Conformal group actions on Cahen-Wallach spaces

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Contents

Abstract

This thesis explores the conformal structure of Cahen-Wallach spaces, and the potential construction of compact conformal quotients of Cahen-Wallach spaces. Along the way, we prove novel results about cocompact group actions, and essential homotheties. We show that any cocompact, properly discontinuous, conformal action on a Cahen-Wallach space of imaginary type must be isometric. And we demonstrate that no cocompact, properly discontinuous, conformal action on a Cahen-Wallach space can centralize an essential transformation. These results are relevant in the study of the compact Lorentzian Lichnerowicz conjecture, as they limit possible counterexamples.

Signed Statement

I certify that this work contains no material which has been accepted for the award of any other degree or diploma in my name, in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text. In addition, I certify that no part of this work will, in the future, be used in a submission in my name, for any other degree or diploma in any university or other tertiary institution without the prior approval of the University of Adelaide and where applicable, any partner institution responsible for the joint-award of this degree.

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Dedication

To my parents

Every ounce of home, support, freedom, teaching, and love you have given me over the past 23 years culminates today in this document. Look how far you brought me.

Chapter 1

Introduction

1.1 Conformal transformations

We take all manifolds to be smooth, Hausdorff, and second countable.

Given a manifold with a semi-Riemannian metric g, isometries are the transformations that preserve the metric under pull-back, i.e. a map ϕ is an isometry if $\phi^* g = g$. These maps preserve rigid structures, not changing distances or angles. Conformal transformations are those that disregard the distance information, but still preserve angles. We describe this as rescaling the metric under pull-back, i.e. a map ϕ is conformal if $\phi^* g = e^{2f} g$ for some function f.

It is natural when talking about conformal transformations, to weaken our structure accordingly: instead of a fixed metric, we take a manifold with a conformal structure, which is a class of semi-Riemannian metrics that are rescalings of each other. In this way a conformal structure provides us with information about angles on the manifold, without giving us any information about distances.

The most obvious example of a conformal transformation are the scalings of Euclidean space: take $(\mathbb{R}^n, g)$, for g the standard scalar product, and consider $\phi(x) = e^s x$. Then the derivative of ϕ at x in direction v is $d\phi|_x(v) = e^s v$, and so $\phi^* g = e^{2s} g$. But this example is actually more specialized. Not only is it conformal, but its scaling is constant. We call such transformations homothetic.

A transformation being conformal for a metric is equivalent to being conformal for its conformal equivalence class. However, a transformation can only be said to be homothetic once a metric in the conformal class has been chosen. This is because a homothety will no longer in general be a homothety after rescaling a metric.

If one wants some examples of conformal transformations, a good place to look is complex analysis. Every holomorphic map of $\mathbb{C}$ is in fact a conformal transformation of $\mathbb{C}$, when thought of as $\mathbb{R}^2$ with the standard scalar product metric.

1.2 Essential conformal structures

Once we have the notion of a conformal transformation, a natural question to ask is

How many semi-Riemannian manifolds admit conformal transformations that are not isometries?

Of course from any example, we could modify it in some trivial way to obtain a 'new' metric (one that is isometric to the original example) also having such a transformation, so we should consider this question up to some equivalence. There are two natural equivalences of semi-Riemannian manifolds we could consider for this question: isometric equivalence classes, or conformal equivalence classes. Here we would like to consider the question up to conformal equivalence, but the question as stated is not well defined for a conformal structure. This is because, as noted in the previous section for homotheties, the property of being an isometry or not depends on the choice of metric in a conformal structure. So to fix this, the notion of essentiality is introduced: A conformal manifold is essential if every metric of the equivalence class has one of these non-isometric conformal transformations.

Definition 1.2.1. Let (M, c) be a conformal manifold. The conformal structure is *essential* if no metric g in c is preserved by the conformal group. In other words, there is no metric g such that every conformal transformation is an isometry of g.

Then the question becomes:

How many conformal manifolds – up to conformal diffeomorphism – have an essential conformal structure?

A conjecture relating to this was made by Lichnerowicz (1964)[1]. His interest was specifically in the compact Riemannian case.

[1] It is not clear if this source is the original conjecture. In it, he seems to make additional assumptions on the curvature that are not mentioned by articles on the subject.

Conjecture (Compact Riemannian Lichnerowicz conjecture). *Let M be a compact Riemannian manifold with an essential conformal structure. Then M is conformal to the sphere $\mathbb{S}^n$ with the standard round metric.*

Frances (2008) provides an excellent history of this conjecture and its generalizations. The summary is that Lichnerowicz was correct: in the Riemannian context, the sphere is the only compact essential structure, as proved separately by Ferrand (1971) and Obata (1971). The first generalization of the conjecture is that a non-compact Riemannian manifold with essential conformal structure is conformal to $\mathbb{R}^n$. Alekseevskii (1972) claimed a proof of this generalization, but a gap in his proof was later discovered. A corrected proof was then provided by Ferrand (1996). We summarize these results in

Theorem (Ferrand (1971), Obata (1971), Ferrand (1996)). *Let (M, g) be a Riemannian manifold of dimension $n \geq 2$. If M has an essential conformal structure, then M is conformal to the sphere $\mathbb{S}^n$ if M is compact, or $\mathbb{R}^n$ if M is not compact.*

The conjecture can be then generalized to semi-Riemannian manifolds, as described by Frances (2015). Then the conjecture becomes that Einstein's universes ($\mathbb{S}_p \times \mathbb{S}_q$ with the positive definite and negative definite standard metrics, respectively), and flat $\mathbb{R}^{p,q}$ are the only essential manifolds. Note that for non-compact spaces, this is immediately not true, as for example, Cahen-Wallach spaces are Lorentzian essential manifolds that are not conformal to $\mathbb{R}^{p,q}$, and in fact, are not even conformally flat. This conjecture is also false in the compact case, and Frances (2005) gave examples of compact manifolds with essential structure, that were different from Einstein's universes. The counterexamples provided by Frances were all conformally flat, which prompted a weaker version: The newest conjecture being that an essential structure must be conformally flat, and so they desired conformally curved counterexamples. Frances (2015) found such counterexamples in all non-Lorentzian signatures, but the question remains open in the Lorentzian signature.

Conjecture (Compact Lorentzian Lichnerowicz conjecture). *Let M be a compact Lorentzian manifold with an essential conformal structure. Then M is conformally flat.*

Frances constructed these examples by taking a strongly essential 1-parameter family of homotheties on a conformally curved locally symmetric semi-Riemannian manifold, then finding a group of homotheties acting properly discontinuously and cocompactly that centralize the essential homotheties. Then the essential homotheties are preserved and so descend to the quotient. Thus demonstrating that the quotient has essential structure.

It is worth noting that each individual homothety in the 1-parameter family Frances considered has a fixed point and thus is essential. This motivates the way we approach the conjecture in this thesis. We explore the possibility of applying this same technique, and so we consider essential homotheties of the conformally curved locally symmetric Cahen-Wallach spaces, and we attempt to find a group of homotheties acting properly discontinuously and cocompactly that centralize the essential homothety.

Cahen-Wallach spaces are non-compact, in general conformally curved, Lorentzian symmetric spaces. We investigate them specifically because, in being symmetric, they have a large isometry group, and thus a large conformal group with which to compactify them. Initially, we might like to study isometric quotients of Cahen-Wallach spaces, these already being classified by work of Kath & Olbrich (2019), however using a theorem of Cahen & Kerbrat (1982), we show that the conformal group of a conformally curved Cahen-Wallach space is equal to its homothetic group. So if we quotient by isometries, our quotient space cannot have any strict conformal transformations induced from the covering space, because they would be homothetic on the quotient, and compact spaces cannot have strict homotheties. So we are forced to consider quotienting by groups not contained within the isometries. So the motivating question is:

Can we find a compact conformal quotient of a Cahen-Wallach space that has essential structure?

1.3 Main results and thesis outline

In response to the question posed at the end of the previous section, we have two primary results, both giving a partial negative answer. They are found in Chapter 5.

In Section 5.2, Theorem 5.2.4, we conclude that for imaginary type Cahen-Wallach spaces, every homothetic quotient is an isometric quotient, thus, since compact spaces cannot have strict homotheties, no compact quotient of imaginary type can have any strict conformal transformations, let alone essential ones.

In Section 5.3, we explore the possibility of constructing a compact quotient with essential structure by selecting our group with which to quotient from the centralizer of an essential homothety. Doing so successfully would then induce the essential homothety on the quotient, thus giving it an essential structure. Theorem 5.3.5 is precisely the statement that this is impossible.

Now we describe the structure of the thesis, note in particular that every chapter features some original results. (The number in brackets represents the number corresponding to the theorem's appearance in the thesis.)

In Chapter 2, we give a background on group actions, describing several basic properties. In Section 2.2, we explore properly discontinuous actions. These are precisely the actions that enable the quotient space to have an induced manifold structure, and reference is provided for this fact. Then in Section 2.3, we explore the notion of a fundamental region with the goal of providing a necessary condition for cocompactness. Here, previous results exist in the case of metric spaces and isometric actions, e.g. Theorem 2.3.12, but it seems there is a lack of theory for more general actions, this motivates Section 2.4, in which we introduce a new notion for fundamental regions.

Definition (2.4.1)**.** Let G be a group acting on a topological space X. Let $R \subset X$ be a fundamental region for the action. Then R is *finitely self adjacent* if for some open neighbourhood U such that $\overline{R} \subset U$, the set $\{g \in G \mid gU \cap U \neq \emptyset\}$ is finite.

Then we prove the following novel result, this result is an analogue of Theorem 2.3.12, in the way that it provides a necessary condition of cocompactness of an action.

Theorem (2.4.2)**.** *Let G be a group acting by homeomorphisms on a locally compact topological space X. Let R be a finitely self adjacent fundamental region for this action. If X/G is compact, then $\overline{R}$ is compact.*

In Chapter 3, we give some background on conformal transformations. In Section 3.2, we present the reader with Proposition 3.2.1, a result of Cahen & Kerbrat (1982), and detail its proof. This result provides a strong restriction on the conformal group of a conformally curved space with parallel Weyl tensor, namely that it is equal to the homothetic group. In particular, this shows that the conformal group of a conformally curved locally symmetric space coincides with its homothetic group. It also has the consequence that for a compact conformally curved space with parallel Weyl tensor, every conformal transformation is actually an isometry.

In Section 3.3, we explore the connection between a homothety being essential, and having a fixed point. This begins with the observation of Corollary 3.3.5 that any homothety with a fixed point must be essential. For complete Riemannian manifolds, the converse of this statement is known to be true, since every strict homothety has a fixed point by the Banach fixed point theorem. However the converse is not so clear in the semi-Riemannian context, though many results are known, particularly in the case of a 1-parameter subgroup of strict homotheties. We give a small sampling of these in Section 3.4. Frances (2007) gives some overview here. Our methods of analysing the converse in the semi-Riemannian setting do not take this 1-parameter subgroup approach, and instead focus on individual transformations. In fact, in doing so, the notion we introduced in Chapter 2 makes a surprising reappearance. We prove

Theorem (3.3.6)**.** *Let ϕ be a strict homothety of a semi-Riemannian manifold. If the action of $\langle\phi\rangle$ admits a finitely self adjacent fundamental region, then ϕ is not essential.*

This leads us naturally to Conjecture 3.3.8, conjecturing that if a homothety has no points with finite orbit, the group action it generates must admit a finitely self adjacent fundamental region. If true, the consequence of this conjecture is that a strict homothety will be essential if and only if it has a point with finite orbit. We are unable to prove this conjecture in general, but we answer it positively in the case of Cahen-Wallach spaces. We define Cahen-Wallach spaces in Definition 4.0.1. These are the manifold $M = \mathbb{R}^{n+2}$, together with the metric

$$g := 2dx^+dx^- + S_{ij}x^ix^j(dx^+)^2 + \delta_{ij}dx^idx^j,$$

where $x^+, x^-, x^1, \dots, x^n$ are global coordinates on M, and where S is a symmetric non-degenerate matrix. Note that by Cahen & Wallach (1970), the isometry class of $CW_{n+2}(S)$ depends only on the eigenvalues of S. When all the eigenvalues of S are negative, we say the Cahen-Wallach space is of imaginary type. Similarly when all the eigenvalues of S are positive, we say the Cahen-Wallach space is of real type. When it is neither real nor imaginary type, the Cahen-Wallach space is of mixed type. Then in Section 4.1, we calculate their Riemannian and Weyl curvatures, in particular describing precisely which Cahen-Wallach spaces are conformally curved. In Section 4.2, we calculate the isometry, homothety, and conformal groups (for conformally curved Cahen-Wallach spaces). In particular, by Theorem 4.2.9 and Proposition 4.2.5, when $CW_{n+2}(S)$ is conformally curved (i.e. whenever S has at least two distinct eigenvalues), then

$$\mathrm{Conf}(CW_{n+2}(S)) \simeq H_n \rtimes (E(1) \times K \times \mathbb{R}),$$

where H_n is the Heisenberg group, $E(1)$ is $\mathbb{Z}_2 \ltimes \mathbb{R}$, the group of Euclidean transformations of $\mathbb{R}$, and K is the centralizer of S in $O(n)$. Note also that the copy of $\mathbb{R}$ inside $E(1)$ acts on the Heisenberg in such a way that $H_n \rtimes \mathbb{R}$ is isomorphic to the generalized oscillator group as in Section 2.1 of Kath & Olbrich (2019).

Then in Section 4.3 we give various sufficient conditions for a Cahen-Wallach homothety to have a fixed point, this being of interest not only because properly discontinuous group actions must be free, but also because homotheties with fixed points are the essential homotheties. Then in Section 4.4 we reach the proof that a strict homothety of Cahen-Wallach space is essential if and only if it has a fixed point.

Theorem (4.4.1)**.** *A strict homothety of a Cahen-Wallach space is essential if and only if it fixes a point.*

In Chapter 5, we address the core motivating question expressed in the previous section. In Section 5.1, we present a short, but essential lemma.

Lemma (5.1.1)**.** *A cyclic group of homotheties of a Cahen-Wallach space cannot act cocompactly.*

In Section 5.2, we prove

Theorem (5.2.4)**.** *A group of homotheties of a Cahen-Wallach space of imaginary type acting properly discontinuously and cocompactly must be contained within the isometries.*

In particular, as a result of this theorem, we observe that for a conformally curved Cahen-Wallach space of imaginary type, any compact quotient with a manifold structure cannot have any conformal transformations that are not isometries. In particular, any such quotient must have an inessential structure. This result, along with the difficulties encountered in non-imaginary type Cahen-Wallach space, are explored in Section 5.4, though in it we are able to construct a compact quotient manifold using strict homotheties acting on an open submanifold. However, this construction has not yielded an essential quotient space.

Finally, in Section 5.3, we prove our main theorem

Theorem (5.3.5)**.** *A group of conformal transformations of a conformally curved Cahen-Wallach space centralizing an essential conformal transformation cannot act properly discontinuously and cocompactly.*

This result is a general, though not quite complete, negative answer to the question posed at the end of the previous section for the case of Cahen-Wallach spaces.

1.4 Outlook

Theorem 5.3.5 leaves us with the following possible avenues for constructing a counterexample to the compact Lorentzian Lichnerowicz conjecture:

- In this thesis, we consider quotients of Cahen-Wallach spaces, but it is possible to consider instead quotients of their open submanifolds. If one could find an open submanifold U, and a group of conformal transformations of U, centralizing an essential transformation, and acting properly discontinuously and cocompactly, then this would construct a counterexample for the Lichnerowicz conjecture. Here it is still important to consider the group as not being contained within the isometries, since a compact isometric quotient of conformally curved submanifold of Cahen-Wallach space has no strict conformal transformation, by Proposition 3.2.1 and Proposition 3.1.10. This method is more faithful to the method used by Frances (2015) for higher signatures, and we explore it briefly in Example 5.4.4. The difficulty in applying this method is that the action of Frances' homotheties are able to scale in every dimension of the manifold, however this is not possible for homotheties of Cahen-Wallach space: it is impossible to scale in the x^+ dimension.

- Cahen-Wallach spaces are just one special type of conformally curved essential Lorentzian manfifold. In particular, they are the symmetric pp-waves. It remains to explore this more general class, and attempt to produce an essential compact quotient manifold from these. Alternatively, one could also consider manifolds that are not indecomposable, e.g. a product of a Cahen-Wallach space with a Riemannian symmetric space.

- We have considered the case that the quotienting group of homotheties preserve the essential homothety by centralizing it. The question remains unanswered of an essential homothety normalizing the quotienting group, as in Proposition 3.5.5. This carries the large additional difficulty of determining the normalizer of an arbitrary subgroup, rather than the centralizer of single elements.

- A single transformation is essential if it preserves no metric in the conformal class. This is a special case of an essential structure, because in general it might be that each transformation fixes some metric, but these fixed metrics differ across the group, so that the group as a whole does not preserve any of them. In this thesis, we have considered essential transformations preserved by a quotienting group, but we could instead consider a collection of transformations that together are essential, but individually are not. Although, this would be difficult, and likely unsuccessful, since no example of a manifold with an essential structure but without essential transformations is known.

- We consider essential transformations on the cover that descend to the quotient. This is justified by the fact that every transformation on the quotient lifts to a transformation on the cover by Proposition 3.5.7, however as noted after Theorem 3.5.10, it need not be the case that the covering transformation is essential if the transformation on the quotient is essential. So the possibility remains to consider transformations that are inessential, but that become essential on the quotient.

For imaginary type Cahen-Wallach spaces, Corollary 5.2.5 tells us that a compact quotient has no strict conformal transformations at all, and so in this case the third, fourth, and fifth avenues are closed.

The question of existence of a strict homothetic compact quotient is left unanswered in real and mixed type, since we have been unable to produce a strict homothetic quotient of Cahen-Wallach spaces of real or mixed type. It seems likely that it is impossible, and we explore this informally in Example 5.4.3, though we are unable to prove that this is impossible.

In an unrelated direction, we leave open Conjecture 3.3.8, which is the question of whether for general semi-Riemannian manifolds, a strict homothety ϕ with no finite orbit points (x such that $\phi^k(x) = x$ for some k) must admit a finitely self adjacent fundamental region. If proven to be true, then this would provide a complete description of essential homotheties: that a strict homothety is essential if and only if it has a point with finite orbit.

Chapter 2

Group actions and orbit spaces

2.1 Basic properties

In this section, we give definitions, and state some known results, including their proofs. In following sections we will often omit the proofs of known results, instead simply citing a source where the proof can be found. We assume the reader has an understanding of groups.

Definition 2.1.1. Let G be a group, and let X be a set. An *action* of G on X is a function $\theta : G \times X \to X$, such that for all $g, h \in G$, $x \in X$,

- $\theta(\mathrm{id}, x) = x$, where id is the group identity.
- $\theta(gh, x) = \theta(g, \theta(h, x))$.

We often use the abbreviation $gx := \theta(g, x)$.

Note that the above only considers left group actions. It suffices to consider the theory of left group actions because every right group action $\hat{\theta} : (x, g) \mapsto xg$ can be realized as a left group action by G^{-1}, $\theta : (g, x) \mapsto xg^{-1}$.

Given a group action, we form the orbit space, or the quotient of X by G.

Definition 2.1.2. Let G be a group acting on a set X. Define the *orbit of an element* $x \in X$

$$[x] := Gx = \{gx \mid g \in G\}.$$

Then define the *quotient*

$$X/G := \{[x] \mid x \in X\}.$$

We define the *projection map*

$$\pi : X \to X/G, \qquad x \mapsto [x].$$

If X is a topological space, then we define a topology on X/G given by the final topology with respect to the projection map π. Note that then π is a continuous function.

We describe a couple of notable properties of a group action that do not require any additional topological structure on G.

Definition 2.1.3. Let G be a group acting on a set X.

- The action is *free* if $gx = x$ for any $x \in X$, $g \in G$, implies $g = \text{id}$.
- If X is a topological space, then the group *acts by homeomorphisms* if for all $g \in G$,

$$\theta_g : X \to X, \qquad x \mapsto gx$$

  is a homeomorphism.

It is useful to talk about groups G that carry an additional topological structure.

Definition 2.1.4. Let G be a group with a topology. We define G to be a *topological group* if the composition map

$$G \times G \to G, \qquad (g, h) \mapsto gh$$

and the inverse map

$$G \to G, \qquad g \mapsto g^{-1}$$

are continuous.

Proposition 2.1.5. *Let G be a topological group. Then G has the discrete topology if and only if* $\{\text{id}\}$ *is an open set*

Proof. Of course if G is discrete, then by definition every subset is open, so $\{\text{id}\}$ must be open.

Let $\{\text{id}\}$ be an open set. For each $h \in G$, consider $G \to G \times G \to G$ given by $g \mapsto (g, h^{-1}) \mapsto gh^{-1}$ is continuous. But the preimage of $\{\text{id}\}$ under this map is $\{h\}$. Hence every singleton set is open, and so G is discrete. □

Once we have this additional structure on G, it is natural to consider topological groups that act on topological spaces in a way that respects both topologies

Definition 2.1.6. Let G be a topological group acting on a topological space X. The action is *continuous* if the group action map θ is continuous.

Proposition 2.1.7. *Let G be a group acting on a topological space X. If G acts continuously, then G acts by homeomorphisms. If G acts by homeomorphisms then G acts continuously when given the discrete topology.*

Proof. Let G act continuously. Then for each g, we see that $\theta_g : x \mapsto gx$ is a continuous function because $\theta_g = \theta \circ \iota_g$, where $\iota_g : x \mapsto (g, x)$. But since $\theta_g^{-1} = \theta_{g^{-1}}$ is also continuous, we conclude that θ_g is a homeomorphism.

Let G act by homeomorphisms, give G the discrete topology. Let $U \subset X$ be open. Consider $\theta^{-1}(U) = \bigcup_{g \in G} \{g\} \times g^{-1}U$. Then $\{g\}$ is open because G is discrete, $g^{-1}U$ is open because G acts by homeomorphisms, then the product of these sets is open, and so $\theta^{-1}(U)$ is the union of open sets, and hence is open itself. □

Proposition 2.1.8. *Let G be a group acting by homeomorphisms on a topological space X. Then the projection map π is open (the image of open sets are open).*

Proof. Let $U \subset X$ be an open set. $\pi(U)$ is open by definition if and only if $\pi^{-1}(\pi(U))$ is open. But since our quotient is formed by a group action, we know the form of $\pi^{-1}(\pi(U))$ precisely as $\bigcup_{g \in G} gU$. Then since each gU is open in X, the union is open. Hence π is an open map. □

We now describe properness of an action. It is a condition that informally makes the action look locally like a compact group acting on our space X. It also turns out to have a significant role in actions on manifolds such that the quotient space is also a manifold. This will be explained in more detail in the next section.

Definition 2.1.9. Let G be a topological group acting on a topological space X. The action is *proper* if the map $\Theta : (g, x) \mapsto (gx, x)$ is proper (i.e. the pre-image of compact sets are compact.)

Proposition 2.1.10. *Let G be a topological group acting on a topological space X. The action is continuous if and only if Θ is continuous*

Proof. Since the codomain of Θ has the product topology, Θ is continuous if and only if $p_1 \circ \Theta$ and $p_2 \circ \Theta$ are continuous, where $p_1, p_2 : X \times X \to X$ are the projections onto the first and second component, respectively. But then note that $p_2 \circ \Theta : (g, x) \mapsto x$ is the projection to the second coordinate, and thus is automatically always continuous. So then since $p_1 \circ \Theta : (g, x) \mapsto gx$ is precisely the map θ, we see that Θ is continuous if and only if θ is continuous. □

Proposition 2.1.11 (Lee 2011, Proposition 12.23)**.** *Let G be a topological group acting continuously on a Hausdorff topological space X. The action is proper if and only if for all compact $K \subset X$, $\{g \mid gK \cap K \neq \emptyset\}$ is compact.*

Proof. Let the action be proper. Let $K \subset X$ be compact. Since Θ is a proper map, $\Theta^{-1}(K \times K)$ is compact. We write this preimage in another form,

$$\Theta^{-1}(K \times K) = \bigcup_{g \in G} (g^{-1}, gK \cap K).$$

Then since the projection map $\pi : G \times X \to G$ is continuous, $\pi(\Theta^{-1}(K \times K) = \{g^{-1} \mid gK \cap K \neq \emptyset\}$ is compact, and so $\{g \mid gK \cap K \neq \emptyset\}$ is compact.

Let $\{g \mid gK \cap K \neq \emptyset\}$ be compact for all compact sets $K \subset X$. Let $L \subset X \times X$ be compact. Then define $\pi_1, \pi_2 : X \times X \to X$ as the projection maps to the first and second component, respectively. Define $K := \pi_1(L) \cup \pi_2(L)$, and note that $K \times K$ is a compact set containing L. Since X is Hausdorff, L is closed, and so $\Theta^{-1}(L)$ is a closed subset of $\Theta^{-1}(K \times K)$, since Θ is continuous by Proposition 2.1.10. So for properness of Θ, it suffices to show that $\Theta^{-1}(K \times K)$ is compact for all compact K. Then we see that

$$\begin{aligned}\Theta^{-1}(K \times K) &= \bigcup_{g \in G} (g^{-1}, gK \cap K) \\ &\subset \{g^{-1} \mid gK \cap K \neq \emptyset\} \times K.\end{aligned}$$

Which by assumption, is a product of compact sets. Then since X is Hausdorff, so $K \times K$ is closed, and thus $\Theta^{-1}(K \times K)$ is closed, we get that $\Theta^{-1}(K \times K)$ is compact. □

2.2 Proper discontinuity

In this section we describe properly discontinuous actions, these are sufficiently nice, so that when we quotient a manifold by a properly discontinuous action, the quotient retains a manifold structure.

Definition 2.2.1. Let G be a group acting on a topological space X. The group action is *properly discontinuous* if it satisfies the following two conditions

- PD1: For each point $x \in X$, there is a neighbourhood U of x such that if gU meets U, i.e. $gU \cap U \neq \emptyset$ for $g \in G$, then $g = \mathrm{id}$.

- PD2: For all pairs of points $x, y \in X$ in different orbits, there are neighbourhoods U of x, and V of y, such that for all $g \in G$, gU and V are disjoint, i.e. $gU \cap V = \emptyset$.

We can equivalently phrase these in terms of the map Θ

- PD1: For each point $x \in X$, there is a neighbourhood U of x such that $\Theta^{-1}(U \times U) = \{\mathrm{id}\} \times U$.

- PD2: For all pairs of points $x, y \in X$ in different orbits, there are neighbourhoods U of x, and V of y, such that $\Theta^{-1}(U \times V)$ is empty.

Defining proper discontinuity in this form (rather than the largely equivalent form of Theorem 2.2.5) has the advantage that the roles of PD1 and PD2 are distinct and clear: PD1 plays the role of making the projection a covering map. In particular, in the case that X is a manifold, this is the reason why the quotient is locally Euclidean. And PD2 is precisely the condition that makes the quotient space Hausdorff.

Theorem 2.2.2 (Lee 2011, Theorem 12.14)**.** *Let G be a group acting effectively and by homeomorphisms on a connected, locally path-connected topological space X. Then the action satisfies PD1 if and only if the projection map π is a covering map.*

Proposition 2.2.3 (Lee 2011, Proposition 12.21)**.** *Let G be a group acting by homeomorphisms on a topological space X. Then the action satisfies PD2 if and only if X/G is Hausdorff.*

Proof. Let X/G be Hausdorff. For each x, y in X. Let U, V be neighbourhoods of $[x]$ and $[y]$ respectively, such that $U \cap V = \emptyset$. Then $\pi^{-1}(U)$ and $\pi^{-1}(V)$ are neighbourhoods of x, y satisfying PD2.

Let the action satisfy PD2. Let x', y' in X/G. Choose representatives x, y in X such that $\pi(x) = x'$, $\pi(y) = y'$. Then let U, V, be neighbourhoods of x, y respectively, satisfying PD2. Then since π is open by Proposition 2.1.8, $\pi(U)$ and $\pi(V)$ are neighbourhoods of x', y' respectively, such that $\pi(U) \cap \pi(V) = \emptyset$. □

Proper discontinuity as a whole has an equivalent formulation in terms of proper actions. First we specify the definition of locally compact used in this thesis

Definition 2.2.4. A topological space X is *locally compact* if each point in X has a compact neighbourhood.

Theorem 2.2.5 (Lee 2011, Chapter 12)**.** *Let G be a topological group acting continuously on a topological space X. If G acts properly discontinuously then G is discrete, and the action is proper and free. If X is locally compact and Hausdorff, then the converse also holds: If G is discrete and acts properly and freely, then G acts properly discontinuously.*

Proof. We first note that in Lee's book we can find most of the proof of this claim, but he fails to note the implications that properly discontinuous implies G being discrete and acting freely. Hence we first prove these two implications, and then provide theorem numbers for the rest of the proof.

Let the action of G on X be properly discontinuous. PD1 immediately gives that the action is free. We show G is discrete: Let U be any open set satisfying PD1, i.e. $\Theta^{-1}(U \times U) = \{\mathrm{id}\} \times U$. Since the action is continuous, and so Θ is continuous by Proposition 2.1.10, we see that $\{\mathrm{id}\} \times U$ is an open set in $G \times X$. Hence $\{\mathrm{id}\}$ is open in G, and so G is discrete by Proposition 2.1.5.

The remainder of the proof can be found scattered among several propositions found in Chapter 12 of Lee's book. First note that Lee defines a 'covering space action' to be the combination of our PD1 and the assumption that the group acts by homeomorphisms. Also keep in mind that the action is PD2 if and only if X/G is Hausdorff by Proposition 2.2.3.

Under the assumption that the action of G on X is properly discontinuous, the action is proper (Lee 2011, Proposition 12.25).

Now assume the action G is discrete, free, and proper, and X is locally compact and Hausdorff. The action satisfies PD2 (Lee 2011, Proposition 12.24), and finally the action is PD1 as in the proof of Theorem 12.26 in Lee (2011). □

Now we describe some lemmas for determining that an action is properly discontinuous. Lemmas 2.2.6 and 2.2.7 (and later Lemmas 2.3.2 and 2.3.3) are used in Section 4 of Kath & Olbrich (2019), though they use a slightly weaker version in which they assume that the space X is locally compact and Hausdorff for all four theorems, and they do so without proof. We provide the proofs here.

Lemma 2.2.6. *Let G be a topological group acting continuously on a topological space X. Let $\Gamma < G$ be a subgroup. If G acts properly discontinuously then Γ acts properly discontinuously.*

Proof. Note that for any $S \subset X \times X$, $\Theta_\Gamma^{-1}(S) \subset \Theta_G^{-1}(S)$, where Θ_Γ and Θ_G are the map Θ for Γ and G, respectively.

The result then follows from the second formulation of proper discontinuity in Definition 2.2.1, that which is in terms of the map Θ. □

Lemma 2.2.7. *Let G be a locally compact topological group acting continuously on a Hausdorff topological space X. Let $\Gamma < G$ be a subgroup such that G/Γ is compact (where Γ acts on G by left multiplication, and G/Γ are the right cosets). If Γ acts properly then G acts properly.*

Proof. Since G is locally compact, we can construct a compact subset H of G (not necessarily a subgroup) such that H surjects onto the quotient G/Γ. Details of the construction of such a set will later be provided by Proposition 2.3.7.

Let Γ act properly. Using Proposition 2.1.11, this is equivalent to the statement that for all compact K, $\{\gamma \in \Gamma \mid \gamma K \cap K \neq \emptyset\}$ is compact. Then using that $G = \Gamma H$. Let $g = \gamma h$. If $gK \cap K \neq \emptyset$, then $h\gamma(hK) \cap hK \neq \emptyset$. Hence since $g = h^{-1}(h\gamma)h$, we have

$$\{g \in G \mid gK \cap K \neq \emptyset\} \subset H^{-1}\{\gamma \in \Gamma \mid \gamma(HK) \cap (HK) \neq \emptyset\}H.$$

Where HK is compact since H and K are compact, and the action is continuous. So $H^{-1}\{\gamma \in \Gamma \mid \gamma(HK) \cap (HK) \neq \emptyset\}H$ is the product of compact sets and hence is compact.

Finally, we conclude that $\{g \in G \mid gK \cap K \neq \emptyset\}$ is closed because it is equal to $\pi_1(\Theta^{-1}(K \times K)$, where K is closed because X is Hausdorff, and the projection map to the first component π_1 is a closed map by the tube lemma. See e.g. (Rotman 1998, Lemma 8.9′).

So $\{g \in G \mid gK \cap K \neq \emptyset\}$ is compact, and hence G acts properly on X. □

In line with our definition of group actions only considering left actions, in Lemma 2.2.7 we take Γ to act on G by left multiplication, so our quotient group is formed with right cosets. If one wishes to consider right actions instead, the lemma applies mutatis mutandis. Note that the left coset and right coset spaces are homeomorphic.

Lemma 2.2.8. *Let G be a topological group acting continuously on a topological space X. Let $f : X \to Y$ be a homeomorphism to a topological space Y. Let G act on Y by $(g, y) \mapsto f(gf^{-1}(y))$.*

Then G acts continuously on Y. and G separately acts freely and properly on X if and only if G acts freely and properly on Y, respectively.

Proof. The action is continuous since it is the composition of the continuous maps $(g, y) \mapsto (g, f^{-1}(y)) \mapsto gf^{-1}(y) \mapsto f(gf^{-1}(y))$. For each of the remaining properties, we need prove only the forward implication, since G acts on X by $f^{-1}(f(gf^{-1}(f(x))))$.

Assume G acts freely on X. Let $f(gf^{-1}(y)) = y$. This gives $f(gx) = f(x)$ for $x := f^{-1}(y)$. Then $gx = x$ by injectivity of f, which implies $g = e$ by freeness on X.

Assume G acts properly on X, i.e. $\Theta(g, x) = (gx, x)$ is a proper map. We note that homeomorphisms are proper maps. Then the corresponding map for Y is proper since it is the composition of proper maps

$$(g, y) \overset{\mathrm{id}\times f^{-1}}{\mapsto} (g, f^{-1}(y)) \overset{\Theta}{\mapsto} (gf^{-1}(y), f^{-1}(y)) \overset{f\times f}{\mapsto} (f(gf^{-1}(y)), y).$$

□

Now the next two theorems detail why proper discontinuity is precisely the right condition to produce quotients with an induced manifold structure, together they describe that a discrete smooth group action gives a quotient manifold structure if and only if it acts properly discontinuously.

Definition 2.2.9. Let $(\tilde{M}, \tilde{g})$ and (M, g) be two semi-Riemannian manifolds. A map $\pi : \tilde{M} \to M$ is a *smooth covering map* if π is smooth, surjective, and each point in $\tilde{M}$ has a neighbourhood U such that $\pi|_U$ is a diffeomorphism.

Theorem 2.2.10 (Lee 2012, Theorem 21.13)**.** *Let M be a connected manifold. Let Γ be a discrete Lie group acting smoothly and properly discontinuously on M. Then the quotient M/Γ is a topological manifold and has a unique smooth structure such that the projection map π is a smooth covering map, and $\pi_*(\pi_1(M, p)) = \pi_1(M/\Gamma, \pi(p))$ for every $p \in M$.*

Theorem 2.2.11. *Let M be a manifold. Let Γ be a discrete Lie group acting faithfully by homeomorphisms on M such that the quotient M/Γ is a topological manifold, and has a smooth structure such that the projection map π is a smooth covering map. Then Γ acts smoothly and properly discontinuously.*

Proof. $\mathrm{Aut}_\pi(M)$ with the discrete topology acts smoothly and properly discontinuously on M (Lee 2012, Proposition 21.12) ($\mathrm{Aut}_\pi(M)$ as defined in Lee is the group of homeomorphisms ϕ of M, such that $\pi \circ \phi = \pi$.) Note that by the definition of M/Γ, it is immediate that Γ is contained within $\mathrm{Aut}_\pi(M)$. Hence if Γ is discrete, then it acts smoothly, and also by Lemma 2.2.6, Γ acts properly discontinuously. □

It is worth noting that a Riemannian manifold has an induced distance metric obtained by taking the infimum of lengths of curves connecting points. And that isometries of the Riemannian manifold are also isometries of this induced distance metric. This process does not work in other signatures, but in the case of a proper action, we can still find a distance metric (by first finding a Riemannian metric) for which the group acts by isometries.

Theorem 2.2.12 (Koszul et al. 1965, Theorem 2). *Let G be a Lie group acting smoothly and properly on a manifold X. Then there exists a Riemannian metric h on X such that G is contained within the isometries of (X, h).*

2.3 Cocompactness

Definition 2.3.1. Let G be a group acting on a topological space X. The action is *cocompact* if X/G is compact.

We start by giving a few lemmas, comparable to Lemmas 2.2.6 to 2.2.8 for proper actions.

Lemma 2.3.2. *Let G be a group acting on a topological space X. Let Γ be a subgroup of G. If Γ acts cocompactly, then G acts cocompactly.*

Proof. The projection map $\pi_G : X \to X/G$ is invariant under Γ, i.e. for $\gamma \in \Gamma$, $\pi_G(\gamma x) = \pi_G(x)$. So we have an induced continuous map $X/\Gamma \to X/G$. Hence since X/Γ is compact, its continuous image X/G is compact. □

Lemma 2.3.3. *Let G be a locally compact topological group acting continuously on a locally compact topological space X. Let Γ be a subgroup of G such that G/Γ is compact. If G acts cocompactly on X, then Γ acts cocompactly on X.*

Proof. Since G is locally compact, by Proposition 2.3.7, there is a compact set $H \subset G$, such that $\Gamma H = G$. Since X is locally compact, by Proposition 2.3.7, there is a compact set $K \subset X$ such that $GK = X$.

But then $\Gamma(HK) = (\Gamma H)K = GK = X$, and so since $\theta : G \times X \to X$ is continuous, $HK = \theta(H \times K)$ is compact. So we have a compact set surjecting onto the quotient by Γ, so Γ acts cocompactly. □

Lemma 2.3.4. *Let G be a group acting on a topological space X. Let $f : X \to Y$ be a homeomorphism to a topological space Y. Let G act on Y by $(g, y) \mapsto fgf^{-1}(y)$. Then X/G is homeomorphic to Y/G. In particular, X/G is compact if and only if Y/G is compact.*

Proof. The map f descends to a map $\hat{f} : X/G \to Y/G$ since f is well defined with respect to G, i.e. for each $g \in G$, $f(gx) = fgf^{-1}f(x) = fgf^{-1}(y)$.

$\hat{f}$ is surjective because f is surjective. Let $\hat{f}([p]) = \hat{f}([q])$, for $p, q \in X$.

Then for some g, $f(p) = fgf^{-1}(f(q)) = fg(q)$. Then since f is injective, we have $p = gq$, and so $[p] = [q]$. Hence $\hat{f}$ is injective.

Then we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle \pi_X} & & \downarrow{\scriptstyle \pi_Y} \\ X/G & \xrightarrow{\hat{f}} & Y/G \end{array}$$

And so, $\hat{f}$ is a homeomorphism because f is a homeomorphism, and π_X, π_Y are topology defining maps. □

Example 2.3.5. We give an example of a properly discontinuous and cocompact action, which is in part a preparation for Example 5.4.1 in the context of Cahen-Wallach spaces.

Consider $X := \mathbb{R} \times \mathbb{C}$. Define $\hat{\Gamma}$ to be the group generated by:

$$\hat{\gamma} : \begin{pmatrix} s \\ z \end{pmatrix} \mapsto \begin{pmatrix} s+1 \\ z \end{pmatrix}, \qquad \hat{\eta} : \begin{pmatrix} s \\ z \end{pmatrix} \mapsto \begin{pmatrix} s \\ z + e^{is\pi/2} \end{pmatrix}.$$

Note that $\hat{\gamma}$ and $\hat{\eta}$ are homeomorphisms of X. Then when given the discrete topology, $\hat{\Gamma}$ acts continuously on X by Proposition 2.1.7. We aim to show that $\hat{\Gamma}$ acts properly discontinuously and cocompactly on X.

Attempting to show this directly is made difficult in particular by the nature of the map $\hat{\eta}$, and how the $\mathbb{C}$ component depends on the $\mathbb{R}$ component. This is where Lemma 2.3.4 is helpful. We simplify the problem by considering the homeomorphism $f : X \to \mathbb{R} \times \mathbb{C}$,

$$f : \begin{pmatrix} s \\ z \end{pmatrix} \mapsto \begin{pmatrix} s \\ ze^{-is\pi/2} \end{pmatrix}, \qquad f^{-1} : \begin{pmatrix} t \\ w \end{pmatrix} \mapsto \begin{pmatrix} t \\ we^{it\pi/2} \end{pmatrix}.$$

Then in line with Lemmas 2.2.8 and 2.3.4, we consider $\Gamma := f\hat{\Gamma}f^{-1}$,

$$\gamma := f\hat{\gamma}f^{-1} : \begin{pmatrix} t \\ w \end{pmatrix} \mapsto \begin{pmatrix} t+1 \\ -iw \end{pmatrix}, \qquad \eta := f\hat{\eta}f^{-1} : \begin{pmatrix} t \\ w \end{pmatrix} \mapsto \begin{pmatrix} t \\ w+1 \end{pmatrix}.$$

The advantage of Γ over $\hat{\Gamma}$, is that we've converted the translation in $\hat{\eta}$ which depended on s, into a constant rotation in γ. In fact, identifying $\mathbb{R} \times \mathbb{C}$ with $\mathbb{R}^3$, we see that Γ is actually a group of Euclidean motions: $\eta = e_2$ is the e_2

translation $((x, y, z) \mapsto (x, y+1, z))$, and γ is the e_1 translation, together with rotation in the second and third coordinate by $-\pi/2$.

Now we consider the subgroup $\Lambda < \Gamma$ given by $\langle \gamma^4, \eta, \gamma\eta\gamma^{-1} \rangle$. Note that $\Lambda = \langle 4e_1, e_2, -e_3 \rangle$, which is a lattice in $\mathbb{R}^3$. In particular since Λ acts cocompactly, Γ must also act cocompactly by Lemma 2.3.2. Λ also acts properly, so we show that Λ is a finite index subgroup so that we may apply Lemma 2.2.7 to conclude that Γ acts properly.

Define $\zeta := \gamma\eta\gamma^{-1} = -e_3$. We observe the following relations:

$$\eta\gamma^2 = \gamma^2\eta^{-1}, \qquad \eta\gamma^{-1} = \gamma^{-1}\zeta.$$

Now for any element of the form $\eta^b\gamma^a$, we can move γ^a left as follows: If a is even, $\eta^b\gamma^a = \gamma^a\eta^{\pm b}$. But if a is odd, $\eta^b\gamma^a = \gamma^{a+1}\eta^{\pm b}\gamma^{-1} = \gamma^a\zeta^{\pm b}$.

So now we consider an arbitrary element of $\hat{\Gamma}$ as

$$\gamma^{a_1}\eta^{b_1}\gamma^{a_2}\eta^{b_2}\ldots\gamma^{a_n}\eta^{b_n}.$$

From what we just showed, this can be reduced down – by moving γ's left – to the form

$$\gamma^{a'}\eta^{c_1}\zeta^{d_1}\ldots\eta^{c_m}\zeta^{d_m}.$$

This element is of the form $\gamma^{a' \mod 4}\lambda$ for some $\lambda \in \Lambda$. Hence Λ is an index 4 subgroup of Γ.

Hence by Lemma 2.2.7, since Λ acts properly, Γ acts properly.

Now we show Γ acts freely: Let $\gamma^i\lambda \in \Gamma$, for $i \in \{0, 1, 2, 3\}$, and $\lambda \in \Lambda$, such that for some (t, w), $\gamma^i\lambda(t, w) = (t, w)$.

Consider the action of $\gamma^i\lambda$ on the first coordinate: $\gamma^i\lambda(t, w) = (t+c, \ldots)$, for $i = c \mod 4$. Then by assumption $c = 0$, and so $i = 0$. But then $\lambda(t, w) = (t, w)$, and since Λ is a lattice and so acts freely, we get $\lambda = \mathrm{id}$, and hence Γ acts freely.

Therefore Γ acts properly discontinuously and cocompactly on $\mathbb{R} \times \mathbb{C}$, and so by Lemmas 2.2.8 and 2.3.4, $\hat{\Gamma}$ acts properly discontinuously and cocompactly on X.

Finding a good sufficient condition for cocompactness is not challenging, and can usually be done explicitly by inspection, applying the following proposition.

Proposition 2.3.6. *Let G be a group acting on a topological space X. Let $K \subset X$ be a compact set such that $\pi(K) = X/G$. Then X/G is compact.*

Proof. The projection map to the quotient π is a continuous function. Then since the continuous image of a compact set is compact, $\pi(K) = X/G$ must be compact. □

For example, to prove that the circle ($\mathbb{R}/\mathbb{Z}$ where $\mathbb{Z}$ acts on $\mathbb{R}$ by addition $(z, r) \mapsto z + r$) is compact, we only need to consider the compact set $[0, 1]$, and observe that $\pi([0, 1]) = \mathbb{R}/\mathbb{Z}$.

In fact, if X is locally compact, then the converse holds.

Proposition 2.3.7. *Let X be a locally compact topological space (in the sense that every point in X has a compact neighbourhood.)*

Let G be a group acting by homeomorphisms on X, such that X/G is compact. Then there is a compact set K such that $\pi(K) = X/G$.

Proof. For each $x \in X$, let K_x be a compact neighbourhood of x. By Proposition 2.1.8, each set $\pi(K_x^\circ)$ is open (where $^\circ$ denotes the interior.)

Then $\{\pi(K_x^\circ)\}_{x \in X}$ is an open cover of X/G. But since X/G is compact, there is a finite subcover, $X/G = \bigcup_{i=1}^m \pi(K_{x_i}^\circ)$.

Then $\{\pi(K_{x_i})\}$ also covers X/G, and so $K := \bigcup_{i=1}^m K_{x_i}$ is a compact set satisfying $\pi(K) = X/G$. $\square$

Despite being a necessary condition for cocompactness, Proposition 2.3.7 is not particularly useful for proving that an action is not cocompact. To see this, consider the following example.

Example 2.3.8. Take $X = \mathbb{R}^2$, acted on by $G = \mathbb{Z}$. Define the group action to be

$$\theta(n, (x, y)) \mapsto (x, y + n).$$

Now we should be able to see that the quotient is a cylinder $X/G = \mathbb{R} \times \mathbb{S}^1$. Suppose that we did not know this, and that we want to prove that the quotient is not compact.

Take the projection $p : (x, y) \mapsto x$. Since for all group elements g, $p(g(x, y)) = p(x, y)$ (precisely because the group elements leave x fixed), p has an induced map on the quotient: $\hat{p} : X/G \to \mathbb{R}$, $\hat{p} : [(x, y)] \mapsto x$. We see that $\hat{p}$ is surjective, and is continuous because p is continuous. Then $\mathbb{R}$ is the continuous image of X/G, so X/G cannot be compact.

But we can modify this example only slightly to get an example that is immune to the proof method just described: Take a homeomorphism γ of $\mathbb{R}^2$ of the form

$$\gamma : \mathbb{R}^2 \to \mathbb{R}^2, \qquad (x, y) \mapsto (\phi(x, y), y + 1).$$

Consider G as the group generated by γ. Because of the introduction of the map ϕ, we have lost the nicely preserved information that we were extorting to show non-cocompactness. So some another technique is required, this is where the notion of a fundamental region will be useful.

Definition 2.3.9. Let G be a group acting on a topological space X. Then an open set $R \subset X$ is a *fundamental region* if

- the members of $\{gR \mid g \in G\}$ are pairwise disjoint, and
- the closure of R surjects onto the quotient, i.e. $\bigcup_{g \in G} g\overline{R} = X$.

Under some additional niceness conditions, fundamental regions provide a necessary test for cocompactness that specializes precisely where the method used in Example 2.3.8 failed.

Definition 2.3.10. Let G be a group acting on a topological space X. The action is *discontinuous* if for all compact $K \subset X$, K meets gK for only finitely many g.

It is worth noticing that by Proposition 2.1.11, for discrete G acting continuously on Hausdorff X, this condition is equivalent to properness.

Definition 2.3.11. Let G be a group acting on a topological space X. Let R be a fundamental region for the action. Then R is *locally finite* if $\{g\overline{R} \mid g \in G\}$ is a locally finite collection of sets.

Theorem 2.3.12 (Ratcliffe 2019, Theorem 6.6.9)**.** *Let G be a group of isometries acting cocompactly and discontinuously on a locally compact metric space (X, d). Let $R \subset X$ be a fundamental region for this action. Then R is locally finite if and only if $\overline{R}$ is compact.*

Ratcliffe has another theorem (Ratcliffe 2019, Theorem 6.6.7) which can be used to aid calculation of the topology of the quotient space. In it he makes the assumption that X is a metric space, but we can reproduce one direction of his proof without this assumption:

Theorem 2.3.13. *Let G be a group acting by homeomorphisms on a topological space X. Let R be a fundamental region for this action. The inclusion $\iota : \overline{R} \to X$ induces a continuous bijection $\kappa : \overline{R}/G \to X/G$, and if R is locally finite, then κ is a homeomorphism.*

Proof. We follow the proof found in Ratcliffe (2019) almost exactly, removing unnecessary references to the metric.

We define $\kappa : Gx \cap \overline{R} \mapsto Gx$. κ is immediately injective. κ is surjective because R is a fundamental region, and so $g\overline{R}$ covers X.

Let $\eta : \overline{R} \to \overline{R}/G$ be the projection map. So κ is continuous if and only if $\kappa \circ \eta$ is continuous. Then $\kappa \circ \eta$ is continuous by the following commutative diagram

$$\begin{array}{ccc} \overline{R} & \xrightarrow{\iota} & X \\ \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \pi} \\ \overline{R}/G & \xrightarrow{\kappa} & X/G \end{array}$$

So κ is a continuous bijection.

Let R be locally finite. We show κ is an open map. Let $\tilde{U} \subset \overline{R}/G$ be open. By definition, this is true if and only if $\eta^{-1}(\tilde{U})$ is open, i.e. $\eta^{-1}(\tilde{U}) = \overline{R} \cap U$, for some open set $U \subset X$. Define

$$W := \bigcup_{g \in G} g(\overline{R} \cap U).$$

Note that $\pi(W) = \pi(\overline{R} \cap U) = \pi \circ \iota(\overline{R} \cap U) = \kappa \circ \eta(\overline{R} \cap U) = \kappa(\tilde{U})$. So since π is open by Proposition 2.1.8, it suffices to prove that W is open.

Let $w \in W$. Since R is locally finite, there is a neighbourhood V of w that meets $\{g\overline{R}\}$ for only finitely many g. In particular, we have

$$V \subset g_1\overline{R} \cup \ldots \cup g_m\overline{R}.$$

If w is not in $g_i\overline{R}$ for an i, we replace V with $V \setminus (g_i\overline{R})$, which is still an open neighbourhood of w. Hence we assume each $g_i\overline{R}$ contains w. Since $g_i^{-1}w$ is in $\overline{R}$, we note that $\eta(g_i^{-1}w) = \eta(w)$, and so $g_i^{-1}w$ is in $\eta^{-1}(\tilde{U}) = \overline{R} \cap U$. Hence in particular, w is in g_iV. So by defining $\tilde{V} := V \cap g_1V \cap \ldots \cap g_mV$, we have that $\tilde{V}$ is an open neighbourhood of w, such that each element $x \in \tilde{V}$ is contained within some $g_i\overline{R}$, and also all g_jV. Hence x is in $g_i(\overline{R} \cap V)$, and so x is in W. Hence W is open, and so κ is an open map. □

Example 2.3.14. Consider $X = \mathbb{R}$, and $G = \mathbb{Z}$. The action of $\mathbb{Z}$ on $\mathbb{R}$ (as an additive subgroup) is cocompact and discontinuous. The quotient $\mathbb{R}/\mathbb{Z}$ is the circle $\mathbb{S}^1$.

We verify Theorem 2.3.12 by noting that $R = (0, 1)$ is a locally finite fundamental region for this action, and that $\overline{R}$ is compact.

Now we consider instead the set $R = \bigcup_{n \in \mathbb{N}}(n + \frac{n}{n+1}, n + \frac{n+1}{n+2})$. We note that since $\overline{\bigcup_{n \in \mathbb{N}}(\frac{n}{n+1}, \frac{n+1}{n+2})} = [0, 1]$, R is a fundamental region for this action, yet $\overline{R}$ is not compact. This is because R is not locally finite, as we show now.

Define $\gamma \in \mathbb{Z}$ to be translation by 1, so that $\gamma^n(x) = x + n$. Observe that all sequences $x_n \in (n + \frac{n}{n+1}, n + \frac{n+1}{n+2})$ satisfy $x_n \in R$, and $\gamma^{-n}(x_n) \to 0$. Then any neighbourhood of 0 meets infinitely many $\gamma^{-n}\overline{R}$, so R is not locally finite.

Note that this example can be strengthened to obtain a connected fundamental region, which is not compact, despite the action being cocompact. We do this by taking $X = \mathbb{R}^2$ and $G = \mathbb{Z}^2$. The quotient X/G is the torus. Then we define $R := \bigcup_{x\in(0,1)} \left(\{x\} \times (\frac{1}{x}, \frac{1}{x}+1)\right)$. R is a connected fundamental region (a fundamental domain) for a properly discontinuous and cocompact action, but $\overline{R}$ is not compact, and R is not locally finite.

Example 2.3.15. We continue Example 2.3.8. So we take an isometry of $\mathbb{R}^2$ of the form

$$\gamma : (x, y) \mapsto (\phi(x, y), y + 1).$$

Take $G = \langle\gamma\rangle$. Of course, we know enough about isometries of $\mathbb{R}^n$ in general, that we could say more about this map γ, but for this motivational example, we shall pretend this information is inaccessible to us.

We attempt to apply Theorem 2.3.12. In this context we prove that $R := \mathbb{R} \times (0, 1)$ is a locally finite fundamental region for this action. $\gamma^n R = \mathbb{R} \times (n, n+1)$, and $\gamma^n \overline{R} = \mathbb{R} \times [n, n+1]$. Then since $\{(n, n+1)\}$ are disjoint and $\{[n, n+1]\}$ cover $\mathbb{R}$, R is a fundamental region. Clearly, $\{g\overline{R}\}$ is locally finite at any point in gR for some g. Otherwise, if p is in the boundary of a $g\overline{R}$, then p is of the form (x, y) for y a natural number. But then we can simply take $p \in U := \mathbb{R} \times (y - 1, y + 1)$, U meets $g\overline{R}$ only for $g = y - 1$ and $g = y$. Hence R is locally finite. So (taking for granted that the action is discontinuous), we apply Theorem 2.3.12 and conclude that this action is not cocompact.

However, what if we take the map ϕ so that γ is not an isometry of any (topology preserving) metric on $\mathbb{R}^2$? Then Theorem 2.3.12 cannot help us. In the next Section we present an original definition and result that can help us even in this context.

2.4 Finitely self adjacent

In Example 2.3.15, we described a theoretical example in which we expect the action to not be cocompact, but in which current fundamental region theory, as described in Ratcliffe (2019), seems insufficient to prove that the action is not cocompact. This motivates a new theorem, inspired by Theorem 2.3.12, that is able to handle more general topological spaces. To do this we introduce a new notion: that of a finitely self adjacent fundamental region.

Definition 2.4.1. Let G be a group acting on a topological space X. Let R be a fundamental region for the action. Then R is *finitely self adjacent* if for some open neighbourhood U such that $\overline{R} \subset U$, the set $\{g \in G \mid gU \cap U \neq \emptyset\}$ is finite.

Theorem 2.4.2. *Let G be a group acting by homeomorphisms on a locally compact topological space X (every point has a compact neighbourhood). Let R be a finitely self adjacent fundamental region for this action. If X/G is compact, then $\overline{R}$ is compact.*

Proof. Let X/G be compact. By Proposition 2.3.7, there is a compact set K such that $\pi(K) = X/G$. We now show that we can cover $\overline{R}$ by a finite union of gK.

Take a finitely self adjacent open neighbourhood U of $\overline{R}$. Define $\hat{g}_1, \ldots, \hat{g}_n$ to be the elements of G such that $\hat{g}_i U$ meets U. Note that $\{gU \mid g \in G\}$ is an open cover for X, and in particular is an open cover of K. So we take a finite subcover of K, $\{g_1 U, \ldots, g_m U\}$.

Now let $x \in \overline{R}$, and $g \in G$ such that $gx \in K$. Since $\{g_i U\}$ cover K, this tells us that for some i, $g_i^{-1} gx \in U$, and hence since U is a finitely self adjacent neighbourhood of $\overline{R}$, it follows that $g_i^{-1} g = \hat{g}_j$ for some j, and so $g = g_i \hat{g}_j$.

But then this implies that $\{\hat{g}_j^{-1} g_i^{-1} K\}$ is a finite covering of $\overline{R}$ by compact sets, and so $\overline{R}$ is compact. □

It should be noted that this theorem does *not* include the converse which is found in Ratcliffe's theorem, which would state – if true – that for a cocompact action, if $\overline{R}$ is compact, then R is finitely self adjacent.

Example 2.4.3. We continue Examples 2.3.8 and 2.3.15 by considering $\mathbb{R}^2$, and $\gamma : (x, y) \mapsto (\phi(x, y), y + 1)$ for some continuous $\phi : \mathbb{R}^2 \to \mathbb{R}$, such that for each y, $\phi_y : x \mapsto \phi(x, y)$ is a homeomorphism. The group $\langle\gamma\rangle$ acts on $\mathbb{R}^2$. We note that $\mathbb{R}^2$ is locally compact, and in fact, admits a compact exhaustion, $K_i := [-i, i]^2$. We showed in Example 2.3.15 that $R := \mathbb{R} \times (0, 1)$ is a fundamental region for the action. Define $U := \mathbb{R} \times (-1/2, 3/2)$. Note that $\gamma^n U = \mathbb{R} \times (n - 1/2, n + 3/2)$, and so if $\gamma^n U \cap U$ is nonempty, then $n \in \{-1, 0, 1\}$. Therefore R is finitely self adjacent. Hence by Theorem 2.4.2, we draw a conclusion that the other methods are unable to in this context: that the action is not cocompact.

In fact, this example extends to a general principal: that a homeomorphism containing a pure translation component can compactify only in that direction, and leaves the other direction fundamentally untouched, no matter how disastrously it acts upon it.

Proposition 2.4.4. *Let X be a locally compact topological space. Consider $X \times \mathbb{R}$. Let $c \in \mathbb{R}$, and let*

$$\phi : X \times \mathbb{R} \to X \times \mathbb{R}, \qquad (x,t) \mapsto (\psi(x,t), t+c),$$

where $\psi : X \times \mathbb{R} \to X$ is any map such that ϕ is a homeomorphism. Then $X \times (0,c)$ is a finitely self adjacent fundamental region for the action of $\langle\phi\rangle$, and so $(X \times \mathbb{R})/\langle\phi\rangle$ is compact if and only if X is compact.

Proof. We define

$$R := X \times (0,c) = \{(x,t) \in X \times \mathbb{R} \mid 0 < t < c\}.$$

and show R is a finitely self adjacent fundamental region. Note that $\phi^m R = X \times (mc, mc+c)$, and hence $\phi^m R$ does not meet R for $m \neq 0$. Similarly, $\phi^m \overline{R} = X \times [mc, mc+c]$, and so $\bigcup_{m\in\mathbb{Z}} \phi^m \overline{R} = X \times \mathbb{R}$. So R is a fundamental region. Define a neighbourhood of $\overline{R}$,

$$U := X \times (-c, 2c).$$

Note that $\phi^m U = X \times (mc - c, mc + 2c)$, and so if $\phi^m U$ meets U, then $m \in \{-2, -1, 0, 1, 2\}$. Therefore R is finitely self adjacent.

Note that $X \times \mathbb{R}$ is locally compact because X and $\mathbb{R}$ are locally compact. Then by Theorem 2.4.2 and Proposition 2.3.7, $(X \times \mathbb{R})/\langle\phi\rangle$ is compact if and only if $X \times [0,c]$ is compact, which is true if and only if X is compact. □

Now we investigate how the properties finitely self adjacent and locally finite are related.

Proposition 2.4.5. *Let G be a group acting by homeomorphisms on a topological space X. Let R be a fundamental region for the action. If R is finitely self adjacent, then for some neighbourhood U of $\overline{R}$, $\{gU \mid g \in G\}$ is a locally finite collection of sets*

Proof. Define U to be the finitely self adjacent neighbourhood of R.

Let $x \in X$. Since R is a fundamental region, there is a $g \in G$, such that $x \in g\overline{R}$. Then gU is a neighbourhood of x. If $\hat{g}U$ meets this neighbourhood of x, then $g^{-1}\hat{g}U \cap U \neq \emptyset$. Since U is finitely self adjacent, there are only finitely many $g^{-1}\hat{g}$ solving this equation, and hence only finitely many $\hat{g}$ such that $\hat{g}U$ meets this neighbourhood of x. Hence $\{gU \mid g \in G\}$ is locally finite. □

But then for each g, we see that $g\overline{R} \subset gU$, and so get the following corollary.

Corollary 2.4.6. *Let G be a group acting by homeomorphisms on a topological space X. Let R be a fundamental region for the action. If R is finitely self adjacent, then R is locally finite.*

If we recall the situation of Proposition 2.4.4, using this corollary and Theorem 2.3.13, we conclude not only that $(X \times \mathbb{R})/G$ is cocompact if and only if X is cocompact, but also we make the intuition preceding it that that the quotient leaves X untouched precise: we have that $(X \times \mathbb{R})/G$ is in fact homeomorphic to $(X \times [0, c])/\sim$, where $\sim$ is an equivalence relation simply gluing the ends together, $X \times \{0\}$ to $X \times \{c\}$, given by ϕ.

Remark 2.4.7. Note that finitely self adjacent is in general a strictly stronger condition that being locally finite. We give an example demonstrating the strictness of this implication in Example 2.4.9.

Remark 2.4.8. The reason for introducing the notion of finitely self adjacent was to have a criterion such that when an action is cocompact, the fundamental region is compact. This is achieved by Theorem 2.4.2, however this is also achieved with the weaker condition of being locally finite in the context of a metric space (Ratcliffe 2019, Theorem 6.6.9). We can also apply the result for locally finite fundamental regions to discrete, proper, and smooth actions on a manifold. This is done by using Theorem 2.2.12 to find a preserved distance metric, so we can apply the locally finite version of the result.

For the applications of this thesis, this observation makes finitely self adjacent fundamental regions less relevant, though they still have value: in Lemma 5.1.1, the use of finitely self adjacent, instead of locally finite, avoids the additional work of showing that $\langle \gamma \rangle$ is discrete and acts properly in the third case. Additionally, the strengthening of the assumption from locally finite to finitely self adjacent seems to have minimal impact in most concrete examples. But in exchange, we gain the ability to make statements of much more generality, assuming only homeomorphisms acting on a locally compact topological space. In particular, there is no need for the assumption of properness of the action.

Example 2.4.9. To finish this chapter, we supply in detail an example of an action and a fundamental region that is locally finite, but not finitely self adjacent. The contents of this example will not be important to later exposition, and may be skipped by the unintrigued reader.

We define an individual room to be the unit square with the corners removed (endowed with the subspace topology in $\mathbb{R}^2$.)

$$B := [0, 1]^2 \setminus \{(0, 0), (1, 0), (0, 1), (1, 1)\}.$$

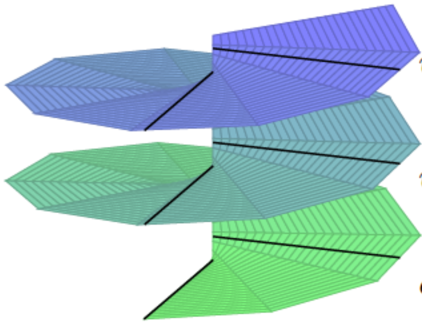


Figure 2.1: If the corners of B were included, every sufficiently small neighbourhood of $(e,(0,1))$ would be a helicoid like this. The labels indicate which copy of B is bounded by each pair of black lines.

Then we take the free group on two generators $F_2 = \langle r, u\rangle$ (r, u should be thought of as the directions right and up, respectively), with the discrete topology, and consider an F_2 number of rooms B by taking $F_2 \times B$, with the product topology. And finally we want to connect the rooms together – sharing a wall if they are adjacent in the free group – by taking $\sim$ to be the equivalence relation generated by

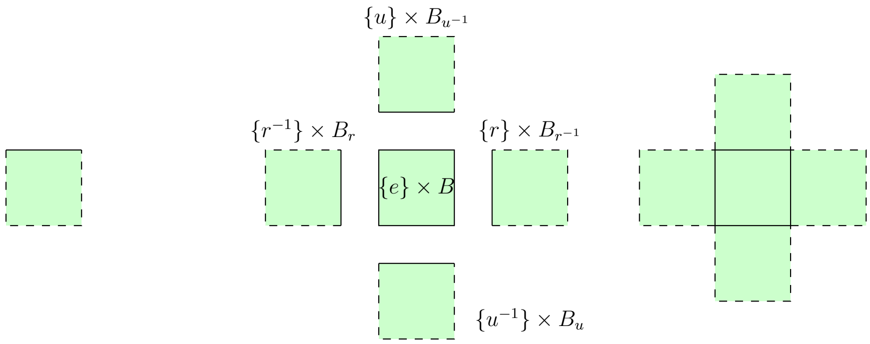
$$(w,(1,y)) \sim (wr,(0,y)), \text{ and } (w,(x,1)) \sim (wu,(x,0)),$$

for all words $w \in F_2$, and $x, y \in [0,1]$. In particular, it's worth noting that each equivalence class contains at most two elements of $F_2 \times B$. This is because no element $(x,y) \in B$ can have both $x \in \{0,1\}$ and $y \in \{0,1\}$ simultaneously. We define (giving X the quotient topology),

$$X := (F_2 \times B)/\sim .$$

We have named X the "free 2-house", emphasizing its free group navigational properties, and the fact that this example could be extended to arbitrary dimension, by taking $F_n \times [0,1]^n$, and removing all $n-2$ dimensional or smaller edges. X can be thought of as an infinite house, where all of its rooms are identically sized squares, but such that the room to the right and up is a different room than the one located up and right. In order for X to be locally Euclidean, it is crucial that we remove the corners of B, otherwise every neighbourhood of the corner will have an open subset homeomorphic to the helicoid (see Fig. 2.1).

We describe a coordinate neighbourhood of each room (see Fig. 2.2). First we define the following subsets of B, and note that they are open in



(a) B_u is the interior of B, together with its upper edge.

(b) U_e (Where e is the empty word.)

(c) $\pi(U_e)$, the coordinate neighbourhood.

Figure 2.2: Coordinate neighbourhoods of X.

the subspace topology of B:

$$B_r := (0,1] \times (0,1), \qquad B_u := (0,1) \times (0,1]$$
$$B_{r^{-1}} := [0,1) \times (0,1), \qquad B_{u^{-1}} := (0,1) \times [0,1)$$

For $w \in F_2$, we take the w room, together with the interior of the rooms adjacent, and their connecting wall:

$$U_w := \{w\} \times B \cup \bigcup_{v \in \{r,u,r^{-1},u^{-1}\}} \{wv\} \times B_{v^{-1}}.$$

Then define the coordinate neighbourhoods to be $\pi(U_w)$, noting that $\pi(U_w)$ is open, because $\pi^{-1}(\pi(U_w))$ is equal to U_w, and so is open.

Then define the coordinate maps $\phi_w : \pi(U_w) \to V \subset \mathbb{R}^2$ as

$$\phi_w : (wv,(x,y)) \mapsto \begin{cases} (x,y) & v = e \\ (x+1,y) & v = r \\ (x-1,y) & v = r^{-1} \\ (x,y+1) & v = u \\ (x,y-1) & v = u^{-1} \end{cases}.$$

ϕ_w is well defined, and is a homeomorphism from X to $\mathbb{R}^2$. The transition functions $\phi_{w'}\phi_w^{-1}$ are simply translations of $\mathbb{R}^2$.

We define our metric h on X by pulling back the standard Riemannian metric on $\mathbb{R}^2$ on each coordinate neighbourhood by the ϕ_w. This choice is well defined on overlaps because the transition functions are translations of $\mathbb{R}^2$, and in particular are isometries.

Now we define our group action by considering a subgroup of the isometries generated by the following elements: For each room we take a group element whose action is given by reflecting the entire manifold through the $y = x$ line contained within that room. We describe the action precicely: First define the automorphism σ of F_2 by swapping the generators, i.e. σ is the unique homomorphism $F_2 \to F_2$ satisfying

$$\sigma(r) = u, \text{ and } \sigma(u) = r.$$

Then for each $w \in F_2$, we define the transformation of X by

$$\begin{aligned} g_w &: (wv, (x, y)) \mapsto (w\sigma(v), (y, x)), \qquad \text{or equivalently,} \\ g_w &: (v, (x, y)) \mapsto (w\sigma(w^{-1}v), (y, x)). \end{aligned}$$

Note that each g_w is of order 2, and in particular the map $F_2 \to \mathrm{Iso}(X, h)$, $w \mapsto g_w$ is *not* a group homomorphism.

We see that g_w is an isometry by calculating on each coordinate neighbourhood

$$\phi_{w\sigma(v)} \circ g_w \circ \phi_{wv}^{-1} : (x, y) \mapsto (y, x),$$

which is an isometry of the standard Riemannian metric on $\mathbb{R}^2$. We define our group G to be the subgroup of isometries of (X, h) generated by $\{g_w\}_{w \in F_2}$. Note that G can also be thought of as acting on F_2 and B separately, this observation will be taken advantage of a few times in the following discussion.

Using the fact that each g_w is of order 2, we can write an arbitrary element g of G in the form

$$g = g_{w_1} \circ g_{w_2} \circ \ldots \circ g_{w_n}.$$

Or, because σ is a bijection on the words, we will choose to write g as

$$g = g_{w_1} \circ g_{\sigma(w_2)} \circ g_{w_3} \circ g_{\sigma(w_4)} \circ \ldots \circ g_{w_n},$$

where we replace w_i with $\sigma(w_i)$ for even i (in particular, if n is even then we should replace the last term with $g_{\sigma(w_n)}$.) The advantage of the second form is that if we consider the action of G on F_2, we can write the image of any word explicitly as

$$g : v \mapsto w_1\sigma(w_1^{-1})w_2\sigma(w_2^{-1}) \ldots w_n\sigma(w_n^{-1})\sigma^n(v).$$

In particular if we define the word $w := w_1\sigma(w_1^{-1})w_2\sigma(w_2^{-1})\dots w_n\sigma(w_n^{-1})$, then g has the simple form

$$g : v \mapsto w\sigma^n(v).$$

The action of g on B can be expressed quite simply as

$$g : (x, y) \mapsto \begin{cases} (x, y) & n \text{ even} \\ (y, x) & n \text{ odd} \end{cases}.$$

Now with these facts about the action established, we present the fundamental region. Define the upper left triangle of each room to be

$$T := \{(x, y) \in B \mid 0 < x < y < 1\}.$$

Then define our candidate region

$$R := \{r^i\}_{i\in\mathbb{Z}} \times T.$$

Our region R is half of each box in a single copy of $\mathbb{Z}$ inside F_2. In the following discussion we prove that R is in fact a fundamental region. Before continuing, we first present some intuition about why R is a fundamental region:

Images of R under G are horizontal or vertical lines of triangles in X. If a given image is a horizontal line, then the triangles are upper left in B, and if the image is a vertical line, then the triangles are lower right in B. Then since each element of F_2 has only two lines meeting it – one horizontal and one vertical – then we see that the images of R cannot meet each other. The missing piece here is that we need to prove that images of R are not simply translations along one of these lines, i.e. we need to discount the possibility that there is a group element sending r^i to r^{i+j} for some j and all i. Seeing that this is impossible can be a challenging task on the level of intuition, so we leave it for the formal proof. To make it so that images of the line cover everywhere, we see that we can get to a desired word by 'walking' the line there, e.g. if $v = r^iu^jr^ku^l$, then the sequence of maps $g_{r^iu^jr^k} \circ g_{r^iu^j} \circ g_{r^i}$ will flip R down the appropriate stems of the free group to arrive at the destination. Now we begin the formal proof that R is a fundamental region:

We prove first that group images of $\overline{R}$ cover X. First note that $g_{r^i}\overline{R} \cup \overline{R}$ contains $\{r^i\} \times B$. Hence it suffices to prove that the action of G on F_2 can send every word v to r^i for some i. We induct on the length of the word. The base case $v = e = r^0$ requires no work. Assume the inductive hypothesis, so take $v = v'a$, with g such that $g(v') = r^i$, and $a \in \{r, u, r^{-1}, u^{-1}\}$. Then note that

$$\begin{aligned} g(v) &= w\sigma^n(v'a) \\ &= w\sigma^n(v')\sigma^n(a) \\ &= r^i\sigma^n(a). \end{aligned}$$

If $\sigma^n(a)$ is r or r^{-1}, then we are done. If $\sigma^n(a)$ is $u^{\pm 1}$ then we simply take

$$g_{r^i}(r^i u^{\pm 1}) = r^{i\pm 1}.$$

So $\{g\overline{R} \mid g \in G\}$ covers X.

Now we prove the gR are disjoint. If gR meets R in $\{r^i\} \times B$, then we have some element $g \in G$ that sends r^i to r^j for some j. Take g in the form discussed after the definition of the group, so that $g : v \mapsto w\sigma^n(v)$.

If n is even, then $g : v \mapsto wv$, and so if $g : r^i \mapsto r^j$, then $w = r^{j-i}$.

If n is odd, then $g : v \mapsto w\sigma(v)$, and so if $g : r^i \mapsto r^j$, then $w = r^j u^{-i}$.

We prove that both cases imply $i = j$. Inspired by advice of Tim Moy, we do this by summing the powers of r and u: Define $S : F_2 \to \mathbb{Z}$ to be the unique group homomorphism such that $S(r) = 1$, and $S(u) = 1$. For example, $S(u^3r^2u^{-1}) = 4$. Observe that for all words v, $S(\sigma(v)) = S(v)$. Then we consider

$$\begin{aligned} S(w) &= S(w_1\sigma(w_1^{-1})w_2\sigma(w_2^{-1})\dots w_n\sigma(w_n^{-1})) \\ &= S(w_1) + S(w_1^{-1}) + S(w_2) + S(w_2^{-1}) + \dots + S(w_n) + S(w_n^{-1}) \\ &= S(w_1) - S(w_1) + \dots + S(w_n) - S(w_n) \\ &= 0. \end{aligned}$$

So for all $g : v \mapsto w\sigma^n(v)$, we have $S(w) = 0$. Therefore when n is even, if $w = r^{j-i}$, then $0 = S(w) = j - i$, and so $j = i$. And when n is odd, if $w = r^j u^{-i}$, then $0 = S(w) = j - i$, and so $i = j$.

Hence if an element of G sends r^i to r^j, then $i = j$. In particular if n is even, then $w = e$, so g is the identity on F_2. But since n is even, g is also the identity on B. Hence g is the identity in G. Alternatively, if n is odd, then $g : v \mapsto r^i u^{-i}\sigma(v)$, but this is the same as $g_{r^i}(v)$. Hence $g = g_{r^i}$ on F_2. But since n is odd, g is the map $(x, y) \mapsto (y, x)$ on B. Hence g is equal to g_{r^i} in G. But since $g_{r^i}T \cap T = \emptyset$, we finally conclude that gR does not meet R.

So R is a fundamental region.

Then finally, we note that R is locally finite, since each coordinate neighbourhood of X meets exactly 6 copies of $g\overline{R}$ (see Fig. 2.3). And that R is

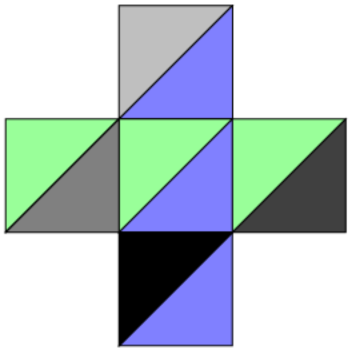
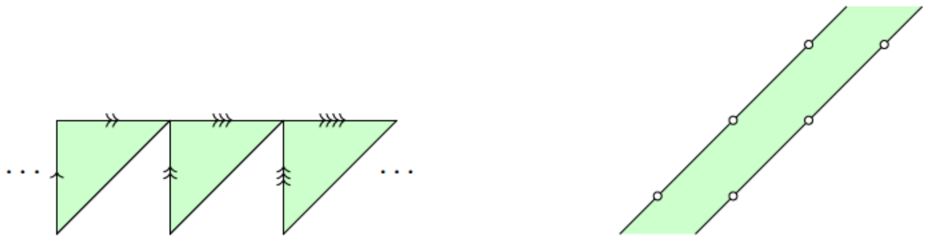


Figure 2.3: Diagram of the six $g\overline{R}$ that meet a coordinate neighbourhood, each colour represents a different g.



(a) The gluing rule on $\overline{R}$, e.g. the 2-arrow edges are identified in the indicated orientation.

(b) The space $\overline{R}/G$. The empty dots indicate missing points on the boundary, they are included now because the triangles here are not being thought of as subsets of B, which already has the corner points missing.

Figure 2.4

not finitely self adjacent, because for all i, $g_{r^i}\overline{R}$ meets $\overline{R}$ in $\{r^i\} \times B$, and so for any neighbourhood U of $\overline{R}$, $g_{r^i}U$ meets U here as well.

We can go further and describe the quotient. By Theorem 2.3.13, X/G is homeomorphic to $\overline{R}/G$. So our quotient is the disjoint union of countably many $\overline{T}$, with a gluing rule on the boundaries (see Fig. 2.4).

It is also worth noting that if needed, this example can be modified quite easily to produce an example for a properly discontinuous action, at the cost of losing connectedness of X: This is done by removing the $y = x$ line in the definition of B. Once we do this, the quotient space is as described in Fig. 2.4, without the boundary.

Chapter 3

Conformal structures

3.1 Definitions and background theorems

In this section, we will present various definitions of conformal geometry, we start with the most basic.

Definition 3.1.1. Let (M, g), (N, h) be semi-Riemannian manifolds. Let $\phi : M \to N$ be a diffeomorphism. Then ϕ is an *isometry* if $\phi^*h = g$. Where the pullback ϕ^*h is defined as

$$\phi^*h|_x(v, w) := h|_{\phi(x)}(d\phi|_x(v), d\phi|_x(w)),$$

for $x \in M$, and v, w tangent vectors at x.

Weakening the notion of an isometry is what gives rise to conformal transformations, homotheties, and conformal geometry in general.

Definition 3.1.2. Let (M, g), (N, h) be semi-Riemannian manifolds. Let $\phi : M \to N$ be a diffeomorphism. Then ϕ is a *conformal transformation* if $\phi^*h = e^{2f}g$, for some function $f : M \to \mathbb{R}$.

If f is constant, then ϕ is a *homothety*.

If f is non-zero, then ϕ is *strict*.

Note that ϕ being a strict homothety or a strict conformal transformation is precisely the condition that ϕ is not an isometry.

Example 3.1.3. Consider $\mathbb{R}^n$ with the standard Euclidean metric. The isometry group is $O(n) \ltimes \mathbb{R}^n$. This can be seen by noting that the translation map for every vector in $\mathbb{R}^n$ is an isometry, and that the isometries fixing a point are exactly the orthogonal matrices. The homothetic group is obtained by adding the scalings $x \mapsto e^s x$ to the isometry group. The conformal group of $\mathbb{R}^n$ is no bigger than its homothetic group, but it is worth noting that if we remove a point from $\mathbb{R}^n$, we gain some new conformal transformations: the inversions, e.g. $x \mapsto ||x||^{-2}x$. These are the maps that send the removed point to a point at infinity. And in fact this leads us to consider the sphere.

The sphere $\mathbb{S}^n$ can be thought of as $\mathbb{R}^n \cup \{\infty\}$, identified using the stereographic projection. Note that the stereographic projection is a conformal transformation, and so we can identify the conformal groups of these spaces. So the conformal group of $\mathbb{S}^n$ is the homothetic group of $\mathbb{R}^n$, together with the inversions.

Another useful identification of $\mathbb{S}^n$ is as embedded in the light cone of $\mathbb{R}^{n+2}$ with standard Lorentzian metric. In fact any section of this light cone represents a conformal rescaling of $\mathbb{S}^n$. So another way to see the conformal group of $\mathbb{S}^n$ is as the transformations of $\mathbb{R}^{n+2}$ that preserve the light cone, i.e. as $O(n+1,1)$.

Conformal geometry and conformal transformations are perhaps more natural to consider on a weakened structure, that of a conformal manifold.

Definition 3.1.4. Let M be a manifold. A *conformal structure* on M is an equivalence class of metrics on M, where two metrics g, h are equivalent if there is a smooth function $f : M \to \mathbb{R}$ such that $g = e^{2f}h$.

A *conformal manifold* is a pair (M, c) where M is a manifold, and c is a conformal structure on M.

A conformal structure can be equivalently described as a ray sub-bundle of the symmetric tensors, where a ray bundle is a principle fiber bundle with structure group $G = \mathbb{R}_{>0}$. The equivalence is simple to see in one direction: An equivalence class of metrics defines a ray sub-bundle by taking $C|_p := \mathbb{R}_{>0} \cdot g|_p$ for any representative g. The other direction requires the existence of a nowhere zero section of the ray bundle, which we can show by showing that the bundle is trivial. This can be done directly using a partition of unity argument on representative metrics of local trivializations, or it can be seen using the following result for principal G-bundles. If K is a maximal compact subgroup of G, then the bundle reduces to a K-bundle (Baum 2009, Satz 2.17). In the case of ray bundles, $G = \mathbb{R}_{>0}$, and the maximal compact subgroup K is trivial. Hence we get a trivial sub-bundle, and so the bundle has a nowhere zero section.

Definition 3.1.5. Let (M, c), $(\hat{M}, \hat{c})$ be conformal manifolds. Let $\phi : M \to \hat{M}$ be a diffeomorphism. Then ϕ is a *conformal transformation* if $\phi^*\hat{c} = c$, where the pullback $\phi^*\hat{c}$ is defined as the conformal class containing $\phi^*\hat{g}$ for a representative metric $\hat{g}$ in $\hat{c}$.

Note that this is well defined, since if any metric in $\hat{c}$ pulls back to a metric in c, then every metric in $\hat{c}$ must do so.

For any semi-Riemannian manifold (M, g), we can consider the induced conformal manifold $(M, [g])$, where $[g]$ is the conformal class containing g. The double use of the term 'conformal transformation' could lead to confusion if the notions are not in some way equivalent. However we note that a diffeomorphism $\phi : M \to N$ is a conformal transformation between (M, g) and $(\hat{M}, \hat{g})$ if and only if it is a conformal transformation between the induced conformal manifolds $(M, [g])$ and $(\hat{M}, [\hat{g}])$.

We do not, and indeed can not define the notion of an isometry nor that of a homothety for conformal manifolds. This is because if a map is an isometry or homothety for a given metric, it will not in general be one for a rescaled metric as well.

Note that the properties of being an isometry, homothety, or a conformal transformation are each closed under composition. We form their respective groups. Let (M, g) be a semi-Riemannian manifold.

Definition 3.1.6. The *isometry group* of (M, g), $\mathrm{Iso}(M, g)$ is the group of isometries from (M, g) to itself.

The *homothetic group* of (M, g), $\mathrm{Homoth}(M, g)$ is the group of homotheties from (M, g) to itself.

The *conformal group* of (M, g), $\mathrm{Conf}(M, g)$ is the group of conformal transformations from (M, g) to itself.

Let (M, c) be a conformal manifold. The conformal group of (M, c), $\mathrm{Conf}(M, c)$ is the group of conformal transformations from (M, c) to itself.

Since the conformal transformations of (M, g) and $(M, [g])$ are identical, their conformal groups are also equal, i.e. $\mathrm{Conf}(M, g) = \mathrm{Conf}(M, [g])$.

We know several facts about the homothetic group, which can aid in its calculation.

Proposition 3.1.7. *Let (M, g) be a semi-Riemannian manifold. Define the map $f : \mathrm{Homoth}(M, g) \to \mathbb{R}$ that sends ϕ such that $\phi^*g = e^{2s}g$ to $f(\phi) = s$.*

Then f is a group homomorphism with kernel $\mathrm{Iso}(M, g)$.

Proof. f is a homomorphism because if $\phi^*g = e^{2s}g$, $\psi^*g = e^{2t}g$, then

$$\begin{aligned}(\phi\psi)^*g &= \psi^*\phi^*g \\ &= \psi^*(e^{2s}g) \\ &= e^{2t}e^{2s}g \\ &= e^{2(s+t)}g.\end{aligned}$$

If $\phi^* g = e^0 g$, then ϕ is an isometry, and so the kernel of f is the isometries. □

Corollary 3.1.8. *Let (M, g) be a semi-Riemannian manifold. Then for some subgroup H of $\mathbb{R}$,* $\mathrm{Homoth}(M, g) = \mathrm{Iso}(M, g) \rtimes H$. *Further, if for each $s \in \mathbb{R}$, there is a h_s such that $h_s^* g = e^{2s} g$, then* $\mathrm{Homoth}(M, g) = \mathrm{Iso}(M, g) \rtimes \mathbb{R}$.

Homotheties are quite rare in general, and in fact for complete Riemannian manifolds, we have a simple but stringent obstruction to admitting strict homotheties.

Remark 3.1.9. Alekseevski (1985) observes that it is well known that a complete Riemannian manifold with a 1-parameter subgroup of strict homotheties, then it is isometric to Euclidean space. The idea is that for a complete Riemannian manifold, we get an induced complete distance metric, such that homotheties of the manifold, are homotheties of the distance metric. Then application of the Banach fixed point theorem yields a fixed point. Then we can apply Theorem 2.3 of Obata (1970).

We also know that compact manifolds cannot have any strict homotheties. This theorem is preexisting (Alekseevski 1985, Corollary 2.1), although this proof seems to ignore questions of orientability. We make these concerns explicit in our proof.

Proposition 3.1.10. *Let (M, g) be a compact semi-Riemannian manifold. Then the homothetic and isometry groups are equal,* $\mathrm{Homoth}(M, g) = \mathrm{Iso}(M, g)$.

Proof. First we consider the case that M is orientable. Let ω be a volume form on M defined by g. Note that $\int_M \omega > 0$ (Lee 2012, Proposition 16.6(c)). Note also that $\phi^* \omega = \det(d\phi)\omega = \pm e^{ns}\omega$. Then (Lee 2012, Proposition 16.6(d))

$$\int_M \omega = \pm \int_M \phi^* \omega = \int_M e^{ns} \omega = e^{ns} \int_M \omega.$$

But then this implies that $s = 0$, and so ϕ is an isometry.

Now we consider non-orientable M, we do this by passing the problem to its orientation covering, $\hat{M}$. The orientation covering is defined as the set of all orientations of tangent spaces of M. We give this a manifold structure such that it is a double covering of M (O'Neill 1983, Corollary 7.9), and note that $\hat{M}$ is orientable (O'Neill 1983, Lemma 7.10).

We show that $\hat{M}$ is compact if M is compact. Cover $\hat{M}$ with open $\hat{U}_\alpha$ trivializing the covering, i.e. $\hat{U}_\alpha = \mathbb{Z}_2 \times U_\alpha$, for U_α open in M. Then for compact M, there is a finite subcover U_i of M. Note also that the closures $\overline{U_i}$ are compact. Then $\overline{\hat{U}_i}$ are a finite compact covering of $\hat{M}$, so $\hat{M}$ is compact.

We define a metric on $\hat{M}$ by $\pi^* g$, where $\pi : \hat{M} \to M$ is the projection map. Let ϕ be a homothety of M. We get an induced map $\hat{\phi} : \hat{M} \to \hat{M}$ defined by

$$\hat{\phi} : [e_1, \ldots, e_n] \mapsto [d\phi(e_1), \ldots d\phi(e_n)].$$

$\hat{\phi}$ can be seen to be a homothety, and $\hat{\phi}$ is an isometry if and only if ϕ is an isometry. We provide details of this in Theorem 3.5.8.

Then $\hat{\phi}$ must be an isometry, because $\hat{M}$ is compact and orientable. So ϕ is an isometry. □

A useful notion of a conformal manifold, is whether it locally has a flat representative metric. Note that this is equivalent locally there being a conformal transformation from the manifold to flat $\mathbb{R}^n$ with the Euclidean metric. The equivalence is simple: if a manifold locally has a flat representative metric, then by definition of flatness, the representative metric is locally isometric to flat $\mathbb{R}^n$. For the converse, we define our representative metric as the pull-back of the euclidean metric by the conformal transformation.

In dimension at least 4, these are true if and only if its Weyl tensor is identically 0 (Schouten 1954, VI.§5) (In dimension 1 or 2, every manifold is conformally flat, and in dimension 3, we consider instead the vanishing of the Cotton tensor).

When any (all) of these are satisfied, we say the manifold is conformally flat. We define conventions for curvature and the Weyl tensor now.

Definition 3.1.11. Let (M, g) be a semi-Riemannian manifold. Let X, Y, Z be vector fields on M. The $(1, 3)$ Riemannian curvature tensor R is defined to be

$$R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

We use the following convention for components of R in local coordinates

$$R_{\alpha\beta}{}^{\delta}{}_{\gamma} := dx^\delta(R(\partial_\alpha, \partial_\beta)\partial_\gamma).$$

The $(0, 2)$ Ricci curvature tensor is defined as

$$Ric_{\alpha\beta} := R_{\alpha\beta} = R_{\mu\alpha}{}^{\mu}{}_{\beta}.$$

And the scalar curvature is

$$s := g^{\mu\nu} R_{\mu\nu}.$$

Definition 3.1.12. Let h and k be two symmetric $(0,2)$ tensors. The Kulkarni-Nomizu product $h \owedge k$ is the $(0,4)$ tensor

$$(h \owedge k)_{\alpha\beta\gamma\delta} := h_{\alpha\gamma}k_{\beta\delta} - h_{\alpha\delta}k_{\beta\gamma} + h_{\beta\delta}k_{\alpha\gamma} - h_{\beta\gamma}k_{\alpha\delta}.$$

Definition 3.1.13. Let (M,g) be a semi-Riemannian manifold. The $(0,4)$ Weyl tensor (also known as the conformal curvature tensor) is

$$W = R - \frac{1}{n-2}\left(Ric - \frac{s}{2(n-1)}g\right) \owedge g$$

Where n is the dimension of M, R is the $(0,4)$ Riemannian curvature tensor, Ric is the Ricci tensor, and s is the scalar curvature.

Note in particular, that the Weyl tensor is a conformal invariant, if $\hat{g} = e^{2f}g$, then the corresponding Weyl tensors agree, $\hat{W} = W$ (Schouten 1954, VI.§5).

3.2 Conformal transformations of manifolds with parallel Weyl tensor

We present here a result of Cahen & Kerbrat (1982), in particular it restricts the conformal group of conformally curved locally symmetric spaces to their homothetic group.

Proposition 3.2.1 (Cahen & Kerbrat 1982, Proposition 2.1)**.** *Let (M,g) be a connected semi-Riemannian manifold of dimension $n \geq 4$, such that its conformal curvature (Weyl) tensor is parallel. Let $U \subset M$ be open, and let $\phi : U \to \phi(U) \subset M$ be a conformal diffeomorphism. Then ϕ is a homothety, or the Weyl tensor is identically zero.*

Proof. Here we present the proof provided by Cahen and Kerbrat, elaborating on some steps, and including a minor simplification.

Let $\phi^*g = e^{2f}g$, for a function $f : U \to \mathbb{R}$. We define $\hat{g} = e^{2f}g$. Let ∇ be the Levi-Civita connection on (M,g). Let $\hat{\nabla}$ be the Levi-Civita connection on $(U,\hat{g})$.

Throughout this proof, let X, Y, Z, T, V be vector fields on U. We define

$$\lambda(X,Y) := \hat{\nabla}_X Y - \nabla_X Y.$$

But then we observe that for metric rescalings, we have

$$\hat{\nabla}_X Y = \nabla_X Y + df(X)Y + df(Y)X - g(X,Y)\nabla f,$$

where ∇f is the gradient of f with respect to g, $\nabla f^\alpha = g^{\mu\alpha} df_\alpha$. And so

$$\lambda(X,Y) = df(X)Y + df(Y)X - g(X,Y)\nabla f.$$

Let W be the Weyl tensor. Since W is parallel, we have

$$\begin{aligned}(\nabla_X W)(Y,Z)T &= \nabla_X(W(Y,Z)T) - W(\nabla_X Y,Z)T - W(Y,\nabla_X Z)T \\ &\quad - W(Y,Z)\nabla_X T \\ &= 0,\end{aligned}$$

and hence

$$W(\nabla_X Y,Z)T + W(Y,\nabla_X Z)T + W(Y,Z)\nabla_X T = \nabla_X(W(Y,Z)T)$$

We need also this equation for $\hat\nabla$. Since W is conformally invariant (Schouten 1954, VI.§5), $\nabla W = 0$, and $e^{2f}g$ is isometric to g, we see also that

$$\begin{aligned}0 &= \phi_*((\nabla_X W)(Y,Z)T)) \\ &= \phi_*\nabla_X(W(Y,Z)T) - \phi_* W(\nabla_X Y,Z)T \\ &\quad - \phi_* W(Y,\nabla_X Z)T - \phi_* W(Y,Z)\nabla_X T \\ &= \hat\nabla_{\phi_* X}(\phi_* W(Y,Z)T) - W(\phi_*\nabla_X Y,\phi_* Z)\phi_* T \\ &\quad - W(\phi_* Y,\phi_*\nabla_X Z)\phi_* T - W(\phi_* Y,\phi_* Z)\phi_*\nabla_X T \\ &= \hat\nabla_{\phi_* X}(W(\phi_* Y,\phi_* Z)\phi_* T) - W(\hat\nabla_{\phi_* X}\phi_* Y,\phi_* Z)\phi_* T \\ &\quad - W(\phi_* Y,\hat\nabla_{\phi_* X}\phi_* Z)\phi_* T - W(\phi_* Y,\phi_* Z)\hat\nabla_{\phi_* X}\phi_* T \\ &= (\hat\nabla_{\phi_* X} W)(\phi_* Y,\phi_* Z)\phi_* T\end{aligned}$$

Then since this holds for all X, Y, Z, T, and ϕ_* is a bijection on the vector fields, we conclude that $\hat\nabla W = 0$, so in particular as above for ∇,

$$W(\hat\nabla_X Y,Z)T + W(Y,\hat\nabla_X Z)T + W(Y,Z)\hat\nabla_X T = \hat\nabla_X(W(Y,Z)T).$$

Now, using the equation $\lambda(X,Y) = \hat\nabla_X Y - \nabla_X Y$, we can see that

$$\begin{aligned}W(\lambda(X,Y)Z)T &+ W(Y,\lambda(X,Z))T + W(Y,Z)\lambda(X,T) \\ &= \hat\nabla_X(W(Y,Z)T) - \nabla_X(W(Y,Z)T) \\ &= \lambda(X,W(Y,Z)T).\end{aligned}$$

Now we expand this using the alternative form $\lambda(X,Y) = df(X)Y + df(Y)X - g(X,Y)\nabla f$ to get

$$\begin{aligned}2X(f)W(Y,Z)T &+ Y(f)W(X,Z)T + Z(f)W(Y,Z)T + T(f)W(Y,Z)X \\ &- g(X,Y)W(\nabla f,Z)T - g(X,Z)W(Y,\nabla f)T - g(X,T)W(Y,Z)\nabla f \\ &+ W^*(Y,Z,T,X)\nabla f - W^*(Y,Z,T,\nabla f)X = 0,\end{aligned} \tag{3.1}$$

where we define $W^*(Y,Z,T,X) := g(W(Y,Z)T,X)$. We will now show that by taking the trace of this with respect to the input X and the output, we will get $(3-n)W^*(Y,Z,T,\nabla f)=0$. To achieve this we shall convert the equation into abstract index notation. We let the X,Y,Z,T inputs correspond to the indices $\alpha,\beta,\gamma,\delta$, respectively, and let the contravariant index correspond to ϵ. Note that $\nabla f^\mu = g^{\mu\nu}df_\nu$, and that $W^*(Y,Z,T,X)$ corresponds to the tensor $W_{\beta\gamma\alpha\delta} = g_{\alpha\mu}W_{\beta\gamma}{}^{\mu}{}_{\delta}$. Then the equation is

$$\begin{aligned} 2df_\alpha W_{\beta\gamma}{}^{\epsilon}{}_{\delta} + df_\beta W_{\alpha\gamma}{}^{\epsilon}{}_{\delta} + df_\gamma W_{\beta\alpha}{}^{\epsilon}{}_{\delta} + df_\delta W_{\beta\gamma}{}^{\epsilon}{}_{\alpha} \\ - g_{\alpha\beta}W_{\mu\gamma}{}^{\epsilon}{}_{\delta}g^{\mu\nu}df_\nu - g_{\alpha\gamma}W_{\beta\mu}{}^{\epsilon}{}_{\delta}g^{\mu\nu}df_\nu - g_{\alpha\delta}W_{\beta\gamma}{}^{\epsilon}{}_{\mu}g^{\mu\nu}df_\nu \\ + g_{\alpha\mu}W_{\beta\gamma}{}^{\mu}{}_{\delta}g^{\epsilon\nu}df_\nu - W_{\beta\gamma}{}^{\mu}{}_{\delta}df_\mu\delta^{\epsilon}_{\alpha} = 0. \end{aligned}$$

Then we trace the α and ϵ indices together. We now tackle the result in parts. First we note that the 2nd, 3rd, and 4th terms on the first line are all 0 because W is trace free. Now we consider the terms on the second line, and show these sum to zero.

$$\begin{aligned} -g_{\alpha\beta}W_{\mu\gamma}{}^{\alpha}{}_{\delta}g^{\mu\nu}df_\nu - g_{\alpha\gamma}W_{\beta\mu}{}^{\alpha}{}_{\delta}g^{\mu\nu}df_\nu - g_{\alpha\delta}W_{\beta\gamma}{}^{\alpha}{}_{\mu}g^{\mu\nu}df_\nu \\ = -W_{\mu\gamma\beta\delta}g^{\mu\nu}df_\nu - W_{\beta\mu\gamma\delta}g^{\mu\nu}df_\nu - W_{\beta\gamma\delta\mu}g^{\mu\nu}df_\nu \\ = -g^{\mu\nu}df_\nu\left(W_{\mu\gamma\beta\delta} + W_{\beta\mu\gamma\delta} + W_{\gamma\beta\mu\delta}\right) \\ = 0 \end{aligned}$$

by the algebraic Bianchi identity. So now all that remains is to consider

$$\begin{aligned} 0 &= 2df_\alpha W_{\beta\gamma}{}^{\alpha}{}_{\delta} + g_{\alpha\mu}W_{\beta\gamma}{}^{\mu}{}_{\delta}g^{\alpha\nu}df_\nu - W_{\beta\gamma}{}^{\mu}{}_{\delta}df_\mu\delta^{\epsilon}_{\alpha} \\ &= 2df_\alpha W_{\beta\gamma}{}^{\alpha}{}_{\delta} + \delta^{\nu}_{\mu}W_{\beta\gamma}{}^{\mu}{}_{\delta}df_\nu - n\, W_{\beta\gamma}{}^{\mu}{}_{\delta}df_\mu \\ &= (3-n)df_\mu W_{\beta\gamma}{}^{\mu}{}_{\delta}. \end{aligned}$$

And hence, since the dimension n is at least 4, we have for vector fields on U,

$$W^*(Y,Z,T,\nabla f)=0.$$

So, using this, we take Eq. (3.1) and product it with V using the metric, and so we get

$$\begin{aligned} 2X(f)W^*(Y,Z,T,V) + Y(f)W^*(X,Z,T,V) + Z(f)W^*(Y,Z,T,V) \\ + T(f)W^*(Y,Z,X,V) + V(f)W^*(Y,Z,T,X) = 0. \end{aligned}$$

Now, assume that ϕ is not a homothety. Then there is an $x\in U$ such that $df|_x \neq 0$. So we may take a tangent vector τ at x such that $df(v) = -1/2$. So for all tangent vectors ξ,η,ζ,σ at x, we have

$$\begin{aligned} W^*|_x(\xi,\eta,\zeta,\sigma) = df(\xi)W^*|_x(\tau,\eta,\zeta,\sigma) + df(\eta)W^*|_x(\xi,\tau,\zeta,\sigma) \\ + df(\zeta)W^*|_x(\xi,\eta,\tau,\sigma) + df(\sigma)W^*|_x(\xi,\eta,\zeta,\tau). \end{aligned}$$

In particular, if ξ, η, ζ, σ are all in the kernel of df, then

$$W^*|_x(\xi, \eta, \zeta, \sigma) = 0.$$

If ξ, η, ζ are in the kernel of df, and $\sigma = \tau$, then

$$W^*|_x(\xi, \eta, \zeta, \tau) = -\frac{1}{2} W^*|_x(\xi, \eta, \zeta, \tau) = 0.$$

And if ξ, ζ are in the kernel of df, and $\eta = \sigma = \tau$, then

$$W^*|_x(\xi, \tau, \zeta, \tau) = -W^*|_x(\xi, \tau, \zeta, \tau) = 0.$$

Hence since $df(\tau)$ spans the image, and so the tangent space at x is $\mathbb{R}\tau \oplus \ker(df)$, we use these equations and the symmetries of W to conclude that $C|_x = 0$.

Then since W is parallel, and M is connected, W must be identically zero. □

Corollary 3.2.2. *Let (M, g) be a compact semi-Riemannian manifold, with non-zero, parallel Weyl curvature. Then the conformal group and the isometry group coincide,* $\mathrm{Conf}(M, g) = \mathrm{Iso}(M, g)$.

Proof. Applying Proposition 3.2.1, together with Proposition 3.1.10 gives the result. □

In particular, we note that for the condition that the Weyl tensor be parallel, it is sufficient that the space be locally symmetric, i.e. the Riemannian curvature tensor is parallel. Hence we can conclude for example that if a manifold if locally symmetric, compact, and conformally curved, then its conformal group and its isometry group coincide.

Note that Proposition 3.2.1 applies if the space is locally symmetric.

Proposition 3.2.3. *Let (M, g) be a locally symmetric semi-Riemannian manifold, i.e. $\nabla R = 0$. Then the Weyl tensor is parallel.*

Proof. Since $\nabla R = 0$, we have $\nabla Ric = 0$ and $ds = 0$. We also know that $\nabla g = 0$. So

$$\begin{aligned}
\nabla_X W = \nabla_X R - \frac{1}{n} \nabla_X \left(\left(Ric - \frac{s}{2(n+1)} g \right) \owedge g \right). \\
= -\frac{1}{n} \left(\nabla_X Ric - \frac{X(s)}{2(n+1)} g - \frac{s}{2(n+1)} \nabla_X g \right) \owedge g \\
- \frac{1}{n} \left(Ric - \frac{s}{2(n+1)} g \right) \owedge \nabla_X g \\
= 0.
\end{aligned}$$

□

Corollary 3.2.4. *Let (M, g) be a connected, locally symmetric, semi-Riemannian manifold of dimension $n \geq 4$. Let $U \subset M$ be open, and let $\phi : U \to \phi(U) \subset M$ be a conformal diffeomorphism. Then ϕ is a homothety, or the Weyl tensor is identically zero.*

3.3 Essential conformal transformations

In Section 1.2, we introduced the notion of an essential conformal structure; a manifold with a conformal class of metrics (M, c) is essential if there is no metric in the conformal class $g \in c$, such that the conformal group of g is equal to the isometry group of g. A natural way to consider such objects is to consider a stronger condition – that of a conformal class having an essential conformal transformation.

Definition 3.3.1. Let (M, c) be a conformal manifold. A conformal transformation $\phi \in \mathrm{Conf}(M, c)$ is *essential* if there is no metric in the equivalence class $g \in c$, such that ϕ is an isometry of g.

This notion is simpler to study than essential structures, because it allows us to consider the way a single element of the conformal group transforms under metric rescalings, rather than having to consider the group in its entirety.

Proposition 3.3.2. *Let (M, c) be a conformal manifold. If M has an essential conformal transformation ϕ, then (M, c) has an essential conformal structure.*

Proof. By definition, for all $g \in c$, ϕ is in $\mathrm{Conf}(M, g)$, but is not in $\mathrm{Iso}(M, g)$. But then for all g, $\mathrm{Conf}(M, g) \neq \mathrm{Iso}(M, g)$. Hence the conformal structure is essential. □

The converse of this proposition is a priori not true, it seems possible that every conformal transformation has a rescaled metric making it an isometry, but that the rescaling is different for each transformation, so making the structure inessential. Although, we have no example of such a structure.

Example 3.3.3. Consider $\mathbb{R}^n$ with the standard Euclidean metric g. Define $\phi : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto e^s x$, for $s \in \mathbb{R}$. ϕ is a homothety of g, $\phi^* g = e^{2s} g$. ϕ also fixes the origin, $\phi(0) = 0$. As we shall show in the next theorem, this information is sufficient to tell us that ϕ is an essential homothety, but we can also see this more informally:

Consider the rescaled metric $\hat{g} = ||x||^{-2} g$. Note that ϕ is an isometry of $\hat{g}$, but also that $\hat{g}$ has a singularity at the origin. This singularity partially justifies the essentiality of ϕ. But also what this demonstrates is that if we remove the origin and consider the open submanifold $M := \mathbb{R}^n \setminus \{0\}$, ϕ is inessential for (M, g), precisely because $\hat{g}$ is defined everywhere on M, and ϕ is an isometry of $(M, \hat{g})$.

Removing points in this way will often cause the submanifold to have fewer conformal transformations (M loses the isometries with a translation component), although it may at times gain inversion transformations (M has conformal transformations that $\mathbb{R}^n$ does not, e.g. $x \mapsto ||x||^{-2}x$), but more crucially, the submanifold will strictly gain metrics in the conformal class, specifically those rescalings with singularities at the removed points, and so runs the risk of losing essentiality.

Now we address essentiality for homotheties. In particular, we show that if a homothety has a fixed point then it is essential, and we give a necessary condition for essentiality in terms of fundamental regions.

Proposition 3.3.4. *Let (M, g) be a semi-Riemannian manifold. Let ϕ be a strict homothety of (M, g) with a finite orbit point, i.e. for some $x \in M$, and for some $k > 0$, $\phi^k(x) = x$. Then ϕ is essential.*

Proof. Let $\phi^* g = e^{2s} g$ for a real number s, we prove the contrapositive.

Assume $\phi^k(x) = x$, and ϕ is inessential: let f be a smooth function on M such that ϕ is an isometry of $e^{2f} g$. Then we evaluate at the fixed point x,

$$(e^{2f} g)|_x = (\phi^k)^*(e^{2f} g)|_x = e^{2f \circ \phi^k}(x)((\phi^k)^* g)|_x = e^{2ks} e^{2f(x)} g|_x = e^{2ks}(e^{2f} g)|_x$$

But then this implies $s = 0$, and so ϕ is an isometry. □

Corollary 3.3.5. *Let (M, g) be a semi-Riemannian manifold. Let ϕ be a strict homothety of (M, g) with a fixed point, $\phi(x) = x$. Then ϕ is essential.*

This proposition provides a sufficient condition for essentiality of a homothety. This next theorem provides a necessary condition for essentiality.

Theorem 3.3.6. *Let (M, g) be a semi-Riemannian manifold. Let ϕ be a strict homothety of (M, g), such that $\langle \phi \rangle$ admits a finitely self adjacent fundamental region. Then ϕ is not essential.*

Proof. Let R be a finitely self adjacent fundamental region. By Proposition 2.4.5, we can take a neighbourhood U of $\overline{R}$, such that $\{\phi^i U\}_{i \in \mathbb{Z}}$ is locally finite. We show that we can construct a partition of unity subordinate to this cover, with the additional property that $f_i = f_{i+1} \circ \phi$.

Take a bump function $\hat{f}_0 \geq 0$, whose support is in U, such that $\hat{f}_0|_{\overline{R}} \equiv 1$. Then for each $i \in \mathbb{Z}$, define

$$\hat{f}_i := \hat{f}_0 \circ \phi^{-i}.$$

Note that the support of $\hat{f}_i$ is contained within $\phi^i(U)$, and $\hat{f}_i|_{\phi^i(\overline{R})} \equiv 1$. Note also that

$$\begin{aligned} \hat{f}_i &= \hat{f}_0 \circ \phi^{-i-1} \circ \phi \\ &= \hat{f}_{i+1} \circ \phi. \end{aligned}$$

Then we define

$$f_i := \hat{f}_i / \left(\sum_{j \in \mathbb{Z}} \hat{f}_j \right)$$

Since the support of $\hat{f}_i$ are locally finite, the sum is finite in a neighbourhood of each point, so f_i is smooth since $\hat{f}_j$ are all smooth. Since $\phi^j(\overline{R})$ cover M, and $\hat{f}_j \geq 0$, we see that $\sum_{j \in \mathbb{Z}} \hat{f}_j$ has no zeros, so f_i is defined everywhere. Hence $\{f_i\}$ is a partition of unity, which satisfies $f_i = f_{i+1} \circ \phi$.

Now we define our rescaling f, so that ϕ is an isometry of $e^{2f}g$. Let $\phi^* g = e^{2s} g$. Define

$$f := -s \sum_{i \in \mathbb{Z}} (i f_i).$$

f is smooth since since each f_i is smooth, and their supports are locally finite. f also has the property that

$$\begin{aligned} f \circ \phi &= -s \sum_{i \in \mathbb{Z}} (i f_i \circ \phi) \\ &= -s \sum_{i \in \mathbb{Z}} ((i-1) f_{i-1} + f_{i-1}) \\ &= -s \sum_{i \in \mathbb{Z}} (i f_i) - s \sum_{i \in \mathbb{Z}} f_i \\ &= f - s. \end{aligned}$$

Hence we see that

$$\begin{aligned} \phi^*(e^{2f} g) &= e^{2f \circ \phi} \phi^* g \\ &= e^{2f \circ \phi} e^{2s} g \\ &= e^{2f} g. \end{aligned}$$

And so ϕ is inessential. □

Remark 3.3.7. The combination of Proposition 3.3.4 and Theorem 3.3.6 give the corollary that if a homothety has a fixed point, it cannot admit a finitely self adjacent fundamental region. This fact can also be seen more directly: Let ϕ be a homeomorphism of a topological space X. If $\phi(x) = x$ for some $x \in X$, then immediately $x \in \overline{R}$. And in particular $\phi^i(x) \in \overline{R}$ for every integer i. Now as long as $\langle\phi\rangle$ is an infinite group (a fact that is always true for strict homotheties, but may not be true for arbitrary homeomorphisms), then we have infinitely many elements ϕ^i of our group such that $\phi^i\overline{R}\cap\overline{R}$ contains x, and so is non empty. So from this we conclude that no fundamental region for this action can be finitely self adjacent (or even locally finite).

These results point us towards a third theorem we would like, that is if we can prove that whenever a strict homothety has no finite orbit points, it must admit a finitely self adjacent fundamental region, then we could conclude that for strict homotheties, essentiality is equivalent to having a fixed point. We are unable to prove this missing piece in general, but it is true for Cahen-Wallach spaces, as we will demonstrate in Theorem 4.4.1. In particular, the crucial argument to the existence of the fundamental region there has already been given in Proposition 2.4.4.

We formalize this in a conjecture

Conjecture 3.3.8. *Let (M, g) be a semi-Riemannian manifold. Let ϕ be a strict homothety of (M, g). If ϕ has no finite orbit points, then the action of $\langle\phi\rangle$ admits a finitely self adjacent fundamental region.*

If this conjecture holds to be true, then we get the immediate corollary that a strict homothety ϕ is essential if and only if ϕ has a finite orbit point, akin to Theorem 4.4.1.

Alekseevski (1985), later extended by Podoksenov (1989), claims a partial answer to this question for Lorentzian manifolds: if the Lorentzian manifold is causal then any 1-parameter subgroup of strict homotheties without fixed points is not essential. However, his proof depends on an earlier paper (Alekseevskii 1972), which Ferrand (1996) claims contains a gap. Ferrand credits the discovery of this gap to R.J. Zimmer and K.R. Gutschera, though the author of this thesis could not verify the source for this discovery. It is also not clear to the author of this thesis if the result in Alekseevski (1985) is impacted by this gap.

The difference between Alekseevsky's claimed result and our conjecture is that our results apply to individual homotheties, rather than relying on an entire 1-parameter subgroup of them, and so applies in more generality.

3.4 Conformal vector fields

Essentiality can be discussed for infinitesimal conformal transformations, which correspond to 1-parameter subgroups of conformal transformations. Many existing results apply here, e.g. the work by Alekseevski (1985), mentioned at the end of the previous section. Doing so has the disadvantage that we lose the ability to focus on individual conformal transformations. This disadvantage is the reason we do not give much focus to the infinitesimal perspective in this thesis. However, in this section, we briefly explore some well known results about infinitesimal transformations.

Definition 3.4.1. Let (M, g) be a semi-Riemannian manifold. Let X be a vector field on M. Then X is a *conformal vector field* if $\mathcal{L}_X g = \sigma g$, where $\mathcal{L}$ is the Lie derivative, and $\sigma : M \to \mathbb{R}$ is a smooth function.

Further, X is a homothetic vector field if σ is constant, and X is a Killing vector field if σ is zero.

Definition 3.4.2. Let (M, g) be a semi-Riemannian manifold. Let ϕ_t be a 1-parameter family of conformal transformations of (M, g). The corresponding conformal vector field is

$$X|_x := \frac{d}{dt}|_{t=0}\phi_t(x)$$

In the previous section, we demonstrated a link between essential homotheties, and homotheties with fixed points. Note that a zero of a conformal vector field is precisely a fixed point of its corresponding family of conformal transformations. Then the infinitesimal perspective gives a similar link for conformal vector fields and their zeros.

Proposition 3.4.3. *Let (M, g) be a semi-Riemannian manifold. Let X be a conformal vector field. At any point such that X is non-zero, there is a neighbourhood U and a rescaled metric $\hat{g}|_U$ such that X is a Killing vector field for $\hat{g}|_U$.*

Proof. Let $\mathcal{L}_X g = \sigma g$. Let X be non-zero at a point. We can find admissible coordinates (U, ϕ) at this point such that $X = \frac{\partial}{\partial x^1}$.

Let $f : U \to \mathbb{R}$ be a solution to

$$X(f) = \frac{\partial f}{\partial x^1} = -\sigma.$$

Then we have that

$$\begin{aligned} \mathcal{L}_X(e^f g) &= (X(f)e^f + e^f \sigma)g \\ &= 0. \end{aligned}$$

Hence X is a Killing vector field for $e^f g$. □

Further, if $g(X, X)$ is nowhere zero, this rescaling can be done globally.

Proposition 3.4.4. *Let (M, g) be a semi-Riemannian manifold. Let X be a conformal vector field. If $g(X, X)$ has no zeros, then there is a rescaled metric $\hat{g}$ such that $\mathcal{L}_X \hat{g} = 0$.*

Proof. Let $\mathcal{L}_X g = \sigma g$. If $g(X, X)$ has no zeros, then we define $\hat{g} := fg$, where

$$f := g(X, X)^{-1}.$$

Then we have that

$$\begin{aligned} X(f) &= -g(X, X)^{-2} X(g(X, X)) \\ &= -g(X, X)^{-2} (\mathcal{L}_X g)(X, X) \\ &= -g(X, X)^{-2} \sigma g(X, X) \\ &= -f\sigma. \end{aligned}$$

Then

$$\begin{aligned} \mathcal{L}_X(fg) &= (X(f) + f\sigma)g \\ &= 0 \end{aligned}$$

Hence X is a Killing vector field for $\hat{g}$

□

Corollary 3.4.5. *Let (M, g) be a Riemannian manifold. Let X be a conformal vector field. Then if X has no zeros, there is a rescaled metric $\hat{g}$ such that $\mathcal{L}_X \hat{g} = 0$.*

The consequence of these theorems is that in regards to essentiality one is only interested in conformal vector fields with zeros, but the question remains how to characterise the essential points (near which the metric cannot be rescaled such that the conformal vector field is Killing) amongst the zeros. In Riemannian signature, the situation is well understood (Moroianu et al. 2011). For indefinite metrics this result fails but several results are known, most importantly the results in Alekseevski (1985) and Podoksenov (1989), by Kühnel & Rademacher (1997, 1994), and the description of the zero set of conformal vector fields by Derdzinski (2011).

The advantage of the study of infinitesimal conformal fields is this wealth of preexisting results and knowledge, though it comes at the cost of being unable to make statements about individual transformations, and instead requires 1-parameter families of transformations. It is for this reason that we shall not consider conformal vector fields in detail for the remainder of this thesis.

3.5 Conformal quotients

In this section, we present some results about quotients of manifolds by conformal transformations, and the conformal structure of such quotients. First, recall from Theorem 2.2.10, that if a group action is properly discontinuous, then the quotient space carries a manifold structure. If we include the assumption that our action is by conformal transformations or isometries, then in addition the quotient space carries a conformal or semi-Riemannian metric structure respectively, i.e. an isometric quotient induces a metric structure, and a conformal quotient induces a conformal structure.

Definition 3.5.1 (O'Neill 1983, Definition 7.11)**.** Let $(\tilde{M}, \tilde{g})$, (M, g) be two semi-Riemannian manifolds. A map $\pi : \tilde{M} \to M$ is a *semi-Riemannian covering map* if π is a smooth covering map and $\pi^* g = \tilde{g}$.

When π is a smooth covering map, the second property as mentioned by O'Neill (O'Neill 1983, Definition 3.60) is equivalent, by the inverse function theorem, to each point of $\tilde{M}$ having a neighbourhood U such that $\phi|_U$ is an isometry.

Theorem 3.5.2 (O'Neill 1983, Corollary 7.12)**.** *Let* $(M, \tilde{g})$ *be a semi-Riemannian manifold. Let* $\Gamma < \mathrm{Iso}(\tilde{g})$ *be a group of isometries acting properly discontinuously. Then there is a unique metric* g *on* M/Γ *such that the projection map* $\pi : M \to M/\Gamma$ *is a semi-Riemannian covering map.*

Proof. By Theorem 2.2.10, it suffices to show there is a unique metric g such that $\pi^* g = \tilde{g}$.

Let q be a point in M/Γ. Let $\pi(p) = q$. Since π is locally a diffeomorphism, there is a neighbourhood U of p, such that $\pi|_U$ is a diffeomorphism. So in order that $\pi^* g = \tilde{g}$, we have no choice but to define

$$g|_q := \pi|_U^{-1*} \tilde{g}|_p.$$

We can rephrase this in the following way. Let v be a tangent vector at q. Since π is locally a diffeomorphism, at each point $p \in M$ such that $\pi(p) = q$, there is a unique tangent vector v_p such that $d\pi(v_p) = v$. In order that $\pi^* g = \tilde{g}$, we define for v, w tangent vectors at q,

$$g|_q(v, w) := \tilde{g}|_p(v_p, w_p).$$

We must check of course that this is well defined. Let $p' = \gamma(p)$, and U' a neighbourhood of p' such that $\pi|_{U'}$ is a diffeomorphism. But note that since

$\pi \circ \gamma^{-1} = \pi$, we have that $\pi|_{U'}^{-1} = \gamma \circ \pi|_{U}^{-1}$. Then, since Γ acts by isometries, we have

$$\begin{aligned}\pi|_{U'}^{-1*} g|_{p'} &= (\gamma \circ \pi|_{U}^{-1})^* g|_{p'} \\ &= \pi|_{U}^{-1*} \gamma^* g|_{p'} = \pi|_{U}^{-1*} g|_{p}.\end{aligned}$$

So g is well defined. We see that g is a smooth metric tensor in particular because it smoothly varies in any neighbourhood $\pi(U)$ of M such that $\pi|_U$ is a diffeomorphism, and these neighbourhoods cover M/Γ. □

We want an analogue of this theorem for conformal transformations of conformal manifolds.

Definition 3.5.3. Let $(\tilde{M}, \tilde{c})$, (M, c) be two conformal manifolds. A map $\pi : \tilde{M} \to M$ is a *conformal covering map* if π is a smooth covering map and each point in $\tilde{M}$ has a neighbourhood U such that $\pi|_U : (U, \tilde{c}|_U) \to (\pi(U), c|_{\pi(U)})$ is a conformal transformation.

We have here taken inspiration from the definition of a semi-Riemannian covering, and as in that definition, we will show now this added condition is equivalent to the statement that $\pi^* c = \tilde{c}$. It is immediate to see that if π is a smooth covering map such that $\pi^* c = \tilde{c}$, then π is a conformal covering map. The converse is also true since for representatives $\tilde{g}, g$, we have that $(\pi^* g)|_{\tilde{U}} = (f\tilde{g})|_{\tilde{U}}$ for some function f defined on $\tilde{U}$. But since a different f for a different $\tilde{U}$ agrees on overlaps, then f is unique, locally smooth, and since $\tilde{U}$'s cover $\tilde{M}$, f is globally defined. Hence $\pi^* c = \tilde{c}$.

Theorem 3.5.4. *Let $(M, \tilde{c})$ be conformal manifold. Let $\Gamma < \mathrm{Conf}(\tilde{c})$ be a group of conformal transformations that acts properly discontinuously. Then there is a unique conformal structure c on M/Γ such that the projection map $\pi : M \to M/\Gamma$ is a conformal covering map.*

Proof. Pick a representative $\tilde{g}$ of $\tilde{c}$. We define our conformal class as a ray bundle. We do this in a similar way as we defined the metric in the proof of Theorem 3.5.2. Let $q \in M/\Gamma$, and $p \in M$ such that $\pi(p) = q$. Since π is a local diffeomorphism, take a neighbourhood U of p such that $\pi|_U$ is a diffeomorphism. Then we have no choice but to define

$$c|_q := \mathbb{R}_{>0} \cdot (\pi|_{U}^{-1*} g|_p).$$

This is well defined, because if $p' = \gamma(p)$, and $\gamma^* g = f \cdot g$, then

$$\begin{aligned} c|_q &= \mathbb{R}_{>0} \cdot (\pi|_{U'}^{-1*} g|_{p'}) \\ &= \mathbb{R}_{>0} \cdot (\pi|_{U}^{-1*} \gamma^* g|_{p'}) \\ &= \mathbb{R}_{>0} \cdot (\pi|_{U}^{-1*} f(p) g|_{p}) \\ &= \mathbb{R}_{>0} \cdot f(p)(\pi|_{U}^{-1*} g|_{p}) \\ &= c|_q. \end{aligned}$$

This is a ray bundle with local trivialization given by the open sets U such that $\pi|_U$ is a diffeomorphism. □

Our primary method of finding transformations of quotients, is to construct our quotient group Γ so that it preserves a transformation of interest ϕ, usually by having Γ and ϕ commute with each other. The resulting transformation will preserve the induced structure. We will show these statements precisely now.

Proposition 3.5.5. *Let Γ be a group acting by homeomorphisms on a topological space $\tilde{X}$. Let $\tilde{\phi} : \tilde{X} \to \tilde{X}$ be a homeomorphism. If $\tilde{\phi}$ is an element in the normalizer of Γ, then there exists a homeomorphism $\phi : \tilde{X}/\Gamma \to \tilde{X}/\Gamma$ such that the following diagram commutes:*

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{\phi}} & \tilde{X} \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \pi} \\ \tilde{X}/\Gamma & \xrightarrow{\phi} & \tilde{X}/\Gamma \end{array} .$$

Proof. Being in the normalizer of Γ is precisely the condition that $\tilde{\phi}\Gamma = \Gamma\tilde{\phi}$, i.e. for all γ in Γ, there is a γ' in Γ such that $\tilde{\phi}\gamma = \gamma'\tilde{\phi}$. Define

$$\phi : [x] \mapsto [\tilde{\phi}(x)].$$

We show ϕ is well defined:

$$\begin{aligned} \phi([\gamma x]) = [\tilde{\phi}(\gamma x)] \\ &= [\gamma' \tilde{\phi}(x)] \\ &= [\tilde{\phi}(x)] \\ &= \phi([x]). \end{aligned}$$

The diagram commutes immediately by definition of ϕ. We have that ϕ is a homeomorphism because $\tilde{\phi}$ is a homeomorphism and π is a topology defining map □

The converse is not a priori true. Since for the induced map ϕ to be well defined, we require only that for each x and γ, there is a γ' such that $\tilde{\phi}(\gamma x) = \gamma'\tilde{\phi}(x)$, noting that the choice of element γ' can vary as x varies Whereas for $\tilde{\phi}$ to be in the normalizer of γ, we require that for all γ, there is a γ' such that for all x, $\tilde{\phi}(\gamma x) = \gamma'\tilde{\phi}(x)$, where γ' does not vary as x varies. It feels like the assumption that $\tilde{\phi}$ is a homeomorphism should eliminate the possibility of γ' varying in x, thus proving the converse, but we are unable to demonstrate that this is the case.

Corollary 3.5.6. *Let ϕ be a homeomorphism of a topological space $\tilde{X}$. Let Γ be a group centralizing ϕ, acting by homeomorphisms on $\tilde{X}$. Then there exists a homeomorphism $\phi : \tilde{X}/\Gamma \to \tilde{X}/\Gamma$ such that the following diagram commutes:*

$$\begin{CD} \tilde{X} @>{\tilde{\phi}}>> \tilde{X} \\ @V{\pi}VV @VV{\pi}V \\ \tilde{X}/\Gamma @>{\phi}>> \tilde{X}/\Gamma \end{CD} \quad .$$

This proposition gives us a condition for when a map on the cover descends to the quotient, the next theorem allows us to lift every map on the quotient to a map on the cover.

Proposition 3.5.7. *Let $\tilde{X}$, X be topological spaces, with $\tilde{X}$ simply connected. Let $\pi : \tilde{X} \to X$ be a covering map. Let $\phi : X \to X$ be a homeomorphism. Then there exists a $\tilde{\phi}$ such that the following diagram commutes:*

$$\begin{CD} \tilde{X} @>{\tilde{\phi}}>> \tilde{X} \\ @V{\pi}VV @VV{\pi}V \\ X @>{\phi}>> X \end{CD} .$$

Proof. We have a map $\phi \circ \pi : \tilde{X} \to X$. Pick any base point $\tilde{x}_0 \in X$, and a destination point $\tilde{y}_0$ such that $\pi(\tilde{y}_0) = \phi \circ \pi(\tilde{x}_0)$.

Then since $\tilde{X}$ is simply connected, there is a unique lift $\tilde{\phi} := \widetilde{\phi \circ \pi} : \tilde{X} \to \tilde{X}$ such that the diagram commutes, and $\phi(\tilde{x}_0) = \tilde{y}_0$ (Hatcher 2002, Proposition 1.33). $\square$

Theorem 3.5.8. *Let $(\tilde{M}, \tilde{g})$, (M, g) be semi-Riemannian manifolds, and let $\pi : \tilde{M} \to M$ be a semi-Riemannian covering map.*

Let $\tilde{\phi}$, ϕ be diffeomorphisms such that the following diagram commutes:

$$\begin{CD} \tilde{M} @>{\tilde{\phi}}>> \tilde{M} \\ @V{\pi}VV @VV{\pi}V \\ M @>{\phi}>> M \end{CD} .$$

Then ϕ is a homothety if and only if $\tilde{\phi}$ is a homothety. Further, ϕ is an isometry if and only if $\tilde{\phi}$ is an isometry.

Proof. Assume $\phi^* g = e^{2s} g$ is a homothety. The commuting diagram gives

$$\tilde{\phi}^* \tilde{g} = \tilde{\phi}^* \pi^* g = \pi^* \phi^* g = \pi^*(e^{2s} g) = e^{2s} \pi^* g = e^{2s} \tilde{g}.$$

So $\tilde{\phi}$ is a homothety. Further, if $s = 0$ so that ϕ is an isometry, then $\tilde{\phi}$ is an isometry.

Assume $\tilde{\phi}^* \tilde{g} = e^{2s} \tilde{g}$ is a homothety. The commuting diagram gives

$$\begin{aligned} \pi^* \phi^* g &= \tilde{\phi}^* \pi^* g \\ &= \tilde{\phi}^* \tilde{g} = e^{2s} \tilde{g} \\ &= \pi^*(e^{2s} g). \end{aligned}$$

When we evaluate this at a point in $\tilde{M}$, we observe that since π is semi-Riemannian covering map, there is a neighbourhood U of the point such that $\pi|_U$ is invertible, and $\pi^{-1}|_{\pi(U)}$ is also an isometry. Hence each point in $\tilde{M}$ has a neighbourhood U, such that $(\phi^* g)|_U = e^{2s} g|_U$, and so ϕ is a homothety. Further, if $s = 0$ so that $\tilde{\phi}$ is an isometry, then ϕ is an isometry. □

Theorem 3.5.9. *Let $(\tilde{M}, \tilde{c})$, (M, c) be conformal manifolds, and let $\pi : \tilde{M} \to M$ be a conformal covering map.*

Let $\tilde{\phi}$, ϕ be diffeomorphisms such that the following diagram commutes:

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{\phi}} & \tilde{M} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow{\phi} & M \end{array} .$$

The ϕ is a conformal transformation if and only if $\tilde{\phi}$ is a conformal transformation.

Proof. The proof is identical to the proof of Theorem 3.5.8.

Assume ϕ is a conformal transformation. The commuting diagram gives

$$\tilde{\phi}^* \tilde{c} = \tilde{\phi}^* \pi^* c = \pi^* \phi^* c = \pi^* c = \pi^* c = \tilde{c}.$$

So $\tilde{\phi}$ is conformal.

Assume $\tilde{\phi}$ is a conformal transformation. The commuting diagram gives

$$\begin{aligned} \pi^* \phi^* c &= \tilde{\phi}^* \pi^* c \\ &= \tilde{\phi}^* \tilde{c} = \tilde{c} \\ &= \pi^* c. \end{aligned}$$

When we evaluate this at a point in $\tilde{M}$, we observe that since π is a conformal covering map, there is a neighbourhood U of the point such that $\pi|_U$ is invertible, and $\pi^{-1}|_{\pi(U)}$ is a conformal transformation. Hence each point in $\tilde{M}$ has a neighbourhood U, such that $(\phi^* c)|_U = c|_U$, and so ϕ is conformal. $\square$

Theorem 3.5.10. *Let* $(\tilde{M}, \tilde{c})$, (M, c) *be conformal manifolds, and let* $\pi : \tilde{M} \to M$ *be a conformal covering map.*

Let $\tilde{\phi}$, ϕ *be diffeomorphisms such that the following diagram commutes:*

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{\phi}} & \tilde{M} \\ \big\downarrow \pi & & \big\downarrow \pi \\ M & \xrightarrow{\phi} & M \end{array}.$$

If $\tilde{\phi}$ *is an essential conformal transformation, then* ϕ *is an essential conformal transformation.*

Proof. Let $\tilde{\phi}$ be a conformal transformation. Then ϕ is a conformal transformation by Theorem 3.5.9. Assume for contrapositive that ϕ is inessential. Let $g \in c$ such that $\phi^* g = g$.

We have $\tilde{\phi}^* \pi^* g = \pi^* \phi^* g = \pi^* g$. So then we have that $\pi^* g$ is an element of $\tilde{c}$, (since $g \in c$, and $\pi^* c = \tilde{c}$) such that $\tilde{\phi}$ is an isometry of $\pi^* g$. So $\tilde{\phi}$ is inessential. $\square$

The converse is not expected to be true in general. If we have a compact quotient with an essential transformation, then there exists a lifted conformal transformation by Theorem 3.5.9. However, this lifted conformal transformation might not be essential. This can be seen by noting that the metric on the covering spaces can scale differently at each point of the fibers, but is more limited on the quotient.

Hence we know that if ϕ is essential, then the lift $\tilde{\phi}$ exists but may not be essential. On the other hand, if $\tilde{\phi}$ is essential and if ϕ exists, then ϕ is essential.

Chapter 4

Cahen-Wallach spaces

We want to consider conformally curved symmetric spaces. In Riemannian signature, there are many types of symmetric spaces to consider, but in Lorentzian signature, the list is much briefer. Cahen & Wallach (1970) classified the Lorentzian symmetric spaces. What we call Cahen-Wallach spaces are the indecomposable, solvable, Lorentzian, symmetric spaces as classified in their paper. A Cahen-Wallach space is in general also conformally curved, as we show later in Proposition 4.1.5.

Definition 4.0.1 (Cahen-Wallach spaces, $CW_{n+2}(S)$)**.** Consider the manifold $\mathbb{R}^{n+2}$ with global coordinates $(x^+, x^1, x^2, \ldots, x^n, x^-)$. We define $\boldsymbol{x} := (x^1, x^2, \ldots, x^n)$. Let S be a $n \times n$ symmetric non-degenerate matrix. Define a metric

$$g|_x := 2dx^+dx^- + d\boldsymbol{x}^\top d\boldsymbol{x} + \boldsymbol{x}^\top S\boldsymbol{x}(dx^+)^2.$$

The Cahen-Wallach space for S is

$$CW_{n+2}(S) := \left(\mathbb{R}^{n+2}, g\right).$$

Every indecomposable, solvable, Lorentzian, symmetric space is a Cahen-Wallach space (Cahen & Wallach 1970, Theorems 4 and 5(a)).

Remark 4.0.2. In terms of the standard coordinate basis for T_xCW: $\partial_\mu := \frac{\partial}{\partial x^\mu}$, ordered by the coordinate ordering, the metric $g|_x$ is the block matrix

$$\begin{pmatrix} \boldsymbol{x}^\top S\boldsymbol{x} & 0 & 1 \\ 0 & I_n & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Where I_n is the $n \times n$ identity matrix. The inverse of $g|_x$ is

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & I_n & 0 \\ 1 & 0 & -\boldsymbol{x}^\top S\boldsymbol{x} \end{pmatrix}.$$

We use Latin e.g. $i, j, k \in \{1, \dots n\}$ to index through the coordinates of $\boldsymbol{x}$, and we use Greek e.g. $\mu, \nu \in \{+, 1, \dots, n, -\}$ to index over all $n+2$ coordinates.

Proposition 4.0.3 (Cahen & Wallach 1970, Lemma 1(iii))**.** *Two Cahen-Wallach spaces $CW_{n+2}(S), CW_{n+2}(S')$ are isometric if and only if the eigenvalues of S and S' coincide up to a positive constant multiple.*

As a result of this proposition, we will whenever convenient assume that S is diagonal. This classification of Cahen-Wallach spaces in terms of the eigenvalues of S motivates a breakdown into types. In line with Kath & Olbrich (2019), we say that a Cahen-Wallach space is of *imaginary type* if all the eigenvalues of S are negative. Similarly, we say it is of real type if all the eigenvalues are positive, and of mixed type if it is neither real nor imaginary type.

4.1 Curvature

In this Section we provide an explicit calculation of Christoffel symbols, the Riemannian curvature tensor, the Weyl conformal curvature tensor, and the parallel vector fields of a Cahen-Wallach space.

We use the shorthand $\partial_\mu := \frac{\partial}{\partial x^\mu}$.

Lemma 4.1.1. *The only non-zero Christoffel symbols of $CW_{n+2}(S)$ are*

$$-\Gamma^i_{++}(x) = \Gamma^-_{+i}(x) = \Gamma^-_{i+}(x) = S_{ij}x^j.$$

Proof. Kozsul's formula for coordinate vectors gives us

$$2g(\nabla_{\partial_\mu}\partial_\nu, \partial_\alpha) = \frac{\partial}{\partial x^\mu}(g_{\nu\alpha}) + \frac{\partial}{\partial x^\nu}(g_{\mu\alpha}) - \frac{\partial}{\partial x^\alpha}(g_{\mu\nu}).$$

Since the only non-constant entry in g is g_{++} with respect to x^i, we see that $g(\nabla_{\partial_\mu}\partial_\nu, \partial_\alpha)$ is non zero if and only if two of μ, ν, α are $+$, and the third is i. There are three ways to choose these, and we calculate them now:

$$2g(\nabla_{\partial_+}\partial_+, \partial_i) = -\frac{\partial}{\partial x^i}(\boldsymbol{x}^\top S\boldsymbol{x}) = -2S_i\boldsymbol{x}.$$

Therefore $\Gamma^i_{++} = -S_i\boldsymbol{x}$.

$$2g(\nabla_{\partial_+}\partial_i, \partial_+) = 2g(\nabla_{\partial_i}\partial_+, \partial_+) = \frac{\partial}{\partial x^i}(\boldsymbol{x}^\top S\boldsymbol{x}) = 2S_i\boldsymbol{x}$$

Therefore $\Gamma^-_{+i} = S_i\boldsymbol{x}$. □

Proposition 4.1.2. *Let R be the Riemannian curvature tensor of $CW_{n+2}(S)$. Then as a $(0,4)$ tensor,*

$$R_{+ij+} = R_{i++j} = -R_{i+j+} = -R_{+i+j} = S_{ij}.$$

And all other components of R are 0.

Note that we are taking $R_{\alpha\beta\gamma\delta} = g_{\gamma\epsilon}R_{\alpha\beta}{}^{\epsilon}{}_{\delta}$, where $R_{\alpha\beta}{}^{\gamma}{}_{\delta}$ is defined as in Definition 3.1.11.

Proof. We take

$$\begin{aligned} R(\partial_\mu, \partial_\nu)\partial_\alpha &= \nabla_{\partial_\mu}\nabla_{\partial_\nu}\partial_\alpha - \nabla_{\partial_\nu}\nabla_{\partial_\mu}\partial_\alpha \\ &= \left(\frac{\partial}{\partial x^\mu}(\Gamma^\beta_{\nu\alpha}) - \frac{\partial}{\partial x^\nu}(\Gamma^\beta_{\mu\alpha})\right)\partial_\beta + \Gamma^\beta_{\nu\alpha}\Gamma^\gamma_{\mu\beta}\partial_\gamma - \Gamma^\beta_{\mu\alpha}\Gamma^\gamma_{\nu\beta}\partial_\gamma \end{aligned}$$

For this to be non-zero, we require first that none of μ, ν, α be $-$, since the Christoffel symbols of the form $\Gamma^\gamma_{\beta-}$ are all zero, and all Christoffel symbols are constant with respect to x^-. The remaining components (after accounting for symmetries of R) are

$$\begin{aligned} R(\partial_+, \partial_i)\partial_+ &= \nabla_{\partial_+}(S_i\boldsymbol{x}\partial_-) - \nabla_{\partial_i}(-S_j\boldsymbol{x}\partial_j) = S_{ij}\partial_j, \\ R(\partial_i, \partial_+)\partial_j &= \nabla_{\partial_i}(S_j\boldsymbol{x}\partial_-) = S_{ij}\partial_-, \\ R(\partial_i, \partial_j)\partial_+ &= \nabla_{\partial_i}(S_j\boldsymbol{x}\partial_-) - \nabla_{\partial_j}(S_i\boldsymbol{x}\partial_-) = (S_{ij} - S_{ji})\partial_- = 0, \\ R(\partial_i, \partial_j)\partial_k &= 0. \end{aligned}$$

The two non-zero terms above give rise to the same (up to symmetries of R) component of R as a $(0,4)$-tensor:

$$R_{+ij+} = S_{ij}.$$

We get the other non-zero terms by using the symmetries of R. □

Proposition 4.1.3. *The only non-zero component of the Ricci curvature tensor of $CW_{n+2}(S)$ is*

$$R_{++}|_x = -\operatorname{trace}(S).$$

In other words,

$$Ric = -\operatorname{trace}(S)(dx^+)^2.$$

Proof. The Ricci curvature tensor is

$$R_{\alpha\beta} = R_{\mu\alpha}{}^{\mu}{}_{\beta} = g^{\mu\nu}R_{\mu\alpha\nu\beta}.$$

Then using the values for the curvature found in Proposition 4.1.2, we see that the only potentially non-zero entries of the Ricci tensor are

$$\begin{aligned} R_{+i} = R_{i+} &= g^{\mu\nu} R_{\mu i \nu +} = g^{+j} S_{ij} = 0, \\ R_{++} &= g^{\mu\nu} R_{\mu + \nu +} = -g^{ij} S_{ij} = -\operatorname{trace}(S), \\ R_{ij} &= g^{\mu\nu} R_{\mu i \nu j} = -g^{++} S_{ij} = 0. \end{aligned}$$

□

Proposition 4.1.4. *The scalar curvature of $CW_{n+2}(S)$ is 0.*

Proof. The scalar curvature is

$$\begin{aligned} s &= g^{\mu\nu} R_{\mu\nu} \\ &= g^{++} R_{++} = 0. \end{aligned}$$

□

Proposition 4.1.5. *Let W be the Weyl tensor of $CW_{n+2}(S)$. Then as a $(0,4)$ tensor,*

$$W_{+ij+} = W_{i++j} = -W_{i+j+} = -W_{+i+j} = S_{ij} - \delta_{ij} \operatorname{trace}(S)/n,$$

where δ is the Kronecker delta. All other components of W are zero.

In particular, W is identically 0 if and only if S is a multiple of the identity matrix. Hence W is not identically 0 if and only if S has at least two distinct eigenvalues.

Proof. As defined in Definition 3.1.13, the Weyl tensor is (taking note of the fact that the dimension of our manifold is $n+2$)

$$W = R - \frac{1}{n}\left(Ric - \frac{s}{2(n+1)} g\right) \owedge g.$$

Where R is the Riemannian curvature, Ric is the Ricci curvature, s is the scalar curvature, and $\owedge$ is the Kulkarni-Nomizu product. By Proposition 4.1.4, $s = 0$. So

$$W = R - \frac{1}{n} Ric \owedge g.$$

We now expand the definition of $\owedge$ in terms of indices $\alpha, \beta, \gamma, \delta$

$$W_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - \frac{1}{n}\left(R_{\alpha\gamma} g_{\delta\beta} - R_{\alpha\delta} g_{\gamma\beta} + R_{\beta\delta} g_{\gamma\alpha} - R_{\beta\gamma} g_{\delta\alpha}\right).$$

First note that for W to be non-zero, at least two of $\alpha, \beta, \gamma, \delta$ must be $+$, since otherwise R and every Ricci term will be zero. But by the skew symmetries of the Weyl tensor, no more than two indices can be $+$ for non-zero terms. Hence the only possible non-zero terms (after accounting for symmetries) are:

$$\begin{aligned}
W_{+ij+} &= R_{+ij+} - \frac{1}{n}\left(R_{+j}g_{+i} - R_{++}g_{ji} + R_{i+}g_{j+} - R_{ij}g_{++}\right)\\
&= S_{ij} - \frac{1}{n}\left(0 + \operatorname{trace}(S)\delta_{ji} + 0 - 0\right)\\
&= S_{ij} - \delta_{ij}\operatorname{trace}(S)/n.\\
W_{+i+-} &= 0 - \frac{1}{n}\left(R_{++}g_{i-} - 0 + 0 - 0\right)\\
&= 0\\
W_{+-+-} &= 0 - \frac{1}{n}\left(R_{++}g_{--} - 0 + 0 - 0\right)\\
&= 0.
\end{aligned}$$

□

Proposition 4.1.6. *Let V be a vector field of $CW_{n+2}(S)$. Then V is parallel if and only if $V = a\partial_-$ for some $a \in \mathbb{R}$.*

Proof. Since V is parallel, $R_{\mu\nu\alpha\beta}V^\beta = 0$ for all μ, ν, α.

Consider $R_{+ij\beta}V^\beta = S_{ij}\delta^+_\beta V^\beta$, where δ is the Kronecker delta. Since this is 0 (and since S_{ij} is non-zero for some i, j), we see that $V^+ = 0$.

Next consider $R_{+i\alpha+}V^\alpha = S_{ij}V^j$. Since this is also 0, and since S is non-degenerate, this implies that $V^j = 0$ for all j.

Therefore $V = a\partial_-$ for some function (not necessarily constant) a.

We finally consider $\nabla_X(a\partial_-) = X(a)\partial_-$ (∂_- is parallel by the Christoffel symbols). This shows that $a\partial_-$ is parallel if and only if $X(a)$ is 0 for all vector fields X, which is true if and only if a is constant. □

4.2 The conformal group of Cahen-Wallach spaces

In this section, we explicitly calculate the conformal group of Cahen-Wallach spaces. We achieve this by first calculating the isometry and homothety groups. In (Kath & Olbrich 2019, Proposition 2.6), Kath and Olbrich provide a less naive approach to the calculation of the isometry group that extorts their construction of Cahen-Wallach spaces as symmetric triples. The advantage of our approach is that it allows us to realize explicitly how the group acts on our space.

Theorem 4.2.1 (Iso($CW_{n+2}(S)$)). *Let U be an open set in $CW_{n+2}(S)$, and let $\phi : U \to \phi(U)$ be a diffeomorphism. Then ϕ is an isometry, $\phi^* g = g$, if and only if*

$$(\beta, b, a, c, A) := \phi : \begin{pmatrix} x^+ \\ \boldsymbol{x} \\ x^- \end{pmatrix} \mapsto \begin{pmatrix} ax^+ + c \\ A\boldsymbol{x} + \beta(x^+) \\ a\left(x^- + b - \left\langle \dot\beta(x^+), A\boldsymbol{x} + \frac{1}{2}\beta(x^+)\right\rangle\right) \end{pmatrix},$$

where $a = \pm 1$, $b, c \in \mathbb{R}$, $A \in Z_{O(n)}(S)$ is an orthogonal matrix commuting with S, and $\beta : \mathbb{R} \to \mathbb{R}^n$, satisfying the second order ODE $\ddot\beta = S\beta$. Where $\ddot\beta$ denotes $\frac{d^2\beta}{d(x^+)^2}$, $\dot\beta$ denotes $\frac{d\beta}{dx^+}$ and $\langle ., . \rangle$ is the standard Euclidean inner product on $\mathbb{R}^n$.

Proof. Let $\phi^* g = g$. We know that the push forward of a parallel vector field by ϕ is parallel, because

$$\nabla_X \phi_* V = \nabla_{\phi_* X'} \phi_* V = \phi_*(\nabla_X V) = 0.$$

(ϕ_* is surjective on on the vector fields because $\phi_*^{-1} = (\phi_*)^{-1}$). And so by Proposition 4.1.6, $\phi_* \partial_- = a\partial_-$ for some $a \in \mathbb{R}$.

We write ϕ^μ as a shorthand for $x^\mu \circ \phi$, and $\boldsymbol{\phi}$ as shorthand for $\boldsymbol{x} \circ \phi$. The previous statement says that $\frac{\partial \phi^\mu}{\partial x^-} = a\delta^\mu_-$.

We consider cases of the following equation for an isometry

$$g_{\alpha\beta} = \left(\frac{\partial \phi^\mu}{\partial x^\alpha}\right)\left(\frac{\partial \phi^\nu}{\partial x^\beta}\right) g_{\mu\nu} \circ \phi.$$

Recall that

$$g = \begin{pmatrix} \boldsymbol{x}^\top S \boldsymbol{x} & 0 & 1 \\ 0 & I_n & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

I.e. $g_{++} = \boldsymbol{x}^\top S\boldsymbol{x}$, $g_{+-} = g_{-+} = 1$, $g_{ij} = \delta_{ij}$, and $g_{\mu\nu} = 0$ otherwise.

$$1 = g_{-+} = a\delta^\mu_- \left(\frac{\partial \phi^\nu}{\partial x^+}\right) g_{\mu\nu} \circ \phi = a\left(\frac{\partial \phi^\nu}{\partial x^+}\right) g_{-\nu} \circ \phi.$$

$$0 = g_{-i} = a\left(\frac{\partial \phi^\nu}{\partial x^i}\right) g_{-\nu} \circ \phi.$$

These two equations tell us that

$$\frac{\partial \phi^+}{\partial x^+} = a^{-1}, \qquad \frac{\partial \phi^+}{\partial x^i} = 0$$

Hence

$$\phi^+(x) = a^{-1}x^+ + c,$$

for some constant $c \in \mathbb{R}$. Next, we have

$$\delta_{ij} = g_{ij} = \sum_k \left(\frac{\partial \phi^k}{\partial x^i}\right)\left(\frac{\partial \phi^k}{\partial x^j}\right) = \left\langle \frac{\partial \boldsymbol{\phi}}{\partial x^i}, \frac{\partial \boldsymbol{\phi}}{\partial x^j} \right\rangle.$$

This tells us in particular that $\{\frac{\partial \boldsymbol{\phi}}{\partial x^i}\}$ span $\mathbb{R}^n$, and that $\frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{x}}$ is an orthogonal matrix at each x.

Then consider

$$\begin{aligned} 0 = &\frac{\partial}{\partial x^k}\left(\left\langle \frac{\partial \boldsymbol{\phi}}{\partial x^i}, \frac{\partial \boldsymbol{\phi}}{\partial x^j} \right\rangle\right) + \frac{\partial}{\partial x^j}\left(\left\langle \frac{\partial \boldsymbol{\phi}}{\partial x^k}, \frac{\partial \boldsymbol{\phi}}{\partial x^i} \right\rangle\right) \\ &- \frac{\partial}{\partial x^i}\left(\left\langle \frac{\partial \boldsymbol{\phi}}{\partial x^j}, \frac{\partial \boldsymbol{\phi}}{\partial x^k} \right\rangle\right). \\ \implies &2\left\langle \frac{\partial^2 \boldsymbol{\phi}}{\partial x^k \partial x^j}, \frac{\partial \boldsymbol{\phi}}{\partial x^i} \right\rangle = 0. \end{aligned}$$

Since this equation holds for all i, j, k, and since $\{\frac{\partial(\phi)}{\partial x^i}\}$ span $\mathbb{R}^n$, we see that

$$\frac{\partial^2 \boldsymbol{\phi}}{\partial x^k \partial x^j} = 0$$

for all j, k. Then this tells us that

$$\boldsymbol{\phi}(x) = A(x^+)\boldsymbol{x} + \beta(x^+),$$

for $A(x^+)$ orthogonal, $\beta(x^+) \in \mathbb{R}^n$.

$$\begin{aligned} 0 = g_{+i} &= a^{-1}\frac{\partial \phi^-}{\partial x^i} + \frac{\partial \boldsymbol{\phi}}{\partial x^+}^{\top}\frac{\partial \boldsymbol{\phi}}{\partial x^i} \\ &= a^{-1}\frac{\partial \phi^-}{\partial x^i} + (\dot{A}\boldsymbol{x} + \dot{\beta})^{\top}A_i. \end{aligned}$$

Then we differentiate this with respect to x^k to get

$$\frac{\partial^2 \phi^-}{\partial x^k \partial x^i} = -a\dot{A}_k^{\top}A_i.$$

But since $\frac{\partial^2\phi^-}{\partial x^k \partial x^i} = \frac{\partial^2\phi^-}{\partial x^i \partial x^k}$, we have $\dot{A}_k^\top A_i = \dot{A}_i^\top A_k$ for all i, k. So, $\dot{A}^\top A = A^\top \dot{A}$. Since A is orthogonal, we can differentiate $A^\top A = I_n$ with respect to x^+ to get $\dot{A}^\top A + A^\top \dot{A} = 0$, and hence $2\dot{A}^\top A = 0$. Then since A is non-degenerate, we conclude $\dot{A} = 0$, and so A has no dependence on x^+.

Now we look back to the g_{+i} equation and conclude that

$$\frac{\partial\phi^-}{\partial x^i} = -a\dot{\beta}^\top A_i.$$

Now finally we consider

$$\boldsymbol{x}^\top S\boldsymbol{x} = g_{++} = a^{-2}(A\boldsymbol{x}+\beta)^\top S(A\boldsymbol{x}+\beta) + 2a^{-1}\frac{\partial\phi^-}{\partial x^+} + \dot{\beta}^\top\dot{\beta}.$$

We take the derivative with respect to x^i to obtain

$$2S_i\boldsymbol{x} = 2a^{-2}A_i^\top S(A\boldsymbol{x}+\beta) + 2a^{-1}\frac{\partial^2\phi^-}{\partial x^i \partial x^+}.$$

Since we know $\frac{\partial\phi^-}{\partial x^i} = -a\dot{\beta}^\top A_i$, we can see that $\frac{\partial^2\phi^-}{\partial x^i \partial x^+} = -a\ddot{\beta}^\top A_i$. So since this holds for all $\boldsymbol{x}$, we see

$$\begin{aligned} S_i &= a^{-2}A^\top S A_i \\ \implies AS &= a^{-2}SA \\ 0 &= 2a^{-2}\beta^\top S A_i - 2\ddot{\beta}^\top A_i \\ \implies a^{-2}\beta^\top S &= \ddot{\beta}^\top. \end{aligned}$$

These tell us that (by taking the trace of the second equation) $a^{-2} = 1$, and in particular $a^{-1} = a$, A is in the centralizer of S, and $\ddot{\beta} = S\beta$.

Now we revisit the g_{++} equation to find the form of ϕ^-.

$$\begin{aligned} 0 &= 2\boldsymbol{x}^\top A^\top S\beta + \beta^\top S\beta + 2a\frac{\partial\phi^-}{\partial x^+} + \dot{\beta}^\top\dot{\beta} \\ \implies -2a\frac{\partial\phi^-}{\partial x^+} &= 2\left\langle\ddot{\beta}, A\boldsymbol{x}\right\rangle + \left\langle\ddot{\beta}, \beta\right\rangle + \left\langle\dot{\beta}, \dot{\beta}\right\rangle \\ &= \frac{\partial}{\partial x^+}\left(2\left\langle\dot{\beta}, A\boldsymbol{x}\right\rangle + \left\langle\dot{\beta}, \beta\right\rangle\right) \end{aligned}$$

So we can finally conclude that

$$\phi^-(x) = ax^- + ab - a\left\langle\dot{\beta}(x^+), A\boldsymbol{x} + \frac{1}{2}\beta(x^+)\right\rangle$$

For some constant $b \in \mathbb{R}$.

To show the converse, we verify that every such map is in fact an isometry. Note that since $(\beta, b, a, c, A) = (0, 0, a, c, I_n) \circ (\beta, 0, 1, 0, I_n) \circ (0, b, 1, 0, A)$, to simplify the calculation slightly we consider the maps with only a β-term, and those with no β-term separately.

In coordinates, we can calculate the pull back by the matrix multiplication. Let $\phi = (\beta, 0, 1, 0, I_n)$.

$$\begin{aligned}
\phi^* g|_x &= d\phi|_x^\top g|_{\phi(x)} d\phi|_x \\
&= \begin{pmatrix} 1 & \dot{\beta}^\top & -\left\langle \ddot{\beta}, \boldsymbol{x} + \frac{1}{2}\beta \right\rangle - \frac{1}{2}\left\langle \dot{\beta}, \dot{\beta} \right\rangle \\ 0 & I_n & -\dot{\beta} \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{\phi}(x)^\top S \boldsymbol{\phi}(x) & 0 & 1 \\ 0 & I_n & 0 \\ 1 & 0 & 0 \end{pmatrix} \\
&\quad \cdot \begin{pmatrix} 1 & 0 & 0 \\ \dot{\beta} & I_n & 0 \\ -\left\langle \ddot{\beta}, \boldsymbol{x} + \frac{1}{2}\beta \right\rangle - \frac{1}{2}\left\langle \dot{\beta}, \dot{\beta} \right\rangle & -\beta^\top & 1 \end{pmatrix} \\
&= \begin{pmatrix} \boldsymbol{\phi}(x)^\top S \boldsymbol{\phi}(x) - \left\langle \ddot{\beta}, \boldsymbol{x} + \frac{1}{2}\beta \right\rangle - \frac{1}{2}\left\langle \dot{\beta}, \dot{\beta} \right\rangle & \dot{\beta}^\top & 1 \\ -\dot{\beta} & I_n & 0 \\ 1 & 0 & 0 \end{pmatrix} \\
&\quad \cdot \begin{pmatrix} 1 & 0 & 0 \\ \dot{\beta} & I_n & 0 \\ -\left\langle \ddot{\beta}, \boldsymbol{x} + \frac{1}{2}\beta \right\rangle - \frac{1}{2}\left\langle \dot{\beta}, \dot{\beta} \right\rangle & -\beta^\top & 1 \end{pmatrix} = \begin{pmatrix} (*) & 0 & 1 \\ 0 & I_n & 0 \\ 1 & 0 & 0 \end{pmatrix}
\end{aligned}$$

We calculate the top left entry of the final matrix here seperately. Keep in mind that S is symmetric, and $\ddot{\beta} = S\beta$.

$$\begin{aligned}
(*) = (\boldsymbol{x} + \beta)^\top S (\boldsymbol{x} + \beta) &- \left\langle \ddot{\beta}, \boldsymbol{x} + \frac{1}{2}\beta \right\rangle - \frac{1}{2}\left\langle \dot{\beta}, \dot{\beta} \right\rangle \\
&+ \dot{\beta}^\top \dot{\beta} - \left\langle \ddot{\beta}, \boldsymbol{x} + \frac{1}{2}\beta \right\rangle - \frac{1}{2}\left\langle \dot{\beta}, \dot{\beta} \right\rangle \\
&= \boldsymbol{x}^\top S \boldsymbol{x} + \langle S\beta, \boldsymbol{x} \rangle + \langle \boldsymbol{x}, S\beta \rangle + \langle S\beta, \beta \rangle - \left\langle \ddot{\beta}, \boldsymbol{x} + \frac{1}{2}\beta \right\rangle \\
&\quad - \frac{1}{2}\left\langle \dot{\beta}, \dot{\beta} \right\rangle + \left\langle \dot{\beta}, \dot{\beta} \right\rangle - \left\langle \ddot{\beta}, \boldsymbol{x} + \frac{1}{2}\beta \right\rangle - \frac{1}{2}\left\langle \dot{\beta}, \dot{\beta} \right\rangle \\
&= \boldsymbol{x}^\top S \boldsymbol{x} + 2\left\langle \ddot{\beta}, \boldsymbol{x} \right\rangle + \left\langle \ddot{\beta}, \beta \right\rangle - 2\left\langle \ddot{\beta}, \boldsymbol{x} + \frac{1}{2}\beta \right\rangle \\
&= \boldsymbol{x}^\top S \boldsymbol{x}.
\end{aligned}$$

So, $\phi^* g = g$.

Now let $\phi = (0, b, a, c, A)$. Keep in mind that $a^2 = 1$, $A^\top A = 1$, and $A^\top S A = S$.

$$\begin{aligned}
\phi^* g|_x &= d\phi|_x^\top g|_{\phi(x)} d\phi|_x \\
&= \begin{pmatrix} a & 0 & 0 \\ 0 & A^\top & 0 \\ 0 & 0 & a \end{pmatrix} \begin{pmatrix} (A\boldsymbol{x})^\top S A\boldsymbol{x} & 0 & 1 \\ 0 & I_n & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & a \end{pmatrix} \\
&= \begin{pmatrix} a^2 \boldsymbol{x}^\top A^\top S A \boldsymbol{x} & 0 & a^2 \\ 0 & A^\top A & 0 \\ a^2 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \boldsymbol{x}^\top S \boldsymbol{x} & 0 & 1 \\ 0 & I_n & 0 \\ 1 & 0 & 0 \end{pmatrix} = g|_x.
\end{aligned}$$

So, $\phi^* g = g$. □

Definition 4.2.2. For any $s \in \mathbb{R}$, and any $U \subset CW_{n+2}(S)$, define the *pure homothety* of U with homothetic factor s to be

$$\begin{aligned}
h_s &: U \to h_s(U), \\
h_s &: \begin{pmatrix} x^+ \\ \boldsymbol{x} \\ x^- \end{pmatrix} \mapsto \begin{pmatrix} x^+ \\ e^s \boldsymbol{x} \\ e^{2s} x^- \end{pmatrix}.
\end{aligned}$$

Lemma 4.2.3. *Let h_s be a pure homothety of $U \subset CW_{n+2}(S)$. Then $h_s^* g = e^{2s} g$.*

Proof. Note that as matrices

$$h_s^* g|_x = dh_s|_x^\top g|_{h_s(x)} dh_s|_x,$$

where

$$\begin{aligned}
dh_s|_x &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^s & 0 \\ 0 & 0 & e^{2s} \end{pmatrix}, \\
g|_{h_s(x)} &= \begin{pmatrix} e^{2s} \boldsymbol{x}^\top S \boldsymbol{x} & 0 & 1 \\ 0 & I_n & 0 \\ 1 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

So,

$$
\begin{aligned}
h_s^* g|_x &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^s & 0 \\ 0 & 0 & e^{2s} \end{pmatrix} \begin{pmatrix} e^{2s}\boldsymbol{x}^\top S\boldsymbol{x} & 0 & 1 \\ 0 & I_n & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^s & 0 \\ 0 & 0 & e^{2s} \end{pmatrix} \\
&= e^{2s} \begin{pmatrix} \boldsymbol{x}^\top S\boldsymbol{x} & 0 & 1 \\ 0 & I_n & 0 \\ 1 & 0 & 0 \end{pmatrix} = e^{2s} g.
\end{aligned}
$$

□

Proposition 4.2.4 (Homoth($CW_{n+2}(S)$)**.** *Let U be open in $CW_{n+2}(S)$, and let $\phi : U \to \phi(U)$ be a diffeomorphism. Then ϕ is a homothety, $\phi^* g = e^{2s} g$, if and only if ϕ is the composition of an isometry and the pure homothety with homothetic factor s,*

$$(\beta, b, a, c, A, s) := \phi = (\beta, b, a, c, A) \circ h_s.$$

Where (β, b, a, c, A) is an isometry as in Theorem 4.2.1.

Proof. By Lemma 4.2.3, for each s, we have $h_s^* g = e^{2s} g$. Then by Corollary 3.1.8, every homothety is of the desired form. □

Proposition 4.2.5 (Conf($CW_{n+2}(S)$)**.** *Let U be an open set in $CW_{n+2}(S)$, and let $\phi : U \to \phi(U)$ be a diffeomorphism.*

Let S have at least two distinct eigenvalues. Then ϕ is a conformal transformation if and only if ϕ is a homothety, i.e.

$$\phi = (\beta, b, a, c, A, s) : \begin{pmatrix} x^+ \\ \boldsymbol{x} \\ x^- \end{pmatrix} \mapsto \begin{pmatrix} ax^+ + c \\ e^s A\boldsymbol{x} + \beta(x^+) \\ a\left(e^{2s} x^- + b - \left\langle \dot\beta(x^+), e^s A\boldsymbol{x} + \frac{1}{2}\beta(x^+) \right\rangle \right) \end{pmatrix}$$

where $a = \pm 1$, $b, c, s \in \mathbb{R}$, $A \in Z_{O(n)}(S)$ is an orthogonal matrix commuting with S, and $\beta : \mathbb{R} \to \mathbb{R}^n$, satisfying the second order ODE $\ddot\beta = S\beta$. Where $\ddot\beta$ denotes $\frac{d^2\beta}{d(x^+)^2}$, and $\langle ., . \rangle$ is the standard Euclidean inner product on $\mathbb{R}^n$.

Proof. When S has at least two distinct eigenvalues, then by Proposition 4.1.5, the Weyl tensor is non-zero. Then since the Weyl tensor is parallel by Proposition 3.2.3, we conclude that every conformal transformation is a homothety by Proposition 3.2.1. □

Now we calculate the group composition.

Proposition 4.2.6. *Let S be symmetric. Let $\beta, \beta' : \mathbb{R} \to \mathbb{R}^n$, such that $\ddot{\beta} = S\beta$, and $\ddot{\beta}' = S\beta'$.*

Define $\omega(\beta, \beta') := \left\langle \beta, \dot{\beta}' \right\rangle - \left\langle \dot{\beta}, \beta' \right\rangle$. Then $\omega(\beta, \beta')$ is constant.

Proof. Take the derivative of ω,

$$\begin{aligned}\dot{\omega}(\beta, \beta') &= \left\langle \dot{\beta}, \dot{\beta}' \right\rangle + \left\langle \beta, \ddot{\beta}' \right\rangle - \left\langle \ddot{\beta}, \beta' \right\rangle - \left\langle \dot{\beta}, \dot{\beta}' \right\rangle \\ &= \langle \beta, S\beta' \rangle - \langle S\beta, \beta' \rangle\end{aligned}$$

Then since S is symmetric, this is 0, and so ω is constant. □

Proposition 4.2.7. *Let (β, b, a, c, A, s), $(\beta', b', a', c', A', s')$ be two homotheties of open sets in $CW_{n+2}(S)$. Let ψ be their composition*

$$\psi := (\beta, b, a, c, A, s) \circ (\beta', b', a', c', A', s').$$

Let ψ be of the form $(\beta_\psi, b_\psi, a_\psi, c_\psi, A_\psi, s_\psi)$. Then the variables defining ψ are given by:

$$\begin{aligned}\beta_\phi &= e^s A\beta' + \beta \circ E_{a',c'} \\ b_\psi &= e^{2s} b' + ba' + \frac{1}{2}\omega(\beta \circ E_{a',c'}, e^s A\beta') \\ a_\psi &= aa' \\ c_\psi &= ac' + c \\ A_\psi &= AA' \\ s_\psi &= s + s',\end{aligned}$$

where ω is as in Proposition 4.2.6, and $E_{a',c'} : \mathbb{R} \to \mathbb{R}$ is the Euclidean transformation of $\mathbb{R}^1$ given by $E_{a',c'} : x \mapsto a'x + c'$.

Proof. For convenience, we remind the reader of the definition of $\phi_{\beta,b,a,c,A,s}$:

$$\phi_{\beta,b,a,c,A,s} : \begin{pmatrix} x^+ \\ \boldsymbol{x} \\ x^- \end{pmatrix} \mapsto \begin{pmatrix} ax^+ + c \\ e^s A\boldsymbol{x} + \beta(x^+) \\ a\left(e^{2s}x^- + b - \left\langle \dot{\beta}(x^+), e^s A\boldsymbol{x} + \frac{1}{2}\beta(x^+) \right\rangle\right) \end{pmatrix}.$$

Then the composition is given by

$$\psi : \begin{pmatrix} x^+ \\ \boldsymbol{x} \\ x^- \end{pmatrix} \mapsto \begin{pmatrix} a(a'x^+ + c') + c \\ \\ e^s A\left(e^{s'} A'\boldsymbol{x} + \beta'(x^+)\right) + \beta(a'x^+ + c') \\ \\ a\left(e^{2s} a'\left(e^{2s'} x^- + b' - \left\langle \dot{\beta}'(x^+), A' e^{s'}\boldsymbol{x} + \frac{1}{2}\beta'(x^+) \right\rangle\right) + b \ldots \right. \\ \left. \ldots - \left\langle \dot{\beta}(a'x^+ + c'), e^s A\left(e^{s'} A'\boldsymbol{x} + \beta'(x^+)\right) + \frac{1}{2}\beta(a'x^+ + c') \right\rangle\right) \end{pmatrix}.$$

From the first two coordinates, every term except b_ψ can be read off and verified with the statement of the proposition immediately.

What remains is to calculate b_ψ, this is done by manipulating ϕ^- into the following form:

$$aa'\left(e^{2s+2s'}x^- + b_\psi - \Big\langle e^s A\dot{\beta}'(x^+) + a'\dot{\beta}(a'x^+ + c'), \ldots \right.$$
$$\left. \ldots e^{s+s'}AA'\boldsymbol{x} + \frac{1}{2}(e^s A\beta'(x^+) + \beta(a'x^+ + c'))\Big\rangle\right).$$

The calculation is simple manipulation of a large number of symbols and is unenlightening, so we do not include it.

We note that $\omega(\beta \circ E_{a',c'}, e^s A\beta')$ is constant with respect to x^+ for two reasons: firstly we expect it to be, in order that the homotheties are a group closed under composition, and secondly because $\frac{\partial^2(\beta \circ E_{a',c'})}{(\partial x^+)^2} = a'^2\ddot{\beta} \circ E_{a',c'} = S\beta \circ E_{a',c'}$, and $\frac{\partial^2(e^s A\beta')}{(\partial x^+)^2} = e^s AS\beta' = S(e^s A\beta')$. So Proposition 4.2.6 applies and tells us that ω is constant. □

Corollary 4.2.8. *Let $\phi = (\beta, b, a, c, A, s)$ be a homothety of $CW_{n+2}(S)$, and let $\psi := \phi^{-1}$. Let ψ be of the form $(\beta_\psi, b_\psi, a_\psi, c_\psi, A_\psi, s_\psi)$. Then the variables defining ψ are given by:*

$$\begin{aligned}
\beta_\psi &= -e^{-s}A^\top \beta \circ E_{a,-ac} \\
b_\psi &= -e^{-2s}ab \\
a_\psi &= a \\
c_\psi &= -ac \\
A_\psi &= A^\top \\
s_\psi &= -s
\end{aligned}$$

Theorem 4.2.9. *The homothety group of a Cahen-Wallach space is isomorphic (as a group) to*

$$\mathrm{Homoth}(CW_{n+2}(S)) \simeq H_n \rtimes (E(1) \times K \times \mathbb{R}),$$

where H_n is the $2n+1$ dimensional Heisenberg group, $E(1) = \mathbb{Z}_2 \ltimes \mathbb{R}$ is the group of Euclidean motions of $\mathbb{R}$, and K is the subgroup $Z_{O(n)}(S)$ of orthogonal matrices centralizing S. Note in particular, that the $\mathbb{R}$ component of $E(1)$ acts on H_n in such a way to make $H_n \rtimes \mathbb{R}$ isomorphic to the generalized oscillator group as in Section 2.1 of Kath & Olbrich (2019).

The isomorphism is given by

$$(\beta, b, a, c, A, s) \mapsto ((\beta(0), \dot{\beta}(0), b), (a, c), A, s).$$

Proof. Define $\hat{\omega}$ a 2-form on $\mathbb{R}^n \times \mathbb{R}^n$, $\hat{\omega} : ((x, y), (x', y')) \mapsto \langle x, y' \rangle - \langle y, x' \rangle$. The $2n + 1$ dimensional Heisenberg group $H_n(\hat{\omega})$ is $\mathbb{R}^{2n+1}$ with group operation ($x \in \mathbb{R}^{2n}, b \in \mathbb{R}$.)

$$(x, b)(x', b') := \left(x + x', b + b' + \frac{1}{2}\hat{\omega}(x, x') \right)$$

Define (β, b) to be the homothety $(\beta, b, 1, 0, I_n, 0)$. Then (β, b) is mapped into $H_n(\hat{\omega})$ by

$$f : (\beta, b) \mapsto (\beta(0), \dot{\beta}(0), b).$$

Then by Proposition 4.2.7, we see that (recalling that ω is constant with respect to x^+, so we may choose to evaluate ω at $x^+ = 0$)

$$\begin{aligned}
&f((\beta, b) \circ (\beta', b')) \\
&= f\left(\beta + \beta', b + b' + \frac{1}{2}\omega(\beta, \beta')(0)\right) \\
&= \left((\beta + \beta')(0), \frac{\partial}{\partial x^+}(\beta + \beta')(0), b + b' + \frac{1}{2}\omega(\beta, \beta')(0)\right) \\
&= \left(\beta(0) + \beta'(0), \dot{\beta}(0) + \dot{\beta}'(0), b + b' + \frac{1}{2}\hat{\omega}\left(\beta(0), \dot{\beta}(0), \beta'(0), \dot{\beta}'(0)\right)\right) \\
&= f(\beta, b)f(\beta', b').
\end{aligned}$$

So f is a homomorphism, and f is bijective by existence and uniqueness of solutions to $\ddot{\beta} = S\beta$ with initial conditions $\beta(0)$, $\dot{\beta}(0)$.

H_n is normal inside Homoth as can be seen by noting that none of a_ψ, c_ψ, A_ψ, or s_ψ depend on β, β', b, or b'.

We define further our map to send $(0, 0, a, c, I_n, 0)$ to $E_{a,c}$, the Euclidean motion $x \mapsto ax + c$ in $E(1)$. This is a homomorphism as can be seen clearly by noting that $(0, 0, a, c, I_n, 0)$ acts on x^+ by isometries of Euclidean $\mathbb{R}$.

Define the image of $(0, 0, 1, 0, A, 0)$ to be A in $Z_{O(n)}(S)$, and the image of $(0, 0, 1, 0, I_n, s)$ is s in $\mathbb{R}$ as in Corollary 3.1.8.

Now we conclude that $E(1)$, K, and $\mathbb{R}$ are all normal in Homoth $/H_n$. We see $E(1)$ is normal because A_ψ and s_ψ do not depend on a, a', c, or c'. Next K is normal because a_ψ, c_ψ, and s_ψ do not depend on A or A'. And finally $\mathbb{R}$ is normal because a_ψ, c_ψ, and A_ψ do not depend on s or s'. □

One thing to note is that Theorem 4.2.9 gives only a group isomorphism, and we have not yet established the topology on Homoth. An obvious choice of topology for Homoth is the compact-open topology, but instead of attempting to calculate this topology, we provide our own topology for Homoth, under the condition that the topology makes Homoth a topological group, and that Homoth with this topology acts continuously on $CW_{n+2}(S)$. This approach is justified by Theorem 2.2.5, which we interpret here to mean that for *any* topology such that the group is a topological group and its action is continuous, subgroups acting properly discontinuously are discrete, and act properly with the subspace topology (and freely, but this last condition has no impact on this discussion of topologies). This theorem gives no special preference to the compact open topology or any other topology.

Definition 4.2.10. We give H_n the standard topology of $\mathbb{R}^{2n+1}$, $E(1)$ the product topology of $\mathbb{Z}_2 \times \mathbb{R}$, where $\mathbb{R}$ has the standard topology, and $\mathbb{Z}_2$ has the discrete topology, K the subspace topology of $O(n)$, and $\mathbb{R}$ the standard topology.

Then we give Homoth the product topology, considering Homoth (topologically) as the space $\mathbb{R}^{2n+1} \times \mathbb{R} \times \mathbb{Z}_2 \times K \times \mathbb{R}$.

Proposition 4.2.11. *Let* Homoth *be the group of homotheties of* $CW_{n+2}(S)$*, endowed with the topology described in Definition 4.2.10. Then* Homoth *is a topological group, and* Homoth *acts continuously on* $CW_{n+2}(S)$*.*

Proof. To show that Homoth is a topological group, we need to show that $g \mapsto g^{-1}$, and $(g, h) \mapsto gh$ are continuous functions.

We identify the space of solutions β to $\ddot{\beta} = S\beta$ with $(\beta(0), \dot{\beta}(0))$ in $\mathbb{R}^{2n}$ (as in Theorem 4.2.9).

By Corollary 4.2.8, the inverse is

$$\phi = (\beta(0), \dot{\beta}(0), b, a, c, A, s) \mapsto \begin{pmatrix} -e^{-s}A^\top \beta(-ac), \\ -ae^{-s}A^\top \dot{\beta}(-ac), \\ -e^{-2s}ab, \\ a, \\ -ac, \\ A^\top, \\ -s \end{pmatrix}$$

Since Homoth has the product topology, we need only show that each component of the inverse function is continuous. But this is clear because all components (except the β component) are just exponentials and polynomials in their inputs. And the β component is continuous by continuity of solutions of ODE's with respect to their initial conditions, see e.g. (Lee 2012, Theorem D.1).

An identical argument shows that the composition map of Proposition 4.2.7 is also continuous. Hence Homoth with this topology is a topological group.

Similarly, the action is continuous because each component of $\theta : (\phi, x) \mapsto \phi(x)$ is again exponentials, polynomials, and solutions to ODE's. $\square$

4.3 Fixed points

In this section, we detail various sufficient conditions for a homothety of a Cahen-Wallach space to have a fixed point. These will be used during the next chapter to show that various actions are not properly discontinuous.

Proposition 4.3.1. *Let $\phi = (\beta, b, a, c, A, s)$ be a strict homothety of $CW_{n+2}(S)$. Then ϕ has a fixed point if and only if $x^+ \mapsto ax^+ + c$ has a fixed point (i.e. $a = -1$ or $c = 0$.)*

In other words, a strict homothety ϕ fails to fix a point if and only if $\phi^+(x) = x^+ + c$ for non-zero c.

Proof. If $ax^+ + c$ has no fixed point, then ϕ immediately cannot fix any point.

Let y^+ be a fixed point, $y^+ = ay^+ + c$, then one can verify that

$$\begin{pmatrix} y^+ \\ -(e^s A - I_n)^{-1}\beta(y^+) \\ -(e^{2s}a - 1)^{-1} a \left(b - \left\langle \dot{\beta}(y^+), -e^s A (e^s A - I_n)^{-1}\beta(y^+) + \frac{1}{2}\beta(y^+) \right\rangle \right) \end{pmatrix}$$

is a fixed point of ϕ. The assumption that ϕ be a strict homothety is crucial in justifying the existence of $(e^{2s}a - 1)^{-1}$ and $(e^s A - I_n)^{-1}$ $\square$

Lemma 4.3.2. *Let $\phi = (\beta, b, a, c, A)$ be an isometry of $CW_{n+2}(S)$, such that $a = -1$. Then ϕ fixes a point if and only if $\boldsymbol{x} \mapsto A\boldsymbol{x} + \beta(c/2)$ fixes a point.*

Proof. Note that $c/2$ is the unique fixed point of $x^+ \mapsto -x^+ + c$, so if $A\boldsymbol{x} + \beta(c/2)$ does not have a fixed point, then ϕ cannot fix any point.

Let $\boldsymbol{y} = A\boldsymbol{y} + \beta(c/2)$, then one can check that

$$\begin{pmatrix} c/2 \\ \boldsymbol{y} \\ -\frac{1}{2} \left(b - \left\langle \dot{\beta}(c/2), A\boldsymbol{y} + \frac{1}{2}\beta(c/2) \right\rangle \right) \end{pmatrix}$$

is a fixed point of ϕ. $\square$

Proposition 4.3.3. *Let $\phi = (a, b, c, A, \beta, s)$ be a homothety of $CW_{n+2}(S)$, such that for some $k > 0$, $\phi^k = \mathrm{id}$. Then ϕ fixes a point.*

Proof. We construct a point y fixed by ϕ.

We start with y^+: If $a = 1$, then $(\phi^k)^+ = x^+ + kc$. This can only fix a point for $k > 0$ if $c = 0$, so we conclude either $c = 0$ or $a = -1$. In either case

$$y^+ := c/2$$

is a fixed point of $x^+ \mapsto ax^+ + c$. From this we also get by Proposition 4.3.1, that ϕ either has a fixed point, or is an isometry. So we assume ϕ is an isometry. Define $\beta_0 := \beta(c/2)$, and $\dot{\beta}_0 := \dot{\beta}(c/2)$.

Next we consider the $\boldsymbol{x}$ component: since $\phi^k(c/2, 0, 0) = (c/2, 0, 0)$ we get that the euclidean motion $E_{A,\beta_0}(\boldsymbol{x}) = A\boldsymbol{x} + \beta_0$ satisfies $E^k_{A,\beta_0} \equiv \mathrm{id}$. In particular, $E^k_{A,\beta_0}(0) = \sum_{i=0}^{k-1} A^i\beta_0 = 0$.

In general any euclidean motion E satisfying $E^k(x) = x$, fixes a point. This fixed point is given by the centre of mass (in our case, $x = 0$,)

$$\boldsymbol{y} := \frac{1}{k}\sum_{i=1}^{k} E^i_{A,\beta_0}(0).$$

Now when $a = -1$, then Lemma 4.3.2 gives us a fixed point of ϕ.

When $a = 1$, to have a fixed point, we require that $b - \left\langle \dot{\beta}_0, A\boldsymbol{y} + \frac{1}{2}\beta_0 \right\rangle = 0$. But since $\phi(y^+, \boldsymbol{y}, \_) = (y^+, \boldsymbol{y}, \ldots)$, we get

$$\phi^k \begin{pmatrix} y^+ \\ \boldsymbol{y} \\ 0 \end{pmatrix} = \begin{pmatrix} y^+ \\ \boldsymbol{y} \\ k(b - \left\langle \dot{\beta}_0, A\boldsymbol{y} + \frac{1}{2}\beta_0 \right\rangle) \end{pmatrix} = \begin{pmatrix} y^+ \\ \boldsymbol{y} \\ 0 \end{pmatrix}.$$

Hence $b - \left\langle \dot{\beta}_0, A\boldsymbol{y} + \frac{1}{2}\beta_0 \right\rangle = 0$, and so any choice of y^-, for example $y^- := 0$ makes $(y^+, \boldsymbol{y}, y^-)$ a fixed point of ϕ. □

Theorem 4.3.4. *Let $\phi_{a,b,c,A,\beta,s}$ be a homothety of $CW_{n+2}(S)$, such that there is a $k > 0$, and $x \in \mathbb{R}^{n+2}$, such that $\phi^k(x) = x$. Then ϕ has a fixed point if either*

- *ϕ is a strict homothety,*
- *k is odd, or*
- *x^+ is fixed by ϕ^+, i.e. $x^+ = ax^+ + c$.*

Otherwise, ϕ^2 has a fixed point.

Proof. By conjugating by an isometry sending x to 0, we may assume that the fixed point of ϕ^k is 0. Note that after this conjugation, the third additional condition becomes the condition that $c = 0$.

We construct a point y fixed by ϕ. If $a = 1$, then $(\phi^k)^+ = x^+ + kc$, and so for ϕ^k to fix a point, c must be 0. So we see that either $a = -1$, or $c = 0$.

Case 1: ϕ is a strict homothety. In particular, by Proposition 4.3.1, since either $a = -1$ or $c = 0$, ϕ fixes a point.

Case 2: k odd. By above either $a = -1$ or $c = 0$. But if $a = -1$, then $(\phi^k)^+ = -x^+ + c$, and since ϕ^k fixes 0, we see that $c = 0$. So k being odd implies that $c = 0$.

Case 3: $c = 0$. We assume ϕ is an isometry, since the alternative is covered by case 1.

Note that 0 is a fixed point of $x^+ \mapsto ax^+$, so we can define $\beta_0 := \beta(0)$, $\dot{\beta}_0 := \dot{\beta}(0)$. Hence

$$\phi^i \begin{pmatrix} 0 \\ \boldsymbol{x} \\ x^- \end{pmatrix} = \begin{pmatrix} 0 \\ E^i_{A,\beta_0}(\boldsymbol{x}) \\ \cdots \end{pmatrix}$$

Where E_{A,β_0} is the euclidean motion on $\mathbb{R}^n$, $\boldsymbol{x} \mapsto A\boldsymbol{x} + \beta_0$. Therefore by assumption that $\phi^k(0) = 0$, we see that $E^k_{A,\beta_0}(0) = 0$. We know that for any euclidean motion E such that E^k fixes a point, E fixes a point given by the centre of mass

$$\boldsymbol{y} := \frac{1}{k}\sum_{i=1}^{k} E^i_{A,\beta_0}(0) = \frac{1}{k}\sum_{i=0}^{k-2}(k-1-i)A^i\beta_0.$$

Now if $a = -1$, then the existence of such a $\boldsymbol{y}$ gives us a fixed point of ϕ by Lemma 4.3.2. So we assume $a = 1$.

Then since $a = 1$, we see that by $\phi^k(0,0,0) = (0,0,0)$,

$$b = \frac{1}{k}\left\langle \dot{\beta}_0, \frac{k}{2}\beta_0 + \sum_{i=0}^{k-1} E^i_{A,\beta_0}(0)\right\rangle = \frac{1}{k}\left\langle \dot{\beta}_0, \frac{k}{2}\beta_0 + \sum_{i=0}^{k-1}(k-i)A^i\beta_0\right\rangle.$$

So we compute $b - \left\langle \dot{\beta}_0, A\boldsymbol{y} + \frac{1}{2}\beta_0 \right\rangle$

$$
\begin{aligned}
&= \left\langle \dot{\beta}_0, \frac{1}{k}\left(\frac{k}{2}\beta_0 + \sum_{i=1}^{k-1}(k-i)A^i\beta_0\right) - A\boldsymbol{y} - \frac{1}{2}\beta_0 \right\rangle \\
&= \left\langle \dot{\beta}_0, \frac{1}{k}\left(\sum_{i=1}^{k-1}(k-i)A^i\beta_0\right) - A\boldsymbol{y} \right\rangle \\
&= \left\langle \dot{\beta}_0, \frac{1}{k}\left(\sum_{i=1}^{k-1}(k-i)A^i\beta_0\right) + \beta_0 - \boldsymbol{y} \right\rangle \\
&= \left\langle \dot{\beta}_0, \frac{1}{k}\left(\sum_{i=1}^{k-1}(k-i)A^i\beta_0\right) + \beta_0 - \frac{1}{k}\sum_{i=0}^{k-2}(k-1-i)A^i\beta_0 \right\rangle \\
&= \left\langle \dot{\beta}_0, \frac{1}{k}\left(A^{k-1}\beta_0 - (k-1)\beta_0 + \sum_{i=1}^{k-2}A^i\beta_0\right) + \beta_0 \right\rangle \\
&= \left\langle \dot{\beta}_0, \frac{1}{k}\sum_{i=0}^{k-1}A^i\beta_0 \right\rangle \\
&= \left\langle \dot{\beta}_0, \frac{1}{k}E^k_{A,\beta_0}(0) \right\rangle \\
&= 0.
\end{aligned}
$$

Therefore $(0, \boldsymbol{y}, 0)$ is fixed by ϕ.

Now we consider ϕ that satisfies none of the three additional conditions. In particular, assume $c \neq 0$. We see right away from $\phi^k(0) = 0$ that $a = -1$, and k is even. So ϕ^2 satisfies the assumptions of the first part of this proof: $(\phi^2)^{k/2}$ fixes 0, and $c_{\phi^2} = 0$. Hence ϕ^2 fixes a point. □

A necessary and sufficient condition for ϕ^2 to fix a point, without ϕ fixing a point, is that

$$
\begin{gathered}
a = -1, \qquad c \neq 0, \qquad A\beta(0) + \beta(c) = 0, \\
\omega(\beta(-x^+ + c), A\beta(x^+)) = 0,
\end{gathered}
$$

and that $\beta(c/2)$ is not in the image of $A - I_n$. Note that an equivalent formulation of the ω condition is that $\left\langle \beta(c), A\dot{\beta}(0) - \dot{\beta}(c) \right\rangle = 0$. An example of such a map is as follows.

Example 4.3.5. This example shows that there are isometries ϕ admitting 2-cycles, without having fixed points:

Consider $CW_{n+2}(S)$, for $n = 1$, and $S = -1$, so that solutions to $\ddot{\beta} = S\beta$ are linear combinations of the trigonometric sin and cos functions. Let

$$\phi = (\sin(x^+), b_\phi, -1, \pi, 1, 0),$$

i.e. $\beta_\phi = \sin(x^+)$, b_ϕ arbitrary, $a_\phi = -1$, $c_\phi = \pi$, $A_\phi = 1$, and $s_\phi = 0$.

$$\phi \begin{pmatrix} x^+ \\ \boldsymbol{x} \\ x^- \end{pmatrix} = \begin{pmatrix} -x^+ + \pi \\ \boldsymbol{x} + \sin(x^+) \\ -\left(x^- + b - \left\langle \cos(x^+), \boldsymbol{x} + \frac{1}{2}\sin(x^+)\right\rangle\right) \end{pmatrix}$$

Note that $\phi^2(0,0,0) = (0,0,0)$. If ϕ were to fix a point, it would be of the form $(\pi/2, \boldsymbol{y}, y^-)$. However, $\phi(\pi/2, \boldsymbol{y}, y^-) = (\pi/2, \boldsymbol{y} + 1, \dots)$, and so ϕ has no fixed point.

4.4 Essential homotheties

In this section we completely characterize essential homotheties of Cahen-Wallach spaces as the homotheties with fixed points. In particular, when a Cahen-Wallach space is conformally curved, this characterizes all essential conformal transformations.

We have established in Corollary 3.3.5, that when a homothety has a fixed point, it is an essential transformation. We also established in Theorem 3.3.6, that when a homothety admits a finitely self adjacent fundamental region, it is inessential. Thus if we can establish that when a homothety has no fixed point, it admits a finitely self adjacent fundamental region, then we can complete the characterization of essential homotheties.

We do this now in the case of Cahen-Wallach spaces.

Theorem 4.4.1. *A strict homothety ϕ of a Cahen-Wallach space is essential if and only if it fixes a point.*

Proof. First, if ϕ has a fixed point, then by Corollary 3.3.5, ϕ is essential.

Let ϕ have no fixed point. Then by Proposition 4.3.1, $\phi^+(x) = x^+ + c$ for some non-zero $c \in \mathbb{R}$. Then by Proposition 2.4.4, the action of $\langle\phi\rangle$ admits a finitely self adjacent fundamental region, namely $\mathbb{R}^{n+1} \times (0, c)$. Then by Theorem 3.3.6, ϕ is essential. □

Note that this theorem considers strict homotheties with fixed points, but in Conjecture 3.3.8, we consider homotheties with finite orbit points. We can consider only fixed points for Cahen-Wallach spaces because a strict homothety of a Cahen-Wallach space has a finite orbit point if and only if it has a fixed point by Theorem 4.3.4. However in general the weakening of the assumption to consider finite orbit points is harmless, since ϕ has a finite orbit point if and only if ϕ^k has a fixed point for some k, and ϕ is essential if and only if ϕ^k is essential.

Remark 4.4.2. As a consequence of this theorem, we reach the perhaps quite surprising fact that essentiality of a homothety can be destroyed simply by composing it with an isometry. This is surprising because it is not the case for Euclidean space, in particular, any isometry composed with a strict homothety will continue to have a fixed point. The key difference is that our pure homotheties have a dimension in which they do not scale, a property not shared by homotheties of Euclidean space.

A pure homothety h_s is essential, since it fixes the origin, but for the pure translation $T_c := (0, 0, 1, c, I_n, 0)$ (where $c \neq 0$), we have that $h_s \circ T_c$ is inessential. We can note this explicitly by finding the fixed metric. Define

$$f(x) := -\frac{s}{c}x^+.$$

Then

$$\begin{aligned}(h_s \circ T_c)^*(e^{2f}g) &= e^{2f\circ h_s \circ T_c}e^{2s}g \\ &= e^{-2\frac{s}{c}(x^+ + c)+2s}g \\ &= e^{2f}g.\end{aligned}$$

Chapter 5

Compact Cahen-Wallach quotients

5.1 Cyclic groups

In this section we provide a simple, yet crucial lemma, which states that a cyclic action of homotheties on $CW_{n+2}(S)$ cannot act cocompactly.

This will be used by both sections to follow. Note that although we do not need or use this theorem in its full strength, this theorem does not need to first assume that the action of $\langle\gamma\rangle$ is properly discontinuous.

Lemma 5.1.1. *Let γ be a homothety of $CW_{n+2}(S)$. Then $\langle\gamma\rangle$ does not act cocompactly.*

Proof. We split into cases on the x^+ component of γ. Let $x^+ \circ \gamma = ax^+ + c$.

Case 1: $a = 1, c = 0$. Define a continuous map

$$f : CW_{n+2}(S) \to \mathbb{R}, \qquad x \mapsto x^+.$$

Note that f is invariant under $\langle\gamma\rangle$, i.e. for all $\gamma^m \in \langle\gamma\rangle$, $f(\gamma^m(x)) = f(x)$, because $x^+ \circ \gamma = x^+$. Then we get an induced continuous function $\tilde{f} : CW_{n+2}(S)/\langle\gamma\rangle \to \mathbb{R}$. $\tilde{f}$ is surjective because f is surjective. Then we have that a continuous image of $CW_{n+2}(S)/\langle\gamma\rangle$ is not a compact set, so $\langle\gamma\rangle$ cannot act cocompactly.

Case 2: $a = -1$. Define a continuous map

$$f : CW_{n+2}(S) \to \mathbb{R}, \qquad x \mapsto |x^+ - c/2|.$$

Consider $f(\gamma(x)) = |(-x^+ + c) - c/2| = |-x^+ + c/2| = f(x)$. Hence f is invariant under $\langle\gamma\rangle$, and so we get an induced continuous function

$\tilde{f} : CW_{n+2}(S)/\langle\gamma\rangle \to \mathbb{R}$. $\tilde{f}$ surjects onto $[0, \infty)$ because f does. Hence $CW_{n+2}(S)/\langle\gamma\rangle$ is not compact.

Case 3: $a = 1, c \neq 0$. Then we are in the situation of Proposition 2.4.4, and so since $\mathbb{R}^{n+1}$ is not compact, $CW_{n+2}(S)/\langle\gamma\rangle$ is not compact. □

5.2 The imaginary case

Note that in this section we make distinctions based on the signs of the eigenvalues of S. We aim to use notation that agrees with Kath & Olbrich (2019). In particular, we denote a positive eigenvalue of S as λ^2, for $\lambda \in \mathbb{R}$, and we denote a negative eigenvalue of S as $-\mu^2$, for $\mu \in \mathbb{R}$.

In this Section we prove the result that any compact, properly discontinuous homothetic quotient of a Cahen-Wallach space of imaginary type is in fact an isometric quotient. Some explicit exploration of why such a quotient is impossible is given in Example 5.4.2. The core idea is to conjugate any one of the strict homotheties of the group into a nice form (in particular, a form that contains no Heisenberg component), then explicitly investigating the behaviour of groups that include such an element. The majority of the work of the proof is contained within the following technical lemma.

Lemma 5.2.1. *Let (β, b, a, c, A, s) be a strict homothety of $CW_{n+2}(S)$. For any $a' = \pm 1$, there is a solution to the equation $\ddot{\beta}' = S\beta'$ such that*

$$e^s A\beta'(x^+) - \beta'(ax^+ + a'c) = -\beta(a'x^+)$$

unless S has a positive eigenvalue λ^2 such that $s = \pm\lambda c$.

Proof. We first restrict to the case that S is a multiple of the identity. We do so by first assuming that S is diagonal by Proposition 4.0.3, then we may consider each eigenspace separately since A commutes with S. So we solve the equation separately in each eigenspace of S. Then we split into two cases.

Case 1: $S = -\mu^2 I_n$, for a real positive number μ. Then

$$\beta(x^+) = \sigma\cos(\mu x^+) + \tau\sin(\mu x^+), \qquad \beta'(x^+) = \sigma'\cos(\mu x^+) + \tau'\sin(\mu x^+),$$

for $\sigma, \tau, \sigma', \tau' \in \mathbb{R}^n$.

Plugging these functions for β, β' into the equation we wish to solve gives

$$e^s A\left(\sigma'\cos(\mu x^+) + \tau'\sin(\mu x^+)\right) - \sigma'\cos(a\mu x^+ + \mu a'c) - \tau'\sin(a\mu x^+ + \mu a'c)$$
$$= -\sigma\cos(a'\mu x^+) - \tau\sin(a'\mu x^+).$$

Then applying the sum of angles formulae for sin and cos, we get

$$\begin{aligned} e^s A \left(\sigma' \cos(\mu x^+) + \tau' \sin(\mu x^+)\right) \\ - \sigma' \left(\cos(a\mu x^+)\cos(a'\mu c) - \sin(a\mu x^+)\sin(a'\mu c)\right) \\ - \tau' \left(\sin(a\mu x^+)\cos(a'\mu c) + \cos(a\mu x^+)\sin(a'\mu c)\right) \\ = -\sigma \cos(a'\mu x^+) - \tau \sin(a'\mu x^+). \end{aligned}$$

Then since $a, a' = \pm 1$, we can ignore a and a' inside of cos, and pull them out of sin, so,

$$\begin{aligned} e^s A \left(\sigma' \cos(\mu x^+) + \tau' \sin(\mu x^+)\right) \\ - \sigma' \left(\cos(\mu x^+)\cos(\mu c) - aa' \sin(\mu x^+)\sin(\mu c)\right) \\ - \tau' \left(a \sin(\mu x^+)\cos(\mu c) + a' \cos(\mu x^+)\sin(\mu c)\right) \\ = -\sigma \cos(\mu x^+) - \tau a' \sin(\mu x^+). \end{aligned}$$

Now we solve for the coefficient of $\cos(\mu x^+)$ and $\sin(\mu x^+)$ to get the pair of equations:

$$\begin{aligned} e^s A\sigma' - \sigma' \cos(\mu c) - \tau' a' \sin(\mu c) = -\sigma, \\ e^s A\tau' + \sigma' aa' \sin(\mu c) - \tau' a \cos(\mu c) = -\tau a'. \end{aligned}$$

Now we can solve this for σ', τ' in terms of any σ, τ if and only if the block matrix

$$M := \begin{pmatrix} e^s A - \cos(\mu c) I_n & -a' \sin(\mu c) I_n \\ aa' \sin(\mu c) I_n & e^s A - a\cos(\mu c) I_n \end{pmatrix}$$

is non-degenerate.

Since the bottom two blocks commute, we get the determinant (Silvester 2000, Theorem 3)

$$\begin{aligned} \det(M) &= \det\left(\left(e^s A - \cos(\mu c) I_n\right)\left(e^s A - a\cos(\mu c) I_n\right) + a' aa' \sin^2(\mu c) I_n\right) \\ &= \det\left(e^{2s} A^2 - (1+a) e^s \cos(\mu c) A + a I_n\right). \end{aligned}$$

We know that A is orthogonal, hence normal, and so is diagonalizable over $\mathbb{C}$. So we conjugate this determinant by the diagonalization to obtain

$$\det(M) = \prod_{i=1}^{n} \left(e^{2s} r_i^2 - (1+a) e^s \cos(\mu c) r_i + a\right),$$

where r_i are the eigenvalues of A over $\mathbb{C}$, and hence satisfy $|r_i| = 1$. Now we can finally guarantee that the determinant of M is non-zero if the quadratic $e^{2s}z^2 - (1+a)e^s\cos(\mu c)z + a$ has no zeros on the unit circle.

If $a = -1$, the zeros are $z = \pm e^{-s}$.

If $a = 1$, the zeros are $z = e^{-s\pm i\mu c}$.

In both cases, solutions fail to lie on the unit circle since $s \neq 0$.

This proves the statement when $S = -\mu^2 I_n$.

Case 2: $S = \lambda^2 I_n$, for a real positive number λ.

The second case follows almost identically, except with hyperbolic trig functions and their identities.

$$\beta(x^+) = \sigma\cosh(\lambda x^+) + \tau\sinh(\lambda x^+), \quad \beta'(x^+) = \sigma'\cosh(\lambda x^+) + \tau'\sinh(\lambda x^+),$$

As in case 1, we eventually derive the matrix which must be non-degenerate

$$M := \begin{pmatrix} e^s A - \cosh(\lambda c)I_n & -a'\sinh(\lambda c)I_n \\ -aa'\sinh(\lambda c)I_n & e^s A - a\cosh(\lambda c)I_n \end{pmatrix}$$

Note the only difference to case 1 at this stage is the minus sign in the bottom left block, this sign cancels when evaluating this determinant and using $\cosh^2(\lambda c) - \sinh^2(\lambda c) = 1$, instead of $\cos^2(\mu c) + \sin^2(\mu c) = 1$. So we arrive at almost the same determinant

$$\det(M) = \prod_{i=1}^{n} \left(e^{2s}r_i^2 - (1+a)e^s\cosh(\lambda c)r_i + a\right)$$

If $a = -1$, the zeros are $z = \pm e^{-s}$.

If $a = 1$, the zeros are $z = e^{-s\pm\lambda c}$.

So finally we have no unit circle zeros unless $S = \lambda^2 I_n$, $a = 1$, $s = \pm\lambda c$. □

Note that having $s = \pm c\lambda$ for some positive eigenvalue λ^2 of S, does not imply the non-existence of the solution β' to this previous theorem. For example, if $\beta \equiv 0$, then $\beta' \equiv 0$ always solves the equation. To conclude non-existence, additional assumptions are needed, for example, a must be equal to 1, and A must have 1 as an eigenvalue.

Proposition 5.2.2. *Let $\phi = (\beta, b, a, c, A, s)$ be a strict homothety of $CW_{n+2}(S)$ such that every positive eigenvalue λ^2 of S, satisfies $s \neq \pm\lambda c$.*

Then ϕ is conjugate by an isometry to an element $\hat{\phi}$ such that $\beta_{\hat{\phi}} \equiv 0$, $b_{\hat{\phi}} = 0$, and $c_{\hat{\phi}} \geq 0$. In particular, $\hat{\phi}$ has no Heisenberg component, and so $\hat{\phi} \in E(1) \times K \times \mathbb{R}$.

Proof. Consider $\phi' = (\beta', b', a', 0, I_n, 0)$, where a' is the sign of c_ϕ, β' is a solution to the equation

$$e^s A\beta'(x^+) - \beta'(ax^+ + a'c) = -\beta(a'x^+),$$

and

$$b' = -\left(e^{2s} - a\right)^{-1}\Bigg(ba' + \frac{1}{2}\omega\left(\beta(a'x^+), e^s A\beta'(x^+)\right) - \frac{1}{2}\omega\left(\beta'(ax^+ + a'x), e^s A\beta'(x^+) + \beta(a'x^+)\right)\Bigg).$$

b' exists since $s \neq 0$, and β' exists by Lemma 5.2.1. Then taking the conjugate $\phi'^{-1}\phi\phi'$, using the formulae in Proposition 4.2.7 and Corollary 4.2.8 gives the result. □

In particular, by taking the inverse of ϕ if necessary, Proposition 5.2.2 allows us (under the assumptions of the proposition) to take our properly discontinuous and cocompact group Γ to contain an element of the highly special form $(0, 0, 1, c, A, s)$, for c, s both non-negative.

Alternatively, if one does not care about the sign of c and s, then the conjugation of the previous proposition requires only conjugation by an element in the Heisenberg.

Proposition 5.2.3. *Let Γ be a group of homotheties of $CW_{n+2}(S)$ such that Γ contains a strict homothety of the form*

$$\gamma = (0, 0, 1, c, A, s).$$

Assume Γ acts properly discontinuously and cocompactly. Then for all elements ϕ of Γ not contained in $\langle\gamma\rangle$, β_ϕ grows exponentially.

Proof. First note that $c \neq 0$, since otherwise $\gamma(0) = 0$. Then if β_ϕ, b_ϕ are both zero, then for any sequence of rational numbers $p_i/q_i \to c'/c$, we have that $\gamma^{p_i}\phi^{-q_i}(0) \to 0$, contradicting PD1. Hence either b_ϕ or β_ϕ is non-zero.

Now consider the sequence

$$\gamma^{-i}\phi\gamma^i(0) = \begin{pmatrix} c \\ e^{-is}\beta_\phi(ic) \\ e^{-2is}(b_\phi - \left\langle \dot{\beta}_\phi(ic), \frac{1}{2}\beta_\phi(ic)\right\rangle) \end{pmatrix}.$$

Since Γ acts properly discontinuously, this sequence cannot accumulate anywhere in $\mathbb{R}^{n+2}$. In particular, we see that either

$$e^{-is}\beta_\phi(ic) \nrightarrow 0, \quad \text{or} \quad e^{-2is}\left\langle \dot{\beta}_\phi(ic), \frac{1}{2}\beta_\phi(ic)\right\rangle \nrightarrow 0.$$

Hence either β or its derivative grows exponentially. Then noting the form of β as a solution to $\ddot{\beta} = S\beta$, these are the same condition, namely that β grows exponentially. □

More can be concluded, but we do so here only informally because it is not necessary for our primary conclusion. Note that if either of the sequences approach 0, then we can see that the other will also approach zero. So in actuality, we just need to consider whether $e^{-ns}\beta_\phi(nc)$ approaches zero.

Consider positive eigenvalues of S. If we take $c > 0$, and $s > 0$, then we could observe that the sequence $e^{-ns}\beta(nc)$ will always approach 0 for solutions $\beta = \sigma e^{-\lambda x}$, thus fundamentally restricting us to solutions $\beta = \sigma e^{\lambda x}$, this is similar to Example 5.4.3.

We also get some minor restriction of the c and s of the γ in the case of $\beta = \sigma e^{\lambda x}$, again by forcing $e^{-ns}\beta_\phi(nc) \nrightarrow 0$, we conclude that $\lambda c - s > 0$, for at least one positive eigenvalue λ^2 of S.

Now, with these informal minor observations dealt with, we arrive at the main theorem of this section.

Theorem 5.2.4. *A group of homotheties of a Cahen-Wallach space of imaginary type acting properly discontinuously and cocompactly must be contained within the isometries.*

Proof. Assume for contradiction that $\hat{\Gamma}$ acts properly discontinuously, cocompactly, and contains a strict homothety. Since our Cahen-Wallach space is of imaginary type, and so every eigenvalue of S is negative, we may apply without restriction Proposition 5.2.2 to get a group Γ conjugate to $\hat{\Gamma}$ such that Γ contains an element of the form $(0, 0, a, c, A, s)$ such that $s \neq 0$. We may of course square this element to obtain an element within Γ of the form $\gamma := (0, 0, 1, c, A, s)$, still with $s \neq 0$. Note that Γ acts properly discontinuously and cocompactly by Lemmas 2.2.8 and 2.3.4. Then we apply Lemma 5.1.1 to conclude that Γ is not cyclic, since Γ acts cocompactly. Now let $\phi \in \Gamma$ be an element such that ϕ is not in $\langle\gamma\rangle$. So we can apply Proposition 5.2.3 to conclude that β_ϕ grows exponentially. But since our Cahen-Wallach space is of imaginary type, all β are trigonometric, and thus bounded. This is a contradiction. □

Corollary 5.2.5. *Let Γ be a group of conformal transformations acting properly discontinuously and cocompactly on a conformally curved Cahen-Wallach space of imaginary type $CW_{n+2}(S)$. Then Γ is a group of isometries. In particular, then $CW_{n+2}(S)/\Gamma$ is a compact, conformally curved, locally Cahen-Wallach space, and so its conformal group is equal to its isometry group.*

Proof. By Proposition 3.2.1, Γ is a group of homotheties. Then using Theorem 5.2.4, Γ is a group of isometries. Then the metric endowed to the quotient by Theorem 3.5.2 is locally isometric to $CW_{n+2}(S)$. In particular, since $CW_{n+2}(S)$ is conformally curved and locally symmetric, the quotient is too. Then by Proposition 3.2.1, the conformal group of the quotient is equal to its homothetic group. But then by Proposition 3.1.10, since the quotient is compact, its homothetic group is equal to its isometry group. □

5.3 Centralizing an essential homothety

We are interested in the possibility of taking an essential homothety on our Cahen-Wallach space, and inducing an essential conformal transformation on the quotient using Theorem 3.5.10. In pursuing this, we consider groups centralizing an essential homothety, using Corollary 3.5.6. In this section, we demonstrate that we cannot act properly discontinuously and cocompactly using a subgroup of the centralizer of an essential homothety of a Cahen-Wallach space. We prove first this result in the special case of the centralizer of a pure homothety, this proof showcases the overall idea of the more general proof in the case of the centralizer of any homothety with a fixed point, but is otherwise entirely optional.

Proposition 5.3.1. *Let $h_t : (x^+, \boldsymbol{x}, x^-) \mapsto (x^+, e^t\boldsymbol{x}, e^{2t}x^-)$ be a strict pure homothety of $CW_{n+2}(S)$. Let Γ be a group of homotheties centralizing h_t, i.e. for all $\gamma \in \Gamma$, $h_t\gamma = \gamma h_t$. Then Γ does not act properly discontinuously and cocompactly.*

Proof. By explicitly evaluating the composition, or by inspecting Proposition 4.2.7, we see that the centralizer of a strict pure homothety in the homotheties is

$$Z_{\text{Homoth}}(h_t) = \{(\beta, b, a, c, A, s) \in \text{Homoth} \mid \beta \equiv 0, b = 0\}.$$

In light of Theorem 4.2.9, we write

$$Z_{\text{Homoth}}(h_t) = E(1) \times K \times \mathbb{R} = (\mathbb{Z}_2 \ltimes \mathbb{R}) \times K \times \mathbb{R}.$$

In other words, an arbitrary element of the centralizer is

$$\gamma = (a, c, A, s) : \begin{pmatrix} x^+ \\ \boldsymbol{x} \\ x^- \end{pmatrix} \mapsto \begin{pmatrix} ax^+ + c \\ e^s A\boldsymbol{x} \\ ae^{2s}x^- \end{pmatrix}.$$

Assume for contradiction that Γ is a subgroup of $Z_{\mathrm{Homoth}}(h_t)$ acting properly discontinuously and cocompactly.

By freeness, all non-identity elements $\gamma \in \Gamma$ have $c \neq 0$, otherwise $\gamma(0) = 0$. Note that by Proposition 4.3.3, Γ has no torsion elements. So we conclude that if $a = -1$, then $x^+ \circ \gamma^2 = x^+$, and so γ^2 has a fixed point. Hence all elements $\gamma \in \Gamma$ have $a = 1$.

Because $a = 1$ for all elements γ, the projection

$$\rho : \Gamma \to \mathbb{R}^2, \qquad (1, c, A, s) \mapsto (c, s)$$

is a homomorphism. We show that ρ is injective: Let $\gamma = (1, 0, A, 0) \in \ker(\rho)$. Then γ is in the compact subgroup K. (Recall that we are endowing the homotheties with the topology described in Definition 4.2.10 and Proposition 4.2.11.) Hence since K is compact, either γ is torsion, or $\langle\gamma\rangle$ is not discrete. If γ is torsion, then Γ has torsion and so does not act freely by Proposition 4.3.3. If $\langle\gamma\rangle$ is not discrete, then Γ does not act properly discontinuously. Since both cases contradict proper discontinuity, ρ must be injective.

Therefore Γ is isomorphic to $\rho(\Gamma)$, a subgroup of $\mathbb{R}^2$. Note that since the homotheties have the product topology, and ρ is a projection map, ρ is an open map. Hence $\rho(\Gamma)$ is discrete in $\mathbb{R}^2$. Note also that by Lemma 5.1.1, Γ is not cyclic since Γ acts cocompactly. Then we conclude that Γ has a subgroup isomorphic to $\mathbb{Z}^2$, i.e. there are two elements $\gamma, \eta \in \Gamma$ such that $\langle\gamma\rangle \cap \langle\eta\rangle = \{\mathrm{id}\}$.

Since c_η is non-zero from earlier in this proof, we may take a sequence of rational numbers p_i/q_i approaching c_γ/c_η. Note that $q_i c_\gamma - p_i c_\eta$ approaches 0, and thus $\gamma^{q_i}\eta^{-p_i}(0) \to 0$, contradicting PD1. □

Now our goal is to adapt this proof to the centralizer of an arbitrary homothety fixing a point. Of course, after conjugating by an appropriate isometry of our Cahen-Wallach space, it suffices to consider the homotheties fixing the origin. To adapt this proof, we first project away the Heisenberg terms that appear in our new centralizer. Remarkably, we can still do this without losing injectivity.

Proposition 5.3.2. *Let $\psi = (\beta, b, a, c, A, s)$ be a strict homothety of $CW_{n+2}(S)$ such that $\psi(0) = 0$. Let $Z_{\mathrm{Homoth}}(\psi)$ be the centralizer of ψ in the homotheties. Then the projection*

$$p : Z_{\mathrm{Homoth}}(\psi) \to E(1) \times K \times \mathbb{R}$$

is an injective homomorphism.

Moreover, let $\gamma_n = (\beta_n, b_n, a_n, c_n, A_n, s_n) \in Z_{\mathrm{Homoth}}(\psi)$ such that $c_n \to 0$. Then $b_n \to 0$, and $\beta_n(0) \to 0$. In particular, if $c_n \to 0$, then $\gamma_n(0) \to 0$.

Proof. The projection p is a homomorphism, since the Heisenberg is normal in the homothety group, and the kernel of p is the Heisenberg group.

Let $\phi \in \ker(p)$, i.e. $a_\phi = 1$, $c_\phi = 0$, $A_\phi = I_n$, $s_\phi = 0$. We aim to show that $b_\phi = 0$, and $\beta_\phi \equiv 0$.

First, observe that since $\psi(0) = 0$, we have that $\beta(0) = 0$, $b = 0$, and $c = 0$.

Since $\phi \in Z(\psi)$, we have $\beta_{\psi\phi} = \beta_{\phi\psi}$, and hence for $\psi = (\beta, b, a, c, A, s)$,

$$e^s A\beta_\phi(x^+) + \beta(x^+) = \beta(x^+) + \beta_\phi(ax^+),$$

which implies

$$e^s A\beta_\phi(x^+) - \beta_\phi(ax^+) = 0.$$

If $a = 1$, then since $s \neq 0$, $(e^s A - I_n)$ is invertible. So $\beta(x^+) \equiv 0$.

If $a = -1$, then we instead find the initial conditions for β_ϕ

$$(e^s A - I_n)\beta_\phi(0) = 0, \qquad (e^s A + I_n)\dot{\beta}_\phi(0) = 0$$

Hence again since $s \neq 0$, these tell us that $\beta_\phi(0) = 0$, and $\dot{\beta}_\phi(0) = 0$, and so $\beta \equiv 0$.

Now let $b_{\psi\phi} = b_{\phi\psi}$, then

$$e^{2s} b_\phi + b = b + ab_\phi,$$

and so $(e^{2s} - a)b_\phi = 0$. Then since $s \neq 0$, $b_\phi = 0$. Therefore p is injective.

To show the second part of the theorem, we state the inverse of p (on the image). Let $(a_\phi, c_\phi, A_\phi, s_\phi)$ be in the image of p. Then consider

$$b_\phi := \frac{1}{2}(e^{2s} - a)^{-1} \left(\omega(\beta_\phi \circ E_{a,0}, e^{s_\phi} A_\phi \beta) - \omega(\beta \circ E_{a_\phi, c_\phi}, e^s A\beta_\phi)\right),$$

and

$$\beta_\phi(x^+) := (e^s A - I_n)^{-1}(e^{s_\phi} A_\phi \beta(x^+) - \beta(a_\phi x^+ + c_\phi)), \quad \text{if } a = 1,$$

or

$$\beta_\phi(0) = 0, \qquad \dot{\beta}_\phi(0) = (e^s A + I_n)^{-1}(e^{s_\phi} A_\phi - a_\phi)\dot{\beta}(0), \quad \text{if } a = -1.$$

Showing that this is the inverse of p requires only verification that $(\beta_\phi, b_\phi, a_\phi, c_\phi, A_\phi, s_\phi)$ commutes with ψ. To aid in the $a = -1$ case, one should keep in mind that since $c = 0$, c_ϕ must be 0, and that to check $\beta_{\psi\phi} = \beta_{\phi\psi}$, one should check the initial conditions of $\beta_{\psi\phi} - \beta_{\phi\psi}$.

Then the conclusion follows by evaluating the formula for b_ϕ, $\beta_\phi(0)$ in the limit as $c_\phi \to 0$. □

Theorem 5.3.3. *Let ψ be a strict homothety of $CW_{n+2}(S)$ that fixes zero. Let Γ be a subgroup of the centralizer of ψ in the homotheties.*

Then Γ does not act properly discontinuously and cocompactly.

Proof. Assume for contradiction that Γ is a subgroup of $Z_{\text{Homoth}}(\psi)$ acting properly discontinuously and cocompactly.

From Proposition 5.3.2, we have an isomorphism between Γ and a subgroup of $E(1) \times K \times \mathbb{R}$. Since the homotheties have the product topology, and p is a projection map, p is an open map. Therefore $p(\Gamma)$ is discrete.

By the second part of Proposition 5.3.2, if we have $\gamma \in \Gamma$ such that $c_\gamma = 0$, then $\gamma(0) = 0$. Then by freeness of Γ, $\gamma = \text{id}$. So for all non-identity γ, $c_\gamma \neq 0$. Note that by Proposition 4.3.3, Γ has no torsion elements. Then we also conclude that for all non-identity elements γ, $a_\gamma = 1$, since otherwise $c_{\gamma^2} = 0$.

Then similarly to Proposition 5.3.1, the projection

$$\rho : p(\Gamma) \to \mathbb{R}^2, \qquad \rho(a_\phi, c_\phi, A_\phi, s_\phi) \mapsto (c_\phi, s_\phi)$$

is an injective homomorphism, because any kernel element $(a_\phi, 0, A_\phi, 0)$ is in a compact subgroup, and hence is the identity, torsion, or $\langle \phi \rangle$ is not discrete. If $\langle \phi \rangle$ is not dicrete, then Γ does not act properly discontinuously. And if ϕ is torsion, then Γ does not act freely by Proposition 4.3.3. Hence if Γ acts properly discontinuously, then ρ is injective.

Then since ρ is again a projection map, and so is an open map, $\rho(p(\Gamma))$ is a discrete subgroup of $\mathbb{R}^2$, and so since Γ is not cyclic by Lemma 5.1.1 (assuming that Γ acts cocompactly,) Γ has a subgroup isomorphic to $\mathbb{Z}^2$.

Let $\gamma, \eta \in \Gamma$ be two elements such that $\langle \gamma \rangle \cap \langle \eta \rangle = \{\text{id}\}$. By earlier in this proof, $c_\eta \neq 0$, and so we take a sequence of rational numbers p_n/q_n approaching c_γ/c_η. Then we consider the sequence $\gamma^{p_n}\eta^{-q_n}$. This is a sequence of elements such that c approaches 0. Hence by the second part in Proposition 5.3.2, $\gamma^{p_n}\eta^{-q_n}(0) \to 0$, contradicting PD1. □

Theorem 5.3.4. *A group of homotheties of a Cahen-Wallach space centralizing an essential homothety cannot act properly discontinuously and cocompactly.*

Proof. By Theorem 4.4.1, the essential homothety centralized by Γ has a fixed point. Since Cahen-Wallach spaces are homogeneous, the isometry group acts transitively. We conjugate the essential homothety and the group Γ by the isometry that sends 0 to the fixed point our essential homothety. Then by Lemmas 2.2.8 and 2.3.4, Γ acts properly discontinuously and cocompactly if and only if its conjugate acts properly discontinuously and cocompactly. So we assume without loss of generality that our homothety fixes 0. Then Γ centralizes a homothety that fixes zero, and so by Theorem 5.3.3, Γ does not act properly discontinuously and cocompactly. $\square$

Then, simple application of Proposition 3.2.1 gives the following result as a consequence.

Theorem 5.3.5. *A group of conformal transformations of a conformally curved Cahen-Wallach space centralizing an essential conformal transformation cannot act properly discontinuously and cocompactly.*

5.4 Some examples

In this section, we give first an example of a compact quotient of a Cahen-Wallach space by isometries, then we attempt to produce examples using groups of homotheties not contained within the isometries. These examples fail to produce a compact quotient, but we include them because their failure is instructive. We then produce an example of a strict homothetic quotient for an open submanifold of a Cahen Wallach space.

Example 5.4.1 (Isometric example)**.** This example is a properly discontinuous and cocompact group of isometries acting on $CW_{n+2}(S)$.

Take $CW_{n+2}(S)$, for $n = 2$, and $S = -I$, so that solutions to $\ddot{\beta} = S\beta$ are of the form $d\cos(x^+) + e\sin(x^+)$, where $d, e \in \mathbb{R}^2$.

Define our group $\Gamma := \langle \gamma, \eta, \zeta \rangle$, where

$$\gamma \begin{pmatrix} x^+ \\ \boldsymbol{x} \\ x^- \end{pmatrix} := \begin{pmatrix} x^+ + \frac{\pi}{2} \\ \boldsymbol{x} \\ x^- \end{pmatrix}, \qquad \eta \begin{pmatrix} x^+ \\ \boldsymbol{x} \\ x^- \end{pmatrix} := \begin{pmatrix} x^+ \\ \boldsymbol{x} \\ x^- + 1 \end{pmatrix},$$

$$\zeta \begin{pmatrix} x^+ \\ \boldsymbol{x} \\ x^- \end{pmatrix} := \begin{pmatrix} x^+ \\ \boldsymbol{x} + \begin{pmatrix} \cos(x^+) \\ \sin(x^+) \end{pmatrix} \\ x^- - \left\langle \begin{pmatrix} -\sin(x^+) \\ \cos(x^+) \end{pmatrix}, \boldsymbol{x} \right\rangle \end{pmatrix}.$$

Note that in the notation of Theorem 4.2.1, $\gamma = (0, 0, 1, \pi/2, I)$, $\eta = (0, 1, 1, 0, I)$, and $\zeta = ((\cos(x^+), \sin(x^+)), 0, 1, 0, I)$

Consider the homeomorphism $f : \mathbb{R}^4 \to CW_4(S)$ given by

$$f \begin{pmatrix} u \\ x \\ y \\ v \end{pmatrix} := \begin{pmatrix} u \\ x \begin{pmatrix} \cos(u) \\ \sin(u) \end{pmatrix} + y \begin{pmatrix} -\sin(u) \\ \cos(u) \end{pmatrix} \\ -v - xy \end{pmatrix},$$

$$f^{-1}\begin{pmatrix} x^+ \\ \boldsymbol{x}^1 \\ \boldsymbol{x}^2 \\ x^- \end{pmatrix} = \begin{pmatrix} x^+ \\ \boldsymbol{x}^1 \begin{pmatrix} \cos(-x^+) \\ \sin(-x^+) \end{pmatrix} + \boldsymbol{x}^2 \begin{pmatrix} -\sin(-x^+) \\ \cos(-x^+) \end{pmatrix} \\ -x^- - (\boldsymbol{x}^1 \cos(-x^+) - \boldsymbol{x}^2 sin(-x^+))\,(\boldsymbol{x}^1 \sin(-x^+) \\ +\boldsymbol{x}^2 \cos(-x^+)) \end{pmatrix}.$$

Then the action on $\mathbb{R}^4$ by Γ is given by

$$f^{-1}\gamma f \begin{pmatrix} u \\ x \\ y \\ v \end{pmatrix} = \begin{pmatrix} u + \frac{\pi}{2} \\ y \\ -x \\ v \end{pmatrix}, \qquad f^{-1}\eta f \begin{pmatrix} u \\ x \\ y \\ v \end{pmatrix} = \begin{pmatrix} u \\ x \\ y \\ v-1 \end{pmatrix},$$

$$f^{-1}\zeta f \begin{pmatrix} u \\ x \\ y \\ v \end{pmatrix} = \begin{pmatrix} u \\ x+1 \\ y \\ v \end{pmatrix}.$$

Compare this example with Example 2.3.5. The conjugation has achieved the same purpose of removing the dependence of the group action on x^+. Γ acts properly discontinuously and cocompactly on $\mathbb{R}^4$ is the same way as Example 2.3.5: Observe that

$$\Lambda := f\left\langle \gamma^4,\ \zeta,\ \gamma^{-1}\zeta\gamma,\ \eta \right\rangle f^{-1} = \langle 2\pi e_1, e_2, e_3, -e_4 \rangle$$

is a subgroup of $f\Gamma f^{-1}$ of index 4, and is a lattice, so Λ acts properly and cocompactly. Hence by Lemmas 2.2.7 and 2.3.2, $f\Gamma f^{-1}$ acts properly and cocompactly. Then we observe that $f\gamma f^{-1}$ and Λ both act freely. Hence $f\Gamma f^{-1}$ acts properly discontinuously and cocompactly on $\mathbb{R}^4$. Then by Lemmas 2.2.8 and 2.3.4, Γ acts properly discontinuously and cocompactly on $CW_{n+2}(S)$.

Example 5.4.2. In Theorem 5.2.4, we proved that it is impossible to produce a homothetic quotient in the imaginary case. This example aims to provide a demonstration of, and intuition for, what goes wrong.

Take $CW_{n+2}(S)$, for $n = 1$, $S = -1$. Solutions to $\ddot{\beta} = S\beta$ are a linear combination of the trigonometric sin and cos functions. We aim to construct a group of homotheties, containing at least one strict homothety, that act properly discontinuously and cocompactly.

We consider the simplest strict homothety without fixed points possible. In line with Proposition 4.3.1, the simplest strict homothety we can include in our quotienting group (while acting freely) is

$$\gamma : \begin{pmatrix} x^+ \\ \boldsymbol{x} \\ x^- \end{pmatrix} \mapsto \begin{pmatrix} x^+ + 1 \\ 2\boldsymbol{x} \\ 4x^- \end{pmatrix}.$$

$\langle \gamma \rangle$ does not act cocompactly. So we now attempt to add a homothety η so that $\langle \gamma, \eta \rangle$ still acts properly discontinuously. By Proposition 2.4.4, the fact that γ has a translation component, means that we need to compactify in the $\boldsymbol{x}$, x^- directions. This is of course achievable only through the introduction of a non-zero β term. In line with this, we define

$$\eta : \begin{pmatrix} x^+ \\ \boldsymbol{x} \\ x^- \end{pmatrix} \mapsto \begin{pmatrix} x^+ \\ \boldsymbol{x} + \beta(x^+) \\ x^- - \left\langle \dot{\beta}(x^+), \boldsymbol{x} + \frac{1}{2}\beta(x^+) \right\rangle \end{pmatrix}.$$

But then we consider

$$\gamma^{-n}\eta\gamma^n \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2^{-n}\beta(n) \\ 4^{-n}\left\langle \dot{\beta}(n), \frac{1}{2}\beta(n) \right\rangle \end{pmatrix}$$

Then since β and $\dot{\beta}$ are both bounded (this is where the assumption that the eigenvalues of S are imaginary plays a crucial role), $\gamma^{-n}\eta\gamma^n(0)$ approaches 0, contradicting PD1. So this example cannot lead to a properly discontinuous and cocompact action.

Example 5.4.3. The previous example demonstrates the issue that arises when our solutions to $\ddot{\beta} = S\beta$ are trigonometric. This example demonstrates the difficulties that continue to arise even in the hyperbolic case.

Take $CW_{n+2}(S)$, for $n = 1$, $S = 1$. Solutions to $\ddot{\beta} = S\beta$ are a linear combination of the hyperbolic trigonometric functions sinh, cosh. Equivalently, and as shall be more useful during this example, β can be written in the form $ke^{x^+} + le^{-x^+}$, where $k, l \in \mathbb{R}$

As before, we take our starting place for the example as the homothety

$$\gamma : \begin{pmatrix} x^+ \\ \boldsymbol{x} \\ x^- \end{pmatrix} \mapsto \begin{pmatrix} x^+ + 1 \\ e\boldsymbol{x} \\ e^2 x^- \end{pmatrix}.$$

Similarly to the previous example, using Proposition 2.4.4, it's important that we have an element η with a non-zero β term. We consider

$$\eta : \begin{pmatrix} x^+ \\ \boldsymbol{x} \\ x^- \end{pmatrix} \mapsto \begin{pmatrix} x \\ \boldsymbol{x} + ke^{x^+} \\ x^- - \left\langle ke^{x^+}, \boldsymbol{x} + \frac{1}{2}ke^{x^+} \right\rangle \end{pmatrix}.$$

We define a function $f : \mathbb{R}^3 \to CW_{n+2}(S)$,

$$f : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ e^x y \\ e^{2x}(z - y^2/2) \end{pmatrix},$$

$$f^{-1} : \begin{pmatrix} x^+ \\ \boldsymbol{x} \\ x^- \end{pmatrix} \mapsto \begin{pmatrix} x^+ \\ e^{-x^+}\boldsymbol{x} \\ e^{-2x^+}(x^- + \boldsymbol{x}^2/2) \end{pmatrix}.$$

Our conjugates are

$$f^{-1}\gamma f : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x+1 \\ y \\ z \end{pmatrix}, \qquad f^{-1}\eta f : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ y+k \\ z \end{pmatrix}.$$

At this stage it looks promising, but we still have a remaining direction to compactify. The issue that occurs in general at this stage is that when we have γ a strict homothety of the simplest form possible without admitting fixed points as in Proposition 4.3.1, introducing an element ζ with a b term won't help us, for the same reason that the bounded trigonometric β caused problems in the previous example: $\gamma^{-i}\zeta\gamma^i(0)$ will approach 0. What this means is that it seems we will require β terms to compactify in $n+1$ directions. As our η demonstrates, it is quite possible to compactify in n of these directions, but compactifying in the last dimension seems impossible as the following demonstrates: Define

$$\zeta : \begin{pmatrix} x^+ \\ \boldsymbol{x} \\ x^- \end{pmatrix} \mapsto \begin{pmatrix} x^+ \\ \boldsymbol{x} + ke^{-x^+} \\ x^- - \left\langle ke^{-x^+}, \boldsymbol{x} + \frac{1}{2}ke^{-x^+} \right\rangle \end{pmatrix}.$$

Then

$$f^{-1}\zeta f : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ y + le^{-2x} \\ z \end{pmatrix}.$$

This ζ term demonstrates two issues: first that our conjugated element fails to act at all on the z direction – which is the direction still remaining to be acted cocompactly on. And second that when we have a homothety in the form of γ, we see immediately that only n of the beta dimensions are able to grow fast enough to avoid $\gamma^{-i}\zeta\gamma^i(0) \to 0$. This is because it is not sufficient that β be exponential, it must grow exponentially in the same direction as γ. So this example too cannot lead to a properly discontinuous and cocompact action.

It is worth noting at this point that this example extends slightly. If we take $S = \lambda^2$ for some $\lambda \in \mathbb{R}$ (Our β must then be adjusted to $ke^{\lambda x^+}$), and $\gamma : (x^+, \boldsymbol{x}, x^-) \mapsto (x^+ + c, e^s \boldsymbol{x}, e^{2s}\boldsymbol{x})$, then an adjustment to the function f is possible to achieve the same result we get here, but only if $s = \pm\lambda c$. Otherwise the conjugate of η retains dependency on x. It's possible that in considering this situation when $s \neq \pm\lambda c$, in the situation of a more elaborate γ, featuring its own β, b terms, or in the situation that η, ζ are more elaborate, that we gain the ability to act cocompactly in the x^- direction without losing proper discontinuity, but it's also possible that doing so only obfuscates the behaviour demonstrated in this example.

Example 5.4.4. In this example, we produce a compact quotient of an open submanifold of Cahen-Wallach spaces. To do so we shall overlook some small technical details on the way, although we will point out the details that are overlooked.

Consider $CW_{n+2}(S)$. Define $U := \mathbb{R}^{n+2} \setminus \{(x^+, 0, 0) \mid x^+ \in \mathbb{R}\}$. We have removed all fixed points of a pure homothety, allowing us to use a pure homothety to compactify: Take $\Gamma := \langle \gamma, \eta \rangle$, for

$$\gamma \begin{pmatrix} x^+ \\ \boldsymbol{x} \\ x^- \end{pmatrix} := \begin{pmatrix} x^+ + 1 \\ \boldsymbol{x} \\ x^- \end{pmatrix}, \qquad \eta \begin{pmatrix} x^+ \\ \boldsymbol{x} \\ x^- \end{pmatrix} := \begin{pmatrix} x^+ \\ 2\boldsymbol{x} \\ 4x^- \end{pmatrix}.$$

The choice of c and s for γ and η are not important. We now show that Γ acts properly discontinuously and cocompactly on U.

Our fundamental region for this action is roughly an annulus in the last $n+1$ dimensions, times a unit interval. Define

$$R := (0,1) \times ((-2,2)^n \times (-4,4)) \setminus [-1,1]^{n+1}.$$

gR does not meet R as follows: First note that γ and η commute. Let $p : (x^+, \boldsymbol{x}, x^-) \mapsto (\boldsymbol{x}, x^-)$, then

$$\begin{aligned} x^+(\gamma^i \eta^j R) &= (i, i+1), \\ p(\gamma^i \eta^j R) &= \left((-2^{j+1}, 2^{j+1})^n \times (-4^{j+1}, 4^{j+1})\right) \setminus [-2^j, 2^j]^{n+1}. \end{aligned}$$

Hence since (for i, j not both zero) $x^+(\gamma^i \eta^j R) \cap x^+(R) = \emptyset$, and $p(\gamma^i \eta^j R) \cap p(R) = \emptyset$, it must be that $\gamma^i \eta^j R \cap R = \emptyset$. Hence $gR \cap R = \emptyset$.

Note that

$$\overline{R} = [0,1] \times ([-2,2]^n \times [-4,4]) \setminus (-1,1)^{n+1}.$$

Then $\gamma^i \eta^j \overline{R}$ cover U (the fact that the line such that $\boldsymbol{x} = 0$, $x^- = 0$ is removed to make U is important here).

So R is a fundamental region. We take a neighbourhood V of $\overline{R}$:

$$V := (-1,2) \times ((-4,4)^n \times (-16,16)) \setminus \left([-\frac{1}{2}, \frac{1}{2}]^n \times [-\frac{1}{4}, \frac{1}{4}] \right).$$

Note that gV meets V only for

$$\{\gamma^i \eta^j \mid i,j \in \{-2,-1,0,1,2\}\}.$$

Hence R is finitely self adjacent. In particular, by Corollary 2.4.6, R is locally finite. Then by Theorem 2.3.13, U/Γ is homeomorphic to $\overline{R}/\Gamma$. Then since $\overline{R}/\Gamma$ is a manifold (we claim it is homeomorphic to $\mathbb{S}^1 \times (\mathbb{S}^n \times \mathbb{S}^1)$, proving this is a technical detail we ignore.), we get by Theorem 2.2.11 that Γ acts properly discontinuously (here we should first justify that Γ is discrete, which is not too difficult given the topology of Definition 4.2.10 and Proposition 4.2.11. However we should also justify why the projection is a smooth covering map, this is another detail we ignore). Also since $\overline{R}$ is compact, Γ also acts cocompactly.

Hence our action on the open submanifold U is properly discontinuous and cocompact. However, the homotheties centralized by Γ are not essential. Namely,

$$Z_{\mathrm{Homoth}}(\Gamma) = \{(0,0,1,c,A,s) \in \mathrm{Homoth}\}.$$

Define

$$f : \begin{pmatrix} x^+ \\ \boldsymbol{x} \\ x^- \end{pmatrix} = (||\boldsymbol{x}||^4 + (x^-)^2)^{-1/2}.$$

Then for $\phi \in Z_{\mathrm{Homoth}}(\Gamma)$,

$$\begin{aligned} \phi^*(fg)|_x &= f(\phi(x))\phi^* g|_x \\ &= (||e^s A\boldsymbol{x}||^4 + (e^{2s}x^-)^2)^{-1/2} e^{2s} g|_x \\ &= (||\boldsymbol{x}||^2 + (x^-)^2)^{-1/2} g|_x \\ &= (fg)|_x. \end{aligned}$$

So ϕ is inessential on U. Note that this same choice of f works for all such ϕ, and thus $Z_{\mathrm{Homoth}}(\Gamma)$ is inessential. In this example we can go further and conclude that the normalizer of Γ is inessential as well, since solving the c_ψ and s_ψ equations of Proposition 4.2.7 in the case that $\phi\gamma^r\eta^t = \gamma^{r'}\eta^{t'}\phi$ implies already that $t = t'$, and that $r' = ar$. Hence we see that the normalizer is simply $\{(0,0,a,c,A,s) \in \mathrm{Homoth}\}$, and the same f as before makes the normalizer inessential.

We stress that this is not a proof that U/Γ has an inessential conformal structure, because as noted after Theorem 3.5.10, it is possible to have an essential transformation on the quotient whose lift is not essential. And as noted after Proposition 3.5.5, a priori a transformation may be preserved without normalizing Γ.

Bibliography

Alekseevski, D. (1985), 'Selfsimilar Lorentzian manifolds', *Ann. Global Anal. Geom.* **3**(1), 59–84.

Alekseevskii, D. V. (1972), 'Groups of conformal transformations of Riemannian spaces', *Math. USSR Sbornik* **18**(2), 285–301.

Baum, H. (2009), *Eichfeldtheorie*, Springer.

Cahen, M. & Kerbrat, Y. (1978), 'Champs de vecteurs conformes et transformations conformes des espaces Lorentziens symétriques', *J. Math. pures et appl.* **57**, 99–132.

Cahen, M. & Kerbrat, Y. (1982), 'Transformations conformes des espaces symétriques pseudo-riemanniens', *Annali di Matematica Pura ed Applicata* **132**, 275–289.

Cahen, M. & Wallach, N. (1970), 'Lorentzian symmetric spaces', *Bull. Amer. Math. Soc.* **76**(3), 585–591.

Derdzinski, A. (2011), 'Zeros of conformal fields in any metric signature', *Class. Quantum Grav.* **28**(075011).

Eschenburg, J.-H. (1997), 'Lecture Notes on Symmetric Spaces'.
URL: *https://myweb.rz.uni-augsburg.de/~eschenbu/symspace.pdf*

Ferrand, J. (1971), 'Transformations conformes et quasi-conformes des variétés riemanniennes compactes', *Mém. Acad. Royale Belgique* **39**, 1–44.

Ferrand, J. (1996), 'The action of conformal transformations on a Riemannian manifold', *Mathematische Annalen* **304**(1), 277–291.

Frances, C. (2005), 'Sur les variétés lorentziennes dont le groupe conforme est essentiel', *Mathematische Annalen* **332**(1), 103–119.

Frances, C. (2007), 'Causal conformal vector fields, and singularities of twistor spinors', *Ann Glob Anal Geom* **32**, 277–295.

Frances, C. (2008), 'Essential conformal structures in Riemannian and Lorentzian geometry', *Recent Developments in pseudo-Riemannian geometry* **4**, 231–260.

Frances, C. (2015), 'About pseudo-Riemannian Lichnerowicz conjecture', *Transformation Groups* **20**, 1015–1022.

Gordon, C. S. & Wilson, E. N. (1986), 'The spectrum of the Laplacian on Riemannian Heisenberg manifolds.', *Michigan Math. J.* **33**(2), 253–271.

Hatcher, A. (2002), *Algebraic Topology*, Cambridge University Press.

Kapovich, M. (2020), 'A note on properly discontinuous actions'.
URL: *https://www.math.ucdavis.edu/~kapovich/EPR/prop-disc.pdf*

Kath, I. & Olbrich, M. (2019), *Compact quotients of Cahen-Wallach spaces*, Vol. 262 of *Memoirs of the American Mathematical Society*, American Mathematical Society.

Kühnel, W. & Rademacher, H. (1994), 'Essential conformal fields in pseudo–Riemannian geometry', *J. Math. Pures et Appl.* **74**(9), 453–581.

Kühnel, W. & Rademacher, H.-B. (1997), 'Essential Conformal Fields in Pseudo-Riemannian Geometry. ii', *J. Math. Sci. Univ. Tokyo* **4**, 649–662.

Kühnel, W. & Rademacher, H.-B. (2008), 'Conformal transformations of pseudo-Riemannian manifolds', *Recent developments in pseudo-Riemannian geometry* pp. 261–298.

Koszul, J., Simha, R. & Sridharan, R. (1965), 'Lectures on Groups of Transformations', Tata Institute of Fundamental Research, Bombay.

Lee, J. M. (2011), *Introduction to Topological Manifolds*, Vol. 202 of *Graduate Texts in Mathematics*, second edn, Springer.

Lee, J. M. (2012), *Introduction to Smooth Manifolds*, Vol. 218 of *Graduate Texts in Mathematics*, second edn, Springer.

Lichnerowicz, A. (1964), 'Sur les transformations conformes d'une variété riemannienne compacte', *C.R. Acad. Sci. Paris* **259**, 697–700.

Moroianu, A., Belgun, F. & Ornea, L. (2011), 'Essential points of conformal vector fields', *Journal of Geometry and Physics* **61**, 589–593.

Neukirchner, T. (2003), 'Solvable pseudo-riemannian symmetric spaces'. **URL:** *https://arxiv.org/abs/math/0301326*

Obata, M. (1970), 'Conformal transformations of Riemannian manifolds', *J. Differential Geometry* **4**, 311–333.

Obata, M. (1971), 'The conjectures on conformal transformations of Riemannian manifolds', *Bulletin of the American Mathematical Society* **77**(2), 265–270.

O'Neill, B. (1983), *Semi-Riemannian Geometry With Applications to Relativity*, Academic press.

Podoksenov, M. N. (1989), 'A Lorentzian manifold with a one-parameter group of homotheties which has a closed isotropic orbit', *Sib Math J* **30**(5), 771–773.

Ratcliffe, J. G. (2019), *Foundations of Hyperbolic Manifolds*, Vol. 149 of *Graduate Texts in Mathematics*, third edn, Springer.

Rotman, J. J. (1998), *An Introduction to Algebraic Topology*, Vol. 119 of *Graduate Texts in Mathematics*, fourth edn, Springer.

Schouten, J. A. (1954), *Ricci-Calculus: An Introduction to Tensor Analysis and Its Geometrical Applications*, Grundlehren der mathematischen Wissenschaften, second edn, Springer.

Silvester, J. R. (2000), 'Determinants of block matrices', *The Mathematical Gazette* **84**(501), 460–467.