_\alpha} - \nabla^{E_\alpha}(L_v - \iota_v\nabla^{E_\alpha}) = [\nabla^{E_\alpha}, L_v] = 0,$$

as ∇^{E_α} is $\mathbb{T}\times U(1)$-invariant. Similarly, $\iota_v(F^{E'_\alpha}) = 0$. We therefore have

$$\iota_v ch_{\nabla^\xi:\mathcal{G}}(\nabla^E,\nabla^{E'})|_{\pi^{-1}(U_\alpha)} = 0.$$

As ∇^{E_α} is $\mathbb{T}\times U(1)$-invariant, clearly $L_v(F^{E_\alpha}) = 0$. The moment

$$\mu_{v+uK}^{E_\alpha} = L_{v+uK} - [\iota_{v+uK}, \nabla^{E_\alpha}].$$

Since $[v,K] = 0$, it is easy to see that

$$L_v\mu_{v+uK}^{E_\alpha} = 0.$$

Now $L_v\pi^* B_\alpha = 0$ and $L_v e^{-2\pi i\theta_\alpha} = -e^{2\pi i\theta_\alpha}$, we have

$$L_v ch_{\nabla^\xi:\mathcal{G}}(\nabla^E,\nabla^{E'})|_{\pi^{-1}(U_\alpha)} = -ch_{\nabla^\xi:\mathcal{G}}(\nabla^E,\nabla^{E'})|_{\pi^{-1}(U_\alpha)}.$$

At last, as (ξ, ∇^ξ) and $(\{U_\alpha\}, H, B_\alpha, A_{\alpha\beta})$ are coupled on M, one has

$$2\pi i\theta_\alpha - \pi^*\iota_K B_\alpha = \Theta|_{\pi^{-1}(U_\alpha)},$$

where Θ is the connection 1-form on $S\xi$. Hence

(3.15)

$$\begin{aligned}
&(d-\iota_v-u\iota_K)ch_{\nabla^\xi:\mathcal{G}}(\nabla^E,\nabla^{E'})|_{\pi^{-1}(U_\alpha)} = \\
=&(d-\iota_v-u\iota_K)\left[\exp(-u^{-1}\pi^*B_\alpha-2\pi i\theta_\alpha)\operatorname{Tr}\left(\exp(-u^{-1}(F^{E_\alpha}+\mu^{E_\alpha}_{v+uK}))-\exp(-u^{-1}(F^{E'_\alpha}+\mu^{E'_\alpha}_{v+uK}))\right)\right] \\
=&\left[(d-\iota_v-u\iota_K)\exp(-u^{-1}\pi^*B_\alpha-2\pi i\theta_\alpha)\right]\operatorname{Tr}\left(-\exp(u^{-1}(F^{E_\alpha}+\mu^{E_\alpha}_{v+uK}))-\exp(-u^{-1}(F^{E'_\alpha}+\mu^{E'_\alpha}_{v+uK}))\right) \\
=&\left[\exp(-u^{-1}\pi^*B_\alpha-2\pi i\theta_\alpha)(-u^{-1}\pi^*dB_\alpha-2\pi i d\theta_\alpha+\iota_K\pi^*B_\alpha)\right] \\
&\cdot\operatorname{Tr}\left(-\exp(u^{-1}(F^{E_\alpha}+\mu^{E_\alpha}_{v+uK}))-\exp(-u^{-1}(F^{E'_\alpha}+\mu^{E'_\alpha}_{v+uK}))\right) \\
=&\left[-u^{-1}\pi^*H-(2\pi i\theta_\alpha-\pi^*\iota_K B_\alpha))\right] \\
&\cdot\left[\exp(-u^{-1}\pi^*B_\alpha-2\pi i d\theta_\alpha)\operatorname{Tr}\left(-\exp(u^{-1}(F^{E_\alpha}+\mu^{E_\alpha}_{v+uK}))-\exp(-u^{-1}(F^{E'_\alpha}+\mu^{E'_\alpha}_{v+uK}))\right)\right] \\
=&\left(-u^{-1}\pi^*H-\Theta\right)ch_{\nabla^\xi:\mathcal{G}}(\nabla^E,\nabla^{E'})|_{\pi^{-1}(U_\alpha)},
\end{aligned}$$

and therefore

$$(d-\iota_v-u\iota_K+\Theta+u^{-1}\pi^*H)ch_{\nabla^\xi:\mathcal{G}}(\nabla^E,\nabla^{E'})|_{\pi^{-1}(U_\alpha)}=0.$$

(ii) Let

$$\nabla^E_t=(1-t)\nabla^E_0+t\nabla^E_1,\ \nabla^{E'}_t=(1-t)\nabla^{E'}_0+t\nabla^{E'}_1$$

and $F^E_t, F^{E'}_t, \mu^E_t, \mu^{E'}_t$ be the corresponding curvatures and momentums.

Let

$$A^{E_\alpha}=\nabla^{E_\alpha}_1-\nabla^{E_\alpha}_0,\ A^{E'\alpha}=\nabla^{E'\alpha}_1-\nabla^{E'\alpha}_0.$$

We have

$$\phi^{-1}_{\alpha\beta}(-u^{-1}(F^{E_\alpha}_t+\mu^{E_\alpha}_{v+uK,t})-u^{-1}\pi^*B_\alpha-2\pi i\theta_\alpha)\phi_{\alpha\beta}=-u^{-1}(F^{E_\beta}_t+\mu^{E_\beta}_{v+uK,t})-u^{-1}\pi^*B_\beta-2\pi i\theta_\beta$$

and

$$\phi^{-1}_{\alpha\beta}(-u^{-1}A^{E_\alpha})\phi_{\alpha\beta}=-u^{-1}A^{E_\beta}.$$

Similarly equalities hold for E'.

Therefore we have

(3.16)

$$\begin{aligned}
&\exp(-u^{-1}\pi^*B_\alpha-2\pi i\theta_\alpha) \\
&\cdot\int_0^1\operatorname{Tr}\left(-u^{-1}A^{E_\alpha}\exp(-u^{-1}(F^{E_\alpha}_t+\mu^{E_\alpha}_{v+uK,t}))+u^{-1}A^{E'_\alpha}\exp(-u^{-1}(F^{E'_\alpha}_t+\mu^{E'_\alpha}_{v+uK,t}))\right)dt
\end{aligned}$$

can be glued together as a global differential form in $\Omega^*(S\xi)[[u,u^{-1}]]$. Denote this form by $cs(\nabla^E_0,\nabla^{E'}_0;\nabla^E_1,\nabla^{E'}_1)$. Since $\iota_v A^{E_\alpha}=0, L_v A^{E_\alpha}=0$, similar to proof of (i), we have

$$\iota_v cs(\nabla^E_0,\nabla^{E'}_0;\nabla^E_1,\nabla^{E'}_1)=0,\ L_v cs(\nabla^E_0,\nabla^{E'}_0;\nabla^E_1,\nabla^{E'}_1)=-cs(\nabla^E_0,\nabla^{E'}_0;\nabla^E_1,\nabla^{E'}_1)$$

and therefore

$$cs(\nabla_0^E, \nabla_0^{E'}; \nabla_1^E, \nabla_1^{E'}) \in \widetilde{\Omega}^*(S\xi)[[u, u^{-1}]].$$

Moreover, by the standard Chern-Simons transgression, we have

(3.17)

$$(d - \iota_v - u\iota_K) \int_0^1 \mathrm{Tr}\left(-u^{-1}A^{E_\alpha}\exp(-u^{-1}(F_t^{E_\alpha} + \mu^{E_\alpha}_{v+uK,t})) + u^{-1}A^{E'_\alpha}\exp(-u^{-1}(F_t^{E'_\alpha} + \mu^{E'_\alpha}_{v+uK,t}))\right) dt$$
$$= \mathrm{Tr}\left(\exp(-u^{-1}(F_1^{E_\alpha} + \mu^{E_\alpha}_{v+uK,1})) - \exp(-u^{-1}(F_1^{E'_\alpha} + \mu^{E'_\alpha}_{v+uK,1}))\right)$$
$$- \mathrm{Tr}\left(\exp(-u^{-1}(F_0^{E_\alpha} + \mu^{E_\alpha}_{v+uK,0})) - \exp(-u^{-1}(F_0^{E'_\alpha} + \mu^{E'_\alpha}_{v+uK,0}))\right).$$

Then similar to (3.15), we see that

$$(d - \iota_v - u\iota_K + \Theta + u^{-1}\pi^*H)cs(\nabla_0^E, \nabla_0^{E'}; \nabla_1^E, \nabla_1^{E'}) = ch_{\nabla^\xi:\mathcal{G}}(\nabla_1^E, \nabla_1^{E'}) - ch_{\nabla^\xi:\mathcal{G}}(\nabla_0^E, \nabla_0^{E'}).$$

□

This theorem shows that $ch_{\nabla^\xi:\mathcal{G}}(\nabla^E, \nabla^{E'})$ is $(d - \iota_v - u\iota_K + \Theta + u^{-1}\pi^*H)$-closed in $\widetilde{\Omega}^*(S\xi)^{\mathbb{T}}[[u, u^{-1}]]$. Theorem 2.3 then tells us that $f^{-1}\left(ch_{\nabla^\xi:\mathcal{G}}(\nabla^E, \nabla^{E'})\right)$ is $(\nabla^\xi - u\iota_K + u^{-1}H)$-closed in $\Omega^*(M, \xi)^{\mathbb{T}}[[u, u^{-1}]]$.

We call

$$CS(\nabla_0^E, \nabla_0^{E'}; \nabla_1^E, \nabla_1^{E'}) := f^{-1}\left(cs(\nabla_0^E, \nabla_0^{E'}; \nabla_1^E, \nabla_1^{E'})\right) \in \Omega^*(M, \xi)[[u, u^{-1}]]$$

the **exotic twisted equivariant Chern-Simons transgression term**. By (3.14) and Theorem 2.3 (formula (2.22)), one has

(3.18) $$Ch_{\nabla^\xi:\mathcal{G}}(\nabla_1^E, \nabla_1^{E'}) - Ch_{\nabla^\xi:\mathcal{G}}(\nabla_0^E, \nabla_0^{E'}) = (\nabla^\xi - u\iota_K + u^{-1}H)CS(\nabla_0^E, \nabla_0^{E'}; \nabla_1^E, \nabla_1^{E'}).$$

We therefore can define the **exotic twisted equivariant Chern character**:

$$Ch_{\nabla^\xi:\mathcal{G}} : K^0_{\mathbb{T}}(M, \nabla^\xi : \mathcal{G}) \to h^*_{\mathbb{T}}(M, \nabla^\xi : H),$$
$$Ch_{\nabla^\xi:\mathcal{G}}(E, E') := \left[f^{-1}\left(ch_{\nabla^\xi:\mathcal{G}}(\nabla^E, \nabla^{E'})\right)\right].$$

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