Directly calculating the glue component of the nucleon in lattice QCD

QCDSF–UKQCD–CSSM Collaborations

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Abstract. We are investigating the direct determination and non-perturbative renormalisation of gluon matrix elements. Such quantities are sensitive to ultra–violet fluctuations, and are in general statistically noisy. To obtain statistically significant results, we extend an earlier application of the Feynman–Hellmann theorem to gluonic matrix elements to calculate a renormalisation factor in the $RI - MOM$ scheme, in the quenched case. This work demonstrates that the Feynman–Hellmann method is capable of providing a feasible option for calculating gluon quantities.

1 Introduction

In this talk we are interested in understanding how the momentum of a nucleon is distributed amongst its constituent particles. The topic of momentum distribution of the nucleon is an open area of investigation, with many other works in the area, including [1–7]. We define $\langle x \rangle_f$ as the total fraction of momentum carried by particle f, so $\langle x \rangle_g$ is the gluon momentum fraction, $\langle x \rangle_u$ the u–quark, and so on. We then expect the separate components to combine into the complete nucleon, so in terms of the momentum fractions,

$$\langle x \rangle_g + \sum_q \langle x \rangle_q = 1 \,. \tag{1}$$

Directly calculating the gluonic contribution to the nucleon momentum via standard 3–point function methods often leads to statistically noisy results, so an alternative method of calculation will be demonstrated here. The work shown here is a direct continuation of that presented in [8].

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Of particular concern is the direct calculation of the renormalisation factor for the gluon momentum fraction, Z_g. Methods to compute this factor can involve computing the corresponding quark renormalisation factor Z_q, and then enforcing the sum rule from eq. (1) in the continuous scheme to obtain Z_g. However, direct computation is also an option of interest to us, particularly through an $RI - MOM$ renormalisation scheme. In order to help deal with the issue of statistical noise, we implement a method based on the Feynman–Hellmann (FH) theorem, referred to as the FH method. From this, the $RI - MOM$ scheme outlined in [9] is used to obtain Z_g.

To begin, noting that we are working in Euclidean space as is typical in lattice calculations, we consider an operator of the form

$$O_{\mu\nu}^{(g)}(\mathbf{x}, \tau) = F_{\mu\alpha}^a(\mathbf{x}, \tau) F_{\nu\alpha}^a(\mathbf{x}, \tau), \tag{2}$$

where

$$F_{\mu\nu}^a(\mathbf{x}, \tau) = \partial_\mu A_\nu^a(\mathbf{x}, \tau) - \partial_\nu A_\mu^a(\mathbf{x}, \tau) - g f^{abc} A_\mu^b(\mathbf{x}, \tau) A_\nu^c(\mathbf{x}, \tau) \tag{3}$$

We also let $O(\tau) = \sum_{\mathbf{x}} O(\mathbf{x}, \tau)$, i.e. we are working at zero momentum transfer. This choice of operator reflects the decomposition of the energy momentum tensor outlined in [10]. We now select the components of the above operator to be analysed, choosing

$$O_g^{(b)} = O_{44}^{(g)} - \frac{1}{3} \sum_{i=1}^{3} O_{ii}^{(g)}, \tag{4}$$

$$O_g^{(b)}(\tau) = \frac{2}{3} \sum_i [\mathcal{E}_i^a(\tau)^2 - \mathcal{B}_i^a(\tau)^2], \tag{5}$$

and

$$\frac{\langle N(\mathbf{p})|O_g^{(b)}|N(\mathbf{p})\rangle}{\langle N(\mathbf{p})|N(\mathbf{p})\rangle} = \frac{1}{2E}\left(-2E^2 - \frac{2}{3}\mathbf{p}^2\right)\langle x\rangle_g, \tag{6}$$

with $\mathcal{E}_i^a(\tau) = F_{i4}^a(\tau)$ and $\mathcal{B}_i^a(\tau) = \frac{1}{2}\epsilon_{ijk}F_{jk}^a(\tau)$, the chromo–electric and chromo–magnetic fields. The label of (b) here aligns with the convention of previous work. The advantage of this operator lies in the fact that, as can be seen from eq. (6), the matrix element does not vanish where $\mathbf{p} = 0$, making calculations more convenient.

2 Feynman–Hellmann Method

The intention behind the FH method is to obtain matrix elements of operators without directly calculating 3–point correlation functions, which tend to suffer from significant statistical noise for gluonic quantities, as demonstrated in [1]. Instead, we introduce a modification to the action, proportionate to a parameter λ:

$$S \to S(\lambda) = S + \lambda \sum_\tau O(\tau). \tag{7}$$

Using this action, gluon 2–point correlation functions $\langle A(p)|A(p)\rangle_\lambda$ are calculated for various values of λ. From here, similar to the process outlined in [3], we have

$$\frac{\partial}{\partial\lambda}\langle A(p)|A(p)\rangle_\lambda\bigg|_{\lambda=0} = -\langle A(p)|: O :|A(p)\rangle. \tag{8}$$

Here : : subtracts the vacuum contributions.

Looking towards the gluonic sector, the usual statement for the Wilson action on the lattice is $\mathrm{Re}\,\mathrm{Tr}_c[1 - U_{\mu\nu}^{\mathrm{Plaq}}] = \frac{1}{4}a^4 g^2 F_{\mu\nu}^a(x)^2 + ...$, and so the corresponding chromo–electric and chromo–magnetic fields definitions are:

$$\frac{1}{2}\mathcal{E}^a(\tau)^2 = \frac{1}{3}\beta \sum_{\mathbf{x},i} \mathrm{Re}\,\mathrm{Tr}_c[1 - U_{i4}^{\mathrm{Plaq}}(\mathbf{x},\tau)]\,, \tag{9}$$

$$\frac{1}{2}\mathcal{B}^a(\tau)^2 = \frac{1}{3}\beta \sum_{\mathbf{x},(i<j)} \mathrm{Re}\,\mathrm{Tr}_c[1 - U_{i4}^{\mathrm{Plaq}}(\mathbf{x},\tau)]\,, \tag{10}$$

where $\beta = 6/g^2$. The modification we apply to the action in eq. (7) is chosen in such a way that the modification can be implemented through anisotropic parameters. We select $\frac{3}{4}O^{(b)}$ as the operator to insert, so we have that

$$\begin{aligned}
S(\lambda) &= \sum_\tau \frac{1}{2}[\mathcal{E}^a(\tau)^2 + \mathcal{B}^a(\tau)^2] + \frac{3}{4}\lambda \sum_\tau O^{(b)}(\tau) \\
&= \sum_\tau \frac{1}{2}[\mathcal{E}^a(\tau)^2 + \mathcal{B}^a(\tau)^2] - \lambda \sum_\tau \frac{1}{2}[-\mathcal{E}^a(\tau)^2 + \mathcal{B}^a(\tau)^2] \\
&= (1+\lambda)\frac{1}{2}\sum_\tau \mathcal{E}^a(\tau)^2 + (1-\lambda)\frac{1}{2}\sum_\tau \mathcal{B}^a(\tau)^2
\end{aligned} \tag{11}$$

where $O^{(b)}(\tau)$ is given in eq. (5). Further details of this process are outlined in [3].

3 Renormalisation Factors

As stated, the primary concern of this investigation is to directly determining the relevant renormalisation factors for the gluon momentum fraction. In general, we expect the quark and gluon sectors to mix when considering different renormalisation schemes. This mixing is denoted by

$$\begin{pmatrix}\langle x\rangle_g \\ \langle x\rangle_q\end{pmatrix}^{\mathrm{R}} = \begin{pmatrix}Z_{gg} & Z_{gq} \\ Z_{qg} & Z_{qq}\end{pmatrix}\begin{pmatrix}\langle x\rangle_g \\ \langle x\rangle_q\end{pmatrix}^{\mathrm{Lat}}\,. \tag{12}$$

We note here that we are only considering the quenched case, as in [1, 2], so we have that $Z_{qg} = 0$. Due to the sum rule in the renormalised scheme, we have

$$\langle x\rangle_g^{\mathrm{R}} + \sum_q \langle x\rangle_q^{\mathrm{R}} = Z_g\,\langle x\rangle_g^{\mathrm{Lat}} + Z_q \sum_q \langle x\rangle_q^{\mathrm{Lat}} = 1\,, \tag{13}$$

with Z_q and Z_g only coupling dependent and given by

$$Z_g = Z_{gg}, \quad \text{and} \quad Z_q = Z_{gq}^R + Z_{qq}^R\,, \tag{14}$$

where R denotes some renormalisation scheme.

To calculate the renormalisation factor Z_g, we first define the propagator and vertex function, $D_\lambda(p)$ and $\Gamma^{(b)}(p)$ respectively, by

$$D_\lambda(p) = \langle A(p)A(-p)\rangle_\lambda\,, \tag{15}$$

$$D_0(p)\Gamma^{(b)}(p)D_0(p) = \left\langle A(p)O^{(b)}A(-p)\right\rangle_0\,. \tag{16}$$

By applying the FH method as in eq. (8), we have that

$$\left\langle A(p)O^{(b)}A(-p)\right\rangle = -\frac{4}{3}\frac{\partial}{\partial\lambda}D_\lambda(p)\bigg|_{\lambda=0}. \tag{17}$$

The factor of $\frac{4}{3}$ here is due to the fact that the process outlined in eq. (11) inserts the operator $\frac{3}{4}O^{(b)}$, and so the shift with λ must be rescaled by this factor. We define renormalisation factors in terms of operators, so

$$A^{\rm R} = Z_3^{1/2}A, \quad \text{and} \quad O^{(b){\rm R}} = Z_g O^{(b)}, \tag{18}$$

$$\implies D_0^{\rm R} = Z_3 D_0, \quad \text{and} \quad \Gamma^{(b){\rm R}} = Z_g Z_3^{-1}\Gamma^{(b)}. \tag{19}$$

The renormalisation scheme being considered here is the $RI-MOM$ scheme as shown in [9], which matches lattice quantities to the respective tree level or Born terms. The tree level propagator $D_\lambda(p)$ for a given λ and Lorenz gauge fixing parameter ξ is given by

$$D_\lambda^{\rm Born}(p)_{\mu\nu}^{ab} = \delta^{ab}\left(\frac{a_{\mu\nu}}{p^2+\lambda(p_4^2-\mathbf{p}^2)} + \frac{b_{\mu\nu}}{(1+\lambda)p^2} + \xi\frac{c_{\mu\nu}}{p^2}\right), \tag{20}$$

for

$$a_{\mu\nu} = \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} - \frac{b_\mu b_\nu}{b^2}, \qquad b_{\mu\nu} = \frac{b_\mu b_\nu}{b^2}, \quad \text{and} \quad c_{\mu\nu} = \frac{p_\mu p_\nu}{p^2}, \tag{21}$$

where $b_\mu = (p_4\mathbf{p}, -\mathbf{p}^2)$. By using the form of $D_\lambda^{\rm Born}$ from eq. (20) to find both $\frac{\partial}{\partial\lambda}D_\lambda^{\rm Born}$ and $(D_0^{\rm Born})^{-1}$ for a given gauge parameter ξ, we can use the statements for $\Gamma^{(b)}(p)$ from eq. (16) and eq. (17) to find that

$$\Gamma^{(b){\rm Born}}(p)_{\mu\nu}^{ab} = \frac{4}{3}\delta^{ab}\left[a_{\mu\nu}(p_4^2-\mathbf{p}^2) + p^2 b_{\mu\nu}\right]. \tag{22}$$

From here we look to construct an equation for Z_g, using eq. (19), eq. (20) and eq. (22). We could begin with

$$-\frac{4}{3}\frac{\partial}{\partial\lambda}D_\lambda(p)\bigg|_{\lambda=0} = D_0^{\rm Born}(p)Z_g^{-1}\Gamma^{(b){\rm Born}}D_0(p), \tag{23}$$

and by taking the trace of both sides, we see that

$$Z_g = \left(\frac{4}{3}\frac{\partial}{\partial\lambda}\,{\rm Tr}(D_\lambda(p))\bigg|_{\lambda=0}\right)^{-1}{\rm Tr}\Big(D_0^{\rm Born}(p)\Gamma^{(b){\rm Born}}(p)D_0(p)\Big)\bigg|_{p^2=\mu^2}, \tag{24}$$

where Tr indicates contraction over all indices (${\rm Tr}\,X = X_{\mu\mu}^{aa}$).

We consider an alternative to eq. (24), for the following reason. Discretisation errors on the lattice are reduced by minimising each p_μ for $\mu = 1, ..., 4$, for a given p^2. Due to this, the most optimal points for examining lattice calculations are at the points where $p_4 = p_i$, i.e. along the momentum space diagonal. However, the trace of the Born vertex function in eq. (22) vanishes in this region, so ${\rm Tr}\,\Gamma^{(b){\rm Born}}(p, p, p, p) = 0$, and so it becomes difficult to examine Z_g here as the signal is suppressed. We may choose to examine a different region of the lattice where the trace of the vertex function does not vanish, or we may instead consider an alternative formula. Opting for the latter, we choose to multiply both sides of eq. (23) by $\Gamma^{(b){\rm Born}}$ before taking the trace of both sides, obtaining

$$Z_g = \left(\frac{4}{3}\frac{\partial}{\partial\lambda}\,{\rm Tr}\Big(D_\lambda(p)\Gamma^{(b){\rm Born}}(p)\Big)\bigg|_{\lambda=0}\right)^{-1}{\rm Tr}\Big(D_0^{\rm Born}(p)\Gamma^{(b){\rm Born}}(p)D_0(p)\Gamma^{(b){\rm Born}}(p)\Big)\bigg|_{p^2=\mu^2}. \tag{25}$$

As ${\rm Tr}\,\Gamma^2$ does not vanish in the region of interest, the desired signal is not suppressed and so this definition is more convenient for a numerical calculation.

4 Results

Calculations for Z_g were conducted on a $24^3 \times 48$ lattice on 1000 configurations, with $\beta = 6.0$ in the Laudau gauge ($\xi = 0$). Three values of the modified action parameter λ are considered, $\lambda = 0, \pm 0.0333$. As such, in total 3000 configurations were generated.
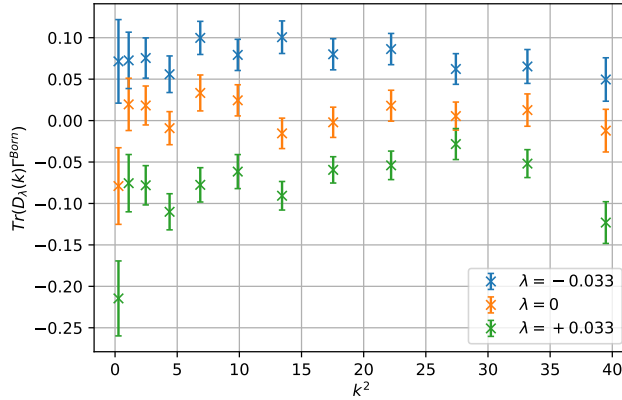

Figure 1. $\text{Tr}\left(D_\lambda(k)\Gamma^{(b)\text{Born}}\right)$ for $k_\mu = (k, k, k, k)/\sqrt{4}$, for 3 values of λ.

All quantities considered here are in units of the lattice spacing. For specificity, we denote momenta by k and p, where $k_\mu = 2\pi n_\mu/N_\mu$ for N_μ the lattice size along axis μ, and $p_\mu = 2\sin\left(k_\mu/2\right)$. To demonstrate the effect of the variation in the action by the λ dependent terms in eq. (11) figure 1 shows $\text{Tr}\left(D_\lambda(k)\Gamma^{(b)\text{Born}}\right)$ against k^2, for 3 values of λ. These points are taken from along the "4D diagonal", or where $k_\mu = (k, k, k, k)/\sqrt{4}$, in order to reduce discretisation effect due to finite lattice spacing. The relative shift in $\text{Tr}\left(D_\lambda(k)\Gamma^{(b)\text{Born}}\right)$ is then obtained, or $\text{Tr}\left(\frac{\partial}{\partial\lambda}D_\lambda(k)\Gamma^{(b)\text{Born}}\right)$ at each value of k^2, and applied to eq. (25) in order to determine Z_g. The result of this calculation is shown in figure 2. We apply a linear fit with
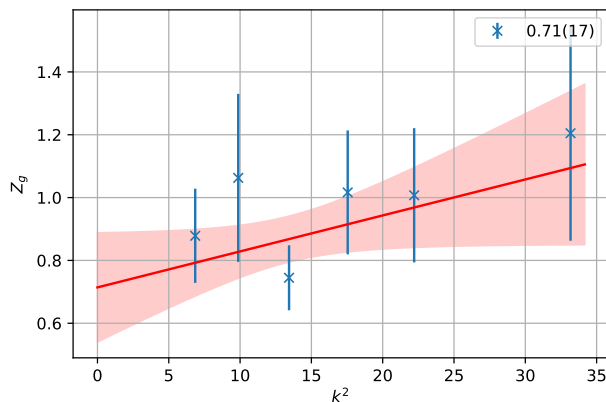

Figure 2. Calculations of Z_g using eq. (25) along $k_\mu = (k, k, k, k)/\sqrt{4}$.

non-zero gradient, $Z_g = Ak^2 + B$, in order to account for discretisation effects for different values of k^2. The shaded region indicates an uncertainty in the fit of 1 standard deviation.

The current analysis finds a value of $Z_g = 0.71(17)$. Further work to reduce statistical effects, e.g. by including more values of k_μ, is being conducted. In [11] a value of $Z_g = 0.748(20)$ is given at $\beta = 6.0$, obtained by considering anisotropic parameters c_σ and c_τ as in [12]. This agrees with the obtained result, though more work is required to improve this calculation.

5 Conclusion

We have demonstrated that the Feynman-Hellmann method provides an alternative to direct calculations of 3–point functions for the renormalisations of gluonic operators. This is achieved throught the generation of several moderately sized (1000 configurations) ensembles of field configurations, each with a differently modified action. Hence so long as ensemble generation is computationally less costly than calculating 3–point correlation functions, as it typically is in the quenched case, this method should provide a reduction in cost. Further analysis is required to determine precise cost to benefit ratios once dynamical quarks are introduced.

The method may be further used by examining the shift in the fermion propagator due to the change in action from eq. (7). This has the potential to provide a determination of Z_{gq}, the contribution to renormalisation from mixing between the quark and gluon sectors.

References

[1] M. Göckeler *et al.*, Nucl. Phys. Proc. Suppl. **53** (1997) 324, arXiv:hep-lat/9608017.

[2] H. B. Meyer *et al.*, Phys. Rev. D **77** (2008) 037501 [arXiv:0707.3225 [hep-lat]].

[3] R. Horsley *et al.* [QCDSF and UKQCD Collaborations], Phys. Lett. B **714** (2012) 312 [arXiv:1205.6410 [hep-lat]].

[4] C. Alexandrou *et al.*, Phys. Rev. D **96** (2017) 054503 [arXiv:1611.06901 [hep-lat]].

[5] Y. B. Yang *et al.* [χQCD Collaboration], Phys. Rev. Lett. **121** (2018) 212001 [arXiv:1808.08677 [hep-lat]].

[6] P. E. Shanahan *et al.*, Phys. Rev. D **99** (2019) 014511 [arXiv:1810.04626 [hep-lat]].

[7] M. Göckeler *et al.* [QCDSF Collaboration], Phys. Rev. D **71** (2005) 114511 [hep-ph/0410187].

[8] R. Horsley *et al.* [QCDSF-UKQCD-CSSM Collaborations] PoS(LATTICE2019)220 (2020) [arXiv:2001.07639 [hep-lat]].

[9] A. J. Chambers *et al.* [QCDSF Collaboration], Phys. Lett. B **740** (2015) 30 [arXiv:1410.3078 [hep-lat]].

[10] X. D. Ji, Phys. Rev. Lett. **74** (1995) 1071 [hep-ph/9410274].

[11] H. B. Meyer, Phys. Rev. D **76** (2007) 101701 [arXiv:0704.1801 [hep-lat]].

[12] J. Engels *et al.*, Nucl. Phys. B **564** (2000) 303 [hep-lat/9905002].

[13] T. R. Haar *et al.*, EPJ Web Conf. **175** (2018) 14011, arXiv:1711.03836 [hep-lat].

[14] R. G. Edwards *et al.*, Nucl. Phys. Proc. Suppl. **140** (2005) 832, arXiv:hep-lat/0409003.