Equivariant Oka theory for Riemann surfaces

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Contents

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Abstract

Oka theory involves the study of deforming continuous maps between complex manifolds into holomorphic maps. Gromov (1989) introduced the class of elliptic manifolds, which satisfy the property that every continuous map from a Stein source into an elliptic target is homotopic to a holomorphic map. Kutzschebauch, Lárusson, and Schwarz (2021) have generalised this theory to the equivariant setting.

Winkelmann (1993) provided a full classification of the pairs of Riemann surfaces for which every continuous map is homotopic to a holomorphic map. Due to the simplicity of the one-dimensional setting, Winkelmann's methods are much more accessible than the techniques introduced by Gromov. Continuing this theme, we generalise Winkelmann's results to the equivariant setting for Riemann surfaces in the case of a Stein source and an elliptic target, avoiding the higher-dimensional techniques used by Kutzschebauch, Lárusson, and Schwarz.

Specifically we show that if G is a finite group acting holomorphically on a noncompact Riemann surface X and $Y = \mathbb{C}, \mathbb{C}^*, \mathbb{C}/\Gamma$ for any lattice $\Gamma \subset \mathbb{C}$, then every G-equivariant continuous map $X \to Y$ is equivariantly homotopic to an equivariant holomorphic map $X \to Y$. We present only partial results for $Y = \mathbb{P}^1$. We show that if G acts effectively on X and $A \subset X$ is the set of points with nontrivial isotropy, then for each equivariant map $f : A \to \mathbb{P}^1$, the set $[X, \mathbb{P}^1]_G^f$ of G-homotopy classes of extensions $X \to \mathbb{P}^1$ of f is a singleton. The problem of whether each G-map $A \to \mathbb{P}^1$ admits an equivariant holomorphic extension is left open.

Chapter 1

Introduction

1.1 Context

This thesis is concerned with Winkelmann's [Win93] classification of pairs of Riemann surfaces (X, Y) for which mappings $X \to Y$ satisfy the homotopy principle, and its extension to the equivariant setting. This problem fits into the more general area of research known as Oka theory, and is motivated by recent progress in the equivariant setting due to Kutzschebauch, Lárusson, and Schwarz [KLS18, KLS21].

For a detailed historical overview of the Oka principle and Oka theory, we refer to the monograph of Forstnerič [For17]. We begin our discussion with the work of Gromov [Gro89]. Gromov was interested in studying the homotopy principle for holomorphic maps, henceforth shortened to the h-principle. More precisely, let X and Y be complex analytic manifolds. Holomorphic maps $X \to Y$ satisfy the *h-principle* if every continuous map $X \to Y$ is homotopic to a holomorphic map. Gromov's main h-principle, which he interpreted as a manifestation of the earlier known Oka principle, was that if X is Stein and Y is elliptic, a property introduced by Gromov, then every continuous map $X \to Y$ is homotopic to a holomorphic map. Following Gromov's reinterpretation, we will use the term *Oka principle* to refer to the holomorphic h-principle when X is Stein.

Related to the Oka principle is the basic Oka property. A complex manifold Y satisfies the *basic Oka property* if every continuous map $X \to Y$ from a reduced Stein space X is homotopic to a holomorphic map. There are stronger Oka properties incorporating approximation and interpolation, motivated by two classical theorems from complex analysis: the Runge approximation theorem, and Weierstrass' theorem on holomorphic functions with prescribed zeros. The various Oka properties involving approximation and interpolation were shown to all be equivalent [For17, Proposition 5.15.1], and collectively are known as *the* Oka property.

A manifold satisfying the Oka property is called an *Oka manifold*. The basic Oka property is not among the equivalent Oka properties: for example, the unit disk Δ satisfies the basic Oka property but is not an Oka manifold. If a complex manifold is elliptic in the sense of Gromov, then it is Oka. The question of whether all Oka manifolds are elliptic was settled in the negative only recently by Kusakabe [Kus20, Corollary 1.4].

Oka theory has also been considered in the equivariant setting. This thesis was originally motivated by the works of Kutzschebauch, Lárusson, and Schwarz [KLS18, KLS21]. In the second paper, Kutzschebauch, Lárusson, and Schwarz introduced the notion of a G-Oka manifold for a reductive complex Lie group G acting holomorphically, and showed that for a finite group G, a Stein G-manifold X, and a G-Oka manifold Y, every continuous G-map $X \to Y$ is G-homotopic to a holomorphic G-map [KLS21, Theorem 4.1]. A similar result was proved for homogeneous spaces Y using different methods in their earlier paper [KLS18].

One of the key assumptions of the main theorem of Kutzschebauch, Lárusson, and Schwarz [KLS21] is that the stabilisers of the G-action on the Stein G-manifold X are all finite. A special case is when G is itself finite. In line with this, we will only consider finite group actions.

We now turn our attention to the one-dimensional setting of Riemann surfaces, which is the main focus of this thesis. We note that a Riemann surface is Stein if and only if it is noncompact (see Forster [For81, Corollary 26.8] for the nontrivial direction), and a Riemann surface is Oka if and only if it is not hyperbolic. In this context, a Riemann surface is hyperbolic if it is holomorphically covered by the unit disk. Hence there are only four types of Oka Riemann surfaces: the complex plane $\mathbb{C}$, the punctured plane $\mathbb{C}^*$, any complex torus $\mathbb{C}/\Gamma$ where $\Gamma \subset \mathbb{C}$ is a lattice, and the Riemann sphere $\mathbb{P}^1$. Also, in the case of Riemann surfaces, the notions of a G-Oka manifold and an Oka G-manifold coincide. This is not true in general [KLS21, Example 2.7].

Winkelmann [Win93] was interested in finding all pairs of Riemann surfaces satisfying the h-principle, beyond the scope of Gromov's main h-principle considering only Stein sources. We state Winkelmann's main result.

Theorem ([Win93, Theorem 1]). *Let M and N be Riemann surfaces. Then every continuous map from M to N is homotopic to a holomorphic map in the following cases:*

(i) *M or N is isomorphic to $\mathbb{C}$ or $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, or $M \simeq \mathbb{P}^1 \not\simeq N$.*

(ii) *M is noncompact and N is isomorphic to $\mathbb{P}^1$, $\mathbb{C}^*$, or a torus.*

(iii) *N is isomorphic to $\Delta^* = \Delta \setminus \{0\}$ and $M \simeq M_0 \setminus \bigsqcup_{i=1}^n K_i$, where M_0 is a compact Riemann surface and each K_i is isomorphic to a nondegenerate closed disk in some local coordinate chart; i.e. M is noncompact, finite type, and without punctures.*

In all other cases there exists a continuous map from M to N which is not homotopic to any holomorphic map.

Case (i) holds due to contractibility arguments. Case (ii) falls under the general framework of Oka theory with the source being Stein and the target being Oka. Case (iii) consists of the exceptions.

While Case (ii) is covered by general Oka theory, Winkelmann provides simple proofs exploiting special features of the explicit manifolds in question. For example, every noncompact Riemann surface M has the homotopy type of a wedge of circles. The Riemann sphere $\mathbb{P}^1$ is simply connected. Hence every continuous map $M \to \mathbb{P}^1$ is homotopic to a constant map for purely topological reasons.

The goal of this thesis is to extend Winkelmann's classification to the equivariant setting. In line with the general framework of Oka theory, we restrict ourselves to the Stein-Oka pairings of Case (ii). On the equivariant side, we remind the reader that we only consider finite group actions. While the equivariant h-principle for Stein-Oka pairings is not a new result because it is covered by the theorems of Kutzschebauch, Lárusson, and Schwarz [KLS18, KLS21], the original contributions of this thesis are instead new and simple proofs primarily using algebraic topology and the theory of Riemann surfaces. We will avoid relying on the heavy machinery of Kutzschebauch, Lárusson, and Schwarz.

More explicitly, first consider the equivariant h-principle for Stein-Oka pairings.

Theorem. *Let X be a noncompact Riemann surface and let Y be an Oka manifold. Suppose that G is a finite group acting holomorphically on X and Y. Suppose further that G acts effectively on X. Every continuous G-map $X \to Y$ is G-homotopic to a holomorphic map.*

Proof (Sketch). Let $\pi : X \to X/G$ be the orbit space projection. The set $A \subset X$ of points with nontrivial isotropy is closed and discrete. Let $Q_0 = A/G$ and $S = X/G \setminus Q_0$. The projection map $q : (X \times Y)/G \to X/G$ sending $[x, y]$ to $[x]$ is an elliptic submersion over S because $q^{-1}S = (\pi^{-1}S \times Y)/G$ is a holomorphic fibre bundle over S with Oka fibres Y. The closed subvariety $Q_0 \subset X/G$ contains all points of X/G over which q fails to be a submersion and the image of the singular locus of $(X \times Y)/G$ because $q^{-1}S$ is a manifold. Thus by a theorem of Forstnerič [For03, Theorem 2.1], for every continuous section $F : X/G \to (X \times Y)/G$ of q that is holomorphic on a neighbourhood of Q_0, there is a homotopy $F_t : X/G \to (X \times Y)/G$ of sections of q agreeing with F on Q_0 for all t, such that $F_0 = F$ and F_1 is holomorphic.

Let $I(X, Y) = \{(x, y) \in X \times Y \mid G_x \subset G_y\}$. Let $M(X, Y) = I(X, Y)/G$. As explained by tom Dieck [tom87, Chapter I, Section 7], continuous G-maps $X \to Y$ are in natural correspondence with sections of the projection $M(X, Y) \to X/G$. Hence given a continuous G-map $f : X \to Y$, we obtain a section F_0 of $M(X, Y) \to X/G$. In the setting of Riemann surfaces, we may safely assume that f is locally constant in some neighbourhood of Q_0. Forstnerič's theorem gives a homotopy F_t of sections of $(X \times Y)/G \to X/G$, which are in fact sections of $M(X, Y) \to X/G$ because the homotopy is fixed on Q_0. The holomorphic map g corresponding to F_1 is G-homotopic to f. □

The proof sketch we have given is in the style of Kutzschebauch, Lárusson, and Schwarz [KLS21, Theorem 4.1], adapted to the simplified setting of Riemann surfaces. Assuming that X is a Riemann surface gives two important simplifications. The first is the low dimensionality. In the general setting, Kutzschebauch, Lárusson, and Schwarz induct over the dimension of certain subvarieties of X/G. A Riemann surface is one-dimensional, so we do not need to deal with the general induction process. The second simplification is that we can always equivariantly deform a continuous G-map $X \to Y$ from a Riemann surface X to be locally constant on a neighbourhood of a discrete set. This allows us to meet the holomorphicity condition required to use Forstnerič's theorem for sections of ramified mappings.

Thus the main black box in our proof sketch is Forstnerič's theorem [For03, Theorem 2.1]; see also the monograph of Forstnerič [For17, Theorem 6.14.6] for an alternative formulation. Forstnerič's theorem is particularly deep, with its proof largely based on a preceding paper by Forstnerič and Prezelj [FP01, Theorem 1.4], which in turn builds on their previous[1] work [FP02]. The origin of this chain of results goes back to Gromov [Gro89, 4.5. Main Theorem], though Gromov's formulation did not have full generality. Further details on this matter can be found in Forstnerič's monograph [For17, Theorem 6.2.2]. We end our discussion of Forstnerič's theorem for ramified mappings by noting that it is the only known Oka principle that does not require the map in question, in our case $q : (X \times Y)/G \to X/G$, to be a submersion.

The point of our proof sketch and its subsequent discussion was to identify the heavy machinery we wished to avoid. Our methods will instead be based on algebraic topology and the theory of Riemann surfaces. We describe the structure of this thesis and our main results in the next section.

1.2 Main results and thesis structure

As we have briefly mentioned, the original contributions of this thesis are mainly new and simple proofs of the equivariant Oka principle for Riemann surfaces. We prove the following equivariant Oka principle.

Theorem. *Let X be a noncompact Riemann surface and let Y be $\mathbb{C}$, $\mathbb{C}^*$ or $\mathbb{C}/\Gamma$ for some lattice $\Gamma \subset \mathbb{C}$. Let G be a finite group acting holomorphically on X and Y. Every continuous G-map $X \to Y$ is G-homotopic to a holomorphic map.*

We prove a full equivariant Oka principle for all Oka Riemann surfaces except $\mathbb{P}^1$. The case of $\mathbb{C}$ is covered by Corollary 4.3.2 and is due to $\mathbb{C}$ being equivariantly contractible with respect to any holomorphic finite group action. The case of $\mathbb{C}^*$ and $\mathbb{C}/\Gamma$ are covered by Theorem 4.3.8.

[1] The apparent inconsistency with the dates is explained by an addendum in the paper [FP02].

The story for the Riemann sphere $\mathbb{P}^1$ is different. The two main problems are that every nontrivial holomorphic finite group action on $\mathbb{P}^1$ is not free, and $\mathbb{P}^1$ does not have a group structure like $\mathbb{C}$, $\mathbb{C}^*$ and $\mathbb{C}/\Gamma$. We only establish the following partial result.

Theorem. *Let X be a noncompact Riemann surface. Suppose that a finite group G acts holomorphically and effectively on X and holomorphically on $\mathbb{P}^1$. Let $A \subset X$ be the set of points with nontrivial isotropy. If $f_0, f_1 : X \to \mathbb{P}^1$ are two continuous G-maps agreeing on A, then f_0 and f_1 are G-homotopic relative A.*

In other words, the equivariant Oka principle for $\mathbb{P}^1$ will hold provided that every equivariant map $A \to \mathbb{P}^1$ extends to a holomorphic equivariant map $X \to \mathbb{P}^1$. Since A is always closed and discrete, we may view this theorem as a reduction from an equivariant Oka principle to an equivariant interpolation theorem.

Other than this reduction theorem, we can establish a full equivariant Oka principle for $\mathbb{P}^1$ when G acts freely on the source. Here and throughout this thesis, given topological spaces X and Y, we let $\mathscr{C}(X, Y)$ be the set of continuous maps $X \to Y$ equipped with the compact open topology. If G is a topological group acting continuously on X and Y, we let $\mathscr{C}_G(X, Y)$ be the space of continuous G-maps $X \to Y$. We analogously define the spaces $\mathscr{O}(X, Y)$ and $\mathscr{O}_G(X, Y)$ of holomorphic maps for complex manifolds X, Y and G.

Theorem. *Let G be a finite group acting holomorphically on a noncompact Riemann surface X and $\mathbb{P}^1$. Let $Q_0 \subset X/G$ be the set of orbits containing points with nontrivial isotropy. Let $S = X/G \setminus Q_0$. There exists a homeomorphism $\mathscr{C}_G(\pi^{-1}S, \mathbb{P}^1) \simeq \mathscr{C}(S, \mathbb{P}^1)$ with respect to the compact open topology restricting to a homeomorphism $\mathscr{O}_G(\pi^{-1}S, \mathbb{P}^1) \simeq \mathscr{O}(S, \mathbb{P}^1)$. In particular, if G acts freely on X, then $\mathscr{O}_G(X, \mathbb{P}^1) \simeq \mathscr{O}(X/G, \mathbb{P}^1)$.*

One may notice the similarity in notation with our earlier proof sketch involving Forstnerič's theorem. In the proof sketch, we stated that $(\pi^{-1}S \times Y)/G \to S$ is a holomorphic fibre bundle with Oka fibres Y. If $Y = \mathbb{P}^1$, then this is a projective bundle over a noncompact Riemann surface and is thus trivial. We only need Forstnerič's theorem to deal with the ramification locus of the orbit space projection $X \to X/G$. If the action on X is free, then there is no ramification and the Oka principle can be deduced from the theory of principal G-bundles.

We now explain the structure of this thesis in more detail. In Chapter 2, we present background material from algebraic topology and Riemann surface theory. The purpose is to introduce the main objects and results required by proofs in the latter chapters. On the algebraic topology side, we introduce relative homotopy groups and the relative Hurewicz theorem for use in equivariant obstruction theory. Covering spaces, fibre bundles and principal bundles are introduced since they are involved with both the plain h-principle, which we review in Chapter 3, and the equivariant Oka principle which we discuss in Chapter 4. On the Riemann surface side, we state some important classification theorems and vanishing theorems. Examples include the classification of Riemann surfaces by their universal covering, and the vanishing of cohomology for a noncompact Riemann surface.

Chapter 3 is our review of Winkelmann's [Win93] classification of the h-principle for Riemann surfaces. The goal of this chapter is to provide a detailed exposition of Winkelmann's arguments. In his paper, after establishing the Oka principle for $\mathbb{C}^*$, Winkelmann observes that for any complex torus T, there is a projection $\tau : \mathbb{C}^* \times \mathbb{C}^* \to T$ that is both a holomorphic map and a homotopy equivalence. Since this is the extent of Winkelmann's argument, we devote a section to investigating such a map. Supposing that $T \simeq \mathbb{C}/\Gamma$ for some lattice $\Gamma \subset \mathbb{C}$, we construct a diagram

$$\begin{array}{ccccc} & & \mathbb{C}^2 & \xrightarrow{\alpha} & \mathbb{C} \\ & \swarrow^{E\times E} & \downarrow p & & \downarrow \pi \\ \mathbb{C}^* \times \mathbb{C}^* & \xleftarrow{(E\times E)^\#} & \mathbb{C}^2/\Gamma & \xrightarrow{\alpha^\#} & \mathbb{C}/\Gamma \end{array}$$

such that $\alpha^\# : \mathbb{C}^2/\Gamma \to \mathbb{C}/\Gamma$ is a holomorphic fibre bundle with contractible fibre $\mathbb{C}$. We call this fibre bundle the *Winkelmann bundle*. The map $(E \times E)^\#$ is a biholomorphism, so we interpret τ to be obtained from the bottom row of this diagram.

Chapter 3 then continues with background material specifically required for Winkelmann's negative cases, which we now state.

Proposition ([Win93, Proposition 1]). *For each of the following pairs of Riemann surfaces M and N, there exists a continuous map from M to N not homotopic to any holomorphic map.*

(i) *M is compact and $N \simeq \mathbb{P}^1$.*

(ii) *M is compact and both N and M are not simply connected.*

(iii) *M is noncompact and not simply connected, and N is hyperbolic, excluding Δ, Δ^*.*

(iv) *$M \simeq M_1 \setminus \{p\}$ for some Riemann surface M_1 with $\pi_1(M) \neq 0$ and $N \simeq \Delta^*$.*

(v) *$\pi_1(M)$ is not finitely generated and $N \simeq \Delta^*$.*

Case (i) is a degree argument. For variety, we follow Saito's [Sai20] novel approach of considering the projective limit $H_q(X||A) = \varprojlim_U H_q(U \setminus A)$ with $U \subset X$ varying over all open sets in X containing the closed set A. We prove a local degree formula in Proposition 3.2.6 which was not provided by Saito.

The following section on hyperbolic geometry is required for Case (iii) and Case (v). We discuss the hyperbolic distance on the upper half plane, and hyperbolic geodesics. Rather than following Winkelmann's original argument for Case (iii), we instead adapt an argument due to Gromov [Gro89] involving the lengths of hyperbolic geodesics. For Case (v), we were unable to complete Winkelmann's original argument because there was difficulty in determining whether a claimed positive uniform lower bound was in fact positive. We reconcile the issue by using the following original result.

Lemma. *Let $z_0 \in \mathbb{H}$. Let d be the hyperbolic distance in $\mathbb{H}$. There exists a positive constant $\lambda > 0$ such that $0 < \lambda \leq d(z_0, z_0 + t)/\log|t|$ for all $t \in \mathbb{R}$ with $|t| > 1$.*

This is Lemma 3.3.8, and Remark 3.4.15 explains its relevance in more detail. Following Winkelmann's original argument, given a sequence (n_j) of positive integers satisfying a certain property, we would need to determine whether the sequence given by $d(z_0, z_0 + n_j)/n_j$ has a positive uniform lower bound. The critical finding is observing that the logarithm resolves this problem unambiguously in the affirmative.

Chapter 4 is concerned with the equivariant Oka principle. Rather than relying on Forstnerič's theorem, we exploit the fact that there are only four types of Oka Riemann surfaces: the complex plane $\mathbb{C}$, the punctured plane $\mathbb{C}^*$, any complex torus $\mathbb{C}/\Gamma$, and the Riemann sphere $\mathbb{P}^1$.

We first determine the holomorphic automorphism groups of $\mathbb{C}$, $\mathbb{C}^*$, and $\mathbb{C}/\Gamma$. These are standard exercises in complex analysis and Riemann surface theory, but we include their proofs for completeness. At the same time, we determine the finite subgroups of $\operatorname{Aut}\mathbb{C}$ and $\operatorname{Aut}\mathbb{C}^*$ up to conjugacy, as these finite subgroups represent the possible holomorphic finite group actions that can occur. For $\mathbb{C}$, this allows us to show that $\mathbb{C}$ is G-contractible for any holomorphic action by a finite group G.

We then turn our attention to the complex structure on the orbit space X/G. For a Riemann surface X equipped with the holomorphic action of a finite group G, there exists a unique complex structure on X/G such that the projection $\pi : X \to X/G$ is holomorphic. Our treatment mainly follows Miranda [Mir95] in the construction of the maps which eventually become coordinate charts for X/G. Diverging from Miranda's chart compatibility argument, we define the structure sheaf $\mathscr{O}_{X/G}$ by letting $\mathscr{O}_{X/G}(U)$ be the ring of continuous functions $f : U \to \mathbb{C}$ such that $f\pi : \pi^{-1}U \to \mathbb{C}$ is holomorphic. We show that $(X/G, \mathscr{O}_{X/G})$ is locally isomorphic as a locally ringed space to a domain in $\mathbb{C}$.

The fact that X/G is noncompact if X is noncompact allows us to reduce the equivariant Oka principle to the plain Oka principle in combination with a homotopy lifting argument. Section 4.3 contains the precise details of how this is achieved, and ends with our previously mentioned equivariant Oka principle for $\mathbb{C}^*$ and $\mathbb{C}/\Gamma$.

We conclude the thesis with Section 4.4, which contains our partial results regarding the equivariant Oka principle for $\mathbb{P}^1$. We give a detailed exposition of the equivariant obstruction theory of tom Dieck [tom87], which is necessary for our theorem reducing the equivariant Oka principle for $\mathbb{P}^1$ to an equivariant holomorphic interpolation theorem. This reduction theorem uses methods only from algebraic topology.

The main object in tom Dieck's equivariant obstruction theory is the equivariant cohomology group $\mathfrak{H}^n_G(X, A; \pi_n Y)$. Fix a G-map $f : A \to Y$ and let $[X, Y]^f_G$ be the set of G-homotopy classes of G-map extensions $X \to Y$ of f. Under certain assumptions, tom Dieck's main result is that there is a bijection of sets $[X, Y]^f_G \simeq \mathfrak{H}^n_G(X, A; \pi_n Y)$.

When G is a finite group acting holomorphically and effectively on a noncompact Riemann surface X and holomorphically on $Y = \mathbb{P}^1$, we can construct an isomorphism

$\mathfrak{H}^2_G(X, A; \pi_2\mathbb{P}^1) \simeq H^2(X/G, A/G)$ with the latter group being ordinary singular cohomology. Since $H^2(X/G, A/G) = 0$ for noncompact X, tom Dieck's theorem implies $[X, \mathbb{P}^1]^f_G \simeq \mathrm{pt}$. Hence we only need an injection, and our exposition of tom Dieck is simplified to account for this.

Our reduction theorem yields two simple cases where we can prove an equivariant Oka principle. The first is for maps $\mathbb{C}^* \to \mathbb{P}^1$ where G acts dihedrally on $\mathbb{C}^*$ and $\mathbb{P}^1$, and the second is when G acts on $\mathbb{P}^1$ by rotations.

We end the chapter and the thesis with Section 4.4.6. Here we prove the equivariant Oka principle for maps $X \to \mathbb{P}^1$ when G acts freely on X. The proof relies only on the theory of principal G-bundles and their associated bundles, along with the triviality of projective bundles over noncompact Riemann surfaces.

1.3 Further directions

We conclude our introduction by discussing three open problems.

Problem 1. Let X be a noncompact Riemann surface. Let G be a finite group acting holomorphically and effectively on X and holomorphically on $\mathbb{P}^1$. Let $A \subset X$ be the set of points with nontrivial isotropy. For each G-map $F_0 : A \to \mathbb{P}^1$, does there exist a holomorphic G-map $F : X \to \mathbb{P}^1$ such that $F|_A = F_0$?

This is the equivariant holomorphic interpolation theorem which would enable us to establish an equivariant Oka principle for $\mathbb{P}^1$. The most natural first step is to consider the induced map on quotients, as shown in the diagram

$$\begin{array}{ccc} A & \xrightarrow{F_0} & \mathbb{P}^1 \\ \downarrow & & \downarrow \\ A/G & \xrightarrow{f_0} & \mathbb{P}^1/G. \end{array}$$

Note that $\mathbb{P}^1/G \simeq \mathbb{P}^1$ for any finite group G acting holomorphically on $\mathbb{P}^1$. Choose $x \in (\mathbb{P}^1/G) \setminus f_0(A/G)$ and identify $(\mathbb{P}^1/G) \setminus \{x\} \simeq \mathbb{C}$. Use Weierstrass' interpolation theorem to produce a nonconstant holomorphic map $f : X/G \to (\mathbb{P}^1/G) \setminus \{x\}$ such that $f|_{A/G} = f_0$. We then need to deal with the lifting problem

$$\begin{array}{ccc} X & \overset{F}{\dashrightarrow} & \mathbb{P}^1 \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle q} \\ X/G & \xrightarrow{f} & \mathbb{P}^1/G. \end{array}$$

The difficulty is that $q : \mathbb{P}^1 \to \mathbb{P}^1/G$ always has ramification, and consequently is never a covering map. If we let $B \subset \mathbb{P}^1/G$ be the set of branch points of $q : \mathbb{P}^1 \to \mathbb{P}^1/G$, then

$\pi^{-1}f^{-1}B \subset X$ is discrete because $f\pi : X \to \mathbb{P}^1/G$ is holomorphic and nonconstant. So we can attempt to find conditions on f for which the covering map lifting condition

$$(f\pi)_*\pi_1(X \setminus \pi^{-1}f^{-1}B, x_0) \subset q_*\pi_1(\mathbb{P}^1 \setminus q^{-1}B, y_0)$$

is satisfied, for appropriately chosen base points $x_0 \in X \setminus \pi^{-1}f^{-1}B$ and $y_0 \in \mathbb{P}^1 \setminus q^{-1}B$.

Problem 2. Let G be a finite group acting holomorphically and effectively on a noncompact Riemann surface X and holomorphically on $\mathbb{P}^1$. Let $Q_0 \subset X/G$ be the set of orbits containing points with nontrivial isotropy. Let $S = X/G \setminus Q_0$. What conditions enable a G-homotopy $\pi^{-1}S \times I \to \mathbb{P}^1$ to be extended to a homotopy $X \times I \to \mathbb{P}^1$?

This is a completely different way to tackle the Oka principle for $\mathbb{P}^1$. While Forstnerič's theorem resolves this question, it is worthwhile trying to find a simpler approach. Our problem involves attempting to extend maps $\pi^{-1}S \to \mathbb{P}^1$ over the closed discrete set $\pi^{-1}Q_0$. Any continuous extension will consequently be holomorphic and equivariant.

To illustrate the stark difference between $\mathscr{C}_G(\pi^{-1}S, \mathbb{P}^1)$ and $\mathscr{C}_G(X, \mathbb{P}^1)$, consider the case when G acts effectively but not freely on both X and $\mathbb{P}^1$. We show in Theorem 4.4.26 that $[X, \mathbb{P}^1]_G \simeq \mathrm{Map}_G(A, \mathbb{P}^1)$. However $[\pi^{-1}S, \mathbb{P}^1]_G = \mathrm{pt}$.

Problem 3. Extend Winkelmann's classification of the h-principle for Riemann surfaces to the equivariant setting beyond the Stein-Oka pairings.

On the affirmative side, the pair $M \simeq \mathbb{P}^1 \not\simeq N$ seems the most difficult to extend because the covering space contractibility argument used in Proposition 3.4.1 does not seem to easily extend to the equivariant setting. Case (iii) of Winkelmann's affirmative cases is interesting. Loosely speaking, the source M is noncompact, of finite type, and without punctures. If we can prove that the noncompact space M/G is also of finite type and without punctures, then we can extend Case (iii) to the equivariant setting.

For the negative cases, the recurring theme in proving the plain h-principle is to first produce a homomorphism between the fundamental groups involved satisfying some property that cannot be achieved by the induced homomorphism of a holomorphic map, then use the property of Eilenberg-Mac Lane spaces to produce a continuous map inducing the initial homomorphism. By homotopy invariance this continuous map is not homotopic to any holomorphic map. Since we discard many continuous maps when we restrict to the equivariant category, it is not clear to what extent the negative cases remain negative.

Chapter 2

Background material

The goal of this chapter is to collect and present some well-known results which form the general background and language required for the subsequent chapters. The two overarching themes of this chapter are algebraic topology and Riemann surface theory.

2.1 Homotopy and homology theory

Many of the established results in algebraic topology are proved for a wide class of spaces known as *CW-complexes*. These are spaces that are constructed by inductively attaching closed n-balls along their boundary. More precisely, set $D^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ and $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$. Define $D^0 = \text{pt}$ and $S^{-1} = \emptyset$. Let (X, A) be a pair of topological spaces. For a family of continuous maps

$$\varphi_j : S^{n-1} \to A, \quad j \in J,$$

if X is a pushout in the diagram

$$\begin{array}{ccc} \coprod_{j\in J} S^{n-1} & \longrightarrow & A \\ \downarrow & & \downarrow \\ \coprod_{j\in J} D^n & \xrightarrow{\phi} & X, \end{array}$$

then we say that X is obtained from A by attaching the family of n-cells $(D^n)_{j\in J}$. The map ϕ is called a characteristic map. A pair (X, A) is called a *relative CW-complex* if there exists a filtration $(X_n)_{n\in\mathbb{Z}}$ of X such that the following properties hold.

1. $A \subset X_0; A = X_n$ for $n < 0$; $X = \bigcup_{n\in\mathbb{Z}} X_n$.

2. For each $n \geq 0$, the space X_n is obtained from X_{n-1} by attaching n-cells.

3. X carries the colimit topology with respect to $(X_n)_{n\in\mathbb{Z}}$.

If $A = \emptyset$, then we say that X is a *CW-complex*. We say that X_n is the n-skeleton of X.

One nice property of CW-complexes is that maps from a CW-complex into a space Y can be constructed cell by cell, and obstructions to extending a map over each n-cell are given by the nth homotopy group of Y. This will become relevant in Section 4.4, except at that point we will also consider group actions. For now, we will develop the language of homotopy groups.

2.1.1 Long exact sequence in homotopy

The first object one usually encounters in homotopy theory is the fundamental group $\pi_1(X, x_0)$ of a pointed space (X, x_0). Let $I = [0, 1]$ be the closed unit interval. Elements of $\pi_1(X, x_0)$ are equivalence classes of loops $(I, \{0, 1\}) \to (X, x_0)$ at x_0 up to homotopy fixed on $\{0, 1\}$. Given two paths $\gamma_1 : I \to X$ and $\gamma_2 : I \to X$ such that $\gamma_1(1) = \gamma_2(0)$, we can form a new path $\gamma_1 * \gamma_2$ by concatenation. Explicitly we define

$$(\gamma_1 * \gamma_2)(t) = \begin{cases} \gamma_1(2t) & \text{if } 0 \le t \le 1/2, \\ \gamma_2(2t-1) & \text{if } 1/2 \le t \le 1. \end{cases}$$

In particular loops can be concatenated. Up to homotopy relative $\{0, 1\}$, concatenation is associative. Identity is the constant map $c_{x_0} : I \to X$ sending the interval to x_0. Given a loop $\gamma : I \to X$ at x_0, the inverse $\gamma^- : I \to X$ is defined $\gamma^-(t) = \gamma(1-t)$.

By identifying endpoints of I, we can instead define $\pi_1(X, x_0)$ as the set $[S^1, *; X, x_0]$ of homotopy classes of maps $S^1 \to X$ sending the base point $* \in S^1$ to $x_0 \in X$. This suggests defining $\pi_n(X, x_0) = [S^n, *; X, x_0]$ as a set. In fact there is a more general definition for pairs of spaces which we now introduce.

Definition 2.1.1. Let (X, A, a_0) be a pair of spaces with base point $a_0 \in A$. Define, for $n \ge 0$, the relative homotopy group $\pi_{n+1}(X, A, a_0)$ to be the set $[D^{n+1}, S^n, *; X, A, a_0]$.

Remark 2.1.2. If $A = a_0$, then every map $(D^{n+1}, S^n, *) \to (X, A, a_0)$ sends S^n to a point. There is a unique induced map from the quotient $D^{n+1}/S^n \simeq S^{n+1}$, generalising the definition $\pi_{n+1}(X, x_0) = [S^{n+1}, *; X, x_0]$.

Remark 2.1.3. For ease of notation, we will often omit base points from our notation and simply write $\pi_n(X)$ or $\pi_n(X, A)$.

The motivation for this definition stems from the desire to have a long exact sequence

$$\ldots \longrightarrow \pi_{n+1}(X, A) \xrightarrow{\partial} \pi_n(A) \xrightarrow{i_*} \pi_n(X) \xrightarrow{j_*} \pi_n(X, A) \longrightarrow \ldots \tag{2.1}$$

where $i : A \to X$ and $j : (X, *) \to (X, A)$ are inclusions, and ∂ is to be determined. If $i_*[f] = 0$, then $f : S^n \to A$ is null homotopic when viewed as a map into X. Hence there exists a continuous map $g : D^{n+1} \to X$ such that $g|S^n = f$. This suggests the definition of $\pi_{n+1}(X, A)$ given, and also suggests that ∂ should be defined $\partial[g] = [g|S^n]$.

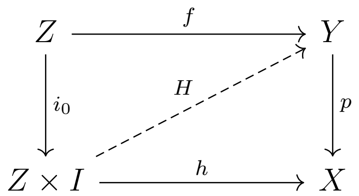
For any map $f : S^n \to X$ of pointed spaces, we precompose by the composition $q : D^n \to D^n/S^{n-1} \simeq S^n$ and compose with j to obtain a map $D^n \to (X, A)$ sending S^{n-1} to $* \in A$; i.e. we define $j_*[f] = [jfq]$.

Let $\Omega^n(X, A)$ be the space of maps $(D^n, S^{n-1}) \to (X, A)$. It can be shown that elements of $\pi_{n+1}(X, A)$ are in one-to-one correspondence with elements of $\pi_1(\Omega^n(X, A))$, and we use this bijection to define a group structure on $\pi_{n+1}(X, A)$ for $n \geq 1$. We consider $\pi_1(X, A)$ as a pointed set without a group structure.

Under these conditions, the sequence in (2.1) becomes exact. For a full exposition, we refer to Whitehead [Whi78, Chapter IV] or tom Dieck [tom08, Chapter 6].

2.1.2 Homotopy lifting and homotopy extension

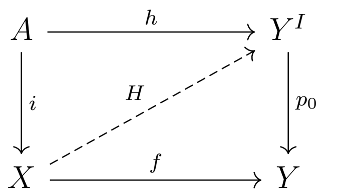
Two important concepts in homotopy theory are homotopy lifting and homotopy extension. A continuous map $p : Y \to X$ satisfies the *homotopy lifting property* with respect to a space Z if for each homotopy $h : Z \times I \to X$ and each map $f : Z \to Y$ lifting $h(-, 0)$, there exists a lift $H : Z \times I \to Y$ of h such that $H(-, 0) = f$. In other words there exists H such that the diagram



commutes, where $i_0 : Z \to Z \times I$ is defined $i_0(z) = (z, 0)$.

If a continuous map $p : Y \to X$ satisfies the homotopy lifting property for all spaces Z, then p is called a *Hurewicz fibration*, or simply a *fibration*. If p satisfies the homotopy lifting property with respect to unit cubes I^n for all n, then p is called a *Serre fibration*. Examples of Serre fibrations include covering spaces and fibre bundles. These will be discussed in Section 2.2 and Section 2.3. The distinction between the two types of fibrations is not important since we only consider homotopy lifting with respect to CW-complexes.

Dually, a continuous map $i : A \to X$ satisfies the *homotopy extension property* with respect to a space Y if for each homotopy $h : A \times I \to Y$ and each map $f : X \to Y$ extending $h(-, 0)$, there exists an extension $H : X \times I \to Y$ of h with $H(-, 0) = f$. In other words there exists H such that the diagram



commutes, where $Y^I = \mathscr{C}(I, Y)$ is the set of continuous maps $I \to Y$ and $p_0 : Y^I \to Y$ is defined $p_0(\gamma) = \gamma(0)$. We note that some conditions are required for there to be a homeomorphism $\mathscr{C}(X \times I, Y) \simeq \mathscr{C}(X, Y^I)$ with respect to the compact open topology. The most common sufficient condition is that all spaces are locally compact Hausdorff.

If a continuous map $i : A \to X$ satisfies the homotopy extension property for all spaces Y, then i is called a *cofibration*. The inclusions $* \hookrightarrow S^n \hookrightarrow D^{n+1}$ are cofibrations, and this allows us to discuss the action of $\pi_1(A)$ on the homotopy sequence of (X, A).

2.1.3 Action of $\pi_1(A)$ on the homotopy sequence of (X, A)

Let $v : I \to X$ be a path and let $f : (S^n, *) \to (X, v(0))$ be a continuous map. Since $* \hookrightarrow S^n$ is a cofibration, the homotopy extension property gives $V_t : S^n \to X$ with $V_0 = f$ and $V_-(*) = v$. This leads to an action of π_1 on higher homotopy via transport.

Proposition 2.1.4. *Let $v : I \to X$ be a loop at $x_0 \in X$, and let $f : (S^n, *) \to (X, x_0)$ be a continuous map. Let $V_t : S^n \to X$ be any homotopy extension of f through v. The assignment $[V_0] \mapsto [V_1]$ depends only on the homotopy class of v, and defines a right group action of $\pi_1(X, x_0)$ on $\pi_n(X, x_0)$.*

Proof. See tom Dieck [tom08, Proposition 6.2.1]. □

A similar result holds in the relative case because the inclusions $* \hookrightarrow S^n \hookrightarrow D^{n+1}$ are cofibrations. Fix $a_0 \in A \subset X$. Let $f : (D^{n+1}, S^n, *) \to (X, A, a_0)$ represent an element of $\pi_{n+1}(X, A, a_0)$, and let $\alpha : * \times I \to A$ be a loop in A at a_0; i.e. the map α represents an element of $\pi_1(A, a_0)$. We first obtain a homotopy $h : S^n \times I \to X$ satisfying $h(-, 0) = f$ and $h(*, -) = \alpha$. Since $f|S^n \to A$ extends to $f : D^{n+1} \to X$, we then obtain a homotopy $H : D^{n+1} \times I \to X$ satisfying $H|S^n \times I = h$ and $H(-, 0) = f$. Define $\alpha_\# : \pi_{n+1}(X, A, a_0) \to \pi_{n+1}(X, A, a_0)$ by $\alpha_\#[f] = [H(-, 1)]$. We omit the proof that $\alpha_\#$ depends only on the homotopy class of α and f, and that the assignment is in fact a group action of $\pi_1(A, a_0)$ on $\pi_{n+1}(X, A, a_0)$. We refer to tom Dieck [tom08, Section 6.2] for a full exposition.

A space X is called *n-simple* if for each $x \in X$, the fundamental group $\pi_1(X, x)$ acts trivially on $\pi_n(X, x)$. The next proposition provides the relevance of this property.

Proposition 2.1.5. *Suppose that X is n-simple and path connected. For each $x_0 \in X$, the map $\pi_n(X, x_0) \to [S^n, X]$ forgetting the base point is a well-defined bijection.*

Proof. The forgetful map is well defined because if two maps $f, g : (S^n, *) \to (X, x_0)$ are homotopic relative $*$, then they are homotopic. Let $f, g : (S^n, *) \to (X, x_0)$ represent elements of $\pi_n(X, x_0)$ that have the same image in $[S^n, X]$. Then there exists a homotopy $H : S^n \times I \to X$ such that $H(-, 0) = f$ and $H(-, 1) = g$. Since $H(*, -) : I \to X$ is a loop at x_0, it represents an element of $\pi_1(X, x_0)$. But H is a homotopy extension of f through $H(*, -)$ and the transport action is trivial, so $[f] = [H(*, -)].[f] = [g]$ in $\pi_n(X, x_0)$.

Let $g : S^n \to X$ represent an element of $[S^n, X]$. Since X is path connected, there exists a path $v : I \to X$ from $g(*)$ to x_0. Use the homotopy extension property to produce a homotopy $H : S^n \times I \to X$ such that $H(-, 0) = g$ and $H(*, -) = v$. Then $H(-, 1)$ represents an element of $\pi_n(X, x_0)$ such that $[g] = [H(-, 1)]$ in $[S^n, X]$. □

Proposition 2.1.6. *The transport action of $\pi_1(A, a_0)$ on $\pi_n(X, A, a_0)$ commutes with the boundary map $\partial : \pi_n(X, A, a_0) \to \pi_n(A, a_0)$.*

Proof. Let $\alpha : I \to A$ be a loop at a_0, let $f : (D^{n+1}, S^n, *) \to (X, A, a_0)$ represent an element of $\pi_n(X, A, a_0)$, let $h : S^n \times I \to X$ satisfy $h(-, 0) = f|S^n$ and $h(*, -) = \alpha$, and let $H : D^{n+1} \times I \to X$ be a homotopy satisfying $H|S^n \times I = h$ and $H(-, 0) = f$. Since $H(-, 1)|S^n = h(-, 1)$, we have $\alpha_\#[f|S^n] = [H(-, 1)|S^n]$ viewing $\alpha_\#$ as an action of $\pi_1(A, a_0)$ on $\pi_n(A, a_0)$. Hence $\partial\alpha_\#[f] = [H(-, 1)|S^n] = \alpha_\#\partial[f]$. □

Proposition 2.1.7. *Let $f : (X, A) \to (Y, B)$ be a map of pointed spaces. Let $\alpha : I \to X$ be a based loop. Then $f_*\alpha_\# = (f\alpha)_\# f_*$.*

Proof. Let $[\xi] \in \pi_{n+1}(X, A)$. Let $h : S^n \times I \to X$ satisfy $h(-, 0) = \xi|S^n$ and $h(*, -) = \alpha$. Let $H : D^{n+1} \times I \to X$ satisfy $H|S^n \times I = h$ and $H(-, 0) = \xi$. Then $fh : S^n \times I \to Y$ satisfies $fh(-, 0) = (f|S^n)_*[\xi|S^n] = f_*(\xi)|S^n$ and $fh(*, -) = f\alpha$ while $fH : D^n \times I \to Y$ satisfies $fH(-, 0) = f_*(\xi)$ and $fH|S^n \times I = fh$. Therefore $\alpha_\#[\xi] = [H(-, 1)]$ and $(f\alpha)_\# f_*[\xi] = [fH(-, 1)] = f_*\alpha_\#[\xi]$. □

2.1.4 Hurewicz homomorphism

The Hurewicz homomorphisms link homotopy groups and homology groups. We state the necessary results without proof. A complete and elementary exposition can be found in tom Dieck [tom08, Chapter 20].

Recall that $H_n(S^n) \simeq \mathbb{Z} \simeq H_n(D^n, S^{n-1})$. Let $q : D^n \to D^n/S^{n-1} \simeq S^n$ be the quotient map, and $\partial : H_n(X, A) \to H_{n-1}(A)$ be the boundary map from the long exact sequence of (X, A). Choose a generator $z_1 \in H_1(S^1)$, then inductively define $z_n \in H_n(S^n)$ and $\widetilde{z}_n \in H_n(D^n, S^{n-1})$ by $\partial\widetilde{z}_n = z_{n-1}$ and $q_*\widetilde{z}_n = z_n$.

Definition 2.1.8. For $n \geq 1$, the *absolute Hurewicz morphism*

$$\varrho : \pi_n(X, a_0) \to H_n(X)$$

is defined by $\varrho(\xi) = f_*(z_n)$ for any representative $f : (S^n, *) \to (X, a_0)$ of ξ.

For $n \geq 2$, the *relative Hurewicz morphism*

$$\varrho : \pi_n(X, A, a_0) \to H_n(X, A)$$

is defined by $\varrho(\xi) = f_*(\widetilde{z}_n)$ for any representative $f : (D^n, S^{n-1}, *) \to (X, A, a_0)$ of ξ.

When $n = 1$, the Hurewicz morphism induces an isomorphism $\pi_1(X, a_0)^{\mathrm{ab}} \to H_1(X)$. In higher degrees, we consider the quotient $\pi_n^{\#}(X, A)$ of $\pi_n(X, A)$ by the subgroup generated by elements of the form $x - x.\alpha$, where $\alpha \in \pi_1(A)$. We then obtain the relative Hurewicz theorem.

Theorem 2.1.9. *Let (X, A) be a CW-pair with connected X and A. For $n \geq 2$, let (X, A) be $(n-1)$-connected. Then $\varrho^{\#} : \pi_n^{\#}(X, A, *) \simeq H_n(X, A)$.*

Proof. tom Dieck [tom08, Theorem 20.1.11]. □

2.2 Manifolds and covering spaces

Definition 2.2.1. Let X be a topological space. A *covering space* of X is a continuous surjection $p : Y \to X$ such that for each $x \in X$, there exists a neighbourhood U of x such that $p^{-1}U = \bigcup_{j \in J} V_j$ for disjoint open sets V_j such that the restriction $p|V_j \to U$ is a homeomorphism for each j.

Definition 2.2.2. Let X be a connected space. A covering space $p : Y \to X$ is *universal* if Y is simply connected.

As previously mentioned, covering spaces are examples of fibrations: they satisfy the homotopy lifting property for all spaces. This property is one of the main ingredients for our equivariant Oka principle in Chapter 4 because the orbit space projection $Y \to Y/G$ is a covering if G acts freely on Y.

While covering spaces are defined for general topological spaces, we focus on smooth manifolds. A *smooth manifold* is a Hausdorff second countable space X with a family $(U_i, \psi_i)_{i \in \Lambda}$ of homeomorphisms $\psi_i : U_i \to \psi_i U_i$ from an open subset U_i of X onto an open subset of $\mathbb{R}^n$ such that $\psi_i \psi_j^{-1} | \psi_j(U_i \cap U_j) \to \psi_i(U_i \cap U_j)$ is a diffeomorphism between open subsets of $\mathbb{R}^n$ for all $i, j \in \Lambda$, and $\bigcup_i U_i = X$. Complex manifolds are defined analogously, and a *Riemann surface* is a connected complex manifold of dimension one.

Theorem 2.2.3. *Every connected smooth manifold X has a universal covering space.*

Proof. Forster [For81, Theorem 5.3]. □

More generally, the necessary and sufficient condition for a space X to have a universal covering is that X is path connected, locally path connected and semilocally simply connected [Hat02, Section 1.3]. Manifolds satisfy these properties.

Theorem 2.2.4. *Let X be a smooth manifold. If $p : Y \to X$ is a covering space, then there exists a unique smooth structure on Y for which p becomes a local diffeomorphism. If X is a Riemann surface, then there exists a unique complex structure on Y for which p becomes a local biholomorphism.*

Proof. See Napier and Ramachandran [NR12, Proposition 10.2.10] for the case of manifolds and Forster [For81, Theorem 4.6] for the case of Riemann surfaces. □

We now list some important lifting properties of covering spaces, starting with the homotopy lifting property.

Proposition 2.2.5. *All covering spaces satisfy the homotopy lifting property.*

Proof. Hatcher [Hat02, Proposition 1.30] □

Covering spaces satisfy a more general lifting property with conditions relating to the fundamental groups of the relevant spaces.

Proposition 2.2.6. *Suppose that $p : (Y, y_0) \to (X, x_0)$ is a covering space, and $f : (Z, z_0) \to (X, x_0)$ is continuous with Z path connected and locally path connected. There exists a lift $(Z, z_0) \to (Y, y_0)$ of f if and only if $f_*\pi_1(Z, z_0) \subset p_*\pi_1(Y, y_0)$.*

Proof. Hatcher [Hat02, Proposition 1.33]. □

In addition to the homotopy lifting property, which is an existence statement, covering spaces also have uniqueness of liftings.

Proposition 2.2.7. *Suppose that $Y \to X$ is a covering space and $f : Z \to X$ is a continuous map from a connected space Z. If two lifts $Z \to Y$ of f agree at a point of Z, then they agree on all of Z.*

Proof. Hatcher [Hat02, Proposition 1.34]. □

Proposition 2.2.8. *Let $p : Y \to X$ be a covering space. If $\gamma, \sigma : I \to Y$ are paths such that $\gamma(0) = \sigma(0)$ and $p\gamma \sim p\sigma$ relative $\{0, 1\}$ in X, then $\gamma \sim \sigma$ relative $\{0, 1\}$ in Y.*

Proof. Let $H : I \times I \to X$ be a homotopy from $p\gamma$ to $p\sigma$ relative $\{0, 1\}$. Lift to a map $\widetilde{H} : I \times I \to Y$ satisfying $p\widetilde{H} = H$ and $\widetilde{H}(0, 0) = \gamma(0)$. Since $p\widetilde{H}(t, 0) = p\gamma(t)$ and $\widetilde{H}(0, 0) = \gamma(0)$, we have $\widetilde{H}(t, 0) = \gamma(t)$ by uniqueness of lifts. Since $p\widetilde{H}(0, s) = p\gamma(0)$ and $\widetilde{H}(0, 0) = \gamma(0)$, we have $\widetilde{H}(0, s) \equiv \gamma(0)$. Since $p\widetilde{H}(t, 1) = p\sigma(t)$ and $\widetilde{H}(0, 1) = \gamma(0) = \sigma(0)$, we have $\widetilde{H}(t, 1) = \sigma(t)$. Finally, since $p\widetilde{H}(1, s) \equiv p\sigma(1)$ and $\widetilde{H}(1, 1) = \sigma(1)$, we have $\widetilde{H}(1, s) \equiv \sigma(1)$. By assumption $\gamma(0) = \sigma(0)$, while $\gamma(1) = \widetilde{H}(1, 1) = \sigma(1)$. Hence $\widetilde{H}$ is a homotopy from γ to σ relative $\{0, 1\}$. □

There is also a correspondence between the subgroups of the fundamental group of a space X and coverings of X. We state one direction of this correspondence.

Proposition 2.2.9. *Suppose that X is path connected, locally path connected and semilocally simply connected. For every subgroup $H \subset \pi_1(X, x_0)$, there exists a covering $p : (Y, y_0) \to (X, x_0)$ such that $p_*\pi_1(Y, y_0) = H$ for a suitably chosen base point $y_0 \in Y$.*

Proof. Hatcher [Hat02, Proposition 1.36]. □

2.3 Fibre bundles

Definition 2.3.1. A *fibre bundle* with fibre F is a triple (E, p, B) consisting of a surjective continuous map $p : E \to B$ such that for each point $b \in B$, there exists a neighbourhood U of b and a homeomorphism $h : p^{-1}U \to U \times F$ such that the diagram

$$\begin{array}{ccc} p^{-1}U & \xrightarrow{h} & U \times F \\ & \searrow^{p} & \downarrow{\scriptstyle \mathrm{pr}_1} \\ & & U \end{array}$$

commutes.

Theorem 2.3.2. *Let $p : E \to B$ be a fibre bundle with fibre F. Choose base points $b_0 \in B$ and $x_0 \in F = p^{-1}(b_0)$. If B is path connected, then there exists a long exact sequence*

$$\cdots \to \pi_n(F, x_0) \to \pi_n(E, x_0) \xrightarrow{p_*} \pi_n(B, b_0) \to \pi_{n-1}(F, x_0) \to \cdots \to \pi_0(E, x_0) \to 0$$

of pointed sets.

Proof. Hatcher [Hat02, Theorem 4.41, Proposition 4.48]. Theorem 4.41 establishes the long exact sequence on homotopy associated to a Serre fibration while Proposition 4.48 establishes that fibre bundles are Serre fibrations. □

Corollary 2.3.3. *If F is contractible, then the projection $p : E \to B$ induces isomorphisms $p_* : \pi_n(E, x_0) \to \pi_n(B, b_0)$ for each $n \geq 1$.* □

Theorem 2.3.4 (Whitehead's theorem)**.** *If a map $f : X \to Y$ between connected CW-complexes induces isomorphisms $f_* : \pi_n(X) \to \pi_n(Y)$ for all $n \geq 1$, then f is a homotopy equivalence.*

Proof. Hatcher [Hat02, Theorem 4.5]. □

Corollary 2.3.5. *Suppose that E and B are connected CW-complexes. If $p : E \to B$ is a fibre bundle with contractible fibre, then $p : E \to B$ is a homotopy equivalence.* □

2.4 Principal G-bundles and associated fibre bundles

Principal G-bundles provide an alternative approach to understanding fibre bundles. For example, the transition functions of a principal G-bundle and any associated fibre bundle coincide. The exposition follows Husemoller [Hus94], though unlike Husemoller, we will insist that all bundles are locally trivial and that groups always act on the left.

Definition 2.4.1. A *principal G-bundle* $p : X \to B$ is a fibre bundle with fibre G such that X is a free G-space, there exists a homeomorphism $f : X/G \to B$ such that $p = f \circ \pi$ where $\pi : X \to X/G$ is the quotient projection, and the local trivialisations $h : p^{-1}U \to U \times G$ are G-equivariant with respect to the action on $U \times G$ defined $g.(b, y) = (b, gy)$.

Remark 2.4.2. One obtains a holomorphic principal G-bundle by modifying the definition in the appropriate manner.

Definition 2.4.3. A *principal bundle morphism* $(X, p, B) \to (X', p', B')$ is a pair of maps $(f, f^{\#})$ such that $f : X \to X'$ is G-equivariant and $f^{\#} : B \to B'$ satisfies $f^{\#} \circ p = p' \circ f$.

Proposition 2.4.4. *Let $B \times G$ be the trivial bundle over B. Every G-automorphism of $B \times G$ over B is of the form $h_g(b, s) = (b, sg(b))$, where $g : B \to G$ is a continuous map, and conversely, such a relation defines a G-automorphism over B.*

Proof. Supposing that $g : B \to G$ is given, the map $h_g : B \times G \to B \times G$ defined by $h_g(b, s) = (b, sg(b))$ is fibre-preserving and equivariant with inverse $h_g^{-1} = h_{g^{-1}}$, where $g^{-1}(b) = g(b)^{-1}$ is the pointwise inverse. Conversely, let $h : B \times G \to B \times G$ be an equivariant automorphism over B. Since $ph = p$, we have $h(b, s) = (b, \mathrm{pr}_2\, h(b, s))$. Define $g : B \to G$ by $g(b) = \mathrm{pr}_2\, h(b, 1)$. Then $h(b, s) = h_g(b, s)$ by equivariance of h. □

Remark 2.4.5. Proposition 2.4.4 gives transition functions for general principal bundles.

Suppose that (X, p, B) is a principal G-bundle and F is a G-space. Equip $X \times F$ with the diagonal action $g(x, y) = (gx, gy)$. Let $X_F = (X \times F)/G$ and let $p_F : X_F \to B$ be the unique map such that

$$\begin{array}{ccc} X \times F & \xrightarrow{\mathrm{pr}_1} & X \\ \downarrow & & \downarrow p \\ X_F & \xrightarrow{p_F} & B \end{array}$$

commutes. We define (X_F, p_F, B) to be the fibre bundle associated with (X, p, B).

Theorem 2.4.6. *The associated fibre bundle (X_F, p_F, B) is locally trivial.*

Proof. For each $b \in B$ there exists an open neighbourhood U of b and an equivariant homeomorphism $h_U : p^{-1}U \to U \times G$ such that $\mathrm{pr}_1\, h_U = p$. Then $\mathrm{pr}_2\, h_U : p^{-1}U \to G$ is equivariant where the action on G is left multiplication. Define $f_U : p^{-1}U \times F \to U \times F$ by $f_U(x, y) = (p(x), \mathrm{pr}_2\, h_U(x)^{-1}y)$. The map f_U is open because p is open and the group action $\varphi : G \times F \to F$ is open; in fact $\varphi(W_1 \times W_2) = \bigcup_{g \in W_1} gW_2$ for any open rectangle $W_1 \times W_2 \subset G \times F$. If $(x, y) \sim (x', y')$ modulo G, then $f_U(x, y) = f_U(x', y')$ by equivariance of $\mathrm{pr}_2\, h_U$. This gives an induced quotient map $f_U^{\#} : p_F^{-1}U \to U \times F$ which one verifies is a bijection, inheriting both continuity and openness from f_U. So $f_U^{\#}$ is a homeomorphism. □

Remark 2.4.7. If we have local trivialisations $h_i : p^{-1}U_i \to U_i \times G$ for the principal G-bundle (X, p, B) with corresponding transition functions $g_{ij}(b) = \mathrm{pr}_2\, h_j \circ h_i^{-1}(b, 1)$, then the transition functions γ_{ij} obtained from $f_i^{\#} \circ (f_j^{\#})^{-1}$ are precisely g_{ij}. Observe that $\gamma_{ij}(b) = \mathrm{pr}_2\, h_i(x)^{-1} \cdot \mathrm{pr}_2\, h_j(x)$ for any $x \in p^{-1}(b)$; this is well defined since $p(x) = p(x')$ if and only if $gx = x'$ for some $g \in G$ because (X, p, B) is a G-bundle. Equivariance gives $\gamma_{ij}(b) = \mathrm{pr}_2\, h_j(\mathrm{pr}_2\, h_i(x)^{-1}.x)$. But $\mathrm{pr}_2\, h_i(x)^{-1}.x = h_i^{-1}(b, 1)$ and so $\gamma_{ij}(b) = g_{ij}(b)$.

Remark 2.4.8. Suppose that (X, p, B) is a principal G-bundle and (X_F, p_F, B) is an associated fibre bundle with fibre F and G-action $\varphi : G \to \operatorname{Aut} F$. Let $(U_i, h_i)_{i\in\Lambda}$ be a local trivialisation for (X, p, B) with corresponding transition functions $g_{ij} : U_i \cap U_j \to G$. By Remark 2.4.7, these also serve as transition functions for (X_F, p_F, B) in that the transitions $\psi_{ij} : U_i \cap U_j \times F \to U_i \cap U_j \times F$ are given by $\psi_{ij}(b, y) = (b, \varphi(g_{ij}(b))(y))$.

2.5 Riemann surfaces

We end our background chapter by presenting some key results on Riemann surfaces.

2.5.1 Classification theorems

Theorem 2.5.1. *Let M be a noncompact Riemann surface. Then M is homotopy equivalent to a wedge sum of circles.*

Proof. Napier and Ramachandran [NR04, Theorem 2.2]. □

Theorem 2.5.2. *Suppose that M is a Riemann surface with $\pi_1(M)$ finitely generated. Then there exists a compact Riemann surface M_0 and a (possibly empty) compact set $K \subset M_0$ that is the union of finitely many disjoint compact sets, each of which is either a singleton or a closed disk in some local holomorphic chart, such that M is biholomorphic to $M_0 \setminus K$.*

Proof. The earliest proof known to the author is by Stout [Sto65, Theorem 8.1]. Another approach via the holomorphic attachment and removal of tubes is left as an exercise by Napier and Ramachandran [NR12, Exercise 5.17.2]. □

Theorem 2.5.3 (Riemann mapping theorem)**.** *Suppose that X is a Riemann surface with $\mathrm{Rh}^1_{\mathscr{O}}(X) = 0$. Then X can be mapped biholomorphically onto either the Riemann sphere $\mathbb{P}^1$, the complex plane $\mathbb{C}$, or the unit disk Δ.*

Proof. Forster [For81, Theorem 27.9]. □

Remark 2.5.4. Here $\mathrm{Rh}^1_{\mathscr{O}}(X)$ is the first holomorphic de Rham cohomology group. Its vanishing means that every closed holomorphic one-form has a holomorphic primitive. This condition holds if X is simply connected [For81, 27.1]. The converse of this statement is the content of the Riemann mapping theorem.

Theorem 2.5.5. (a) *The Riemann sphere $\mathbb{P}^1$ is a holomorphic covering for only $\mathbb{P}^1$.*

(b) *The complex plane $\mathbb{C}$ is a holomorphic covering for $\mathbb{C}$, the punctured plane $\mathbb{C}^*$, and all complex tori.*

(c) *Every other Riemann surface has the unit disk Δ as a holomorphic covering.*

Proof. Forster [For81, Theorem 27.12]. □

Remark 2.5.6. We will refer to Riemann surfaces covered by the disk as *hyperbolic* and the other Riemann surfaces as *Oka*.

Having established the classification of Riemann surfaces by their universal covering, we will take this opportunity to introduce Eilenberg-Mac Lane spaces. In particular we show that every Riemann surface other than $\mathbb{P}^1$ is Eilenberg-Mac Lane.

Definition 2.5.7. Suppose that X is a path connected topological space such that $\pi_n(X) \simeq G$ and $\pi_q(X) = 0$ for $q \neq n$. Then X is said to be *Eilenberg-Mac Lane*, denoted $K(G, n)$.

Proposition 2.5.8. *Let $p : (Y, y_0) \to (X, x_0)$ be a covering space. The induced map $p_* : \pi_n(Y, y_0) \to \pi_n(X, x_0)$ is an isomorphism for all $n \geq 2$.*

Proof. Hatcher [Hat02, Proposition 4.1]. □

Corollary 2.5.9. *Let $X \not\simeq \mathbb{P}^1$ be a Riemann surface. Then X is $K(\pi_1(X), 1)$.*

Proof. The Riemann mapping theorem implies that X has a contractible covering, and so all higher homotopy groups vanish. □

Proposition 2.5.10. *Let X be a connected CW-complex and let Y be $K(G, 1)$. Then every homomorphism $\pi_1(X, x_0) \to \pi_1(Y, y_0)$ is induced by a map $(X, x_0) \to (Y, y_0)$ that is unique up to homotopy relative x_0.*

Proof. Hatcher [Hat02, Proposition 1B.9]. □

2.5.2 Vanishing theorems

Theorem 2.5.11. *Let X be a noncompact Riemann surface. Then $H^1(X, \mathscr{O}) = 0$.*

Proof. Forster [For81, Theorem 26.1]. □

Theorem 2.5.12. *Let X be a noncompact Riemann surface and $GL(n, \mathscr{O})$ the sheaf of invertible $n \times n$ matrices with coefficients in $\mathscr{O}$. Then $H^1(X, GL(n, \mathscr{O})) = 0$. Equivalently every vector bundle over a noncompact Riemann surface is trivial.*

Proof. Forster [For81, Theorem 30.4, Corollary 30.5]. □

Corollary 2.5.13. *Every projective bundle over a noncompact Riemann surface is trivial.*

Proof. Note that $H^1(X, GL(n, \mathscr{O})) = 0$ implies $H^1(X, SL(n, \mathscr{O})) = 0$ since if (g_{ij}) is a cocycle in $SL(n, \mathscr{O})$ with respect to an open covering (U_i), we can first obtain a splitting $g_{ij} = g_i g_j^{-1}$ in $GL(n, \mathscr{O})$ with $1 = \det g_{ij} = \det g_i (\det g_j)^{-1}$ on $U_i \cap U_j$, and then replace g_i with $g_i d_i$ where d_i is the identity matrix with the (1,1)-entry replaced with $1/\det g_i$.

The short exact sequence of sheaves $0 \to \mathbb{Z}/2\mathbb{Z} \to SL(n, \mathscr{O}) \to PSL(n, \mathscr{O}) \to 0$ induces a long exact sequence of cohomology which gives

$$H^1(X, SL(n, \mathscr{O})) = 0 \to H^1(X, PSL(n, \mathscr{O})) \to H^2(X, \mathbb{Z}/2\mathbb{Z}).$$

Sheaf cohomology with coefficients in an abelian group coincides with singular cohomology for paracompact and locally contractible spaces, in particular Riemann surfaces. Every noncompact Riemann surface is homotopy equivalent to a wedge of circles, and singular cohomology is a homotopy invariant. Hence $H_2(X, \mathbb{Z}) \simeq \bigoplus H_2(S^1, \mathbb{Z}) = 0$. The homotopy type of a noncompact Riemann surface also implies that $H_1(X, \mathbb{Z})$ is free, so a special case of the universal coefficients theorem [tom08, Proposition 11.9.4] implies that $H^2(X, \mathbb{Z}/2\mathbb{Z}) \simeq \mathrm{Hom}(H_2(X, \mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) = 0$. □

Chapter 3

Winkelmann's classification of the homotopy principle

Winkelmann [Win93] classified the h-principle for maps between pairs of Riemann surfaces. First Winkelmann establishes the pairs of Riemann surfaces for which every continuous map is homotopic to a holomorphic map.

Theorem 3.0.1 ([Win93, Theorem 1]). *Let M and N be Riemann surfaces. Then every continuous map from M to N is homotopic to a holomorphic map in the following cases:*

(i) *M or N is isomorphic to $\mathbb{C}$ or $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, or $M \simeq \mathbb{P}^1 \not\simeq N$.*

(ii) *M is noncompact and N is isomorphic to $\mathbb{P}^1$, $\mathbb{C}^*$, or a torus.*

(iii) *N is isomorphic to $\Delta^* = \Delta \setminus \{0\}$ and $M \simeq M_0 \setminus \bigsqcup_{i=1}^n K_i$, where M_0 is a compact Riemann surface and each K_i is isomorphic to a nondegenerate closed disk in some local coordinate chart; i.e. M is noncompact, finite type, and without punctures.*

In all other cases there exists a continuous map from M to N which is not homotopic to any holomorphic map.

Winkelmann then describes the negative cases.

Proposition 3.0.2 ([Win93, Proposition 1]). *For each of the following pairs of Riemann surfaces M and N, there exists a continuous map from M to N not homotopic to any holomorphic map.*

(i) *M is compact and $N \simeq \mathbb{P}^1$.*

(ii) *M is compact and both N and M are not simply connected.*

(iii) *M is noncompact and not simply connected, and N is hyperbolic, excluding Δ, Δ^*.*

(iv) *$M \simeq M_1 \setminus \{p\}$ for some Riemann surface M_1 with $\pi_1(M) \neq 0$ and $N \simeq \Delta^*$.*

(v) *$\pi_1(M)$ is not finitely generated and $N \simeq \Delta^*$.*

This is an exhaustive list of pairs, as demonstrated by the following table.

<table>
<tr><th></th><th>(Source; Target)</th><th>Y/N</th><th>Reason</th></tr>
<tr><td rowspan="5">Compact source</td><td>$(\mathbb{P}^1; \mathbb{P}^1)$</td><td>N</td><td>P1(i)</td></tr>
<tr><td>$(\mathbb{P}^1; \text{not } \mathbb{P}^1)$</td><td>Y</td><td>T1(i)</td></tr>
<tr><td>$(\text{Not SC}; \mathbb{C}, \Delta)$</td><td>Y</td><td>T1(i)</td></tr>
<tr><td>$(\text{Not SC}; \mathbb{P}^1)$</td><td>N</td><td>P1(i)</td></tr>
<tr><td>(Not SC; Not SC)</td><td>N</td><td>P1(ii)</td></tr>
<tr><td>Noncompact SC source</td><td>$(\mathbb{C}, \Delta; \text{ Anything})$</td><td>Y</td><td>T1(i)</td></tr>
<tr><td rowspan="2">Excluding Δ, Δ^* from targets</td><td>(Noncompact not SC; Oka)</td><td>Y</td><td>T1(ii)</td></tr>
<tr><td>(Noncompact not SC; Not Oka)</td><td>N</td><td>P1(iii)</td></tr>
<tr><td rowspan="2">Δ, Δ^* target</td><td>$(\text{Anything}; \Delta)$</td><td>Y</td><td>T1(i)</td></tr>
<tr><td>$(\pi_1(M) \text{ not FG}; \Delta^*)$</td><td>N</td><td>P1(v)</td></tr>
<tr><td rowspan="2">$\pi_1(M)$ finitely generated</td><td>$(\text{Exists puncture, not SC}; \Delta^*)$</td><td>N</td><td>P1(iv)</td></tr>
<tr><td>$(\text{No puncture}; \Delta^*)$</td><td>Y</td><td>T1(iii)</td></tr>
</table>

The abbreviation SC is short for simply connected. The abbreviation T1 refers to Theorem 3.0.1, which was originally Theorem 1 of Winkelmann's paper [Win93]. The abbreviation P1 refers to Proposition 3.0.2, which was originally Proposition 1 of Winkelmann's paper.

The existence of a puncture on a Riemann surface M with $\pi_1(M)$ finitely generated refers to the construction in Theorem 2.5.2. The condition that a Riemann surface is Oka is equivalent to a Riemann surface being not hyperbolic. Any simply connected compact Riemann surface is biholomorphic to the Riemann sphere $\mathbb{P}^1$.

The goal of this chapter is to provide a full and detailed exposition of Winkelmann's arguments, or alternative approaches for the same result. While most of the author's contribution for this chapter is expository, Lemma 3.3.8 is an original result used to address a discrepancy in Winkelmann's original argument that the author was unable to resolve; this is addressed in Remark 3.4.15.

3.1 The Winkelmann bundle

In proving the Oka principle for a complex torus T, Winkelmann[1] observed that there is a map $\tau : \mathbb{C}^* \times \mathbb{C}^* \to T$ which is both holomorphic and a homotopy equivalence. We detail the calculations behind obtaining such a map.

[1]We thank the anonymous examiner for pointing out that similar constructions have previously appeared in the literature. For example, Hartshorne [Har70, Chapter VI, Example 3.2] attributes such a construction to Jean-Pierre Serre.

For any complex torus T, there exists $\omega \in \mathbb{H}$ such that $T \simeq \mathbb{C}/\Gamma$, where $\Gamma = \mathbb{Z} + \omega\mathbb{Z}$ is a lattice in $\mathbb{C}$. The lattice acts on $\mathbb{C}$ by translation $(m + \omega n).z = z + m + \omega n$. Define a Γ-action on $\mathbb{C}^2$ by $(m + \omega n).(z, w) = (z + n, w + m)$. We obtain a commutative diagram

$$\begin{array}{ccccc} & & \mathbb{C}^2 & \xrightarrow{\alpha} & \mathbb{C} \\ & \overset{E\times E}{\swarrow} & \downarrow p & & \downarrow \pi \\ \mathbb{C}^* \times \mathbb{C}^* & \xleftarrow{(E\times E)^{\#}} & \mathbb{C}^2/\Gamma & \xrightarrow{\alpha^{\#}} & \mathbb{C}/\Gamma \end{array}$$

determined by the maps $E : \mathbb{C} \to \mathbb{C}^*$ and $\alpha : \mathbb{C}^2 \to \mathbb{C}$ defined $E(z) = \exp(2\pi i z)$ and $\alpha(z, w) = w + \omega z$. The maps p and π are the quotient projections, while $(E \times E)^{\#}$ and $\alpha^{\#}$ are the induced quotient maps; notice that α is Γ-equivariant and $(E \times E)^{\#}$ is a homeomorphism by the open mapping theorem in one complex variable. In fact $\mathbb{C}^2/\Gamma$ realises $\mathbb{C}^* \times \mathbb{C}^*$ as a quotient of its holomorphic universal covering $\mathbb{C}^2$ under the holomorphic covering map $E \times E$; this stems from $E : \mathbb{C} \to \mathbb{C}^*$ being a universal covering. The map $(E \times E)^{\#}$ becomes a biholomorphism and $\alpha^{\#}$ becomes holomorphic.

Theorem 3.1.1. *The map* $\alpha^{\#} : \mathbb{C}^2/\Gamma \to \mathbb{C}/\Gamma$ *is a fibre bundle with fibre* $\mathbb{C}$.

Proof. Let $[b_0] \in \mathbb{C}/\Gamma$. Take any open neighbourhood V of b_0 such that no two points of V are equivalent modulo Γ. The set $U = \pi V$ is an open neighbourhood of $[b_0]$, the restriction $\pi|V \to U$ is a homeomorphism, and $(\alpha^{\#})^{-1}U = p\alpha^{-1}V$. Note that $\alpha^{-1}V$ has no two points equivalent modulo Γ. Therefore $p|\alpha^{-1}V \to (\alpha^{\#})^{-1}U$ is a homeomorphism, and we may define a continuous map $h_U : (\alpha^{\#})^{-1}U \to U \times \mathbb{C}$ via

$$h_U([z, w]) = (\pi(w + \omega z), \mathrm{pr}_1(p|_{\alpha^{-1}V})^{-1}([z, w]));$$

note that π is Γ-invariant so h_U is well defined. The map $U \times \mathbb{C} \to (\alpha^{\#})^{-1}U$ defined $h_U^{-1}([b], z) = [z, (\pi|_V)^{-1}([b]) - \omega z]$ is a continuous inverse. Hence $\alpha^{\#} : \mathbb{C}^2/\Gamma \to \mathbb{C}/\Gamma$ is a fibre bundle with fibre $\mathbb{C}$. □

We now use the theory of principal bundles to provide an alternative proof of local trivialisation. We first observe that $\pi : \mathbb{C} \to \mathbb{C}/\Gamma$ is a principal Γ-bundle because it is a covering map. We then examine an associated fibre bundle with respect to a certain Γ-action on $\mathbb{C}$, and investigate its relation to the Winkelmann bundle $\alpha^{\#} : \mathbb{C}^2/\Gamma \to \mathbb{C}/\Gamma$. The advantage of this approach is that the local trivialisation of the Winkelmann bundle is reduced to establishing the local trivialisation of $(\mathbb{C}, \pi, \mathbb{C}/\Gamma)$, which is slightly easier.

Lemma 3.1.2. *The triple* $(\mathbb{C}, \pi, \mathbb{C}/\Gamma)$ *is a principal* Γ*-bundle.*

Proof. For $[b] \in \mathbb{C}/\Gamma$, let V be a neighbourhood of b such that no two elements of V are equivalent modulo Γ. Set $U = \pi V$ and observe that $\pi|V \to U$ is a homeomorphism. Define $h_U : \pi^{-1}U \to U \times \Gamma$ by $h_U(z) = ([z], z - (\pi|_V)^{-1}\pi(z))$. Then h_U is continuous and equivariant with continuous inverse $h_U^{-1}([z], \gamma) = \gamma + (\pi|_V)^{-1}([z])$. □

Define a Γ-action on $\mathbb{C}$ by $(m+\omega n).v = v+n$. This gives a Γ-action on $\mathbb{C}^2$ defined by $(m+\omega n).(u,v) = (u+m+\omega n, v+n)$ and induces an associated fibre bundle

$$\begin{array}{ccc} \mathbb{C}^2 & \xrightarrow{\mathrm{pr}_1} & \mathbb{C} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathbb{C}^2/\Gamma & \xrightarrow{\mathrm{pr}_1^{\#}} & \mathbb{C}/\Gamma. \end{array}$$

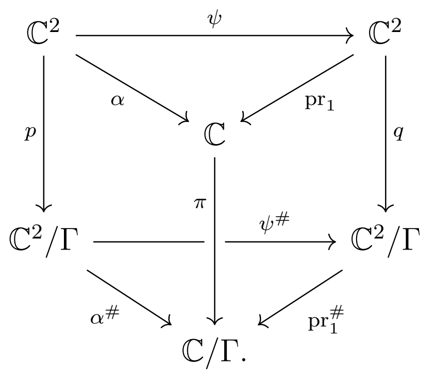
Define $\psi : \mathbb{C}^2 \to \mathbb{C}^2$ by $\psi(z,w) = (w+\omega z, z)$. Since ψ is an equivariant homeomorphism with respect to the action $(m+\omega n).(z,w) = (z+n, w+m)$ on the source and the action $(m+\omega n).(u,v) = (u+m+\omega n, v+n)$ on the target, it descends to a homeomorphism on quotients. We obtain a commuting square

$$\begin{array}{ccc} \mathbb{C}^2 & \xrightarrow{\psi} & \mathbb{C}^2 \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ \mathbb{C}^2/\Gamma & \xrightarrow{\psi^{\#}} & \mathbb{C}^2/\Gamma \end{array}$$

and observe that the next triangle

$$\begin{array}{ccc} \mathbb{C}^2 & \xrightarrow{\psi} & \mathbb{C}^2 \\ & {\scriptstyle \alpha}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_1} & \\ & \mathbb{C} & \end{array}$$

commutes. Surjectivity of the quotient projections is enough to give an isomorphism of fibre bundles through the commutative diagram



Corollary 3.1.3. *The Winkelmann bundle is locally trivial.*

Proof. Use Theorem 2.4.6 and the bundle isomorphism $\psi^{\#} : \mathbb{C}^2/\Gamma \to \mathbb{C}^2/\Gamma$. □

Remark 3.1.4. The Winkelmann bundle is isomorphic as a bundle to an associated fibre bundle where $\varphi : \Gamma \to \operatorname{Aut}\mathbb{C}$ is defined $\varphi(m+\omega n)(z) = z+n$. By Remark 2.4.8, the transition functions for the Winkelmann bundle take values in $\mathbb{Z}$.

3.2 Degrees

The degree is a homotopy invariant of continuous maps between compact oriented topological manifolds with the property that holomorphic maps always have nonnegative degree. Thus the existence of any map with negative degree is enough to violate the h-principle between pairs of compact Riemann surfaces. We follow the approach of Saito [Sai20] which we found to be the most interesting. Since the degree has only a minor role in violating the h-principle for maps from a compact Riemann surface into the Riemann sphere, we will mainly cite results without replicating the proofs. However we do establish a local degree formula in Proposition 3.2.6 which was not explicitly given by Saito.

Let X be a topological space and $A \subset X$ a closed subset. Define $M(X, A)$ to be the set of open sets in X containing A. Define

$$H_q(X||A) = \varprojlim_{U \in M(X,A)} H_q(U \setminus A).$$

For any $U \in M(X, A)$, we have a decomposition $X = U \cup (X \setminus A)$ which gives rise to a Mayer-Vietoris sequence with boundary maps $\delta_{A,U} : H_q(X) \to H_{q-1}(U \setminus A)$. Define $\delta_A : H_q(X) \to H_{q-1}(X||A)$ as the inverse limit of these maps. If $x \in A$ is isolated, then there is a canonical map $H_q(X||x) \to H_q(X||A)$. If Y is Hausdorff and $f : X \to Y$ is a continuous map, we have a canonical map

$$\begin{aligned} f_* : H_q(X||f^{-1}(y)) &= \varprojlim_{U \in N(X, f^{-1}(y))} H_q(U \setminus f^{-1}(y)) \\ \to \varprojlim_{V \in N(Y,y)} H_q(f^{-1}(V) \setminus f^{-1}(y)) &\xrightarrow{f_*} \varprojlim_{V \in N(Y,y)} H_q(V \setminus \{y\}) = H_q(Y||y). \end{aligned}$$

The composition of these two maps gives $f_* : H_q(X||x) \to H_q(Y||y)$. If U is a neighbourhood of x and V a neighbourhood of y such that $f(U) \subset V$ and $U \cap f^{-1}(y) = \{x\}$, then we have a commutative diagram

$$\begin{array}{ccc} H_q(X||x) & \xrightarrow{f_*} & H_q(Y||y) \\ \downarrow & & \downarrow \\ H_q(U \setminus \{x\}) & \xrightarrow{f_*} & H_q(V \setminus \{y\}) \end{array} \tag{3.1}$$

where the vertical maps are projections.

Proposition 3.2.1. *Let X be a Hausdorff space and $q \geq 0$.*

1. *Let $A = \{x_1, \ldots, x_n\}$ be a finite subset of X. The direct sum*

$$\bigoplus_{x \in A} H_q(X||x) \to H_q(X||A)$$

of the canonical morphisms is an isomorphism. The map $\delta_A : H_{q+1}(X) \to H_q(X||A)$ *is a composition of this isomorphism and the direct sum of* $\delta_x : H_{q+1}(X) \to H_q(X||x)$.

2. *Let* Y *be a Hausdorff space and* $f : X \to Y$ *a continuous map. Let* $y \in Y$ *and suppose that the preimage* $f^{-1}(y)$ *consists of finitely many points* $x_1, \ldots, x_n$. *The diagram*

$$\begin{array}{ccc} H_{q+1}(X) & \xrightarrow{\bigoplus_{i=1}^{n} \delta_{x_i}} & \bigoplus_{i=1}^{n} H_q(X||x_i) \\ {\scriptstyle f_*}\downarrow & & \downarrow {\scriptstyle \sum_{i=1}^{n} f_*} \\ H_{q+1}(Y) & \xrightarrow{\delta_y} & H_q(Y||y) \end{array} \tag{3.2}$$

commutes.

Proof. Saito [Sai20, Proposition 8.3.2]. □

Proposition 3.2.1.2 and (3.1) essentially provide us with a local degree formula analogous to Hatcher [Hat02, Proposition 2.30], which we prove in Proposition 3.2.6.

Proposition 3.2.2. *Let* $c = (a, b) \in \mathbb{R}^2$. *Define a one-form over* $\mathbb{R}^2 \setminus \{c\}$ *by*

$$\alpha_c = \frac{-(y-b)\,dx + (x-a)\,dy}{(x-a)^2 + (y-b)^2}.$$

The linear functional $n(-, c)$ *sending a* C^2 *chain* $\gamma \in Z_1(\mathbb{R}^2 \setminus \{c\})_{C^2}$ *to*

$$n(\gamma, c) = \frac{1}{2\pi} \int_\gamma \frac{-(y-b)\,dx + (x-a)\,dy}{(x-a)^2 + (y-b)^2}$$

induces an isomorphism

$$n(-, c) : H_1(\mathbb{R}^2 \setminus \{c\}) \to \mathbb{Z}. \tag{3.3}$$

Proof. Saito [Sai20, Proposition 5.6.1]. □

Proposition 3.2.3. *Let* X *be an oriented surface and* $x \in X$. *There exists an isomorphism*

$$n(-, x) : H_1(X||x) \to \mathbb{Z} \tag{3.4}$$

uniquely satisfying the following condition: if $p : U \to V$ *is a positively oriented coordinate neighbourhood of* x, *and if* $H_1(X||x) \to H_1(U \setminus \{x\})$ *is the projection, then* $n(-, x) : H_1(X||x) \to \mathbb{Z}$ *is the composition*

$$H_1(X||x) \to H_1(U \setminus \{x\}) \xrightarrow{p_*} H_1(V \setminus \{p(x)\}) \xrightarrow{n(-,p(x))} \mathbb{Z}, \tag{3.5}$$

where $n(-, p(x))$ *is the winding number (3.3).*

Proof. Saito [Sai20, Proposition 8.5.4]. □

Define the linear map $T_x : H_2(X) \xrightarrow{\delta_x} H_1(X||x) \xrightarrow{n(-,x)} \mathbb{Z}$. If $p : U \to V$ is a positively oriented coordinate neighbourhood of x, then T_x is given by the composition

$$H_2(X) \xrightarrow{\delta_{x,U}} H_1(U \setminus \{x\}) \xrightarrow{p_*} H_1(V \setminus \{c\}) \xrightarrow{n(-,c)} \mathbb{Z}.$$

Proposition 3.2.4. *Let X be a connected oriented surface. The map $T_x : H_2(X) \to \mathbb{Z}$ does not depend on x.*

Proof. Saito [Sai20, Proposition 8.5.7]. □

Finally Saito defines the fundamental class of a compact oriented surface X, and the degree of a continuous map $f : X \to Y$ between compact oriented surfaces.

Definition 3.2.5 ([Sai20, Definition 8.5.8])**.** Let X be a connected compact oriented surface.

1. If $H_2(X)$ is a free group of rank 1 with basis c, and if for arbitrary $x \in X$ the linear map $T_x : H_2(X) \to \mathbb{Z}$ sends c to 1, then c is called the *fundamental class* of X, denoted $[X]$.

2. Suppose that Y is also a compact connected oriented surface and $f : X \to Y$ a continuous map. If the fundamental classes $[X]$ and $[Y]$ exist, then the integer n determined by $f_*[X] = n[Y]$ is called the *degree* of f, denoted $\deg f$.

By Proposition 3.2.4, we just require the existence of some $x \in X$ such that $T_x(c) = 1$. Unfortunately Saito does not actually prove the existence of the fundamental class. Since T_x is given locally as the composition $n(-, p(x)) \circ p_* \circ \delta_{x,U}$ with the latter two maps being isomorphisms, it suffices to show that there exists a positively oriented coordinate neighbourhood U of x such that $\delta_{x,U}$ is an isomorphism. If we suppose that U is a positively oriented contractible neighbourhood of x, the Mayer-Vietoris exact sequence is

$$H_2(X \setminus \{x\}) \to H_2(X) \xrightarrow{\delta_{x,U}} H_1(U \setminus \{x\}) \to H_1(X \setminus \{x\}),$$

where $H_1(U \setminus \{x\}) \to H_1(X \setminus \{x\})$ is induced by inclusion. If X is a Riemann surface, then $X \setminus \{x\}$ is noncompact and has the homotopy type of a wedge of circles. Hence $H_2(X \setminus \{x\}) = 0$. If X is also compact, the inclusion $U \setminus \{x\} \to X \setminus \{x\}$ induces the zero map on homology because X is compact and oriented. Take a finite oriented triangulation of X such that x is contained in the interior of a triangle τ in U. The boundary of τ is a generator for $H_1(U \setminus \{x\})$ while being the boundary of the triangulation with τ omitted, hence zero in $H_1(X \setminus \{x\})$. By exactness $\delta_{x,U}$ is surjective and therefore an isomorphism.

On the assumption that the fundamental classes of compact Riemann surfaces X and Y exist, Saito [Sai20, Proposition 9.3.1.1] proves that holomorphic maps $f : X \to Y$ satisfy $\deg f \geq 1$.

We now prove the local degree formula.

Proposition 3.2.6. *Let X and Y be connected compact oriented smooth surfaces and $f : X \to Y$ a continuous map. Suppose that there exists $y \in Y$ with $f^{-1}(y) = \{x_1, \ldots, x_m\}$ finite; that (U_i, U_i', p_i) are positively oriented disjoint coordinate neighbourhoods of x_i such that U_i is homeomorphic to a disk; that (V, V', q) is a positively oriented coordinate neighbourhood of y such that V is homeomorphic to a disk; and that $f|U_i \to V$ is a homeomorphism for each U_i. Let* $\deg f_i \in \{\pm 1\}$ *be the integer such that the diagram*

$$\begin{array}{ccc}
H_1(U_i \setminus \{x_i\}) & \xrightarrow{\quad f_* \quad} & H_1(V \setminus \{y\}) \\
\downarrow{\scriptstyle p_{i*}} & & \downarrow{\scriptstyle q_*} \\
H_1(U_i' \setminus \{p(x)\}) & & H_1(V' \setminus \{q(y)\}) \\
\downarrow{\scriptstyle n(-,p(x_i))} & & \downarrow{\scriptstyle n(-,q(y))} \\
\mathbb{Z} & \xrightarrow{\quad \times \deg f_i \quad} & \mathbb{Z}
\end{array}$$

commutes. Then $\deg f = \sum_{i=1}^m \deg f_i$.

Proof. By definition $\deg f = T_y \circ f_*[X]$ and $T_y = n(-, y) \circ \delta_y$. By Proposition 3.2.3, since (V, V', q) is positively oriented, a local expression for $n(-, y)$ is $n(-, q(y)) \circ q_* \circ \mathrm{pr}_V$. Diagram (3.2) shows that $\delta_y \circ f_* = \sum_{i=1}^m f_* \circ \bigoplus_{i=1}^m \delta_{x_i}$. Together the expressions give

$$T_y \circ f_* = \sum_{i=1}^{m} n(-, q(y)) \circ q_* \circ \mathrm{pr}_V \circ f_* \circ \delta_{x_i}.$$

Since the U_i are disjoint, we may apply commutativity of (3.1) to get

$$T_y \circ f_* = \sum_{i=1}^{m} n(-, q(y)) \circ q_* \circ f_* \circ \mathrm{pr}_{U_i} \circ \delta_{x_i}.$$

Linearity along with the definition of $\deg f_i$ implies that

$$T_y \circ f_* = \sum_{i=1}^{m} \deg f_i \cdot (n(-, p(x_i)) \circ p_{i*} \circ \delta_{x_i, U_i}).$$

The latter expression is a local formula for T_{x_i} since the U_i are positively oriented. Hence applying both sides to the fundamental class $[X]$ gives $\deg f = \sum_{i=1}^m \deg f_i$. □

We now construct a map $S^2 \to S^2$ of degree -1 using cubical singular homology, as defined in Serre [Ser51], Massey [Mas91], and Saito [Sai20]. As shown by Saito [Sai20, Proposition 8.5.9], the class $[\sigma] \in H_2(S^2)$ of the parametrisation $\sigma : I^2 \to S^2$ defined

$$\sigma(s, t) = (\sin \pi s \cos 2\pi t, \sin \pi s \sin 2\pi t, \cos \pi s)$$

is a fundamental class for S^2, when S^2 is equipped with the orientation defined by the stereographic projection $(x, y, z) \mapsto (x/(1+z), y/(1+z))$ from the south pole.

Define $\omega : I^3 \to S^2$ by

$$\omega(s,t,u) = \begin{cases} \sigma(s, u-t) & \text{if } t \leq u, \\ \sigma(s,0) & \text{if } u \leq t. \end{cases}$$

Applying the cubical boundary operator gives

$$\partial_3\omega(s,t) = \omega(1,s,t) + \sigma(s,t) + \sigma(s,1-t) - \omega(0,s,t) - \sigma(s,0) - \sigma(s,0).$$

Notice that $\omega(0,s,t) \equiv (0,0,1)$ and $\omega(1,s,t) \equiv (0,0,-1)$. Hence these two maps along with $\sigma(s,0)$ are degenerate. Thus $[\sigma(s,1-t)] = -[\sigma(s,t)]$ in $H_2(X)$. Define $f : S^2 \to S^2$ by $f(x,y,z) = (x,-y,z)$; then $f_*[\sigma] = [\sigma(s,1-t)] = -[\sigma]$, and hence $\deg f = -1$.

The final step is to construct a map $X \to S^2$ with nonzero degree.

Lemma 3.2.7. *Let D^n denote the closed unit ball in $\mathbb{R}^n$ and let $S^n = \partial D^{n+1}$. There exists a homeomorphism $D^n/S^{n-1} \to S^n$.*

Proof. May [May99, Chapter 13.2] provides an example of such. We verify the fact that it is a homeomorphism by providing an explicit inverse $g^\#$. Let p_N, p_S denote the north and south pole of S^n. Let $y = (y_1, \ldots, y_n) \in D^n$. Define a map $f : D^n \to S^n$ by

$$f(y_1, \ldots, y_n) = (u(\|y\|)y_1, \ldots, u(\|y\|)y_n, 2\|y\| - 1)$$

where $u(t) = \sqrt{1-(2t-1)^2}$. Since $f(S^{n-1}) = p_N$, there is an induced quotient map $f^\# : D^n/S^{n-1} \to S^n$. Excluding poles from S^n, define $g : S^n \setminus \{p_N, p_S\} \to D^n$ by

$$g(x_1, \ldots, x_{n+1}) = \left(\frac{x_{n+1}+1}{2} \frac{x_1}{\sqrt{x_1^2 + \cdots + x_n^2}}, \ldots, \frac{x_{n+1}+1}{2} \frac{x_n}{\sqrt{x_1^2 + \cdots + x_n^2}} \right).$$

Observe that g is an inverse to the restriction of f to $D^n \setminus (S^{n-1} \cup \{0\})$. Hence $f^\# : D^n/S^{n-1} \to S^n$ is a continuous bijection from a compact space to a Hausdorff space, and is therefore a homeomorphism. □

Lemma 3.2.8. *Let X be a compact oriented surface. There exists a map $X \to S^2$ with degree -1.*

Proof. Let $x_0 \in X$. Take a coordinate neighbourhood $\psi : U \to V$ centred at x_0 such that the unit disk Δ in $\mathbb{R}^2$ is relatively compact in V. Define $g : X \to S^2$ by $g(x) = p_N$ for $x \in X \setminus \psi^{-1}(\Delta)$ and $g(x) = f \circ \psi(x)$ for $x \in \psi^{-1}(\overline{\Delta})$ with f as defined in Lemma 3.2.7. The map g is continuous by the pasting lemma since there is agreement on the overlap $\psi^{-1}(\partial\Delta)$ of the closed cover of X. For any nondegenerate disk $\Delta' \Subset \Delta$, the map $g|\psi^{-1}(\Delta')$ is a homeomorphism onto an open subset of S^2. Proposition 3.2.6 implies $\deg g \in \{\pm 1\}$. If necessary, we can compose with the map $(x,y,z) \mapsto (x,-y,z)$ which has degree -1, and the composition is a map $X \to S^2$ with degree -1. □

3.3 Hyperbolic geometry

We develop hyperbolic geometry over the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ for the purpose of violating the h-principle for maps into hyperbolic Riemann surfaces. The main goal of this first part is to prove that every hyperbolic surface excluding Δ, Δ^* is the quotient of $\mathbb{H}$ by a subgroup of deck transformations containing a hyperbolic element.

Proposition 3.3.1. *The map*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \phi_A(z) = \frac{az+b}{cz+d}$$

sending $A \in SL(2,\mathbb{R})$ to $\phi_A \in \operatorname{Aut}(\mathbb{H})$ induces an isomorphism $PSL(2,\mathbb{R}) \to \operatorname{Aut}(\mathbb{H})$.

Proof. Napier-Ramachandran [NR12, Theorem 5.8.3]. □

Definition 3.3.2. An element $[A] \in PSL(2,\mathbb{R})$ is *elliptic* if $|\operatorname{Tr} A| < 2$, *parabolic* if $|\operatorname{Tr} A| = 2$ and *hyperbolic* if $|\operatorname{Tr} A| > 2$.

Notice that this definition is independent of representative. An automorphism ϕ_A of $\mathbb{H}$ is elliptic, parabolic or hyperbolic according to its corresponding representative $[A] \in PSL(2,\mathbb{R})$. The trace classification of automorphisms coincides with the classification of automorphisms by the number of fixed points on the boundary.

In fact, let $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$. If $c = 0$, then $ad = 1$. The fixed points are the z satisfying $a^2 z + ab = z$. Observe that ∞ is always a fixed point in this situation. If $a = \pm 1$ and $b \neq 0$, then ∞ is the only fixed point. If $a = \pm 1$ and $b = 0$, then T is the identity and every point is fixed. If $a \neq \pm 1$, then $z = -ab/(a^2 - 1)$ is also a fixed point.

When $c = 0$, the trace is $a + 1/a$. Solving the appropriate quadratic in a, one observes that $|a + 1/a| \geq 2$ with equality if and only if $a = \pm 1$.

If $c \neq 0$, then fixed points are given by the quadratic formula

$$z = \frac{a - d \pm \sqrt{(a+d)^2 - 4}}{2c}.$$

Hence there are two fixed points on $\mathbb{R}$ if and only if $|a + d| > 2$ and $c \neq 0$.

A hyperbolic surface may be considered as a quotient of the universal covering $\mathbb{H}$ by a group of deck transformations $G < \operatorname{Aut}(\mathbb{H})$. Nontrivial deck transformations act without fixed points, so the elements of G must satisfy $|a + d| \geq 2$; i.e. deck transformations are never elliptic.

Proposition 3.3.3. *Suppose that $G < \operatorname{Aut}(\mathbb{H})$ is a group of deck transformations containing only parabolic elements and the identity. Then $\mathbb{H}/G \simeq \Delta$ or $\mathbb{H}/G \simeq \Delta^*$.*

Proof. If G is trivial, then $\mathbb{H}/G \simeq \mathbb{H}$. Suppose that there exists a parabolic element $T \in G$ with fixed point $x_0 \in \mathbb{R}$. The map $g(z) = 1/(x_0 - z)$ sends x_0 to ∞ and hence $gTg^{-1}(z) = z + t$ for some nonzero $t \in \mathbb{R}$. Replacing G by gGg^{-1}, we may assume that G contains $T = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ for $t \neq 0$. For any other parabolic $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have $|a+cnt+d| = 2$ for all $n \in \mathbb{N}$; but $|a+d| = 2$ and $t \neq 0$ so $c = 0$ since $n|ct| - |a+d| \leq 2$ for all $n \in \mathbb{N}$ by the reverse triangle inequality. Hence G is represented by elements $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ with $x \in \mathbb{R}$. By discreteness and nontriviality of G, there exists $t_0 > 0$ such that the automorphism $\tau(z) = z + t_0$ generates G. The automorphism $\sigma(z) = z/t_0$ satisfies $\sigma\tau\sigma^{-1}(z) = z + 1$. But $\mathbb{H}/G \simeq \mathbb{H}/\sigma G \sigma^{-1} \simeq \Delta^*$. Hence the only hyperbolic surfaces whose group of deck transformations have no hyperbolic elements are $\mathbb{H} \simeq \Delta$ and Δ^*. □

3.3.1 The hyperbolic metric

We mainly follow the exposition of Keen and Lakic [KL07]. The result we require is that closed geodesics on a hyperbolic surface cannot be shortened by a homotopy. This is derived from the property that holomorphic self-maps of the unit disk, and equivalently the upper half plane, are distance decreasing with respect to the hyperbolic metric on those surfaces.

Semicircles orthogonal to the real axis are called *geodesics* of $\mathbb{H}$. For any two points $p, q \in \mathbb{H}$, there exists a unique geodesic containing both p and q; if $\operatorname{Re} p = \operatorname{Re} q = x_0$, then we consider the geodesic $\{ix_0t \mid t \in \mathbb{R} \geq 0\}$ joining x_0 to ∞. If $a, b \in \mathbb{R}$ are the points of the geodesic meeting $\mathbb{R}$, then we define the hyperbolic distance between p and q to be

$$d(p, q) = \left| \log \frac{(q-a)(p-b)}{(b-q)(a-p)} \right|. \tag{3.6}$$

If $a = \infty$ or $b = \infty$, then we first multiply the numerator and denominator by $1/a$ or $1/b$.

Remark 3.3.4. Keen and Lakic define the hyperbolic distance on $\mathbb{H}$ to be $d(p, q)/2$. Their derivation begins on the disk with the construction of a hyperbolic density $\rho(z)$ invariant under automorphisms $A : \Delta \to \Delta$, expressed by the equation $\rho(A(z))|A'(z)| = \rho(z)$. They use the normalisation condition $\rho(0) = 1$, while acknowledging that $\rho(0) = 2$ is often found in the literature. Transporting the construction to the half plane with $\rho(0) = 2$ gives $\rho_{\mathbb{H}}(p) = 1/\operatorname{Im} p$, which leads to the commonly found [Kob98, FM12] Poincaré metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

We have introduced the hyperbolic distance on the upper half plane via (3.6) since we intend to calculate using the explicit formula. We will take for granted the following properties of the hyperbolic distance.

Proposition 3.3.5. *The hyperbolic distance is invariant under automorphisms of* $\mathbb{H}$.

Proof. Keen and Lakic [KL07, p. 44]. This is due to the argument inside the logarithm, the *cross ratio*, being invariant under Möbius transformations. □

Theorem 3.3.6 (Schwarz-Pick lemma). *If* $f : \mathbb{H} \to \mathbb{H}$ *is a holomorphic self-map of the upper half plane, then* f *is distance decreasing with respect to the hyperbolic metric. Explicitly* $d(f(z), f(w)) \leq d(z, w)$ *for all* $z, w \in \mathbb{H}$.

Proof. Keen and Lakic [KL07, Theorem 3.2.2]. Though their proof is carried out for holomorphic self-maps of the unit disk, an analogous statement for the half plane holds since we can map the half plane isometrically and biholomorphically onto the disk with the Möbius transformation $z \mapsto (z - i)/(z + i)$. □

Proposition 3.3.3 implies that every hyperbolic surface X excluding Δ and Δ^* has a hyperbolic deck transformation. If $p : \mathbb{H} \to X$ is a covering and $h : \mathbb{H} \to \mathbb{H}$ is a hyperbolic deck transformation of p, then there is a unique geodesic joining the two fixed points of h, called the *geodesic axis* of h.

Proposition 3.3.7. *Suppose that* A *is a hyperbolic automorphism of* $\mathbb{H}$ *with fixed points* $a, b \in \mathbb{R} \cup \{\infty\}$. *Let* L *be the geodesic axis of* A. *Then* $\inf_{z\in\mathbb{H}} d(z, A(z))$ *is achieved for all* $z \in L$ *but not for any other* $z \in \mathbb{H}$.

Proof. Keen and Lakic [KL07, Proposition 2.4.1]. Note that L is preserved by A. □

We will use the next result, due to the author, to show that maps $M \to \Delta^*$ from Riemann surfaces M of infinite type violate the h-principle.

Lemma 3.3.8. *Let* $z_0 = x_0 + iy_0 \in \mathbb{H}$. *Let* d *be the hyperbolic distance on* $\mathbb{H}$. *There exists a positive constant* $\lambda > 0$ *such that* $0 < \lambda \leq d(z_0, z_0 + t)/\log|t|$ *for all* $t \in \mathbb{R}$ *with* $|t| > 1$.

Proof. The geodesic joining z_0 and $z_0 + t$ is defined by the equation $(x - x_0 - t/2)^2 + y^2 = t^2/4 + y_0^2$. The points of the geodesic meeting $\mathbb{R}$ are $(x_0 + 1/2) \pm \sqrt{t^2/4 + y_0^2}$. Hence the distance between z_0 and $z_0 + t$ is

$$d(z_0, z_0 + t) = 2\left|\log\left(\frac{t}{2y_0} + \sqrt{1 + \left(\frac{t}{2y_0}\right)^2}\right)\right|.$$

Since $x + \sqrt{1 + x^2} \to \infty$ as $x \to \infty$ and $x + \sqrt{1 + x^2} \to 0$ as $x \to -\infty$, the distance satisfies $d(z_0, z_0 + t) \to \infty$ as $t \to \pm\infty$. Hence the limit of $d(z_0, z_0 + t)/\log|t|$ is indeterminate as $t \to \pm\infty$. The ratio $d(z_0, z_0 + t)/\log|t|$ is symmetric in t for $|t| > 1$, so we will just investigate the behaviour as $t \to \infty$. In this case we can ignore the absolute values.

The derivative of $\log(x + \sqrt{1 + x^2})$ is $1/\sqrt{1 + x^2}$ and the derivative of $\log x$ is $1/x$. Hence $d(z_0, z_0 + t)/\log t \to 2$ as $t \to \infty$ by l'Hôpital. On the other hand $d(z_0, z_0 + t)$

is bounded and nowhere zero in a neighbourhood of $t = 1$ and $\log t \to 0$ from above as $t \to 1$ from above, so $d(z_0, z_0 + t)/\log t \to \infty$ as $t \to 1$ from above.

Set $k = 1/(2y_0)$. For $t > 1$, we calculate the derivative

$$2\frac{d}{dt}\frac{\log(kt + \sqrt{1 + (kt)^2})}{\log t} = 2\frac{kt \log t - \sqrt{1 + (kt)^2}\log(kt + \sqrt{1 + (kt)^2})}{t\sqrt{1 + (kt)^2}(\log t)^2}.$$

The expression being differentiated is $d(z_0, z_0 + t)/\log t$ with $k = 1/(2y_0)$. Examining the numerator of the derivative, each local minimum of $d(z_0, z_0+t)/\log t$ occurs at t satisfying

$$kt \log t = \sqrt{1 + (kt)^2}\log(kt + \sqrt{1 + (kt)^2}).$$

Thus local minima of $d(z_0, z_0 + t)/\log t$, if any exist, are of the form

$$2\frac{\log(kt + \sqrt{1 + (kt)^2})}{\log t} = \frac{2}{\sqrt{1 + 1/(kt)^2}}.$$

The function $\sqrt{1 + 1/(kt)^2}$ is decreasing for $t \geq 1$ and hence achieves its maximum at $t = 1$ on this interval. We deduce that

$$2\frac{\log(kt + \sqrt{1 + (kt)^2})}{\log t} \geq \frac{2}{\sqrt{1 + 1/k^2}} > 0.$$

for all $t > 1$. Since $k = 1/(2y_0)$, we conclude by symmetry that $\lambda = 2/\sqrt{1 + 4y_0^2} > 0$ is a lower bound for $d(z_0, z_0 + t)/\log |t|$ for all $|t| > 1$. □

3.3.2 Hyperbolic geodesics

Gromov [Gro89] presented an argument for why maps $X \to Y$ do not satisfy the h-principle when Y is a hyperbolic domain of the complex plane and X is a complex analytic manifold such that $H^1(X; \mathbb{Z}) \neq 0$. We wish to present this argument for the case of Riemann surfaces. The proof makes essential use of the Schwarz-Pick lemma as well as the result that closed geodesics on a hyperbolic surface cannot be shortened by a homotopy. The latter result is the focus of this section.

Let d be the hyperbolic distance on $\mathbb{H}$. Define the length of a path $\gamma : I \to \mathbb{H}$ by

$$\ell(\gamma) = \sup \sum_{j=1}^{k} d(\gamma(t_{j-1}), \gamma(t_j)) \tag{3.7}$$

where the supremum is taken over all partitions $0 = t_0 < \cdots < t_k = 1$ of the interval I. If γ is the concatenation of paths $\gamma_1, \ldots, \gamma_k$, then $\ell(\gamma) = \sum_{j=1}^{k} \ell(\gamma_j)$ by the triangle inequality. If γ is a path from a to b, then $d(a, b) \leq \ell(\gamma)$ by taking the trivial partition $0 = t_0 < t_1 = 1$. From the definition, it is not immediately clear that there exist nontrivial paths with finite length. However, it turns out that all C^1 paths have finite length.

Proposition 3.3.9. *Let d be the hyperbolic distance on $\mathbb{H}$. For all $z_0 \in \mathbb{H}$, we have*

$$\lim_{z\to 0} \frac{d(z_0, z_0+z)}{|z|} = \frac{1}{\operatorname{Im} z_0}.$$

Proof. Keen and Lakic [KL07, p. 45]. □

This gives the integral formula for the hyperbolic length of a C^1 curve $\gamma : I \to \mathbb{H}$.

Theorem 3.3.10. *Let $\gamma : I \to \mathbb{H}$ be C^1. Then*

$$\ell(\gamma) = \int_0^1 \frac{|\gamma'(t)|}{\operatorname{Im}\gamma(t)}\, dt.$$

In particular, the length of a C^1 curve is finite.

Proof (Sketch). Combine Proposition 3.3.9 with the standard argument showing equivalence between the integral formula for the arclength of a C^1 curve and length by polygonal approximation in (3.7) where d is instead taken to be the Euclidean distance. □

Let X be a hyperbolic surface covered by $p : \mathbb{H} \to X$. Suppose that $\gamma : I \to X$ is a path and $\widetilde{\gamma} : I \to \mathbb{H}$ any lift. Define $\ell(\gamma) = \ell(\widetilde{\gamma})$. This is well defined because lifts differ by a deck transformation, and deck transformations of p are isometries with respect to the hyperbolic metric. In particular deck transformations are length preserving. If $\alpha : S^1 \to X$ is a closed curve on X, then $\ell(\alpha^n) = |n|\ell(\alpha)$. When dealing with closed curves, we will often precompose by the universal covering $E : \mathbb{R} \to S^1$ defined $E(t) = \exp(2\pi i t)$.

Definition 3.3.11. Let X be a hyperbolic surface covered by $p : \mathbb{H} \to X$. A closed curve $\alpha : S^1 \to X$ is a *closed geodesic* if every lift of $\alpha E : \mathbb{R} \to X$ is a geodesic in $\mathbb{H}$.

The condition that every lift of αE is geodesic is equivalent to the existence of a geodesic lift of αE. This definition is not the conventional definition of closed geodesic, but it suffices for our purposes. By this definition, the hyperbolic surfaces without closed geodesics are precisely Δ and Δ^*, per Proposition 3.3.3 and the next result.

Proposition 3.3.12. *Let X be a hyperbolic surface and $p : \mathbb{H} \to X$ a covering. Suppose that there exists a hyperbolic deck transformation $h : \mathbb{H} \to \mathbb{H}$ of p. Take any point a on the geodesic axis of h. Define $\gamma : I \to \mathbb{H}$ as the segment along the geodesic axis from a to $h(a)$. Then the closed curve $\alpha : S^1 \to X$ identified with the loop $p\gamma : I \to X$ is a closed geodesic. Conversely, if $\alpha : S^1 \to X$ is a closed curve and every lift $\beta : \mathbb{R} \to \mathbb{H}$ of $\alpha E : \mathbb{R} \to X$ is a geodesic, then the deck transformation sending $\beta(0)$ to $\beta(1)$ is hyperbolic.*

Proof. If $\beta : \mathbb{R} \to \mathbb{H}$ is the lift determined by $\beta(0) = a$, then $\beta\tau_n|_I = h^n\gamma$ where $\tau_n : \mathbb{R} \to \mathbb{R}$ is the translation $\tau_n(t) = t + n$. So $\beta(\mathbb{R})$ is the geodesic axis of h. If $\beta' : \mathbb{R} \to \mathbb{H}$ is the lift determined by $\beta'(0) = b \in p^{-1}p(a)$, then there exists a unique deck transformation

$j : \mathbb{H} \to \mathbb{H}$ such that $j(a) = b$. Then $j\beta(0) = \beta'(0)$ and $j\beta = \beta'$. Deck transformations map geodesics to geodesics, so β' is a geodesic. Hence every lift of α is a geodesic.

Conversely, suppose that $\alpha : S^1 \to X$ is a closed curve such that every lift $\beta : \mathbb{R} \to \mathbb{H}$ is a geodesic. The points $\beta(0)$ and $\beta(1)$ define a nontrivial deck transformation h sending $\beta(0)$ to $\beta(1)$. Then $h\beta = \beta\tau_1$ because both cover αE and $h\beta(0) = \beta\tau_1(0)$. Hence h fixes the geodesic $\beta(\mathbb{R})$ and so h has two fixed points on $\partial\mathbb{H}$. □

Remark 3.3.13. The deck transformation h determined by $\beta(0), \beta(1)$ is the identity if and only if $\beta(0) = \beta(1)$. But h is nontrivial since if h was trivial, then β would have compact image and thus would not be a geodesic.

The geodesic segment in Proposition 3.3.12 can be constructed to satisfy the additional property that it is C^1 with nowhere zero derivative. Geodesic segments parametrised in this manner have the distance minimising property.

Proposition 3.3.14. *Let $a, b \in \mathbb{H}$. Suppose that $\gamma : I \to \mathbb{H}$ is a C^1 path along the geodesic joining a to b with nowhere zero derivative. Then $\ell(\gamma) = d(a, b)$.*

Proof (Sketch). A simple calculation of the integral formula shows that if $y_0 > 0$ and $\gamma_0(t) = i(1 + t(y_0 - 1))$ is the linear path along the imaginary axis from i to iy_0, then $\ell(\gamma_0) = d(i, iy_0)$. For general $p, q \in \mathbb{H}$, there exists $T \in \mathrm{Aut}(\mathbb{H})$ such that $T(p) = i$ and $\mathrm{Re}\, T(q) = 0$. Let γ_1 be the linear path along the imaginary axis from i to $T(q)$. Since automorphisms of $\mathbb{H}$ map geodesics into geodesics, the path $T^{-1}\gamma_1$ has the same image as γ, is C^1, and has nowhere zero derivative. Since the integral arclength is independent of parametrisation for such paths, we have $d(p, q) = d(i, T(q)) = \ell(\gamma_1) = \ell(\gamma)$. □

Theorem 3.3.15. *Let X be a hyperbolic surface covered by $p : \mathbb{H} \to X$. Let $\alpha : S^1 \to X$ be a C^1 closed geodesic with nowhere zero derivative. If $\beta : S^1 \to X$ is homotopic to α via $f : S^1 \times I \to X$, then $\ell(\alpha) \leq \ell(\beta)$.*

Proof. Fix $a \in p^{-1}(f(0, 0))$. Suppose that $F : \mathbb{R} \times I \to X$ is a lift of $f(E \times 1)$ determined by $F(0, 0) = a$. The point $F(1, 0)$ determines a deck transformation h_F sending a to $F(1, 0)$. The maps $h_F F(0, -)$ and $F(1, -)$ both lift $f(1, -)$ with $h_F F(0, 0) = F(1, 0)$. Hence $h_F F(0, s) = F(1, s)$ for all $s \in I$, in particular $h_F F(0, 1) = F(1, 1)$. The deck transformation h_F is hyperbolic with geodesic axis $F(\mathbb{R}, 0)$ because α is a closed geodesic. Since $F(-, 1)|_I$ is a lift of $\beta E | I \to X$ and is a path from $F(0, 1)$ to $F(1, 1) = h_F F(0, 1)$, Proposition 3.3.7 implies that

$$\ell(\alpha) = d(a, h_F(a)) = \inf_{z \in \mathbb{H}} (z, h_F(z)) \leq d(F(0, 1), F(1, 1)) \leq \ell(\beta). \qquad \square$$

3.4 The homotopy principle for Riemann surfaces

3.4.1 Affirmative cases

Proposition 3.4.1 (Theorem 3.0.1(i)). *Let M or N be isomorphic to $\mathbb{C}$ or Δ, or let $M \simeq \mathbb{P}^1 \not\simeq N$. Every continuous map $f : M \to N$ is homotopic to a holomorphic map.*

Proof. If M is either $\mathbb{C}$ or Δ, then $H(x,t) = f((1-t)x)$ defines a homotopy from f to the constant map $f(0)$. If N is either $\mathbb{C}$ or Δ, then $H(x,t) = (1-t)f(x)$ defines a homotopy from f to the constant zero map. If $N \not\simeq \mathbb{P}^1$, its universal covering $p : \widetilde{N} \to N$ is either $\mathbb{C}$ or Δ. If M is $\mathbb{P}^1$, we have a lift $\widetilde{f} : M \to \widetilde{N}$ by simple connectivity. Then $H(x,t) = p((1-t)\widetilde{f}(x))$ is a homotopy from f to the constant map $p(0)$. In each case, every continuous map is null homotopic, hence homotopic to a holomorphic map. □

Remark 3.4.2. For the last case $M \simeq \mathbb{P}^1$ and $N \not\simeq \mathbb{P}^1$, all we require from M is simple connectivity. But there are only three simply connected Riemann surfaces up to biholomorphism and the other two are contractible.

We consider the case $M \to \mathbb{P}^1$, where M is noncompact. We require two well-known results from general topology.

Lemma 3.4.3. *Let $(X_\alpha)_{\alpha\in\Lambda}$ be a family of topological spaces and Y a topological space. The canonical map $u : \coprod_{\alpha\in\Lambda}(X_\alpha \times Y) \to (\coprod_{\alpha\in\Lambda} X_\alpha) \times Y$ is a homeomorphism.* □

Lemma 3.4.4. *If $\sim$ is an equivalence relation on X with projection $p : X \to X/{\sim}$ and $F : X \times I \to Y$ is a homotopy such that $x \sim x'$ implies $F(x,t) = F(x',t)$ for all t, then F induces a homotopy $G : X/{\sim} \times I \to Y$ such that $G(p \times 1) = F$.*

Proof. Switzer [Swi02, Proposition 0.8]. □

Proposition 3.4.5 (Theorem 3.0.1(ii), Riemann sphere). *Let M be noncompact. Every continuous map $M \to \mathbb{P}^1$ is homotopic to a holomorphic map.*

Proof. Since a noncompact Riemann surface is homotopy equivalent to a wedge sum of circles, we may equivalently consider maps from a wedge sum of circles into $\mathbb{P}^1$. This consists of an arbitrary collection of loops in $\mathbb{P}^1$ emanating from a single point, so the idea is that simple connectivity of $\mathbb{P}^1$ allows us to contract all such loops to a point.

Suppose that $(X_\alpha, x_\alpha)_{\alpha\in\Lambda}$ is a collection of pointed spaces. Define the wedge sum

$$\bigvee_{\alpha\in\Lambda}(X_\alpha, x_\alpha) = \coprod_{\alpha\in\Lambda}(X_\alpha, x_\alpha)\Big/(x_\alpha \sim x_\beta)_{(\alpha,\beta)\in\Lambda^2}.$$

Let $X_\alpha \simeq S^1$ for all α. Let $\iota_\alpha : (X_\alpha, x_\alpha) \to \coprod_{\alpha\in\Lambda}(X_\alpha, x_\alpha)$ be the canonical inclusion and $\pi : \coprod_{\alpha\in\Lambda}(X_\alpha, x_\alpha) \to \bigvee(X_\alpha, x_\alpha)$ the quotient map. Suppose $g : \bigvee_{\alpha\in\Lambda}(X_\alpha, x_\alpha) \to \mathbb{P}^1$

is a continuous map. By definition of the final topologies on the quotient and coproduct, we obtain a family of continuous maps $g_\alpha = g \circ \pi \circ \iota_\alpha : (X_\alpha, x_\alpha) \to \mathbb{P}^1$ for each $\alpha \in \Lambda$. Since $\mathbb{P}^1$ is simply connected, for each α, there exists a homotopy $F_\alpha : (X_\alpha, x_\alpha) \times I \to \mathbb{P}^1$ relative $\{x_\alpha\}$ from g_α to the constant map $g_\alpha(x_\alpha) = g([x_\alpha])$, with this latter constant map independent of α. There exists a canonical map $\widetilde{F} : \coprod_{\alpha\in\Lambda}((X_\alpha, x_\alpha) \times I) \to \mathbb{P}^1$ such that $\widetilde{F} \circ \iota_\alpha = F_\alpha$ for all α. Via the inverse of the canonical homeomorphism supplied by Lemma 3.4.3, we replace $\widetilde{F}$ by $F = \widetilde{F} \circ u^{-1} : (\coprod_{\alpha\in\Lambda}(X_\alpha, x_\alpha)) \times I \to \mathbb{P}^1$.

By Lemma 3.4.4 we get a continuous map $G : \bigvee_{\alpha\in\Lambda}(X_\alpha, x_\alpha) \times I \to \mathbb{P}^1$ defined $G([x], t) = F(x, t)$ since $F(x_\alpha, t) = F(x_\beta, t)$ for all $(\alpha, \beta) \in \Lambda^2$ and all $t \in I$. For all $\alpha \in \Lambda$ and all $x \in X_\alpha$, we have $G([x], 0) = g([x])$, and $G([x], 1) = g([x_\alpha])$. For all $t \in I$, we have $G([x_\alpha], t) = g([x_\alpha])$. Hence $G : \bigvee_{\alpha\in\Lambda}(X_\alpha, x_\alpha) \times I \to \mathbb{P}^1$ is a homotopy from g to the constant map $g([x_\alpha])$ relative $[x_\alpha]$, and every continuous map is homotopic to a (holomorphic) constant map. □

The h-principle for the punctured plane uses the vanishing theorem $H^1(M, \mathscr{O}) = 0$ for noncompact Riemann surfaces M. The two subsequent affirmative cases eventually reduce to the h-principle for the punctured plane.

Proposition 3.4.6 (Theorem 3.0.1(ii), punctured plane)**.** *Let M be noncompact. Every continuous map $f : M \to \mathbb{C}^*$ is homotopic to a holomorphic map.*

Proof. There exists an open cover (U_j) of M such that $f|U_j = \exp(2\pi i\lambda_j)$ for continuous functions $\lambda_j : U_j \to \mathbb{C}^*$. For example take U_j to be simply connected and obtain λ_j as a continuous lifting of $f|U_j$ to the universal covering $\mathbb{C} \to \mathbb{C}^*$ defined $z \mapsto \exp(2\pi iz)$. On $U_j \cap U_k$ we have $\exp(2\pi i(\lambda_j - \lambda_k)) = 1$ and hence $\lambda_j - \lambda_k = c_{jk}$ for continuous functions $c_{jk} : U_j \cap U_k \to \mathbb{Z}$. Viewing $(c_{jk}) \in Z^1(\mathcal{U}, \mathbb{Z})$ as a cocycle with values in $\mathscr{O}$ and noting that $H^1(M, \mathscr{O}) = 0$ for noncompact M, we obtain a holomorphic splitting $c_{jk} = g_j - g_k$. Then $\exp(2\pi i g_k) = \exp(2\pi i g_j)$ on $U_j \cap U_k$ because c_{jk} takes values in $\mathbb{Z}$. This gives a global section $g \in \mathscr{O}^*(M)$ with $g|U_j = \exp(2\pi i g_j)$, and moreover f is homotopic to g via $F : M \times I \to \mathbb{C}^*$ defined $F(x, t) = \exp(2\pi i[(1 - t)\lambda_j(x) + tg_j(x)])$ on U_j. □

Proposition 3.4.7 (Theorem 3.0.1(ii), complex torus)**.** *Let M be noncompact and T a complex torus. Every continuous map $f : M \to T$ is homotopic to a holomorphic map.*

Proof. As mentioned in the introduction of this thesis and in the first section of this chapter, there exists a map $\tau : \mathbb{C}^* \times \mathbb{C}^* \to T$ which is holomorphic and a homotopy equivalence. If $r : T \to \mathbb{C}^* \times \mathbb{C}^*$ is a continuous map such that $\tau r \sim \mathrm{id}_T$, then $f \sim \tau(rf)$. The map $rf : M \to \mathbb{C}^* \times \mathbb{C}^*$ can be viewed as a pair of continuous maps $f_1, f_2 : M \to \mathbb{C}^*$, each of which is homotopic to a holomorphic map $g_1, g_2 : M \to \mathbb{C}^*$ by Proposition 3.4.6. The map $G = (g_1, g_2) : M \to \mathbb{C}^* \times \mathbb{C}^*$ is holomorphic and homotopic to rf. Hence $f \sim \tau G$, and f is homotopic to a holomorphic map. □

Proposition 3.4.8 (Theorem 3.0.1(iii))**.** *If M is noncompact, finite type, and without punctures, then every continuous map $f : M \to \Delta^*$ is homotopic to a holomorphic map.*

Proof. By Theorem 2.5.2, there exists a compact surface M_0, at least one and at most finitely many disjoint coordinate charts $\psi_i : U_i \to \mathbb{C}$, and disks $\Delta \Subset \psi_i(U_i)$ such that $M \simeq M_0 \setminus \bigcup_i \psi_i^{-1}(\overline{\Delta_i})$. We construct a larger surface $M' \Supset M$ for which the inclusion $M \hookrightarrow M'$ is a homotopy equivalence.

Take disks Δ' and Δ'' centred at the origin such that $\Delta' \Subset \Delta \Subset \Delta'' \Subset \psi(U_i)$. Deformation retract $\overline{\Delta''}\backslash\overline{\Delta'}$ and $\overline{\Delta''}\backslash\overline{\Delta}$ onto $\partial\Delta''$ in each coordinate chart and pull this back to M_0. Paste this together with the identity on $M_0 \setminus \bigcup_i \psi_i^{-1}(\Delta'')$, noting that there is agreement on the overlap $\bigcup_i \psi_i^{-1}(\partial\Delta'')$. Letting $M' = M_0 \setminus \bigcup_i \psi_i^{-1}(\overline{\Delta'})$ and $M'' = M_0 \setminus \bigcup_i \psi_i^{-1}(\Delta'')$, we observe that the topological space M'' is a strong deformation retract of the Riemann surfaces M and M'. Thus the inclusion of M into M' is a homotopy equivalence. Hence any continuous map $f : M \to \Delta^*$ is homotopic to the restriction of a continuous map $F : M' \to \mathbb{C}^*$. By Proposition 3.4.6, there exists a holomorphic map $G : M' \to \mathbb{C}^*$ such that $F \sim G$. Since $G(\overline{M})$ is compact in $\mathbb{C}^*$, there exists $\lambda > 0$ such that $\lambda G(\overline{M}) \subset \Delta^*$. □

3.4.2 Negative cases

The recurring theme of the negative cases is the special property of Eilenberg-Mac Lane spaces (Proposition 2.5.10) that if M is a connected CW-complex and N is Eilenberg-Mac Lane, then every morphism $\pi_1(M, x_0) \to \pi_1(N, y_0)$ is induced by a continuous map $f : (M, x_0) \to (N, y_0)$. This is used in all negative cases except the first when $N = \mathbb{P}^1$.

Lemma 3.4.9 (Proposition 3.0.2(i))**.** *Let M be a compact Riemann surface and $N = \mathbb{P}^1$. There exists a continuous map not homotopic to any holomorphic map.*

Proof. By Lemma 3.2.8 there exists a map $M \to \mathbb{P}^1$ of degree -1. But holomorphic maps have nonnegative degree and degree is a homotopy invariant. □

Lemma 3.4.10 (Proposition 3.0.2(ii))**.** *Let M and N be Riemann surfaces such that M is compact, and both M and N are not simply connected. There exists a continuous map $f : M \to N$ not homotopic to any holomorphic map.*

Proof. Fix base points $x_0 \in M$ and $y_0 \in N$. Since M is compact and not simply connected, we have $H_1(M) \simeq \mathbb{Z}^{2g}$ where $g > 0$ is the genus of M. Since N is not simply connected, the fundamental group $\pi_1(N, y_0)$ is nontrivial. So there exists a morphism $H_1(M) \to \pi_1(N, y_0)$ sending one of the generators of $H_1(M)$ to an nontrivial element $\alpha \in \pi_1(N, y_0)$ and the rest of the generators to zero. Precomposing by the morphism $\pi_1(M, x_0) \to H_1(M)$ sending loops to cycles gives a morphism $\rho : \pi_1(M, x_0) \to \pi_1(N, y_0)$ with image $\langle\alpha\rangle < \pi_1(N, y_0)$.

There exists a covering $p : N_1 \to N$ such that $p_*\pi_1(N_1, y_1) = \langle\alpha\rangle$ by Proposition 2.2.9. Since N is Eilenberg-Mac Lane, the morphism $\rho : \pi_1(M, x_0) \to \pi_1(N, y_0)$ is induced by a continuous map $f : (M, x_0) \to (N, y_0)$, not homotopic to the constant map because ρ is nontrivial. By construction $f_*\pi_1(M, x_0) = \langle\alpha\rangle = p_*\pi_1(N_1, y_1)$, so there exists a lift $\widetilde{f} : (M, x_0) \to (N_1, y_1)$ by Proposition 2.2.6. Since $\pi_1(N_1, y_1)$ is nontrivial and cyclic, the Riemann surface N_1 must be noncompact. Hence holomorphic maps $M \to N_1$ are constant because M is compact. Therefore f is not homotopic to any holomorphic map because f is not homotopic to the constant map. □

The following argument is essentially by Gromov [Gro89, Example 0.1(c)].

Lemma 3.4.11 (Proposition 3.0.2(iii))**.** *Suppose that N is a hyperbolic Riemann surface excluding Δ and Δ^*. If M is a noncompact Riemann surface with $\pi_1(M) \neq 0$, then there exists a continuous map $M \to N$ not homotopic to any holomorphic map.*

Proof. The noncompact Riemann surfaces M with $\pi_1(M) \neq 0$ consist of the punctured plane and noncompact hyperbolic Riemann surfaces excluding Δ. Any holomorphic map $\mathbb{C}^* \to N$ is constant by lifting to universal coverings and applying Liouville's theorem, yet there exist continuous maps $\mathbb{C}^* \to N$ that are not null homotopic. So we only need to be concerned with hyperbolic sources M. In this case, any holomorphic map $f : M \to N$ lifts to a holomorphic map $\widetilde{f} : \mathbb{H} \to \mathbb{H}$ which is distance decreasing by the Schwarz-Pick lemma. So $\ell(f\gamma) \leq \ell(\gamma)$ for any path $\gamma : I \to M$ and any holomorphic map $f : M \to N$.

Fix base points $x_0 \in M$ and $1 \in S^1$. If M is noncompact, then it has the homotopy type of a wedge of circles. If M is also not simply connected, then $\pi_1(M, x_0)$ is a nontrivial free group. Hence we can define a morphism $\pi_1(M, x_0) \to \pi_1(S^1, 1)$ sending a generator representative $\sigma : (I, 0) \to (M, x_0)$ of $\pi_1(M, x_0)$ to the generator of $\pi_1(S^1, 1)$ represented by $E|_I(t) = \exp(2\pi i t)$, and the rest of the generators of $\pi_1(M, x_0)$ to the identity. Since S^1 is Eilenberg-Mac Lane, there exists a continuous map $g : (M, x_0) \to (S^1, 1)$ realising this morphism. Since every continuous loop is homotopic to a smooth loop relative $\{0, 1\}$, we may assume that $\ell(\sigma) < \infty$. By construction $g\sigma \sim E$ relative $\{0, 1\}$.

By the assumptions on N and by Proposition 3.3.12, there exists a closed geodesic $\alpha : S^1 \to N$. Take $n \in \mathbb{Z}$ such that $\ell(\alpha^n) = |n|\ell(\alpha) > \ell(\sigma)$ noting that $\ell(\alpha) > 0$. We claim that $\alpha^n g : M \to N$ is a continuous map not homotopic to any holomorphic map. For if $f : M \to N$ is any continuous map such that $\alpha^n g \sim f$, then $\alpha^n g\sigma \sim f\sigma$. But $\alpha^n g\sigma \sim \alpha^n E$ relative $\{0, 1\}$ and α^n is a closed geodesic, so $\ell(\alpha^n) \leq \ell(f\sigma)$ by Theorem 3.3.15. Then $\ell(\sigma) < |n|\ell(\alpha) \leq \ell(f\sigma)$ which implies that f is not holomorphic. □

Lemma 3.4.12 (Proposition 3.0.2(iv))**.** *Let M_1 be a Riemann surface and $p \in M_1$. Suppose that $M = M_1 \backslash \{p\}$ and that M is not simply connected. There exists a continuous map $f : M \to \Delta^*$ not homotopic to any holomorphic map.*

Proof. Any holomorphic map $g : M \to \Delta^*$ extends to a holomorphic map $\widetilde{g} : M_1 \to \Delta$ by Riemann's removable singularities theorem. So if M_1 is compact, then g is constant. But there are continuous maps $f : M \to \Delta^*$ that are not homotopic to a constant map because M is noncompact and not simply connected while Δ^* is Eilenberg-Mac Lane.

Suppose that M_1 is noncompact. Then $H_2(M_1) = 0$ since M_1 has the homotopy type of a wedge of circles. Let U be a neighbourhood of p biholomorphic to a disk. The

Mayer-Vietoris exact sequence applied to the decomposition $M_1 = M \cup U$ is

$$0 \to H_1(U \setminus \{p\}) \to H_1(M) \to H_1(M_1) \to 0.$$

Since $H_1(M_1)$ is free, the sequence splits and we obtain a surjection

$$(\iota_*)^* : \mathrm{Hom}(H_1(M), \pi_1(\Delta^*, y_0)) \to \mathrm{Hom}(H_1(U \setminus \{p\}), \pi_1(\Delta^*, y_0)) \tag{D}$$

given by the pullback of the pushforward of the inclusion $\iota : U \setminus \{p\} \to M$; we choose and fix $y_0 \in \Delta^*$ arbitrarily. If A and B are groups with B abelian and $q_A : A \to A_{ab} = A/[A, A]$ the projection onto the commutator subgroup quotient, then there is a bijection

$$\mathrm{Hom}(A, B) \to \mathrm{Hom}(A_{ab}, B) \tag{A}$$

sending a map $f : A \to B$ to the induced quotient map $f^{\#} : A_{ab} \to B$. The inverse is precomposition by the quotient projection q_A. If Z is another group and $\psi : Z \to A$ is an isomorphism, then there is a bijection

$$\psi^* : \mathrm{Hom}(A, B) \to \mathrm{Hom}(Z, B). \tag{I}$$

Finally if (X, x_0) is a connected CW-complex and (Y, y_0) is Eilenberg-Mac Lane, then there is a bijection

$$[(X, x_0), (Y, y_0)] \to \mathrm{Hom}(\pi_1(X, x_0), \pi_1(Y, y_0)) \tag{EM}$$

sending a homotopy class $[f]$ of pointed maps to $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$. From these maps we can consider the diagram

$$\begin{array}{ccc}
\mathrm{Hom}(H_1(M), \pi_1(\Delta^*, y_0)) & \xrightarrow{D} & \mathrm{Hom}(H_1(U \setminus \{p\}), \pi_1(\Delta^*, y_0)) \\
{\scriptstyle I}\uparrow & & \uparrow{\scriptstyle I} \\
\mathrm{Hom}(\pi_1(M, x_0)_{ab}, \pi_1(\Delta^*, y_0)) & & \mathrm{Hom}(\pi_1(U \setminus \{p\}, x_0)_{ab}, \pi_1(\Delta^*, y_0)) \\
{\scriptstyle A}\uparrow & & \uparrow{\scriptstyle A} \\
\mathrm{Hom}(\pi_1(M, x_0), \pi_1(\Delta^*, y_0)) & & \mathrm{Hom}(\pi_1(U \setminus \{p\}, x_0), \pi_1(\Delta^*, y_0)) \\
{\scriptstyle EM}\uparrow & & \uparrow{\scriptstyle EM} \\
[(M, x_0), (\Delta^*, y_0)] & \dashrightarrow & [(U \setminus \{p\}, x_0), (\Delta^*, y_0)]
\end{array} \tag{3.8}$$

where we choose and fix $x_0 \in U \setminus \{p\}$ arbitrarily. Since the diagram

$$\begin{array}{ccc}
\pi_1(U \setminus \{p\}, x_0) & \xrightarrow{\iota_*} & \pi_1(M, x_0) \\
\downarrow & & \downarrow \\
\pi_1(U \setminus \{p\}, x_0)_{ab} & & \pi_1(M, x_0)_{ab} \\
\downarrow & & \downarrow \\
H_1(U \setminus \{p\}) & \xrightarrow{\iota_*} & H_1(M)
\end{array}$$

commutes, letting the dashed arrow in (3.8) be restriction $[f] \mapsto [f \circ \iota]$ makes (3.8) commute. Moreover this map is surjective and the base points can be chosen arbitrarily, so every continuous map $U \setminus \{p\} \to \Delta^*$ is homotopic to the restriction of a continuous map $f : M \to \Delta^*$. If $z : U \to \Delta$ is a local complex coordinate centred at p, then restricting $\overline{z} : U \to \Delta$ to $U \setminus \{p\}$ and taking $f : M \to \Delta^*$ to be any homotopic extension gives a continuous map not homotopic to any holomorphic map. □

Proposition 3.4.13 (Proposition 3.0.2(v))**.** *Let M be a Riemann surface such that $\pi_1(M)$ is not finitely generated. There exists a continuous map $f : M \to \Delta^*$ not homotopic to any holomorphic map.*

Proof. Fix $x_0 \in M$. If $\pi_1(M, x_0)$ is not finitely generated, then M is noncompact. Since M is noncompact with $\pi_1(M, x_0)$ not finitely generated, its universal covering is the upper half plane $\mathbb{H}$. Let $p : (\mathbb{H}, a) \to (M, x_0)$ be the universal covering with $a \in p^{-1}(x_0)$ fixed. On the other side, let $q : (\mathbb{H}, i) \to (\Delta^*, e^{-2\pi})$ be the universal covering $q(z) = \exp(2\pi i z)$.

The fundamental group $\pi_1(M, x_0)$ is free because M has the homotopy type of a wedge of circles. For $j \in \mathbb{N}$, let γ_j represent distinct generators of $\pi_1(M, x_0)$. Each curve $\gamma_j : (I, 0) \to (M, x_0)$ lifts to a path $\widetilde{\gamma}_j : (I, 0) \to (\mathbb{H}, a)$ starting at a and ending at $p_j \in \mathbb{H}$. Let d_j be the hyperbolic distance between a and p_j.

Let $\alpha(t) = \exp(2\pi i t) \exp(-2\pi)$ be a generator of $\pi_1(\Delta^*, e^{-2\pi})$. Let $\mu_k : I \to \mathbb{R}$ be the multiplication map $\mu_k(t) = kt$. Any sequence (n_j) in $\mathbb{Z}$ defines a group homomorphism by $\gamma_j \mapsto \alpha\mu_{n_j}$ because $\pi_1(M, x_0)$ is free. Choose $n_j > \exp(j d_j)$ so that $d_j / \log n_j < 1/j$. There exists a continuous map $f : (M, x_0) \to (\Delta^*, e^{-2\pi})$ inducing the corresponding homomorphism $\pi_1(M, x_0) \to \pi_1(\Delta^*, e^{-2\pi})$ since Δ^* is Eilenberg-Mac Lane.

Lift $fp : (\mathbb{H}, a) \to (\Delta^*, e^{-2\pi})$ to $\widetilde{f} : (\mathbb{H}, a) \to (\mathbb{H}, i)$ satisfying $fp = q\widetilde{f}$. Precomposing by $\widetilde{\gamma}_j$, we have $q(\widetilde{f}\widetilde{\gamma}_j) = f\gamma_j \sim \alpha\mu_{n_j}$ relative $\{0, 1\}$ by definition of f. If we define $\widetilde{\alpha}_j : I \to \mathbb{H}$ by $\widetilde{\alpha}_j(t) = i + t n_j$, then $q\widetilde{\alpha}_j = \alpha\mu_{n_j}$. Hence $q\widetilde{\alpha}_j \sim q(\widetilde{f}\widetilde{\gamma}_j)$ relative $\{0, 1\}$ and $\widetilde{f}\widetilde{\gamma}_j(0) = i = \widetilde{\alpha}_j(0)$. Therefore $\widetilde{\alpha}_j \sim \widetilde{f}\widetilde{\gamma}_j$ relative $\{0, 1\}$ by Proposition 2.2.8, which implies that $\widetilde{f}(p_j) = \widetilde{f}(a) + n_j$.

Suppose that there exists a holomorphic map $g : M \to \Delta^*$ and a homotopy $H : M \times I \to \Delta^*$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. Lift $H(p \times 1) : \mathbb{H} \times I \to \Delta^*$ to the homotopy $F : \mathbb{H} \times I \to \mathbb{H}$ determined by $F(a, 0) = i$. The point $F(p_j, 0)$ determines a deck transformation h_F^j of $q : \mathbb{H} \to \Delta^*$ sending i to $F(p_j, 0)$. The maps $h_F^j F(a, -)$ and $F(p_j, -)$ both lift $H(x_0, -)$ with $h_F^j F(a, 0) = F(p_j, 0)$. Hence $h_F^j F(a, s) = F(p_j, s)$ for all $s \in I$. In particular $h_F^j F(a, 1) = F(p_j, 1)$.

Note that $qF(z, 0) = H(p(z), 0) = fp(z)$ and $q\widetilde{f}(z) = fp(z)$ with $F(a, 0) = i = \widetilde{f}(a)$. Hence $F(z, 0) = \widetilde{f}(z)$. By definition $h_F^j \widetilde{f}(a) = h_F^j(i) = F(p_j, 0) = \widetilde{f}(p_j) = \widetilde{f}(a) + n_j$. Deck transformations are determined by a point, so we deduce that $h_F^j(z) = z + n_j$.

Define $\widetilde{g}(z) = F(z, 1)$. Then $\widetilde{g}$ is holomorphic because $q\widetilde{g} = gp$ with both p and q locally biholomorphic. We have $\widetilde{g}(p_j) = F(p_j, 1) = h_F^j F(a, 1) = \widetilde{g}(a) + n_j$. By Lemma 3.3.8, there exists a positive constant $\lambda > 0$ dependent on $\widetilde{g}(a)$ and independent of j such that

$0 < \lambda \leq d(\widetilde{g}(a), \widetilde{g}(a) + n_j)/\log n_j$ for all $j \in \mathbb{N}$. Since holomorphic functions are distance decreasing with respect to the hyperbolic metric, we have $0 < \lambda \leq d_j/\log n_j < 1/j$ for all $j \in \mathbb{N}$, a contradiction. Hence there is no holomorphic map homotopic to f. □

Remark 3.4.14. One might wonder why we need infinitely many generators when we have obtained the bounds $0 < \lambda \leq d_j/\log n_j$ with λ independent of j and with n_j chosen arbitrarily after the d_j have already been determined; it seems as if we just need to choose one of the n_j to be large enough to undercut λ. The reason is that λ depends on $\widetilde{g}(a)$, which depends on f, which is only determined after a sequence (n_j) has been chosen.

Remark 3.4.15. Winkelmann's proof defines λ by the equation $d(\widetilde{g}(a), \widetilde{g}(a) + n_j) = \lambda n_j$ and appeals to the inequality $0 < \lambda \leq d_j/n_j$. But here λ is not independent of j since $d(\widetilde{g}(a), \widetilde{g}(a) + n_j)/n_j = \lambda$ decreases monotonically with respect to increasing n_j, and in fact goes to zero as n_j goes to ∞. All we obtain is a sequence of positive numbers λ_j bounded above by d_j/n_j. It is not clear whether there exists a positive uniform lower bound for the λ_j, so we do not see how to complete Winkelmann's original argument.

The uniform lower bound λ satisfying $0 < \lambda \leq d_j/\log n_j$ produced in Lemma 3.3.8, an original result due to the author, allows us to circumvent this problem.

Chapter 4

The equivariant Oka principle

We investigate the equivariant Oka principle for maps from a noncompact Riemann surface to an Oka Riemann surface. There are only four types of Oka Riemann surfaces: the complex plane $\mathbb{C}$, the punctured plane $\mathbb{C}^*$, any complex torus $\mathbb{C}/\Gamma$, and the Riemann sphere $\mathbb{P}^1$. The automorphism groups of these surfaces are well known, and thus we can understand which finite group actions can occur on these surfaces.

When a group G acts freely and properly discontinuously on a space Y, the quotient projection $Y \to Y/G$ is a covering map. A finite group always acts properly discontinuously on a Hausdorff space, so the equivariant h-principle reduces to the plain h-principle in the case of a free action. In our situation we have to deal with actions which are not necessarily free. However, it turns out that finite group actions on $\mathbb{C}^*$ and $\mathbb{C}/\Gamma$ can be passed to a quotient on which the group acts compatibly with the group operation of $\mathbb{C}^*$ and $\mathbb{C}/\Gamma$, and for which the quotient projection is a covering map. In this case, we use the averaging trick to produce an equivariant homotopy from a plain homotopy, then use the homotopy lifting property to bring this back to the original surface.

For the complex plane $\mathbb{C}$, we prove the stronger result that $\mathbb{C}$ is G-contractible with respect to any action of a finite group G. By contrast, the Riemann sphere $\mathbb{P}^1$ is considerably more difficult because no nontrivial group acts freely on $\mathbb{P}^1$. In this case we only have partial results such as when G acts freely on the source or when the action of G on $\mathbb{P}^1$ is conjugate to a rotation and effective on the source.

4.1 Finite group actions on Oka Riemann surfaces

We start by determining the possible holomorphic finite group actions on $\mathbb{C}$, $\mathbb{C}^*$ and $\mathbb{C}/\Gamma$ up to conjugacy. We first determine the holomorphic automorphism groups of each surface. The calculation is a standard exercise in complex analysis for $\mathbb{C}$ and $\mathbb{C}^*$, and Riemann surface theory for $\mathbb{C}/\Gamma$. We provide proofs for completeness. By considering the automorphism group of each surface, we then determine group actions by looking at finite subgroups. The simplicity of the automorphism groups allows us to extend the plain h-principle to an equivariant h-principle.

4.1.1 The complex plane $\mathbb{C}$

Proposition 4.1.1. $\operatorname{Aut}\mathbb{C} = \{z \mapsto az + b \mid a \in \mathbb{C}^*, b \in \mathbb{C}\}$.

Proof. Suppose that $f : \mathbb{C} \to \mathbb{C}$ is an automorphism. Consider the automorphism $g(z) = f(z) - f(0)$. Then $g(0) = 0$ and $1/g$ is holomorphic on $\mathbb{C}^*$. The isolated singularity at the origin cannot be essential. If it was, then $\overline{1/gU} = \mathbb{C}$ for every punctured neighbourhood U of zero by Casorati-Weierstrass. Take U to be open and bounded. Let V be an open set such that $U \cap V = \emptyset$. Pick $z_0 \in V$. By denseness we can find a sequence (z_n) in U such that $1/g(z_n) \to 1/g(z_0)$. Continuity of g^{-1} gives $z_n \to z_0$ which contradicts our selection of $z_0 \in V$ with $U \cap V = \emptyset$. Thus $1/g$ has a pole of finite order at the origin, which means that g has a zero of finite order at the origin. Since g is entire, we deduce that g is a polynomial by taking a Taylor expansion at the origin.

If $\deg g > 1$, then the derivative g' is nonconstant and thus has a root w by the fundamental theorem of algebra. This would imply that g fails to be injective in any neighbourhood of w. But g is nonconstant so $\deg g > 0$. Since $g(0) = 0$, we deduce that $g(z) = az$ for some $a \in \mathbb{C}^*$. Thus $f(z) = az + b$ where $a \in \mathbb{C}^*$ and $b = f(0) \in \mathbb{C}$.

Therefore every holomorphic automorphism of $\mathbb{C}$ is affine. Conversely any affine map $f(z) = az + b$ with $a \neq 0$ is an automorphism with inverse $f^{-1}(z) = a^{-1}z + (-ba^{-1})$. $\square$

Remark 4.1.2. The automorphism group of $\mathbb{C}$ can be identified with the semidirect product $\operatorname{Aut}\mathbb{C} \simeq \mathbb{C}^* \ltimes \mathbb{C}$.

Proposition 4.1.3. *Let $\varphi : G \to \operatorname{Aut}\mathbb{C}$ be a holomorphic action by a finite group. There exists $n > 0$ such that φG is conjugate to the rotation group $\langle e^{2\pi i/n} z\rangle$ of order n.*

Proof. We can consider the projection $p : \operatorname{Aut}\mathbb{C} \to \mathbb{C}^*$ in the split exact sequence

$$0 \longrightarrow \mathbb{C} \xrightarrow{b \mapsto z+b} \operatorname{Aut}\mathbb{C} \underset{az \leftarrow a}{\overset{az+b \mapsto a}{\rightleftarrows}} \mathbb{C}^* \longrightarrow 1.$$

The restricted projection $p|K \to \mathbb{C}^*$ is injective whenever $K < \operatorname{Aut}\mathbb{C}$ is finite. For if $g_1(z) = az + b_1$ and $g_2(z) = az + b_2$ are two elements of K, then the composition $g_1 g_2^{-1} = z + b_1 - b_2$ is contained in K. Since K is finite it contains no nontrivial translations. Hence $b_1 = b_2$ and $g_1 = g_2$.

The image $\varphi G < \operatorname{Aut}\mathbb{C}$ is finite. Hence $p|\varphi G \to \mathbb{C}^*$ is injective and φG is isomorphic to a cyclic group because finite subgroups of $\mathbb{C}^*$ are cyclic.

Assume that $|\varphi G| = n > 1$. Let $g(z) = az + b$ be a generator of φG. Then $a = \exp(2\pi i k/n)$ for some k such that $\gcd(k, n) = 1$, so $a \neq 1$. Let $\sigma(z) = z + b/(a-1)$. Conjugating gives $\sigma g \sigma^{-1}(z) = az$. Hence $\sigma\varphi G\sigma^{-1}$ is the group of $2\pi k/n$ rotations about the origin. $\square$

4.1.2 The punctured plane $\mathbb{C}^*$

Proposition 4.1.4. $\operatorname{Aut}\mathbb{C}^* = \{z \mapsto az^{\varepsilon} \mid a \in \mathbb{C}^*, \varepsilon \in \{\pm 1\}\}$.

Proof. Any automorphism $f : \mathbb{C}^* \to \mathbb{C}^*$ has an isolated singularity at the origin. This cannot be essential by Casorati-Weierstrass. Otherwise $\overline{fU} = \mathbb{C}$ for all punctured neighbourhoods U of the origin. Take U to be open and bounded. Let V be an open set such that $U \cap V = \emptyset$. Pick $z_0 \in V$. There exists a sequence (z_n) in U such that $f(z_n) \to f(z_0)$ by denseness. Continuity of the inverse f^{-1} implies $z_n \to z_0$, contradicting our selection of $z_0 \in V$ with $U \cap V = \emptyset$.

Hence f has either a removable singularity or a pole at the origin. If the singularity is removable, then f extends to an automorphism $\widetilde{f}$ of $\mathbb{C}$. For this it suffices to show that $\widetilde{f}(0) = 0$. Take any sequence (z_n) in $\mathbb{C}^*$ such that $z_n \to 0$. Then $\widetilde{f}(z_n) \to \widetilde{f}(0)$ by continuity. If $\widetilde{f}(0) \neq 0$, then there exists $w \in \mathbb{C}^*$ such that $f(w) = \widetilde{f}(0)$. Since $\widetilde{f}(z_n) = f(z_n)$ for all n, we get $f(z_n) \to f(w)$. Applying f^{-1} gives $z_n \to w$ by continuity. Uniqueness of limits implies $w = 0$, contradicting the selection $w \in \mathbb{C}^*$.

Therefore $\widetilde{f}(z) = az$ for some $a \in \mathbb{C}^*$ because $\widetilde{f}$ is an automorphism of $\mathbb{C}$ satisfying $\widetilde{f}(0) = 0$. Thus $f(z) = az$ when f has a removable singularity at zero. If f has a pole, then $1/f$ has a removable singularity. Hence $(1/f)(z) = az$, and so $f(z) = a^{-1}z^{-1}$. □

Remark 4.1.5. The automorphism group of $\mathbb{C}^*$ can be identified with the semidirect product $\operatorname{Aut}\mathbb{C}^* \simeq \mathbb{C}^* \rtimes \{\pm 1\}$.

Proposition 4.1.6. *Let $\varphi : G \to \operatorname{Aut}\mathbb{C}^*$ be a holomorphic action by a finite group. There exists $n > 0$ such that φG is conjugate to either the rotation group $\langle e^{2\pi i/n}z\rangle$ of order n or the dihedral group $\langle e^{2\pi i/n}z, 1/z\rangle$ of order $2n$.*

Proof. There is a projection morphism $p : \operatorname{Aut}\mathbb{C}^* \to \{\pm 1\}$. For any finite group action $\varphi : G \to \operatorname{Aut}\mathbb{C}^*$, we can consider the kernel of the restricted projection $\rho = p|\varphi G \to \{\pm 1\}$. Let $|\ker \rho| = n$. Either φG has order n or $2n$. In the former case φG is a rotation group consisting of nth roots of unity. In the latter case φG is a dihedral group consisting of nth roots of unity along with a reflection $(a, -1)$ with $a \in \mathbb{C}^*$. If $b \in \mathbb{C}^*$ satisfies $b^2 = 1/a$, then $(b, 1)(a, -1)(b^{-1}, 1) = (b, 1)(a(b^{-1})^{-1}, 1) = (ab^2, -1) = (1, -1)$. Hence in this case the action is dihedral and the reflection can be taken to be $1/z$. □

Remark 4.1.7. If $|\ker \rho| = |G| = n$, then $\mathbb{C}^*/G \simeq \mathbb{C}^*$ via the map $f(z) = z^n$. If $|G| = 2n$, then $\mathbb{C}^*/G \simeq \mathbb{C}$ via the map $f(z) = (z^n + z^{-n})/2$. If pulled back to the universal covering by $q(z) = \exp(2\pi i z)$, then we have a branched covering $(f \circ q)(z) = \cos(2\pi n z)$.

Remark 4.1.8. The fixed points of the dihedral action occur when z is a $2n$th root of unity. In fact, consider the preimage $z^n + z^{-n} = 2c$. Now $(z^n)^2 - 2cz^n + 1 = 0$, and so $(z^n - c)^2 = -1 + c^2$. If $c^2 \neq 1$, then there are two branches giving n roots each, so we get an orbit of full size $2n$. Else $c^2 = 1$, which implies $c = \pm 1$ and $z^{2n} = 1$. In the case where

$z^n = 1$, we have $f(z) = 1$. When $z^n = -1$, we have $f(z) = -1$. Hence to get a covering space action with respect to the dihedral group, we need to consider $\mathbb{C}^* \setminus \mathcal{W}_{2n} \to \mathbb{C} \setminus \{\pm 1\}$, where $\mathcal{W}_{2n}$ is the set of $2n$th roots of unity.

4.1.3 Complex torus

To clarify the automorphism group of a complex torus, we begin by investigating maps between tori.

Proposition 4.1.9. *Let $f : \mathbb{C}/\Gamma \to \mathbb{C}/\Gamma'$ be a nonconstant holomorphic map between complex tori. Let $\pi : \mathbb{C} \to \mathbb{C}/\Gamma$ and $\pi' : \mathbb{C} \to \mathbb{C}/\Gamma'$ be the quotient projections. There exists a lift of $f\pi$, every lift $\widetilde{f} : \mathbb{C} \to \mathbb{C}$ of $f\pi$ is holomorphic, and consequently is of the form $\widetilde{f}(z) = \alpha z + \beta$ for some $\alpha \in \mathbb{C}^*$ satisfying $\alpha\Gamma \subset \Gamma'$ and some $\beta \in \mathbb{C}$, and α is independent of the choice of lift.*

Proof. There exists a lift of $f\pi$ because $\pi' : \mathbb{C} \to \mathbb{C}/\Gamma'$ is a covering space. Every lift $\widetilde{f}$ is holomorphic because $\pi'\widetilde{f} = f\pi$, and π' is a local biholomorphism. For any $\omega \in \Gamma$, we can consider the translation $T_\omega(z) = z + \omega$. Since $\pi'\widetilde{f}T_\omega = f\pi T_\omega = f\pi = \pi'\widetilde{f}$, the continuous function $\widetilde{f}T_\omega - \widetilde{f}$ maps into the discrete set Γ', and is hence constant. Therefore $\widetilde{f}'T_\omega - \widetilde{f}' = 0$. This argument holds for arbitrary $\omega \in \Gamma$, so $\widetilde{f}'$ is a bounded entire function. By Liouville $\widetilde{f}'$ is constant, so $\widetilde{f}(z) = \alpha z + \beta$ for some $\alpha \in \mathbb{C}^*$ and $\beta \in \mathbb{C}$; note that $\alpha \in \mathbb{C}^*$ because f is nonconstant. The equality $\pi'\widetilde{f}T_\omega(0) = \pi'\widetilde{f}(0)$ implies $\alpha\omega \in \Gamma'$. Hence $\alpha\Gamma \subset \Gamma'$.

Suppose that $\widetilde{f_1}(z) = \alpha_1 z + \beta_1$ and $\widetilde{f_2}(z) = \alpha_2 + \beta_2$ satisfy $\pi'\widetilde{f_1} = f\pi = \pi'\widetilde{f_2}$. Since $\pi'(\beta_1) = \pi'\widetilde{f_1}(0) = \pi'\widetilde{f_2}(0) = \pi'(\beta_2)$, we have $\beta_1 - \beta_2 \in \Gamma'$. Letting $T_{\beta_1 - \beta_2}(z) = z + \beta_1 - \beta_2$ be the translation, uniqueness of lifts implies that $T_{\beta_1-\beta_2}\widetilde{f_2} = \widetilde{f_1}$, so $\alpha_1 = \alpha_2$. □

Corollary 4.1.10. *Suppose that $f : \mathbb{C}/\Gamma \to \mathbb{C}/\Gamma'$ is a nonconstant holomorphic map between tori satisfying $f\pi(0) = \pi'(z_0)$. Then f is a biholomorphism if and only if there exists $\alpha \in \mathbb{C}^*$ such that $\alpha\Gamma = \Gamma'$, and $f\pi = \pi'\widetilde{f}$ for $\widetilde{f}(z) = \alpha z + z_0$.*

Proof. By the assumption on f there exists unique $\alpha \in \mathbb{C}^*$ such that $\alpha\Gamma \subset \Gamma'$ and $\widetilde{f}(z) = \alpha z + z_0$ lifts $f\pi$. If f is a biholomorphism, then $\widetilde{f}^{-1}(z) = \alpha^{-1}z - \alpha^{-1}z_0$ lifts $f^{-1}\pi'$ and satisfies $f^{-1}\pi'(0) = \pi(-\alpha^{-1}z_0)$. Suppose that $\omega' \in \Gamma'$. Since π is a homomorphism and $\pi'(\omega) = \pi'(0)$, the equation $f^{-1}\pi'(\omega) = \pi(\alpha^{-1}\omega - \alpha^{-1}z_0)$ implies $\alpha^{-1}\Gamma' \subset \Gamma$. Hence $\Gamma' \subset \alpha\Gamma \subset \Gamma'$, and $\Gamma' = \alpha\Gamma$.

Conversely, suppose that $\widetilde{f}(z) = \alpha z + z_0$ lifts $f\pi$ and $\alpha\Gamma = \Gamma'$. Then $\Gamma = (1/\alpha)\Gamma'$. Since $\pi\widetilde{f}^{-1}(z+\omega') = \pi(z/\alpha - z_0/\alpha + \omega'/\alpha) = \pi\widetilde{f}^{-1}(z)$ for all $\omega' \in \Gamma'$ because $(1/\alpha)\Gamma' = \Gamma$, there is an induced holomorphic map $g : \mathbb{C}/\Gamma' \to \mathbb{C}/\Gamma$ satisfying $g\pi' = \pi\widetilde{f}^{-1}$. Applying f on the left of both sides gives $fg\pi' = \pi'$. Applying g on the left of both sides of $\pi'\widetilde{f} = f\pi$ gives $gf\pi = \pi$. Since π and π' are both epimorphisms, the right cancellative property implies that $gf = 1_{\mathbb{C}/\Gamma}$ and $fg = 1_{\mathbb{C}/\Gamma'}$. Hence f is a biholomorphism with inverse g. □

We provide a converse to the preceding result.

Proposition 4.1.11. *Let $f : \mathbb{C} \to \mathbb{C}$ be an affine map $f(z) = \alpha z + z_0$. For lattices $\Gamma, \Gamma' \subset \mathbb{C}$, let $\pi : \mathbb{C} \to \mathbb{C}/\Gamma$ and $\pi' : \mathbb{C} \to \mathbb{C}/\Gamma'$ be the quotient projections. There exists a unique map $f^{\#} : \mathbb{C}/\Gamma \to \mathbb{C}/\Gamma'$ making the diagram*

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{f} & \mathbb{C} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ \mathbb{C}/\Gamma & \xrightarrow{f^{\#}} & \mathbb{C}/\Gamma' \end{array}$$

commute if and only if $\alpha\Gamma \subset \Gamma'$. Conversely, if $f : \mathbb{C} \to \mathbb{C}$ is an arbitrary holomorphic function inducing a map $f^{\#} : \mathbb{C}/\Gamma \to \mathbb{C}/\Gamma'$ on quotients, then f is affine.

Proof. Let $f(z) = \alpha z + z_0$ be affine and let $\omega \in \Gamma$. We have $\pi' f(z+\omega) = \pi'(\alpha z + z_0 + \alpha\omega)$. We claim $\pi' f(z+\omega) = \pi' f(z)$ if and only if $\alpha\omega \in \Gamma'$. In fact, if $\pi' f(z+\omega) = \pi' f(z)$, then $\pi'(\alpha z + z_0 + \alpha\omega) = \pi'(\alpha z + z_0)$ and $\alpha\omega \in \Gamma'$. If $\alpha\omega \in \Gamma'$, then $\pi'(\alpha z + z_0 + \alpha\omega) = \pi'(\alpha z + z_0)$. Hence $\alpha\Gamma \subset \Gamma'$ if and only if $\pi' f$ is Γ-invariant, if and only if there exists a unique induced map $f^{\#} : \mathbb{C}/\Gamma \to \mathbb{C}/\Gamma'$ on quotients.

Existence and uniqueness of liftings implies every holomorphic function $f : \mathbb{C} \to \mathbb{C}$ which passes to the quotient is affine. In fact if $f^{\#} : \mathbb{C}/\Gamma \to \mathbb{C}/\Gamma'$ satisfies $f^{\#}\pi = \pi' f$, then $f^{\#}\pi(0) = \pi' f(0)$. Lift $f^{\#}$ to a map $g : \mathbb{C} \to \mathbb{C}$ of the form $g(z) = \alpha z + f(0)$ with $\alpha\Gamma \subset \Gamma'$ using Proposition 4.1.9. Recall that g is unique with the property that $\pi' g = f^{\#}\pi$ and $g(0) = f(0)$. Hence $g = f$ and f is affine. □

Every automorphism $f \in \mathrm{Aut}(\mathbb{C}/\Gamma)$ is induced by a map $\widetilde{f}(z) = \alpha z + z_0$ where $\alpha\Gamma = \Gamma$, and $f\pi(0) = \pi(z_0)$. Let $f, g \in \mathrm{Aut}(\mathbb{C}/\Gamma)$. Suppose that $\widetilde{f}(z) = \alpha z + z_0$ and $\widetilde{g}(z) = \beta z + z_1$ lift f and g. Then $\alpha\Gamma = \Gamma$ and $\beta\Gamma = \Gamma$, so $\alpha\beta\Gamma = \Gamma$ and $\widetilde{g}\widetilde{f}(z) = \alpha\beta z + \beta z_0 + z_1$ passes to a map on the quotient with $\pi\widetilde{g} = g\pi$ and $\pi\widetilde{f} = f\pi$. This implies $\pi\widetilde{g}\widetilde{f} = gf\pi$. Hence $\widetilde{g}\widetilde{f}$ lifts gf.

In the other direction, suppose that $g, f : \mathbb{C} \to \mathbb{C}$ are affine maps defined $g(z) = \beta z + z_1$ and $f(z) = \alpha z + z_0$ with $\alpha\Gamma = \Gamma$ and $\beta\Gamma = \Gamma$. Then $\alpha\beta\Gamma = \Gamma$, so $gf(z) = \alpha\beta z + \beta z_0 + z_1$ passes to a map $(gf)^{\#}$ on the quotient satisfying $(gf)^{\#}\pi = \pi g f = g^{\#}\pi f = g^{\#} f^{\#}\pi$. Hence $(gf)^{\#} = g^{\#} f^{\#}$ by surjectivity of π.

We summarise the correspondence between automorphisms of $\mathbb{C}$ and automorphisms of complex tori.

Proposition 4.1.12. *Define* $\mathrm{Aut}_0(\mathbb{C}/\Gamma) = \{\alpha \in \mathbb{C}^* \mid \alpha\Gamma = \Gamma\}$. *There is a group isomorphism*

$$\chi : \mathrm{Aut}(\mathbb{C}/\Gamma) \to \mathrm{Aut}_0(\mathbb{C}/\Gamma) \ltimes \mathbb{C}/\Gamma$$

defined by $\chi(f) = (\alpha, \pi(z_0))$ where $\widetilde{f}(z) = \alpha z + z_0$ is any lift of f.

Proof. The assignment $f \mapsto \chi(f)$ is independent of the choice of lift because α is uniquely determined, and the lifts $\widetilde{f}$ of f all differ by an element of Γ. But π is Γ-invariant.

There is a holomorphic action of $\mathrm{Aut}_0(\mathbb{C}/\Gamma)$ on $\mathbb{C}/\Gamma$ defined by $\alpha.[z] = [\alpha z]$. Now suppose that $g \in \mathrm{Aut}(\mathbb{C}/\Gamma)$ and that $\widetilde{g}(z) = \beta z + z_1$ lifts g. Then $\widetilde{g}\widetilde{f}(z) = \alpha\beta z + \beta z_0 + z_1$ is a lift of gf. Hence $\chi(gf) = (\alpha\beta, \pi(\beta z_0 + z_1)) = (\alpha\beta, \beta.\pi(z_0) + \pi(z_1)) = \chi(g)\chi(f)$. Hence χ is a homomorphism.

Suppose that $\chi(f) = (1, \pi(0))$. By definition f lifts to $\widetilde{f}(z) = z$, and $\pi\widetilde{f} = f\pi$. But $\widetilde{f} = 1_{\mathbb{C}}$, so $1_{\mathbb{C}/\Gamma}\pi = f\pi$ and $1_{\mathbb{C}/\Gamma} = f$ since π is surjective. Therefore χ is injective. Suppose that $(\alpha, c) \in \mathrm{Aut}_0(\mathbb{C}/\Gamma) \ltimes \mathbb{C}/\Gamma$. Let $c = \pi(z_0)$. Define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = \alpha z + z_0$. Then $f^\# : \mathbb{C}/\Gamma \to \mathbb{C}/\Gamma$ is an automorphism with inverse $(f^\#)^{-1} = (f^{-1})^\#$. The map $f^\#$ lifts back to f, so $\chi(f^\#) = (\alpha, \pi(z_0)) = (\alpha, c)$. Therefore χ is surjective. □

Remark 4.1.13. One can consider $\mathrm{Aut}_0(\mathbb{C}/\Gamma)$ to be the set of automorphisms of $\mathbb{C}/\Gamma$ fixing the origin. We have defined it as a subset of $\mathbb{C}^*$ to point out that all such automorphisms are induced by multiplication maps. Miranda [Mir95, Chapter III, Proposition 1.12] identifies the different possibilities for $\mathrm{Aut}_0(\mathbb{C}/\Gamma)$. This classification is not relevant for our purposes.

4.2 Quotients of Riemann surfaces by holomorphic finite group actions

Following Miranda [Mir95], we show that if X is a Riemann surface and G is a finite group acting holomorphically and effectively on X, the orbit space X/G has a unique complex structure for which the projection $X \to X/G$ is a holomorphic map. There is no loss of generality in assuming that G acts effectively. If the action $\varphi : G \to \mathrm{Aut}\, X$ is not effective, then we consider the action of φG on X, noting that $X/\varphi G = X/G$.

Our presentation differs slightly from Miranda in that we define a structure sheaf $\mathscr{O}_{X/G}$ on X/G first and then show that $(X/G, \mathscr{O}_{X/G})$ is locally isomorphic as a locally ringed space to a domain in $\mathbb{C}$, as opposed to producing an atlas from charts by checking compatibility.

4.2.1 Obtaining suitable chart domains

We first cite three propositions from Miranda regarding the existence of neighbourhoods with nice properties with respect to the group action at each point. The existence of such neighbourhoods depends upon the result that points with nontrivial isotropy do not accumulate, and the stabiliser subgroups are finite cyclic.

Proposition 4.2.1. *Let G be a group acting holomorphically and effective on a Riemann surface X, and fix a point $p \in X$. Suppose that the stabiliser subgroup G_p is finite. Then*

G_p is a finite cyclic group. In particular, if G is finite, then all stabiliser subgroups are finite cyclic subgroups.

Proof. Miranda [Mir95, Chapter III, Proposition 3.1]. □

Proposition 4.2.2. *Let G be a finite group acting holomorphically and effectively on a Riemann surface X. The points of X with nontrivial isotropy do not accumulate.*

Proof. Miranda [Mir95, Chapter III, Proposition 3.2]. □

Remark 4.2.3. Miranda's statement of the proposition is just that the points with nontrivial isotropy form a discrete set, but the proof uses the identity theorem to obtain a contradiction in the event that there exists a sequence of points with nontrivial isotropy converging to a point in X.

Proposition 4.2.4. *Let G be a finite group acting holomorphically and effectively on a Riemann surface X. Fix $p \in X$. There exists an open neighbourhood U of p such that:*

(i) *$gU = U$ for every $g \in G_p$;*

(ii) *$gU \cap U = \emptyset$ for every $g \notin G_p$;*

(iii) *if $x \in U$ fixed by some nontrivial element of G_p, then $x = p$.*

Proof. Miranda [Mir95, Chapter III, Proposition 3.3]. □

Remark 4.2.5. The open neighbourhoods U can be constructed such that they lie in a coordinate chart of p.

These open neighbourhoods are used to get coordinate charts on the orbit space X/G. Our presentation will now slightly diverge from Miranda's. Instead of checking compatibility of the charts as Miranda does, we will instead propose a structure sheaf for the orbit space X/G, and then show that X/G is locally isomorphic as a locally ringed space to a domain in $\mathbb{C}$. These two approaches are equivalent.

4.2.2 Structure sheaf of X/G

Let $\pi : X \to X/G$ be the quotient projection onto the orbit space. Since G is finite, the orbit space X/G is Hausdorff. There is an induced morphism $\pi^* : \mathscr{C}_{X/G} \to \pi_*\mathscr{C}_X$ between sheaves of continuous complex-valued functions. Since the preimage of a subsheaf under a sheaf morphism is a subsheaf, we define $\mathscr{O}_{X/G} = (\pi^*)^{-1}\pi_*\mathscr{O}_X$. Explicitly, for each open set $U \subset X/G$, there is a ring homomorphism $\pi^*(U) : \mathscr{C}_{X/G}(U) \to \pi_*\mathscr{C}_X(U)$ which allows us to set $\mathscr{O}_{X/G}(U) = \pi^*(U)^{-1}\pi_*\mathscr{O}_X(U)$.

By definition the restriction of the induced morphism $\pi^* : \mathscr{C}_{X/G} \to \pi_*\mathscr{C}_X$ to $\mathscr{O}_{X/G}$ induces a morphism of subsheaves $\mathscr{O}_{X/G} \to \pi_*\mathscr{O}_X$. Hence $\pi : X \to X/G$ is holomorphic provided that $(X/G, \mathscr{O}_{X/G})$ is a Riemann surface. The holomorphic functions on $U \subset X/G$ are continuous functions $f : U \to \mathbb{C}$ such that $f\pi : \pi^{-1}U \to \mathbb{C}$ is holomorphic. Conversely, if $U \subset X$ is π-saturated, then any G-invariant holomorphic function $f : U \to \mathbb{C}$ passes to a continuous function $f^\# : \pi U \to \mathbb{C}$ uniquely satisfying the property that $f^\#\pi = f$. Hence every G-invariant holomorphic function corresponds uniquely to a holomorphic function from the quotient.

4.2.3 Construction of charts

We show that $(X/G, \mathscr{O}_{X/G})$ is a Riemann surface. For this we need charts to serve as local isomorphisms. The construction follows Miranda.

For any point $p \in X$, we take a coordinate chart (U, z) centred at p from Proposition 4.2.4. For each $g \in G_p$, we consider the holomorphic function $zg : U \to \mathbb{C}$. Define $h : U \to \mathbb{C}$ by $h(x) = \prod_{g \in G_p} zg(x)$. Applying any $g \in G_p$ permutes the factors in the product of h, so h is G_p-invariant. Also h has multiplicity $m = |G_p|$. Letting $q : U \to U/G_p$ be the quotient projection, there is an induced map $h^\# : U/G_p \to \mathbb{C}$ from the quotient satisfying $h^\# q = h$. Since h has multiplicity m, there are m preimages of $h(x)$ for each $x \neq p$. These preimages get identified in U/G_p, so $h^\#$ is injective. Since h is nonconstant and holomorphic, it is an open map. Hence $h^\#$ is an open map. A similar argument with $q : U \to U/G_p$ and $\pi|U \to U/G$ gives an injective open map $\pi^\# : U/G_p \to U/G$ satisfying $\pi^\# q = \pi|_U$.

The composition $h^\#(\pi^\#)^{-1} : \pi U \to hU$ is a homeomorphism from an open subset of X/G to an open subset of $\mathbb{C}$. We will show that there is an induced isomorphism of subsheaves $\mathscr{O}_{hU} \to (h^\#(\pi^\#)^{-1})_*(\mathscr{O}_{X/G}|_{\pi U})$; we require the following result.

Proposition 4.2.6. *Let $f : X \to Y$ and $g : Y \to Z$ be functions defined on open sets $X, Y, Z \subset \mathbb{C}$, nonconstant on each connected component. If f and $g \circ f$ are holomorphic, and g is continuous, then g is holomorphic on $f(X)$.*

Proof. Note that $f(X)$ is open by the open mapping theorem. Let $w_0 = f(z_0) \in f(X)$. If $f'(z_0) \neq 0$, then there exists an open neighbourhood U of z_0 such that f is invertible on U and $(f|_U)^{-1}$ is holomorphic. Then $g|_{f(U)} = (g \circ f) \circ (f|_U)^{-1}$ is holomorphic on $f(U)$ by the chain rule. If $f'(z_0) = 0$, then there exists a neighbourhood U of z_0 such that $f'(z) \neq 0$ for all $z \in U \setminus \{z_0\}$. So $f(U)$ is a neighbourhood of w_0 such that g is holomorphic on $f(U) \setminus \{w_0\}$. Since g is continuous on $f(U)$, Riemann's removable singularities theorem implies that g is holomorphic on $f(U)$. □

Remark 4.2.7. The assumption that g is continuous is not needed, but the assumption greatly simplifies the proof since we can immediately apply Riemann's theorem.

Proposition 4.2.8. *Let X be a Riemann surface and G a finite group acting holomorphically and effectively on X. Let (U, z) be a coordinate neighbourhood centred at p such* *that $gU \cap U = \emptyset$ for $g \notin G_p$, that $gU = U$ for $g \in G_p$ and that the only point in U fixed by any nontrivial element of G_p is p. The map $\widetilde{h} : \pi^{-1}\pi U \to \mathbb{C}$ defined by $\widetilde{h}(x) = h(g^{-1}x)$ if $x \in gU$ is well defined and holomorphic.*

Proof. If $x \in g_1U \cap g_2U$, then $g_2^{-1}g_1U \cap U \neq \emptyset$ so $g_2^{-1}g_1 \in G_p$. Hence if $x \in g_1U \cap g_2U$, then $h(g_2^{-1}x) = h(g_2^{-1}g_1g_1^{-1}x) = h(g_1^{-1}x)$ because h is G_p-invariant. Hence $\widetilde{h}$ is well defined. The assumptions on U imply that $\pi^{-1}\pi U = \coprod_{g\in G/G_p} gU$ for some system of coset representatives for G/G_p. The gU give a disjoint open cover for $\pi^{-1}\pi U$ with $\widetilde{h}$ holomorphic on each component. □

Proposition 4.2.9. *Let $h : U \to \mathbb{C}$ be defined $h(x) = \prod_{g\in G_p} zg(x)$, and let $q : U \to U/G_p$ and $\pi : X \to X/G$ be quotient maps onto orbit spaces. Let $h^\# : U/G_p \to \mathbb{C}$ and $\pi^\# : U/G_p \to U/G$ be the induced maps uniquely satisfying $h^\# q = h$ and $\pi^\# q = \pi$. Then $h^\#$ and $\pi^\#$ are injective open maps, and the homeomorphism $h^\#(\pi^\#)^{-1} : \pi U \to hU$ induces an isomorphism of subsheaves $\mathscr{O}_{hU} \to (h^\#(\pi^\#)^{-1})_*(\mathscr{O}_{X/G}|_{\pi U})$.*

Proof. We have already established the relevant topological properties of the maps involved, so we only need to prove the isomorphism of subsheaves. Also, since $h^\#(\pi^\#)^{-1} : \pi U \to hU$ is a homeomorphism, the induced map $\mathscr{C}_{hU} \to (h^\#(\pi^\#)^{-1})_*(\mathscr{C}_{X/G}|_{\pi U})$ is already an isomorphism on each open set $W \subset hU$. Thus the proof further reduces to showing that this isomorphism restricts appropriately to the holomorphic subsheaves.

The first step is to show that restricting $\mathscr{C}_{hU} \to (h^\#(\pi^\#)^{-1})_*(\mathscr{C}_{X/G}|_{\pi U})$ to $\mathscr{O}_{hU}$ gives a map into $(h^\#(\pi^\#)^{-1})_*(\mathscr{O}_{X/G}|_{\pi U})$. This amounts to showing that given a holomorphic function $f : W \to \mathbb{C}$ defined on an open subset $W \subset hU$, the composition $f\circ(h^\#(\pi^\#)^{-1})$ is in $(h^\#(\pi^\#)^{-1})_*(\mathscr{O}_{X/G}|_{\pi U})(W)$, which, after unwinding all definitions, amounts to showing that $f \circ h^\#(\pi^\#)^{-1} \circ \pi \in \mathscr{O}_X(\pi^{-1}\pi^\#(h^\#)^{-1}W)$. Since f is a holomorphic function between two open subsets of $\mathbb{C}$, it clearly suffices to show that $h^\#(\pi^\#)^{-1} \circ \pi \in \mathscr{O}_X(\pi^{-1}\pi U)$.

Consider $h^\#(\pi^\#)^{-1} \circ \pi|_{\pi^{-1}\pi U}$. If $x \in \pi^{-1}\pi U$, then there exists $g \in G$ such that $x \in gU$. Then $g^{-1}x \in U$, and G-invariance of π gives $(h^\#(\pi^\#)^{-1} \circ \pi)(x) = \widetilde{h}(x)$. Hence $h^\#(\pi^\#)^{-1} \in \mathscr{O}_{X/G}(\pi U)$, and this implies that the induced morphism on sheaves satisfies $\mathscr{O}_{hU} \to (h^\#(\pi^\#)^{-1})_*(\mathscr{O}_{X/G}|_{\pi U})$.

We now need to show that the inverse sends $(h^\#(\pi^\#)^{-1})_*(\mathscr{O}_{X/G}|_{\pi U})$ into $\mathscr{O}_{hU}$. For this, we need to show that for each open set $W \subset hU$ and each $f \in \mathscr{O}_{X/G}(\pi^\#(h^\#)^{-1}W)$, the function $f \circ (\pi^\#(h^\#)^{-1}|_W) : W \to \mathbb{C}$ is holomorphic.

Suppose that $W \subset hU$ and $f \in (h^\#(\pi^\#)^{-1})_*(\mathscr{O}_{X/G}|_{\pi U})(W)$. By definition, this implies $f \in (\pi^*)^{-1}\pi_*\mathscr{O}_X(\pi^\#(h^\#)^{-1}W)$, and so $f \circ (\pi|_{\pi^{-1}\pi^\#(h^\#)^{-1}W}) \in \mathscr{O}_X(\pi^{-1}\pi^\#(h^\#)^{-1}W)$. The intersection $\pi^{-1}\pi^\#(h^\#)^{-1}W \cap U$ is $h^{-1}W$. In fact if $x \in \pi^{-1}\pi^\#(h^\#)^{-1}W \cap U$, then $\pi(x) = (\pi^\#(h^\#)^{-1}) \circ h(x) \in \pi^\#(h^\#)^{-1}W$. Since $\pi^\#(h^\#)^{-1}$ is injective, we have $h(x) \in W$, implying that $x \in h^{-1}W$. Conversely, if $x \in h^{-1}W$, then $x \in U$ and the above argument implies $x \in \pi^{-1}\pi^\#(h^\#)^{-1}hh^{-1}W \subset \pi^{-1}\pi^\#(h^\#)^{-1}W$. Since $W \subset hU$, the restriction $h|_{h^{-1}W}$ is surjective onto W.

Therefore the restriction of $f \circ (\pi|_{\pi^{-1}\pi^{\#}(h^{\#})^{-1}W})$ to U is

$$f \circ (\pi|_{h^{-1}W}) = (f \circ \pi^{\#}(h^{\#})^{-1}|_W) \circ (h|_{h^{-1}W}).$$

Pick $w \in W$. There exists some $x \in h^{-1}W$ such that $h(x) = w$. Choose a coordinate neighbourhood (Z, ζ) of x. Precomposing by ζ^{-1} gives a holomorphic function $f \circ \pi \circ \zeta^{-1} = (f \circ \pi^{\#}(h^{\#})^{-1}|_{hZ}) \circ (h \circ \zeta^{-1})$ on ζZ. Hence $f \circ \pi^{\#}(h^{\#})^{-1}|_{hZ}$ is holomorphic by Proposition 4.2.6. □

Theorem 4.2.10. *The pair $(X/G, \mathscr{O}_{X/G})$ is a Riemann surface.*

Proof. Take a cover $(U_i, z_i)_{i\in\Lambda}$ of X by coordinate neighbourhoods described in Proposition 4.2.4. On each, define $h_i : U_i \to \mathbb{C}$ as in Proposition 4.2.9. The collection $(h_i^{\#}(\pi^{\#})^{-1} : \pi U_i \to h_i U_i)_{i\in\Lambda}$ defines a Riemann surface structure on $(X/G, \mathscr{O}_{X/G})$. □

Proposition 4.2.11. *A Riemann surface structure on X/G for which $\pi : X \to X/G$ becomes holomorphic is uniquely determined.*

Proof. If $\mathscr{O}'_{X/G}$ is another structure on X/G for which $\pi : X \to X/G$ is holomorphic, then Proposition 4.2.6 implies that $\mathrm{id}_{X/G}$ is a holomorphic map because $\pi = \mathrm{id}_{X/G} \circ \pi$. □

Remark 4.2.12. Miranda [Mir95, Chapter III, Theorem 3.4] establishes chart compatibility by instead considering the different cases when a chart for X/G is centred at a point with multiplicity $m = 1$ or multiplicity $m \geq 2$. The two-out-of-three rule (Proposition 4.2.6), which we required for Proposition 4.2.9, does not make an appearance.

4.3 The equivariant Oka principle for $\mathbb{C}$, $\mathbb{C}^*$, $\mathbb{C}/\Gamma$

We are now able to establish the equivariant Oka principle for $\mathbb{C}$, $\mathbb{C}^*$ and $\mathbb{C}/\Gamma$. We first dispense with $\mathbb{C}$ because this case is rather simple with $\mathbb{C}$ being G-contractible for every holomorphic action by a finite group G. Theorem 4.3.8 establishes the equivariant Oka principle for $\mathbb{C}^*$ and $\mathbb{C}/\Gamma$ by combining covering space theory with an averaging argument. The interval I is always equipped with the trivial G-action.

Proposition 4.3.1. *The complex plane $\mathbb{C}$ is G-contractible for every holomorphic action by a finite group G.*

Proof. Suppose that G is finite and $\varphi : G \to \operatorname{Aut}\mathbb{C}$ is an action on $\mathbb{C}$. Then φG is cyclic, and there exists $g \in G$ such that $\varphi(g)$ generates φG. If φG is nontrivial, then $\varphi(g) = az + b$ for $a \neq 1$. There is a unique fixed point $z_0 = b/(1-a)$ for $\varphi(g)$. Let $\sigma(z) = z - z_0$ be the translation sending the fixed point to zero. Define the homotopy $H : \mathbb{C} \times I \to \mathbb{C}$ by

$$H(z,t) = (1-t)z + tz_0 = z - t\sigma(z).$$

Supposing that $|\varphi G| = n$, we have $\sigma\varphi(g)\sigma^{-1} = \mu_n$, where $\mu_n(z) = \omega_n z$ is multiplication by some primitive root of unity ω_n. Exploiting the linearity of μ_n, one may verify that $H(\varphi(g)^k(z), t) = \varphi(g)^k(H(z,t))$, giving an equivariant homotopy from the identity to the constant map z_0. □

Corollary 4.3.2. *Let G be a finite group acting holomorphically on a Riemann surface X and the complex plane $\mathbb{C}$. Every continuous G-map $X \to \mathbb{C}$ or $\mathbb{C} \to X$ is G-homotopic to a holomorphic G-map.* □

Remark 4.3.3. Since every finite subgroup of $\operatorname{Aut}\Delta$ is conjugate to a rotation group, Proposition 4.3.1 and Corollary 4.3.2 hold with Δ in place of $\mathbb{C}$.

For the case of $\mathbb{C}^*$ and $\mathbb{C}/\Gamma$, we first require some technical results. The automorphism groups of $\mathbb{C}^*$ and $\mathbb{C}/\Gamma$ are $\mathbb{C}^* \rtimes \{\pm 1\}$ and $\operatorname{Aut}_0(\mathbb{C}/\Gamma) \ltimes \mathbb{C}/\Gamma$. The $\{\pm 1\}$ and $\operatorname{Aut}_0(\mathbb{C}/\Gamma)$ automorphisms are homomorphisms with respect to the group operations of $\mathbb{C}^*$ and $\mathbb{C}/\Gamma$. Thus we say that these automorphisms act *linearly* on $\mathbb{C}^*$ and $\mathbb{C}/\Gamma$ in the sense that both become G-modules with respect to these actions. While $\mathbb{C}^*$ is a multiplicative group and the term G-module is typically reserved for an additive group, the discrepancy is only formal since $(zw)^\varepsilon = z^\varepsilon w^\varepsilon$ for $\varepsilon \in \{\pm 1\}$.

We start by explaining how we can reduce an action on $\mathbb{C}^*$ or $\mathbb{C}/\Gamma$ to the linear part.

Proposition 4.3.4. *Suppose that X is a G-space and $N \lhd G$ is a normal subgroup. The orbit space X/N is a G-space with the induced action, and the quotient projection $q : X \to X/N$ is equivariant.*

Proof. The induced action of G on X/N is defined $g[x] = [gx]$. If $x \sim x'$, then there exists $n \in N$ such that $nx = x'$. Since $gx' = gnx = (gng^{-1})gx$, normality implies $gng^{-1} \in N$ so $gx \sim gx'$ and the action is well defined. We have $e[x] = [ex] = [x]$ and $(gh)[x] = [ghx] = g[hx] = g(h[x])$. Finally $q(gx) = [gx] = g[x] = gq(x)$. □

Corollary 4.3.5. *Suppose that G is a finite group acting holomorphically on a Riemann surface Y and that $N \lhd G$ is a normal subgroup such that its image in $\operatorname{Aut} Y$ acts freely on Y. The quotient map $Y \to Y/N$ is an equivariant holomorphic covering with respect to the induced action of G on Y/N.* □

Proposition 4.3.6. *Suppose that Y is $\mathbb{C}^*$ or $\mathbb{C}/\Gamma$, that G acts holomorphically on Y and that $N \lhd G$ is the kernel of the action composed with the semidirect product projection. The quotient space Y/N inherits a group operation from Y and is a G-module with respect to the induced action.*

Proof. Suppose that Y is $\mathbb{C}^*$. Define $[z][w] = [zw]$. If $z \sim z'$ and $w \sim w'$, then there exists $a, b \in N \subset \mathbb{C}^*$ such that $az = z'$ and $bw = w'$. Then $z'w' = azbw = (ab)zw$ so $z'w' \sim zw$. If $(a,\varepsilon) \in \operatorname{Aut}\mathbb{C}^*$, where $(a,\varepsilon).z = az^\varepsilon$, the induced action on $\mathbb{C}^*/N$ is $(a,\varepsilon).([z][w]) = [a(zw)^\varepsilon] = [(zw)^\varepsilon] = [z^\varepsilon][w^\varepsilon] = [az^\varepsilon][az^\varepsilon] = (a,\varepsilon).[z](a,\varepsilon).[w]$.

Suppose that Y is $\mathbb{C}/\Gamma$. Define $[z]+[w]=[z+w]$. If $z \sim z'$ and $w \sim w'$, then there exists $a, b \in N \subset \mathbb{C}/\Gamma$ such that $z+a=z'$ and $w+b=w'$. Then $z'+w'=z+w+(a+b)$ so $z'+w' \sim z+w$. If $(\alpha, c) \in \mathrm{Aut}(\mathbb{C}/\Gamma)$ where $(\alpha, c).z=\alpha.z+c$, the induced action on $(\mathbb{C}/\Gamma)/N$ is $(\alpha, c).([z]+[w])=[\alpha.(z+w)+c]=[\alpha.z+\alpha.w]=(\alpha, c).[z]+(\alpha, c).[w]$. □

The final result is that the lift of an equivariant homotopy is equivariant if the initial map of the lift is equivariant. The proposition does not deal with existence of lifts, since this is covered by the plain homotopy lifting property.

Proposition 4.3.7. *Let X and Y be G-spaces. Suppose that $p: Y \to X$ is an equivariant covering map. Suppose that Z is a connected G-space and $H_0: Z \times I \to X$ is an equivariant homotopy. If $H_1: Z \times I \to Y$ is a lift of H_0 and $H_1(-, 0): Z \to Y$ is equivariant, then $H_1: Z \times I \to Y$ is equivariant.*

Proof. Consider the two maps $H_1 g, gH_1: Z \times I \to Y$. Since $p: Y \to X$ is equivariant and H_1 lifts H_0, we have $pgH_1=gpH_1=gH_0=H_0 g=pH_1 g$. For any $z \in Z$, since $H_1(-, 0)$ is equivariant, we have $gH_1(z, 0)=H_1(gz, 0)$. By uniqueness of lifts, this implies $gH_1=H_1 g$ as maps $Z \times I \to Y$. □

We now arrive at the main theorem.

Theorem 4.3.8. *Let X be a noncompact Riemann surface and let Y be $\mathbb{C}^*$ or $\mathbb{C}/\Gamma$ for some lattice $\Gamma \subset \mathbb{C}$. Let G be a finite group acting holomorphically on X and Y. Every continuous G-map $f: X \to Y$ is G-homotopic to a holomorphic G-map.*

Proof. The automorphism groups of $\mathbb{C}^*$ and $\mathbb{C}/\Gamma$ are $\mathbb{C}^* \rtimes \{\pm 1\}$ and $\mathrm{Aut}_0(\mathbb{C}/\Gamma) \ltimes \mathbb{C}/\Gamma$. Hence given the action $G \to \mathrm{Aut}\, Y$ of a finite group G, there is a projection to $\{\pm 1\}$ or $\mathrm{Aut}_0(\mathbb{C}/\Gamma)$; the kernel N of the action composed with this projection is represented by $\mathbb{C}^*$ in $\mathrm{Aut}\, \mathbb{C}^*$ or $\mathbb{C}/\Gamma$ in $\mathrm{Aut}(\mathbb{C}/\Gamma)$. The $\mathbb{C}^*$ and $\mathbb{C}/\Gamma$ part of the action, consisting of rotation and translation, acts freely on $\mathbb{C}^*$ and $\mathbb{C}/\Gamma$. Hence in each case, there is an equivariant covering space projection $Y \to Y/N$ by Corollary 4.3.5. If $Y=\mathbb{C}^*$, then $Y \simeq Y/N$ via a power map. If $Y=\mathbb{C}/\Gamma$, then Y/N has genus 1 by the Riemann-Hurwitz formula [Mir95, p. 81-82]; it might be that $Y \not\simeq Y/N$ but this is not important. In each case Y/N is a G-module with respect to the induced action of G by Proposition 4.3.6.

The equivariant map $f: X \to Y$ passes to an equivariant map $f_0: X/N \to Y/N$ making the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle q} \\ X/N & \xrightarrow{f_0} & Y/N \end{array} \tag{4.1}$$

commute. Via the plain h-principle, since X/N is noncompact, there exists a homotopy

$$H_0: X/N \times I \to Y/N, \tag{4.2}$$

not necessarily equivariant, such that $H_0(-,0) = f_0$ and $H_0(-,1)$ is holomorphic. Since Y/N is a G-module, we can apply the equivariant projection to get an equivariant homotopy $H_1 : X/N \times I \to Y/N$; in fact we define

$$H_1(x,t) = \begin{cases} \prod_{g\in G} gH_0(g^{-1}x,t) & \text{if } Y = \mathbb{C}^*, \\ \sum_{g\in G} gH_0(g^{-1}x,t) & \text{if } Y = \mathbb{C}/\Gamma. \end{cases} \tag{4.3}$$

The problem is that $H_1(-,0) = f_0^{|G|}$ or $H_1(-,0) = |G|f_0$: the map is scaled incorrectly. However in both cases, the incorrect scaling arises from a G-equivariant covering map $q' : Y/N \to Y/N$ defined by

$$q'(z) = \begin{cases} z^{|G|} & \text{if } Y = \mathbb{C}^*, \\ |G|z & \text{if } Y = \mathbb{C}/\Gamma. \end{cases} \tag{4.4}$$

By the homotopy lifting property, there exists $H_2 : X/N \times I \to Y/N$ making the diagram

$$\begin{array}{ccc} X/N \times \{0\} & \xrightarrow{f_0} & Y/N \\ \big\downarrow & \overset{H_2}{\nearrow} & \big\downarrow q' \\ X/N \times I & \xrightarrow{H_1} & Y/N \end{array} \tag{4.5}$$

commute. Proposition 4.3.7 implies that H_2 is equivariant. Since $q' : Y/N \to Y/N$ is a local biholomorphism, we deduce that $H_2(-,1)$ is holomorphic. Finally, we again invoke the homotopy lifting property and Proposition 4.3.7 to obtain an equivariant homotopy $H_3 : X \times I \to Y$ making the diagram

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{f} & Y \\ \big\downarrow & \overset{H_3}{\nearrow} & \big\downarrow q \\ X \times I & \xrightarrow{H_2\circ(\pi\times \mathrm{id}_I)} & Y/N \end{array} \tag{4.6}$$

commute. Since $q : Y \to Y/N$ is a local biholomorphism, the map $H_3(-,1)$ is holomorphic. Hence every continuous G-map $X \to Y$ is G-homotopic to a holomorphic G-map. □

Remark 4.3.9. Theorem 4.3.8 is an averaging trick in two parts. Normally the averaging trick takes place in a G-module where multiplication by $|G|$ is invertible, so (4.3) and (4.4) would be done in one step. We are not immediately able to average out the equivariant projection in (4.3) because we need to be careful about how we take $|G|$th roots of holomorphic maps or how we divide by $|G|$ in a complex torus.

4.4 The equivariant Oka principle for $\mathbb{P}^1$

Our main result for $\mathbb{P}^1$ is a partial result that reduces the equivariant Oka principle to an equivariant holomorphic interpolation theorem. Let X denote a noncompact Riemann surface equipped with an effective and holomorphic action of a finite group G and let $A \subset X$ be the set of points of X with nontrivial isotropy. Recall that A is closed and discrete by Proposition 4.2.2, and is also G-invariant. For a given G-map $f : A \to \mathbb{P}^1$, let $[X, \mathbb{P}^1]_G^f$ be the set of equivalence classes of continuous G-map extensions $X \to \mathbb{P}^1$ of $f : A \to \mathbb{P}^1$ that are G-homotopic. Using methods purely from algebraic topology, we show that $[X, \mathbb{P}^1]_G^f$ is a singleton, meaning that any two G-map extensions of f can be equivariantly deformed into the other, such that the deformation is fixed on A. Thus the equivariant Oka principle holds if we can show that for any equivariant map $A \to \mathbb{P}^1$, there exists an equivariant holomorphic extension $X \to \mathbb{P}^1$; this problem is left open.

Independently from this reduction, we are also able to establish the equivariant Oka principle using other methods when the action on the source is free.

4.4.1 Equivariant CW-complexes

We follow tom Dieck [tom87, Chapter II, Section 1]. Though tom Dieck considers a compact Lie group G, we will take G to be finite. Let $D^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ be the closed unit ball in $\mathbb{R}^n$ and $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$ the unit sphere in $\mathbb{R}^n$. Equip these spaces with the trivial G-action. Set $D^0 = \mathrm{pt}$ and $S^{-1} = \emptyset$. The basic building blocks of an equivariant CW-complex are the G-spaces $G/H \times D^n$, where $H < G$ is a subgroup, the group G acts on itself and the set G/H of left cosets by left multiplication, and $G/H \times D^n$ is equipped with the diagonal action.

Let A be a G-space. Fix an integer $n \geq 0$. For a family $(H_j)_{j\in J}$ of subgroups of G, along with G-maps

$$\varphi_j : G/H_j \times S^{n-1} \to A, \tag{4.7}$$

we consider pushout diagrams

$$\begin{array}{ccc} \coprod_{j\in J} G/H_j \times S^{n-1} & \longrightarrow & A \\ \big\downarrow & & \big\downarrow \\ \coprod_{j\in J} G/H_j \times D^n & \xrightarrow{\ \phi\ } & X. \end{array} \tag{4.8}$$

In this case, we say that X is obtained from A by attaching the family of equivariant n-cells $(G/H_j \times D^n)_{j\in J}$ of type $(G/H_j)_{j\in J}$. The map ϕ is called a characteristic map.

Definition 4.4.1. Let (X, A) be a pair of G-spaces. An *equivariant CW-decomposition* of (X, A) consists of a filtration $(X_n)_{n\in\mathbb{Z}}$ of X satisfying the following properties.

1. $A \subset X_0$; $A = X_n$ for $n < 0$; $X = \bigcup_{n\in\mathbb{Z}} X_n$.

2. For each $n \geq 0$, the space X_n is obtained from X_{n-1} by attaching equivariant n-cells.

3. X carries the colimit topology with respect to $(X_n)_{n\in\mathbb{Z}}$.

Definition 4.4.2. If (X_n) is an equivariant CW-decomposition of (X, A), then (X, A) is called a *relative equivariant CW-complex* with respect to the filtration (X_n). If $A = \emptyset$, then X is called an *equivariant CW-complex.* The subspace X_n is called the *n-skeleton* of (X, A).

We immediately sideline this definition to work with a substitute property.

Definition 4.4.3. Let X be a G-space and a plain CW-complex. The group G acts *cellularly* on X if the following properties hold.

(i) For each $g \in G$ and each open cell E of X, the translation gE is an open cell of X.

(ii) If $gE = E$, then the map $E \to E$ defined by $x \mapsto gx$ is the identity.

The next two propositions explain how we can use the concept of a cellular action to obtain an equivariant cell structure on a Riemann surface X equipped with a holomorphic action by a finite group G.

Proposition 4.4.4. *Suppose that X is a CW-complex with filtration $(X_n)_{n\in\mathbb{Z}}$, equipped with a cellular G-action. Then X is an equivariant CW-complex with n-skeleton X_n.*

Proof. See tom Dieck [tom87, Chapter II, Proposition 1.15]. □

Proposition 4.4.5. *Let X be a Riemann surface equipped with the effective and holomorphic action of a finite group G. Let $A \subset X$ be the set of points with nontrivial isotropy. There exists an equivariant relative CW-decomposition of* (X, A).

Proof. Every second countable surface admits a triangulation [AS60, Chapter I, §8]. Take a triangulation of X/G such that A/G is contained in the set of vertices of the triangulation. Lift the triangulation along the branched cover $X \to X/G$ to produce a triangulation of X on which G acts cellularly. Applying Proposition 4.4.4 gives a filtration which can be used to realise (X, A) as a relative equivariant CW-complex. When A is nonempty, instead of attaching all 0-cells to the empty set to form X_0, we instead attach all points with trivial isotropy to A. □

Finally we relate cellular actions to the boundary map and Hurewicz map.

Proposition 4.4.6. *Suppose that G acts cellularly on a CW-complex (X_n). For each n, viewing $g \in G$ as a homeomorphism $g : (X_{n+1}, X_n) \to (X_{n+1}, X_n)$, the boundary map $\partial : \pi_{n+1}(X_{n+1}, X_n, *) \to \pi_n(X_n, *)$ commutes with g_* in the diagram*

$$\begin{array}{ccc} \pi_{n+1}(X_{n+1}, X_n, x_0) & \xrightarrow{g_*} & \pi_{n+1}(X_{n+1}, X_n, gx_0) \\ \downarrow \partial & & \downarrow \partial \\ \pi_n(X_n, x_0) & \xrightarrow{g_*} & \pi_n(X_n, gx_0) \end{array}$$

and the Hurewicz map $\varrho : \pi_{n+1}(X_{n+1}, X_n, *) \to H_{n+1}(X_{n+1}, X_n)$ commutes with g_* in the diagram

$$\begin{array}{ccc} \pi_{n+1}(X_{n+1}, X_n, x_0) & \xrightarrow{g_*} & \pi_{n+1}(X_{n+1}, X_n, gx_0) \\ \downarrow \varrho & & \downarrow \varrho \\ H_{n+1}(X_{n+1}, X_n) & \xrightarrow{g_*} & H_{n+1}(X_{n+1}, X_n). \end{array}$$

Proof. Let $f : (D^{n+1}, S^n, *) \to (X_{n+1}, X_n, x_0)$ represent an element of $\pi_{n+1}(X_{n+1}, X_n, x_0)$. The boundary diagram commutes since $(g \circ f)|S^n = g \circ (f|S^n)$. Let $\widetilde{z}_{n+1}$ be a generator of $H_{n+1}(D^{n+1}, S^n)$. The Hurewicz diagram commutes since $(g \circ f)_*(\widetilde{z}_{n+1}) = g_*(f_*(\widetilde{z}_{n+1}))$. □

4.4.2 Extending maps across cells equivariantly

Fix $n \geq 1$. Let Y be n-simple and $(n-1)$-connected. Let (X, A) be a relative equivariant CW-complex with free action on $X \setminus A$. Let $f : A \to Y$ be a G-map. The main goal of this subsection is to explain how to extend f to an equivariant map $X \to Y$.

Lemma 4.4.7 and Lemma 4.4.8 are technical results used implicitly by tom Dieck in his arguments. We state and prove these explicitly for completeness. Theorem 4.4.10 is proved by tom Dieck [tom87, Chapter II, Theorem 3.15] though we make the exposition of Theorem 4.4.10(i) more explicit.

Lemma 4.4.7. *Let G be a discrete group and $f : G \times S^n \to Y$ an equivariant map. Then f restricts to a null homotopic map $f|\{1\} \times S^n \to Y$ if and only if there exists an equivariant extension $\widetilde{f} : G \times D^{n+1} \to Y$ of f.*

Proof. Let $H : S^n \times I \to Y$ satisfy $H(x, 0) = f(1, x)$ and $H(x, 1) = y_0$ for all $x \in S^n$. Define $\widetilde{f} : G \times D^{n+1} \to Y$ by

$$\widetilde{f}(g, x) = \begin{cases} gH\left(\dfrac{x}{\|x\|}, 1 - \|x\|\right) & \text{if } x \neq 0 \\ gy_0 & \text{if } x = 0. \end{cases}$$

Note that $\widetilde{f}$ is clearly continuous away from $(g, 0)$, so we just need to verify continuity at $(g, 0)$. Let V be a neighbourhood of gy_0. For all $x \in S^n$, the set $H^{-1}g^{-1}V$ is a neighbourhood of $(x, 1)$. So for each x, there exists $U_x \subset S^n$ and $0 < \varepsilon_x < 1$ such that $U_x \times (1 - \varepsilon_x, 1] \subset H^{-1}g^{-1}V$. By compactness of S^n, we reduce to finitely many $U_1, \ldots, U_m$ covering S^n and take $\varepsilon = \min_{1 \le i \le m} \varepsilon_i$. Thus $S^n \times (1 - \varepsilon, 1] \subset H^{-1}g^{-1}V$. Consider the open ball $B(0, \varepsilon) \subset D^{n+1}$. If $x \in B(0, \varepsilon) \setminus \{0\}$, then $1 - \|x\| > 1 - \varepsilon$ so $\widetilde{f}(g, x) = gH(x/\|x\|, 1 - \|x\|) \in V$. By definition $\widetilde{f}(g, 0) = gy_0 \in V$. Since G is discrete, the set $\{g\} \times B(0, \varepsilon) \subset \widetilde{f}^{-1}V$ is a neighbourhood of $(g, 0)$, and $\widetilde{f}$ is continuous at $(g, 0)$.

Conversely, suppose that $\widetilde{f} : G \times D^{n+1} \to Y$ satisfies $\widetilde{f}|G \times S^n \to Y = f$. The map $H : S^n \times I \to Y$ defined $H(x, t) = \widetilde{f}(1, x(1 - t))$ satisfies $H(x, 0) = \widetilde{f}(1, x) = f(1, x)$ and $H(x, 1) \equiv \widetilde{f}(1, 0)$. Thus $f(1, -) : S^n \to Y$ is null homotopic. □

Lemma 4.4.8. *Let $G_j = \{j\} \times G$. If each map $f_{n-1}\varphi_j : G_j \times S^{n-1} \to X_{n-1} \to Y$ restricts to a null homotopic map $S^{n-1} \to Y$, then there exists a G-map extension $f_n : X_n \to Y$.*

Proof. For each $f_{n-1}\varphi_j$, we obtain a G-map extension $\widetilde{f}_{n-1,j} : G_j \times D^n \to Y$ of $f_{n-1}\varphi_j$ by Lemma 4.4.7. Hence the diagram

$$\begin{array}{ccc} \coprod_j G_j \times S^{n-1} & \longrightarrow & X_{n-1} \\ \downarrow & & \downarrow \\ \coprod_j G_j \times D^n & \longrightarrow & Y \end{array}$$

commutes, yielding a G-map extension $f_n : X_n \to Y$ of f_{n-1} since X_n is a pushout. □

Remark 4.4.9. The group action does not play a deep role in either of the technical lemmas given. In particular, Lemma 4.4.7 reduces the extension problem to the nonequivariant setting by solving the problem with respect to the identity of G.

We now arrive at the main extension theorem for the subsection, which can be found in tom Dieck [tom87, Chapter II, Theorem 3.15]. Eventually we want to consider homotopy classes of extensions, but for now we can only say that equivariant extensions to the n-skeleton are equivariantly homotopic on the $(n-1)$-skeleton.

Theorem 4.4.10. *Fix $n \ge 1$. Let Y be n-simple and $(n-1)$-connected. Let (X, A) be a relative equivariant CW-complex with free action on $X \setminus A$. Let $f : A \to Y$ be a G-map.*

(i) *The map $f : A \to Y$ extends to a G-map $X_n \to Y$. Any two equivariant extensions are G-homotopic relative A on X_{n-1}.*

(ii) *Let $k : A \times I \to Y$ be a G-homotopy from $f_0 = k(-, 0)$ to $f_1 = k(-, 1)$ and let $F_0, F_1 : X_n \to Y$ be equivariant extensions of f_0 and f_1. There exists a G-homotopy $K : X_{n-1} \times I \to Y$ from $F_0|X_{n-1}$ to $F_1|X_{n-1}$ extending k.*

Proof. (i) Let $f : A \to Y$ be equivariant. Suppose that the attaching maps for $k = 0$ are given in the pushout diagram

$$\begin{array}{ccc} & & A \\ & & \downarrow \\ \coprod_{j\in J} G_j & \longrightarrow & X_0, \end{array} \tag{4.9}$$

noting that $S^{-1} = \emptyset$ and $D^0 = \mathrm{pt}$. We can define an equivariant map $\coprod_{j\in J} G_j \to Y$ by arbitrarily prescribing values on the identity components $\coprod_{j\in J}\{1\}$, and this will induce an equivariant map $X_0 \to Y$ extending $f : A \to Y$ since Diagram 4.9 is a pushout.

For the extension to X_1 and X_2, apply Lemma 4.4.8 to $G_j \times S^{k-1} \to X_{k-1} \to Y$ in the cases $k = 1, 2$. To deduce that these compositions restrict to null homotopic maps $S^{k-1} \to Y$, it suffices to observe that Y is 1-connected. Hence all maps $S^0 \to Y$ and $S^1 \to Y$ are null homotopic.

For the homotopy relative A on X_{n-1} between two maps $f_0, f_1 : X_n \to Y$ extending f, we consider the relative equivariant CW-complex $(X_n \times I, X_n \times \partial I \cup A \times I)$ along with the map $F : X_n \times \partial I \cup A \times I \to Y$ defined by f_i on $X_n \times \{i\}$ and by $f \times \mathrm{id}_I : A \times I \to Y$ on $A \times I$. The n-skeleton of $(X_n \times I, X_n \times \partial I \cup A \times I)$ is $X_n \times \partial I \cup X_{n-1} \times I$.

(ii) Apply (i) to $(X_{n-1} \times I, X_{n-1} \times \partial I \cup A \times I)$. □

Corollary 4.4.11. *Let X be a Riemann surface and $Y = \mathbb{P}^1$. Suppose that G is a finite group acting holomorphically and effectively on X, and holomorphically on Y. Let $A \subset X_0 \subset X_1 \subset X_2 = X$ be an equivariant triangulation of X, where X_k is the k-skeleton and A is the set of points with nontrivial isotropy. Each equivariant map $f : A \to Y$ admits an equivariant extension to X, and any two such extensions are G-homotopic relative A on X_1.*

Proof. Take $Y = \mathbb{P}^1$ in Theorem 4.4.10. □

The ultimate goal of this section is to show that, given any G-map $f : A \to \mathbb{P}^1$, the set $[X, \mathbb{P}^1]_G^f$ of G-homotopy classes of G-map extensions of f is a singleton. Corollary 4.4.11 allows us to show that any two G-map extensions $f_0, f_1 : X \to \mathbb{P}^1$ of any G-map $A \to \mathbb{P}^1$ are G-homotopic relative A on the 1-skeleton X_1. The remaining task is to extend this to a G-homotopy on the whole surface $X = X_2$. For this, we require equivariant obstruction theory.

4.4.3 Overview of equivariant obstruction theory

We follow tom Dieck [tom87, Chapter II, Section 3]. While tom Dieck takes G to be a compact Lie group, we will take G to be finite so we need not consider the quotient G/G_0 by the identity component G_0. Let (X, A) be a relative equivariant CW-complex with a free G-action on $X \setminus A$; in other words X_n is obtained from X_{n-1} by attaching n-cells of type G. The filtration (X_n) gives a cellular chain complex $C_*(X, A)$ via

$$\cdots \to H_{n+1}(X_{n+1}, X_n) \xrightarrow{d} H_n(X_n, X_{n-1}) \to \dots \tag{4.10}$$

where the homology is ordinary singular homology with coefficients in $\mathbb{Z}$. The boundary map d is the composition of $H_{n+1}(X_{n+1}, X_n) \to H_n(X_n)$ and $H_n(X_n) \to H_n(X_n, X_{n-1})$ in the long exact sequences of the pairs (X_{n+1}, X_n) and (X_n, X_{n-1}).

The cellular G-action on the pair (X_n, X_{n-1}) induces a G-action on $H_n(X_n, X_{n-1})$ by functoriality. Then $H_n(X_n, X_{n-1})$ becomes a $\mathbb{Z}G$-module and (4.10) becomes a complex of $\mathbb{Z}G$-modules. If M is another $\mathbb{Z}G$-module, we can consider the cochain complex

$$\mathrm{Hom}_{\mathbb{Z}G}(C_*(X, A), M) = C^*_G(X, A; M). \tag{4.11}$$

We are then interested in the cohomology groups of this complex, denoted

$$IH^*_G(X, A; M). \tag{4.12}$$

The specific $\mathbb{Z}G$-module we are interested in is $M = \pi_2(\mathbb{P}^1, y_0) \simeq \mathbb{Z}$. Since $\mathbb{P}^1$ is simply connected, it is 2-simple. Also $\mathbb{P}^1$ is path connected. Hence the map $\pi_2(\mathbb{P}^1, y_0) \to [S^2, \mathbb{P}^1]$ forgetting the base point is a bijection by Proposition 2.1.5, and there is a well-defined action of G on $\pi_2(\mathbb{P}^1, y_0)$. Since homotopy classes of continuous maps $S^2 \to \mathbb{P}^1$ are classified by degree, the action of G on $\pi_2(\mathbb{P}^1, y_0)$ is trivial since G acts on $\mathbb{P}^1$ by holomorphic automorphisms.

Let $f_0 : A \to Y$ be a G-map. Let $[X, Y]^{f_0}_G$ be the set of G-homotopy classes relative A of maps $f : X \to Y$ satisfying $f|A = f_0$. The main theorem tom Dieck proves is:

Theorem 4.4.12. *Let Y be an n-simple and $(n-1)$-connected space. There is a bijection $[X_n, Y]^{f_0}_G \to IH^n_G(X, A; \pi_n Y)$.*

Proof. See tom Dieck [tom87, Chapter II, Theorem 3.17]. □

The n-simple assumption is needed to ensure that the map $\pi_n(Y, y_0) \to [S^n, Y]$ forgetting the base point is a bijection. We specialise to $Y = \mathbb{P}^1$ and $n = 2$. When G is a finite group acting holomorphically on Y, and X is a noncompact Riemann surface admitting a holomorphic and effective action of G, the cohomology group $\mathfrak{H}^2_G(X, A; \pi_2 Y)$ is known.

Proposition 4.4.13. *Suppose that X is a Riemann surface equipped with an effective and holomorphic action of a finite group G, and that G acts holomorphically on $\mathbb{P}^1$. Let $A \subset X$ be the set of points with nontrivial isotropy. Then*

$$\mathfrak{H}^2_G(X, A; \pi_2 \mathbb{P}^1) \simeq H^2(X/G, A/G; \mathbb{Z})$$

as $\mathbb{Z}G$-modules, where G acts trivially on $H^2(X/G, A/G; \mathbb{Z})$.

Proof. We obtain an equivariant cell structure on X by first taking a triangulation of the quotient X/G that lifts to an equivariant triangulation of X along the branched covering $X \to X/G$ as in Proposition 4.4.5. Let $(e^n_\lambda)_{\lambda\in\Lambda}$ be the n-cells in the triangulation of X/G. Then for $n > 0$, the n-cells of X can be labelled $(g, e^n_\lambda)_{(g,\lambda)\in G\times\Lambda}$. The n-cells $(e^n_\lambda)_{\lambda\in\Lambda}$ form a $\mathbb{Z}G$-basis for $H_n((X/G)_n, (X/G)_{n-1})$ equipped with the trivial G-action and the n-cells $(1, e^n_\lambda)_{\lambda\in\Lambda}$ form a $\mathbb{Z}G$-basis for $H_n(X_n, X_{n-1})$. Elements of $C^n(X/G, A/G; \mathbb{Z})$ and $C^n_G(X, A; \pi_2\mathbb{P}^1)$ are equivariant functions on these bases with values in $\mathbb{Z}$.

Since G acts holomorphically on $\mathbb{P}^1$, the induced action on $\pi_2\mathbb{P}^1 \simeq \mathbb{Z}$ is trivial. Identify $\varphi \in C^n_G(X, A; \mathbb{Z})$ with the map determined by $\overline{\varphi}(e_\lambda) = \varphi(1, e_\lambda)$. This assignment is equivariant. Conversely for $\varphi \in C^n(X/G, A/G; \mathbb{Z})$, let $\overline{\varphi}$ be the map determined by $\overline{\varphi}(1, e_\lambda) = \varphi(e_\lambda)$. Since $\overline{(-)} : C^n_G(X, A; \mathbb{Z}) \to C^n(X/G, A/G; \mathbb{Z})$ defines a chain map on equivariant cochains for $n > 0$, we get an isomorphism $IH^2_G(X, A; \mathbb{Z}) \simeq H^2(X/G, A/G; \mathbb{Z})$ on cohomology, the latter being isomorphic to ordinary singular cohomology. □

Finally this gives the reduction theorem we aim to prove.

Corollary 4.4.14. *With the assumptions of Proposition 4.4.13, suppose in addition that X is noncompact. Then $[X, Y]^{f_0}_G = \mathrm{pt}$ for each equivariant map $f_0 : A \to Y$.*

Proof. If X is noncompact, then $H^2(X/G, A/G; \mathbb{Z}) = 0$. So Theorem 4.4.12 establishes $[X, Y]^{f_0}_G = \mathrm{pt}$ for each initial G-map $f_0 : A \to Y$. Continuity is a trivial consideration since A is closed and discrete. □

Remark 4.4.15. Corollary 4.4.11 allowed us to prove that any two G-map extensions $X \to Y$ of $f_0 : A \to Y$ are homotopic on the 1-skeleton X_1. With the use of equivariant obstruction theory in Corollary 4.4.14, we are now able to show that any two G-map extensions of f_0 are homotopic on the whole surface $X = X_2$.

We will not rely fully on Theorem 4.4.12. Since $H^2(X/G, A/G; \mathbb{Z}) = 0$, all we need is an injection. Therefore our approach will focus specifically on establishing the claim that two maps $X \to \mathbb{P}^1$ agreeing on the set A of points with nontrivial isotropy are homotopic, and we will explain tom Dieck's obstruction theory only to the extent required to establish this result.

In Section 4.4.4 we examine a certain map $c^{n+1} : [X_n, Y]_G \to C^{n+1}_G(X, A; \pi_n Y)$ sending homotopy classes of equivariant maps to equivariant cochains, fleshing out tom Dieck's exposition where appropriate. While this map has made no appearance in the theorems cited thus far, the cochain map and its properties are central to Theorem 4.4.12. We use this cochain map in Section 4.4.5 to reduce the equivariant Oka principle for $\mathbb{P}^1$ to an equivariant holomorphic interpolation theorem. Our focus is on the cochain map and thus our exposition differs from the more general approach of tom Dieck. We end with Section 4.4.6 wherein we prove a full equivariant Oka principle for $\mathbb{P}^1$ when the action is free on the source.

4.4.4 The cochain map

We continue following tom Dieck [tom87, Chapter II, Section 3]. Let (X, A) be a relative equivariant CW-complex with free action on $X \setminus A$, and denote its k-skeleton by X_k. Fix an integer $n \geq 1$ and assume that Y is path connected and n-simple, so that $\pi_1(Y, y)$ acts trivially on $\pi_n(Y, y)$. The n-simple assumption means that the map $\pi_n(Y, y) \to [S^n, Y]$ forgetting the base point is a bijection, so we omit the base point from our notation $\pi_n Y$.

In this subsection, we are mainly concerned with properties of the cochain map $c^{n+1} : [X_n, Y]_G \to C_G^{n+1}(X, A; \pi_n Y)$ detailed in the following proposition. Recall from Section 2.1 that for a pair (X, A), there is a right action of $\pi_1(A)$ on $\pi_n(X, A)$ commuting with the boundary map. Letting $\pi_n^{\#}(X, A)$ be the quotient of $\pi_n(X, A)$ by the subgroup generated by elements of the form $x - x.\alpha$ with $\alpha \in \pi_1(A)$, the relative Hurewicz theorem (Theorem 2.1.9) states that the Hurewicz map $\pi_n(X, A, *) \to H_n(X, A)$ induces an isomorphism $\pi_n^{\#}(X, A) \simeq H_n(X, A)$.

Proposition 4.4.16. *Suppose that (X_{n+1}, X_n) is n-connected. Let $\varrho : \pi_{n+1}(X_{n+1}, X_n) \to H_{n+1}(X_{n+1}, X_n)$ be the Hurewicz map. Let $h : X_n \to Y$ be a continuous G-map. The map $c^{n+1} : [X_n, Y]_G \to C_G^{n+1}(X, A; \pi_n Y)$ given by the assignment $c^{n+1}(h) = (h_*\partial)^{\#}(\varrho^{\#})^{-1}$ is well defined.*

Proof. Since (X_{n+1}, X_n) is n-connected, we may apply the relative Hurewicz theorem. Since the action of $\pi_1 Y$ on higher homotopy is trivial by the n-simple assumption, Proposition 2.1.6 and Proposition 2.1.7 imply that the quotient map $q : \pi_{n+1}(X_{n+1}, X_n) \to \pi_{n+1}^{\#}(X_{n+1}, X_n)$ satisfies $\ker q \subset \ker h_*\partial$ for any $h : X_n \to Y$. In particular we obtain a map $(h_*\partial)^{\#}(\varrho^{\#})^{-1} : H_{n+1}(X_{n+1}, X_n) \to \pi_n(Y)$ with the relevant maps as defined in the commuting diagram

$$\begin{array}{ccccccc} H_{n+1}(X_{n+1}, X_n) & \xleftarrow{\varrho} & \pi_{n+1}(X_{n+1}, X_n) & \xrightarrow{\partial} & \pi_n(X_n) & \xrightarrow{h_*} & \pi_n(Y) \\ & \nwarrow^{\varrho^{\#}} & \downarrow q & \nearrow_{(h_*\partial)^{\#}} & & & \\ & & \pi_{n+1}^{\#}(X_{n+1}, X_n), & & & & \end{array} \tag{4.13}$$

noting that $\varrho^{\#}$ is an isomorphism by the relative Hurewicz theorem.

Since G acts cellularly on X, the boundary map and Hurewicz map are equivariant in the sense of Proposition 4.4.6. Also h is equivariant. Since $(\varrho^{\#})^{-1}\varrho = q$, we observe that

$$(h_*\partial)^{\#}(\varrho^{\#})^{-1} g_* \varrho = h_*\partial g_* = g_*(h_*\partial)^{\#}(\varrho^{\#})^{-1}\varrho. \tag{4.14}$$

By the right cancellative property of the surjective map ϱ, Equation (4.14) implies that $c^{n+1}(h) = (h_*\partial)^{\#}(\varrho^{\#})^{-1}$ is equivariant. Moreover $c^{n+1}(h)$ is independent of homotopy class because the induced map $h_* : \pi_n(X_n) \to \pi_n(Y)$ is independent of homotopy class. □

Proposition 4.4.17. *The sequence*

$$[X_{n+1}, Y]_G \to [X_n, Y]_G \xrightarrow{c^{n+1}} C_G^{n+1}(X, A; \pi_n Y)$$

is exact: a G-map $h : X_n \to Y$ extends equivariantly to X_{n+1} if and only if $c^{n+1}(h) = 0$.

Proof. Let $\Phi = (\Phi_j) : \coprod_j G \times (D^{n+1}, S^n) \to (X_{n+1}, X_n)$ be a characteristic map. Set $\phi_j = \Phi_j(1, -)$. Letting $\widetilde{z}_{n+1}$ be a generator of $H_{n+1}(D^{n+1}, S^n)$, each $(\phi_j)_* \widetilde{z}_{n+1}$ is a basis element of $H_{n+1}(X_{n+1}, X_n)$ viewed as a G-module. The Hurewicz map sends ϕ_j to $(\phi_j)_* \widetilde{z}_{n+1}$, so $(\varrho^\#)^{-1}(\phi_j)_* \widetilde{z}_{n+1} = [\phi_j]$. Hence $c^{n+1}(h)((\phi_j)_* \widetilde{z}_{n+1}) = h\phi_j | S^n$.

If h admits an equivariant extension $H : X_{n+1} \to Y$, then $h\Phi_j | G \times S^n \to Y$ extends equivariantly to $H\Phi_j : G \times D^{n+1} \to Y$. Hence $h\phi_j | S^n \to Y$ is null homotopic by Lemma 4.4.7. This implies $c^{n+1}(h)((\phi_j)_* \widetilde{z}_{n+1}) = h\phi_j | S^n$ is zero in $\pi_n(Y)$ for each j and hence $c^{n+1}(h) = 0$. Conversely if $c^{n+1}(h) = 0$, then each $h\phi_j | S^n$ is null homotopic, and so h admits an extension to X_{n+1} by Lemma 4.4.8. □

Remark 4.4.18. Proposition 4.4.17 deals with extensions of maps $X_n \to Y$ to the $(n+1)$-skeleton X_{n+1}. The reason we care about the $(n+1)$-skeleton is because we are concerned with homotopy classes of maps. We start with a space (X, A) admitting a relative equivariant CW-decomposition with $X = X_n$. To discuss homotopy, we pass to the relative complex $(X_n \times I, X_n \times \partial I \cup A \times I)$. The $(n+1)$-skeleton of this cylinder is $\widehat{X}_{n+1} = X_n \times I$ while the n-skeleton is $\widehat{X}_n = X_n \times \partial I \cup X_{n-1} \times I$.

Remark 4.4.19. Theorem 4.4.10 tells us under certain conditions on (X, A) and Y that given a G-map $f : A \to Y$, any two G-map extensions $f_0, f_1 : X_n \to Y$ are G-homotopic on the $(n-1)$-skeleton X_{n-1}. A G-homotopy on X_{n-1} from f_0 to f_1 is precisely a G-map from $\widehat{X}_n = X_n \times \partial I \cup X_{n-1} \times I$ to Y. A G-map extension $\widehat{X}_{n+1} \to Y$ to the $(n+1)$-skeleton is then a G-homotopy from f_0 to f_1, and this is our principal concern.

Lemma 4.4.20. *For each $[h] \in [X_n, Y]_G$, the cochain $c^{n+1}(h)$ is a cocycle.*

Proof. The differential map $d_{n+1} : H_{n+1}(X_{n+1}, X_n) \to H_n(X_n, X_{n-1})$ is given by the composition of the boundary map $\partial_{n+1} : H_{n+1}(X_{n+1}, X_n) \to H_n(X_n)$ with the map $p_n : H_n(X_n) \to H_n(X_n, X_{n-1})$ induced from the quotient map $C_n(X_n) \to C_n(X_n)/C_n(X_{n-1})$. The differential $\delta^n : C_G^n(X, A; \pi_n Y) \to C_G^{n+1}(X, A; \pi_n Y)$ is defined by precomposition, and so $\delta^{n+1}(c^{n+1}(h)) = (h_* \partial_{n+1})^\# (\varrho_{n+1}^\#)^{-1} p_{n+1} \partial_{n+2}$.

Commutativity of the diagram

$$\begin{array}{ccccccc}
H_{n+2}(X_{n+2}, X_{n+1}) & \xleftarrow{\varrho_{n+2}} & \pi_{n+2}(X_{n+2}, X_{n+1}) & & & & \\
\downarrow \partial_{n+2} & & \downarrow \partial_{n+2} & & & & \\
H_{n+1}(X_{n+1}) & \xleftarrow{\varrho_{n+1}} & \pi_{n+1}(X_{n+1}) & & & & \\
\downarrow p_{n+1} & & \downarrow j_{n+1} & & & & \\
H_{n+1}(X_{n+1}, X_n) & \xleftarrow{\varrho_{n+1}} & \pi_{n+1}(X_{n+1}, X_n) & \xrightarrow{\partial_{n+1}} & \pi_n(X_n) & \xrightarrow{h_*} & \pi_n(Y)
\end{array}$$

implies that $\delta^{n+1}(c^{n+1}(h)) \circ \varrho_{n+2} = h_* \partial_{n+1} j_{n+1} \partial_{n+2} = 0$, noting that $\partial_{n+1} j_{n+1} \partial_{n+2} = 0$. Since ϱ_{n+2} is surjective, we have $\delta^{n+1}(c^{n+1}(h)) = 0$. □

Remark 4.4.21. Having established that $c^{n+1}(h)$ is a cocycle, we can understand it up to coboundary via the cohomology group $\mathfrak{H}_G^{n+1}(X, A; \pi_n Y)$.

Let $f_0, f_1 : X_n \to Y$ be G-maps and let $k : X_{n-1} \times I \to Y$ be a G-homotopy from f_0 to f_1 on X_{n-1}. Consider the relative G-CW-complex $(\widehat{X}, \widehat{A}) = (X_n \times I, X_n \times \partial I \cup A \times I)$ with n-skeleton $\widehat{X}_n = X_n \times \partial I \cup X_{n-1} \times I$ and $(n+1)$-skeleton $\widehat{X}_{n+1} = X_n \times I$. The attaching maps $\widehat{\phi}_j : G \times (D^{m+1}, S^m) \to (\widehat{X}_{m+1}, \widehat{X}_m)$ for $m \leq n$ are obtained from the attaching maps $\phi_j : G \times (D^m, S^{m-1}) \to (X_m, X_{m-1})$ by fixing a homeomorphism $\psi : (D^{m+1}, S^m) \to (D^m \times I, D^m \times \partial I \cup S^{m-1} \times I)$ and setting $\widehat{\phi}_j = (\phi_j \times \mathrm{id}_I)(\mathrm{id}_G \times \psi)$. Explicitly $\widehat{\phi}_j(g, x) = (\phi_j(g, \mathrm{pr}_1 \psi(x)), \mathrm{pr}_2 \psi(x))$. The maps f_0, k, f_1 define a map $F : \widehat{X}_n \to Y$. This gives an obstruction cocycle

$$c^{n+1}(F) \in \mathrm{Hom}_{\mathbb{Z}G}(H_{n+1}(\widehat{X}_{n+1}, \widehat{X}_n), \pi_n Y)$$

There is an equivariant isomorphism $\sigma_{\widetilde{z}_1} : H_n(X_n, X_{n-1}) \to H_{n+1}(\widehat{X}_{n+1}, \widehat{X}_n)$ that induces an isomorphism $\mathfrak{H}_G^n(X, A; \pi_n Y) \simeq \mathfrak{H}_G^{n+1}(\widehat{X}, \widehat{A}; \pi_n Y)$ on cohomology, both of which are referred to as the suspension isomorphism. The map $\sigma_{\widetilde{z}_1}$ arises from the external homology product as detailed in Dold [Dol72, Chapter VII, Section 2] and tom Dieck [tom08, Section 9.8], while a full exposition of the induced map on cohomology can be found in tom Dieck [tom87, Chapter II, (3.5)-(3.9)].

Composing with the suspension isomorphism $\sigma_{\widetilde{z}_1} : H_n(X_n, X_{n-1}) \to H_{n+1}(\widehat{X}_{n+1}, \widehat{X}_n)$ gives an element

$$c^{n+1}(F) \circ \sigma_{\widetilde{z}_1} =: d(f_0, k, f_1) \in \mathrm{Hom}_{\mathbb{Z}G}(H_n(X_n, X_{n-1}), \pi_n Y)$$

called the equivariant difference cochain.

We will now cite an important result relating $c^{n+1}(f_0)$ and $c^{n+1}(f_1)$. The proof is given in full detail by tom Dieck, and comes down to diagram chases involving the suspension isomorphism.

Proposition 4.4.22. *Suppose that $f_0, f_1 : X_n \to Y$ are G-maps that are G-homotopic on X_{n-1} via $k : X_{n-1} \times I \to Y$. Then $\delta^n d(f_0, k, f_1) = c^{n+1}(f_0) - c^{n+1}(f_1)$.*

Proof. See tom Dieck [tom87, Chapter II, Lemma 3.14]. □

There are four parts to the equivariant difference cochain: the initial G-map f_0, the G-homotopy k on X_{n-1}, the final G-map f_1, and the equivariant cochain $\xi = d(f_0, k, f_1)$. Normally we start with the first three as input data and end up with the cochain as output. The next proposition, which acts as the final ingredient for our reduction argument, takes the initial G-map f_0, the G-homotopy k, and the equivariant cochain ξ as input, then produces a final G-map f_1 for which $d(f_0, k, f_1) = \xi$.

Proposition 4.4.23. *For each G-map $f_0 : X_n \to Y$, each G-homotopy $k : X_{n-1} \times I \to Y$ on X_{n-1} satisfying $k(-, 0) = f_0|X_{n-1}$, and each equivariant cochain $\xi \in C^n_G(X, A; \pi_n Y)$, there exists a G-map $f_1 : X_n \to Y$ such that $f_1|X_{n-1} = k(-, 1)$ and $d(f_0, k, f_1) = \xi$.*

Proof. We adapt Whitehead [Whi78, Chapter V, Lemma 5.10, Lemma 5.12] to the equivariant setting. Let $E_0 = D^n \times 0 \cup S^{n-1} \times I \simeq D^n$ and $E_1 = D^n$. Glue these together by identifying ∂E_1 with $S^{n-1} \times 1 \subset E_0$. Then $E_0 \cup E_1 \simeq S^n$, and $E_0 \hookrightarrow E_0 \cup E_1$ is a cofibration. Let $\xi \in C^n_G(X, A; \pi_n Y)$ be an equivariant cochain. For each attaching map $\phi_j : G \times (D^n, S^{n-1}) \to (X_n, X_{n-1})$, we obtain a basis element $e_j \in H_n(X_n, X_{n-1})$. Then $\xi(e_j) \in \pi_n Y$. For each j, take any representative $h_{0j} : E_0 \cup E_1 \to Y$ of $\xi(e_j)$.

The maps $f^1_{0j}(x) = f_0\phi_j(1, x)$ for $x \in D^n$ and $k^1_j(x, t) = k(\phi_j(1, x), t)$ for $x \in S^{n-1}$ together define a map $F_{0j} : E_0 \to Y$ such that $F_{0j}|E_0 \cap E_1 = k^1_j(-, 1)$. Since E_0 is contractible and Y is path connected, there exists a homotopy $E_0 \times I \to Y$ from $h_{0j}|E_0$ to F_{0j}. Extending to $E_0 \cup E_1 \times 0$ via h_{0j} and applying the homotopy extension property of the cofibration $E_0 \hookrightarrow E_0 \cup E_1$ to the diagram

$$\begin{array}{ccc} E_0 \cup E_1 \times 0 \cup E_0 \times I & \longrightarrow & Y \\ \big\downarrow & \overset{H_j}{\nearrow} & \\ E_0 \cup E_1 \times I & & \end{array}$$

gives a homotopy $H_j : E_0 \cup E_1 \times I \to Y$ such that $H_j(-, 0) = h_{0j}$ and $H_j(-, 1)|E_0 = F_{0j}$. Define $f_{1j} : G \times D^n \to Y$ by setting $f_{1j}(g, x) = gH_j(x, 1)$, viewing $(E_1, E_1 \cap E_0)$ as (D^n, S^{n-1}). If $x \in S^{n-1} = E_1 \cap E_0$, then $f_{1j}(g, x) = gF_{0j}(x)$ because $x \in E_0$. Now observe that $gF_{0j}(x) = gk^1_j(x, 1)$ because $x \in E_1$. Finally, equivariance of k and ϕ_j give $gk^1_j(x, 1) = k(\phi_j(g, x), 1)$. We conclude that $f_{1j}(g, x) = k(\phi_j(g, x), 1)$ for each j if $x \in S^{n-1}$, which implies that the diagram

$$\begin{array}{ccc} \coprod_j G \times S^{n-1} & \xrightarrow{(\phi_j)} & X_{n-1} \\ \big\downarrow & & \big\downarrow k(-,1) \\ \coprod_j G \times D^n & \xrightarrow{(f_{1j})} & Y \end{array}$$

commutes. Since X_n is a pushout, we obtain a G-map $f_1 : X_n \to Y$ uniquely determined by the equations $f_1|X_{n-1} = k(-, 1)$ and $f_1\phi_j = f_{1j}$ for all j. By construction $H_j(-, 0) = h_{0j}$ and $H_j(-, 1) = d(f_0, k, f_1)(e_j)$. In fact $H_j(-, 1)|E_0 = F_{0j}$, which is defined by $f_0\phi_j(1, -)$ and $k(\phi_j(1, -) \times \mathrm{id}_I)$. Then $H_j(-, 1)|E_1 = f_{1j}(1, -) = f_1\phi_j(1, -)$. Thus H_j is a homotopy that establishes $d(f_0, k, f_1)(e_j) = \xi(e_j)$ in $\pi_n Y$. Hence $d(f_0, k, f_1) = \xi$. □

4.4.5 Reduction to interpolation

Our main result regarding the equivariant Oka principle for $\mathbb{P}^1$ is a reduction to a holomorphic interpolation theorem.

Theorem 4.4.24. *Let X be a noncompact Riemann surface. Suppose that a finite group G acts holomorphically and effectively on X and holomorphically on $Y = \mathbb{P}^1$. Let $A \subset X$ be the set of points with nontrivial isotropy. If $f_0, f_1 : X \to Y$ are two continuous G-maps agreeing on A, then f_0 and f_1 are G-homotopic relative A.*

Proof. By Proposition 4.4.5, the pair (X, A) admits a relative G-CW structure. Now consider the relative G-CW-complex $(\widehat{X}, \widehat{A}) = (X \times I, X \times \partial I \cup A \times I)$. Since f_0 and f_1 agree on A, there exists a G-homotopy $X_{n-1} \times I \to Y$ relative A from f_0 to f_1 by Corollary 4.4.11. View this as a G-map $F_0 : \widehat{X}_2 \to Y$ on the 2-skeleton $\widehat{X}_2 = X_2 \times \partial I \cup X_1 \times I$.

Apply the cochain map $c^3 : [\widehat{X}_2, Y] \to C^3_G(\widehat{X}, \widehat{A}; \pi_2 Y)$, recalling that $c^3(F_0) = 0$ if and only if F_0 admits a G-map extension to $\widehat{X}_3 = X \times I$. As it turns out, in general, we can only guarantee that $c^3(F_0) = 0$ in $\mathfrak{H}^3_G(\widehat{X}, \widehat{A}; \pi_2 Y)$, which is to say that $c^3(F_0)$ is a coboundary. This result follows from the isomorphisms

$$\mathfrak{H}^3_G(\widehat{X}, \widehat{A}; \pi_2 Y) \simeq \mathfrak{H}^2_G(X, A; \pi_2 Y) \simeq H^2(X/G, A/G; \mathbb{Z}) = 0.$$

The first isomorphism is the suspension isomorphism while the second isomorphism is proved in Proposition 4.4.13. From this isomorphism, there exists $\xi \in C^2_G(\widehat{X}, \widehat{A}; \pi_2 Y)$ such that $c^3(F_0) = \delta^2 \xi$. Now we use Proposition 4.4.23 to produce a G-map $F_1 : \widehat{X}_2 \to Y$ such that $d(F_0, k, F_1) = \xi$, where $k : \widehat{X}_1 \times I \to Y$ is the constant G-homotopy.

Proposition 4.4.22 gives $\delta^2 d(F_0, k, F_1) = c^3(F_0) - c^3(F_1)$. Since $\delta^2 \xi = c^3(F_0)$, we deduce that $c^3(F_1) = 0$. Hence Proposition 4.4.17 implies that F_1 extends equivariantly to $\widehat{X}_3 = \widehat{X} = X \times I$, yielding a G-homotopy from f_0 to f_1. □

Corollary 4.4.25. *If every G-map $A \to Y$ extends to a holomorphic G-map $X \to Y$, then every continuous G-map $X \to Y$ is G-homotopic to a holomorphic map.*

Proof. Let $f : X \to Y$ be a continuous G-map. Take a holomorphic G-map $g : X \to Y$ agreeing with f on A. Theorem 4.4.24 establishes that f is G-homotopic to g. □

The structure can be further specified if we insist that G acts effectively on the source and the target.

Theorem 4.4.26. *Let X be a noncompact Riemann surface. Let G be a finite group acting holomorphically and effectively on X and $\mathbb{P}^1$. Let $A \subset X$ be the set of points of X with nontrivial isotropy. There exists a bijection of sets*

$$[X, Y]_G = [X, Y]^A_G \simeq \mathrm{Map}_G(A, Y)$$

between equivariant homotopy classes of maps $[X, Y]^A_G$ relative A and equivariant maps $\mathrm{Map}_G(A, Y)$. Furthermore, every equivariant homotopy of maps $X \to Y$ is fixed on A.

Proof. Any two G-maps agreeing on A are G-homotopic relative A by Theorem 4.4.24. So the assignment $[X,Y]_G^A \to \mathrm{Map}_G(A,Y)$ given by $[f] \mapsto f|_A$ is injective. Surjectivity comes from being able to extend any G-map $A \to Y$ to X equivariantly via Theorem 4.4.10.

When G acts effectively on the target, the set of $y \in Y$ such that $|G_y| > 1$ is discrete. Suppose that $a \in A$ so $|G_a| > 1$. Let $H : X \times I \to Y$ be any equivariant homotopy. Fix $t_0 \in I$. Take a neighbourhood V of $H(a,t_0)$ such that $H(a,t_0) \in V$ is the only point in V with nontrivial isotropy. The preimage $U = H(a,-)^{-1}V$ is a neighbourhood of t_0 such that $|G_{H(a,t)}| \geq |G_a| > 1$ for all $t \in U$. Since the only point in V with nontrivial isotropy is $H(a,t_0)$, we deduce that $H(a,t) = H(a,t_0)$ for all $t \in U$. Thus $H(a,-) : I \to Y$ is locally constant and hence constant. Therefore H is fixed on A, and $[X,Y]_G = [X,Y]_G^A$. □

The assumption that G acts effectively on X is required to ensure that G acts freely on $X \setminus A$, which is the main assumption underpinning all of the equivariant obstruction theory established thus far. If we do not assume that G acts effectively on the source, then we can produce trivial examples where $\mathscr{C}_G(X,\mathbb{P}^1) = \emptyset$.

Proposition 4.4.27. *Let D_n act dihedrally on $\mathbb{P}^1$ via the automorphisms $z \mapsto \exp(2\pi i/n)z$ and $z \mapsto 1/z$. Let X be any noncompact Riemann surface equipped with the trivial action of D_n. Then $\mathscr{C}_G(X,\mathbb{P}^1) = \emptyset$.*

Proof. An equivariant map ϕ must satisfy $\phi(x) = \phi(gx) = g\phi(x)$ for all $g \in D_n$. So the existence of an equivariant map implies the existence of a point of $\mathbb{P}^1$ fixed by all of D_n. No such point exists: the only points fixed by $z \mapsto \exp(2\pi i/n)$ are 0 and ∞, but these are swapped by $z \mapsto 1/z$. □

Remark 4.4.28. More generally, suppose that X and Y are G-spaces. If X has the trivial G-action and no point of Y is fixed by G, then $\mathscr{C}_G(X,Y) = \emptyset$.

Remark 4.4.29. Theorem 4.4.26 also includes interpolation on the discrete set A as a necessity due to equivariance. In the nonequivariant setting, an elementary proof of the Oka principle with interpolation for Stein-Oka pairings of Riemann surfaces was given by Crawford [Cra14, Theorem 3.1.5]. The method of proof is similar. First Crawford shows that maps $X \to \mathbb{P}^1$ agreeing on a discrete set A are homotopic relative A, a similar result to our Theorem 4.4.24. However, in the nonequivariant setting, there is a holomorphic interpolation theorem for maps $X \to \mathbb{P}^1$ with X noncompact. Let $f : X \to \mathbb{P}^1$ be continuous. Since the restriction $f|A \to \mathbb{P}^1$ always fails to be surjective because A is countable, Crawford takes $x \in \mathbb{P}^1 \setminus fA$ and invokes the Weierstrass interpolation theorem to obtain a holomorphic map $g : X \to \mathbb{C} \simeq (\mathbb{P}^1 \setminus \{x\}) \hookrightarrow \mathbb{P}^1$ agreeing with f on A.

We end the section with two special situations where we can establish an equivariant interpolation result, and hence an equivariant Oka principle by Theorem 4.4.24.

Proposition 4.4.30. *Set $\omega_n = \exp(2\pi i/n)$. Let $G = \langle \omega_n z, 1/z \rangle$ act on $\mathbb{C}^*$ and $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. Let $A = \mathcal{W}_{2n}$ be the $2n$th roots of unity, the points of $\mathbb{C}^*$ with nontrivial isotropy. For every G-map $A \to \mathbb{P}^1$, there exists a holomorphic G-map extension $\mathbb{C}^* \to \mathbb{P}^1$. Thus every continuous G-map $\mathbb{C}^* \to \mathbb{P}^1$ is G-homotopic to a holomorphic map.*

Proof. An equivariant map $f : A \to \mathbb{P}^1$ is determined by $f(1)$ and $f(\omega_{2n})$ because $\mathcal{W}_{2n}$ splits into the even and odd $2n$th roots of unity when we quotient by G. Also, since f is equivariant, we must have $G_x \subset G_{f(x)}$. Observe that, for $0 \le k < n$, the stabiliser of $\pm\omega_{2n}^k$ is $\{z, \omega_n^k/z\}$. Conversely the group $\{z, \omega_n^k/z\}$ fixes only $\pm\omega_{2n}^k$. Hence f sends $\{\pm\omega_{2n}^k\}$ into itself. So there are four equivariant maps $A \to \mathbb{P}^1$, determined by the assignments

$$\begin{aligned}(1, \omega_{2n}) &\mapsto (1, \omega_{2n})\\ (1, \omega_{2n}) &\mapsto (1, -\omega_{2n})\\ (1, \omega_{2n}) &\mapsto (-1, \omega_{2n})\\ (1, \omega_{2n}) &\mapsto (-1, -\omega_{2n}).\end{aligned}$$

These assignments are realised by the equivariant holomorphic maps

$$\begin{aligned}z &\mapsto z\\ z &\mapsto 1/z^{n-1}\\ z &\mapsto -1/z^{n-1}\\ z &\mapsto -z.\end{aligned}$$

□

Proposition 4.4.31. *Let G be a finite group acting holomorphically on a noncompact Riemann surface X and the Riemann sphere $\mathbb{P}^1$, such that G fixes exactly two points $z_1, z_2 \in \mathbb{P}^1$. Let $A \subset X$ be a closed discrete G-invariant subset and $f_0 : A \to \mathbb{P}^1$ a nonconstant G-map. There exists an equivariant holomorphic map $X \to \mathbb{P}^1$ extending f_0.*

Proof. When G fixes exactly two points of $\mathbb{P}^1$, it is conjugate to a rotation group and the action can be linearised. In this setting, we may use the averaging trick combined with the plain Weierstrass theorem for maps into $\mathbb{P}^1$.

Let $s \in \mathrm{Aut}(\mathbb{P}^1)$ be a transformation sending z_1 to 0 and z_2 to ∞. The image of the conjugate group sGs^{-1} is generated by $\omega_n z$, where $\omega_n = \exp(2\pi i/n)$. Let $f : X \to \mathbb{P}^1$ be a holomorphic map such that $f|_A = s \circ f_0$. In particular f is nonconstant, so we can consider it as an element of the $\mathbb{C}$-algebra $\mathscr{M}(X)$ of meromorphic functions over X. Define $F : X \to \mathbb{P}^1$ by

$$F = \frac{1}{|G|} \sum_{g \in G} (sgs^{-1})^{-1} \circ f \circ g.$$

The map $s^{-1} \circ F : X \to \mathbb{P}^1$ is holomorphic and G-equivariant, restricting to f_0 on A. □

Remark 4.4.32. The nonconstant assumption is no real loss of generality. If $f_0 : A \to \mathbb{P}^1$ is constant, then there exists $y_0 \in \mathbb{P}^1$ such that $gy_0 = gf_0(x) = f_0(gx) = y_0$ for all $g \in G$ and all $x \in A$. So the constant map $c_{y_0} : X \to \mathbb{P}^1$ is an equivariant holomorphic map restricting to f_0 on A.

4.4.6 Free action on the source

When G acts freely on the source, the equivariant Oka principle follows from the theory of principal G-bundles.

Theorem 4.4.33. *Let X be a noncompact Riemann surface. Let G be a finite group acting holomorphically on X and $\mathbb{P}^1$. Let $Q_0 \subset X/G$ be the set of orbits containing points with nontrivial isotropy. Let $S = X/G \setminus Q_0$. There exists a homeomorphism $\mathscr{C}_G(\pi^{-1}S, \mathbb{P}^1) \simeq \mathscr{C}(S, \mathbb{P}^1)$ with respect to the compact open topology restricting to a homeomorphism $\mathscr{O}_G(\pi^{-1}S, \mathbb{P}^1) \simeq \mathscr{O}(S, \mathbb{P}^1)$. In particular, if G acts freely on X, then $\mathscr{O}_G(X, \mathbb{P}^1) \simeq \mathscr{O}(X/G, \mathbb{P}^1)$.*

Proof. The covering space $\pi^{-1}S \to S$ is a principal G-bundle, so we may construct the associated bundle $E = (\pi^{-1}S \times \mathbb{P}^1)/G \to S$ with fibre $\mathbb{P}^1$. Supposing that (U_i, l_i) is a holomorphic local trivialisation of $\pi^{-1}S \to S$, we obtain a holomorphic local trivialisation (U_i, h_i) of $(\pi^{-1}S \times \mathbb{P}^1)/G \to S$ with $h_i[x, y] = ([x], \mathrm{pr}_2\, l_i(x)^{-1}y)$. Since the trivialisations $\mathrm{pr}_2\, l_i : \pi^{-1}U_i \to G$ are equivariant, each transition $(\mathrm{pr}_2\, l_i)^{-1} \cdot (\mathrm{pr}_2\, l_j)$ is G-invariant and induces a map $U_i \cap U_j \to G$ from the quotient. Since every projective bundle over a noncompact Riemann surface is trivial by Corollary 2.5.13, we obtain a holomorphic splitting $t_i : U_i \to PSL(2, \mathbb{C})$ of $(\mathrm{pr}_2\, l_i)^{-1} \cdot (\mathrm{pr}_2\, l_j) : U_i \cap U_j \to G$. The maps $(\mathrm{pr}_2\, l_i) \cdot t_i\pi : \pi^{-1}U_i \to PSL(2, \mathbb{C})$ glue to form a global map $T : \pi^{-1}S \to PSL(2, \mathbb{C})$ equivariant with respect to G acting on $PSL(2, \mathbb{C})$ by left multiplication. The homeomorphism $\mathscr{C}_G(\pi^{-1}S, \mathbb{P}^1) \simeq \mathscr{C}(S, \mathbb{P}^1)$ is defined as follows. We send $F \in \mathscr{C}_G(\pi^{-1}S, \mathbb{P}^1)$ to the map $S \to \mathbb{P}^1$ induced by $T(x)^{-1}(F(x))$. In the other direction, we send $f \in \mathscr{C}(S, \mathbb{P}^1)$ to the map $\pi^{-1}S \to \mathbb{P}^1$ defined by $T(x)(f[x])$. □

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