Langlands Duality in the Generic Fibres of Classical Hitchin Fibrations

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Contents

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Dedication

The author dedicates this thesis to the memory and undeniable legacy of Stephanie Anne Flint. A mother, wife, athlete, extrovert, and loved community member of Wallaroo. In short, a beautiful soul who left this earth too soon. Hopefully, heaven has been kind and provided a generous supply of Bird in Hand sparkling wine to enjoy.

In Loving Memory
of
Stephanie Anne Flint

31/03/1961 ∼ 16/02/2017

Abstract

We study L-twisted endomorphisms of holomorphic vector bundles over a compact Riemann surface. In particular, we generalise Hitchin's computation of the generic fibres of the Hitchin fibration to the more general L-twisted case, which presents new challenges since we lose properties intrinsic to the canonical bundle. However, we impose a basepoint-free condition on L to use classical theorems, such as Bertini's theorem, to ensure the generic spectral curves are either smooth or have mild singularities. Moreover, we compute the generic fibres for each classical simple Lie group and GL_n. Akin to Hitchin's result, the fibres are torsors of abelian varieties, and we demonstrate Langlands duality in the Hitchin fibration by proving the generic fibres corresponding to Langland's dual groups are indeed dual abelian varieties.

The thesis consists of two parts. In Part I, we review the necessary background of complex algebraic geometry. Chapter 1 provides an account of the rudiments of divisors and holomorphic vector bundles. Chapter 2 extends the work of Chapter 1 and restricts to the case of compact Riemann surfaces, which lays the foundation for Higgs bundles. Chapter 3 reviews the necessary definitions and details surrounding complex abelian varieties, which we need to study the generic fibres of the Hitchin fibration and the duality.

In Part II, we compute the generic fibres of the Hitchin fibrations. Chapter 4 computes GL_n and type A_n Hitchin fibrations, and demonstrates the self-duality in GL_n, and duality between SL_n and PGL_n. Chapter 5 computes the generic fibres for Sp_{2n} and SO_{2n+1} and demonstrates the Langlands duality in the generic fibres. Chapter 6 computes the generic fibres for SO_{2n} and demonstrates the self-duality.

In the SO_{2n+1} and SO_{2n} cases, generic spectral curves are singular. We resort to normalising the spectral curve in the SO_{2n} case, and in the SO_{2n+1} case, we use explicit local calculations and Hecke modifications. As far as the author is aware, this is a new approach to the SO_{2n+1} computation, as other methods resort to extension classes.

In Appendix A, we recount some results in classical Lie theory for complex simple Lie groups and provide a computation of the Langlands dual group in each classical case.

Appendix B consists of a self-contained proof of a property specific to abelian varieties that one could take as a given throughout the project.

Introduction

This thesis answers a classification problem concerning Higgs bundles from the perspective of algebraic geometry. Before introducing the problem it is profitable to understand where Higgs bundles originate. In 1987, Nigel Hitchin's study of self-dual Yang Mills equations on compact Riemann surfaces in [Hit87a] lead to the celebrated *Hitchin equations*, which are fundamental in Yang Mills theory. Solving the Hitchin equations lead to the notion of a *Higgs bundle*, a pair (E, ϕ) where E is holomorphic vector bundle over a compact Riemann surface C, and ϕ is an auxiliary Higgs field, i.e., a K_C-twisted endomorphism of E where $K_C = T^*C$. Further, changing the complex reductive Lie group dictating the Hitchin equations leads to Higgs bundles with more structure. For example, if the structure group is given by SL_n, then the Higgs bundles (E, ϕ) satisfies the further condition that $\det(E) \cong \mathcal{O}_C$ and $\mathrm{tr}(\phi) = 0$. In the same paper, for each classical G, Hitchin constructed a moduli space of Higgs bundles $\mathcal{M}_G$ subject to a stability and topological condition. The space has a rich geometric structure and has been a powerful tool that enables the study of one field using techniques and tools from a different field. Most notably, the moduli space was used by Simpson and Corlette in [Cor88, Sim92, Sim91] to establish the celebrated nonabelian Hodge correspondence, which relates a certain family of Higgs bundles, flat connections, and semisimple representations of the fundamental group $\pi_1(C)$ thereby relating algebraic geometry, differential geometry, and representation theory.

In the same year Nigel Hitchin published a paper [Hit87b] where he introduced *spectral curves* associated to Higgs bundles, which is a generalised notion of eigenvalues. Also, he defined a morphism $h : \mathcal{M}_G \to B$ that has since been coined the *Hitchin fibration*. Here B is a priori an algebraic variety parameterising the space of spectral curves associated to G-Higgs bundles and two Higgs bundles belong to the same fibre if and only if they share the same spectral curve. Hitchin proved that the Hitchin fibration is an algebraic completely integrable system in the sense of Hamiltonian mechanics. Consequently, the generic fibres are torsors of complex abelian varieties and are also Lagrangian submanifolds. Following on from this result Hitchin began classifying the generic fibres in classical cases. For example, it was shown that for GL_n the generic fibres are torsors of Jacobian varieties, and for SO_{2n} the generic fibres are torsors of Prym varieties associated to an étale double covering.

In 2003, Hausel and Thaddeus conducted the analogous classification for SL_n and PGL_n and found that the generic fibres are dual Prym varieties, see [HT03]. Five years later Hitchin proved that the generic fibres for SO_{2n+1} and Sp_{2n} are dual Prym varieties as well, see [Hit07]. All of these results showed that the Hitchin fibration is deeply intertwined with Langlands duality since passing to the Langlands dual group amounts to dualising the underlying abelian variety for each generic fibre. In 2011, Donagi and Pantev proved in [DP12] that Langlands duality interacts with the Hitchin fibration for arbitrary G. That is, given a complex reductive Lie group G the underlying abelian variety of the generic fibres of the G-Hitchin fibration is dual to the underlying abelian variety of the generic fibres of the ${}^LG$-Hitchin fibration. Their result is the most general and as such required incredibly sophisticated techniques and new objects. However, their results only demonstrate that the varieties are dual and do not inform the reader what the varieties are explicitly. Not knowing what the varieties are explicitly has motivated mathematicians to continue computing the fibres for different G. For instance, in 2021 Mukhopadhyay and Wentworth computed the generic fibres expliclty for the Langlands dual groups Spin_{2n} and PSO_{2n}, and Spin_{2n+1} and $\mathrm{PSp}_{2\mathrm{n}}$, respectively, see [MW21].

The aforementioned computations in the classical cases have details missing that are left to the reader to fill-in. Many of the classifications rely on intrinsic properties of the canonical bundle K_C and do not explain to the reader how the classifications generalise to the case where the canonical bundle is replaced by an arbitrary holomorphic line bundle $L \to C$. Some mathematicians have outlined the analogous classifications in some classical cases for when K_C is replaced by L, e.g., [BNR89]. However, again most the details are missing. Our goal in this research project is to give a complete classification of the generic fibres of the G-Hitchin fibration for each classical G and for when K_C is replaced by L. Moreover, we compute the number of connected components in each case as well as the dimension of the fibres. Further, we demonstrate Langlands duality in this more general Hitchin fibration by showing the generic fibres of the G and ${}^LG$ Hitchin fibrations are dual abelian varieties for each classical G.

The thesis consists of two parts. Part I provides a detailed overview of complex algebraic geometry establishing results that are needed for the classifications. Chapter 1 reviews divisors and holomorphic vector bundles over complex manifolds and establishes how divisors canonically define a holomorphic line bundle. Chapter 2 specialises the results established in Chapter 1 to Riemann surfaces and introduces the celebrated Riemann-Roch theorem, which is a fundamental result relating the topological index and analytical index of the canonical differential operator. Chapter 3 gives a detailed overview of complex abelian varieties including establishing the dual abelian variety and defining Jacobian and Prym varieties. The results in Chapter 3 lay the foundation to study the generic fibres.

Part II consists of the classification problems using the results established in Part I. Chapter 4 introduces G-Higgs bundles and the G-Hitchin fibration where the Higgs field is L-valued opposed to K_C-valued. Here, G is an arbitrary complex reductive Lie group. Moreover, Chapter 4 defines spectral curves, a special type of one-dimensional complex analytic subvariety of the total space of L, and establishes the spectral curve correspondence, which details the relationship between generic spectral curves, which are smooth for GL_n and type A_n, and Higgs bundles. In Chapter 4, the spectral curve correspondence is used to classify the generic fibres of the Hitchin fibration for when $G = \mathrm{GL}_n, \mathrm{SL}_n$, and PGL_n. The generic fibres in GL_n are torsors of a Jacobian variety, which is self-dual, and the underlying abelian variety for the generic fibres for SL_n and PGL_n are dual Prym varieties, which verifies Langlands duality for GL_n and type A_n. Chapter 5 expands on results from Chapter 4 and considers the case where $G = \mathrm{SO}_{2n+1}$ and Sp_{2n}. However, for SO_{2n+1} generic spectral curves are not smooth, which presents a problem. To overcome this issue we resort to local calculations and Hecke modifications near each singularity, which, to the best of the authors knowledge, is a new method used in the B_n classification. Generic spectral curves are smooth for the Sp_{2n} case and it is shown that the underlying abelian varieties for the generic fibres of the SO_{2n+1} and Sp_{2n} Hitchin fibrations are dual Prym varieties, which verifies Langlands duality in this instance. Chapter 6 concludes the thesis by classifying the generic fibres for the SO_{2n} case and the underlying abelian variety of the generic fibres is a Prym variety associated to an étale double cover, which is self-dual. This completes Langlands duality in the Hitchin fibration in each classical case.

The thesis also contains two appendices. Appendix A provides a computation of Langlands duality for each classical Lie group, and Appendix B is a self-contained section providing an overview of positive line bundles on complex tori.

Part I

Complex Algebraic Geometry

Chapter 1

Rudiments of Divisors and Holomorphic Vector Bundles

This chapter introduces the necessary notions in complex geometry used in the research project. We begin by giving an overview of complex differential forms and discussing the canonical splitting of n-forms into their bidegree components. Moreover, we establish Dolbeault cohomology, the complex analogue of de Rham cohomology. Next, we introduce divisors, which lie at the heart of complex algebraic geometry. In particular, the objects of interest for this project are spectral curves, introduced in Chapter 4, which is a specific type of divisor. Since spectral curves define a divisor in a complex surface, we do not distinguish between Cartier and Weil divisors since they agree on smooth varieties. Then, we introduce holomorphic vector bundles and define their most natural characteristic class, the first Chern class. Also, we establish how divisors relate to holomorphic line bundles. Finally, we discuss the rudiments of a linear system of divisors, including Bertini's theorem, which enables us to compute generic spectral curves.

We will follow Griffiths and Harris' *Principles of Algebraic Geometry* [GH94] for divisors and holomorphic vector bundles; and Wells' *Differential Analysis of Complex Manifolds* [Wel08] for complex differential forms. However, there are several references for complex geometry, including, but not limited to, Huybrechts' *Introduction to Complex Geometry* [Huy05], Kobayashi and Nomizu's *Foundations of Differential Geometry* [KN96]. For a detailed account of an algebraic overview of complex manifolds, see Voisin's *Hodge theory and Complex Algebraic Geometry* [Voi03].

1.1 Complex Differential Forms

Let X be a complex manifold of complex dimension n. Suppose $(U,(z^1,\dots,z^n))$ is a local holomorphic chart in X, and write $z^j = x^j + iy^j$ for real coordinates $(x^1,\dots,x^n,y^1,\dots,y^n)$. Then we define complex valued one-forms $\mathrm{d}z^j = \mathrm{d}x^j + i\,\mathrm{d}y^j$ and $\mathrm{d}\overline{z}^j = \mathrm{d}x^j - i\,\mathrm{d}y^j$. Dual to these complex-valued one-forms are the complex-valued vector fields

$$\partial_j := \frac{\partial}{\partial z^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} - i\frac{\partial}{\partial y^j}\right) \quad \text{and} \quad \overline{\partial}_j := \frac{\partial}{\partial \overline{z}^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} + i\frac{\partial}{\partial y^j}\right).$$

The collection of all ∂_j and $\overline{\partial}_j$ form a frame for the vector bundle $TU^{\mathbb{C}} := TU \otimes_{\mathbb{R}} \mathbb{C}$ where TU is the real tangent bundle. Moreover, we may define $T^{1,0}U := \operatorname{span}\{\partial_j\}$ and $T^{0,1}U := \operatorname{span}\{\overline{\partial}_j\}$ and by the chain rule this local splitting is independent of choice of local holomorphic coordinates and thus, defines a canonical splitting

$$TX^{\mathbb{C}} = T^{1,0}X \oplus T^{0,1}X.$$

The *holomorphic tangent bundle of* X is the vector bundle $T^{1,0}X$ and the *anti-holomorphic tangent bundle of* X is the vector bundle $T^{0,1}X$. In Section 1.3 we will justify the use of the word holomorphic by showing $T^{1,0}X$ defines a holomorphic vector bundle. Notice that there is a real isomorphism $T^{1,0}X \cong_{\mathbb{R}} T^{0,1}X$ given by complex conjugating, i.e., $\overline{T^{1,0}X} = T^{0,1}X$. Moreover, the composition

$$TX \to TX^{\mathbb{C}} \xrightarrow{\mathrm{pr}_{1,0}} T^{1,0}X$$

where $TX \to TX^{\mathbb{C}}$ is the canonical inclusion and $TX^{\mathbb{C}} = T^{1,0}X \oplus T^{0,1}X \xrightarrow{\mathrm{pr}_{1,0}} T^{1,0}X$ is the projection map, defines a real isomorphism $TX \cong_{\mathbb{R}} T^{1,0}X$.

Similarly, the complex-valued one-forms $\mathrm{d}z^j$ and $\mathrm{d}\overline{z}^j$ define a canonical splitting

$$T^*X^{\mathbb{C}} = T^*_{1,0}X \oplus T^*_{0,1}X,$$

where $T^*_{1,0}X$ is the *holomorphic cotangent bundle of* X and $T^*_{0,1}X$ is the *anti-holomorphic cotangent bundle of* X. Now we will explore the ramification of the canonical splitting of the complexified cotangent bundle with complex differential forms on X.

Let $\Omega^k(X)$ denote the space of smooth complex-valued differential k-forms on X, i.e., $\Omega^k(X) := \Gamma(X, \bigwedge^k T^*X^{\mathbb{C}})$. If $\Omega^{1,0} := \Gamma(X, T^*_{1,0}X)$ and $\Omega^{0,1} := \Gamma(X, T^*_{0,1}X)$, then by the previous discussion we obtain a canonical decomposition

$$\Omega^1(X) = \Omega^{1,0}(X) \oplus \Omega^{0,1}(X).$$

More generally, the splitting $T^*X^{\mathbb{C}} = T^*_{1,0}X \oplus T^*_{0,1}X$ induces a splitting

$$\bigwedge^k T^*X^{\mathbb{C}} = \bigoplus_{p+q=k} \left(\bigwedge^p T^*_{1,0}X \otimes \bigwedge^q T^*_{0,1}X\right).$$

Hence, if $\Omega^{p,q}(X) := \Gamma(X, \bigwedge^p T^*_{1,0}X \otimes \bigwedge^q T^*_{0,1}X)$, then there is a natural splitting

$$\Omega^k(X) = \bigoplus_{p+q=k} \Omega^{p,q}(X).$$

An element of $\Omega^{p,q}(X)$ is called a *(p,q)-form on X*.

The differential extends complex linearly to an operator $\mathrm{d}\colon \Omega^k(X) \to \Omega^{k+1}(X)$ such that $\mathrm{d}\circ\mathrm{d} = 0$. One may naturally ask what happens to d under the previously described splitting. In other words, how does d act on $\Omega^{p,q}(X)$ where $p+q=k$. To understand this we resort to a local calculation in a frame. Recall that in the coordinate neighbourhood $(U,(z^1,\dots,z^n))$ where $z^j = x^j + iy^j$ the complex-valued differential forms $\{\mathrm{d}z^1, \mathrm{d}z^2, \dots, \mathrm{d}z^n\}$ form a frame for $T^*_{1,0}(X)$ over U and $\{\mathrm{d}\overline{z}^1,\dots,\mathrm{d}\overline{z}^n\}$ form a frame for $T^*_{0,1}(X)$ over U. Then, $\{\mathrm{d}z^I \wedge \mathrm{d}\overline{z}^J\}$ where $|I| = p$ and $|J| = q$ and I and J are strictly increasing form a frame for $\wedge^{p,q} T^*(X)^{\mathbb{C}}$ over U. Thus, $s \in \Omega^{p,q}(X)$ is of the form

$$s = a_{IJ}\, \mathrm{d}z^I \wedge \mathrm{d}\overline{z}^J$$

over U where $a_{IJ} \in \Omega^0(U)$. Since $\mathrm{d}\circ\mathrm{d} = 0$ we see

$$\mathrm{d}s = \mathrm{d}a_{IJ} \wedge \mathrm{d}z^I \wedge \mathrm{d}\overline{z}^J.$$

However,

$$\begin{aligned} \mathrm{d}a_{IJ} &= \frac{\partial a_{IJ}}{\partial x^j}\,\mathrm{d}x^j + \frac{\partial a_{IJ}}{\partial y^j}\,\mathrm{d}y^j \\ &= \frac{\partial a_{IJ}}{\partial z^j}\,\mathrm{d}z^j + \frac{\partial a_{IJ}}{\partial \overline{z}^j}\,\mathrm{d}\overline{z}^j, \end{aligned}$$

hence,

$$\mathrm{d}s = \frac{\partial a_{IJ}}{\partial z^j} dz^j \wedge \mathrm{d}z^I \wedge \mathrm{d}\overline{z}^J + \frac{\partial a_{IJ}}{\partial \overline{z}^j}\,\mathrm{d}\overline{z}^j \wedge \mathrm{d}z^I \wedge \mathrm{d}\overline{z}^J.$$

The first component is a $(p+1,q)$ form and the second is a $(p,q+1)$ form. Therefore, by defining (in the given local frame)

$$\partial = \frac{\partial}{\partial z^j}\,\mathrm{d}z^j \quad \text{and} \quad \overline{\partial} = \frac{\partial}{\partial \overline{z}^j}\,\mathrm{d}\overline{z}^j$$

it follows that there is a canonical splitting of differential operators $\mathrm{d} = \partial + \overline{\partial}$ where $\partial : \Omega^{p,q}(X) \to \Omega^{p+1,q}$ and $\overline{\partial} : \Omega^{p,q}(X) \to \Omega^{p,q+1}(X)$. The differential operator $\overline{\partial}$ captures holomorphic information on the complex manifold X and is an indispensable tool in complex geometry. In essence, the operator $\overline{\partial}$ is a generalisation of the famous *Cauchy-Riemann operator* $\frac{\partial}{\partial \overline{z}}$ in single-variable complex analysis. We have provided an outline of the necessary machinery for our purposes, and for an an extensive treatment of this general theory the reader should consult [Wel08, Section 3].

Definition 1.1.1. (Dolbeault Operator) The differential operator $\overline{\partial} : \Omega^{p,q}(X) \to \Omega^{p,q+1}(X)$ is called the *Dolbeault operator* on X.

Remark 1.1.2. From the foregoing discussion it is clear that the Dolbeault operator depends on the complex structure of X. That is, the Dolbeault operators of two distinct complex manifolds with the same underlying real differentiable manifold need not agree.

A smooth complex-valued function $f \in \Omega^0(X)$ is holomorphic if and only if $\overline{\partial} f = 0$, and more generally $s \in \Omega^{p,0}(X)$ is holomorphic if and only if $\overline{\partial} s = 0$. The identity $\mathrm{d} \circ \mathrm{d} = 0$ implies $\overline{\partial}^2 = 0$ and $\partial^2 = 0$ and $\overline{\partial}\partial + \partial\overline{\partial} = 0$. In particular, $(\Omega^{p,\bullet}, \overline{\partial})$ defines a cochain complex for each $p \geq 0$. That is,

$$\Omega^{p,0}(X) \xrightarrow{\overline{\partial}} \Omega^{p,1}(X) \xrightarrow{\overline{\partial}} \Omega^{p,2}(X) \xrightarrow{\overline{\partial}} \cdots$$

defines a cochain complex. Thus, we can take the cohomology of the cochain sequence, which is the desired *Dolbeault cohomology*.

When studying complex differential forms and splitting the complexified tangent bundle, the notion of an (almost) complex structure naturally arises. Then, one may ask when a real differentiable manifold admits a complex structure. Pursuing this problem has birthed a wealth of new techniques and theorems, most notably the Newlander-Nirenberg theorem. However, since we will only be working with complex manifolds, we do not pursue this problem, and the curious reader may consult [Wel08, Hor90] for more details.

1.1.1 Dolbeault Cohomology

We may now define the desired *Dolbeault cohomology*, which is an important holomorphic invariant on X.

Definition 1.1.3. The (p, q)-th *Dolbeault cohomology group* of X is defined to be

$$\mathrm{H}^{p,q}_{\overline{\partial}}(X) := \mathrm{H}^q((\Omega^{p,\bullet}(X), \overline{\partial})).$$

Notice that $(\Omega^{\bullet,q}(X), \partial)$ defines a cochain complex for $q \geq 0$, but since conjugating gives a real isomorphism $\Omega^{p,q}(X) \cong_{\mathbb{R}} \Omega^{q,p}(X)$ it follows that $\mathrm{H}^{p,q}_{\overline{\partial}}(X) \cong \mathrm{H}^{q,p}_{\partial}(X)$ and thus, there is no new information obtained.

Dolbeault cohomology for complex manifolds mirrors de Rham cohomology for smooth manifolds. One important foundational results for de Rham cohomology is the Poincaré lemma, and indeed there is an analogue for complex manifolds called Dolbeault's lemma.

Lemma 1.1.4 (Dolbeault's lemma)**.** *Let P be any polydisc in $\mathbb{C}^n$. Suppose $\alpha \in \Omega^{p,q}(P)$ is $\overline{\partial}$-closed, i.e. $\overline{\partial}\alpha = 0$. Then there exists $\beta \in \Omega^{p,q-1}$ such that $\overline{\partial}\beta = \alpha$.*

moreover, there is the well-known de Rham isomorphism theorem that states the de Rham cohomology coincides with the sheaf cohomology valued in the locally constant sheaf $\mathbb{R}$. Indeed, there is an analogous result for Dolbeault cohomology.

Theorem 1.1.5 (Dolbeault Isomorphism Theorem)**.** *There is an isomorphism*

$$\mathrm{H}^{p,q}_{\overline{\partial}}(X) \cong \mathrm{H}^q(X, \Omega^p_X)$$

where $\mathcal{O}_X(\Omega^p_X)$ *is the sheaf of holomorphic* $(p,0)$*-forms.*

The standard sheaf theoretic proof of the Dolbeault isomorphism theorem distills down to showing there exists a fine resolution of the sheaf $\mathcal{O}_X(\Omega^p_X)$ given by

$$0 \to \mathcal{O}_X(\Omega^p_X) \to \Omega^{p,0} \xrightarrow{\overline{\partial}} \Omega^{p,1} \xrightarrow{\overline{\partial}} \cdots$$

where $\Omega^{p,q}$ denotes the sheaf of smooth (p,q)-forms on X. The proof of this follows from Dolbeault's lemma. For more details, see [GH94, pp 44]. Notice that $\mathcal{O}_X(\Omega^0_X) \cong \mathcal{O}_X$, which establishes a useful corollary.

Corollary 1.1.6. *By the Dolbeault isomorphism theorem there is an isomorphism*

$$\mathrm{H}^{0,q}_{\overline{\partial}}(X) \cong \mathrm{H}^q(X, \mathcal{O}_X).$$

In particular, if $q > n$*, then* $\mathrm{H}^q(X, \mathcal{O}_X) = 0$.

1.2 Divisors

1.2.1 Analytic Varieties

To establish divisors we will recall the necessary theory and results concerning *analytic varieties*. We will largely be following [GH94, Chapter 0: Section 1].

Let $\mathcal{O}_n := \mathcal{O}_{\mathbb{C}^n,0}$, i.e. $\mathcal{O}_n$ denotes the ring of holomorphic functions defined in some neighbourhood of $0 \in \mathbb{C}^n$. In sheaf-theoretic terms $\mathcal{O}_n$ denotes the stalk at 0 of the structure sheaf $\mathcal{O}_{\mathbb{C}^n}$ of $\mathbb{C}^n$. Now, we will recall the elementary results surrounding *Weierstrass polynomials*, which is needed to work with divisors. Assuming $n > 1$, if $(z_1, \ldots, z_n)$ are coordinates in $\mathbb{C}^n$, then setting $z := (z_1, \ldots, z_{n-1})$ and $w := z_n$ we see (z, w) define coordinates in $\mathbb{C}^n$.

Definition 1.2.1. A *Weierstrass polynomial* in w is a holomorphic function

$$f(z,w) = w^d + a_1(z)w^{d-1} + \cdots + a_d(z),$$

where each a_j is holomorphic in z and $a_j(0) = 0$. The *degree* of f is defined to be d.

Proposition 1.2.2 (Weierstrass Preparation Theorem [Huy05, Proposition 1.1.6])**.** *Suppose* $f(z,w)$ *is holomorphic in some neighbourhood of the origin* $0 \in \mathbb{C}^n$*, and* $f(0,w) \not\equiv 0$*. Then, in some neighbourhood of the origin we may factor* f *uniquely by*

$$f = g \cdot h,$$

where g *is a Weierstrass polynomial in* w *of degree* d*, and* $h(0) \neq 0$.

Proposition 1.2.3 ([Huy05, Proposition 1.1.15])**.** *The local ring $\mathcal{O}_n$ is a unique factorisation domain (UFD).*

Proposition 1.2.4. *If f and g are relatively prime in $\mathcal{O}_n$, then f and g are relatively prime in $\mathcal{O}_{\mathbb{C}^n,z}$ where $||z|| \leq \epsilon$ for sufficiently small $\epsilon > 0$.*

Proof. We may assume that both f and g are holomorphic in the polydisc $B_{\epsilon'}(0)$ for sufficiently small $\epsilon' > 0$. By an affine change of coordinates we may assume both f and g are regular with respect to $w := z_n$ and are both Weierstrass polynomials in w. For each $z' \in \mathbb{C}^n$ such that $||(z', w)|| < \epsilon'$ we have $f(z', z_n) \not\equiv 0$ in w. Since $\mathcal{O}_n$ is a UFD we may write

$$\alpha f + \beta g = \gamma \tag{1.1}$$

where $\alpha, \beta \in \mathcal{O}_{n-1}[w]$ are relatively prime and $\gamma \in \mathcal{O}_{n-1}$, and by continuity (1.1) holds in some ϵ-neighbourhood of $0 \in \mathbb{C}^n$ where $\epsilon \leq \epsilon'$. Assume there is some $||z_0|| \leq \epsilon$ such that $f(z_0) = g(z_0) = 0$ and $f, g \in \mathcal{O}_{\mathbb{C}^n,z_0}$ share a common factor $h \in \mathcal{O}_{\mathbb{C}^n,z_0}$ with $h(z_0) = 0$. Then, by (1.1) since h divides f and g it follows that h divides γ, hence $h \in \mathcal{O}_{n-1}$. However, this implies $h(z', w)$ vanishes identically in w, which implies $f(z', w)$ vanishes identically in w, a contradiction. □

Proposition 1.2.5 (Weierstrass division theorem [Huy05, Proposition 1.1.17])**.** *Let $g(z, w)$ belong to $\mathcal{O}_{n-1}[w]$ be a Weierstrass polynomial of degree k in w. Then, for $f \in \mathcal{O}_n[w]$ we have*

$$f = g \cdot h + r$$

with $r(z, w)$ a polynomial of degree strictly less than k in w.

Corollary 1.2.6 (Weak Nullstellensatz)**.** *If $f(z, w) \in \mathcal{O}_n$ is irreducible and $h \in \mathcal{O}_n$ vanishes on $\{f(z, w) = 0\}$, then f divides h in $\mathcal{O}_n$.*

Proof. By an affine change of coordinates we may assume that $f(z, w)$ is a Weierstrass polynomial of degree k with respect to w. Since f is irreducible it follows that f and $\frac{\partial f}{\partial w}$ are relatively prime $\mathcal{O}_{n-1}[w]$. Hence, we may write

$$\alpha f + \beta \frac{\partial f}{\partial w} = \gamma$$

where $\alpha, \beta \in \mathcal{O}_{n-1}[w]$ are relatively prime and $\gamma \in \mathcal{O}_{n-1}$ is not identically zero. We note that if u is a multiple root of f, then $\frac{\partial f}{\partial w}(u) = 0$ and hence, $\gamma(u) = 0$. Therefore, $f(z, w)$ has k distinct roots whenever $\gamma(z) \neq 0$. By the Weierstrass division theorem

$$h = f \cdot g + r$$

where $r \in \mathcal{O}_{n-1}[w]$ and $\deg_w(r) < k$. Now, for any z_0 outside the locus $\{\gamma = 0\}$ we see $f(z_0, w)$ has at least k distinct roots in w, and hence, $h(z_0, w)$ has at least k-distinct roots in w, and it follows that $r \equiv 0$. Thus, $h = f \cdot g$. □

Now, fix two complex manifolds X and Y of dimensions n and m respectively. We will outline the necessary results of subvarieties of complex manifolds for our purposes. We will define the Jacobian of a holomorphic function between two complex manifolds and then state the needed results.

Suppose $(z_1, \ldots, z_n)$ are local coordinates on X centred at p, and $(w_1, \ldots, w_m)$ are local coordinates on Y centred at q, and suppose $f : X \to Y$ is a holomorphic map such that $f(p) = q$. We define the *Jacobian of* f to be the matrix

$$J(f) = \left[\frac{\partial w_\alpha}{\partial z_j}\right]_{\alpha=1,\ldots,m}^{j=1,\ldots,n}.$$

Although the Jacobian depends on the choice of coordinates, the rank of the Jacobian is independent of choice of coordinates by the chain rule. Now, we may define *complex submanifolds* and *analytic subvarieties* of X.

Definition 1.2.7. A *complex submanifold* S of X is a subset $S \subset X$ either the zero locus of a collection $f = (f_1, \ldots, f_k)$ of k holomorphic functions with $\operatorname{rank}(J(f)) = k$ or the image of an open subset $U \subset \mathbb{C}^{n-k}$ under a holomorphic map $f : U \to X$ with $\operatorname{rank}(J(f)) = n - k$. In either case the *dimension* of S is $n - k$.

Remark 1.2.8. The two definitions are in fact equivalent by the celebrated implicit function theorem.

Definition 1.2.9. An *analytic subvariety* V of X is a subset $V \subset X$ given locally as the zero locus of a finite collection of holomorphic functions. A point $p \in V$ is called *smooth* if V is a complex submanifold of X near p. A point that is not smooth is called a *singular point*. We define V_{sing} to be the set of singular points of V called the *singular locus* of V, and we define the *smooth locus* of V by $V^* := V \setminus V_{\text{sing}}$. Finally, V is *smooth* or *non-singular* if $V = V^*$.

Now, we will state two propositions regarding important properties of analytic subvarieties. The proofs of each proposition can be found in [GH94, pp 21 - 22].

Proposition 1.2.10. *The singular locus V_{sing} of an analytic subvariety $V \subset X$ is contained in an analytic subvariety of X not equal to V.*

Proposition 1.2.11. *A complex analytic subvariety $V \subset X$ is irreducible if and only if the smooth locus V^* is connected.*

We conclude this section with some facts about *analytic hypersurfaces*. An analytic subvariety $V \subset X$ is called an *analytic hypersurface* if at each $p \in V$, the variety V is locally the zero locus of one non-zero holomorphic function f called the *local defining function for V at p*. Note, by weak Nullstellensatz if g is defined near p and vanishes on V, then f divides g. Moreover, f is unique up to multiplication by a holomorphic function non-vanishing at p.

Suppose $p \in V$ and in a local coordinate chart centred at p, V is the zero locus of some irreducible holomorphic function $f \in \mathcal{O}_n$, then V is irreducible at p. Indeed if $V = V_1 \cup V_2$ where V_1, V_2 are proper closed analytic subvarieties of V, then there exists $f_1, f_2 \in \mathcal{O}_n$ where f_1 vanishes identically on V_1, but not V_2, and f_2 vanishes identically on V_2, but not V_1. Since f vanishes identically on both V_1 and V_2 it follows by weak Nullstellensatz that f divides $f_1 f_2$. However, since f is irreducible, f either divides f_1 or f divides f_2, i.e., $V \subset V_1$ or $V \subset V_2$, a contradiction.

If instead V is defined as the zero locus of a non-zero holomorphic function $f \in \mathcal{O}_n$ not necessarily irreducible, then since $\mathcal{O}_n$ is a UFD we may write

$$f = f_1 \cdots f_k$$

where each $f_i \in \mathcal{O}_n$ is irreducible, and this factorisation is unique up to multiplication by a holomorphic function non-vanishing at 0. By defining V_i to be the zero locus of f_i we see that locally

$$V = V_1 \cup \cdots \cup V_k$$

with each V_i irreducible. Hence, locally near each point an analytic hypersurface can be written uniquely as the union of finitely many irreducible analytic hypersurfaces.

1.2.2 Definition and Constructions

Fix a complex manifold X of dimension n. We may now define divisors.

Definition 1.2.12. A *divisor* D on X is a locally finite formal linear combination of irreducible analytical hypersurfaces

$$D = \sum_i a_i V_i$$

where $a_i \in \mathbb{Z}$.

Remark 1.2.13. If X is compact, then a locally finite sum is finite. Moreover, if X is a Riemann surface, then an irreducible analytic hypersurface is simply a point.

If $D = \sum_{i=1}^n a_i V_i$ is a finite linear combination divisor on X, then we define the *degree* of D to be $\deg(D) := \sum_{i=1}^n a_i \in \mathbb{Z}$. The set of divisors on X carries a natural abelian additive group structure and we denote this group by $\mathrm{Div}(X)$ called the *class divisor group on* X. We call a divisor $D = \sum_i a_i V_i$ *effective* if $a_i \geq 0$ for each i, and write $D \geq 0$. Note, an analytic hypersurface $V \subset X$ may be realised as a divisor by $V = \sum_i V_i$ where the V_i are the irreducible components of V.

Now, we will show how a non-identically zero meromorphic function $f : X \to \mathbb{C}$ defines a divisor on X. To define the divisor we need some basic notions. Suppose $V \subset X$ is an irreducible analytic hypersurface. Let $p \in V$ and suppose f is a local defining function of V at p. If g is another holomorphic function defined near p we define the *order of g along V at p*, denoted $\mathrm{ord}_{V,p}(g)$ to be the largest integer a such that in the ring $\mathcal{O}_{X,p}$

$$g = f^a \cdot h$$

with f^a and h relatively prime. If g and g' are irreducible in $\mathcal{O}_{X,p}$, then by Proposition 1.2.4 we see g and g' are relatively prime in $\mathcal{O}_{X,q}$ for q near p. Since V is irreducible it follows that $\mathrm{ord}_{V,p}(g)$ is independent of $p \in V$, i.e., $\mathrm{ord}_{V,p}(g) = \mathrm{ord}_{V,q}(g)$ for $p, q \in V$. Thus, we define the *order of g along V*, denoted $\mathrm{ord}_V(g)$ to be $\mathrm{ord}_{V,p}(g)$ for any $p \in V$. It easily follows from the definition that if g and h are two holomorphic functions defined near a common point that

$$\mathrm{ord}_V(gh) = \mathrm{ord}_V(g) + \mathrm{ord}_V(h).$$

If f is a meromorphic function locally given by $f = g/h$ where g and h are coprime, then we define the *order of f along V* to be

$$\mathrm{ord}_V(f) := \mathrm{ord}_V(g) - \mathrm{ord}_V(h).$$

If $\mathrm{ord}_V(f) = a > 0$, then we say f has a *zero of order a along V*, and if $\mathrm{ord}_V(f) = -a < 0$, then we say f has a *pole of order a along V*.

Now, if $f : X \to \mathbb{C}$ is a meromorphic function we define the *divisor associated to f* by

$$(f) := \sum_V \mathrm{ord}_V(f)V.$$

1.2.3 Sheaf-Theoretic Description

We conclude the section on divisors by providing a sheaf-theoretic description of divisors. Let $\mathcal{M}_X^*$ denote the sheaf of non-identically zero meromorphic functions on X, and let $\mathcal{O}_X^*$ denote the sheaf of nowhere vanishing holomorphic functions on X. Then $\mathrm{Div}(X)$ may be canonically identified with the space of global sections of the quotient sheaf $\mathcal{M}_X^*/\mathcal{O}_X^*$, i.e., $\mathrm{H}^0(X, \mathcal{M}_X^*/\mathcal{O}_X^*)$. Indeed, if $\{f\}$ is a global section of $\mathcal{M}_X^*/\mathcal{O}_X^*$, then we may choose an open cover of X with $f_\alpha := f|_{U_\alpha} \not\equiv 0$ and $\frac{f_\alpha}{f_\beta} \in \mathcal{O}_X^*(U_{\alpha\beta})$ where $U_{\alpha\beta} := U_\alpha \cap U_\beta$. Since $\frac{f_\alpha}{f_\beta}$ is holomorphic and nowhere zero it follows that $\mathrm{ord}_V(f_\alpha) = \mathrm{ord}_V(f_\beta)$ for each irreducible analytic hypersurface with $V \cap U_{\alpha\beta} \neq \varnothing$. Therefore, we may define the divisor

$$D := \sum_V \mathrm{ord}_V(f_\alpha)V$$

where $V \cap U_\alpha \neq \varnothing$.

Conversely, suppose $D = \sum_i a_i V_i$ is a divisor on X, then we may choose an open cover $\{U_\alpha\}$ such that in each U_α every V_i appearing in D has local defining function $g_{i\alpha} \in \mathcal{O}_X(U_\alpha)$. Set

$$f_\alpha := \prod_i g_{i\alpha}^{a_i} \in \mathcal{M}_X^*(U_\alpha).$$

Then the collection $\{f_\alpha\}$ patch together to define a global section of $\mathcal{M}_X^*/\mathcal{O}_X^*$. Of course, we need to prove these constructions are mutual inverses.

Proposition 1.2.14. *There is a canonical isomorphism*

$$\mathrm{Div}(X) \cong \mathrm{H}^0(X, \mathcal{M}_X^*/\mathcal{O}_X^*).$$

Proof. It suffices to show the foregoing constructions are mutual inverses. Suppose $\{f\}$ is a global section of $\mathcal{M}_X^*/\mathcal{O}_X^*$ and suppose we choose an open cover $\{U_\alpha\}$ and construct the divisor $D = \sum_V \mathrm{ord}_V(f)V$ as before. Then, for each irreducible analytic hypersurface $V \subset X$ such that $V \cap U_\alpha \neq \varnothing$ there are local defining functions $g_{V\alpha}$ for each V and we set

$$h_\alpha = \prod_V g_{V\alpha}^{\mathrm{ord}_V(f_\alpha)}.$$

Recall that $\{h_\alpha\}$ patch together to form a global section $\{h\}$ of $\mathcal{M}_X^*/\mathcal{O}_X^*$, and since clearly $h_\alpha = c_\alpha f_\alpha$ for some $c_\alpha \in \mathcal{O}_X^*(U_\alpha)$ it follows that $\{h\}$ and $\{f\}$ define the same section.

Conversely, suppose $D = \sum_i a_i V_i$ is a divisor on X and we choose an open cover $\{U_\alpha\}$ of X such that for each U_α, every V_i appearing in D has local defining functions $g_{i\alpha}$. Setting $f_\alpha := \prod_i g_{i\alpha}^{a_i}$ allows us to define the divisor $D' = \sum_V \mathrm{ord}_V(f_\alpha)V$. Clearly, the only irreducible analytic hypersurfaces $V \subset X$ with $\mathrm{ord}_V(f_\alpha) \neq 0$ are the V_i appearing in D, Moreover, if follows by definition of each f_α that $\mathrm{ord}_{V_i}(f_\alpha) = a_i$ and thus, $D = D'$. □

1.3 Holomorphic Vector Bundles

Let X be a complex manifold of dimension n. A smooth complex vector bundle $\pi : E \to X$ is called a *holomorphic vector bundle* if the total space of E may be endowed with the structure of a complex manifold such that $\pi : E \to X$ is holomorphic. Equivalently, $E \to X$ is holomorphic if E admits a trivialising open cover $\{U_\alpha\}$ such that the transition functions $g_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{GL}_r(\mathbb{C})$ are holomorphic.

The standard constructions of new vector bundles from old applies in the holomorphic case, e.g., if $E \to X$ and $F \to X$ are holomorphic vector bundles, then the vector bundles $E^*, \mathrm{Hom}(E,F), E \oplus F, E \otimes F, \det(E)$ are all holomorphic. The reader may easily verify these claims by considering the transition functions in the construction and noticing they are all holomorphic.

We will now give justification to the name *holomorphic tangent bundle* used in Section 1.1. If $(\{U_\alpha\}, \{\phi_\alpha\})$ is a holomorphic atlas for X, then the transition maps $\phi_{\alpha\beta} := \phi_\alpha \circ \phi_\beta^{-1}$

are biholomorphisms, hence the maps $g_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{GL}_n(\mathbb{C})$ given by $g_{\alpha\beta} := D(\phi_{\alpha\beta})$ define a holomorphic 1-cocycle, which are the transition functions for $T^{1,0}(X)$. Hence, the dual bundle $T^*_{1,0}(X)$ is also a holomorphic vector bundle, which we call the *holomorphic cotangent bundle*. From this construction there is an important holomorphic vector bundle that we will now define.

Definition 1.3.1. The bundle of *holomorphic p-forms* denoted Ω^p_X is defined to be $\Omega^p_X := \bigwedge^p T^*_{1,0}(X)$ for $0 \leq p \leq n$. Moreover, the *canonical bundle* denoted K_X is the bundle of holomorphic n-forms, i.e., $K_X := \det(T^*_{1,0}(X)) = \Omega^n_X$.

Remark 1.3.2. This definition of holomorphic tangent and cotangent bundles are independent of choice of holomorphic coordinates so $T^*_{1,0}(X)$ and $T^{1,0}(X)$ are holomorphic invariants on X. Note, if X is a Riemann surface, then the canonical bundle K_X is simply the holomorphic cotangent bundle.

One may naturally ask if the notion of holomorphic extends to sections, and indeed it does. Let $U \subset X$ be an open subset. A *holomorphic section* of a holomorphic vector bundle $E \to X$ over U is a section $s : U \to E|_U$ of the underlying complex vector bundle, that is a holomorphic map. If $\{U_\alpha\}$ is a trivialising open cover of E with holomorphic transition functions $\{g_{\alpha\beta}\}$, then the holomorphic section $s : U \to E|_U$ is a family of holomorphic maps $\{s_\alpha : U \cap U_\alpha \to \mathbb{C}^r\}$ where $U \cap U_\alpha \neq \varnothing$ such that on the overlap $U \cap U_\alpha \cap U_\beta$

$$s_\alpha = g_{\alpha\beta} s_\beta.$$

Note, holomorphic sections of E naturally define a sheaf, $\mathcal{O}_X(E)$, where $\mathcal{O}_X(E)(U) := \{s \in \Gamma(U, E) \mid s \text{ is holomorphic}\}$ for each open subset $U \subset X$.

Notation 1.3.3. We will use roman text to denote holomorphic vector bundles, e.g. $E \to X$. However, to simplify notation we will use calligraphic font to denoted the sheaf of sections, i.e., $\mathcal{E} := \mathcal{O}_X(E)$. Moreover, if we wish to write the sheaf of sections of $E|_U \to U$ where $U \subset X$ is an open subset we will write $\mathcal{E}|_U := \mathcal{O}_U(E)$. Note, if there is likely any ambiguity to arise we will use the standard $\mathcal{O}_X(E)$ notation, e.g., the sheaf of sections of a tensor product since $\mathcal{E} \otimes \mathcal{F} \neq \mathcal{O}_X(E \otimes F)$ in general. Moreover, we will denote the sheaf of meromorphic section of E over X by $\mathcal{M}_X(E)$, i.e., $\mathcal{M}_X(E) := \mathcal{M}_X \otimes_{\mathcal{O}_X} \mathcal{E}$.

A *homomorphism* of holomorphic vector bundles is a homomorphism $\psi : E \to F$ of smooth complex vector bundles such that ψ is a holomorphic map between the complex manifolds E and F, and ψ is an *isomorphism* if $\psi(x) : E_x \to F_x$ is an isomorphism of complex vector spaces for each $x \in X$.

Holomorphic vector bundles possess unique properties that are not shared by their smooth complex counterparts. For instance, we will see that two holomorphic vector bundles may be isomorphic as smooth complex vector bundles, but need not be isomorphic as holomorphic vector bundles.

From an algebro-geometric point-of-view, holomorphic vector bundles are interesting due to their intimate relationship with locally-free sheaf of $\mathcal{O}_X$-modules.

Definition 1.3.4. Suppose $(X, \mathcal{O}_X)$ is a ringed space. A *sheaf of $\mathcal{O}_X$-modules* is a sheaf $\mathcal{F} \to X$ of abelian groups on X such that $\mathcal{F}(U)$ has the structure of a $\mathcal{O}_X(U)$-module for each open subset $U \subset X$, and if $V \subset U$ then the diagram

$$\begin{tikzcd} \mathcal{F}(U) \times \mathcal{O}_X(U) \arrow[r] \arrow[d, "\mathrm{res} \times \mathrm{res}"] & \mathcal{F}(U) \arrow[d, "\mathrm{res}"] \\ \mathcal{F}(V) \times \mathcal{O}_X(V) \arrow[r] & \mathcal{F}(V) \end{tikzcd}$$

commutes. The horizontal maps are the module action.

To every holomorphic vector bundle $E \to X$, the sheaf of sections $\mathcal{E}$ defines a locally-free sheaf of $\mathcal{O}_X$-modules. Indeed, $\mathcal{E}(U)$ carries the structure of a $\mathcal{O}_X(U)$-module for each open subset $U \subset X$ given by

$$\mathcal{O}_X(U) \times \mathcal{E}(U) \ni (f, s) \mapsto fs \in \mathcal{E}(U).$$

The action clearly commutes with restriction, and if $\mathrm{rank}(E) = r$, then since E is locally trivial we may choose a sufficiently small open set U near each point such that $\mathcal{E}|_U \cong {\mathcal{O}_U}^{\oplus r}$. Remarkably, every locally-free sheaf of $\mathcal{O}_X$-modules may be realised as the sheaf of sections of some holomorphic vector bundle, thus there is a canonical bijection between holomorphic vector bundles and locally-free sheaves of $\mathcal{O}_X$-modules. The reader should consult [Huy05, Proposition 2.2.19] for more details.

Thus, we may define the *cohomology* of a holomorphic vector bundle $E \to X$ to be the sheaf cohomology of the associated locally-free sheaf of $\mathcal{O}_X$-modules, i.e., $\mathrm{H}^p(X, E) := \mathrm{H}^p(X, \mathcal{E})$ for $p \geq 0$. A classical hard analytic result surrounding sheaf cohomology of holomorphic vector bundles is the fact that over a compact complex manifold X the sheaf cohomology groups are finite dimensional vector spaces.

Theorem 1.3.5 ([Hor90])**.** *Suppose X is compact and let $E \to X$ be a holomorphic vector bundle. Then, the sheaf cohomology groups $\mathrm{H}^p(X, E)$ are finite dimensional complex vector spaces for $p \geq 0$.*

Remark 1.3.6. If X is not compact then the sheaf cohomology groups need not be finite dimensional, e.g., $\mathrm{H}^0(\mathbb{C}, \mathcal{O}_{\mathbb{C}})$.

Notation 1.3.7. When X is compact and $E \to X$ defines a holomorphic vector bundle we denote the dimension of $\mathrm{H}^p(X, E)$ by $h^p(X, E)$, i.e., $h^p(X, E) := \dim \mathrm{H}^p(X, E) < \infty$.

1.3.1 Picard Group and First Chern Class

Suppose now we restrict our attention to holomorphic line bundles over our complex manifold X. The set of isomorphism classes of holomorphic line bundles over X admits a natural abelian group structure with tensor product. Indeed, if L and L' are two holomorphic line bundles over X, then $L \otimes L' \cong L' \otimes L$ is another holomorphic line bundle over X, and the trivial holomorphic line bundle $\mathcal{O}_X$ satisfies $\mathcal{O}_X \otimes L \cong L$. Moreover, there is a canonical isomorphism $L \otimes L^* \cong \mathrm{End}(L)$, and a holomorphic line bundle if trivial if and only if it admits a global nowhere vanishing holomorphic section. Notice that $\mathrm{id} : X \to \mathrm{End}(L)$ defined by $\mathrm{id}(x) : L_x \to L_x$ is such a section, hence $\mathrm{End}(L) \cong \mathcal{O}_X$. Thus, the dual line bundle L^* defines an inverse with respect to tensor product, i.e., $L^{-1} := L^*$.

Definition 1.3.8. The *Picard group of* X denoted $\mathrm{Pic}(X)$ is the group of isomorphism classes of holomorphic line bundles on X with group operation given by tensor product.

Remark 1.3.9. In the language of sheaf of $\mathcal{O}_X$-modules, $\mathrm{Pic}(X)$ is the group of invertible sheaves. However, we can define $\mathrm{Pic}(X)$ as the group of invertible sheaves for more general ringed spaces.

The relationship between holomorphic line bundles and their transition functions define a canonical isomorphism $\mathrm{Pic}(X) \cong \check{\mathrm{H}}^1(X, \mathcal{O}_X^*)$, and over any sufficiently nice topological space the Čech cohomology and sheaf cohomology agree, hence there is a canonical isomorphism $\mathrm{Pic}(X) \cong \mathrm{H}^1(X, \mathcal{O}_X^*)$. From now on we will interchange the two groups without mention.

We remark that similar reasoning shows that there is a group of isomorphism classes of smooth complex line bundles under tensor product that can be identified with $\mathrm{H}^1(X, \mathcal{C}_X^*)$ where $\mathcal{C}_X^*$ denotes the sheaf of nowhere vanishing smooth complex-valued functions on X.

Now we may define the *first Chern class* of a vector bundle (smooth complex or holomorphic) on X. The first Chern class is an example of a *characteristic class* and is one of the most fundamental characteristic classes in complex algebraic geometry. To define the first Chern class consider the short exact sequence of sheaves

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2\pi i} \mathcal{C}_X \xrightarrow{\exp} \mathcal{C}_X^* \to 0.$$

Passing to the long exact sequence in sheaf cohomology gives

$$\cdots \to \mathrm{H}^1(X, \mathcal{C}_X) \to \mathrm{H}^1(X, \mathcal{C}_X^*) \xrightarrow{\delta} \mathrm{H}^2(X, \mathbb{Z}) \to \mathrm{H}^2(X, \mathcal{C}_X) \to \cdots \tag{1.2}$$

Denoting the connecting homomorphism δ by c_1 enables us to define the first Chern class of a smooth complex line bundle.

Definition 1.3.10. The *first Chern class*[1] of a smooth complex line bundle L on X is second integral cohomology class $c_1(L) \in \mathrm{H}^2(X, \mathbb{Z})$.

[1] There are higher Chern classes for higher rank vector bundles

Since the sheaf $\mathcal{C}_X$ is fine, $\mathrm{H}^p(X, \mathcal{C}_X) = 0$ for each $p > 0$, so (1.2) reduces to the exact sequence

$$0 \to \mathrm{H}^1(X, \mathcal{C}_X^*) \xrightarrow{c_1} \mathrm{H}^2(X, \mathbb{Z}) \to 0.$$

Hence, the first Chern class classifies smooth complex line bundles up to isomorphism. If instead we considered the standard exponential exact sequence of sheaves

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 0$$

we may proceed in the same way to define the first Chern class of a holomorphic line bundle. Of course, we can forget the holomorphic structure of $L \in \mathrm{Pic}(X)$ and consider L as a smooth complex line bundle, and indeed the first Chern classes agree. To see this consider the homomorphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "\cdot 2\pi i"] \arrow[d, equal] & \mathcal{O}_X \arrow[r, "\exp"] \arrow[d] & \mathcal{O}_X^* \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\cdot 2\pi i"] & \mathcal{C}_X \arrow[r, "\exp"] & \mathcal{C}_X^* \arrow[r] & 0
\end{tikzcd}$$

where the vertical arrows are the canonical inclusions. By passing to the long exact sequences it follows that we obtain the commutative square

$$\begin{tikzcd}
\mathrm{Pic}(X) \arrow[r, "c_1"] \arrow[d] & \mathrm{H}^2(X, \mathbb{Z}) \arrow[d, equal] \\
\mathrm{H}^1(X, \mathcal{C}_X^*) \arrow[r, "c_1"] & \mathrm{H}^2(X, \mathbb{Z})
\end{tikzcd}$$

which shows the first Chern classes agree. We may define the first Chern class for any rank holomorphic vector bundle.

Definition 1.3.11. The *first Chern class* of a holomorphic vector bundle $E \to X$ is defined to be the first Chern class of the determinant line bundle, i.e. $c_1(E) := c_1(\det(E))$.

The first Chern class enables us to define an important class of holomorphic vector bundles, which are called *positive vector bundles*. Before defining positive vector bundles, we require a proposition from Griffiths and Harris.

Proposition 1.3.12 ([GH94, pp. 142]). *Let $L \in \mathrm{Pic}(X)$ be a given holomorphic line bundle and suppose ∇ is a connection on L. Then under the de Rham isomorphism theorem $\mathrm{H}^2(C, \mathbb{C}) \cong \mathrm{H}^2_{dR}(X)$ the first Chern class may be represented by the closed 2-form $\frac{i}{2\pi} F_\nabla$ where F_∇ denotes the curvature of ∇.*

Definition 1.3.13. Suppose that X is compact and $E \to X$ is a holomorphic vector bundle. Then, by considering the first Chern class as an element of $c_1(E) \in \mathrm{H}^2_{dR}(X)$ we say E is *positive* if there is a closed positive $(1,1)$-form ω representing $c_1(E)$, i.e., $c_1(E) = [\omega]$.

1.3.2 Divisors Constructing Holomorphic Line Bundles

Let D be a divisor on X. Recall that under the canonical isomorphism $\mathrm{Div}(X) \cong \mathrm{H}^0(X, \mathcal{M}_X^*/\mathcal{O}_X^*)$ we may characterise the divisor by an open cover $\{U_\alpha\}$ of X with local defining functions $f_\alpha \in \mathcal{M}_X^*(U_\alpha)$ such that $\frac{f_\alpha}{f_\beta} \in \mathcal{O}_X^*(U_{\alpha\beta})$. Hence, the holomorphic functions $g_{\alpha\beta} : U_{\alpha\beta} \to \mathbb{C}^*$ given by $g_{\alpha\beta} = \frac{f_\alpha}{f_\beta}$ define a holomorphic 1-cocycle $\{g_{\alpha\beta}\}$, which in turn defines a holomorphic line bundle, denoted $\mathcal{O}_X(D)$. Of course, we need to verify that this construction is well-defined. Suppose $\{f'_\alpha\}$ are different local defining functions for D, then notice $c_\alpha := \frac{f'_\alpha}{f_\alpha} \in \mathcal{O}_X^*(U_\alpha)$ and hence,

$$g'_{\alpha\beta} := \frac{f'_\alpha}{f'_\beta} = g_{\alpha\beta} \cdot \frac{c_\alpha}{c_\beta}.$$

It follows that $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ define isomorphic line bundles, verifying $\mathcal{O}_X(D)$ is well-defined. Now, consider two quick lemmas proving two important properties of this construction.

Lemma 1.3.14. *The assignment*

$$\mathrm{Div}(X) \ni D \mapsto \mathcal{O}_X(D) \in \mathrm{Pic}(X)$$

defines a homomorphism.

Proof. Let $D, D' \in \mathrm{Div}(X)$ be given. Suppose $\{U_\alpha\}$ is an open cover of X such that D and D' have local defining functions $\{f_\alpha\}$ and $\{f'_\alpha\}$ respectively. Notice that $D + D'$ has local defining functions $\{f_\alpha \cdot f'_\alpha\}$, hence $\mathcal{O}_X(D+D')$ has transition functions $\frac{f_\alpha}{f_\beta} \cdot \frac{f'_\alpha}{f'_\beta}$, which are transition functions for $\mathcal{O}_X(D) \otimes \mathcal{O}_X(D')$, i.e. $\mathcal{O}_X(D + D') \cong \mathcal{O}_X(D) \otimes \mathcal{O}_X(D')$. □

Lemma 1.3.15. *A divisor $D \in \mathrm{Div}(X)$ defines the trivial holomorphic line bundle if and only if D is the divisor of some meromorphic function, i.e., $\mathcal{O}_X(D) \cong \mathcal{O}_X$ if and only if $D = (f)$ for some non-identically zero meromorphic function $f : X \to \mathbb{C}$.*

Proof. Suppose $D = (f)$. Then, for some open cover $\{U_\alpha\}$ the divisor D has defining functions $f_\alpha := f|_{U_\alpha}$. Hence, $\mathcal{O}_X(D)$ has transition functions $g_{\alpha\beta} = \frac{f_\alpha}{f_\beta} = 1$, i.e. $\mathcal{O}_X(D)$ has constant transition functions equal to 1, so $\mathcal{O}_X(D) \cong \mathcal{O}_X$.

Conversely, suppose $\mathcal{O}_X(D) \cong \mathcal{O}_X$. Let $\{U_\alpha\}$ be an open cover of X such that D has local defining functions $\{f_\alpha\}$. Then $\mathcal{O}_X(D)$ has transition functions $g_{\alpha\beta} := \frac{f_\alpha}{f_\beta}$ and since $\mathcal{O}_X(D) \cong \mathcal{O}_X$ there exists $h_\alpha \in \mathcal{O}_X^*(U_\alpha)$ such that

$$g_{\alpha\beta} = \frac{f_\alpha}{f_\beta} = \frac{h_\alpha}{h_\beta}.$$

Thus, on $U_{\alpha\beta}$ we see $f_\alpha h_\alpha^{-1} = f_\beta h_\beta^{-1}$ and so we may define a global non-identically zero meromorphic function on X by $f(x) := f_\alpha(x) h_\alpha^{-1}(x)$ whenever $x \in U_\alpha$ and it easily follows that $D = (f)$. □

We say that two divisors D, D' on X are *linearly equivalent* and write $D \sim D'$ if $D = D' + (f)$ for some global non-identically zero meromorphic function $f : X \to \mathbb{C}$. Note, linear equivalence is an equivalence relation and by Lemma 1.3.15 we see $D \sim D'$ if and only if $\mathcal{O}_X(D) \cong \mathcal{O}_X(D')$.

One may naturally ask if every holomorphic line bundle comes from a divisor, i.e., if $L \in \mathrm{Pic}(X)$, then $L \cong \mathcal{O}_X(D)$ for some $D \in \mathrm{Div}(X)$. In general this is not the case. To see why consider the short exact sequence of sheaves

$$0 \to \mathcal{O}_X^* \to \mathcal{M}_X^* \to \mathcal{M}_X^*/\mathcal{O}_X^* \to 0.$$

Then passing to the long exact sequence in sheaf cohomology we obtain

$$\mathrm{H}^0(X, \mathcal{M}_X^*) \to \mathrm{Div}(X) \to \mathrm{Pic}(X) \to \mathrm{H}^1(X, \mathcal{M}_X^*) \to \cdots$$

where $\mathrm{H}^0(X, \mathcal{M}_X^*) \to \mathrm{Div}(X)$ is the assignment $f \mapsto (f)$, and $\mathrm{Div}(X) \to \mathrm{Pic}(X)$ is the assignment $D \mapsto \mathcal{O}_X(D)$. By exactness, $\mathrm{Div}(X) \to \mathrm{Pic}(X)$ is surjective if and only if $\mathrm{Pic}(X) \to \mathrm{H}^1(X, \mathcal{M}_X^*)$ is the zero map, which is not the case in general. Moreover, by exactness the kernel of the map $\mathrm{Div}(X) \to \mathrm{Pic}(X)$ is precisely the image of the map $\mathrm{H}^0(X, \mathcal{M}_X^*) \to \mathrm{Div}(X)$ and quotienting out $\mathrm{Div}(X)$ by the image is exactly identifying divisors up to linear equivalence. Thus, $\mathrm{Div}(X)/_\sim$ canonically defines a subgroup of $\mathrm{Pic}(X)$ by the first isomorphism theorem.

Now, we will give an alternative necessary and sufficient condition for $\mathrm{Div}(X) \to \mathrm{Pic}(X)$ to be surjective, which is intrinsic to $\mathrm{Pic}(X)$. First, we will recall the definition of a meromorphic section of a holomorphic line bundle.

Definition 1.3.16. Let $L \in \mathrm{Pic}(X)$ and let $\{U_\alpha\}$ be a trivialising open cover such that L has transition functions $\{g_{\alpha\beta}\}$. Suppose $U \subset X$ is an open subset. A *meromorphic section* s of L over U is a family of meromorphic functions $s_\alpha : U \cap U_\alpha \to \mathbb{C}$ where $U \cap U_\alpha \neq \varnothing$ such that

$$s_\alpha = g_{\alpha\beta} s_\beta.$$

Equivalently, a meromorphic section s of L is a section of the sheaf $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M}_X = \mathcal{M}_X(L)$.

Proposition 1.3.17. *Suppose $L \in \mathrm{Pic}(X)$. Then, $L \cong \mathcal{O}_X(D)$ for some $D \in \mathrm{Div}(X)$ if and only if L admits a global non-identically zero meromorphic section.*

Proof. Suppose $L \cong \mathcal{O}_X(D)$ for some $D \in \mathrm{Div}(X)$. Then we may choose an open cover $\{U_\alpha\}$ for X such that D has local defining functions $\{f_\alpha\}$ and L has transition functions $g_{\alpha\beta} := \frac{f_\alpha}{f_\beta}$. Then $\{f_\alpha\}$ patches together to define a global non-identically zero meromorphic section of L.

Conversely, suppose L is a holomorphic line bundle with a global not-identically zero meromorphic section s. Suppose $\{U_\alpha\}$ is a trivialising open cover such that L has transition functions $\{g_{\alpha\beta}\}$. Then the meromorphic section s is a family of meromorphic functions

$s_\alpha \in \mathcal{M}_X^*(U_\alpha)$ such that $s_\alpha = g_{\alpha\beta}s_\beta$. In particular, $\frac{s_\alpha}{s_\beta} \in \mathcal{O}_X^*(U_{\alpha\beta})$, hence for any irreducible analytic hypersurface $V \subset X$ with $V \cap U_{\alpha\beta} \neq \varnothing$ we have $\mathrm{ord}_V(s_\alpha) = \mathrm{ord}_V(s_\beta)$. Thus, we may define a divisor $(s) \in \mathrm{Div}(X)$ by

$$(s) = \sum_V \mathrm{ord}_V(s_\alpha)V$$

where $V \cap U_\alpha \neq \varnothing$. Set $D = (s)$, then clearly D has local defining functions $\{s_\alpha\}$ with respect to the open cover $\{U_\alpha\}$ and $\mathcal{O}_X(D)$ has transition functions $\{g_{\alpha\beta}\}$. Therefore, $L \cong \mathcal{O}_X(D)$. □

1.4 Linear System of Divisors

Now, we will introduce the notion of *linear system of divisors* on our complex manifold X. Suppose $D = \sum_i a_iV_i$ is a divisor on X. Then we define

$$|D| := \{E \in \mathrm{Div}(X) \,|\, E \sim D;\ E \geq 0\},$$

that is, $|D|$ denotes the set of effective divisors that are linearly equivalent to D. Moreover, let

$$\mathcal{L}(D) := \{f \in \mathcal{M}^*(X) \,|\, D + (f) \geq 0\} \cup \{0\}.$$

Recall that the holomorphic line bundle $\mathcal{O}_X(D)$ has a global meromorphic section s_0 such that $(s_0) = D$. The section s_0 defines an important correspondence.

Lemma 1.4.1. *Suppose $D = \sum_i a_iV_i$ is a divisor on X. Then there is a one-to-one correspondence*

$$\mathcal{L}(D) \xrightarrow{\otimes s_0} \mathrm{H}^0(X, \mathcal{O}_X(D)).$$

Proof. Let $f \in \mathcal{L}(D)$ be given. If $f = 0$, then $fs_0 = 0 \in \mathrm{H}^0(X, \mathcal{O}_X(D))$. Otherwise, since $(s_0) = D$ and $D + (f) \geq 0$ it follows that $(fs_0) \geq 0$, hence $fs_0 \in \mathrm{H}^0(X, \mathcal{O}_X(D))$. On the other hand, given a holomorphic section $s \in \mathrm{H}^0(X, \mathcal{O}_X(D))$ notice that $f_s := \frac{s}{s_0}$ defines a meromorphic function on X. If $s = 0$, then $f_s = 0$. Otherwise, it follows $(f_s) + D \geq 0$. These assignments are clearly inverses and establishes the desired correspondence. □

Now we will prove an important correspondence for X compact.

Proposition 1.4.2. *Suppose X is compact and $D = \sum_i a_iV_i$ is a divisor on X. Then there is a one-to-one correspondence*

$$|D| \cong \mathbb{P}(\mathcal{L}(D)).$$

Proof. Consider the map

$$\psi : \mathbb{P}(\mathcal{L}(D)) \to |D|$$

induced by the assignment $\mathcal{L}(D)^* \ni f \mapsto D + (f) \in |D|$. The map is well-defined since $(f) = (\lambda f)$ for every $\lambda \in \mathbb{C}^*$. To see that ψ is surjective notice that for $E \in |D|$ we have $E = D + (f) \geq 0$ where $f \in \mathcal{M}^*(X)$. Hence, $f \in \mathcal{L}(D)$ and thus, $\psi(f) = E$. To see that the map is injective suppose $\psi(f) = \psi(g)$, i.e. $D+(f) = D+(g)$. It follows that $\left(\frac{f}{g}\right) = 0$, so the meromorphic function $\frac{f}{g}$ has no poles or zeros and since X is compact this implies $\frac{f}{g}$ is a non-zero constant, i.e. $f = \lambda g$ for some $\lambda \in \mathbb{C}^*$. Therefore, f and g define the same element in $\mathbb{P}(\mathcal{L}(D))$. $\square$

Combining Lemma 1.4.1 and Proposition 1.4.2 shows $|D| \cong \mathbb{P}(\mathrm{H}^0(X, \mathcal{O}_X(D)))$ when X is compact. Under this identification we may define linear system of divisors, which does not require X to be compact.

Definition 1.4.3. A *linear system of divisors* on X is a family of effective divisors on X corresponding to a linear subspace of $\mathbb{P}(\mathrm{H}^0(X, L))$ for some holomorphic line bundle $L \to X$. A linear system is called *complete* if the system is of the form $|D|$ for some divisor on X.

Suppose $\mathcal{D} = \{D_\lambda\}_{\lambda \in \mathbb{P}^n}$ is a linear system of divisors. We define the *base locus* of the system $\mathcal{D}$ to be the common intersection $\bigcap_{\lambda \in \mathbb{P}^n} D_\lambda$, which we denote by $\mathcal{B}$. To compute the base locus notice that if $\lambda_0, \ldots, \lambda_n$ are linearly independent in $\mathbb{P}^n$, then

$$\mathcal{B} = D_{\lambda_0} \cap \cdots \cap D_{\lambda_n}.$$

We conclude this section by referencing a fundamental result concerning linear system of divisors called Bertini's theorem, which we swipe from [GH94, pp 137] verbatim.

Theorem 1.4.4 (Bertini's theorem)**.** *The generic element of a linear system is smooth away from the base locus of the system.*

In other words, the singular locus of a generic element in a linear system is contained in the base locus of the system. In particular, if the base locus of a linear system is empty, then a generic element is smooth.

We have stated the bare minimum result necessary for our purposes. For an extensive treatment of linear system of divisors on a compact Riemann surface the reader should consult [Mir95, Chapter V. Section 3].

1.5 Important Theorems

The following theorems are classical and required for our purposes.

Theorem 1.5.1 (Serre Duality)**.** *Suppose X is a compact complex manifold of dimension n, and let $E \to X$ be a holomorphic vector bundle. Then $\mathrm{H}^p(X, E) \cong \mathrm{H}^{n-p}(X, K_X \otimes E^*)^*$ for every $p \geq 0$.*

Theorem 1.5.2 (Kodaira Embedding Theorem)**.** *Suppose X is a compact complex manifold. Then a holomorphic line bundle $L \to X$ is positive if and only if L is ample, i.e., if a tensor power of L defines an embedding of X into projective space. In other words, a compact complex manifold X can be embedded into some projective space $\mathbb{P}^N$ if and only if X admits a positive line bundle.*

Theorem 1.5.3 (Chow's Theorem)**.** *Any complex submanifold $X \subset \mathbb{P}^n$ can be realised as the zero set of finitely many homogeneous polynomials, i.e., every complex submanifold of $\mathbb{P}^n$ is algebraic.*

Chapter 2

Algebraic Geometry on Compact Riemann Surfaces

This chapter elaborates further on the results in Chapter 1 when the complex manifold in question is a compact Riemann surface. Early in the chapter, we introduce the degree of a holomorphic vector bundle and a foundational theorem called the Riemann-Roch theorem. We use the Riemann-Roch theorem to prove several facts. For example, the Riemann-Roch theorem implies that every holomorphic line bundle comes from a divisor, which gives an alternative description of the Picard group. We also define a topological invariant of holomorphic vector bundles called the degree, which is unique to the setting of compact Riemann surfaces. The degree provides enormous insight into the relationship between divisors and holomorphic line bundles and their sections. We conclude the section by rephrasing the language of a linear system of divisors to holomorphic line bundles and studying the sections of holomorphic line bundles whose base locus is empty, i.e., basepoint-free. When classifying Higgs bundles by their spectral data in Chapters 4, 5, and 6 we will only consider basepoint-free holomorphic line bundles.

There are several textbooks on the fundamentals of compact Riemann surfaces. For an algebraic overview the reader can consult Farkas and Kra's *Riemann Surfaces* [FK92], and Forster's *Lectures on Riemann Surfaces* [For81]. Narasimhan's *Compact Riemann surfaces* [Nar92] gives an extensive treatment of holomorphic vector bundles, and Varolin's *Riemann Surfaces by way of Complex Analytic Geometry* [Var11] has a more analytic flavour.

2.1 Holomorphic Vector Bundles on Compact Riemann Surfaces

Throughout this section C denotes a fixed compact Riemann surface with genus g. Recall that the genus g is equal to half the first Betti number, which is a topological invariant.

Over a compact Riemann surface holomorphic vector bundles exude several properties that are not seen in higher dimensions. For instance, the kernel of a holomorphic vector bundle homomorphism that does not have constant rank induces another holomorphic vector bundle, which is not true in general for higher dimensions.

Proposition 2.1.1. *Let E and F be holomorphic vector bundles over a Riemann surface X. Let $f : E \to F$ be a vector bundle homomorphism and let $\mathcal{A}$ be the kernel of the sheaf map $f : \mathcal{E} \to \mathcal{F}$. Then the following is true*

(i) *There exists a holomorphic subbundle $A \subseteq E$ such that $\mathcal{A} = \mathcal{O}_X(A)$ as subsheaves of $\mathcal{E}$;*

(ii) *There is a dense open subset $U \subseteq X$ such that $f|_U$ has constant rank and $\mathcal{A}|_U = \ker(f|_U)$.*

Proof. We may as well assume $f \neq 0$ else the result is trivial. Let $x \in X$ be given. Then, $\mathcal{E}_x$ and $\mathcal{F}_x$ are free $\mathcal{O}_x$-modules, i.e., $\mathcal{E}_x \cong \mathcal{O}_x^{\oplus n}$ and $\mathcal{F}_x \cong \mathcal{O}_x^{\oplus m}$ where n and m are the ranks of E and F respectfully. Hence, we may view $f_x : \mathcal{O}_x^{\oplus n} \to \mathcal{O}_x^{\oplus m}$. Since $\mathcal{O}_x$ is a PID we may choose a basis for $\mathcal{O}_x^{\oplus n}$ and $\mathcal{O}_x^{\oplus m}$ over $\mathcal{O}_x$ such that f_x is in Smith normal form, i.e.,

$$f_x = \begin{bmatrix} \alpha_1 & 0 & 0 & & \cdots & & 0 \\ 0 & \alpha_2 & 0 & & \cdots & & 0 \\ 0 & 0 & \ddots & & & & 0 \\ \vdots & & & \alpha_r & & & \vdots \\ & & & & 0 & & \\ & & & & & \ddots & \\ 0 & & & \cdots & & & 0 \end{bmatrix}$$

where α_i divides α_{i+1}. Notice that $\alpha_i = g_i z^{e_i}$ where $g_i \in \mathcal{O}_x^*$ and so by another change of basis we may write

$$f_x = \begin{bmatrix} z^{e_1} & 0 & 0 & & \cdots & & 0 \\ 0 & z^{e_2} & 0 & & \cdots & & 0 \\ 0 & 0 & \ddots & & & & 0 \\ \vdots & & & z^{e_r} & & & \vdots \\ & & & & 0 & & \\ & & & & & \ddots & \\ 0 & & & \cdots & & & 0 \end{bmatrix}$$

where $0 \leq e_1 \leq \cdots \leq e_r$. Thus, given $c = (c_1, \ldots, c_n) \in \mathcal{O}_x^{\oplus n}$ we see $f_x c = 0$ if and only if $c_1 = \cdots = c_r = 0$. Therefore, choosing a holomorphic coordinate chart (W_x, z) centred at x and choosing analytic representatives $c_i(z)$ for the germ c_i where $i = r+1, \dots, n$ it follows that $\mathcal{A}|_{W_x} = \{(0, \dots, 0, c_{r+1}(z), \dots, c_n(z))\}$, which is the sheaf associated to the vector bundle $W_x \times \mathbb{C}^{n-r} \to W_x$. To prove this globally defines a subbundle of E we need to verify that the rank is independent of $x \in X$. Suppose we choose $y \in W_x$ not equal to x, then the identical argument gives

$$f_y = \begin{bmatrix} z^{h_1} & 0 & 0 & & \cdots & & 0 \\ 0 & z^{e_2} & 0 & & \cdots & & 0 \\ 0 & 0 & \ddots & & & & 0 \\ \vdots & & & z^{h_t} & & & \vdots \\ & & & & 0 & & \\ & & & & & \ddots & \\ 0 & & & \cdots & & & 0 \end{bmatrix}$$

where $0 \leq h_1 \leq \cdots \leq h_t$. Then, choosing a holomorphic coordinate chart (W_y, z') centred at $y \in X$ it follows that $\mathcal{A}|_{W_y} = \{(0, \dots, 0, c_{t+1}(z'), \dots, c_n(z'))\}$, which is sheaf associated to $W_y \times \mathbb{C}^{n-t} \to W_y$. However, $W_x \cap W_y \neq \varnothing$ and over the intersection the different descriptions of $\mathcal{A}$ must agree, i.e., $r = t$, and since X is connected this proves that $\mathcal{A}$ is the sheaf associated to a holomorphic subbundle $A \subset E$.

To see (ii) note that $\{W_x\}_{x \in X}$ defines an open cover for X. Over $W_x \setminus \{x\}$ the map f_x has constant rank, but the rank may not agree at x. Let

$$U = \{y \in X \mid f \text{ has constant rank in a neighbourhood of } y\}$$

then it is clear that $X \setminus U$ is discrete and $U \subseteq X$ is dense, and moreover, $f|_U$ has constant rank and $A|_U = \ker(f|_U)$. □

2.1.1 Degree of a Vector Bundle on a Compact Riemann Surface

Since C is a compact Riemann surface $\mathrm{H}^2(C, \mathbb{Z}) \cong \mathbb{Z}$ where the isomorphism comes from pairing elements of $\mathrm{H}^2(C, \mathbb{Z})$ with the canonical fundamental class $[C] \in \mathrm{H}_2(C, \mathbb{Z})$, which is determined by the natural orientation of C. To define the degree of a smooth complex vector bundle we will first define the degree of a smooth complex line bundle. Recall that every smooth complex line bundle $L \to C$ has a canonical characteristic class, namely the first Chern class $c_1(L) \in \mathrm{H}^2(C, \mathbb{Z})$. Hence, pairing $c_1(L)$ with $[C]$ defines an integer, i.e.,

$$\langle c_1(L), [C] \rangle = \int_C c_1(L) \in \mathbb{Z},$$

The integer obtained is the *degree* of L denoted $\deg(L)$. Since for a general smooth complex vector bundle $E \to C$ the first Chern class is given by $c_1(E) = c_1(\det(E))$ we may define the degree of E to be the degree of $\det(E)$.

Definition 2.1.2. Let $E \to C$ be a smooth complex vector bundle. Then the *degree* of E is the integer defined by

$$\deg(E) := \int_C c_1(E) \in \mathbb{Z}.$$

In other words, $\deg(E) := \deg(\det(E))$.

Remark 2.1.3. Since the first Chern class of a holomorphic vector bundle and the underlying smooth complex vector bundle agree the degree of a holomorphic vector bundle is defined in the same manner. Moreover, since the first Chern class defines a homomorphism $c_1 : \mathrm{Pic}(C) \to \mathrm{H}^2(C, \mathbb{Z})$ it is clear that $\deg : \mathrm{Pic}(C) \to \mathbb{Z}$ defines a homomorphism.

Since the sheaf $\mathcal{C}_C$ is fine recall that the long exact sequence in sheaf cohomology associated to the exponential short exact sequence is given by

$$0 \to \mathrm{H}^1(C, \mathcal{C}_C^*) \xrightarrow{c_1} \mathrm{H}^2(C, \mathbb{Z}) \to 0.$$

Thus, identifying $\mathrm{H}^2(C, \mathbb{Z}) \cong \mathbb{Z}$ gives the short exact sequence

$$0 \to \mathrm{H}^1(C, \mathcal{C}_C^*) \xrightarrow{\deg} \mathbb{Z} \to 0.$$

Therefore, the degree classifies smooth complex line bundles up to isomorphism.

2.1.2 Riemann-Roch Theorem

The Riemann-Roch theorem is a deep classical theorem that has been at the cornerstone of complex algebraic geometry for the majority of the fields existence. Here, we will present the Riemann-Roch theorem for holomorphic vector bundles over a compact Riemann surface C, but note that the theorem exists in higher dimensions. A proof of the Riemann-Roch theorem can be found in any textbook on compact Riemann surfaces, e.g., [Var11, Chapter 13].

Theorem 2.1.4 (Riemann-Roch theorem)**.** *Let $E \to C$ be a rank r holomorphic vector bundle over a compact Riemann surface C with genus g. Then,*

$$h^0(C, E) - h^1(C, E) = \deg(E) + r(1 - g).$$

Remark 2.1.5. The left-hand-side is an analytical invariant and the right-hand-side is a topological invariant. In fact the Riemann-Roch theorem is a special case of the celebrated Atiyah-Singer index theorem since left-hand-side is the analytical index for an elliptic differential operator and the right-hand-side is the topological index.

Corollary 2.1.6. *The first sheaf cohomology of the structure sheaf has complex dimension g, i.e., $h^1(C, \mathcal{O}_C) = g$. Hence, the genus is an analytical invariant.*

Proof. Recall that elements of $\mathrm{H}^0(C, \mathcal{O}_C)$ correspond to global holomorphic functions, which are constant since C is compact, i.e., $\mathrm{H}^0(C, \mathcal{O}_C) \cong \mathbb{C}$. Thus, $h^0(C, \mathcal{O}_C) = 1$ and since $\deg(\mathcal{O}_C) = 0$ the result follows from the Riemann-Roch theorem. □

Corollary 2.1.7. *The degree of the canonical bundle K_C is equal to $2g-2$, i.e., $\deg(K_C) = 2g - 2$.*

Proof. By Serre duality $h^0(C, K_C) = h^1(C, \mathcal{O}_C) = g$ and $h^1(C, K_C) = h^0(C, \mathcal{O}_C) = 1$. Thus, by the Riemann-Roch theorem

$$g - 1 = \deg(K_C) + 1 - g,$$

which shows $\deg(K_C) = 2g - 2$. □

2.2 Holomorphic Line Bundles on compact Riemann Surfaces

From the statement of the Riemann-Roch theorem, if $L \to C$ is a holomorphic line bundle with $\deg(L) \geq g$, then $h^0(C, L) \geq 1$, and hence, L admits a global non-identically zero holomorphic section. In fact, the Riemann-Roch theorem implies that every holomorphic line bundle over a compact Riemann surface admits a global non-identically zero meromorphic section. A similar argument to the proof of Lemma 1.4.1 shows that given a divisor D on C and $L \in \mathrm{Pic}(C)$ the space $\mathrm{H}^0(C, L(D))$ may be viewed as the space consisting of global meromorphic sections s of L such that $(s) + D \geq 0$, and the zero section.

Proposition 2.2.1. *Every holomorphic line bundle $L \to C$ admits a global non-identically zero meromorphic section.*

Proof. Choose sufficiently many points $x_1, \ldots, x_m \in C$ so that $\deg(L(D)) \geq g$ where $D = x_1 + \ldots + x_m$. Then $L(D)$ admits a global non-identically zero holomorphic section, which defines a global non-identically zero meromorphic section of L. □

Recall from Proposition 1.3.17 that a holomorphic line bundle $L \to C$ comes from a divisors $D \in \mathrm{Div}(C)$, i.e., $L \cong \mathcal{O}_C(D)$ if and only if L admits global non-identically zero meromorphic section, and thus, we immediately obtain the following corollary.

Corollary 2.2.2. *Every holomorphic line bundle on C comes from a divisor, and thus, the class divisor group modulo linear equivalence is canonically isomorphic to the Picard group, i.e., $\mathrm{Div}(C)/_\sim \cong \mathrm{Pic}(C)$.*

Remark 2.2.3. Since every holomorphic line bundle $L \to C$ comes from a divisor we may use an alternate form of induction when proving formulas regarding holomorphic line bundles on C. Namely, we will prove the result for $\mathcal{O}_C$, then assume the result is true for L and prove the result is true for $L(p)$ and $L(-p)$ where $p \in C$ is a point divisor.

2.2.1 Different Descriptions of the Degree

One may naturally ask if the degree of a holomorphic line bundle agrees with the degree of the divisor defining the line bundle. This turns out to be the case and in order to do so we will use connections.

Proposition 2.2.4. *Let $L \to C$ be a holomorphic line bundle and suppose s is a global non-identically zero holomorphic section of L. Then, the number of zeros of s counting multiplicity is equal to the degree of L.*

Proof. Let $L \to C$ be a smooth complex line bundle and suppose s is a global section of L with isolated zeros $z_1, \ldots, z_n$ with multiplicity $m_1, \ldots, m_n$ respectively. To compute the degree of L recall from Proposition 1.3.12 that the first Chern class may be represented by a multiple of the curvature form of a connection. Hence, we will construct a connection to compute $\deg(L)$. For each $j = 1, \ldots, n$ choose an open neighbourhood B_j of z_j that is diffeomorphic to an open ball such that $L|_{B_j} \cong \mathcal{O}_{B_j}$ and $\overline{B_j} \cap \overline{B_k} = \varnothing$ for $j \neq k$. Set $U_j = B_j$ and let $U_0 = C \setminus \{z_1, \ldots, z_n\}$, then $\{U_0, U_1, \ldots, U_n\}$ forms an open cover for C. Now, choose nowhere vanishing holomorphic section e_j of L over B_j and define a connection ∇_j on $L|_{U_j}$ by $\nabla_j(e_j) = 0$ for $j = 1, \ldots, n$. Then, set $e_0 = s$ and define ∇_0 on $L|_{U_0}$ by $\nabla_0 e_0 = 0$. Let $\{\rho_j\}$ be partitions of unity subordinate to the given open cover, then we may define a connection ∇ on L by $\nabla := \sum_{j=0}^n \rho_j \nabla_j$. To compute the curvature F_∇ it suffices to compute the curvature over U_0 since $\overline{U_0} = C$. Since $\nabla_0 e_0 = 0$ notice that

$$\nabla e_0 = \sum_{j=0}^{n} \rho_j \nabla_j e_0 = \sum_{j=1}^{n} \rho_j \nabla_j s.$$

Over U_j the section s may be written as $s = f_j e_j$ for some holomorphic function $f_j : U_j \to \mathbb{C}$ that has a zero of order m_j at z_j. Thus,

$$\nabla e_0 = \sum_{j=1}^{n} \rho_j \, \mathrm{d}f_j \otimes e_j = \sum_{j=1}^{n} \rho_j \frac{\mathrm{d}f_j}{f_j} \otimes s.$$

Hence, the connection is of the form $\nabla = \mathrm{d} + a$ where $a := \sum_{j=1}^n \rho_j \frac{\mathrm{d}f_j}{f_j}$, and thus, the curvature of ∇ is equal to $\mathrm{d}a$ since $a \wedge a = 0$. Now,

$$\mathrm{d}a = \sum_{j=1}^{n} \mathrm{d}\left(\rho_j \frac{\mathrm{d}f_j}{f_j}\right)$$

Note, each $\rho_j \frac{\mathrm{d}f_j}{f_j}$ is supported on an annulus A_j where $\rho_j = 1$ on the boundary. Thus,

$$\frac{i}{2\pi} \int_{A_j} \mathrm{d}\left(\rho_j \frac{\mathrm{d}f_j}{f_j}\right) = \frac{i}{2\pi} \int_{\partial A_j} \rho_j \frac{\mathrm{d}f_j}{f_j} = \frac{1}{2\pi i} \oint \frac{\mathrm{d}f_j}{f_j} = m_j$$

where the first equality follows from Stoke's theorem and the last equality is the argument principle. It follows that

$$\frac{i}{2\pi}\int_C F_\nabla = \sum_{j=1}^{n} m_j,$$

which proves the result. $\square$

Corollary 2.2.5. *Suppose $L \to C$ is a holomorphic line bundle such that $L \cong \mathcal{O}_C(D)$ for some $D \in \mathrm{Div}(C)$. Then,* $\deg(L) = \deg(D)$.

Proof. We may write $D = D_1 - D_2$ where $D_1, D_2 \geq 0$. Then $\deg(D) = \deg(D_1) - \deg(D_2)$ and $L \cong \mathcal{O}_C(D_1) \otimes \mathcal{O}_C(D_2)^*$. Thus, it suffices to show $\deg(\mathcal{O}_C(D_i)) = \deg(D_i)$. In other words, we can assume without loss of generality that D is effective. Now, let s_0 be the standard section of $\mathcal{O}_C(D)$, i.e., $(s_0) = D$. The degree of D is precisely the number of zeros of s_0 counting multiplicity, and by Proposition 2.2.4 the number of zeros of s_0 counting multiplicity is equal to $\deg(\mathcal{O}_C(D))$, and hence, $\deg(D) = \deg(\mathcal{O}_C(D))$. $\square$

Corollary 2.2.6. *If $L \to C$ is a holomorphic line bundle with* $\deg(L) < 0$*, then L has no non-identically zero global holomorphic sections, i.e.,* $\mathrm{H}^0(C, L) = 0$. *Moreover, if* $\deg(L) > 2g - 2$*, then* $\mathrm{H}^1(C, L) = 0$.

Proof. The first statement is immediate from Proposition 2.2.4 and the second statement follows from Serre duality since $\deg(K_C \otimes L^*) < 0$. $\square$

2.2.2 Base Points of Holomorphic Line Bundles over Compact Riemann Surfaces

Since every holomorphic line bundle over C comes from a divisor we may reformulate the base locus of linear system of divisors from (1.4) to that of holomorphic line bundles.

Let $L \cong \mathcal{O}_C(D)$ be a holomorphic line bundle over C defined by $D \in \mathrm{Div}(C)$. Recall that $|D| \cong \mathbb{P}(\mathrm{H}^0(C, L))$ and the base locus, $\mathcal{B}$, of $|D|$ is the collection of points in C common to all divisors in $|D|$. In the language of sections the base locus is the collection of $p \in C$ such that every holomorphic section s of L vanishes at p, i.e., $s(p) = 0$. If $|D| = \varnothing$, i.e., $\mathrm{H}^0(C, L) = \{0\}$, then every $p \in C$ belongs to the base locus. This motivates the definition for the basepoint of a holomorphic line bundle.

Definition 2.2.7. Let $L \to C$ be a holomorphic line bundle. A *basepoint* of L is a point $p \in C$ such that $s(p) = 0$ for every $s \in \mathrm{H}^0(C, L)$. In other words, the collection of all basepoints forms the base locus. If L has no basepoints, i.e., for every $p \in C$ there exists a section $s \in \mathrm{H}^0(C, L)$ such that $s(p) \neq 0$, then L is called *basepoint-free.*

For our purposes we will be interested in basepoint-free holomorphic line bundles. Consider now a sufficient condition for a holomorphic line bundle to be basepoint-free.

Lemma 2.2.8. *If* $L \in \mathrm{Pic}(C)$ *such that* $\deg(L) \geq 2g$, *then* L *is basepoint-free.*

Proof. Let $p \in C$ be given. Recall that $\mathrm{H}^0(C, L(-p))$ corresponds to sections of L that vanish at p, and hence, it suffices to show the canonical inclusion $\mathrm{H}^0(C, L(-p)) \to \mathrm{H}^0(C, L)$ is not surjective. Consider the short exact sequence of sheaves

$$0 \to \mathcal{O}_C(L(-p)) \xrightarrow{\otimes s_0} \mathcal{O}_C(L) \xrightarrow{\mathrm{ev}_p} \mathcal{O}_p(L) \to 0$$

where s_0 denotes the standard section of $\mathcal{O}_C(p)$. Passing to the long exact sequence in sheaf cohomology yields

$$0 \to \mathrm{H}^0(C, L(-p)) \to \mathrm{H}^0(C, L) \to L_p \to \mathrm{H}^1(C, L(-p)) \to \cdots .$$

Since $\deg(L(-p)) \geq 2g - 1 > 2g - 2$ we have $\mathrm{H}^1(C, L(-p)) = 0$. Thus, we obtain the short exact sequence

$$0 \to \mathrm{H}^0(C, L(-p)) \to \mathrm{H}^0(C, L) \to L_p \to 0.$$

The result follows from exactness. □

The condition $\deg(L) \geq 2g$ is sufficient for L to be basepoint-free but it is not necessary. For instance, $\mathcal{O}_C$ is basepoint-free since we can take a global nowhere vanishing section, and for $g \geq 1$ the canonical bundle K_C is basepoint-free.

Proposition 2.2.9. *Suppose* $g \geq 1$, *then* K_C *is basepoint-free.*

Proof. Suppose on the contrary that K_C has a basepoint $p \in C$. Then, $\mathrm{H}^0(C, K_C(-p)) \cong \mathrm{H}^0(C, K_C)$, and thus, $h^0(C, K_C(-p)) = g$. Hence, by Serre duality $h^1(C, \mathcal{O}_C(p)) = g$. Now, by the Riemann-Roch theorem

$$h^0(C, \mathcal{O}_C(p)) = h^1(C, \mathcal{O}_C(p)) + \deg(p) + 1 - g = 2.$$

Therefore, there exists a meromorphic function that only has a simple pole at $p \in C$ and no other poles so $C \cong \mathbb{P}^1$, i.e., $g = 0$, which is a contradiction. □

Lemma 2.2.10. *Let* $L \to C$ *be a basepoint-free holomorphic line bundle. Then,* L *has sections* s_1 *and* s_2 *that share no zeros.*

Proof. Since L is basepoint-free there exists $s_1 \in \mathrm{H}^0(C, L)$, which is not identically zero. Let $D_1 := (s_1)$, i.e., D_1 is the divisor associated to s_1. Consider

$$V := \{s \in \mathrm{H}^0(C, L) \mid s(p) = 0 \text{ for some } p \in D_1\}.$$

In other words, V is the space of sections that share a zero with s_1. Recall that

$$\mathrm{H}^0(C, L(-p)) \cong \{s \in H^0(C, L) \mid s(p) = 0\}$$

and hence,

$$V \cong \bigcup_{p \in D_1} \mathrm{H}^0(C, L(-p)).$$

Since L is basepoint-free each $\mathrm{H}^0(C, L(-p))$ defines a hyperplane in $\mathrm{H}^0(C, L)$ and it follows that $V \neq \mathrm{H}^0(C, L)$. □

Lemma 2.2.11. *Let $L \to C$ be a basepoint-free holomorphic line bundle, and suppose $s_1, s_2 \in \mathrm{H}^0(C, L)$ are sections that share no zeros. Then, there are complex numbers $a, b \in \mathbb{C}$ such that $as_1 + bs_2$ only has simple zeros.*

Proof. Consider the function $f : C \to \mathbb{P}^1$ defined by

$$f(p) = [-s_2(p) : s_1(p)].$$

Since s_1 and s_2 are holomorphic it follows that f is holomorphic. Moreover, f is not constant for if f is constant it follows that s_1 and s_2 are proportional, which contradicts that they share no zeros. Thus, $f : C \to \mathbb{P}^1$ defines a branched cover of compact Riemann surfaces. Now, $p \in f^{-1}[a, b]$ if and only if $(as_1 + bs_2)(p) = 0$, i.e., $f^{-1}[a, b] = Z(as_1 + bs_2)$. By a linear change of coordinates it suffices to study the case $[a, b] = [1, 0]$. Suppose $p \in f^{-1}[1, 0]$, then $s_2(p) \neq 0$ and $s_1(p) = 0$. Choose a local trivialisation of L and choose local holomorphic coordinate z centred at p such that $s_1(z) = z^m$. We claim that m is the ramification index of f at p. Since $s_2(0) \neq 0$, we see $s_2(z) \neq 0$ by shrinking the trivialisation if necessary and it follows that

$$f(z) = \left[1 : -\frac{z^m}{s_2(z)}\right].$$

Choosing local coordinates w centred at [1,0] we see

$$f(z) = -\frac{z^m}{s_2(z)} = z^m g(z)$$

for some holomorphic function $g(z)$ such that $g(0) \neq 0$, which proves the claim. Therefore, choosing $[a, b] \in \mathbb{P}^1$ that is not a branch point of $f : C \to \mathbb{P}^1$ it follows that $as_1 + bs_2 \in \mathrm{H}^0(C, L)$ only has simple zeros. □

Corollary 2.2.12. *Basepoint-free holomorphic line bundles over a compact Riemann surface admit sections whose zeros are simple.*

Chapter 3

Complex Abelian Varieties

This chapter introduces complex abelian varieties, which are complex tori with the structure of projective varieties. First, we recall the elementary theory of complex tori necessary for our purposes. In particular, we introduce the dual complex torus and dual abelian variety, which we need to study Langlands duality in the generic fibres of classical Hitchin fibrations. More specifically, in Chapters 4, 5, and 6 we compute the generic fibres of the G-Hitchin fibration for each classical simple Lie group G, which are abelian varieties and prove the generic fibres of the corresponding ${}^LG$-Hitchin fibration is the dual abelian variety. Not every complex torus is an abelian variety, so we study line bundles on complex tori to determine when an abelian variety structure exists. In particular, we reference the Riemann-bilinear relations, which are necessary and sufficient conditions for a complex torus to admit a projective variety structure. Next, we introduce Jacobian and Prym varieties, which are two types of abelian varieties that appear in the classification of the generic fibres of the Hitchin fibration. We prove that the Jacobian variety of a curve is self-dual, i.e., isomorphic to its dual abelian variety and its dimension is equal to the genus of the curve. Finally, we compute the dimension of a general Prym variety, an abelian variety naturally associated with a branched cover of compact Riemann surfaces, and we show that the Prym variety associated with an étale double cover is self-dual. Since we are working in the field of complex numbers, the theory is well understood, and we will be following the canonical reference of Birkenhake and Lange [BL04], which contains analytical arguments. For a more algebraic treatment that generalises the theory over arbitrary fields, consult *Abelian Varieties* by David Mumford [Mum70].

3.1 Rudiments of Complex Tori

3.1.1 Definitions, Isogenies, and the Stein Factorisation

Suppose V is a complex vector space of dimension n, and $\Gamma \subset V$ is a *lattice*, i.e., Γ embeds in X as free abelian group of rank $2n$. Then a *complex torus* is the quotient space $X = V/\Gamma$ endowed with the quotient topology. Since $\Gamma \subset V$ is discrete the action of Γ is locally finite, hence X is a (connected) complex manifold and $\dim(X) = n$. Topologically, a complex tori is a product of finitely many S^1, so complex tori are compact. Moreover, complex tori carry a natural abelian group structure compatible with the complex structure, and thus, complex tori define a complex Lie group. Thus, complex tori are (connected) complex compact abelian Lie groups. Conversely, a standard classification in Lie theory says that every connected complex abelian Lie group is the product of a torus and affine space, and hence, every connected compact complex abelian Lie group is a complex torus. In the complex setting it turns out one can drop the abelian condition since it is automatic, i.e., every connected compact complex Lie group is abelian.

Lemma 3.1.1. *Suppose X is a connected compact complex Lie group. Then, X is abelian.*

Proof. It suffices to prove the commutator $[x, y] = x^{-1}y^{-1}xy$ is trivial for every $x, y \in X$. Let $U \subsetneq X$ be a coordinate patch for the identity element $1 \in X$. Since $[x, 1] = 1$ for every $x \in X$, and the commutator map $[-,-] : X \times X \to X$ is continuous, we can choose open neighbourhoods V_x and W_x for x and 1 respectively such that $[V_x, W_x] \subseteq U$. Since X is compact finitely many V_x cover X, i.e., $X = \bigcup_{j=1}^m V_{x_j}$. Let $W = \bigcap_{j=1}^m W_{x_j}$ denote the corresponding intersection; then $[X, W] \subset U$. For every $y \in W$ the assignment $[-, y] : X \to X$ is holomorphic and since $[X, W] \subseteq U$ we may take U to be a coordinate patch, so $[-, y]$ is a bounded holomorphic function over X, i.e., $[-, y]$ is constant. Since $[1, y] = 1$ for every $y \in W$ it follows that $[X, W] = 1$, which implies $[X, X] = 1$ since W is open and non-empty. $\square$

In summary, every connected compact complex Lie group is a complex torus. Therefore, complex tori are indispensable objects when studying (complex) Lie theory.

For any given complex torus $X = V/\Gamma$ the canonical projection map $\mathrm{pr} : V \to X$ defines a universal cover of X whose kernel is canonically isomorphic to Γ, and hence, $\pi_1(X) \cong \Gamma$. Thus, the fundamental group of a complex torus is canonically isomorphic to its lattice, and since lattices are abelian it follows by Hurewicz theorem that $\mathrm{H}_1(X, \mathbb{Z}) \cong \Gamma$. A *homomorphism* of complex tori is a homomorphism with respect to their complex Lie group structure, and since every connected compact complex Lie group is a complex torus we immediate obtain the following the proposition.

Proposition 3.1.2. *Let $f : X \to X'$ be a homomorphism of complex tori. Then the image, $f(X)$, defines a complex subtorus of X'; and* $\ker(f)$ *defines a closed subgroup of*

X where the connected component containing the identity, $(\ker f)_0$, *has finite index in* $\ker(f)$, *and defines a complex subtorus of X.*

Now, we will show that a holomorphic map $h : X \to X'$ between two complex tori is precisely the composition of a translation and a homomorphism. In what proceeds we denote translating by x_0 by $t_{x_0} : X \ni x \mapsto x + x_0 \in X$.

Proposition 3.1.3. *Let $X = V/\Gamma$ and $X' = V'/\Gamma'$ be complex tori and suppose $h : X \to X'$ is a holomorphic map. Then*

(i) *There is a unique homomorphism $f : X \to X'$ such that $h = t_{h(0)}f$;*

(ii) *There is a unique $\mathbb{C}$-linear map $F : V \to V'$ with $F(\Gamma) \subset \Gamma'$ inducing the homomorphism $f : X \to X'$.*

Proof. Consider the holomorphic map $f = t_{-h(0)}h$. The map $V \xrightarrow{\pi} X \xrightarrow{f} X' = V \xrightarrow{f\pi} X'$ has a unique lift, $F : V \to V'$, to the universal covering space. Since $f\pi = \pi' F$ it follows that $F(\Gamma) \subset \Gamma'$. Moreover, $F(v + \lambda) - F(v) \in \Gamma'$ for every $v \in V$ and $\lambda \in \Gamma$ and since Γ' is discrete the continuous map $V \ni v \mapsto F(v + \lambda) - F(v) \in \Gamma'$ is constant. By considering $v = 0$ it follows that $F(v + \lambda) = F(v) + F(\lambda)$, which shows the partial derivatives of F are $2g$-fold periodic. Thus, the partial derivatives of F are holomorphic and bounded, and hence, constant. Therefore, F is a $\mathbb{C}$-homomorphism and it follows that f is a homomorphism. The uniqueness of f is clear, and by the universal property, F is unique. □

Corollary 3.1.4. *If $h : X \to X'$ is a holomorphic map such that $h(0) = 0$, then h defines a homomorphism.*

The unique map $F : V \to V'$ that induces f is called the *analytic representation* and there is a natural assignment

$$\rho_a : \mathrm{Hom}(X, X') \ni f \mapsto F \in \mathrm{Hom}_{\mathbb{C}}(V, V').$$

Moreover, the restriction of F to Γ is $\mathbb{Z}$-linear and determines F, and hence f, uniquely. We call $F|_\Gamma$ the *rational representation* and we obtain the injective assignment

$$\rho_r : \mathrm{Hom}(X, X') \ni f \mapsto F|_\Gamma \in \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \Gamma')$$

There is a special type of homomorphism of complex tori that is paramount in the theory of abelian varieties, namely an *isogeny*.

Definition 3.1.5. An *isogeny* between two complex tori X and X' is a surjective homomorphism $f : X \to X'$ that has a finite kernel. When an isogeny between two complex tori exists we say the tori are *isogenous* and write $X \simeq X'$.

Of course, a homomorphism $f : X \to X'$ between two complex tori is an isogeny if and only if f is surjective and $\dim(X) = \dim(X')$. If $\Lambda \subset X$ is a finite subgroup, then the quotient space X/Λ defines a complex torus and the projection map $\pi : X \to X/\Lambda$ is an isogeny. Conversely, it is clear that every isogeny is of this type up to isomorphism.

We conclude the subsection by introducing the *Stein factorisation*. Suppose $f : X \to X'$ is a surjective homomorphism between two complex tori. Since $(\ker f)_0 \subset X$ defines a complex subtorus there is a canonical factorisation into a surjective homomorphism $g : X \to X/(\ker f)_0$ whose kernel is a complex torus, and an isogeny $h : X/(\ker f)_0 \to X'$, called the *Stein factorisation* of f

$$\begin{tikzcd} & X/(\ker f)_0 \arrow[dr, "h"] & \\ X \arrow[ur, "g"] \arrow[rr, "f"] & & X' \end{tikzcd}$$

Note, $\ker(h) \cong \pi_0(\ker(f))$, and moreover, if f is an isogeny, then $(\ker f)_0$ is a point, g is the identity map, and $f = h$.

3.1.2 Period Matrix

Let $X = V/\Gamma$ be a complex torus of dimension g. Choose a basis $e_1, \dots, e_g$ for V and a basis $\lambda_1, \dots, \lambda_{2g}$ for Γ. We may express each λ_i in terms of the basis for V, i.e., $\lambda_i = \sum_{j=1}^{n} \lambda_{ji} e_j$. The matrix

$$\Pi = \begin{bmatrix} \lambda_{1,1} & \cdots & \cdots & \lambda_{1,2g} \\ \vdots & \ddots & \ddots & \vdots \\ \lambda_{g,1} & \cdots & \cdots & \lambda_{g,2g} \end{bmatrix}$$

determines the complex tori X completely and is called the *period matrix*. Note also that in this basis $X = \mathbb{C}^g/\Pi\mathbb{Z}^{2g}$. The period matrix is dependent on the choice of bases.

3.1.3 Dual Complex Torus

There are different definitions for the *dual complex torus* that are equivalent. We will provide the definition most relevant for our purposes. Let $X = V/\Gamma$ be a complex torus of dimension g. Consider the complex vector space $\overline{V}^* = \mathrm{Hom}_{\overline{\mathbb{C}}}(V, \mathbb{C})$, i.e., the space of $\mathbb{C}$-antilinear maps. The underlying real vector space is canonically isomorphic to $\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$. To see this, consider the homomorphism $\overline{V}^* \ni l \mapsto \Im(l) \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ with explicit inverse $\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R}) \ni k \mapsto -k(i\bullet) + ik(\bullet) \in \overline{V}^*$. Under this isomorphism it is clear that the map

$$\overline{V}^* \times V \ni (l, v) \mapsto \Im(l(v)) \in \mathbb{R} \tag{3.1}$$

defines a $\mathbb{R}$-bilinear non-degenerate form. Now, the free abelian group $\Gamma^* := \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{Z})$ canonically defines a lattice in the vector space $\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R}) \cong_{\mathbb{R}} \overline{V}^*$. Indeed,

$$\mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathrm{Hom}_{\mathbb{R}}(\Gamma \otimes_{\mathbb{Z}} \mathbb{R}, \mathbb{R}) \cong \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$$

where $\Gamma \otimes_{\mathbb{Z}} \mathbb{R} \cong V$ since Γ defines a lattice in V. Using the non-degenerate form (3.1) we may restate the lattice.

Lemma 3.1.6. *The free abelian group* $\widehat{\Gamma} := \{l \in \overline{V}^* \mid \langle l, \Gamma \rangle \subseteq \mathbb{Z}\}$ *is canonically isomorphic to* $\Gamma^* := \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{Z})$*, and hence,* $\widehat{\Gamma}$ *defines a lattice in* $\overline{V}^*$.

Proof. Under the real isomorphism $\overline{V}^* \cong_{\mathbb{R}} \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$, we see

$$\widehat{\Gamma} = \{k \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R}) \mid k(\Gamma) \subseteq \mathbb{Z}\}.$$

Now, consider the homomorphisms

$$\Gamma^* \ni f \mapsto f \otimes_{\mathbb{Z}} 1 \in \widehat{\Gamma}$$

and

$$\widehat{\Gamma} \ni g \mapsto g|_{\Gamma} \in \Gamma^*.$$

Since (3.1) is non-degenerate it is clear that $(f \otimes_{\mathbb{Z}} 1)|_{\Gamma} = f$ for every $f \in \Gamma^*$, and $g = (g|_{\Gamma} \otimes_{\mathbb{Z}} 1)$ for every $g \in \widehat{\Gamma}$, and thus, the homomorphisms are mutual inverses. □

Thus, we define the *dual complex torus* of X to be the complex torus

$$X^\vee := \overline{V}^* / \widehat{\Gamma}.$$

Note, we may also define the dual complex torus by $X^\vee = \overline{V}^* / \Gamma^*$ and hereon out we will use the definition most convenient to the application. By double anti-duality and since (3.1) is non-degenerate, Γ is the dual lattice of $\widehat{\Gamma}$, and hence, $(X^\vee)^\vee \cong X$. Now that we have established the dual complex torus we will mention some classical results.

Proposition 3.1.7. *Suppose* $X = V/\Gamma$ *is a complex torus. The dual complex torus,* $X^\vee$*, is canonically isomorphic to the group of characters of* Γ*, i.e.,* $X^\vee \cong \mathrm{Hom}(\Gamma, U(1))$.

Proof. Consider the homomorphism

$$\overline{V}^* \ni l \mapsto \exp(2\pi i \langle l, \bullet \rangle) \in \mathrm{Hom}(\Gamma, U(1)).$$

Since (3.1) is non-degenerate, the assignment is surjective, and kernel is precisely $\widehat{\Gamma}$. Therefore, $\overline{V}^* / \widehat{\Gamma} \cong \mathrm{Hom}(\Gamma, U(1))$, i.e., $X^\vee \cong \mathrm{Hom}(\Gamma, U(1))$. □

Recall that the lattice of the complex torus is precisely the fundamental group. Thus, the dual complex torus may equivalently be defined as the space of $U(1)$-representations of the fundamental group, i.e., $\widehat{X} = \mathrm{Hom}(\pi_1(X), U(1))$. By the well-known Appell-Humbert theorem, there is a canonical identification between the space of $U(1)$-representations of the fundamental group of X and the moduli space of holomorphic line bundles on X with trivial first Chern class, i.e., $\mathrm{Hom}(\pi_1(X), U(1)) \cong \mathrm{Pic}^0(X)$. Thus, there is a canonical isomorphism $X^\vee \cong \mathrm{Pic}^0(X)$. Note, this shows that $\mathrm{Pic}^0(X)$ carries the structure of a complex torus. In (3.4) we will prove that $\mathrm{Pic}^0(C)$ carries the structure of a complex tori where C is a compact Riemann surface.

Suppose $f : X_1 \to X_2$ is a homomorphism between two complex tori, $X_i = V_i/\Gamma_i$, with analytical representation $\rho_a(f) = F$. The anti-dual map $F^* : \overline{V_2}^* \to \overline{V_1}^*$ satisfies $F^*\widehat{\Gamma_2} \subset \widehat{\Gamma_1}$, and hence, induces a homomorphism, $f^\vee : X_2^\vee \to X_1^\vee$, called the *dual homomorphism.* If $g : X_2 \to X_3$ is another homomorphism then it is easy to see that $(gf)^\vee = f^\vee g^\vee$. Moreover, it is clear that $(\mathrm{id}_X)^\vee = \mathrm{id}_{X^\vee}$, and thus, dualising defines an involutive contravariant functor on the category of complex tori.

We will now prove an important result regarding complex tori, namely, that the dual sequence associated to a short exact sequence of complex tori is exact. First, we require computing the homotopy groups of complex tori.

Lemma 3.1.8. *Let $X = V/\Gamma$ be a complex torus of dimension g. Then the homotopy groups $\pi_0(X)$ and $\pi_j(X)$ are trivial for $j \geq 2$ and $\pi_1(X) \cong \mathbb{Z}^{2g}$.*

Proof. Complex tori are connected, so $\pi_0(X) \cong 1$, and we already established that $\pi_1(X) \cong \Gamma$, which is a free abelian group of rank $2g$. Recall that the canonical projection map $\mathrm{pr} : V \to X$ is the universal covering, and hence, there is a canonical isomorphism $\pi_j(V) \cong \pi_j(X)$ for $j \geq 2$. However, V is contractible and thus, $\pi_j(X) \cong 1$ for $j \geq 2$. □

Proposition 3.1.9. *Let $X_i = V_i/\Gamma_i$ be complex tori for $i = 1, 2, 3$, and suppose there is a short exact sequence*

$$0 \to X_1 \to X_2 \to X_3 \to 0.$$

Then, the dual sequence

$$0 \to X_3^\vee \to X_2^\vee \to X_1^\vee \to 0$$

defines another short exact sequence.

Proof. By Lemma 3.1.8 the associated exact sequence in homotopy is given by

$$0 \to \Gamma_1 \to \Gamma_2 \to \Gamma_3 \to 0.$$

Now, applying the contravariant functor $\mathrm{Hom}(\bullet, U(1))$ gives the exact sequence

$$0 \to \mathrm{Hom}(\Gamma_3, U(1)) \to \mathrm{Hom}(\Gamma_2, U(1)) \to \mathrm{Hom}(\Gamma_1, U(1)) \to \mathrm{Ext}^1_{\mathbb{Z}}(\Gamma_3, U(1)) \to \cdots.$$

However, Γ_3 defines a free abelian group, and hence, $\operatorname{Ext}^1_{\mathbb{Z}}(\Gamma_3, U(1)) = 0$. Moreover, by Proposition 3.1.7, there is a canonical isomorphism $X_i^\vee \cong \operatorname{Hom}(\Gamma_i, U(1))$ for $i = 1, 2, 3$, and thus,

$$0 \to X_3^\vee \to X_2^\vee \to X_1^\vee \to 0$$

defines a short exact sequence. □

Lemma 3.1.10. *Suppose $h : X \to Y$ is an isogeny of complex tori. Then the dual homomorphism, $h^\vee : Y^\vee \to X^\vee$, is an isogeny too, and $\ker(h^\vee) \cong \operatorname{Hom}(\ker(h), U(1))$, i.e., $\ker(h^\vee)$ is Pontryagin dual to $\ker(h)$.*

Proof. Since h is an isogeny if and only if h is surjective and $\dim(X) = \dim(Y)$ we may assume without loss of generality that $X = V/\Gamma_X$ and $Y = V/\Gamma_Y$, and that the analytic representation is given by id_V, i.e., $\rho_a(h) = \mathrm{id}_V$. Hence, the analytic representation of the dual homomorphism is given by $\mathrm{id}_{\overline{V}^*}$, so $h^\vee$ defines an isogeny. Now, consider the short exact sequence

$$0 \to \ker(h) \to X \xrightarrow{h} Y \to 0.$$

Passing to the exact sequence in homotopy gives

$$0 \to \Gamma_X \to \Gamma_Y \to \ker(h) \to 0.$$

Applying the contravariant functor $\operatorname{Hom}(\bullet, U(1))$ gives the exact sequence

$$0 \to \operatorname{Hom}(\ker(h), U(1)) \to \operatorname{Hom}(\Gamma_Y, U(1)) \to \operatorname{Hom}(\Gamma_X, U(1)) \to \operatorname{Ext}^1_{\mathbb{Z}}(\ker(h), U(1)) \to \cdots$$

Recall that $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_n, U(1)) \cong U(1)/nU(1) \cong 1$, and hence, by the fundamental theorem of finitely generated abelian groups, $\operatorname{Ext}^1_{\mathbb{Z}}(\ker(h), U(1)) = 0$. Thus, by Proposition 3.1.7, we obtain the short exact sequence

$$0 \to \operatorname{Hom}(\ker(h), U(1)) \to Y^\vee \xrightarrow{h^\vee} X^\vee \to 0.$$

By exactness, $\ker(h^\vee) \cong \operatorname{Hom}(\ker(h), U(1))$. □

Corollary 3.1.11. *Suppose $h : X \to Y$ is an isogeny of complex tori. Then h is injective if and only if $h^\vee$ is injective.*

Proposition 3.1.12. *Let $f : X_1 \to X_2$ be a surjective homomorphism of complex tori. Then, $\ker(f)$ is connected if and only if the dual homomorphism, $f^\vee$ is injective.*

Proof. Let $f = hg$ be the Stein factorisation. Recall that $\pi_0(\ker(f)) \cong \ker(h)$, so $\ker(f)$ is connected if and only if h is injective. Hence, by Corollary 3.1.11 we see $\ker(f)$ is connected if and only if $h^\vee$ is injective. Now, $f^\vee = g^\vee h^\vee$ and assuming $g^\vee$ is injective, $h^\vee$

is injective if and only if $f^\vee$ is injective. Thus, it suffices to prove $g^\vee$ is injective. Consider the short exact sequence of complex tori

$$0 \to (\ker f)_0 \to X_1 \xrightarrow{g} X_1/(\ker f)_0 \to 0.$$

By Proposition 3.1.9,

$$0 \to (X_1/(\ker f)_0)^\vee \xrightarrow{g^\vee} X_1^\vee \to (\ker f)_0^\vee \to 0$$

defines a short exact sequence, and thus, $g^\vee$ is injective. □

3.2 Holomorphic Line Bundles on a Complex Torus

We wish to introduce the necessary results concerning holomorphic line bundles on complex tori to define abelian varieties. In particular, in this section we will show that the first Chern class of a holomorphic line bundle can be canonically identified with a Hermitian form that allows one to describe positive line bundles as line bundles whose first Chern class define a positive definite Hermitian form.

3.2.1 Factors of Automorphy

Now, we will introduce factors of automorphy for complex tori. The general theory applies to an arbitrary complex manifold, but we only need the theory for complex tori. Let $X = V/\Gamma$ be a complex tori and recall that $\pi_1(X) \cong \Gamma$. Hence, Γ has a natural left action on V. We wish to describe holomorphic line bundles $L \to X$ such that $\mathrm{pr}^* L \cong \mathcal{O}_V$ in terms of the cohomology associated to the left action of Γ on V. Here pr denotes the projection map $\mathrm{pr} : V \to X$. It turns out that $\mathrm{Pic}(V) = \{\mathcal{O}_V\}$ [1], and hence, we will use factors of automorphy to describe $\mathrm{Pic}(X)$ completely.

Before introducing factors of automorphy note that the action of Γ on V induces a $\pi_1(X)$-module structure on $\mathcal{O}_V^*(V)$.

Definition 3.2.1. A *factor of automorphy* is a holomorphic map $f : \Gamma \times V \to \mathbb{C}^*$ satisfying

$$f(\lambda + \mu, v) = f(\lambda, \mu + v) f(\mu, v)$$

for all $\lambda, \mu \in \pi_1(X)$ and $v \in V$. In other words, f is a 1-cocycle of Γ with values in $\mathcal{O}_V^*(V)$, i.e., $f \in Z^1(\Gamma, \mathcal{O}_V^*(V))$.

[1] It is well known that $\mathrm{H}^1(V, \mathcal{O}_V) = 0$, and since V is contractible, $\mathrm{H}^2(V, \mathbb{Z}) = 0$. Hence $\mathrm{H}^1(V, \mathcal{O}_V^*) = 0$ since $\mathrm{H}^1(V, \mathcal{O}_V) \to \mathrm{H}^1(V, \mathcal{O}_V^*) \xrightarrow{c_1} \mathrm{H}^2(V, \mathbb{Z})$ is exact

A factor of automorphy $f : \Gamma \times V \to \mathbb{C}^*$ defines a holomorphic line bundle on X. Indeed, Γ acts on the trivial holomorphic line bundle $V \times \mathbb{C} \to V$ by

$$\Gamma \times (V \times \mathbb{C}) \ni (\lambda, (v, t)) \mapsto (\lambda v, f(\lambda, v)t) \in V \times \mathbb{C}.$$

The action is free and properly discontinuous, and thus,

$$L = (V \times \mathbb{C})/\Gamma$$

defines a complex manifold, and the projection map $\pi : L \to X$ induced by the projection $V \times \mathbb{C} \to V$ defines a holomorphic line bundle. This construction defines a homomorphism

$$Z^1(\Gamma, \mathcal{O}_V^*(V)) \to \mathrm{Pic}(X).$$

In fact, the homomorphism is surjective and the kernel of the homomorphism is precisely $B^1(\Gamma, \mathcal{O}_V^*(V))$, which gives a canonical isomorphism $\mathrm{H}^1(\Gamma, \mathcal{O}_V^*(V)) \cong \mathrm{Pic}(X)$. We refer the reader to [BL04, Appendix B] for the details.

Proposition 3.2.2 ([BL04, Proposition B.1])**.** *The homomorphism* $Z^1(\Gamma, \mathcal{O}_V^*(V)) \to \mathrm{Pic}(X)$ *induces a canonical isomorphism*

$$\mathrm{H}^1(\Gamma, \mathcal{O}_V^*(V)) \cong \mathrm{Pic}(X).$$

For the rest of the section we will identify $\mathrm{Pic}(X) \cong \mathrm{H}^1(\Gamma, \mathcal{O}_V^*(V))$.

3.2.2 Néron-Severi Group

Let $f \in \mathrm{H}^1(\Gamma, \mathcal{O}_V^*(V))$ be a factor of automorphy with corresponding holomorphic line bundle $L \in \mathrm{Pic}(X)$. Notice that $f = \exp(2\pi i g)$ for some $g : \Gamma \times V \to \mathbb{C}$ holomorphic in V. The following theorem states there there is a canonical isomorphism $\mathrm{H}^2(X, \mathbb{Z}) \to \bigwedge^2 \mathrm{Hom}(\Gamma, \mathbb{Z})$ that allows one to compute $c_1(L) \in \mathrm{H}^2(X, \mathbb{Z})$ with respect to f.

Theorem 3.2.3 ([BL04, Theorem 2.1.2])**.** *There is a canonical isomorphism* $\mathrm{H}^2(X, \mathbb{Z}) \to \bigwedge^2 \mathrm{Hom}(\Gamma, \mathbb{Z})$ *that sends the first Chern class* $c_1(L) \in \mathrm{H}^2(X, \mathbb{Z})$ *to the alternating form*

$$E_L(\lambda, \mu) = g(\mu, v + \lambda) + g(\lambda, v) - g(\lambda, v + \mu) - g(\mu, v)$$

for all $\lambda, \mu \in \Gamma$ *and* $v \in V$.

In fact, we may extend this characterisation to real alternating forms on the vector space V.

Proposition 3.2.4 ([BL04, Proposition 2.1.6])**.** *Let* $E : V \times V \to \mathbb{R}$ *be a real alternating form. Then the following are equivalent*

(i) *There exists a holomorphic line bundle* $L \in \mathrm{Pic}(X)$ *such that* E *represents* L*, i.e.,* $E|_{\Gamma\times\Gamma} = E_L$*;*

(ii) *The alternating form is integral on the lattice and* E *is compatible with the complex structure, i.e.,* $E(\Gamma, \Gamma) \subseteq \mathbb{Z}$ *and* $E(iv, iw) = E(v, w)$ *for every* $v, w \in V$*.*

Finally, we may extend the characterisation to Hermitian forms on V.

Lemma 3.2.5. *There is a one-to-one correspondence between the set of Hermitian forms* H *on* V *and the set of real-valued alternating forms* E *on* V *satisfying* $E(iv, iw) = E(v, w)$*, which is given by*

$$E(v, w) = \Im(H(v, w)) \qquad \text{and} \qquad H(v, w) = E(iv, w) + iE(v, w)$$

for every $v, w \in V$*.*

Corollary 3.2.6. *The first Chern class* $c_1(L) \in \mathrm{H}^2(X, \mathbb{Z})$ *associated to* $L \in \mathrm{Pic}(X)$ *is canonically identified to a Hermitian form* H *on* V *with* $\Im(H(\Gamma, \Gamma)) \subseteq \mathbb{Z}$*.*

Now, we may define the Néron-Severi group associated to X and give a characterisation in terms of the foregoing results.

Definition 3.2.7. The *Néron-Severi group* of X, denoted $\mathrm{NS}(X)$, is the subgroup $\mathrm{NS}(X) \subseteq \mathrm{H}^2(X, \mathbb{Z})$ defined by the image of the first Chern class $c_1 : \mathrm{Pic}(X) \to \mathrm{H}^2(X, \mathbb{Z})$.

Note that $\mathrm{NS}(X)$ is canonically identified with the set of Hermitian forms H on V whose imaginary component is integral on the lattice, i.e., $\Im(H(\Gamma, \Gamma)) \subseteq \mathbb{Z}$. Under this identification we may define a positive definite line bundle.

Definition 3.2.8. A holomorphic line bundle $L \in \mathrm{Pic}(X)$ is called *positive definite* if the Hermitian form H on V associated to $c_1(L)$ is positive definite.

One may naturally ask if positive definite line bundles and positive line bundles on complex tori are the same. This is indeed the case. The proof requires the use of hermitian metrics and some Hodge theory so we divert the reader to Appendix B for the proof.

Suppose $E = \Im(H)$ is the real-valued alternating form associated to $c_1(L)$ and that the complex torus X has dimension g. Then, by the elementary divisor theorem, we may choose a basis $\lambda_1, \ldots, \lambda_g, \mu_1, \ldots, \mu_g$ for Γ such that in the basis

$$E = \begin{bmatrix} 0 & D \\ -D & 0 \end{bmatrix}$$

where $D = \mathrm{diag}(d_1, \ldots, d_g)$ where $d_i \geq 0$ and d_i divides d_{i+1} for $i = 1, \ldots, g-1$. We call the g-tuple $(d_1, \ldots, d_g)$ of L the *type of* L, and we call the basis $\lambda_1, \ldots, \lambda_g, \mu_1, \ldots, \mu_g$ the symplectic basis of Γ for L. In [BL04, Corollary 4.4.6] it is shown that the type of the line bundle only depends on the first Chern class.

3.3 Abelian Varieties

Historically, a *complex abelian variety* is a complex torus that carries the structure of a complex projective variety. By the Kodaira embedding theorem, a complex torus X can be embedded in projective space if and only if X admits a positive line bundle. However, a line bundle on X is positive if and only if it is positive definite, and thus, X can be embedded in projective space if and only if it admits a positive definite line bundle. Moreover, by Chow's theorem a complex submanifold of complex projective space is an algebraic variety. Therefore, we may give an equivalent definition of an abelian variety.

Definition 3.3.1. A complex torus X is a (*polarisable*) *abelian variety* if X admits a positive definite holomorphic line bundle L. The first Chern class of such a line bundle L is called a *polarisation* and the pair (X, H) where $H = c_1(L)$ is called a *polarised abelian variety.*

Since $\phi_L = \phi_N$ whenever $c_1(L) = c_1(N)$ we define the *type* of a polarisation to be the type of the underlying line bundle, and if a polarisation $H = c_1(L)$ has type $(1, \ldots, 1)$ we call the polarisation *principal.* In [BL04, pp. 99] it is shown that H principal implies $\phi_L : X \to X^\vee$ is an isomorphism.

A *homomorphism* of abelian varieties is a map $f : (X', H') \to (X, H)$ where $f : X \to X'$ is a homomorphism of complex tori and $f^*H = H'$. Note that necessarily the homomorphism f has finite kernel for if $f : X' \to X$ had an infinite kernel, then the Hermitian form f^*H would be degenerate. Moreover, if $f : X' \to X$ is a homomorphism of complex tori with finite kernel and $H = c_1(L)$ defines a polarisation on X, i.e., (X, H) is a polarised abelian variety, then $L' = f^*(L)$ defines a polarisation on X called the *induced polarisation.* Consequently, any complex tori isogenous to an abelian variety is an abelian variety, and if $Y \subseteq X$ is a complex subtorus, then by consider the inclusion map $i : Y \to X$ we see i^*L defines a polarisation on Y, and hence, Y defines an abelian variety.

Now, we claim that the dual complex torus to an abelian variety is an abelian variety too. Let $(X, c_1(L))$ be an abelian variety. By identifying $X^\vee \cong \mathrm{Pic}^0(X)$ and noting that $c_1(t_x^*L) = c_1(L)$ where $t_x : X \to X$ is translation by $x \in X$ we may define

$$\phi_L : X \ni x \mapsto t_x^*L \otimes L^{-1} \in X^\vee.$$

The map ϕ_L defines an isogeny of complex tori, and the proof is non-trivial. For instance, to define a homomorphism one must show that $t_{x+y}^*L \cong t_x^*L \otimes t_y^*L \otimes L^{-1}$. Since we only need the result for our purposes we direct the reader to [BL04, Section 2.4] for the proof. Thus, the dual complex torus to an abelian variety is an abelian variety as claimed. We call the dual complex torus to an abelian variety the *dual abelian variety.*

3.3.1 Riemann Bilinear Relations

The Riemann bilinear relations give necessary and sufficient conditions for a complex torus X to admit a polarisation. Moreover, the Riemann bilinear relations are phrased in terms of a period matrix. Choose a basis $e_1, \ldots, e_g$ for V and $\lambda_1, \ldots, \lambda_{2g}$ for Γ. Then, $X = \mathbb{C}^g/\Pi\mathbb{Z}^{2g}$ where $\Pi \in \mathrm{Mat}_{g\times 2g}(\mathbb{C})$ is the period matrix with respect to the chosen bases.

Theorem 3.3.2 (Riemann Bilinear Relations [BL04, Theorem 4.2.1]). *The complex torus X is an abelian variety if and only if there is a non-degenerate alternating matrix $A \in \mathrm{Mat}_{2g}(\mathbb{Z})$ such that*

(i) $\Pi A^{-1}\Pi^t = 0$;

(ii) $i\Pi A^{-1}\overline{\Pi}^t > 0$.

The matrix A corresponds to an alternating form $E : \Gamma \times \Gamma \to \mathbb{Z}$ on the lattice, which extends to $E : \mathbb{C}^g \times \mathbb{C}^g \to \mathbb{R}$. Define $H : \mathbb{C}^g \times \mathbb{C}^g \to \mathbb{C}$ by

$$H(u, v) = E(iu, v) + iE(u, v)$$

then condition (i) is equivalent to saying that H is Hermitian, and (ii) is equivalent to H being positive definite.

3.4 Jacobian Variety

In this section we will introduce a principally polarised abelian variety called the *Jacobian variety*. Although Jacobian varieties can be defined in more general settings, we will restrict our attention to Jacobian varieties associated to a compact Riemann surface C with genus g since that suffices for our purposes. Before introducing the Jacobian variety consider the exact sequence in sheaf cohomology given by the standard exponential sequence

$$\cdots \to \mathrm{H}^1(C,\mathbb{Z}) \to \mathrm{H}^1(C,\mathcal{O}_C) \to \mathrm{H}^1(C,\mathcal{O}_C^*) \xrightarrow{c_1} \mathrm{H}^2(C,\mathbb{Z}) \to \cdots .$$

Since C is compact, $\mathrm{H}^0(C,\mathcal{O}_C) \cong \mathbb{C}$ and $\mathrm{H}^0(C,\mathcal{O}_C^*) \cong \mathbb{C}^*$, so the map $\mathrm{H}^0(C,\mathcal{O}_C) \xrightarrow{\exp} \mathrm{H}^0(C,\mathcal{O}_C^*)$ becomes $\mathbb{C} \xrightarrow{\exp} \mathbb{C}^*$, which is surjective. Moreover, since C is complex one-dimensional, $\mathrm{H}^2(C,\mathcal{O}_C) = 0$ by the Dolbeault isomorphism theorem. Therefore,

$$0 \to \mathrm{H}^1(C,\mathbb{Z}) \to \mathrm{H}^1(C,\mathcal{O}_C) \to \mathrm{H}^1(C,\mathcal{O}_C^*) \xrightarrow{c_1} \mathrm{H}^2(C,\mathbb{Z}) \to 0$$

defines an exact sequence. Thus, we obtain the short exact sequence

$$0 \to \mathrm{H}^1(C,\mathcal{O}_C)/\mathrm{H}^1(C,\mathbb{Z}) \to \mathrm{Pic}(C) \xrightarrow{\deg} \mathbb{Z} \to 0.$$

Hence, the space $\mathrm{H}^1(C, \mathcal{O}_C)/\mathrm{H}^1(C, \mathbb{Z})$ may be realised as the moduli space of degree zero holomorphic line bundles, which we denote by $\mathrm{Pic}^0(C)$. Now, $\mathrm{H}^1(C, \mathcal{O}_C) \cong \mathbb{C}^g$ and $\mathrm{H}^1(C, \mathbb{Z}) \cong \mathbb{Z}^{2g}$, so $\mathrm{Pic}^0(C) \cong \mathbb{C}^g/\mathbb{Z}^{2g}$, and hence, one may naturally ask if $\mathrm{Pic}^0(C)$ defines a complex torus, which is to say $\mathrm{H}^1(C, \mathbb{Z}) \to \mathrm{H}^1(C, \mathcal{O}_C)$ embeds as a lattice. This turns out to be the case.

Proposition 3.4.1. *The moduli space of a degree zero holomorphic line bundles on C naturally inherits the structure of a complex torus, i.e., $\mathrm{Pic}^0(C)$ defines a complex torus.*

Proof. The map $\mathrm{H}^1(C, \mathbb{Z}) \to \mathrm{H}^1(C, \mathcal{O}_C)$ factors through $\mathrm{H}^1(C, \mathbb{R})$, i.e., $\mathrm{H}^1(C, \mathbb{Z}) \to \mathrm{H}^1(C, \mathbb{R}) \to \mathrm{H}^1(C, \mathcal{O}_C)$. By the universal coefficient theorem, $\mathrm{H}^1(C, \mathbb{R}) \cong \mathrm{H}^1(C, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$, and hence, the map $\mathrm{H}^1(C, \mathbb{Z}) \to \mathrm{H}^1(C, \mathbb{R})$ clearly embeds as a lattice. Thus, we are left to prove that the map $\mathrm{H}^1(C, \mathbb{R}) \to \mathrm{H}^1(C, \mathcal{O}_C)$ is a real isomorphism. Since $\dim_{\mathbb{R}}(\mathrm{H}^1(C, \mathcal{O}_C)) = 2g$ and $\dim_{\mathbb{R}}(\mathrm{H}^1(C, \mathbb{R})) = 2g$ it suffices to prove the map is injective. By the Dolbeault isomorphism theorem, $\mathrm{H}^1(C, \mathcal{O}_C) \cong \mathrm{H}^{0,1}_{\overline{\partial}}(C)$, and by the de Rham isomorphism theorem $\mathrm{H}^1(C, \mathbb{R}) \cong \mathrm{H}^1_{dR}(C)$, and hence, the map of interest becomes $\mathrm{H}^1_{dR}(C) \to \mathrm{H}^{0,1}_{\overline{\partial}}(C)$. This map is induced as follows: let ω be a d-closed real-valued 1-form, which can be realised as an element of $T^*C^{\mathbb{C}}$ by $\omega \otimes_{\mathbb{C}} 1$. Under the canonical splitting we may write $\omega = u + \overline{u}$ where u is a $(1,0)$-form. Then, the map $\mathrm{H}^1_{dR}(C) \to \mathrm{H}^{0,1}_{\overline{\partial}}(C)$ is induced from the assignment $u + \overline{u} \mapsto \overline{u}$. To see that this map is injective suppose $\overline{u} = \overline{\partial} f$ for some smooth complex-valued function f, so $u = \partial \overline{f}$. Since $u + \overline{u}$ is d-closed it follows that $\overline{\partial}\partial(\overline{f} - f) = 0$. Writing $\overline{f} - f = ig$ for some real-valued smooth function g we see $i\overline{\partial}\partial g = 0$, and hence $ig\overline{\partial}\partial g = 0$. Integrating over C and applying Stokes' theorem gives

$$\int_C i(\overline{\partial} g \wedge \partial g) = 0,$$

which implies $\partial g = 0$ since the assignment $\Omega^{1,0}(C) \ni \tau \mapsto i\overline{\tau} \wedge \tau \in \Omega^{1,1}_{\mathbb{R}}(C)$ is positive definite. Hence, $\Im(f)$ is constant so we may assume without loss of generality that f is real-valued, and thus, $u + \overline{u} = \mathrm{d}f$. Therefore, the kernel of $u + \overline{u} \mapsto \overline{u}$ is $\overline{\partial}$-exact exactly when $u + \overline{u}$ is d-exact, which is to say the induced map $\mathrm{H}^1_{dR}(C) \to \mathrm{H}^{0,1}_{\overline{\partial}}(C)$ is injective, and hence, an isomorphism. $\square$

Thus, $\mathrm{Pic}^0(C)$ canonically defines a complex torus, which enables us to define the Jacobian variety of a compact Riemann surface.

Definition 3.4.2. Suppose C is a compact Riemann surface of genus g. Then the *Jacobian variety* of C, denoted $\mathrm{Jac}(C)$ is defined to be the moduli space of degree 0 holomorphic line bundle on C, i.e., $\mathrm{Jac}(C) := \mathrm{Pic}^0(C)$, which defines a complex torus of dimension g.

Strictly speaking $\mathrm{Jac}(C)$ should be called the Jacobian *torus* since we have not seen that $\mathrm{Jac}(C)$ defines an abelian variety. To see that $\mathrm{Jac}(C)$ defines an abelian variety we will show that the dual complex torus to the Jacobian satisfies the Riemann bilinear relations. The dual complex torus to the Jacobian variety is called the *Albanese variety*, denoted $\mathrm{Alb}(C)$, i.e., $\mathrm{Alb}(C) := \mathrm{Jac}(C)^\vee$.

3.4.1 Albanese Variety

By setting $V := \mathrm{H}^1(C, \mathcal{O}_C)$ and $\Gamma := \mathrm{H}^1(C, \mathbb{Z})$ we see $\mathrm{Jac}(C) = V/\Gamma$. Hence, the Albanese variety is given by $\mathrm{Alb}(C) = \overline{V}^*/\Gamma^*$. Now, by twice applying the Dolbeault isomorphism theorem we see

$$\overline{V}^* = \overline{\mathrm{H}^1(C, \mathcal{O}_C)}^* \cong \overline{\mathrm{H}^{0,1}_{\overline{\partial}}(C)}^* = \mathrm{H}^{1,0}_{\overline{\partial}}(C)^* \cong \mathrm{Hom}_{\mathbb{C}}(\mathrm{H}^0(C, K_C), \mathbb{C}). \tag{3.2}$$

Moreover, by the universal coefficient theorem, $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}^1(C,\mathbb{Z}),\mathbb{Z}) \cong \mathrm{H}_1(C,\mathbb{Z})$, i.e., $\Gamma^* \cong \mathrm{H}_1(C,\mathbb{Z})$. Thus, the Albanese variety is given by $\mathrm{Alb}(C) \cong \mathrm{Hom}_{\mathbb{C}}(\mathrm{H}^0(C, K_C), \mathbb{C})/\mathrm{H}_1(C,\mathbb{Z})$. However, we need to unwind the identifications to see how $\mathrm{H}_1(C,\mathbb{Z})$ embeds as a lattice inside $\mathrm{Hom}_{\mathbb{C}}(\mathrm{H}^0(C, K_C), \mathbb{C})$. To do so we will consider complex vector spaces as real vector spaces with a complex structure, i.e., an endomorphism that squares to $-\mathrm{id}$, and prove the necessary results.

Let W be a real vector space and let J be a complex structure on W, then $W \otimes_{\mathbb{R}} \mathbb{C} \cong W^{1,0} \oplus W^{0,1}$ where $W^{1,0}$ and $W^{0,1}$ denote the i and $-i$ eigenspaces respectively. There is a canonical isomorphism $(W, J) \cong (W^{1,0}, i)$ given by

$$(W, J) \ni w \mapsto w^{1,0} := \frac{1}{2}(1 - iJ)w \in (W^{1,0}, i)$$

with explicit inverse

$$(W^{1,0}, i) \ni t \mapsto t + \overline{t} \in (W, J).$$

Lemma 3.4.3. *The real vector space* $\mathrm{Hom}_{\mathbb{R}}(W, \mathbb{R})$ *equipped with the complex structure* J^t *is isomorphic to the complex vector space* $\mathrm{Hom}_{\mathbb{C}}(W^{1,0}, \mathbb{C})$ *with explicit isomorphism* $(\mathrm{Hom}_{\mathbb{R}}(W, \mathbb{R}), J^t) \in \lambda \mapsto \mu \in \mathrm{Hom}_{\mathbb{C}}(W^{1,0}, \mathbb{C})$ *where*

$$\mu(v) := \frac{1}{2}(\lambda(v + \overline{v}) - i\lambda(iv - i\overline{v}))$$

where $v \in W^{1,0}$.

Proof. Let $W^* = \mathrm{Hom}_{\mathbb{R}}(W, \mathbb{R})$ denote the real dual vector space. Then J^t forms a complex structure on W^* by $(J^t\lambda)(v) = \lambda(Jv)$ where $\lambda \in W^*$ and $v \in W$. Now,

$$W^* \otimes_{\mathbb{R}} \mathbb{C} = (W^*)^{1,0} \oplus (W^*)^{0,1}.$$

However, $W^* \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{Hom}_{\mathbb{R}}(W, \mathbb{C})$, and thus, $(W^*)^{1,0} = \{\lambda \in \mathrm{Hom}_{\mathbb{R}}(W, \mathbb{C}) \mid J^t\lambda = i\lambda\}$. That is, $\lambda \in (W^*)^{1,0}$ if and only if $\lambda(Jv) = i\lambda(v)$ for every $v \in W$, or, equivalently, $\lambda : (W, J) \to (\mathbb{C}, i)$ defines a $\mathbb{C}$-linear homomorphism. Since $(W, J) \cong (W^{1,0}, i)$ one sees that $(W^*)^{1,0} = \mathrm{Hom}_{\mathbb{C}}((W, J), \mathbb{C}) \cong \mathrm{Hom}_{\mathbb{C}}(W^{1,0}, \mathbb{C})$, and thus, $(W^*, J^t) \cong \mathrm{Hom}_{\mathbb{C}}(W^{1,0}, \mathbb{C})$. We are left to show this is the desired isomorphism. Indeed, the isomorphism $(W^*, J^t) \cong \mathrm{Hom}_{\mathbb{C}}((W, J), \mathbb{C})$ is given by

$$(W^*, J^t) \ni \lambda \mapsto \lambda^{1,0} \in \mathrm{Hom}_{\mathbb{C}}((W, J), \mathbb{C}).$$

Moreover, the isomorphism $\mathrm{Hom}_{\mathbb{C}}((W,J),\mathbb{C}) \cong \mathrm{Hom}_{\mathbb{C}}(W^{1,0},\mathbb{C})$ is given by

$$\mathrm{Hom}_{\mathbb{C}}((W,J),\mathbb{C}) \ni \ell \mapsto \mu \in \mathrm{Hom}_{\mathbb{C}}(W^{1,0},\mathbb{C})$$

where $\mu(v) = \ell(v+\overline{v})$ for every $v \in W^{1,0}$. Combining the two isomorphisms, i.e., setting $\ell = \lambda^{1,0}$ shows $\mu(v) = \frac{1}{2}(1-iJ^t)\lambda(v+\overline{v})$. In particular, since $Jv = iv$ and $J\overline{v} = -i\overline{v}$ it follows that

$$\mu(v) = \frac{1}{2}(\lambda(v+\overline{v}) - i\lambda(iv - i\overline{v})). \tag{3.3}$$

□

Recall that the complex vector space $\mathrm{H}^1(C,\mathcal{O}_C)$ has underlying real vector space $\mathrm{H}^1(C,\mathbb{R})$, and hence, $V = (\mathrm{H}^1(C,\mathbb{R}), I)$ where I is the complex structure, so $\overline{V}^* = (\mathrm{Hom}_{\mathbb{R}}(\mathrm{H}^1(C,\mathbb{R}),\mathbb{R}), -I^t)$. Therefore, the real vector space of interest is $W = \mathrm{H}^1(C,\mathbb{R})$, and from (3.2) one sees that $W^{1,0} = \mathrm{H}^0(C,K_C)$. We now have the necessary machinery to deduce how the lattice is embedded in the Albanese variety.

Lemma 3.4.4. *The Albanese variety is given by* $\mathrm{Alb}(C) \cong \mathrm{Hom}_{\mathbb{C}}(\mathrm{H}^0(C,K_C),\mathbb{C})/\mathrm{H}_1(C,\mathbb{Z})$ *where* $\mathrm{H}_1(C,\mathbb{Z})$ *is embedded inside* $\mathrm{Hom}_{\mathbb{C}}(\mathrm{H}^0(C,K_C),\mathbb{C})$ *by*

$$\mathrm{H}_1(C,\mathbb{Z}) \ni [\gamma] \mapsto \left([\omega] \mapsto \int_\gamma \omega\right) \in \mathrm{Hom}_{\mathbb{C}}(\mathrm{H}^0(C,K_C),\mathbb{C})$$

Proof. By the universal coefficient theorem $\mathrm{H}_1(C,\mathbb{R}) \cong \mathrm{H}_1(C,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$, and hence, $\mathrm{H}_1(C,\mathbb{Z})$ canonically embeds inside $\mathrm{H}_1(C,\mathbb{R})$ as a lattice via $\mathrm{H}_1(C,\mathbb{Z}) \ni [\gamma] \mapsto [\gamma] \otimes_{\mathbb{Z}} 1 \in \mathrm{H}_1(C,\mathbb{R})$. Moreover, from the natural pairing of cohomology and homology there is a canonical isomorphism $\mathrm{H}_1(C,\mathbb{R}) \cong \mathrm{Hom}_{\mathbb{R}}(\mathrm{H}^1(C,\mathbb{R}),\mathbb{R})$. Explicitly, the isomorphism is given by

$$\mathrm{H}_1(C,\mathbb{R}) \ni [\tau] \mapsto \left([\sigma] \mapsto \int_\tau \sigma\right) \in \mathrm{Hom}_{\mathbb{R}}(\mathrm{H}^1(C,\mathbb{R}),\mathbb{R}).$$

Hence, $\mathrm{H}_1(C,\mathbb{Z})$ canonically embeds as a lattice inside $\mathrm{Hom}_{\mathbb{R}}(\mathrm{H}^1(C,\mathbb{R}),\mathbb{R})$ by

$$\mathrm{H}_1(C,\mathbb{Z}) \ni [\gamma] \mapsto \left([\sigma] \mapsto \int_\gamma \sigma\right) \in \mathrm{Hom}_{\mathbb{R}}(\mathrm{H}^1(C,\mathbb{R}),\mathbb{R}).$$

Fix $[\gamma] \in \mathrm{H}_1(C,\mathbb{Z})$ and consider the homomorphism $\lambda : \mathrm{H}^1(C,\mathbb{R}) \to \mathbb{R}$ that sends

$$[\sigma] \mapsto \int_\gamma \sigma.$$

Then, under the isomorphism (3.3) we may realise λ belonging to $\mathrm{Hom}_{\mathbb{C}}(\mathrm{H}^0(C,K_C),\mathbb{C})$ by

$$\mu(\omega) = \frac{1}{2}\left(\int_\gamma (\omega + \overline{\omega}) - i\int_\gamma (i\omega - i\overline{\omega})\right) = \int_\gamma \omega.$$

In other words, $\mathrm{H}_1(C,\mathbb{Z})$ embeds as a lattice inside $\mathrm{Hom}_{\mathbb{C}}(\mathrm{H}^0(C,K_C),\mathbb{C})$ via

$$\mathrm{H}_1(C,\mathbb{Z}) \ni [\gamma] \mapsto \left(\omega \mapsto \int_\gamma \omega\right) \in \mathrm{Hom}_{\mathbb{C}}(\mathrm{H}^0(C,K_C),\mathbb{C}).$$

□

3.4.2 Jacobian and Albanese Varieties are Isomorphic

Over a compact Riemann surface C the Jacobian and Albanese varieties are canonically isomorphic, which we will now show using divisors. Indeed, fix $b \in C$ and consider $a \in C$ as a point divisor. Then, choosing a real curve γ from b to a enables us to define the homomorphism

$$\mathrm{H}^0(C,K_C) \ni \omega \mapsto \int_\gamma \omega \in \mathbb{C}.$$

However, the choice of γ is not unique and the value is not independent of choice of γ. However, if γ' is another real curve from b to a then the difference $\gamma - \gamma' = \sigma$ only depends on the homotopy, and hence, homology, i.e., σ is unique up to $\mathrm{H}_1(C,\mathbb{Z})$. Moreover, $\mathrm{H}_1(C,\mathbb{Z})$ embeds as a lattice inside $\mathrm{Hom}_{\mathbb{C}}(\mathrm{H}^0(C,K_C),\mathbb{C})$ via $\mathrm{H}_1(C,\mathbb{Z}) \ni [\gamma] \mapsto \left(\omega \mapsto \int_\gamma \omega\right) \in \mathrm{Hom}_{\mathbb{C}}(\mathrm{H}^0(C,K_C),\mathbb{C})$. Therefore, the assignment $a \mapsto \left(\omega \mapsto \int_\gamma \omega \mod \mathrm{H}_1(C,\mathbb{Z})\right)$ is well-defined and valued in $\mathrm{Alb}(C)$. For brevity, we will set $\int_b^a \omega := \int_\gamma \omega \mod \mathrm{H}_1(C,\mathbb{Z})$. Denote the map by u_b, then by decreeing $u_b\left(\sum_{i=1}^N x_i - y_i\right) = \sum_{i=1}^N u_b(x_i) - u_b(y_i)$ we obtain a well defined homomorphism $u_b : \mathrm{Div}^0(C) \to \mathrm{Alb}(C)$.

Lemma 3.4.5. *The assignment $u_b : \mathrm{Div}^0(C) \to \mathrm{Alb}(C)$ is independent of choice of $b \in C$, and hence, defines a homomorphism $u : \mathrm{Div}^0(C) \to C$, which is called the Abel-Jacobi map.*

Proof. Let $D = \sum_{i=1}^N (x_i - y_i)$ be a given element of $\mathrm{Div}^0(C)$. Then, since

$$u_b(x_i - y_i)(\omega) = \int_b^{x_i} \omega - \int_b^{y_i} \omega = \int_b^{x_i} \omega + \int_{y_i}^{b} \omega = \int_{y_i}^{x_i} \omega$$

for $i = 1, \ldots, N$, it follows that

$$u_b(D)(\omega) = \int_{y_1}^{x_1} \omega + \cdots + \int_{y_N}^{x_N} \omega,$$

which is indepent of $b \in C$. □

The well-known Jacobi inversion theorem states that the Abel-Jacobi map is surjective, and the well-known Abel theorem states that D is linearly equivalent to zero if and only if $u(D)(\omega) = 0$ for every $\omega \in \mathrm{Div}^0(C)$ Therefore, by the first isomorphism theorem $\mathrm{Div}^0(C)/_\sim \cong \mathrm{Alb}(C)$, or, equivalently, $\mathrm{Jac}(C) \cong \mathrm{Alb}(C)$.

Theorem 3.4.6. *The Jacobian variety and Albanese variety of C are isomorphic, i.e., $\mathrm{Jac}(C) \cong \mathrm{Alb}(C)$.*

3.4.3 Period Matrix

We will compute the period matrix of the Albanese variety in a basis and show that the period matrix satisfies the Riemann bilinear relations, thus showing the Albanese variety defines an abelian variety, and in particular, the Jacobian variety defines an abelian variety.

Let $\lambda_1, \ldots, \lambda_{2g}$ be a basis for $\mathrm{H}_1(C, \mathbb{Z})$ that has intersection matrix

$$A^{-1} := \begin{bmatrix} 0 & -1_g \\ 1_g & 0 \end{bmatrix}.$$

Notice that $A \in \mathrm{Mat}_{2g}(\mathbb{Z})$ is a non-degenerate alternating matrix. Let $\omega_1, \ldots, \omega_g$ be a basis for $\mathrm{H}^0(C, K_C)$, and denote the dual basis by $l^1, \ldots, l^g$, i.e., $l^i(\omega_j) = \delta_j{}^i$. Then, with respect to the homology basis and the basis $l_1, \ldots, l_g$ for $\mathrm{Hom}_{\mathbb{C}}(\mathrm{H}^0(C, K_C), \mathbb{C})$ the period matrix is given by

$$\Pi = \begin{bmatrix} \int_{\lambda_1} \omega_1 & \cdots & \cdots & \int_{\lambda_{2g}} \omega_1 \\ \vdots & \ddots & \ddots & \vdots \\ \int_{\lambda_1} \omega_g & \cdots & \cdots & \int_{\lambda_{2g}} \omega_g \end{bmatrix}. \tag{3.4}$$

By the Riemann bilinear relations, to prove $\mathrm{Alb}(C)$ defines an abelian variety it is necessary and sufficient to show

$$\Pi A^{-1} \Pi^t = 0 \qquad \text{and} \qquad i \Pi A^{-1} \overline{\Pi}^t > 0.$$

Recall that the basis $\lambda_1, \ldots, \lambda_{2g}$ canonically defines a basis for the real vector space $\mathrm{Hom}_{\mathbb{C}}(\mathrm{H}^0(C, K_C), \mathbb{C})$. The period matrix Π in (3.4) is the matrix of the $\mathbb{C}$-linear map $\mathrm{Hom}_{\mathbb{C}}(\mathrm{H}^0(C, K_C), \mathbb{C}) \to \mathrm{Hom}_{\mathbb{C}}(\mathrm{H}^1_{dR}(C, \mathbb{C}), \mathbb{C})$ in the bases $l^1, \ldots, l^g$ and $\lambda_1, \ldots, \lambda_{2g}$ where $\mathrm{Hom}_{\mathbb{C}}(\mathrm{H}^1_{dR}(C, \mathbb{C}), \mathbb{C}) \cong \mathrm{H}_1(C, \mathbb{C})$ by the de Rham isomorphism theorem. Let $\psi_1, \ldots, \psi_{2g}$ be the basis for $\mathrm{H}^1_{dR}(C, \mathbb{C})$ dual to $\mathrm{H}_1(C, \mathbb{C})$, i.e., $\int_{\lambda_i} \psi_j = \delta_j{}^i$. Then, the dual map $\mathrm{H}^1_{dR}(C, \mathbb{C}) \to \mathrm{H}^0(C, K_C)$ is given by Π^t with respect to the bases $\psi_1, \ldots, \psi_{2g}$ and $\omega_1, \ldots, \omega_g$. Hence,

$$\omega_i = \sum_{j=1}^{2g} \left(\int_{\lambda_j} \omega_i \right) \psi_j$$

for $i = 1, \ldots, g$. Now, let $P : \mathrm{H}_1(C, \mathbb{C}) \to \mathrm{H}^1_{dR}(C, \mathbb{C})$ denote the Poincaré duality isomorphism. Since the intersection form in homology is Poincaré dual to the wedge product in de Rham cohomology

$$(\lambda_i \cdot \lambda_j) = \int_C P(\lambda_i) \wedge P(\lambda_j).$$

Notice that $P(\lambda_1), \ldots, P(\lambda_{2g})$ defines a basis for $\mathrm{H}^1_{dR}(C, \mathbb{C})$. We will determine the relationship between this basis and $\psi_1, \ldots, \psi_{2g}$. Suppose $P(\lambda_j) = \sum_{i=1}^{2g} a_{ij}\psi_i$. Then,

$$a_{ij} = \sum_{t=1}^{2g} a_{tj} \left(\int_{\lambda_i} \psi_t \right) = \int_{\lambda_i} P(\lambda_j) = \int_C P(\lambda_i) \wedge P(\lambda_j) = (\lambda_i \cdot \lambda_j) = \begin{bmatrix} 0 & -1_g \\ 1_g & 0 \end{bmatrix}_{ij}.$$

From this relationship it follows that

$$\int_C \psi_i \wedge \psi_j = \begin{bmatrix} 0 & -1_g \\ 1_g & 0 \end{bmatrix}_{ij}$$

for $i, j = 1, \ldots, 2g$. Then, we see

$$\begin{aligned} i \int_C \omega_j \wedge \overline{\omega_k} &= i \sum_{s,t=1}^{2g} \left(\int_{\lambda_s} \omega_i \right) \left(\int_{\lambda_t} \overline{\omega_j} \right) \int_C \psi_s \wedge \psi_t \\ &= i \sum_{s,t=1}^{2g} \Pi_{is} \begin{bmatrix} 0 & -1_g \\ 1_g & 0 \end{bmatrix}_{st} \overline{\Pi}_{jt} \\ &= i \left(\Pi A^{-1} \overline{\Pi}^t \right)_{jk}. \end{aligned}$$

Therefore, given an arbitrary holomorphic 1-form $\omega = \sum_{k=1}^{g} a_k \omega_k$

$$i \int_C \omega \wedge \overline{\omega} = \begin{bmatrix} a_1 & \cdots & a_g \end{bmatrix} \left(i \Pi A^{-1} \overline{\Pi}^t \right) \begin{bmatrix} \overline{a_1} \\ \vdots \\ \overline{a_g} \end{bmatrix}.$$

Since $i \int_C \omega \wedge \overline{\omega} \geq 0$ with equality if and only if $\omega = 0$ it follows that $i\Pi A^{-1}\overline{\Pi}^t > 0$, which verifies one condition. To see $\Pi A^{-1} \Pi^t = 0$ notice that $\omega_j \wedge \omega_k = 0$ since $\omega_j \wedge \omega_k$ defines a holomorphic 2-form on a compact Riemann surface, and thus

$$\int_C \omega_j \wedge \omega_k = \left(\Pi A^{-1} \Pi^t \right)_{jk} = 0.$$

Therefore, $\Pi A \Pi^t = 0$, so $\mathrm{Alb}(C)$ defines an abelian variety.

Proposition 3.4.7. *The Albanese variety of a compact Riemann surface defines an abelian variety, and hence, the Jacobian variety of an abelian variety defines an abelian variety.*

A divisor Θ on $\mathrm{Alb}(C)$ (resp. $\mathrm{Jac}(C)$) whose associated line bundle $\mathcal{O}_{\mathrm{Alb}(C)}(\Theta)$ (resp. $\mathcal{O}_{\mathrm{Jac}(C)}(\Theta)$) defines the canonical polarisation associated to A is called a *theta divisor*.

3.5 Prym Varieties

3.5.1 Norm Map and Definitions

To define Prym varieties we first need to recall the *norm map* associated to a branched cover between two compact Riemann surfaces. Throughout this section $\pi : C \to C'$ denotes a branched cover between two compact Riemann surfaces. Suppose $K(C)$ and $K(C')$ denote the function fields of C and C' respectively, then the norm map associated to $\pi : C \to C'$ is the map $\mathrm{Nm}_\pi : K(C) \to K(C')$ defined by $\mathrm{Nm}_\pi(f)(x) = \prod_{\pi(y)=x} f(y)$. Clearly, if f is not identically zero, then $\mathrm{Nm}_\pi(f)$ is not identically zero, i.e., $\mathrm{Nm}_\pi : K(C)^* \to K(C')^*$. We wish to extend the norm map to divisors in a canonical way. The map $\pi : C \to C'$ defines a canonical map $\mathrm{Div}(C) \to \mathrm{Div}(C')$ given by $\pi\left(\sum_{i=1}^N n_i x_i\right) = \sum_{i=1}^N n_i \pi(x_i)$, which we claim is the canonical extension of Nm_π to divisors.

Lemma 3.5.1. *The norm map* $\mathrm{Nm}_\pi : K(C)^* \to K(C')^*$ *is compatible with the divisor map* $\pi : \mathrm{Div}(C) \to \mathrm{Div}(C')$*, i.e.,* $\pi((f)) = (\mathrm{Nm}_\pi(f))$.

Proof. To prove $\pi((f)) = (\mathrm{Nm}_\pi(f))$ we are required to show $\mathrm{ord}_p(f) = \mathrm{ord}_{\pi(p)}(\mathrm{Nm}_\pi(f))$ for every $p \in C$. Let $a = \mathrm{ord}_p(f)$. There are two cases to check, namely when p is not a ramification point and when p is a ramification point. First, suppose p is not a ramification point. Then we may choose local coordinates w and z centred at p and $\pi(p)$ such that $w = z$. Then, $f(w) = w^a g(w)$ where $g(w) \neq 0$. Hence, we may assume without loss of generality that $f(w) = w^a$. Since $\mathrm{Nm}_\pi(w^a) = \mathrm{Nm}_\pi(w)^a$ it suffices to check $f(w) = w$. Now,

$$\mathrm{Nm}_\pi(w)(z) = w,$$

which shows $\mathrm{ord}_p(w) = \mathrm{ord}_{\pi(p)}(\mathrm{Nm}_\pi(w))$. Suppose now that p is a ramification point with ramification index m. Now, we choose coordinates w and z centred at p and $\pi(p)$ such that $z = w^m$. The argument shows we only need to check $f(w) = w$. Thus,

$$\mathrm{Nm}_\pi(w)(z) = w(\xi w) \cdots (\xi^{m-1} w) = \xi^{\frac{1}{2}m(m-1)} w^m$$

where $\xi = \exp(2\pi i/m)$. Since $z = w^m$ the result follows. $\square$

Therefore, the map $\mathrm{Div}(C) \to \mathrm{Div}(C')$ preserves linear equivalence, and hence, induces a map $\mathrm{Pic}(C) \to \mathrm{Pic}(C')$, which is the desired norm map.

Definition 3.5.2. The *norm map* associated to $\pi : C \to C'$ is the assignment $\mathrm{Nm}_{C/C'} : \mathrm{Pic}(C) \to \mathrm{Pic}(C')$ defined on divisors by

$$\mathrm{Nm}_{C/C'}\left(\sum_{i=1}^N n_i x_i\right) = \sum_{i=1}^N n_i \pi(x_i).$$

Remark 3.5.3. The norm map defines a group homomorphism, and preserves degree. Therefore, by restricting to $\mathrm{Jac}(C) \subset \mathrm{Pic}(C)$ the norm map defines a group homomorphism $\mathrm{Nm}_{C/C'} : \mathrm{Jac}(C) \to \mathrm{Jac}(C')$, which some authors define to be the norm map. Moreover, since $\pi : C \to C'$ is surjective it is clear that $\mathrm{Nm}_{C/C'}$ is surjective too.

Definition 3.5.4. The *Prym variety* associated to the branched cover $\pi : C \to C'$, denoted by $\mathrm{Prym}(C, C')$ is the connected component of the kernel of the norm map $\mathrm{Nm}_{C/C'} : \mathrm{Jac}(C) \to \mathrm{Jac}(C')$, i.e., $\mathrm{Prym}(S, C) := \ker(\mathrm{Nm}_{C/C'})_0$.

Remark 3.5.5. Some authors call these varieties *generalised* Prym variety since a Prym variety is classically defined with respect to a double cover. However, we are adopting the modern convention and simply calling them Prym varieties.

Since $\mathrm{Prym}(C, C')$ defines a complex subtorus of the abelian variety $\mathrm{Jac}(C)$ notice that $\mathrm{Prym}(C, C')$ defines an abelian variety. Now, in general the kernel of a norm map is not connected. However, we can compute number of connected component of $\ker(\mathrm{Nm}_{C/C'})$.

Lemma 3.5.6. *The number of connected components of* $\ker(\mathrm{Nm}_{C/C'})$ *is equal to the cardinality of the cokernel of* $\pi_* : \pi_1(\mathrm{Jac}(C)) \to \pi_1(\mathrm{Jac}(C'))$.

Proof. Consider the fibration

$$\ker(\mathrm{Nm}_{C/C'}) \to \mathrm{Jac}(C) \xrightarrow{\mathrm{Nm}_{C/C'}} \mathrm{Jac}(C').$$

Then, passing to the long exact sequence in homotopy gives the exact sequence

$$\cdots \to \pi_1(\mathrm{Jac}(C)) \xrightarrow{\pi_*} \pi_1(\mathrm{Jac}(C')) \to \pi_0(\ker(\mathrm{Nm}_{C/C'})) \to 1.$$

The result follows follow by exactness. □

3.5.2 Pullback is Dual to the Norm map

Since $\mathrm{Nm}_{C/C'} : \mathrm{Jac}(C) \to \mathrm{Jac}(C')$ is a homomorphism of complex tori recall that there exists a unique dual map $\mathrm{Nm}^\vee_{C/C'} : \mathrm{Alb}(C') \to \mathrm{Alb}(C)$. By identifying $\mathrm{Jac}(C) \cong \mathrm{Alb}(C)$ and $\mathrm{Jac}(C') \cong \mathrm{Alb}(C')$ under the Abel-Jacobi maps we claim that the pullback map $\pi^* : \mathrm{Jac}(C') \to \mathrm{Jac}(C)$ defines the dual homomorphism to $\mathrm{Nm}_{C/C'} : \mathrm{Jac}(C) \to \mathrm{Jac}(C')$. To prove this claim we will prove that the diagram

$$\begin{array}{ccc} \mathrm{Div}^0(C) & \xrightarrow{\mathrm{Nm}_{C/C'}} & \mathrm{Div}^0(C') \\ {\scriptstyle u_C}\downarrow & & \downarrow{\scriptstyle u_{C'}} \\ \mathrm{Alb}(C) & \xrightarrow{(\pi^*)^\vee} & \mathrm{Alb}(C') \end{array} \tag{3.5}$$

commutes where u_C and $u_{C'}$ are the Abel-Jacobi maps. Divisors of the form $x-y$ where $x, y \in C$ generate $\mathrm{Div}^0(C)$, and thus, it suffices to prove the diagram commutes for $x-y$. To do so we will compute the analytic representation for $(\pi^*)^\vee$. Recall that $\rho_a((\pi^*)^\vee)$ is dual to $\rho_a(\pi^*)$. Recall that $\rho_a(\pi^*) : \mathrm{H}^1(C', \mathcal{O}_{C'}) \to \mathrm{H}^1(C, \mathcal{O}_C)$, then by applying the dolbeault isomorphism theorem it follows that $\rho_a(\pi^*) : \mathrm{H}^{0,1}_{\overline{\partial}}(C') \to \mathrm{H}^{0,1}_{\overline{\partial}}(C)$ is precisely pullback of 1-forms. Therefore, $\rho_a((\pi^*)^\vee) : \mathrm{Hom}_{\mathbb{C}}(\mathrm{H}^0(C, K_C), \mathbb{C}) \to \mathrm{Hom}_{\mathbb{C}}(\mathrm{H}^0(C', K_{C'}), \mathbb{C})$ is dual to pulling back 1-forms.

Let γ be a real curve from x to y in C and let γ' be a real curve from $\pi(x)$ to $\pi(y)$ in C'. Consider the homomorphism $F : \mathrm{H}^0(C, K_C) \to \mathbb{C}$ defined by $F(\omega) = \int_\gamma \omega$. Fix $\omega' \in \mathrm{H}^0(C', K_{C'})$. Then

$$\rho_a((\pi^*)^\vee)(F)(\omega') = \int_\gamma \pi^*(\omega') = \int_{\pi(\gamma)} \omega'.$$

Since γ' and $\pi(\gamma)$ are real curves from $\pi(a)$ to $\pi(b)$ it follows that $\omega' \mapsto \int_{\pi(\gamma)} \omega'$ and $\omega' \mapsto \int_{\gamma'} \omega'$ define the same element in $\mathrm{Alb}(C')$ namely $\omega \mapsto \int_{\pi(y)}^{\pi(x)} \omega'$. In other words (3.5) commutes, which proves the claim.

Proposition 3.5.7. *Let $\pi : C \to C'$ be a branched cover of compact Riemann surfaces and suppose $\mathrm{Nm}_{C/C'} : \mathrm{Jac}(C) \to \mathrm{Jac}(C')$ is the associated norm map. Then, the dual map to $\mathrm{Nm}_{C/C'}$ after identifying $\mathrm{Jac}(C) \cong \mathrm{Alb}(C)$ and $\mathrm{Jac}(C') \cong \mathrm{Alb}(C')$ via the Abel-Jacobi maps is precisely the pullback map π^*, i.e., $\mathrm{Nm}_{C/C'}^\vee = \pi^*$.*

Now, recall from Proposition 3.1.12 that the kernel of a homomorphism of complex tori is connected if and only if the dual homomorphism is injective. Hence, we immediately obtain the following lemma.

Lemma 3.5.8. *Let $\pi : C \to C'$ be a branched cover of compact Riemann surfaces. Then, $\ker(\mathrm{Nm}_{C/C'}) = \mathrm{Prym}(C, C')$ if and only if $\pi^* : \mathrm{Jac}(C) \to \mathrm{Jac}(C)$ is injective.*

Explicitly checking $\pi^* : \mathrm{Jac}(C') \to \mathrm{Jac}(C)$ is injective is not straightforward from the definition. Fortunately, there is a necessary and sufficient condition for $\pi^* : \mathrm{Jac}(C') \to \mathrm{Jac}(C)$ to be injective, which is easier to check.

Proposition 3.5.9 ([BL04, Proposition 11.4.3])**.** *The homomorphism $\pi^* : \mathrm{Jac}(C') \to \mathrm{Jac}(C)$ is not injective if and only if π factorises via a cyclic étale covering π' of degree $n \geq 2$*

$$\begin{tikzcd} C \arrow[rr, "\pi"] \arrow[dr, "\pi''"'] & & C' \\ & C'' \arrow[ur, "\pi'"'] & \end{tikzcd}$$

Corollary 3.5.10. *Suppose $\pi : C \to C'$ is a branched cover of compact Riemann surfaces. Then $\mathrm{Prym}(C, C') = \ker(\mathrm{Nm}_{C/C'})$ if and only if π does not factorise through a cyclic étale covering of degree $n \geq 2$.*

3.5.3 Dimension of Prym Variety

To compute the dimension of the Prym variety $\mathrm{Prym}(C, C')$ associated to the branched cover $\pi : C \to C'$ of compact Riemann surfaces we require the following lemma.

Lemma 3.5.11. *Let $X = V/\Gamma$ be a complex torus of dimension g and consider the map $n_X : X \to X$ that sends $x \mapsto nx$ where $n \in \mathbb{N}$. Then $\ker(n_X) \cong \mathbb{Z}_n^{2g}$.*

Proof. Suppose $x \in \ker(n_X)$, i.e., $nx \in \Gamma$, or, equivalently, $x \in \frac{1}{n}\Gamma$. Then, clearly $\ker(n_X) \cong \left(\frac{1}{n}\Gamma\right)/\Gamma$. Moreover,

$$\left(\frac{1}{n}\Gamma\right)/\Gamma \cong \Gamma/n\Gamma \cong \mathbb{Z}_n^{2g},$$

hence, $\ker(n_X) \cong \mathbb{Z}_n^{2g}$. □

Proposition 3.5.12. *The complex tori $\mathrm{Jac}(C') \times \mathrm{Prym}(C, C')$ and $\mathrm{Jac}(C)$ are isogenous.*

Proof. Consider the homomorphism $\psi : \mathrm{Jac}(C') \times \mathrm{Prym}(C, C') \to \mathrm{Jac}(C)$ defined by

$$\psi(x, y) = f^*(x) + y.$$

Suppose $(x, y) \in \ker(\psi)$, i.e., $f^*(x) = -y$. Applying $\mathrm{Nm}_{C/C'}$ it follows that $2x = 0$, i.e., $x \in \mathrm{Jac}(C')[2]$. Then, clearly, $2y = 0$, i.e., $y \in \mathrm{Prym}(C, C')[2]$. By Lemma 3.5.11 the kernel of ψ is finite. Now, we claim that ψ is surjective. Let $y \in \mathrm{Jac}(C)$ be given. Let $x = \mathrm{Nm}_{C/C'}(y)$. Writing $2y = f^*(x) + (2y - f^*(x))$ we see

$$\mathrm{Nm}_{C/C'}(2y - f^*(x)) = 2\,\mathrm{Nm}_{C/C'}(y) - 2x = 2x - 2x = 0.$$

Thus, if v denotes the number of connected components of $\ker(\mathrm{Nm}_{C/C'})$ then $v(2y - f^*(x)) \in \mathrm{Prym}(C, C')$. It follows that $2vy \in f^*(x) + \mathrm{Prym}(C, C') \subset \mathrm{im}(\psi)$, and since $\mathrm{Jac}(C)$ is a complex torus it is a divisible group, which shows ψ is surjective. □

Corollary 3.5.13. *The dimension of $\mathrm{Prym}(C, C')$ is equal to the difference of $g(C)$ and $g(C')$, i.e., $\dim(\mathrm{Prym}(C, C')) = g(C) - g(C')$.*

Proof. By Proposition 3.5.12 it immediately follows that

$$\dim(\mathrm{Prym}(C, C')) = \dim(\mathrm{Jac}(C)) - \dim(\mathrm{Jac}(C'))$$

since isogenous complex tori have the same dimension and

$$\dim(\mathrm{Jac}(C') \times \mathrm{Prym}(C, C')) = \dim(\mathrm{Jac}(C')) + \dim(\mathrm{Prym}(C, C')).$$

However, the dimension of the Jacobian variety of a compact Riemann surface is equal to the genus of the surface and the result follows. □

3.5.4 Prym Variety of an Étale Double Cover

Suppose now that $\pi : C \to C'$ is a étale double cover of compact Riemann surfaces. In this section we will show that the associated Prym variety $\mathrm{Prym}(C, C')$ is a principally polarised abelian variety, which shows $\mathrm{Prym}(C, C') \cong \mathrm{Prym}(C, C')^\vee$. More specifically, we will provide a topological proof by showing that the restriction of the alternating form associated to the canonical polarisation of $\mathrm{Jac}(C)$ to $\mathrm{Prym}(C, C')$ defines a principal polarisation. The topological approach is classical and in the vein of the thesis, however some mathematicians such as David Mumford have given purely algebraic proofs that can be generalised to fields of different characteristics see [Mum74, Sections 1-3].

Let Θ and Θ' be theta divisors for $\mathrm{Jac}(C)$ and $\mathrm{Jac}(C')$ respectively. Since $\pi : C \to C'$ is a double cover there is a canonically associated involution $\iota : C \to C$ not the identity characterised by $\pi(\iota(x)) = \pi(x)$ for every $x \in C$. The involution canonically extends to an involution $\widetilde{\iota} : \mathrm{Jac}(C) \to \mathrm{Jac}(C)$.

Lemma 3.5.14. *The involution $\widetilde{\iota}$ acts as $+1$ on $\pi^* \mathrm{Jac}(C')$ and -1 on $\mathrm{Prym}(C, C')$.*

Proof. Let $x \in \mathrm{Jac}(C)$. Then, notice that

$$\pi^* \mathrm{Nm}_{C/C'}(x) = \pi^*(\pi(x)) = x + \widetilde{\iota}(x). \tag{3.6}$$

Hence, if $x \in \ker(\mathrm{Nm}_{C/C'})$, then $\widetilde{\iota}(x) = -x$. Now, let $y \in \mathrm{Jac}(C')$. Recall that

$$\mathrm{Nm}_{C/C'} \pi^*(y) = 2y,$$

and notice that $\pi^* \mathrm{Nm}_{C/C'} \pi^*(y) = 2\pi^*(y)$. However, by (3.6) we see

$$\pi^* \mathrm{Nm}_{C/C'} \pi^*(y) = \pi^*(y) + \widetilde{\iota}(\pi^*(y))$$

so it follows that $\widetilde{\iota}(\pi^*(y)) = \pi^*(y)$. □

Corollary 3.5.15. *The Prym variety may be described by* $\mathrm{Prym}(C, C') = \ker(1 + \widetilde{\iota})_0$.

Identifying the Jacobian variety and Albanese variety via the Abel-Jacobi map, i.e., $\mathrm{Jac}(C) \cong \mathrm{H}^0(C, K_C)^*/\mathrm{H}_1(C, \mathbb{Z})$ the induced action of ι on $\mathrm{H}^0(C, K_C)^*$ and $\mathrm{H}_1(C, \mathbb{Z})$ is the analytic representation $\rho_a(\widetilde{\iota})$ and the rational representation $\rho_r(\widetilde{\iota})$ respectively. Let $\mathrm{H}^0(C, K_C)^- \subset \mathrm{H}^0(C, K_C)$ denote the (-1)-eigenspace for the induced action of ι on $\mathrm{H}^0(C, K_C)$ and let $\mathrm{H}_1(C, \mathbb{Z})^- := \ker(1 + \rho_r(\widetilde{\iota}))$. Then, by Corollary 3.5.15

$$\mathrm{Prym}(C, C') = (\mathrm{H}^0(C, K_C)^-)^*/\mathrm{H}_1(C, \mathbb{Z})^-.$$

Let g and g' denote the genus of C and C' respectively. Since π is unramified, by Riemann-Hurwitz, $\chi(C) = 2\chi(C')$, and hence, $g = 2g' - 1$. By Corollary 4.19 it immediately follows that $\dim \mathrm{Prym}(C, C') = g' - 1$. Therefore, setting $t = g' - 1$,

$$\dim(\mathrm{Jac}(C)) = 2t + 1, \quad \dim(\mathrm{Jac}(C')) = t + 1, \quad \dim \mathrm{Prym}(C, C') = t.$$

Choose a homology basis $\lambda_0, \ldots, \lambda_t, \mu_0, \ldots, \mu_t$ for $\mathrm{H}_1(C', \mathbb{Z})$ with intersection matrix $\begin{bmatrix} 0 & -1_{t+1} \\ 1_{t+1} & 0 \end{bmatrix}$. The Riemann bilinear relations implies that

$$\begin{bmatrix} 0 & -1_{t+1} \\ 1_{t+1} & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1_{t+1} \\ -1_{t+1} & 0 \end{bmatrix}$$

describes the matrix of the alternating form defining the canonical polarisation with respect to the given homology basis. In particular, the homology basis defines a symplectic basis of the lattice $\mathrm{H}_1(C', \mathbb{Z})$ in $\mathrm{Hom}_{\mathbb{C}}(\mathrm{H}^0(C, K_C), \mathbb{C})$.

From a topological perspective $\pi : C \to C'$ is a connected double cover so by the classification of covering spaces, up to homeomorphism, $\pi : C \to C'$ is determined by an index 2 subgroup of $\pi_1(C')$ that is unique up to conjugacy. Index 2 subgroups correspond to the kernel of homomorphisms of $\pi_1(C')$ onto $\mathbb{Z}_2$. Since $\mathbb{Z}_2$ is abelian every group homomorphism $\pi_1(C') \to \mathbb{Z}_2$ unique factorises through the abelianisation $\pi_1(C')_{\mathrm{ab}} \cong \mathrm{H}_1(C, \mathbb{Z})$, i.e.,

$$\mathrm{Hom}(\pi_1(C'), \mathbb{Z}_2) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_1(C', \mathbb{Z}), \mathbb{Z}_2).$$

By the universal coefficient theorem of cohomology $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_1(C', \mathbb{Z}), \mathbb{Z}_2) \cong \mathrm{H}^1(C, \mathbb{Z}_2)$. Now, by Poincaré duality there is a canonical isomorphism $\mathrm{H}^1(C, \mathbb{Z}_2) \cong \mathrm{H}_1(C, \mathbb{Z}_2)$. Therefore, up to homeomorphism., the cover $\pi : C \to C'$ is determined by a non-zero element of $\mathrm{H}_1(C', \mathbb{Z}_2)$. It turns out that we may assume that this element is the element of λ_0 in $\mathrm{H}_1(C', \mathbb{Z}_2)$, which we will now prove.

Lemma 3.5.16. *Every element of* $\mathrm{H}^1(C', \mathbb{Z}_2)$ *is in the image of a primitive element of* $\mathrm{H}^1(C', \mathbb{Z})$.

Proof. Consider the short exact sequence

$$1 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}_2 \to 1.$$

Passing to the long exact sequence in sheaf cohomology yields the long exact sequence

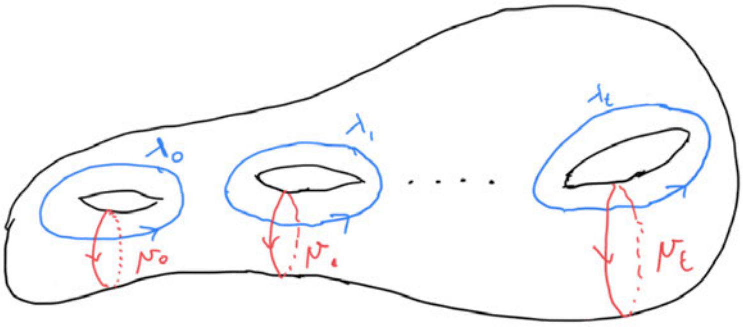
$$1 \to \mathrm{H}^1(C', \mathbb{Z}) \xrightarrow{2} \mathrm{H}^1(C', \mathbb{Z}) \to \mathrm{H}^1(C', \mathbb{Z}_2) \to \mathrm{H}^2(C', \mathbb{Z}) \to \cdots$$

The $\mathbb{Z}$-module $\mathrm{H}^1(C', \mathbb{Z}_2)$ is torsion whereas $\mathrm{H}^1(C', \mathbb{Z})$ is torsion-free, and thus, the homomorphism $\mathrm{H}^1(C', \mathbb{Z}_2) \to \mathrm{H}^2(C', \mathbb{Z})$ is the zero homomorphism. Therefore, we obtain the short exact sequence

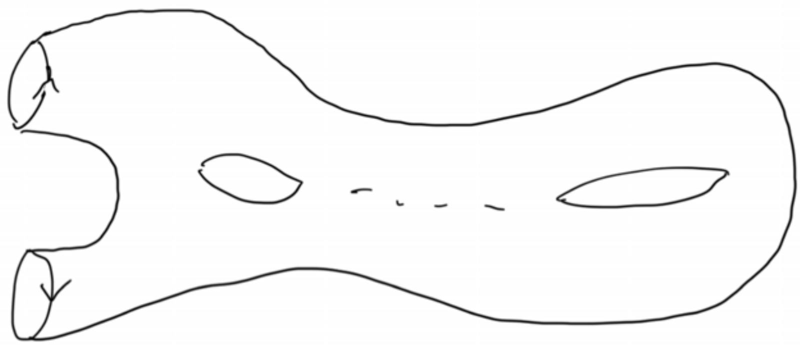
$$1 \to \mathrm{H}^1(C', \mathbb{Z}) \xrightarrow{2} \mathrm{H}^1(C', \mathbb{Z}) \to \mathrm{H}^1(C', \mathbb{Z}_2) \to 0.$$

The desired result now follows. $\square$

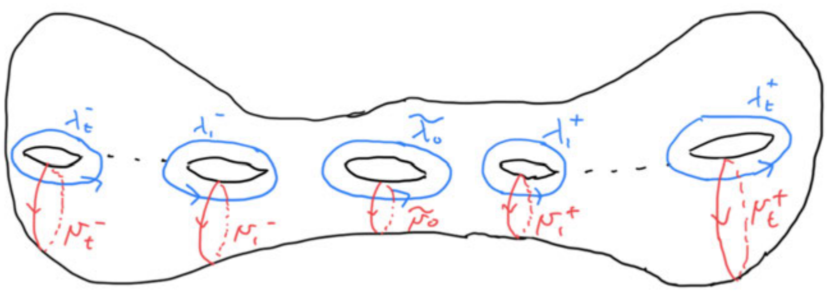
Topologically we may describe $\pi : C \to C'$ as follows: consider the following diagram of C'



We cut C' along μ_0 obtaining



Then C is given by gluing together two copies of C' together with the upper and lower boundary of μ_0 reversed



This yields a homology basis

$$\widetilde{\lambda}_0, \lambda_i^+, \lambda_i^-, \widetilde{\mu}_0, \mu_i^+, \mu_i^- \tag{3.7}$$

for $\mathrm{H}_1(C,\mathbb{Z})$ where $i = 1, \dots, t$ that has intersection matrix $\begin{bmatrix} 0 & -1_{2t+1} \\ 1_{2t+1} & 0 \end{bmatrix}$, and thus, forms a symplectic basis. The involution ι acts by interchanging the two copies of C', and hence, $\iota(\lambda_i^{\pm}) = \lambda_i^{\mp}$ and $\iota(\mu_i^{\pm}) = \mu_i^{\mp}$. Moreover, it is clear that $\iota(\widetilde{\lambda}_0) = \lambda_0$ since the orientation does not change. Also $\iota(\widetilde{\mu}_0) = \widetilde{\mu}_0$ since $\widetilde{\mu}_0 - \iota(\widetilde{\mu}_0)$ defines a boundary. Now,

$$\alpha_i := \lambda_i^+ - \lambda_i^- \quad \text{and} \quad \beta_i = \mu_i^+ - \mu_i^-; \quad i = 1, \dots, t$$

forms a basis for $\mathrm{H}_1(C,\mathbb{Z})^-$. Let $E : \mathrm{H}_1(C,\mathbb{Z}) \times \mathrm{H}_1(C,\mathbb{Z}) \to \mathbb{Z}$ denote the canonical polarisation of $\mathrm{Jac}(C)$. We will compute the polarisation restricted to the sublattice $\mathrm{H}_1(C,\mathbb{Z})^-$.

Lemma 3.5.17. *The alternating form $E : \mathrm{H}_1(C,\mathbb{Z}) \times \mathrm{H}_1(C,\mathbb{Z}) \to \mathbb{Z}$ restricted to the basis α_i, β_i is given by*

$$E(\alpha_i, \beta_j) = 2\delta_{ij}, \quad E(\alpha_i, \alpha_j) = E(\beta_i, \beta_j) = 0$$

for $1 \le i, j \le t$.

Proof. This is a straightforward but tedious calculation. We may realise $\alpha := \alpha_i$ and $\beta := \beta_j$ as a column vector where by abuse of notation $\alpha_{1+i} = 1$, $\alpha_{1+i+t} = -1$, and $\alpha_k = 0$ otherwise. Similarly, $\beta_{2+j+2t} = 1$, $\beta_{2+j+3t} = -1$, and $\beta_k = 0$ otherwise. With respect to the homology basis (3.7) we see $E_{2t+1+i,i} = -1$, $E_{i,2t+1+i} = 1$, and $E_{k,l} = 0$ otherwise. Then,

$$\alpha^t E \beta = \sum_{k=1}^{4t+2} \alpha_k (E\beta)_k = \sum_{k=1}^{4t+2} \alpha_k \left(\sum_{i=1}^{4t+2} E_{k,l}\beta_l \right) = \sum_{k=1}^{4t+2} \alpha_k E_{k,2+j+2t} - \sum_{k=1}^{4t+2} \alpha_k E_{k,2+j+3t}.$$

However,

$$\sum_{k=1}^{4t+2} \alpha_k E_{k,2+j+2t} - \sum_{k=1}^{4t+2} \alpha_k E_{k,2+j+3t} = E_{1+i,2+j+2t} - E_{1+i+t,2+j+2t} - E_{1+i,2+j+3t} + E_{1+i+t,2+j+3t}.$$

One can easily verify that

$$E_{1+i,2+j+2t} - E_{1+i+t,2+j+2t} - E_{1+i,2+j+3t} + E_{1+i+t,2+j+3t} = 2\delta_{ij}.$$

Therefore, $E(\alpha_i, \beta_j) = 2\delta_{ij}$, and a similar calculation shows $E(\alpha_i, \alpha_j) = E(\beta_i, \beta_j) = 0$. □

Thus, the canonical polarisation on $\mathrm{Jac}(C)$ induces a polarisation on $\mathrm{Prym}(C, C')$ that is twice a principle polarisation Ξ, i.e., $i_0^* c_1(\mathcal{O}_C(\Theta)) = 2\Xi$ where $i_0 : \mathrm{Prym}(C, C') \to \mathrm{Jac}(C)$ denotes the inclusion map. Hence, $(\mathrm{Prym}(C, C'), \Xi)$ is a principally polarised abelian variety, which establishes the following theorem.

Theorem 3.5.18. *The Prym variety $\mathrm{Prym}(C, C')$ that is associated to an étale double cover $\pi : C \to C'$ is a principally polarised abelian variety, and thus, $\mathrm{Prym}(C, C') \cong \mathrm{Prym}(C, C')^\vee$.*

Part II

Classical Higgs Bundles and Spectral Curves

Chapter 4

Classification of Generic Fibres of GL_n and type A_n Hitchin Fibration

In 1987, Nigel Hitchin researched the Yang-Mills equations extending the work of Atiyah and Bott [AB83] by considering a conformal invariance in two dimensions. Hitchin's work in [Hit87a] led him to *Higgs bundles* where he first studied the case of rank 2 Higgs bundles with an odd degree and a fixed determinant, and he showed that the corresponding moduli space is a finite dimensional hyperkähler manifold. In the same year, Hitchin introduced a canonical fibration associated with Higgs bundles that mathematicians have since coined the *Hitchin fibration*, and he classified the generic fibres in [Hit87b]. However, his classification for SO_{2n+1}-Higgs bundles was incorrect, which we will discuss in Chapter 5. We may view a fibre of the Hitchin fibration as a collection of Higgs bundles that correspond to the same *spectral curve*. A spectral curve is a complex analytic variety that generalises the notion of an eigenvalue to a twisted endomorphism of a holomorphic vector bundle, see [BNR89].

Higgs bundles have since appeared in several deep theorems, most notably in the proof of the nonabelian Hodge correspondence [Cor88, Sim92, Sim91], and further work by Carlos Simpson [Sim94a, Sim94b].

This chapter extends the classification of the generic fibres of the Hitchin fibration for $\mathrm{GL}_n, \mathrm{SL}_n$, and PGL_n to a more general type of Higgs bundles that sees the canonical bundle replaced by an arbitrary basepoint-free holomorphic line bundle. Moreover, we derive *relative duality*, which describes the dual vector bundle associated with the pushforward of a line bundle. Relative duality provides insight into the GL_n case and additional cases considered in Chapters 5 and 6. When working with a non-specified line bundle, we lose properties intrinsic to the canonical bundle, which presents an obstacle. However, the basepoint-free assumption ensures that a generic choice of spectral curve is smooth. The generic fibres of the Hitchin fibration in this more general setting are torsors of abelian varieties, and we demonstrate Langlands duality in the Hitchin system. We show that a generic fibre of the GL_n Hitchin fibration is a torsor of a Jacobian variety of a curve, which agrees with ${}^L\mathrm{GL}_n \cong \mathrm{GL}_n$. Moreover, we show that the generic fibres of the SL_n and PGL_n Hitchin fibrations are dual abelian varieties, which agrees with ${}^L\mathrm{SL}_n \cong \mathrm{PGL}_n$.

We divert the reader to Appendix A for proof of the duality ${}^L\mathrm{SL}_n \cong \mathrm{PGL}_n$, and the reader should consult [Fre10] to see proof that ${}^L\mathrm{GL}_n \cong \mathrm{GL}_n$.

4.1 Classification of General Linear Higgs bundles

Throughout this chapter fix a compact Riemann surfaces C with genus $g \geq 2$, let $q : L \to C$ be a fixed basepoint-free holomorphic line bundle with positive degree, and let Y denote the total space of L. Also, in this section, powers of line bundles represent tensor product and *not* direct sum.

4.2 Spectral Curves

Before defining spectral curves recall that when L is pulled back over its total space Y there exists a section $\lambda \in \mathrm{H}^0(Y, L)$ called the *tautological section*.

Definition 4.2.1. Consider the holomorphic line bundle L^n over Y where L^n denotes the n-th tensor power of L over Y. Then a *spectral curve* S is the zero locus of a section $p(\lambda) \in \mathrm{H}^0(Y, L^n)$ of the form

$$p(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n$$

where $a_i \in \mathrm{H}^0(C, L^i)$ for $i = 1, 2, \ldots, n$.

By definition, a spectral curve defines a complex analytic subvariety of Y, possibly reducible or non-reduced, which has codimension 1, i.e., S defines a divisor in Y. The spectral curve has a natural scheme structure with structure sheaf $\mathcal{O}_S := \mathcal{O}_Y|_S$. Since $S \subset Y$ there is a canonical holomorphic map $\pi : S \to C$, which we will prove defines a finite morphism, i.e., $\pi_*\mathcal{O}_S$ is a finitely generated $\mathcal{O}_C$-module. First, we require a short lemma.

Lemma 4.2.2. *Let* $\mathcal{O}_{\mathbb{P}^n}(d)$ *denote the* d*-th tensor power of the hyperplane bundle* $\mathcal{O}_{\mathbb{P}^n}(1)$. *Then, there is a canonical isomorphism*

$$\mathrm{Sym}^d(\mathbb{C}^{n+1*}) \cong \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)).$$

Proof. Let $z_0, \ldots, z_n$ denote Euclidean coordinates on $\mathbb{C}^{n+1}$. Let F be a given d-linear form on $\mathbb{C}^{n+1*}$. Then, by restriction F induces a global section σ_F of $\mathcal{O}_{\mathbb{P}^n}(d)$, i.e., over a point $x \in \mathbb{P}^n$

$$\sigma_F(x) = F|_{\{\lambda x\}}.$$

Since we are restricting F to one line at a time, it is clear that $\sigma_F = 0$ if and only if F is alternating in any two factors. Hence, there is an induced injective map

$$\mathrm{Sym}^d(\mathbb{C}^{n+1*}) \to \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)).$$

To see that this map is surjective, let σ be a given global section of $\mathcal{O}_{\mathbb{P}^n}(d)$. Then, the quotient σ/σ_F is a meromorphic function on $\mathbb{P}^n$ where σ_F is the section of $\mathcal{O}_{\mathbb{P}^n}(d)$ corresponding to a degree d homogeneous polynomial F. Let $\mathrm{pr} : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ denote the canonical projection map and define

$$G' := \mathrm{pr}^* \left(\frac{\sigma}{\sigma_F} \right).$$

Now, G' has a simple pole along the divisor defined by $F = 0$ in $\mathbb{C}^{n+1} \setminus \{0\}$ and is holomorphic elsewhere, thus

$$G = G' \cdot F$$

is holomorphic everywhere on $\mathbb{C}^{n+1} \setminus \{0\}$. Therefore, by Hartog's extension theorem G extends to an entire holomorphic function on $\mathbb{C}^{n+1}$. Since $G'(\lambda z) = G'(z)$ and $F(\lambda z) = \lambda^d F(z)$ for every $z \in \mathbb{C}^{n+1}$ and $\lambda \in \mathbb{C}$ it follows that

$$G(\lambda z) = \lambda^d G(z).$$

Thus, if $\iota : \mathbb{C} \ni t \mapsto (t\mu_0, \ldots, t\mu_n) \in \mathbb{C}^{n+1}$ is a given line through the origin in $\mathbb{C}^{n+1}$, the pullback $\iota^* G$ is either identically zero or has a zero of order d at $t = 0$ and a pole of order d at $t = \infty$. Thus,

$$\iota^* G = \mu t^d$$

for some $\mu \in \mathbb{C}$ and it follows from the power series expansion about the origin that G is a homogeneous polynomial of degree d. Therefore, $\sigma = \sigma_G$, which proves surjectivity. □

Proposition 4.2.3. *Let $\pi : S \to C$ be a spectral curve defined by the section*

$$p(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_n.$$

Then the map

$$\mathcal{O}_C \oplus L^{-1} \oplus \cdots \oplus L^{-(n-1)} \to \pi_* \mathcal{O}_S$$

given by

$$(c_0, c_1, \ldots, c_{n-1}) \mapsto c_0 + c_1 \lambda + \cdots + c_{n-1} \lambda^{n-1}$$

is an isomorphism.

Although this result has been established in [BNR89, pp. 173] their proof is purely algebraic and requires a lot of background knowledge. Here, we will give an elementary complex analytic proof.

Proof. Let $\overline{Y} = \mathbb{P}(\mathcal{O}_C \oplus L)$ be the projectivisation of L. Then $\overline{Y}$ is a $\mathbb{P}^1$-bundle over C which contains Y as an open subset. Let $\mathcal{O}_{\overline{Y}}(1)$ denote the associated hyperplane bundle. The complement $Y_\infty := \overline{Y} - Y$ is a divisor in $\overline{Y}$ whose corresponding line bundle is $\mathcal{O}_{\overline{Y}}(1)$. Hence, there is a section s of $\mathcal{O}_{\overline{Y}}(1)$ whose zero locus is Y_∞. Since s does not vanish over Y there is a uniquely determined trivialisation $\mathcal{O}_{\overline{Y}}(1)|_Y \cong \mathcal{O}_Y$ that sends $s|_Y$ to 1.

By Lemma 4.2.2 the global sections of $\mathcal{O}_{\overline{Y}}(1)$ correspond to linear maps on $\mathcal{O}_C \oplus L$, that is, to global sections of $\mathrm{Hom}(\mathcal{O}_C \oplus L, \mathcal{O}_C)$. The section s is the homomorphism $\mathcal{O}_C \oplus L \to \mathcal{O}_C$ given by projection onto the first factor. Similarly, a L-valued section of $\mathcal{O}_{\overline{Y}}(1)$ corresponds to a global section of $\mathrm{Hom}(\mathcal{O}_C \oplus L, L)$. Let $\widetilde{\lambda}$ denote the section of $L \otimes \mathcal{O}_{\overline{Y}}(1)$ that corresponds to the homomorphism $\mathcal{O}_C \oplus L \to L$ given by projection onto the second factor.

We claim that under the trivialisation $\mathcal{O}_{\overline{Y}}(1)|_Y \cong \mathcal{O}_Y$ that sends $s|_Y$ to 1, the section $\widetilde{\lambda}$ is sent to λ. To see this, note that Y corresponds to the set of points in $\overline{Y}$ of the form $[1, y]$ where $y \in Y$. By definition, $\widetilde{\lambda}(0, y) = y$, which is precisely the tautological section.

Now, we define

$$\widetilde{p}(\widetilde{\lambda}) = \widetilde{\lambda}^n + a_1 \widetilde{\lambda}^{n-1} s + \cdots + a_n s^n,$$

which is a section of $L^n \otimes \mathcal{O}_{\overline{Y}}(n)$ over $\overline{Y}$. Moreover, $\widetilde{p}(\widetilde{\lambda})|_Y = p(\lambda)$ under the trivialisation $\mathcal{O}_{\overline{Y}}(1)|_Y \cong \mathcal{O}_Y$.

Next, we claim that S is the zero locus of $\widetilde{p}(\widetilde{\lambda})$. Since $\widetilde{p}(\widetilde{\lambda})|_Y = p(\lambda)$ under the trivialisation, it suffices to show that $\widetilde{p}(\widetilde{\lambda})$ is non-zero on Y_∞. However, $s|_{Y_\infty} = 0$ so it follows that $\widetilde{p}(\widetilde{\lambda})|_{Y_\infty} = \widetilde{\lambda}^n|_{Y_\infty}$, and since Y_∞ is given by points of the form $[0, y]$ with $y \in Y$, $y \neq 0$ we see $\widetilde{\lambda}([0, y]) = y$, which proves the claim.

Since S is the zero locus of $\widetilde{p}(\widetilde{\lambda})$ there is a short exact sequence of sheaves on $\overline{Y}$

$$0 \to \mathcal{O}_{\overline{Y}}(L^{-n}(-1)) \xrightarrow{\widetilde{p}(\widetilde{\lambda})} \mathcal{O}_{\overline{Y}}(n-1) \to \mathcal{O}_S(n-1) \to 0.$$

However, since $S \subset Y$, using the trivialisation $\mathcal{O}_{\overline{Y}}(1)|_Y \cong \mathcal{O}_Y$ one sees $\mathcal{O}_S(n-1) \cong \mathcal{O}_S$. Hence, the above sequence can be re-written as

$$0 \to \mathcal{O}_{\overline{Y}}(L^{-n}(-1)) \xrightarrow{\widetilde{p}(\widetilde{\lambda})} \mathcal{O}_{\overline{Y}}(n-1) \to \mathcal{O}_S \to 0.$$

Now, applying the direct image functor π_* along with the right derived functors, one obtains a long exact sequence of sheaves on $\overline{Y}$

$$0 \to \pi_*\mathcal{O}_{\overline{Y}}(L^{-n}(-1)) \xrightarrow{\widetilde{p}(\widetilde{\lambda})} \pi_*\mathcal{O}_{\overline{Y}}(n-1) \to \pi_*\mathcal{O}_S \to R^1\pi_*\mathcal{O}_{\overline{Y}}(L^{-n}(-1)) \to \cdots \qquad (4.1)$$

By the projection formula

$$\pi_*\mathcal{O}_{\overline{Y}}(L^{-n}(-1)) \cong L^{-n} \otimes \pi_*\mathcal{O}_{\overline{Y}}(-1) = 0.$$

Next, we claim $R^1\pi_*\mathcal{O}_{\overline{Y}}(L^{-n}(-1)) = 0$. Recall that $R^1\pi_*\mathcal{O}_{\overline{Y}}(L^{-n}(-1))$ is the sheaf associated to the presheaf

$$U \mapsto \mathrm{H}^1(\pi^{-1}(U), \mathcal{O}_{\overline{Y}}(L^{-n}(-1))).$$

For every $y \in C$ note that $\pi^{-1}(y) \cong \mathbb{P}^1$ and $L|_{\pi^{-1}(y)} \cong \mathcal{O}_{\mathbb{P}^1}$, hence

$$\mathrm{H}^1(\pi^{-1}(y), \mathcal{O}_{\overline{Y}}(L^{-n}(-1))) \cong \mathrm{H}^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)).$$

By Serre duality $\mathrm{H}^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) \cong \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1))^* = 0$. Therefore, by Grauert's base change theorem [BPVdV84, Theorem 8.5 (iv)], we have $R^1\pi_*\mathcal{O}_{\overline{Y}}(L^{-n}(-1)) = 0$. Also, by Lemma 4.2.2 notice that

$$\pi_*\mathcal{O}_{\overline{Y}}(n-1) \cong \mathrm{Sym}^{n-1}((\mathcal{O}_C \oplus L)^*) \cong \mathcal{O}_C \oplus L^{-1} \oplus \cdots \oplus L^{-(n-1)}.$$

Thus, the long exact sequence of sheaves in (4.1) descends to an isomorphism

$$\mathcal{O}_C \oplus L^{-1} \oplus \cdots \oplus L^{-(n-1)} \to \pi_*\mathcal{O}_S. \tag{4.2}$$

It remains to show that the above isomorphism is the one in the statement of the proposition. The isomorphism

$$\mathcal{O}_C \oplus L^{-1} \oplus \cdots \oplus L^{-(n-1)} \cong \pi_*\mathcal{O}_{\overline{Y}}(n-1)$$

is given by

$$(c_0, c_1, \ldots, c_{n-1}) \mapsto c_0 s^n + c_1 s^{n-1}\widetilde{\lambda} + \cdots + c_{n-1}\widetilde{\lambda}^{n-1}$$

and the isomorphism $\pi_*\mathcal{O}_{\overline{Y}}(n-1) \to \pi_*\mathcal{O}_S$ is given by restricting to S. Applying the unique trivialisation $\mathcal{O}_{\overline{Y}}(-1)|_Y \cong \mathcal{O}_Y$ that sends $s|_Y$ to 1 and $\widetilde{\lambda}$ to λ reduces the isomorphism in (4.2) to

$$(c_0, c_1, \ldots, c_{n-1}) \to c_0 + c_1\lambda + \cdots + c_{n-1}\lambda^{n-1}.$$

□

In general, the pushforward of a vector bundle is a torsion-free sheaf, and under a branched cover, the singular set has codimension at least two. Hence, over a Riemann surface, the singular set is empty, and thus, the pushforward of a vector bundle defines a vector bundle.

Whenever S is smooth as a scheme, one sees that S defines a Riemann surface, and in this instance $\pi : S \to C$ defines a n-fold branched cover of Riemann surfaces. Since the morphism is finite and non-constant S is in fact a compact Riemann surface. It is not immediately clear that smooth spectral curves are connected, however, this turns out to be the case, which we will prove after introducing Higgs bundles. We will now prove that generic spectral curves are smooth.

Lemma 4.2.4. *Generic spectral curves are smooth.*

Proof. Consider sections of the line bundle L^n over Y of the form

$$s = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n$$

where $a_i \in \mathrm{H}^0(C, L^i)$ for $i = 1, \ldots, n$. Allowing the a_i to vary the zero set of s forms a linear system of divisors on Y. Since λ^n belongs to the system, a basepoint of the system necessarily belongs to the zero section $\lambda = 0$. Then, a basepoint of the system is a basepoint for L^n over C. However, L^n over C is basepoint-free [1], and thus, the base locus of the system is empty. Therefore, by Bertini's theorem, a generic divisor of the system is smooth, i.e., generic spectral curves are smooth. □

Smooth spectral curves play a vital role in the desired classifications of Higgs bundles subject to endowed structures from matrix Lie groups. In the case of GL_n, SL_n, PGL_n, and Sp_{2n}, we will prove that generic spectral curves are smooth. In the SO_{2n} and SO_{2n+1} cases, we will see that singularities are unavoidable, and hence, developing tools to study the local structure of spectral curves is profitable.

Let $\pi : S \to C$ be a spectral curve that is not necessarily smooth and let $x \in C$. Since the eigenvalues are generically distinct away from branch points suppose x is a branch point. Now, choose local coordinates (U, z) centred at x such that U is biholomorphic to the unit disc Δ. Moreover, since branch points are discrete, we may assume that $x \in U$ is the only branch point. Suppose the spectral curve is defined by

$$p(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n$$

where $a_i \in \mathrm{H}^0(C, L^i)$. Then, S is locally defined by

$$p(\lambda, z) = \lambda^n + a_1(z)\lambda^{n-1} + \cdots + a_n(z) \in \mathcal{O}_x[\lambda].$$

For $z \in \Delta^* := \Delta \setminus \{0\}$ the eigenvalues are distinct so let $\lambda_1(z), \ldots, \lambda_n(z)$ denote the corresponding distinct eigenvalues, i.e., $p(\lambda_i(z), z) = 0$. Thus,

$$p(\lambda, z) = (\lambda - \lambda_1(z)) \cdots (\lambda - \lambda_n(z)).$$

We will now show that the eigenvalues $\lambda_1(z), \ldots, \lambda_n(z)$ vary holomorphically as $z \in \Delta^*$ varies.

Lemma 4.2.5. *The eigenvalues $\lambda_1(z), \ldots, \lambda_n(z)$ vary holomorphically as $z \in \Delta^*$ varies.*

Proof. Holomorphicity is a local property so we fix $z_0 \in \Delta^*$, and let $i \in \{1, \ldots, n\}$. By choosing an open ball $z_0 \in B \subset \Delta^*$ it follows that the following assignment is holomorphic

$$\mathbb{C} \times B \ni (\lambda, z) \mapsto p(\lambda, z) \in \mathbb{C}.$$

[1] Since the line bundle L is basepoint-free, for every $x \in C$ there exists $s \in \mathrm{H}^0(C, L)$ such that $s(x) \neq 0$. Then $s^n(x) \neq 0$ and $s^n \in \mathrm{H}^0(C, L^n)$, so L^n is basepoint-free.

Now, choose an open ball V such that $\lambda_i(z_0) \in V$ and $\lambda_j(z_0) \notin \overline{V}$ for each $j \neq i$. By the argument principle
$$\frac{1}{2\pi i}\int_{\partial V}\frac{p_\lambda(\lambda,z)}{p(\lambda,z)}d\lambda = n(z)$$
where $n(z)$ denotes of zeros of $p(\lambda, z)$ in V. Since $n(z)$ is continuous and integer-valued, $n(z)$ is constant and since $n(z_0) = 1$ it follows that $n(z) = 1$ for each $z \in B$. By shrinking B if necessary we may assume $\lambda_i(z) \in V$ for every $z \in B$, and thus, by the residue theorem
$$\frac{1}{2\pi i}\int_{\partial V}\frac{\lambda p_\lambda(\lambda,z)}{p(\lambda,z)}d\lambda = \lambda_i(z).$$
Since the left-hand side is holomorphic in z, we see $\lambda_i(z)$ is holomorphic in z. □

By the Riemann monodromy theorem, the zeros $\lambda_1(z), \dots, \lambda_n(z)$ extend to a single-valued holomorphic function over any simply connected subset of Δ^*. Moreover, by the identity theorem, the extended holomorphic functions again satisfy
$$p(\lambda_i(z), z) = 0$$
for $i = 1, \dots, n$. This is called the *heredity property* of functional relations under analytic continuations. Now, analytically continuing each $\lambda_i(z)$ in a loop γ containing 0 gives extended functions $\lambda_i^*(z)$, which are zeros of $p(\lambda, z)$ by the heredity property. If $\lambda_i(z) \neq \lambda_j(z)$ then $\lambda_i^*(z) \neq \lambda_j^*(z)$. To see this, notice that if $\lambda_i^*(z) = \lambda_j^*(z)$ then by continuing along the reverse path we see $\lambda_i(z) = \lambda_j(z)$. Therefore, $\lambda_i^*(z) = \lambda_{\tau(i)}(z)$ for some permutation $\tau \in S_n$ called the *monodromy*.

We will now prove that the orbits of monodromy exactly correspond to the irreducible factors of $p(\lambda, z)$ in $\mathcal{O}_x[\lambda]$. Before giving the proof, we recall the well-known fact that as a consequence of the Weierstrass preparation theorem, the ring $\mathcal{O}_x[\lambda]$ is a unique factorisation domain (UFD).

Proposition 4.2.6. *The orbits of monodromy exactly correspond to the irreducible factors of $p(\lambda, z)$ in $\mathcal{O}_x[\lambda]$.*

Proof. Let $\lambda_1(z), \dots, \lambda_n(z)$ denote the zeros of $p(\lambda, z)$ as before, and let τ denote the monodromy. By relabelling if necessary we may assume $\lambda_1(z), \dots, \lambda_k(z)$ corresponds to an orbit of τ. Define
$$g(\lambda, z) = \prod_{i=1}^{k}(\lambda - \lambda_i(z)) = \lambda^k + b_1(z)\lambda^{k-1} + \cdots + b_k(z).$$
By definition each $b_j(z)$ is invariant under τ, and hence, each $b_j(z)$ are holomorphic inside Δ^*. Now, we claim that each $b_j(z)$ is bounded in a neighbourhood of the origin. To prove this it is sufficient to show that each λ_i is bounded in a neighbourhood of the origin.

Choose $M > 0$ such that each coefficient $a_j(z)$ satisfies $|a_j(z)| \leq M$ for all $|z| \leq 1$. Consider the disc D centred at 0 with radius $M+1$, then for all $\lambda \in \partial D$, i.e. $|\lambda| = M+1$ we see

$$|a_1(z)\lambda^{n-1} + \cdots + a_n(z)| \leq M(1 + |\lambda| + \cdots + |\lambda|^{n-1}) = M\frac{|\lambda|^n - 1}{|\lambda| - 1} = (M+1)^n - 1.$$

Therefore, $|\lambda|^n > |a_1(z)\lambda^{n-1} + \cdots + a_n(z)|$ for every $\lambda \in \partial D$, thus by Rouché theorem

$$|\lambda_i(z)| \leq M+1$$

for every $|z| \leq 1$. Therefore, each $b_j(z)$ is bounded in a neighbourhood of the origin. Thus, by the Riemann extension theorem, each $b_j(z)$ is holomorphic over Δ and thus, $g(\lambda, z) \in \mathcal{O}_x[\lambda]$. Hence, each orbit of monodromy defines a factor of $p(\lambda, z)$ in $\mathcal{O}_x[\lambda]$. Moreover, since τ permutes the zeros of $g(\lambda, z)$ it is clear that g is irreducible in $\mathcal{O}_x[\lambda]$, hence each orbit of monodromy defines an irreducible factor of $p(\lambda, z)$. Of course,

$$p(\lambda, z) = (\lambda - \lambda_1(z)) \cdots (\lambda - \lambda_n(z))$$

and since the orbits of monodromy partition the set of zeros of $p(\lambda, z)$, it is clear that the orbit of monodromy factor $p(\lambda, z)$ into irreducible polynomials. Finally, since $\mathcal{O}_x[\lambda]$ is a UFD, it follows that the irreducible factors exactly corresponds to the orbits of monodromy. $\square$

Corollary 4.2.7. *Let $\pi : S \to C$ be a spectral curve and suppose that locally about a point $y \in S$ the spectral curve is a disjoint union of irreducible curves $S = S_1 \cup \cdots \cup S_m$. Then, the polynomial $p(\lambda)$ defining S factors via $p = p_1 \cdots p_m$ where p_i is the polynomial defining S_i.*

Proof. Let $x = \pi(y)$. Choose local coordinates (U, z) centred at $x \in C$ such that U is biholomorphic to the unit disc Δ. Then a loop γ in Δ^* containing 0 lifts to a loop in the spectral curve. Hence, the lifted loop must lie in a path component of the spectral curve, and thus, the loop lies completely in S_i for some i. Consequently, the monodromy sends eigenvalues in S_j to S_j for $j = 1, \ldots, m$ so the monodromy τ factors as $\tau = \tau_1 \cdots \tau_m$ where τ_j is in the induced monodromy on S_j. Therefore, by Proposition 4.2.6 the polynomial p defining S factors via $p = p_1 \cdots p_m$ where p_j is the polynomial defining S_j. $\square$

4.3 General Linear Higgs Bundles

General linear Higgs bundles are the simplest examples of *G-Higgs bundles* where G is a complex reductive group. Since we will study more general Higgs bundles, consider the definition of G-Higgs bundles.

Definition 4.3.1. Let G be a complex reductive group, and let P be a principal G-bundle. Then a *G-Higgs bundle* is a pair (P, Φ) where $\Phi : ad(P) \to L \otimes ad(P)$ is a holomorphic vector bundle map. Here $ad(P)$ denotes the associated adjoint bundle $ad(P) := P \times_G \mathfrak{g}$. The holomorphic map Φ is called the *Higgs field*.

Remark 4.3.2. Some authors may call $\Phi : ad(P) \to L \otimes ad(P)$ an L-twisted endomorphism of $ad(P)$ and only call Φ a Higgs field when $L = K_C$. However, we will simply use the term Higgs field in either case.

Remark 4.3.3. When $G = \mathrm{GL}_n$ recall that principal GL_n-bundles are canonically associated to rank n holomorphic vector bundles. Hence, a GL_n-Higgs bundle is a pair (E, ϕ) where E is a rank n holomorphic vector bundle and $\phi : E \to L \otimes E$ is a holomorphic vector bundle map. Moreover, if $G \subset \mathrm{GL}_n$ is a matrix subgroup, then a G-Higgs bundle is a pair (E, ϕ) reflecting the structure of G.

4.3.1 Characteristic Polynomial

Let (E, ϕ) be a GL_n-Higgs bundle. To define the characteristic polynomial we first need to define the *trace* of a Higgs field.

Definition 4.3.4. The Higgs field $\phi : E \to L \otimes E$ can be identified as a homomorphism $\phi : E \otimes E^* \to L$. Then the *trace* of ϕ denoted $\mathrm{tr}(\phi)$ is the image of the identity section under ϕ, which is a section of L.

Now, the *characteristic coefficients* of ϕ are the sections $a_i \in \mathrm{H}^0(C, L^i)$ defined by $a_i = (-1)^i \,\mathrm{tr}(\wedge^i \phi)$ for $i = 0, \ldots, n$. By the Cayley-Hamilton theorem $\sum_{i=0}^n a_i \phi^{n-i} = 0$. It is not immediately clear what the generalisation of the diagonalisation map should be since the Higgs field is L-valued. However, by pulling the Higgs field back to Y we may use the tautological section $\lambda \in \mathrm{H}^0(Y, q^*L)$, which enables us to define the characteristic polynomial of ϕ.

Definition 4.3.5. The *characteristic polynomial* of $\phi : E \to L \otimes E$ is given by the section $p(\lambda) \in \mathrm{H}^0(Y, q^*L^n)$ defined by

$$p(\lambda) = \lambda^n + q^*(a_1)\lambda^{n-1} + \cdots + q^*(a_n)$$

where $a_i \in \mathrm{H}^0(C, L^i)$ are the characteristic coefficients.

Remark 4.3.6. Notice that for every $\ell \in Y$

$$p(\lambda)(\ell) = \ell^n + a_1(q(\ell))\ell^{n-1} + \cdots + a_n(q(\ell))$$

defines the standard characteristic polynomial for $\phi(q(\ell)) : E_{q(\ell)} \to L_{q(\ell)} \otimes E_{q(\ell)}$.

Hence, the characteristic polynomial $p(\lambda)$ is equal to $\wedge^n(\lambda - q^*\phi)$, i.e.,

$$p(\lambda) = \det(\lambda - q^*\phi).$$

Notation 4.3.7. When writing out the characteristic polynomial we will omit the pull-back symbols and write

$$p(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n.$$

Notice that the zero locus of the characteristic polynomial of (E, ϕ) defines a spectral curve $\pi : S \to C$. One may naturally ask if every spectral curve is associated with a Higgs bundle. It turns out that this is indeed the case, and first, we will prove that every smooth spectral curve has an associated Higgs bundle.

Proposition 4.3.8. *Suppose $\pi : S \to C$ be a smooth spectral curve defined by $p(\lambda) \in \mathrm{H}^0(Y, L^n)$. Let $N \to S$ be a given holomorphic line bundle. Then the pair $(\pi_* N, \pi_*(\lambda))$ where λ is the tautological section $\lambda : N \to \pi^*(L) \otimes N$ defines a Higgs bundle on C whose associated spectral curve is S.*

Proof. Recall that $\pi_* N$ defines a rank n holomorphic vector bundle over C. Moreover, it is clear that $\pi_*(\lambda)$ defines a Higgs field since, by the projection formula, $\pi_*(\pi^*(L) \otimes N) \cong L \otimes \pi_*(N)$. Thus, $(\pi_* N, \pi_*(\lambda))$ defines a Higgs bundle over C. Generically away from the branch points, λ has n distinct values, hence $\pi_*(\lambda)$ has n distinct eigenvalues. Therefore, the minimum polynomial of $\pi_*(\lambda)$ has degree at least n and since the characteristic polynomial of $\pi_*(\lambda)$ has degree n it follows that the minimal polynomial and characteristic polynomials agree. Denote the characteristic polynomial of $\pi_*(\lambda)$ by $m(\lambda)$. Now, S is defined as the zero locus of $p(\lambda)$, and thus, $m(\lambda)$ divides $p(\lambda)$. Clearly, $p(\lambda)$ divides $m(\lambda)$ and therefore, $p(\lambda) = m(\lambda)$, i.e., $(\pi_* N, \pi_*(\lambda))$ has spectral curve $\pi : S \to C$. $\square$

The above proposition in fact implies that a smooth spectral curve S is connected.

Lemma 4.3.9. *A smooth spectral curve S is connected.*

Proof. Suppose that S is disconnected, i.e., $S = S_1 \cup S_2$ where $S_1 \cap S_2 = \varnothing$. Note that S_1 and S_2 are smooth schemes. The structure sheaf on S is the sum of the structure sheaves of S_1 and S_2, i.e., $\mathcal{O}_S = \mathcal{O}_{S_1} \oplus \mathcal{O}_{S_2}$. Moreover, the tautological section of L over S is given by $\lambda = \lambda_1 \oplus \lambda_2$ where λ_1 and λ_2 are the tautological sections of L over S_1 and S_2 respectively. It follows that $(\pi_*\mathcal{O}_S, \pi_*\lambda) = (\pi_*\mathcal{O}_{S_1}, \pi_*\lambda_1) \oplus (\pi_*\mathcal{O}_{S_2}, \pi_*\lambda_2)$ where by Proposition 4.3.8 the Higgs bundle $(\pi_*\mathcal{O}_S, \pi_*\lambda)$ has spectral curve S, and $(\pi_*\mathcal{O}_{S_1}, \pi_*\lambda_1)$ and $(\pi_*\mathcal{O}_{S_2}, \pi_*\lambda_2)$ have spectral curve S_1 and S_2 respectively. The direct sum decomposition of $(\pi_*\mathcal{O}_S, \pi_*\lambda)$ implies $p(\lambda) = p_1(\lambda_1)p_2(\lambda_2)$ where p, p_1, p_2 are the polynomials defining S, S_1, S_2 respectively. Suppose p_1 and p_2 has degree k and l respectively. Then, the resultant of p_1 and p_2, $\mathrm{Res}(p_1, p_2)$, defines a holomorphic section of L^{kl}, which has positive degree since $\deg(L) > 0$, and hence, $\mathrm{Res}(p_1, p_2)$ vanishes somewhere. This implies p_1 and p_2 share a common zero, i.e., S_1 and S_2 intersect, a contradiction. $\square$

4.3.2 Higgs Bundles With Smooth Spectral Curve

By Proposition 4.3.8 a holomorphic line bundle over a smooth spectral curve defines a Higgs bundle over C in a natural way. We will now see that every Higgs bundle (E, ϕ) with a smooth spectral $\pi : S \to C$ curve constructs, in a natural way, a holomorphic line bundle over S. To obtain such a bundle, we will show that the Higgs field $\phi : E \to L \otimes E$ whose corresponding spectral curve is smooth is regular.

Lemma 4.3.10. *Suppose $\phi : E \to L \otimes E$ is a Higgs field whose corresponding spectral curve is smooth. Then the map ϕ is regular. In other words, for each $x \in C$, $\phi(x)$ has only one Jordan block for each eigenvalue. Equivalently, the eigenspaces of $\phi(x)$ are one-dimensional.*

Proof. For the sake of contradiction, suppose that there is some $x \in C$ such that $\phi(x)$ has an eigenvalue $\lambda_0 \in L_x$ whose eigenspace has dimension at least 2. Then we may choose linearly independent eigenvectors e_1, e_2. Extend to a basis $e_1, \dots, e_n$ of E_x, and extend the basis to a local holomorphic frame $e_1(z), \dots, e_n(z)$ of E. Here z is a local coordinate on C centred at x. Now, we choose a local trivialisation of L so that λ_0 can be viewed as a constant $\lambda_0 \in \mathbb{C}$. By shrinking the local trivialisation if necessary we may view the tautological section λ of L over Y as a holomorphic function, and we set $w = \lambda - \lambda_0$. Hence, (w, z) define local coordinates for Y centred at $\lambda_0 \in L_x$. Consider the characteristic polynomial of ϕ in the local coordinates

$$p(w, z) = \lambda^n + a_1(z)\lambda^{n-1} + \cdots + a_n(z).$$

Then the spectral curve S is locally defined about $(w, z) = (0, 0)$ by $p(w, z) = 0$. Viewing $(\lambda - \phi(z))e_i$ as a locally defined holomorphic function for $i = 1, 2$ we see

$$(\lambda - \phi(z))e_i(z)|_{(0,0)} = (\lambda_0 - \phi(0))e_i(0) = 0.$$

Hence, $(\lambda - \phi(z))e_i(z)$ vanishes at $(0, 0)$ for $i = 1, 2$. However, notice that

$$(\lambda - \phi(z))e_1 \wedge \cdots \wedge (\lambda - \phi(z))e_n = p(w, z)\, e_1 \wedge \cdots \wedge e_n.$$

Thus, $p(w, z)$ vanishes to at least second-order at $(0, 0)$, which contradicts the smoothness of the spectral curve. □

By Lemma 4.3.10 for each $p \in S$ the $\lambda(p)$-eigenspaces of $\phi(p)$ are 1-dimensional. Thus, $A := \ker(\lambda - \phi)$ defines a holomorphic line bundle over S, which is called the *spectral line bundle*. Of course, to obtain A we could have taken the subbundle induced by viewing ϕ as a sheaf map, however knowing that ϕ is a regular map is profitable in the classification. Thus, from the Higgs bundle (E, ϕ), we have constructed a natural holomorphic line over S. By Proposition 4.3.8 pushing forward A defines a Higgs bundle $(\pi_* A, \pi_*(\lambda))$ over C and one may ask if this Higgs bundle is isomorphic to (E, ϕ). In general, this is not true. However, twisting A by the *ramification divisor* of π constructs a Higgs bundle isomorphic to (E, ϕ). We will introduce the ramification divisor of π along with some elementary results and properties.

Definition 4.3.11. Let $f : X \to X'$ be a branched cover of compact Riemann surfaces. Then the *ramification divisor of* f is defined to be

$$R_f := \sum_{p \in X} (m_p - 1)\, p$$

where m_p denotes the ramification index at $p \in X$.

Remark 4.3.12. When no ambiguity is likely to arise, we omit the subscript and denote the ramification divisor by R.

Lemma 4.3.13. *Let $f : X \to X'$ be a branched cover of compact Riemann surfaces with ramification divisor R. Then there is a canonical isomorphism $K_X \otimes f^*(K_{X'}^{-1}) \cong \mathcal{O}_X(R)$.*

Proof. The differential of $f : X \to X'$ is a holomorphic bundle map $\mathrm{d}f : TX \to f^*TX'$, or, equivalently, a section of $(TX)^* \otimes f^*TX' = K_X \otimes f^*(K_{X'}^{-1})$ where we view the tangent bundles as the holomorphic tangent bundles. It suffices to show $(\mathrm{d}f) = R$. Let $p \in X$ be given and suppose p has ramification index m. Choose local coordinates w on X and z on X' centred at p and $f(p)$ respectively such that $z = f(w) = w^m$. Hence, $\mathrm{d}f(\partial_w) = mw^{m-1}\partial_z$. Thus, if $m = 1$, i.e. p is not a ramification point, then $\mathrm{d}f$ is non-vanishing. However, if $m > 1$, i.e. p is a ramification point, then df has a zero of order $m - 1$ at p. Therefore, $(\mathrm{d}f) = R$. □

Lemma 4.3.14. *Let $\pi : S \to C$ be a smooth spectral curve with ramification divisor R. Then there is a canonical isomorphism*

$$K_Y \cong \pi^*(L^{n-1}K_C)$$

hence an isomorphism

$$\mathcal{O}_S(R) \cong \pi^*(L^{n-1}).$$

Proof. Since the total space, Y, of L is a fibre bundle, the tangent bundle TY fits into an exact sequence:

$$0 \to \pi^*(L) \to TY \to \pi^*(TC) \to 0.$$

By taking determinants, there is a canonical isomorphism:

$$\det(TY) \cong \pi^*(L) \otimes \pi^*(TC).$$

Dualising the above isomorphism one obtains

$$K_Y \cong \pi^*(L^{-1}K_C).$$

Since the spectral curve is the zero locus of a section of $\pi^*(L^n)$, by the adjunction formula one obtains $K_S \cong K_Y|_S \otimes \pi^*(L^n)$, i.e.

$$K_S \cong \pi^*(L^{n-1}K_C). \tag{4.3}$$

By Lemma 4.3.13 and (4.3) we see

$$\mathcal{O}_S(R) \cong \pi^*(L^{n-1}).$$

$\square$

These elementary isomorphisms provide the necessary machinery to derive the desired correspondence. Another useful property of Higgs bundles with smooth spectral curves is the fact that an automorphism is a scalar multiple of the identity. To prove this, we first require a lemma.

Lemma 4.3.15. *Let (E, ϕ) be a Higgs bundle with smooth spectral curve $\pi : S \to C$ defined by $p(\lambda)$. Then the Higgs bundle (E, ϕ) has no non-zero ϕ-invariant proper holomorphic subbundles.*

Proof. Suppose the holomorphic vector bundle E has a proper non-zero ϕ-invariant subbundle $F \subset E$, i.e. $\phi(F) \subset L \otimes F$. Thus, $\phi^F := \phi|_F$ defines a Higgs field over F so (F, ϕ^F) is a Higgs bundle. Now, suppose (F, ϕ^F) has characteristic polynomial $q(\lambda)$. Then by the Cayley-Hamilton theorem it follows that $q(\lambda)$ divides $p(\lambda)$. However, by Corollary 4.2.7 it follows that $\pi : S \to C$ is not smooth, which is a contradiction $\square$

Proposition 4.3.16. *Let (E, ϕ) be a Higgs bundle with a smooth spectral curve S. Then,* $\mathrm{Aut}((E, \phi)) \cong \mathbb{C}^*$.

Proof. Let $f : E \to E$ be an automorphism of (E, ϕ), i.e., $f \circ \phi = \phi \circ f$. Then, the characteristic polynomial of f is given by

$$\det(\lambda \operatorname{id}_E - f) = \lambda^n + c_1\lambda^{n-1} + \cdots + c_n.$$

Since f is an honest endomorphism, the characteristic coefficients are constant, which implies that the eigenvalues $\lambda_1, \ldots, \lambda_n$ are constant. Let μ be one eigenvalue and let F be the μ-generalised eigenbundle, which is non-zero. Notice that F is ϕ-invariant, and hence, by Lemma 4.3.15, $F = E$. Thus, μ is the only eigenvalue, and we claim that $f = \mu \operatorname{id}_E$. Let $N \subset E$ be the subbundle induced by the kernel of the sheaf map $\mu \operatorname{id}_E - f : \mathcal{E} \to \mathcal{E}$. The bundle is non-zero since μ is an eigenvalue and since f commutes with ϕ it follows that N is ϕ-invariant, and thus, $N = E$, i.e., $f = \mu \operatorname{id}_E$. $\square$

Along with these isomorphisms, we also require *relative duality.*

4.4 Relative Duality

In this section we will prove relative duality for a branched cover of compact Riemann surfaces, which describes dualising the pushforward of a vector bundle under a branched cover. For our purposes, we only need the result for line bundles.

Theorem 4.4.1 (Relative Duality). *Let $f : X \to X'$ be a branched cover of compact Riemann surfaces with ramification divisor R. Let $N \to X$ be a given holomorphic line bundle. Then there is a canonical isomorphism of vector bundles*

$$f_*(N)^* \cong f_*(N^*(R)).$$

In other words, the dual of the pushforward of a line bundle is isomorphic to first dualising the line bundle, twisting by R, then applying the direct image functor.

Hitchin outlined a proof of relative duality in [Hit16]. Here, we will fill-in the details except we give a slightly different non-degenerate pairing. Namely, we will use the canonical pairing of sections and apply the *trace map*. To prove non-degeneracy we will then resort to using Euler's formula in a basis. Moreover, the proof of relative duality requires Proposition 4.2.3. We will prove that every branched cover of compact Riemann surfaces is locally a spectral curve.

Lemma 4.4.2. *Every branched cover of compact Riemann surfaces is locally a spectral curve.*

Proof. Suppose $f : X \to X'$ is a branched cover of compact Riemann surfaces. Away from ramification points, f is locally a biholomorphism. Moreover, ramification points are isolated, and hence the set of ramification points is discrete. Thus, it suffices to prove the claim locally about a single ramification point. Let $p \in X$ be a given ramification point with ramification index m. Then, choose local coordinates w on X and z on X' centred at p and $f(p)$ respectively such that $z = f(w) = w^m$ and the only ramification point is p, and the only branch point is $f(p)$. Then, in local coordinates, X is defined as the zero locus of $w^m - z$. Hence, X is locally a spectral curve. $\square$

The desired isomorphism is a non-degenerate dual pairing

$$(\ ,\ ) : f_*(N) \otimes f_*(N^*(R)) \to \mathcal{O}_{X'}.$$

Since non-degeneracy is a local property we may assume without loss of generality that $f : X \to X'$ is a spectral curve and X is defined by $p(\lambda)$. To construct the pairing consider the definition of the trace map.

Definition 4.4.3. Let $f : X \to X'$ be a branched cover of compact Riemann surfaces. Denote the function field of X and X' by $K(X)$ and $K(X')$ respectively. The *trace map* denoted $\mathrm{Tr}_{X/X'} : K(X) \to K(X')$ is defined by

$$\mathrm{Tr}_{X/X'}(g)(y) = \sum_{x \in f^{-1}(y)} g(x)$$

for every $y \in X'$.

Remark 4.4.4. The trace map sends meromorphic functions on X to meromorphic functions on X'. Moreover, from the definition, it is clear that for open $V \subset X'$, the trace map sends meromorphic functions over $f^{-1}(V)$ to meromorphic functions over V. Therefore, one may regard the trace map as a sheaf map

$$\mathrm{Tr}_{X/X'} : f_*\mathcal{M}_X \to \mathcal{M}_{X'}.$$

Here, $\mathcal{M}_X$ and $\mathcal{M}_{X'}$ denotes the sheaf of meromorphic functions on X and X' respectively.

The sheaf $f_*\mathcal{M}_X$ is a sheaf of $\mathcal{M}_{X'}$-modules where for each open set $U \subset X'$ the module action $\mathcal{M}_{X'}(U) \times \mathcal{M}_X(f^{-1}U) \to \mathcal{M}_X(f^{-1}U)$ is given by $(h, g) \mapsto hg$ where $(hg)(y) := h(f(y))g(y)$ for every $y \in f^{-1}U$. In fact, $f_*\mathcal{M}_X$ defines a sheaf of $\mathcal{M}_{X'}$-algebras.

Lemma 4.4.5. *The trace map* $\mathrm{Tr}_{X/X'} : f_*\mathcal{M}_X \to \mathcal{M}_{X'}$ *is a homomorphism of sheaves of* $\mathcal{M}_{X'}$*-modules.*

Proof. The trace map is clearly compatible with restriction and preserves additivity so we are left to show the trace map commutes with the $\mathcal{M}_{X'}(U)$ action for each open $U \subset X'$. Let $U \subset X'$ be a given open subset and let $h \in \mathcal{M}_{X'}(U)$ and $g \in \mathcal{M}_X(f^{-1}U)$. Then, for every $x \in U$

$$\mathrm{Tr}_{X/X'}(gh)(x) = \sum_{y \in f^{-1}(x)} (gh)(y) = h(x) \sum_{y \in f^{-1}(x)} h(y) = g(x)\,\mathrm{Tr}_{X/X'}(h)(x).$$

□

The trace map extends to sections of vector bundles in an obvious manner. Suppose $E \to X'$ is a holomorphic vector bundle, then $f^*E \to X$ defines a holomorphic vector bundle, and the trace map has the following form

$$\mathrm{Tr}_{X/X'} : f_*(\mathcal{M}_X(f^*(E))) \to \mathcal{M}_{X'}(E). \tag{4.4}$$

Now, focusing on the case that $N = \mathcal{O}_X$ we may choose a non-empty open subset $U \subset X'$ trivialising $f_*\mathcal{O}_X$. Hence, $f_*\mathcal{O}_X(U)$ is a free $\mathcal{O}_{X'}(U)$-module of rank n. By Lemma 4.4.2 we may shrink U if necessary so that locally $f : X \to X'$ is a spectral curve. Suppose X is locally defined by $p(\lambda)$. We may use the canonical isomorphism in Proposition 4.2.3 to define a basis for $f_*\mathcal{O}_X(U)$ over $\mathcal{O}_{X'}(U)$. Namely, by considering

$$(0, \ldots, 1, \ldots, 0) \mapsto \lambda^i$$

we see $1, \lambda, \ldots, \lambda^{n-1}$ forms a basis for $f_*\mathcal{O}_X(U)$ over $\mathcal{O}_{X'}(U)$. Moreover, we have seen $(\partial_\lambda p(\lambda)) = R$ where $R = R_f$. Thus, $1/\partial_\lambda p(\lambda), \lambda/\partial_\lambda p(\lambda), \ldots, \lambda^{n-1}/\partial_\lambda p(\lambda)$ forms a basis for $f_*\mathcal{O}_X(R)(U)$ over $\mathcal{O}_{X'}(U)$. In the following proposition, we compute the basis under the trace map.

Proposition 4.4.6 (Euler's Formula)**.** *The basis,* $1/\partial_\lambda p(\lambda), \ldots, \lambda^{n-1}/\partial_\lambda p(\lambda)$ *of* $f_*\mathcal{O}_X(R)(U)$ *over* $\mathcal{O}_{X'}(U)$ *satisfies*

$$\mathrm{Tr}_{X/X'}\left(\frac{\lambda^j}{\partial_\lambda p(\lambda)}\right) = \begin{cases} 0 & \text{for } 1 \leq j \leq n-2 \\ 1 & \text{for } j = n-1 \end{cases}$$

Proof. By continuity, it suffices to compute away from branch points. Let $x \in X$ be a generic point. Then the zeros of $p(\lambda, x)$ are distinct, which we denote by $\lambda_1, \ldots, \lambda_n$. By definition

$$\mathrm{Tr}_{X/X'}\left(\frac{\lambda^j}{\partial_\lambda p(\lambda)}\right)(x) = \sum_{i=1}^{n} \frac{\lambda_i^j}{\partial_\lambda p(\lambda_i)}$$

for $j = 1, \ldots, n-1$. Let w be local coordinates on X and let γ be a closed contour containing each zero $\lambda_1, \ldots, \lambda_n$. By a straightforward application of the residue theorem one sees

$$\frac{1}{2\pi i} \oint_\gamma \frac{w^j}{p(w)} dw = \sum_{i=1}^{n} \frac{\lambda_i^j}{\partial_\lambda p(\lambda_i)}.$$

Suppose now that $\gamma = \{w \mid |w| = M\}$ for sufficiently large M, then $|p(w)| \geq rM^n$ where r is some non-zero constant. Now, approximating the integral

$$\left| \frac{1}{2\pi i} \oint_\gamma \frac{w^j}{p(w)} \right| \leq \frac{1}{2\pi}(2\pi M)\frac{M^j}{rM^n} = \frac{M^{j+1}}{rM^n} = \frac{1}{r}M^{j+1-n}.$$

If $1 \leq j \leq n-2$, then $M^{j+1-n} \to 0$ as $M \to \infty$ from which it follows that

$$\mathrm{Tr}_{X/X'}\left(\frac{\lambda^j}{p(\lambda)}\right) = 0.$$

If $j = n-1$ notice that

$$\partial_\lambda p(\lambda) = n\lambda^{n-1} + \text{lower order terms},$$

and by the previous argument it follows that

$$\frac{1}{2\pi i}\oint_\gamma \frac{\partial_w p(w)}{p(w)}dw = \frac{n}{2\pi i}\oint_\gamma \frac{w^{n-1}}{p(w)}dw.$$

By the argument principle, the left-hand side is equal to n, and thus,

$$\frac{1}{2\pi i}\oint_\gamma \frac{w^{n-1}}{p(w)}dw = 1.$$

Therefore,

$$\mathrm{Tr}_{X/X'}\left(\frac{\lambda^{n-1}}{\partial_\lambda p(\lambda)}\right) = 1.$$

□

Corollary 4.4.7. *Let $f : X \to X'$ be a branched cover of compact Riemann surfaces. Then, the trace of a meromorphic function on X with poles along the ramification R is a holomorphic function over X', i.e.*

$$\mathrm{Tr}_{X/X'} : f_*\mathcal{O}_X(R) \to \mathcal{O}_{X'}$$

Proof. The proof follows from Proposition 4.4.6 since the trace map sends a basis to holomorphic functions. □

Consequently, restricting the trace map in (4.4) to meromorphic sections s of $f^*(E)$ such that $(s) \geq -R$ the map becomes

$$\mathrm{Tr}_{X/X'} : f_*(f^*(\mathcal{E})(R)) \to \mathcal{E}. \tag{4.5}$$

Notation 4.4.8. When $f : X \to X'$ is a branched cover with ramification divisor R and $A \to X$ is a holomorphic vector bundle we will denote the locally free sheaf of $\mathcal{O}_X$-modules associated to $A(R)$ by $\mathcal{A}(R)$, i.e., $\mathcal{A}(R) := \mathcal{O}_X(A(R))$. Note, this is not the same as $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{O}_X(R)$.

Before giving the proof of relative duality, we require one more lemma.

Lemma 4.4.9. *Suppose $f : X \to X'$ is a branched cover of compact Riemann surfaces. Let F and F' be holomorphic vector bundles over X', and let $h : F \to F'$ be a holomorphic bundle map. Then the diagram*

$$\begin{tikzcd} f_*(f^*(\mathcal{F})(R)) \arrow[r, "f^*h"] \arrow[d, "\mathrm{Tr}_{X/X'}"'] & f_*(f^*(\mathcal{F}')(R)) \arrow[d, "\mathrm{Tr}_{X/X'}"] \\ \mathcal{F} \arrow[r, "h"] & \mathcal{F}' \end{tikzcd}$$

commutes.

Proof. Let s be a given section of $f_*(f^*F(R))$, and let $x \in C$. Then,

$$h(\mathrm{Tr}_{X/X'}(s))(x) = h \sum_{y \in f^{-1}(x)} s(y) = \sum_{y \in f^{-1}(x)} h(s(y)),$$

which is the definition of $\mathrm{Tr}_{X/X'}(f^*h(s))(x)$. $\square$

Now we give the proof of relative duality.

Proof of Relative Duality. Consider the canonical pairing between $f_*(N)$ and $f_*(N^*(R))$, which pairs sections forming a section of $f_*(\mathcal{O}_X(R))$, i.e., $\langle h, g\rangle \mapsto h \otimes g$. By Lemma 4.4.9 applying the trace map $\mathrm{Tr}_{X/X'} : f_*(\mathcal{O}_X(R)) \to \mathcal{O}_{X'}$ to the pairing $h \otimes g$ gives an $\mathcal{O}_{X'}$-bilinear form $\mathrm{Tr}_{X/X'}(h \otimes g)$. Therefore, there is an induced pairing on the underlying vector bundles

$$(\ ,\ ) : f_*(N) \otimes f_*(N^*(R)) \to \mathcal{O}_{X'} \tag{4.6}$$

To deduce the desired duality, it suffices to prove the pairing is non-degenerate. Since non-degeneracy is a local property, we may assume without loss of generality that $N = \mathcal{O}_X$. Choose an open set $U \subset X'$ as before that trivialises $f_*\mathcal{O}_X$. Recall that $1, \lambda, \ldots, \lambda^{n-1}$ forms a basis for $(f_*\mathcal{O}_X)(U)$ over $\mathcal{O}_{X'}(U)$, and $1/\partial_\lambda p(\lambda), \ldots, \lambda^{n-1}/\partial_\lambda p(\lambda)$ forms a basis for $f_*(\mathcal{O}_X(R))$ over $\mathcal{O}_{X'}(U)$. By Euler's formula notice that

$$\mathrm{Tr}_{X/X'}\left(\frac{\lambda^{i+j}}{\partial_\lambda p(\lambda)}\right) = \begin{cases} 0 & \text{for } i+j < n-1 \\ 1 & \text{for } i+j = n-1 \\ * & \text{for } i+j > n-1 \end{cases}.$$

Hence, the intersection matrix is given by

$$\begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & * \\ \vdots & \iddots & \iddots & \vdots \\ 1 & * & \ldots & * \end{pmatrix}.$$

Thus, the intersection matrix has determinant ± 1. Therefore the pairing in (4.6) is non-degenerate and thus, $f_*(N)^* \cong f_*(N^*(R))$. $\square$

4.5 Classification

Recall that if (E, ϕ) is a GL_n-Higgs bundle with smooth spectral curve $\pi : S \to C$, then the spectral line bundle is defined to $A := \ker(\lambda - \pi^*\phi)$. We now claim that $\pi_*(A(R)) \cong E$ where R denotes the ramification divisor of $\pi : S \to C$. Notice that A is a rank 1 subbundle of π^*E. Hence, we define a sheaf map $\psi : \pi_*\mathcal{A}(R) \to \mathcal{E}$ by the restriction of the trace map, i.e., the unique map such that the following map commutes

$$\begin{tikzcd} \pi_*\mathcal{A}(R) \arrow[r] \arrow[dr, "\psi"'] & \pi_*(\pi^*\mathcal{E}(R)) \arrow[d, "\mathrm{Tr}_{S/C}"] \\ & \mathcal{E} \end{tikzcd}.$$

To prove $\pi_*(A(R)) \cong E$, it suffices to show that the sheaf map ψ is an isomorphism.

Proposition 4.5.1. *The sheaf map $\psi : \pi_*\mathcal{A}(R) \to \mathcal{E}$ is an isomorphism.*

Proof. Since $\mathrm{rank}(\pi_*(A(R))) = n = \mathrm{rank}(E)$ it suffices to prove ψ is injective. Suppose $\{U_k\}$ is an open cover of C that trivialises $\pi_*(A(R))$ and such that $\pi^{-1}(U_k)$ trivialises A. Let $U \in \{U_k\}$ and suppose $\psi(U)(s) = 0$, i.e., $\mathrm{Tr}_{S/C}(s) = 0$. Since $\phi : E \to L \otimes E$ is a homomorphism it follows that $\phi^i \,\mathrm{Tr}_{S/C}(s) = 0$ for $i = 0, \dots, n-1$. By Lemma 4.4.9 it follows that $\mathrm{Tr}_{S/C}((\pi^*\phi)^i s) = 0$ for $i = 0, \dots, n-1$. However, $A = \ker(\lambda - \pi^*(\phi))$ and it follows that $\mathrm{Tr}_{S/C}(\lambda^i s) = 0$. Recall that $1/\partial_\lambda p(\lambda), \lambda/\partial_\lambda p(\lambda), \dots, \lambda^{n-1}/\partial_\lambda p(\lambda)$ forms a basis for $\mathcal{A}(R)(\pi^{-1}U)$ for $\mathcal{O}_C(U)$. Hence, we may write

$$s = \sum_{j=0}^{n-1} \frac{\lambda^j}{\partial_\lambda p(\lambda)} s_j$$

for some $s_j \in \mathcal{O}_C(U)$. Since $\mathrm{Tr}_{S/C}$ is a homomorphism of sheaves of $\mathcal{O}_C$-modules

$$\mathrm{Tr}_{S/C}(\lambda^i s) = \sum_{j=0}^{n-1} \mathrm{Tr}_{S/C}\left(\frac{\lambda^{i+j}}{\partial_\lambda p(\lambda)}\right) s_j = 0.$$

By Euler's formula it follows that $s_{n-1-i} = 0$ for $i = 0, \dots, n-1$, and thus, $s = 0$. Therefore, ψ is injective and hence, ψ is an isomorphism. $\square$

Therefore, there is an isomorphism of vector bundles $\pi_*(A(R)) \cong E$. Thus, a Higgs bundle (E, ϕ) with smooth spectral curve $\pi : S \to C$ defines, in a natural way, a holomorphic line bundle $A(R) \to S$. Moreover, pushing forward $A(R)$ recovers E, i.e., $\pi_*(A(R)) \cong E$. Recall from Proposition 4.3.8 that every holomorphic line bundle $N \to S$ canonically defines a Higgs bundle $(\pi_*N, \pi_*\lambda)$ whose spectral curve is $\pi : S \to C$. We now claim that there is a canonical isomorphism of Higgs bundles $(E, \phi) \cong (\pi_*(A(R)), \pi_*\lambda)$ where $\lambda : A(R) \to \pi^*(L) \otimes A(R)$ denotes the tautological section.

Lemma 4.5.2. *Suppose (E, ϕ) is a GL_n-Higgs bundle with spectral curve $\pi : S \to C$. Let $A = \ker(\lambda - \phi)$ denote the spectral line bundle, and denote the ramification divisor of π by R. Then $A(R)$ defines a holomorphic line bundle over S, and the Higgs bundle $(\pi_*(A(R)), \pi_*\lambda)$ over C is isomorphic to (E, ϕ).*

Proof. In Proposition 4.5.1 the sheaf map $\psi : \pi_*(\mathcal{A}(R)) \to \mathcal{E}$ defined by the trace map gives an isomorphism $\pi_* A(R) \cong E$ of vector bundles. Thus, it suffices to show that this isomorphism commutes with the Higgs fields. Let $U \subseteq C$ be a given open subset. Then for every $s \in \mathcal{A}(R)(\pi^{-1}(U))$ it follows that $(\pi_*\lambda)s = \lambda s$. Therefore, since $A := \ker(\lambda - \pi^*\phi)$, one sees

$$\psi((\pi_*\lambda)s) = \mathrm{Tr}_{S/C}(\lambda s) = \mathrm{Tr}_{S/C}((\pi^*\phi)s) = \phi\,\mathrm{Tr}_{S/C}(s)$$

where the last equality follows from Lemma 4.4.9. □

Finally, suppose instead that we started with a holomorphic line bundle $N \to S$ over a smooth spectral curve $\pi : S \to C$ and constructed the Higgs bundle $(E, \phi) := (\pi_* N, \pi_*\lambda)$ over C. Then, computing the spectral line bundle $A := \ker(\lambda - \pi^*\phi)$ and twisting by the ramification divisor R of π, we obtain a holomorphic line bundle $A(R) \to S$. Now we will show the holomorphic line bundles are isomorphic, i.e., $N \cong A(R)$.

Lemma 4.5.3. *Suppose $\pi : S \to C$ is a smooth spectral curve with ramification divisor R, and let $N \to S$ be a holomorphic line bundle. Then the Higgs bundle $(E, \phi) := (\pi_* N, \pi_*\lambda)$ where $\lambda : N \to \pi^* L \otimes N$ denotes the tautological section defines a holomorphic line bundle $A(R) \to S$ that is isomorphic to N.*

Proof. For brevity set $\widetilde{N} := A(R)$. By Lemma 4.5.2 the Higgs bundle $(\widetilde{E}, \widetilde{\phi}) := (\pi_*\widetilde{N}, \pi_*\widetilde{\lambda})$, where $\widetilde{\lambda} : \widetilde{N} \to \pi^*(L) \otimes \widetilde{N}$ denotes the tautological section, is isomorphic to (E, ϕ), which is induced by the trace map $\psi : \widetilde{E} \to E$. Moreover, $\psi : \widetilde{E} \to E$ is an isomorphism of $\pi_*\mathcal{O}_S$-algebras. By Proposition 4.2.3 it follows that $\pi_*\mathcal{O}_S$ is generated by $\mathcal{O}_C$ and the tautological section λ. We see that ψ is an isomorphism of $\mathcal{O}_C$-modules that commutes with λ. Let $\{U_i\}$ be an open cover of C such that N and $\widetilde{N}$ are trivialised over each pre-image $\widehat{U}_i := \pi^{-1}(U_i)$. Let $g_{ij}, \widetilde{g_{ij}} : \widehat{U}_i \cap \widehat{U}_j \to \mathcal{O}_S^*$ denote the transition functions of N and $\widetilde{N}$ respectively. Let ψ_i denote ψ in the trivialising neighbourhood $\widehat{U}_i$. Then, over $\widehat{U}_i \cap \widehat{U}_j$, the following diagram commutes

$$\begin{array}{ccc} (\pi_*\mathcal{O}_S)(U_i \cap U_j) & \xrightarrow{\psi_i|_{U_i\cap U_j}} & (\pi_*\mathcal{O}_S)(U_i \cap U_j) \\ \widetilde{g_{ij}}\Big\uparrow & & \Big\uparrow g_{ij} \\ (\pi_*\mathcal{O}_S)(U_i \cap U_j) & \xrightarrow{\psi_j|_{U_i\cap U_j}} & (\pi_*\mathcal{O}_S)(U_i \cap U_j) \end{array}$$

Hence, the following diagram commutes too

$$\begin{array}{ccc} \mathcal{O}_S(\widehat{U}_i \cap \widehat{U}_j) & \xrightarrow{\psi_i} & \mathcal{O}_S(\widehat{U}_i \cap \widehat{U}_j) \\ \widetilde{g_{ij}}\Big\uparrow & & \Big\uparrow g_{ij} \\ \mathcal{O}_S(\widehat{U}_i \cap \widehat{U}_j) & \xrightarrow{\psi_j} & \mathcal{O}_S(\widehat{U}_i \cap \widehat{U}_j) \end{array}$$

Therefore, $g_{ij} = \psi_i \widetilde{g_{ij}} \psi_j^{-1}$ and thus, $N \cong \widetilde{N}$ □

In summary, we have derived the following classification known as the spectral curve correspondence.

Theorem 4.5.4 (Spectral Curve Correspondence)**.** *Suppose* (E, ϕ) *is a* GL_n*-Higgs bundle with smooth spectral curve* $\pi : S \to C$ *and let* R *denote the ramification divisor of* π*. Then* $A := \ker(\lambda - \phi)$ *defines a holomorphic line bundle over* S *called the spectral curve line bundle, and* E *is isomorphic to* $\pi_*(A(R))$*. Moreover, the Higgs field* $\phi : E \to L \otimes E$ *is induced by the tautological section* $\lambda : A(R) \to \pi^*(L) \otimes A(R)$*.*

Conversely, given any holomorphic line bundle N *over* S*, if we set* $E = \pi_* N$ *and let* $\phi : E \to L \otimes E$ *be the map induced by* $\lambda : N \to \pi^*(L) \otimes N$*, then* (E, ϕ) *is a Higgs bundle over* C *with spectral curve* S*. Furthermore, these assignments are mutual inverses. Hence, the isomorphism classes of Higgs bundles with spectral curve* S *is in one-to-one correspondence with* $\mathrm{Pic}(S)$*.*

The spectral curve correspondence lays the foundation for further classifications when Lie groups endow extra structure on the Higgs bundles. In the following section, we use the spectral curve correspondence to classify the generic fibres of the general linear Hitchin fibration.

4.6 Hitchin Fibration

Now, we will introduce the *Hitchin fibration* in this setting. To define the Hitchin fibration, we require a moduli space of Higgs bundles. The construction of moduli spaces is highly non-trivial, and constructing the desired moduli space would entail a separate research project. Hence, we will introduce the notion of *stability* and provide a reference for the construction of the desired moduli space. Recall that the *slope* of a holomorphic vector bundle E denoted $\mu(E)$ is defined to be $\mu(E) := \deg(E)/\operatorname{rank}(E)$.

Definition 4.6.1. A Higgs bundle (E, ϕ) is *stable* (resp. *semi-stable*) if for every non-trivial proper subbundle $F \subset E$ such that $\phi(F) \subseteq L \otimes F$, one has

$$\mu(F) < \mu(E) \quad \text{(resp. } \leq \text{)}.$$

In 1987, Nigel Hitchin constructed the moduli space of stable GL_n-Higgs bundles subject to a fixed rank and degree for when the Higgs fields are K_C-valued endomorphisms. For our purposes, we require the analogous moduli space for when K_C is replaced by an arbitrary non-trivial basepoint-free holomorphic line bundle L. Fortunately, in 1991, Nitin Nitsure constructed such moduli space and the concerning reader may find the construction in [Nit91]. In fact, he constructed the moduli space in the more general setting where the compact Riemann surface is a smooth projective variety defined over an arbitrary algebraically closed field and the vector bundles are algebraic. Moreover, he constructed the space subject to semi-stability and not stability. Of course, to consider semi-stability instead one needs the notion of S-equivalence. However, since Higgs bundles with smooth spectral curves have no non-zero proper holomorphic subbundles that are invariant under the Higgs field they are vacuously stable. Hence, we will only consider stable Higgs bundles in which case we can avoid S-equivalence. More generally, if G is a complex reductive Lie group then we can consider the the moduli space, $\mathcal{M}_G$, of G-Higgs bundles subject to a fixed topological class. The concerning reader may consult [Sch08, Theorem 2.8.1.2] for explicit construction of such moduli space. Note, when G is a matrix Lie group the topological class is the degree. Consider now the definition of the G-Hitchin fibration.

Definition 4.6.2. Let $p_1, \dots, p_k$ be a homogeneous basis for the algebra of invariant polynomials of the Lie algebra $\mathfrak{g}$, and denote the degree of p_i by d_i, which are called the *fundamental degrees of* $\mathfrak{g}$ Then, the *Hitchin fibration* is the map

$$h : \mathcal{M}_G \to \bigoplus_{i=1}^{k} \mathrm{H}^0(C, L^{d_i})$$

defined by

$$(E, \phi) \mapsto (p_1(\phi), \dots, p_k(\phi)).$$

The affine space $\bigoplus_{i=1}^{k} \mathrm{H}^0(C, L^{d_i})$ is called the *G-Hitchin Base.*

Remark 4.6.3. Since the Hitchin base only depends on the Lie algebra $\mathfrak{g}$ it is clear that if G and G' are complex reductive Lie groups such that $\mathfrak{g} \cong \mathfrak{g}'$, then the G-Hitchin base and G'-Hitchin base agree.

The GL_n-Hitchin fibration has a simple description: if the characteristic polynomial of (E, ϕ) is given by

$$\lambda^n + a_1\lambda^{n-1} + \cdots + a_n$$

then $(a_1, \dots, a_n)$ forms a homogeneous basis for the algebra of invariant polynomials of the Lie algebra $\mathfrak{gl}_n$, hence

$$h(E, \phi) = (a_1, \dots, a_n).$$

Consequently, the Hitchin base becomes $\bigoplus_{i=1}^{n} \mathrm{H}^0(C, L^i)$.

Notice that a point $(a_1, \dots, a_n) \in \bigoplus_{i=1}^{n} \mathrm{H}^0(C, L^i)$ canonically defines a spectral curve. Also, from Lemma 4.2.4 notice that the generic points in $\bigoplus_{i=1}^{n} \mathrm{H}^0(C, L^i)$ are precisely those corresponding to smooth spectral curves. Hence we will use spectral curve correspondence to classify the generic fibres of the Hitchin fibration. Now, we will prove that the map is surjective, from which we reference that the map is proper.

Proposition 4.6.4 ([Sch08, Proposition 2.8.1.4])**.** *The G-Hitchin fibration is a proper map for any complex reductive Lie group G. Consequently, the Hitchin fibration has a closed image.*

Note, Nitsure in [Nit91] gave a simpler proof that the Hitchin fibration is proper for the case $G = \mathrm{GL}_n$.

Proposition 4.6.5. *The* GL_n*-hitchin fibration is surjective.*

Proof. By the spectral curve correspondence, the generic points in $\bigoplus_{i=1}^n \mathrm{H}^0(C, L^i)$ lie in the image of h. Moreover, by Bertini's theorem, the generic points form a dense open subset. Thus, the image contains an open dense subset, and by Proposition 4.6.4 the image is closed. Therefore, h is surjective. □

Corollary 4.6.6. *Let* $\pi : S \to C$ *be a spectral curve not necessarily smooth. Then the spectral curve is associated with some Higgs bundle* (E, ϕ).

Proof. Suppose the spectral curve $\pi : S \to C$ is defined by

$$p(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n$$

where $a_i \in \mathrm{H}^0(C, L^i)$. Then, $(a_1, \ldots, a_n) \in \bigoplus_{i=1}^n \mathrm{H}^0(C, L^i)$ so by Proposition 4.6.5 there is a GL_n Higgs bundle (E, ϕ) such that $h(E, \phi) = (a_1, \ldots, a_n)$. Therefore, the Higgs bundle (E, ϕ) has spectral curve $\pi : S \to C$. □

4.6.1 Dimension of GL_n-Hitchin Base.

In this subsection we will compute the dimension of the GL_n-Hitchin base for when $\deg(L) \geq 2g$. We are making the assumption that $\deg(L) \geq 2g$ to ensure that L is basepoint-free and that $\mathrm{H}^1(C, L^i) = 0$ for $i = 1, \ldots, n$ so that we may apply Riemann-Roch. Without the assumption it is not clear that $\mathrm{H}^1(C, L^i) = 0$ for each i, and hence, $h^0(C, L^i)$ would be in terms of $h^1(C, L^i)$, which doesn't provide insight. Thus, by assuming $\deg(L) \geq 2g$ the dimension $h^0(C, L^i)$ will only be in terms of the topological invariants g and $\deg(L)$. Now, by the Riemann-Roch theorem

$$h^0(C, L^i) = i \deg(L) + 1 - g.$$

Hence,

$$\sum_{i=1}^n h^0(C, L^i) = \sum_{i=1}^n (i \deg(L) + 1 - g),$$

thus,

$$\dim \bigoplus_{i=1}^n \mathrm{H}^0(C, L^i) = \frac{n(n+1)}{2} \deg(L) + n(1 - g). \tag{4.7}$$

Now, we will compute the dimension of the Hitchin base for when $L = K_C$, which is basepoint-free and the classical case considered by Hitchin. For $i > 2$ notice that $\deg(K^i) \geq 2g - 2$ and thus, $\mathrm{H}^1(C, K_C^i) = 0$. By the Riemann-Roch theorem

$$h^0(C, K_C^i) = i(2g - 2) + 1 - g.$$

Moreover, by Serre duality $h^0(C, K_C) = g$, and hence,

$$\sum_{i=1}^{n} h^0(C, K_C^i) = g + \left(\sum_{i=1}^{n} i(2g-2)\right) + (n-1)(1-g).$$

It follows that

$$\dim \bigoplus_{i=1}^{n} \mathrm{H}^0(C, K_C^i) = 1 + (g-1)n^2. \tag{4.8}$$

4.6.2 Generic Fibres of GL_n-Hitchin Fibration

Before applying the spectral curve correspondence to classify the generic fibres of the Hitchin fibration, notice that the Higgs bundles in the moduli space $\mathcal{M}^{n,d}$ has a fixed degree. Hence, it is clear that not every holomorphic line bundle over a smooth spectral curve will belong to the fibre since the degrees may not agree. Thus, in the following proposition, we will derive a formula that computes the degree of the pushforward of a line bundle.

Proposition 4.6.7. *Let $f : X \to X'$ be a branched cover of compact Riemann surfaces. Then for every holomorphic line bundle $N \to X$, there is an isomorphism*

$$\det(f_* N) \cong \mathrm{Nm}_{X/X'}(N) \otimes \det(f_* \mathcal{O}_X).$$

Proof. We proceed by induction as per Remark 2.2.3. The base case is immediate since $\mathrm{Nm}_{X/X'}(\mathcal{O}_X) = \mathcal{O}_{X'}$. Let $N \to X$ be a given holomorphic line bundle. Assume the formula is true for N, i.e.

$$\det(f_* N) \cong \mathrm{Nm}_{X/X'}(N) \otimes \det(f_* \mathcal{O}_X)$$

then it suffices to prove the formula holds for $N(p)$ and $N(-p)$ for arbitrary $p \in X$. To see the formula holds for $N(p)$ consider the short exact sequence of sheaves

$$0 \to \mathcal{O}_X N \to \mathcal{O}_X N(p) \to \mathcal{O}_p(N) \to 0.$$

Applying the direct image functors gives the following long exact sequence of sheaves

$$0 \to f_* \mathcal{O}_X(N) \to f_* \mathcal{O}_X(N(p)) \to f_* \mathcal{O}_p(N) \to R^1 f_* \mathcal{O}_X(N) \to \cdots \tag{4.9}$$

The right derived functor $R^1 f_* \mathcal{O}_X(N)$ is the sheaf associated to the presheaf $U \mapsto \mathrm{H}^1(f^{-1}(U), N)$. Let $x' \in X'$ then since $f^{-1}(x')$ is finite it follows that $\mathrm{H}^1(f^{-1}(x'), N) = 0$. Therefore, by Grauert's base change theorem it follows that $R^1 f_* \mathcal{O}_X(N) = 0$. Thus, the long exact sequence of sheaves in (4.9) descends to the short exact sequence

$$0 \to f_* \mathcal{O}_X(N) \to f_* \mathcal{O}_X(N(p)) \to f_* \mathcal{O}_p(N) \to 0.$$

Therefore, taking the determinant of the map $f_*\mathcal{O}_X(N) \to f_*\mathcal{O}_X(N(p))$ induces a canonical isomorphism

$$\det(f_*N(p)) \cong \det(f_*N) \otimes \mathcal{O}_{X'}(f(p)). \tag{4.10}$$

By applying the inductive hypothesis one sees

$$\det(f_*N(p)) \cong \mathrm{Nm}_{X/X'}(N) \otimes \det(f_*\mathcal{O}_X) \otimes \mathcal{O}_{X'}(f(p)).$$

Since $\mathcal{O}_{X'}(f(p)) = \mathrm{Nm}_{X/X'}(\mathcal{O}_X(p))$ it follows that

$$\det(f_*N(p)) \cong \mathrm{Nm}_{X/X'}(N(p)) \otimes \det(f_*\mathcal{O}_X),$$

which is the desired formula for $N(p)$. The proof of the formula for $N(-p)$ by considering the short exact sequence

$$0 \to \mathcal{O}_X(N(-p)) \to \mathcal{O}_X(N) \to \mathcal{O}_p(N) \to 0$$

instead. □

Corollary 4.6.8. *Let $f : X \to X'$ be a branched cover of compact Riemann surfaces. Then, for every holomorphic line bundle $N \to X$, the degree of the pushforward is given by*

$$\deg(f_*N) = \deg(N) + \deg(f_*\mathcal{O}_X).$$

Proof. By Proposition 4.6.7 the degree of the pushforward is given by

$$\deg(f_*N) = \deg(\mathrm{Nm}_{X/X'}(N)) + \deg(f_*\mathcal{O}_X).$$

Since the norm map is degree preserving we see $\deg(\mathrm{Nm}_{X/X'}(N)) = \deg(N)$ and the result follows immediately. □

Of course, to compute the degree of the pushforward, one needs $\deg(f_*\mathcal{O}_X)$. However, for a smooth spectral curve $\pi : S \to C$ recall from Proposition 4.2.3 that there is an isomorphism

$$\pi_*\mathcal{O}_S \cong \mathcal{O}_C \oplus L^{-1} \oplus \cdots \oplus L^{-(n-1)}.$$

Hence, by taking determinants, we see

$$\det(\pi_*\mathcal{O}_S) \cong L^{-n(n-1)/2}. \tag{4.11}$$

Thus, the degree of $\pi_*\mathcal{O}_S$ is given by

$$\deg(\pi_*\mathcal{O}_S) = -\frac{n(n-1)}{2}\deg(L). \tag{4.12}$$

Therefore, for a given holomorphic line bundle $N \to S$, combining Corollary 4.6.8 and (4.12) gives

$$\deg(\pi_* N) = \deg(N) - \frac{n(n-1)}{2}\deg(L).$$

Now, suppose $\deg(\pi_* N) = d$, then it follows that

$$\deg(N) = d + \frac{n(n-1)}{2}\deg(L). \tag{4.13}$$

Thus, under the spectral curve correspondence, for the Higgs bundle $(\pi_* N, \pi_* \lambda)$ to belong to $\mathcal{M}^{n,d}$ it is necessary that $\deg(N) = r$ where $r = d + n(n-1)/2\ \deg(L)$. For the Higgs bundle to belong to the moduli space $\mathcal{M}^{n,d}$ we require the Higgs bundle to be stable. We may now classify the generic fibres of the Hitchin fibration.

Theorem 4.6.9. *The generic fibres of the Hitchin fibration* $h : \mathcal{M}^{n,d} \to \bigoplus_{i=1}^n \mathrm{H}^0(C, L^i)$ *have one connected component and are torsors of a Jacobian variety.*

Proof. Recall that the generic points in $\bigoplus_{i=1}^n \mathrm{H}^0(C, L^i)$ are precisely the points corresponding to the smooth spectral curves. Hence, the fibre at a generic element that defines smooth spectral curve $\pi : S \to C$ is precisely isomorphism classes of rank n, degree d stable Higgs bundles with smooth spectral curve S. Ignoring the degree condition, by the spectral curve correspondence, the isomorphism classes of Higgs bundles with smooth spectral curve S is in one-to-one correspondence with $\mathrm{Pic}(S)$. Finally, by (4.13) the holomorphic line bundles over S with degree

$$r = d + \frac{n(n-1)}{2}\deg(L)$$

are precisely the line bundles whose corresponding Higgs bundle have degree d. Therefore, the fibre is in one-to-one correspondence with $\mathrm{Pic}^r(S)$, which is connected and a torsor of $\mathrm{Jac}(S)$. □

4.6.3 Dimension of Generic Fibres

Let $\pi : S \to C$ be a smooth spectral curve. The dimension of the abelian variety $\mathrm{Jac}(S)$ is equal to the genus of S, i.e., $\dim \mathrm{Jac}(S) = g(S)$. Since the generic fibre of the GL_n-Hitchin fibration corresponding to S is a torsor of $\mathrm{Jac}(S)$ it follows that the dimension of the fibre, viewed as an algebraic variety is equal to $g(S)$. To compute the genus of S we will apply Riemann-Hurwitz to $\pi : S \to C$. Recall that $\mathcal{O}_S(R) \cong \pi^*(L^{n-1})$ where $R = R_\pi$ denotes the ramification divisor of $\pi : S \to C$. Therefore, by Riemann-Hurwitz

$$2g(S) - 2 = n(2g-2) + n(n-1)\deg(L).$$

Solving for $g(S)$ gives

$$\dim \operatorname{Jac}(S) = n(g-1) + \frac{n(n-1)}{2}\deg(L) + 1. \tag{4.14}$$

Note, if $L = K_C$, then by substituting $\deg(L) = 2g - 2$ it follows that

$$\dim \operatorname{Jac}(S) = 1 + n^2(g-1). \tag{4.15}$$

We see from (4.8) and (4.15) that dimension of the generic fibres and Hitchin base are the same for when $L = K_C$, which suggests that the Hitchin fibration is a *Lagrangian fibration*. This is in fact the case for GL_n and the classical simple Lie groups since the moduli space has a canonical symplectic structure, which can be understood via deformation theory. However, we will not investigate this and we direct the reader to [Hit87b] for more information.

4.7 Generic Fibres of type A_n-Hitchin Fibrations

In this final section, we derive the SL_n and PGL_n spectral curve correspondences and then classify the generic fibres of the Hitchin fibration for type A_n. First, consider the definition of SL_n-Higgs bundles.

4.7.1 SL_n-Higgs bundles and SL_n-Hitchin Fibration

Special linear Higgs bundles are the first example of G-Higgs bundles with $G \subset \mathrm{GL}_n$. A SL_n-*Higgs bundle* is a pair (E, ϕ) where E is a rank n holomorphic vector bundle such that $\wedge^n E \cong \mathcal{O}_C$, and $\phi : E \to L \otimes E$ is a holomorphic vector bundle map such that $\operatorname{tr}\phi = 0$.

Notice for a given SL_n-Higgs bundle (E, ϕ) the a_1 term in the characteristic polynomial is identically zero since the trace of the Higgs field is identically zero. That is, the characteristic polynomial is given by

$$\det(\lambda - \phi) := \lambda^n + a_2\lambda^{n-2} + \cdots + a_n. \tag{4.16}$$

Now, we will show that generic spectral curves defined by sections of the form in (4.16) are smooth.

Lemma 4.7.1. *Generic spectral curves defined by a section $p(\lambda) \in \mathrm{H}^0(Y, L^n)$ of the form*

$$p(\lambda) = \lambda^n + a_2\lambda^{n-2} + \cdots + a_n$$

where $a_i \in \mathrm{H}^0(Y, L^i)$ for $i = 2, \ldots, n$ are smooth.

Proof. Consider sections of the line bundle L^n over Y of the form:

$$s = \lambda^n + a_2\lambda^{n-2} + \cdots + a_n$$

where $a_i \in \mathrm{H}^0(C, L^i)$ for $i = 2, \ldots, n$. Allowing the a_i to vary the zero set of s forms a linear system of divisors on Y. Since λ^n belongs to the system, a basepoint of the system necessarily belongs to the zero section $\lambda = 0$. Then, a basepoint of the system is a basepoint for L^n over C. However, L^n over C is basepoint-free, and thus, the base locus of the system is empty. Therefore, by Bertini's theorem, a generic divisor of the system is smooth. $\square$

Suppose $\pi : S \to C$ is a smooth spectral curve defined by a section of L^n over Y of the form in (4.16). Then, by the spectral curve correspondence, every holomorphic line bundle $N \to S$ defines a Higgs bundle $(\pi_* N, \pi_*\lambda)$ whose spectral curve is S. In particular, since the a_1 term defining the spectral curve is identically zero it follows that $\mathrm{Tr}(\pi_*\lambda) = 0$. Therefore, $(\pi_* N, \pi_*\lambda)$ defines a SL_n-Higgs bundles whose corresponding spectral curve is S if and only if $\det(\pi_* N) \cong \mathcal{O}_C$. Recall from Proposition 4.6.7 that

$$\det(\pi_* N) \cong \mathrm{Nm}_{S/C}(N) \otimes \det(\pi_*\mathcal{O}_S).$$

Moreover, from (4.11) recall that

$$\det(\pi_*\mathcal{O}_S) \cong L^{-n(n-1)/2}.$$

Therefore, $\det(\pi_* N) \cong \mathcal{O}_C$ if and only if

$$\mathrm{Nm}_{S/C}(N) \cong L^{n(n-1)/2}. \tag{4.17}$$

Since the norm map $\mathrm{Nm}_{S/C} : S \to C$ is surjective, we may choose a holomorphic line bundle $M \to S$ such that

$$\mathrm{Nm}_{S/C}(M) = L^{n(n-1)/2}.$$

Consequently, we may express the holomorphic line bundle $N \to S$ in (4.17) via

$$N \cong U \otimes M$$

where $U \to S$ is a holomorphic line bundle. It follows that $\mathrm{Nm}_{S/C}(U) \cong \mathcal{O}_C$, that is $U \in \ker(\mathrm{Nm}_{S/C})$. Now, we claim that $\ker(\mathrm{Nm}_{S/C}) = \mathrm{Prym}(S, C)$, and thus, $N \to S$ belongs to a torsor of the Prym variety $\mathrm{Prym}(S, C)$.

Proposition 4.7.2. *The kernel of the norm map* $\mathrm{Nm}_{S/C}$ *is connected, i.e.,* $\ker(\mathrm{Nm}_{S/C}) = \mathrm{Prym}(S, C)$.

Proof. By Corollary 3.5.10 it suffices to prove $\pi : S \to C$ does not factorise through a cyclic étale covering of degree $m \geq 2$. Suppose on the contrary $\pi : S \to C$ factorises through a cyclic étale covering $h : X \to C$ of degree $m \geq 2$, i.e., $\pi = hg$ where $g : S \to X$ is a branched cover of say degree k, so $n = km$. Recall by the adjunction formula that $\mathcal{O}_S(R_\pi) \cong \pi^*(L^{n-1})$. Moreover, $\mathcal{O}_S(R_\pi) \cong \mathcal{O}_S(R_g)$ since h is unramified. Indeed, by Lemma 4.3.13 we see $\mathcal{O}_S(R_\pi) \cong K_S \otimes \pi^*(K_C^{-1})$ and $\mathcal{O}_S(R_g) \cong K_S \otimes g^*(K_X^{-1})$. Also, since $\mathcal{O}_X(R_h) \cong \mathcal{O}_X$ it follows that $h^*(K_C) \cong K_X$, and hence,

$$\mathcal{O}_S(R_g) \cong K_S \otimes g^*h^*(K_C^{-1}) \cong K_S \otimes \pi^*(K_C^{-1}) \cong \mathcal{O}_S(R_\pi).$$

Next, we claim that $\mathcal{O}_S(R_g) \cong g^*(A^{k-1})$ where $A = h^*(L^m)$. To see this let Z denote the total space of A and notice that S is the zero locus of a section of $g^*(A^k)$. Since the total space, Z, of A is a fibre bundle the tangent bundle fits into the short exact sequence

$$0 \to g^*(A) \to TZ \to g^*(TX) \to 0.$$

By taking determinants and then dualising there is a canonical isomorphism

$$K_Z \cong g^*(A^{-1}K_X).$$

Now, by the adjunction formula $K_S \cong K_Z|_S \otimes g^*(A^k)$, and hence, $K_S \cong g^*(A^{k-1}K_X)$. Thus,

$$\mathcal{O}_S(R_g) \cong K_S \otimes g^*(K_X^{-1}) \cong g^*(A^{k-1}).$$

Since $\mathcal{O}_S(R_g) \cong \mathcal{O}_S(R_\pi)$, their degrees coincide and since $\deg(L) \neq 0$ it follows that $m = 1$, which is a contradiction. Therefore, $\mathrm{Prym}(S, C) = \ker(\mathrm{Nm}_{S/C})$. □

Conversely, suppose (E, ϕ) is a SL_n-Higgs bundle with smooth spectral curve $\pi : S \to C$, then, by the spectral curve correspondence, (E, ϕ) corresponds to the holomorphic line bundle $A(R) \to S$ where $A := \ker(\lambda - \phi)$ and R denotes the ramification divisor of $\pi : S \to C$. Then the same argument as before shows that

$$\mathrm{Nm}_{S/C}(A(R)) \cong L^{n(n-1)/2},$$

which belongs to a torsor of $\mathrm{Prym}(S, C)$. Since S is smooth the same argument as in Lemma 4.5.2 and Lemma 4.5.3 shows the assignments are still mutual inverses. Therefore, we have established the SL_n-spectral curve correspondence.

Theorem 4.7.3 (SL_n-Spectral Curve Correspondence)**.** *Isomorphism classes of* SL_n*-Higgs bundles with smooth spectral curve* $\pi : S \to C$ *are in one-to-one correspondence with a torsor of the abelian variety* $\mathrm{Prym}(S, C)$.

Now, we will apply the SL_n-spectral curve correspondence to classify the generic fibres of the SL_n-Hitchin fibration, which we now introduce. First, let $\mathcal{M}_{\mathrm{SL}_n}$ denote the moduli space of stable SL_n-Higgs bundles. Note that for a SL_n-Higgs bundle (E, ϕ) the determinant bundle is trivial, i.e., $\wedge^n E \cong \mathcal{O}_C$. Therefore, $\deg(E) = 0$ and thus, $\mathcal{M}_{\mathrm{SL}_n} \subset \mathcal{M}^{n,0}$.

In the case of SL_n-Higgs bundles, it is well-known that the coefficients of the characteristic polynomial in (4.16) form a homogeneous basis for the invariant polynomials for the Lie algebra $\mathfrak{sl}_n$. Thus, the SL_n-Hitchin fibration is the map

$$h : \mathcal{M}_{\mathrm{SL}_n} \to \bigoplus_{i=2} \mathrm{H}^0(C, L^i)$$

defined by

$$h(E, \phi) = (a_2, \ldots, a_n).$$

where

$$\det(\lambda - \phi) = \lambda^n + a_2\lambda^{n-2} + \cdots + a_n.$$

Similar to the GL_n-case, the SL_n-Hitchin fibration is surjective.

4.7.2 Dimension of A_n-Hitchin base

Notice that the A_n-Hitchin base is the GL_n-Hitchin base without $\mathrm{H}^0(C, L)$. Recall that $h^0(C, L^i) = i\deg(L) + 1 - g$ for every $i > 1$ where we assume that $\deg(L) \geq 2g$. Thus,

$$\sum_{i=2}^{n} h^0(C, L^i) = \sum_{i=2}^{n} (i\deg(L) + 1 - g),$$

i.e.,

$$\dim \bigoplus_{i=2}^{n} \mathrm{H}^0(C, L^i) = \deg(L)\left(\frac{n(n+1)}{2} - 1\right) + (n-1)(1-g). \tag{4.18}$$

We may also compute the dimension of the Hitchin base for $L = K_C$, and since there is no $\mathrm{H}^0(C, L)$ it suffices to enter $\deg(L) = 2g - 2$ into (4.18). By substituting and simplifying it follows that

$$\dim \bigoplus_{i=2}^{n} \mathrm{H}^0(C, K_C^i) = (n^2 - 1)(g - 1)$$

4.7.3 Generic Fibres of SL_n-Hitchin fibration

Using the SL_n-spectral curve correspondence, we will compute the generic fibres of the SL_n-Hitchin fibration.

Theorem 4.7.4. *The generic fibres of the* SL_n*-Hitchin fibration*

$$h : \mathcal{M}_{\mathrm{SL}_n} \to \bigoplus_{i=2}^{n} \mathrm{H}^0(C, L^i)$$

are torsors of a Prym variety.

Proof. By Lemma 4.7.1 the generic points in $\bigoplus_{i=2}^{n} \mathrm{H}^0(C, L^i)$ are precisely the points corresponding to smooth spectral curves. Hence, the fibre at a generic element that defines smooth spectral curve $\pi : S \to C$ is precisely isomorphism classes of stable SL_n-Higgs bundles with smooth spectral curve S. By Lemma 4.3.15 each Higgs bundle with spectral curve S is stable. Finally, by the SL_n-spectral curve correspondence, the isomorphism classes of SL_n-Higgs bundles with smooth spectral curve S are in one-to-one correspondence with a torsor $\mathrm{Prym}(S, C)$. □

4.7.4 Dimension of Generic fibre of SL_n-Hitchin Fibration

Let $\pi : S \to C$ be a smooth SL_n-spectral curve. The spectral curve corresponds to a generic fibre of the SL_n-Hitchin fibration that is a torsor of $\mathrm{Prym}(S, C)$. Thus, the dimension of each generic fibre when viewed as an algebraic variety is equal to $\dim \mathrm{Prym}(S, C)$. Recall that $\dim \mathrm{Prym}(S, C) = g(S) - g$. We have computed $g(S)$ in (4.14), and thus,

$$\dim \mathrm{Prym}(S, C) = (n-1)(g-1) + \frac{n(n-1)}{2} \deg(L). \tag{4.19}$$

Now, setting $L = K_C$ it follows that

$$\dim \mathrm{Prym}(S, C) = (n^2 - 1)(g - 1).$$

4.7.5 PGL_n-Higgs bundles and PGL_n-Hitchin Fibration

The group PGL_n does not have a canonical embedding into GL_n. Hence, before defining PGL_n-Higgs bundles, we will prove that every PGL_n-bundle over the compact Riemann surface C lifts to a GL_n-bundle over C.

Lemma 4.7.5. *Every PGL_n-bundle over C lifts to a GL_n-bundle over C.*

Proof. Consider the short exact sequence of groups

$$1 \to \mathbb{C}^* \to \mathrm{GL}_n \xrightarrow{\mathrm{pr}} \mathrm{PGL}_n \to 1. \tag{4.20}$$

Let $\mathcal{U} = \{U_i\}$ be an open cover of C, and let $\{g_{ij}\}$ be a holomorphic 1-cocycle valued in PGL_n, i.e., $g_{ij} : U_i \cap U_j \to \mathrm{PGL}_n$ such that $g_{ij}^{-1} = g_{ji}$ and $g_{ij} g_{jk} = g_{ik}$. For each g_{ij} choose holomorphic lifts $\widetilde{g}_{ij}$ to GL_n, i.e., $\widetilde{g}_{ij} : U_i \cap U_j \to \mathrm{GL}_n$ such that $\mathrm{pr}(\widetilde{g}_{ij}) = g_{ij}$. Notice that $\mathrm{pr}(\widetilde{g}_{ij}\widetilde{g}_{jk}) = g_{ik}$, hence

$$\widetilde{g}_{ij}\widetilde{g}_{jk} = c_{ijk}\widetilde{g}_{ik}$$

for some $c_{ijk} \in \mathbb{C}^*$. Since $\mathbb{C}^*$ is a central extension of groups in (4.20) it follows that c_{ijk} defines a 2-cocycle valued in $\mathbb{C}^*$. From the standard exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_C \to \mathcal{O}_C^* \to 0$$

passing to the long exact sequence in sheaf cohomology gives

$$\cdots \to \mathrm{H}^2(C, \mathcal{O}_C) \to \mathrm{H}^2(C, \mathcal{O}_C^*) \to \mathrm{H}^3(C, \mathbb{Z}) \to \cdots$$

By the Dolbeault isomorphism theorem

$$\mathrm{H}^2(C, \mathcal{O}_C) \cong \mathrm{H}^{0,2}_{\overline{\partial}}(C) = 0$$

since C is a Riemann surface. Moreover, $\mathrm{H}^3(C, \mathbb{Z}) = 0$, and thus, $\mathrm{H}^2(C, \mathcal{O}_C^*) = 0$. Therefore, the 2-cocycle c_{ijk} is a 2-coboundary so $c_{ijk} = c_{ij}c_{ik}^{-1}c_{jk}$ for some holomorphic 1-cocycle $\{c_{ij}\}$ valued in $\mathbb{C}^*$. Setting $g'_{ij} = c_{ij}^{-1}\widetilde{g}_{ij}$ it follows that g'_{ij} defines a holomorphic 1-cocycle valued in GL_n. Moreover, since $\mathrm{pr}(g'_{ij}) = g_{ij}$ the result follows. □

Remark 4.7.6. For an arbitrary complex manifold X, it is not necessarily true that a PGL_n bundle lifts to a GL_n bundle.

From the short exact sequence of groups in (4.20), we may pass to their corresponding sheaves. Then, pushing out to sheaf cohomology gives

$$0 \to \mathrm{H}^1(C, \mathcal{O}_C^*) \to \mathrm{H}^1(C, \mathcal{O}_C(\mathrm{GL}_n)) \to \mathrm{H}^1(C, \mathcal{O}_C(\mathrm{PGL}_n)) \to \cdots$$

By Lemma 4.7.5 every PGL_n bundle lifts to a GL_n bundle, hence the sequence becomes

$$0 \to \mathrm{H}^1(C, \mathcal{O}_C^*) \to \mathrm{H}^1(C, \mathcal{O}_C(\mathrm{GL}_n)) \to \mathrm{H}^1(C, \mathcal{O}_C(\mathrm{PGL}_n)) \to 0.$$

Therefore, the lift is unique, modulo twisting by a holomorphic line bundle over C. Now, we may define PGL_n-Higgs bundles.

Definition 4.7.7. A PGL_n*-Higgs bundle* is a class of GL_n-Higgs bundle (E, ϕ) where $\operatorname{tr}\phi = 0$ defined up to the equivalence relation $(E, \phi) \sim (E \otimes B, \phi \otimes \mathrm{id}_B)$ where $B \to C$ is a holomorphic line bundle.

Let $[(E, \phi)]$ be a given PGL_n-Higgs bundle. Since we define the Higgs field up to twisting by the identity, it is clear that the characteristic polynomial remains unchanged, i.e. $\mathrm{char}(\phi) = \mathrm{char}(\phi \otimes \mathrm{id})$. Thus, the notion of a spectral curve associated with $[(E, \phi)]$ is well-defined. By Lemma 4.7.1 generic spectral curves are smooth, hence suppose (E, ϕ) has smooth spectral curve $\pi : S \to C$. By the spectral curve correspondence (E, ϕ) corresponds to a holomorphic line bundle $N \to S$. Let $B \to C$ be a holomorphic line bundle, then by the projection formula

$$\pi_*(N \otimes \pi^*(B)) \cong \pi_*(N) \otimes B \cong E \otimes B. \tag{4.21}$$

Therefore, when identifying $(E, \phi) \sim (E \otimes B, \phi \otimes \mathrm{id}_B)$ we identify $N \sim N \otimes \pi^*(B)$, which established the PGL_n-Hitchin fibration.

Theorem 4.7.8 (PGL_n-spectral curve correspondence)**.** *Isomorphism classes of* PGL_n *Higgs bundles with smooth spectral curve* $\pi : S \to C$ *are in one-to-one correspondence with* $\operatorname{Pic}(S)/\pi^* \operatorname{Pic}(C)$.

To understand the PGL_n-Hitchin fibration notice that $\mathfrak{pgl}_n \cong \mathfrak{sl}_n$. Hence, analogous to the SL_n case, the coefficients in the characteristic polynomial form a homogeneous basis for the invariant polynomials of $\mathfrak{pgl}_n$. Therefore, the PGL_n-hitchin fibration is the map

$$h : \mathcal{M}_{\mathrm{PGL}_n} \to \bigoplus_{i=2}^{n} \mathrm{H}^0(C, L^i)$$

defined by

$$h([E,\phi]) = (a_2, \dots, a_n).$$

Since a generic point in $\bigoplus_{i=2}^{n} \mathrm{H}^0(C, L^i)$ defines a smooth spectral curve, we will use the PGL_n-spectral curve correspondence to classify the generic fibres of the PGL_n-Hitchin fibration. Before providing the classification we will study $\operatorname{Pic}(S)/\pi^* \operatorname{Pic}(C)$ where $\pi : S \to C$ is a smooth spectral curve defined by

$$p(\lambda) = \lambda^n + a_2\lambda^{n-2} + \cdots + a_n.$$

Lemma 4.7.9. *The group* $\operatorname{Pic}(S)/\pi^* \operatorname{Pic}(C)$ *is precisely* n *copies of the algebraic variety* $\operatorname{Jac}(S)/\pi^* \operatorname{Jac}(C)$.

Proof. Consider the following commutative diagram

$$\begin{tikzcd}
& 0 & 0 & 0 & \\
0 \arrow[r] & \operatorname{Jac}(S)/\pi^* \operatorname{Jac}(C) \arrow[r] \arrow[u] & \operatorname{Pic}(S)/\pi^* \operatorname{Pic}(C) \arrow[r] \arrow[u] & \mathbb{Z}_n \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \operatorname{Jac}(S) \arrow[r] \arrow[u, "\mathrm{pr}"] & \operatorname{Pic}(S) \arrow[r, "\deg"] \arrow[u, "\mathrm{pr}"] & \mathbb{Z} \arrow[r] \arrow[u, "\mathrm{mod}\ n"] & 0 \\
0 \arrow[r] & \operatorname{Jac}(C) \arrow[r] \arrow[u, "\pi^*"] & \operatorname{Pic}(C) \arrow[r, "\deg"] \arrow[u, "\pi^*"] & \mathbb{Z} \arrow[r] \arrow[u, "n"] & 0 \\
& 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Each column is exact and the bottom two rows are exact so by the nine lemma the first row is exact, which implies the result. $\square$

Lemma 4.7.10. *The algebraic variety* $\operatorname{Jac}(S)/\pi^* \operatorname{Jac}(C)$ *is isomorphic to the abelian variety* $\operatorname{Prym}(S,C)^\vee$.

Proof. Since $\ker(\mathrm{Nm}_{S/C}) = \mathrm{Prym}(S, C)$ consider the short exact sequence

$$0 \to \mathrm{Prym}(S, C) \to \mathrm{Jac}(S) \xrightarrow{\mathrm{Nm}_{S/C}} \mathrm{Jac}(C) \to 0.$$

Dualising the short exact sequence and identifying $\mathrm{Jac}(S) \cong \mathrm{Alb}(S)$, $\mathrm{Jac}(C) \cong \mathrm{Alb}(C)$, and $\mathrm{Nm}^{\vee}_{S/C} = \pi^*$ gives the short exact sequence

$$0 \to \mathrm{Jac}(C) \xrightarrow{\pi^*} \mathrm{Jac}(S) \to \mathrm{Prym}(S, C)^{\vee} \to 0.$$

The result immediately follows. □

Theorem 4.7.11. *The generic fibres of the* PGL_n*-Hitchin fibration*

$$h_{\mathrm{PGL}_n} : \mathcal{M}_{\mathrm{PGL}_n} \to \bigoplus_{i=2}^{n} \mathrm{H}^0(C, L^i)$$

are comprised of n*-copies of the abelian variety* $\mathrm{Prym}(S, C)^{\vee}$.

Proof. Recall that the generic points in $\bigoplus_{i=2}^{n} \mathrm{H}^0(C, L^i)$ are precisely the points corresponding to the smooth spectral curves. Hence, the fibre at a generic element that defines smooth spectral curve $\pi : S \to C$ is precisely isomorphism classes of PGL_n-Higgs bundles with smooth spectral curve S. By the PGL_n-spectral curve correspondence, the isomorphism classes of Higgs bundles with smooth spectral curve S is in one-to-one correspondence with $\mathrm{Pic}(S)/\pi^* \mathrm{Pic}(C)$. However, by Lemma 4.7.9 the group $\mathrm{Pic}(S)/\pi^* \mathrm{Pic}(C)$ is precisely n-copies of $\mathrm{Jac}(S)/\pi^* \mathrm{Jac}(C)$. Moreover, by Lemma 4.7.10, $\mathrm{Jac}(S)/\pi^* \mathrm{Jac}(C) \cong \mathrm{Prym}(S, C)^{\vee}$. Therefore, generic fibres of the PGL_n-Hitchin fibration are comprised of n-copies of $\mathrm{Prym}(S, C)^{\vee}$. □

The classification of the generic fibres of the SL_n-Hitchin fibration and PGL_n-Hitchin fibration demonstrate the duality in the Hitchin system since the abelian varieties are dual. Since an abelian variety has the same dimension as its dual the dimension of each component of each generic fibre is given by (4.19). One may notice that a generic fibre in the SL_n-Hitchin fibration has one connected component whereas a generic fibre in the PGL_n-Hitchin fibration has n-connected component. The difference in the number of connected components may be explained via *gerbes* where in particular the SL_n fibre has n gerbes each of which correspond to a connected component in the PGL_n fibre. We will not investigate this further, however the reader may consult [HT03] for more details.

Chapter 5

Type B_n and C_n Higgs bundles and Hitchin Fibration

Upon realising the classification of the generic fibres of the SO_{2n+1} Hitchin fibration was incorrect, Hitchin corrected the computation in [Hit07] and subsequently realised Langlands duality between the generic fibres of the Sp_{2n} and SO_{2n+1} Hitchin fibrations. In the correction, Hitchin resorted to sophisticated techniques, extension classes, and intrinsic properties of the canonical bundle.

In this chapter, we compute the generic fibres of the Sp_{2n} and SO_{2n+1} Hitchin fibrations for a basepoint-free positive degree holomorphic line bundle. Similar to the GL_n case, generic spectral curves are smooth for Sp_{2n}. Since Sp_{2n} is a closed subgroup of GL_{2n}, we may apply the GL_{2n}-spectral curve correspondence to obtain a holomorphic line bundle on the smooth spectral curve and deduce further structure on the line bundle induced by Sp_{2n}. The generic fibres of the Sp_{2n} Hitchin fibration are torsors of a Prym variety. In contrast to Sp_{2n}, generic spectral curves for SO_{2n+1} are not smooth. However, a generic SO_{2n+1} spectral curve canonically defines a smooth Sp_{2n} spectral curve, which allows us to use the Sp_{2n} computation. To pass to the Sp_{2n} computation, we give an explicit correspondence between SO_{2n+1} Higgs bundles and Sp_{2n} Higgs bundles. The main obstacle in establishing the correspondence is the singularities in the SO_{2n+1} spectral curve. Instead of resorting to extension classes and high-powered machinery, we follow the ideas in [BS19] and give a *Hecke modification* of the SO_{2n+1} bundle and resort to explicit local computations. For details about the concept of a Hecke modification, the reader may consult [Kam11, KW07]. Essentially, we modify a holomorphic vector bundle locally about a point to obtain a new vector bundle, and we achieve this by identifying the vector bundle with a particular sheaf.

After establishing the correspondence, we prove that at the level of the generic fibres, this amounts to passing from the Prym variety in the Sp_{2n} to its dual Prym variety, which realises Langlands duality in the Hitchin fibration between Sp_{2n} and SO_{2n+1} since ${}^L\mathrm{Sp}_{2n} \cong \mathrm{SO}_{2n+1}$. For proof that ${}^L\mathrm{Sp}_{2n} \cong \mathrm{SO}_{2n+1}$ see Appendix A.

5.1 Sp_{2n}-Higgs bundles and Sp_{2n}-spectral curve correspondence

Throughout this chapter fix a compact Riemann surfaces C with genus $g \geq 2$. Let $L \to C$ be a fixed basepoint-free holomorphic line bundle with positive degree, and let Y denote the total space of L. Also, in this section, powers of line bundles represent tensor product and *not* direct sum.

Principal Sp_{2n}-bundles over C correspond to rank $2n$-holomorphic vector bundles over C equipped with a symplectic form. Hence, Sp_{2n}-*Higgs bundles* are triples (E, ϕ, ω) where $E \to C$ is a rank $2n$-holomorphic vector bundle, $\phi : E \to L \otimes E$ is a holomorphic vector bundle homomorphism, and $\omega : E \otimes E \to \mathcal{O}_C$ is a symplectic form such that

$$\omega(\phi v, w) + \omega(v, \phi w) = 0. \tag{5.1}$$

Recall that the eigenvalues of a Higgs field are generically distinct. Hence, to understand the characteristic polynomial of $\phi : E \to L \otimes E$ let $A \in \mathfrak{sp}_{2n}$ have distinct eigenvalues λ_i with corresponding eigenvectors $v_i \in \mathbb{C}^{2n}$. Then,

$$\lambda_i \omega(v_i, v_j) = \omega(A v_i, v_j) = -\lambda_j \omega(v_i, v_j).$$

Hence,

$$(\lambda_i + \lambda_j)\omega(v_i, v_j) = 0. \tag{5.2}$$

Therefore, by (5.2) we see $\lambda_i = -\lambda_j$ or $\omega(v_i, v_j) = 0$. Since ω is non-degenerate it follows that eigenvalues occur in opposite pairs, i.e., if λ_i is an eigenvalue, then $-\lambda_i$ is an eigenvalue too. Thus, the characteristic polynomial of A has the form

$$\det(x - A) = x^{2n} + a_2 x^{2n-2} + \cdots + a_{2n}.$$

Moreover, it is well-known that the polynomials $a_2, \ldots, a_{2n}$ forms a homogeneous basis for the invariant polynomials of $\mathfrak{sp}_{2n}$. Therefore, the characteristic polynomial for ϕ pulled back to Y has the form

$$\det(\lambda - \phi) = \lambda^{2n} + a_2 \lambda^{2n-2} + \cdots + a_{2n} \tag{5.3}$$

where $a_{2i} \in \mathrm{H}^0(C, L^{2i})$ for $i = 1, \ldots, n$. Observe that a spectral curve defined by an equation as in (5.3) possesses a canonical involution $\sigma(\lambda) = -\lambda$. Now we will prove smooth symplectic spectral curves exist, i.e., spectral curves defined by polynomials of the form in (5.3).

Lemma 5.1.1. *Generic symplectic spectral curves are smooth.*

Proof. Consider sections of the line bundle L^{2n} over Y of the form:

$$s = \lambda^{2n} + a_2\lambda^{2n-2} + \cdots + a_{2n}$$

where $a_{2i} \in \mathrm{H}^0(C, L^{2i})$ for $i = 1, \ldots, n$. Allowing the a_{2i} to vary the zero set of s forms a linear system of divisors on Y. Since λ^{2n} belongs to the system, a basepoint of the system necessarily belongs to the zero section $\lambda = 0$. Then, a basepoint of the system is a basepoint for L^{2n} over C. However, L^{2n} over C is basepoint-free, and thus, the base locus of the system is empty. Therefore, by Bertini's theorem, a generic divisor of the system is smooth, i.e., generic symplectic spectral curves are smooth. □

Fix a Sp_{2n}-Higgs bundle (E, ϕ, ω) with smooth spectral curve $\pi : S \to C$ with involution $\sigma(\lambda) = -\lambda$. Ignoring the symplectic structure, by the GL_{2n}-spectral curve correspondence (E, ϕ) corresponds, up to isomorphism, to a holomorphic line bundle $N \to S$ such that $(E, \phi) \cong (\pi_* N, \pi_* \lambda)$ where $\lambda : N \to \pi^*(L) \otimes N$ is the tautological section. Now, we will deduce further structure on N using the symplectic form $\omega : E \otimes E \to \mathcal{O}_C$.

Lemma 5.1.2. *Under the* GL_{2n}*-spectral curve correspondence the Higgs bundle* $(E, -\phi)$ *corresponds to* $\sigma^*(N)$.

Proof. The Higgs bundle (E', ϕ') over C where $E' = \pi_*(\sigma^*(N))$ and ϕ' is induced by the tautological section $\lambda : \sigma^*(N) \to \pi^*(L) \otimes \sigma^*(N)$ corresponds to $\sigma^*(N)$. Thus, it suffices to prove $(E, -\phi) \cong (E', \phi')$. Let $U \subset C$ be a given open subset, then

$$\pi_*(\mathcal{E})(U) = \mathcal{N}(\pi^{-1}(U)),$$

likewise

$$\pi_*(\mathcal{E}')(U) = \sigma^*(\mathcal{N})(\pi^{-1}(U)).$$

Since σ preserves $\pi^{-1}(U)$, there is a canonical isomorphism of sections

$$\mathcal{N}(\pi^{-1}(U)) \ni s \mapsto \sigma^* s \in \sigma^*(\mathcal{N})(\pi^{-1}(U)). \tag{5.4}$$

Since $\sigma(\lambda) = -\lambda$ it is clear that

$$\sigma^*\lambda(s) = -\lambda\sigma^*(s) \tag{5.5}$$

for every $s \in \sigma^*(\mathcal{N})(\pi^{-1}(U))$. Equation (5.4) shows that $E \cong E'$ and (5.5) shows that the isomorphism commutes with the Higgs fields. Therefore, $(E, -\phi) \cong (E', \phi')$. □

Consider also the definition of the *dual Higgs bundle* and its correspondence under the GL_{2n}-spectral curve correspondence.

Definition 5.1.3. The *dual Higgs bundle* of (E, ϕ, ω) is the pair (E^*, ϕ^*) where E^* denotes the dual holomorphic vector bundle and $\phi^* : E^* \to L \otimes E^*$ denotes the transpose of ϕ.

Lemma 5.1.4. *Under the* GL_{2n} *spectral curve correspondence the Higgs bundle* (E^*, ϕ^*) *corresponds to* $N^*(R)$ *where* R *denotes the ramification divisor of* $\pi : S \to C$.

Proof. The Higgs bundle (E', ϕ') over C where $E' = \pi_*(N^*(R))$ and ϕ' is induced by the tautological section $\lambda : N^*(R) \to \pi^*(L) \otimes N^*(R)$ corresponds to $N^*(R)$. Thus, it suffices to prove $(E^*, \phi^*) \cong (E', \phi)$. Recall from relative duality that there is a non-degenerate pairing

$$(\, , \,) : E \otimes E' \to \mathcal{O}_C$$

defined by

$$(a, b) = \mathrm{Tr}_{S/C}(a \otimes b).$$

Hence, there is an induced isomorphism $E' \cong E^*$. Let $U \subset C$ be a given open subset of C. Let $a \in \mathcal{N}(\pi^{-1}(U))$ and $b \in (\mathcal{N}^*(R))(\pi^{-1}(U))$. Let $\widetilde{a} \in \mathcal{E}(U)$ be the section of E corresponding to a under the isomorphism $E \cong \pi_*(N)$. Then, notice that $\phi\widetilde{a} = \lambda a$ and $\phi' b = \lambda b$. Therefore,

$$(\phi\widetilde{a}, b) = \mathrm{Tr}_{S/C}((\lambda a) \otimes b) = \mathrm{Tr}_{S/C}(a \otimes (\lambda b)) = (a, \phi' b).$$

Thus, the isomorphism $E' \cong E^*$ from relative duality commutes with the Higgs fields, i.e., $(E^*, \phi^*) \cong (E', \phi')$. □

Since the symplectic form $\omega : E \otimes E \to \mathcal{O}_C$ is non-degenerate, there is an induced isomorphism $E \cong E^*$. Moreover, from (5.1) it follows that ω induces an isomorphism $(E, -\phi) \cong (E^*, \phi^*)$. By Lemma 5.1.2 the Higgs bundle $(E, -\phi)$ corresponds to $\sigma^*(N)$, and by Lemma 5.1.4 the Higgs bundle (E^*, ϕ^*) corresponds to $N^*(R)$. Therefore, the symplectic form induces an isomorphism

$$\theta : \sigma^*(N) \to N^*(R), \tag{5.6}$$

or, equivalently, a global nowhere vanishing section of $\sigma^*(N^*) \otimes N^*(R)$. Therefore, we have established one direction in the desired correspondence.

Proposition 5.1.5. *Let* (E, ϕ, ω) *be a* Sp_{2n}*-Higgs bundle with smooth spectral curve* $\pi : S \to C$ *and canonical involution* $\sigma(\lambda) = -\lambda$. *Suppose that under the* GL_{2n}*-spectral curve correspondence the Higgs bundle* (E, ϕ) *corresponds to the holomorphic line bundle* $N \to S$. *Then, the symplectic form* $\omega : E \otimes E \to \mathcal{O}_C$ *induces an isomorphism* $\theta : \sigma^*(N) \to N^*(R)$ *where* R *denotes the ramification divisor of* $\pi : S \to C$. *Equivalently,* ω *induces a nowhere vanishing section* θ *of* $\sigma^*(N^*) \otimes N^*(R)$.

Remark 5.1.6. Since S is a compact Riemann surface, the isomorphism $\theta : \sigma^*(N) \to N^*(R)$ is unique up to scale and moreover, scaling $\omega : E \otimes E \to \mathcal{O}_C$ scales the isomorphism θ accordingly.

We will now derive the opposite assignment in the desired correspondence. Let $M \to S$ be a holomorphic line bundle equipped with an isomorphism $\tau : \sigma^*(M) \to M^*(R)$. Then, $(V, \xi) = (\pi_* M, \pi_* \lambda)$ defines a GL_{2n}-Higgs bundle over C. Now, we define a bilinear form $\mu : V \otimes V \to \mathcal{O}_C$ as follows: let $U \subset C$ be a given open subset, then for given sections $x, y \in \mathcal{O}_S(M)(\pi^{-1}(U))$ we define

$$\mu(a, b) = \mathrm{Tr}_{S/C}(a \otimes \tau\sigma^* b). \tag{5.7}$$

The pairing on the right-hand side in (5.7) is precisely the pairing form relative duality, hence μ is $\mathcal{O}_C$-bilinear. Since τ is nowhere vanishing it follows that $\mu : V \otimes V \to \mathcal{O}_C$ is non-degenerate.

Lemma 5.1.7. *The bilinear form $\mu : V \otimes V \to \mathcal{O}_C$ defined in* (5.7) *is compatible with the Higgs field $\xi : V \to L \otimes V$.*

Proof. Let $a, b \in \mathcal{O}_S(M)(\pi^{-1}(U))$ be given sections. By definition, $\xi a = \lambda a$ and $\xi b = \lambda b$. Recall from (5.5) that $\lambda\sigma^* = -\sigma^*\lambda$. Hence,

$$\mu(\xi a, b) = \mathrm{Tr}_{S/C}((\lambda a) \otimes (\tau\sigma^* b)) = -\mathrm{Tr}_{S/C}(a \otimes (\tau\sigma^*\lambda b)) = -\mu(a, \xi b).$$

□

Consider the following lemma that shows there is a canonical lift of σ to $\sigma^*(M^*) \otimes M^*(R)$.

Lemma 5.1.8. *There is a canonical lift $\widehat{\sigma}^*$ of the involution σ to $\sigma^*(M^*) \otimes M^*(R)$.*

Proof. By definition $\sigma^*(M^*) \otimes M^*(R) \cong \sigma^*(M^*) \otimes M^* \otimes \mathcal{O}_S(R)$. Moreover, by Lemma 4.3.14 there is a canonical isomorphism $\mathcal{O}_S(R) \cong \pi^*(L^{2n-1})$. Let $p \in S$ be a given point. By definition $\pi(\sigma(p)) = \pi(p)$, and it follows that

$$\sigma^*(\pi^*(L^{2n-1}))_p = \pi^*(L^{2n-1})_p.$$

Therefore, σ lifts to $\mathcal{O}_S(R)$ as the identity. For $\sigma^*(M^*) \otimes M^*$ there is a canonical involution defined by

$$M^*_{\sigma(p)} \otimes M^*_p \ni (a, b) \mapsto (b, a) \in M^*_p \otimes M^*_{\sigma(p)}. \tag{5.8}$$

Combining (5.8) with the identity on $\mathcal{O}_S(R)$ defines the lift of σ denoted $\widehat{\sigma}^*$, which is an involution by construction. □

The trace map $\mathrm{Tr}_{S/C}$ is defined fibrewise, and σ permutes the fibres so it is clear that $\mathrm{Tr}_{S/C}(\sigma^* s) = \mathrm{Tr}_{S/C}(s)$. To determine a necessary and sufficient condition for μ to be skew-symmetric notice that

$$\mu(b, a) = \mathrm{Tr}_{S/C}\,\sigma^*(b \otimes \tau\sigma^* a) = \mathrm{Tr}_{S/C}(\sigma^* b \otimes \widehat{\sigma}^*\tau a) = \mathrm{Tr}_{S/C}(a \otimes \widehat{\sigma}^*\tau\sigma^* b). \tag{5.9}$$

By (5.9) we see $\mu : E \otimes E \to \mathcal{O}_C$ is skew-symmetric if and only if $\widehat{\sigma}^*(\tau) = -\tau$. Fortunately, over a compact Riemann surface this condition is immediate.

Lemma 5.1.9. *Every nowhere vanishing section f of $\sigma^*(M^*) \otimes M^*(R)$ satisfies*

$$\widehat{\sigma}^*(f) = -f.$$

Proof. Let f be a nowhere vanishing section of $\sigma^*(M^*) \otimes M^*(R)$. Since S is a compact Riemann surface the section is unique up to scale. Hence, $\widehat{\sigma}^*(f) = uf$ for some $u \in \mathbb{C}$. Since $\widehat{\sigma}^*$ is an involution it follows that $u = \pm 1$. Notice that the fixed points of σ are precisely when $\lambda = 0$ and $a_{2n} := \det(\phi)$ vanishes. Since $\deg(L) > 0$ implies $\deg(L^{2n}) > 0$ it is clear that a_{2n} vanishes and thus, fixed points of σ exist. Let $p \in S$ be a fixed point of σ. Then, $\widehat{\sigma^*}(p)f(p) = \epsilon f(p)$ for some $\epsilon \in \mathbb{C}$ and moreover,

$$uf(p) = (\widehat{\sigma}^* f)(p) = \epsilon f(p).$$

Since f is nowhere vanishing it follows that $\epsilon = u$. Thus, it suffices to prove $\epsilon = -1$. Since $\sigma(\lambda) = -\lambda$ we may choose local coordinate w centred at p such that $\sigma(w) = -w$ from which it follows that $\epsilon = -1$. ☐

Therefore, $\widehat{\sigma}^*(\tau) = -\tau$ and consequently, (V, ξ, μ) defines a Sp_{2n}-Higgs bundle with spectral curve $\pi : S \to C$. Thus, we have established the second direction in the correspondence.

Proposition 5.1.10. *Suppose $\pi : S \to C$ is a smooth symplectic spectral curve with involution $\sigma(\lambda) = -\lambda$. Let $M \to S$ be a holomorphic line bundle equipped with an isomorphism $\tau : \sigma^*(M) \to M^*(R)$. Then, under the GL_{2n}-spectral curve correspondence, the Higgs bundle $(V, \xi) = (\pi_* M, \pi_* \lambda)$ corresponds to M. Moreover, the isomorphism τ defines a symplectic form $\mu : V \otimes V \to \mathcal{O}_C$ compatible with $\xi : V \to L \otimes V$ by*

$$\mu(a, b) = \mathrm{Tr}_{S/C}(a \otimes \tau\sigma^* b).$$

In particular, the triple (V, ξ, μ) defines a Sp_{2n}-Higgs bundle.

Of course, the symplectic form $\mu : V \otimes V \to \mathcal{O}_C$ depends on the isomorphism τ, which is unique up to scale. Suppose instead we choose the isomorphism $c\tau : \sigma^*(M) \to M^*(R)$ and let $\mu' : V \otimes V \to \mathcal{O}_C$ be the corresponding symplectic form. Then, see that

$$\mu'(a, b) = \mathrm{Tr}_{S/C}(a \otimes (c\tau)\sigma^* b) = c\,\mathrm{Tr}_{S/C}(a \otimes \tau\sigma^* b) = c\mu(a, b),$$

that is, $\mu' = c\mu$. Choosing a holomorphic square root $c^{1/2}$ of c clearly defines a vector bundle isomorphism $c^{1/2} : V \to V$. Moreover, since the Higgs field ξ is a homomorphism we see $c^{1/2}\xi = \xi c^{1/2}$. Also, since μ is $\mathcal{O}_C$-bilinear we see

$$\mu'(a, b) = \mu(c^{1/2}a, c^{1/2}b).$$

Therefore, $c^{1/2}$ induces an isomorphism $(V, \xi, \mu) \cong (V, \xi, \mu')$. Thus, the construction in Proposition 5.1.10 is independent of choice of isomorphism $\tau : \sigma^*(M) \to M^*(R)$.

Since the construction in Proposition 5.1.5 is independent of scaling the symplectic form, it follows by the GL_{2n}-spectral curve correspondence that the assignments in Proposition 5.1.5 and Proposition 5.1.10 are mutual inverses. In summary, we have established the following one-to-one correspondence.

Theorem 5.1.11. *There is a one-to-one correspondence between* Sp_{2n}*-Higgs bundles* (E, ϕ, ω) *with smooth spectral curve* $\pi : S \to C$ *and holomorphic line bundles* $N \to S$ *such that* $\sigma^*(N) \cong N^*(R)$ *where* $\sigma : S \to S$ *denotes the canonical involution and* R *denotes the ramification divisor of* $\pi : S \to C$.

We now study the set of holomorphic line bundles $N \to S$ such that $\sigma^*(N) \cong N^*(R)$. Since $\sigma : S \to S$ is an involution, the group generated by σ is finite and trivially defines a properly discontinuous action. Thus, $T := S/\sigma$ defines a compact Riemann surface. We can see explicitly that T defines a spectral curve. Namely, T is the zero locus of the following polynomial:

$$\eta^n + a_1\eta^{n-1} + \cdots + a_n.$$

Here, η denotes the tautological section of L^2 and $a_i := a_{2i}$ from the polynomial defining S. Thus, T defines a smooth spectral curve.

There is a canonical double cover $p : S \to T$ given by projection. Notice that p is branched precisely when σ has fixed points. Recall from the proof of Lemma 5.1.9 that σ has fixed points, and thus, $p : S \to T$ is a branched double cover. Now, $\pi : S \to C$ factorises through T via.

$$\begin{tikzcd} S \arrow[r, "p"] \arrow[dr, "\pi"'] & T \arrow[d, "q"] \\ & C \end{tikzcd} \tag{5.10}$$

Lemma 5.1.12. *The pullback map* $p^* : \mathrm{Pic}(T) \to \mathrm{Pic}(S)$ *is injective.*

Proof. Recall from Proposition 3.5.9 that p^* is not injective if and only if p factorises through a cyclic étale covering of degree at least 2. Since p is a double cover, p factorises through a cyclic étale covering of degree at least 2 if and only if p is étale. However, p is branched and not étale. □

Corollary 5.1.13. *The kernel of* $\mathrm{Nm}_{S/T} : \mathrm{Pic}(S) \to \mathrm{Pic}(T)$ *is connected, i.e.,*

$$\ker(\mathrm{Nm}_{S/T}) = \mathrm{Prym}(S, T).$$

Lemma 5.1.14. *Let* $A \to S$ *be a holomorphic line bundle. Then,*

$$p^*(\mathrm{Nm}_{S/T}(A)) \cong A \otimes \sigma^*(A).$$

Proof. It suffices to prove the isomorphism for point divisors. Let x be a point divisor in S. Since $\sigma : S \to S$ is a biholomorphism, the pullback σ^* preserves degrees, so since σ is an involution it is clear that $\sigma^*(x) = \sigma(x)$. Hence, $x + \sigma^*(x) = x + \sigma(x)$. Since p is a double cover and $p(x) = p(\sigma(x))$ it follows that

$$p^*(\mathrm{Nm}_{S/T}(x)) = p^*(p(x)) = x + \sigma(x).$$

Therefore, $p^*(\mathrm{Nm}_{S/T}(x)) = x + \sigma^*(x)$. □

Proposition 5.1.15. *The set of holomorphic line bundles $N \to S$ such that $\sigma^*(N) \cong N^*(R)$ is a torsor of the abelian variety* $\mathrm{Prym}(S, T)$.

Proof. By Lemma 5.1.14 we see $p^*(\mathrm{Nm}_{S/T}(N)) \cong N \otimes \sigma^*(N)$. Since $\sigma^*(N) \cong N^*(R)$ it follows that

$$p^*(\mathrm{Nm}_{S/T}(N)) \cong \mathcal{O}_S(R).$$

Recall from Lemma 4.3.14 that $\mathcal{O}_S(R) \cong \pi^*(L^{2n-1})$. From (5.10) it follows that $\pi^*(L^{2n-1}) \cong p^*q^*(L^{2n-1})$, hence

$$p^*(\mathrm{Nm}_{S/T}(N)) \cong p^*(q^*(L^{2n-1})).$$

By Lemma 5.1.12 the pullback p^* is injective, and thus,

$$\mathrm{Nm}_{S/T}(N) \cong q^*(L^{2n-1}).$$

Since $\mathrm{Nm}_{S/T}$ is surjective, we may choose a holomorphic line bundle $M \to S$ such that $\mathrm{Nm}_{S/T}(M) = q^*(L^{2n-1})$. Then $N \cong U \otimes M$ for some holomorphic line bundle $U \to S$, and it follows that $\mathrm{Nm}_{S/T}(U) \cong \mathcal{O}_S$, which proves the result. □

In summary, we have established the Sp_{2n}-spectral curve correspondence.

Theorem 5.1.16 (Sp_{2n}-spectral curve correspondence)**.** *Isomorphism classes of* Sp_{2n}-*Higgs bundles with smooth spectral curve $\pi : S \to C$ is a torsor of the abelian variety* $\mathrm{Prym}(S, T)$ *where $T := S/\sigma$ and $\sigma : S \to S$ denotes the canonical involution.*

5.2 Sp_{2n}-Hitchin Fibration

Let (E, ϕ, ω) be a Sp_{2n}- Higgs bundle. Recall that the characteristic polynomial over Y takes the form

$$\det(\lambda - \phi) = \lambda^{2n} + a_2\lambda^{2n-2} + \cdots + a_{2n} \tag{5.11}$$

where $a_{2i} \in \mathrm{H}^0(C, L^{2i})$ for $i = 1, \ldots, n$. It is well known that the coefficients of the characteristic polynomial forms a homogeneous basis for the invariant polynomials of $\mathfrak{sp}_{2n}$. Therefore, the Sp_{2n}-Hitchin fibration is the map

$$h : \mathcal{M}_{\mathrm{Sp}_{2n}} \to \bigoplus_{i=1}^{n} \mathrm{H}^0(C, L^{2i})$$

defined by

$$h(E,\phi,\omega) = (a_2,\dots,a_{2n}).$$

Similar to the previous cases, the Sp_{2n}-Hitchin fibration is surjective.

5.2.1 Dimension of C_n Hitchin Base

Again we assume $\deg(L) \geq 2g$ so that $\mathrm{H}^1(C, L^{2i}) = 0$ for $i = 1, \dots, n$. Now, by the Riemann-Roch theorem

$$h^0(C, L^{2i}) = 2i\deg(L) + 1 - g.$$

Thus,

$$\sum_{i=1}^{n} h^0(C, L^{2i}) = \sum_{i=1}^{n} (2i\deg(L) + 1 - g).$$

By simplifying the right-hand-side it follows that

$$\dim \bigoplus_{i=1}^{n} \mathrm{H}^0(C, L^{2i}) = n(n+1)\deg(L) + n(1-g). \tag{5.12}$$

Now, notice that $\mathrm{H}^1(C, K_C^{2i}) = 0$, and thus, to compute the dimension of the C_n Hitchin base for when $L = K_C$ we may substitute $\deg(L) = 2g - 2$ into (5.12). Hence,

$$\dim \bigoplus_{i=1}^{n} \mathrm{H}^0(C, K_C^{2i}) = n(2n+1)(g-1).$$

5.2.2 Computing Generic Fibres of Sp_{2n}-Hitchin Fibration

A point in $\bigoplus_{i=1}^{n} \mathrm{H}^0(C, L^{2i})$ canonically defines a symplectic spectral curve. By Lemma 5.1.1 generic symplectic spectral curves are smooth. Hence, we may use the Sp_{2n}-spectral curve correspondence to classify the generic fibres of the Sp_{2n}-Hitchin fibration.

Theorem 5.2.1. *The generic fibres of the* Sp_{2n}*-Hitchin fibration*

$$h : \mathcal{M}_{\mathrm{Sp}_{2n}} \to \bigoplus_{i=1}^{n} \mathrm{H}^0(C, L^{2i})$$

are torsors of Prym varieties.

Proof. Let $(a_2,\dots,a_{2n}) \in \bigoplus_{i=1}^{n} \mathrm{H}^0(C, L^{2i})$ be a generic point. Recall that the corresponding spectral curve $\pi : S \to C$ is smooth and possesses a canonical involution $\sigma(\lambda) = -\lambda$. Therefore, $h^{-1}(a_2,\dots,a_{2n}) \subset \mathcal{M}_{\mathrm{Sp}_{2n}}$ are the isomorphism classes of Sp_{2n}-Higgs bundles with smooth spectral curve S. By the Sp_{2n}-spectral curve correspondence, $h^{-1}(a_2,\dots,a_{2n})$ is in one-to-one correspondence with a torsor of $\mathrm{Prym}(S, S/\sigma)$. □

5.2.3 Dimension of Generic Fibres of Sp_{2n}-Hitchin Fibration

Let $\pi : S \to C$ be a smooth Sp_{2n}-spectral curve with canonical involution σ. The spectral curve corresponds to a generic fibre of the Sp_{2n}-Hitchin fibration that is a torsor of $\mathrm{Prym}(S, S/\sigma)$. Thus, the dimension of fibre viewed as an algebraic variety is equal to $\dim \mathrm{Prym}(S, S/\sigma)$. Recall that $\dim \mathrm{Prym}(S, C) = g(S) - g(S/\sigma)$. Now, $\pi : S \to C$ is a $2n$-branched cover and moreover $\mathcal{O}_S(R) \cong \pi^*(L^{2n-1})$. Thus, by Riemann-Hurwitz

$$2g(S) - 2 = 2n(2g-2) + 2n(2n-1)\deg(L).$$

Solving for $g(S)$ gives

$$g(S) = 2n(g-1) + n(2n-1)\deg(L) + 1. \tag{5.13}$$

Now, the the projection map $\mathrm{pr} : S \to S/\sigma$ is a degree 2 branched with precisely $2n\deg(L)$ ramification points each with ramification index 2. Therefore, by Riemann-Hurwitz

$$2g(S) - 2 = 2(2g(S/\sigma) - 2) + 2n\deg(L)$$

where $g(S/\sigma)$ denotes the genus of S/σ. Solving for $g(S/\sigma)$ and substituting (5.13) into $g(S)$ gives

$$g(S/\sigma) = n(g-1) + n(n-1)\deg(L) + 1.$$

Thus, taking by taking the difference $g(S) - g(S/\sigma)$

$$\dim \mathrm{Prym}(S, S/\sigma) = n(g-1) + n^2\deg(L).$$

Setting $L = K_C$ gives

$$\dim \mathrm{Prym}(S, S/\sigma) = n(2n+1)(g-1).$$

5.3 SO_{2n+1}-Higgs Bundles and SO_{2n+1}-Spectral Curve Correspondence

Principal SO_{2n+1}-bundles over C correspond to rank $(2n+1)$-holomorphic vector bundles over C equipped with an $\mathcal{O}_C$-bilinear, symmetric, non-degenerate form. Hence, SO_{2n+1}-*Higgs bundles* are triples (V, ϕ, ω) where $V \to C$ is a rank $(2n+1)$-holomorphic vector bundle, $\phi : V \to L \otimes V$ is a holomorphic vector bundle homomorphism, and $Q : V \otimes V \to \mathcal{O}_C$ is an $\mathcal{O}_C$-bilinear, symmetric, non-degenerate form such that

$$Q(\phi v, w) + Q(v, \phi w) = 0. \tag{5.14}$$

The same argument as the Sp_{2n} case shows that generically the eigenvalues are distinct where the non-zero eigenvalues occur in opposite pairs and 0 is an eigenvalue. Therefore, the characteristic polynomial of ϕ is of the form

$$\det(\lambda - \phi) = \lambda(\lambda^{2n} + a_2\lambda^{2n-2} + \cdots + a_{2n})$$

where $a_{2i} \in \mathrm{H}^0(C, L^{2i})$. Notice that $p(\lambda) := \lambda^{2n} + a_2\lambda^{2n-2} + \cdots + a_{2n}$ is of the form in (5.3) and defines a symplectic spectral curve. Hence, we may choose a_{2i} such that $p(\lambda)$ defines a smooth symplectic spectral curve $\pi : S \to C$. The spectral curve comes equipped with a canonical involution $\sigma : S \to S$ defined by $\sigma(\lambda) = -\lambda$.

Since $\deg(L) > 0$ the section a_{2n} vanishes somewhere, hence the spectral curve defined by (V, ϕ, Q) is not smooth with singularities at $\lambda = 0$ and $a_{2n} = 0$. To classify generic SO_{2n+1}-spectral curves we will use the zero eigenbundle to establish a correspondence to generic Sp_{2n}-spectral curves along with extra spectral data. The correspondence is straightforward away from the singularities of the SO_{2n+1}-spectral curve and we will first establish the correspondence in the matrix case, then extend the correspondence to the open Riemann surface $C \setminus D$ where $D = (a_{2n})$. Finally, we extend the correspondence over D by explicit local calculations.

5.3.1 Vector Space Case

Let (V, ϕ, Q) be a SO_{2n+1}-triple, i.e., V is a complex vector space of dimension $2n + 1$, $\phi \in \mathfrak{so}(V)$, and $Q : V \otimes V \to \mathbb{C}$ is a symmetric, non-degenerate, bilinear form. Now, the eigenvalue $\lambda = 0$ has multiplicity one and hence, the nullspace $V_0 := \ker(\phi)$ is one-dimensional. Thus, the complex vector space $E := V/V_0$ has dimension $2n$, and the endomorphism ϕ induces an automorphism $\Phi : E \to E$ defined by

$$\Phi(v + V_0) = \phi(v) + V_0.$$

To simplify notation we will let v denote a coset in E and use $\widetilde{v}$ to denote a representative of v in V. We define $\omega : E \otimes E \to \mathbb{C}$ by

$$\omega(v, w) = Q(\phi(\widetilde{v}), \widetilde{w}).$$

Since $V_0 = \ker(\phi)$ and $\phi \in \mathfrak{so}(V)$ it follows easily that ω is well-defined and skew-symmetric. Moreover, since Q is bilinear it follows that ω is bilinear too. Notice that ω is non-degenerate. To see this suppose $\omega(v, w) = 0$ for every $w \in E$, then $Q(\phi(\widetilde{v}), \widetilde{w}) = 0$ for every $\widetilde{w} \in V$. Since Q is non-degenerate, $\phi(\widetilde{v}) = 0$ and hence, $\widetilde{v} \in V_0$ so $v = 0$ in E. Thus, (E, ω) defines a $2n$-dimension complex symplectic vector space. Now, a straightforward calculation shows $\Phi \in \mathfrak{sp}(V)$

$$\omega(\Phi(v), w) = Q(\phi(\widetilde{\Phi(v)}), \widetilde{w}) = Q(\phi^2(\widetilde{v}), \widetilde{w}) = -Q(\phi(\widetilde{v}), \phi(\widetilde{w})) = -\omega(v, \Phi(w)).$$

Therefore, (E, Φ, ω) defines a Sp_{2n}-triple establishing the following construction.

Construction 5.3.1. Suppose (V,ϕ,Q) is a SO_{2n+1}-triple. Then $E := V/\ker(\phi)$ defines a complex vector space of dimension $2n$, and ϕ induces an automorphism $\Phi : E \to E$. Moreover, the map $\omega : E \otimes E \to \mathbb{C}$ given by $\omega(v,w) = Q(\phi(\widetilde{v}),\widetilde{w})$ defines a symplectic form and $\Phi \in \mathfrak{sp}(V)$ and thus, (E,Φ,ω) defines a Sp_{2n}-triple.

To establish the reverse construction we will construct a canonical non-zero vector $v_0 \in V_0$. Let $\alpha \in \wedge^2 V^*$ be defined by $\alpha(v,w) := Q(\phi(v),w)$. Note α is different that ω since α is not defined at the level of cosets. The n-th exterior power of α defines an element $\alpha^n \in \wedge^{2n} V^*$. Choose an orthonormal basis $e^0,\ldots,e^{2n}$ of V^* with respect to Q. Recall that if v and w are eigenvectors with distinct eigenvalues then $Q(v,w) \neq 0$ if and only if the eigenvalues are opposite pairs. Hence, in the eigenspace decomposition

$$V = V_0 \oplus V_{\lambda_1} \oplus V_{-\lambda_1} \oplus \cdots \oplus V_{\lambda_n} \oplus V_{-\lambda_n}$$

the bilinear form Q pairs V_{λ_j} and $V_{-\lambda_j}$. Moreover, since the matrix

$$\begin{bmatrix} 0 & -\lambda_j \\ \lambda_j & 0 \end{bmatrix}$$

has eigenvalues $\pm i\lambda_j$, it follows that α may be represented by

$$\alpha = i\lambda_1\, e^1 \wedge e^2 + \cdots + i\lambda_n\, e^{2n-1} \wedge e^{2n}$$

and hence,

$$\alpha^n = n! i^n \lambda_1 \cdots \lambda_n\, e^1 \wedge \cdots \wedge e^{2n}.$$

The volume form $\nu := e^0 \wedge \cdots \wedge e^{2n}$ is by definition SO_{2n+1}-invariant. Contracting the basis $e^0, e^1, \ldots, e^{2n}$ with respect to Q defines an orthonormal basis $e_0,\ldots,e_{2n}$ for V, and $e^j(e_i) = \delta_i{}^j$. It is easy to see that contracting the vector ν over the vector $v_0 := i^n\lambda_1 \cdots \lambda_n\, e_0$ gives

$$i_{v_0}(\nu) = i^n \lambda_1 \cdots \lambda_n\, e^1 \wedge \cdots \wedge e^{2n}$$

thus, $\alpha^n = n! i_{v_0}(\nu)$.

Lemma 5.3.2. *The vector $v_0 = i^n\lambda_1 \cdots \lambda_n\, e_1 \wedge \cdots \wedge e_n$ belongs to the kernel of ϕ and is non-zero.*

Proof. It is clear that $v_0 \neq 0$. Since the volume form ν is SO_{2n+1}-invariant, $\phi \cdot \nu = 0$, and straightforward calculation shows $\phi \cdot \alpha = 0$

$$(\phi\cdot\alpha)(v,w) = -\alpha(\phi(v),w) - \alpha(v,\phi(w)) = -Q(\phi^2(v),w) - Q(\phi(v),\phi(w)) = 0.$$

It follows by induction that $\phi \cdot \alpha^n = 0$, hence $\phi \cdot i_{v_0}(\nu) = 0$. One can easily verify that

$$\phi \cdot i_{v_0}(\nu) = i_{\phi(v_0)}(\nu) + i_{v_0}(\phi\cdot\nu)$$

and it follows that $i_{\phi(v_0)}(\nu) = 0$. Since ν is non-degenerate, $\phi(v_0) = 0$. □

Notice that the induced automorphism Φ has eigenvalues $\pm\lambda_j$ for $j = 1, \ldots, n$ and hence, the norm of v_0 under Q is given by

$$Q(v_0, v_0) = Q(i^n\lambda_1 \cdots \lambda_n e_0, i^n\lambda_1 \cdots \lambda_n e_0) = (-1)^n\lambda_1^2 \cdots \lambda_n^2 = \det(\Phi). \tag{5.15}$$

Now we may give the reverse construction. Suppose (E, Φ, ω) is a Sp_{2n}-triple. We define a complex vector space V of dimension $2n + 1$ by $V = E \oplus \mathbb{C}$ and we define an endomorphism $\phi : V \to V$ by $\phi(v, w) = (\Phi(v), 0)$. Also, using (5.15) we define a symmetric bilinear form $Q : V \otimes V \to \mathbb{C}$ by

$$Q((v, w), (v, w)) = \omega(v, \Phi(v)) + w^2 \det(\Phi)$$

that extends to all points by the polarisation identity. The triple (V, ϕ, Q) defines a SO_{2n+1}-triple. To see this notice that since ω is non-degenerate and $\det(\Phi) \neq 0$ it follows that Q is non-degenerate and using the polarisation identity it is straightforward to see that

$$Q(\phi(v, w), (v', w')) = -Q((v, w), \phi(v', w')).$$

Construction 5.3.3. Suppose (E, Φ, ω) is a Sp_{2n}-triple. Then $V = E \oplus \mathbb{C}$ defines a complex vector space of dimension $2n{+}1$ and the $\phi : V \to V$ defined by $\phi(v, w) = (\Phi(v), 0)$ is an endomorphism of V. Moreover, the symmetric bilinear form $Q : V \otimes V \to \mathbb{C}$ defined by

$$Q((v, w), (v, w)) = \omega(v, \Phi(v)) + w^2 \det(\Phi)$$

and the polarisation identity is non-degenerate and ϕ-compatible, i.e., $Q(\phi(v, w), (v', w')) = -Q((v, w), \phi(v', w)')$. Hence, (V, ϕ, Q) defines a SO_{2n+1}-triple.

Now, we will verify Construction 5.3.1 and Construction 5.3.3 are mutual inverses providing the desired correspondence.

Lemma 5.3.4. *Suppose (V, ϕ, Q) is a SO_{2n+1}-triple, and let (E, Φ, ω) be the Sp_{2n}-triple obtained by* Construction 5. 3. 1. *Then, applying* Construction 5. 3. 3 *to (E, Φ, ω) recovers (V, ϕ, Q).*

Proof. Let (V', ϕ', Q') be the SO_{2n+1}-triple obtained by applying Consturction 5.3.3 to (E, Φ, ω). Consider the homomorphism $\widehat{\phi} : E \to V$ defined by $\widehat{\phi}(v) = \phi(\widetilde{v})$, which is well-defined. Then, we may define a homomorphism $\phi : V' \to V$ by $\psi(v, w) = \widehat{\phi}(v) + wv_0$. Notice that $(v, w) \in \ker(\psi)$ if and only if $\phi(\widetilde{v}) = -wv_0$, i.e., $\phi(\widetilde{v}) \in V_0$, hence $v = 0$ and $w = 0$. Thus, ψ is injective and since $\dim(V) = \dim(V')$ we see that ψ is an isomorphism. From $\phi \circ \widehat{\phi} = \widehat{\phi} \circ \Phi$ it is straightforward to see $\phi \circ \psi = \psi \circ \phi'$. Finally,

$$\begin{aligned}
Q(\psi(v, w), \psi(v, w)) &= Q(\widehat{\phi}(v) + wv_0, \widehat{\phi}(v) + wv_0) \\
&= Q(\widehat{\phi}(v), \widehat{\phi}(v)) + w^2 \det(\Phi) \\
&= \omega(v, \Phi(v)) + w^2 \det(\Phi) \\
&= Q'((v, w), (v, w)).
\end{aligned}$$

Therefore, the isomorphism ψ induces an isomorphism $(V, \phi, Q) \cong (V', \phi', Q')$. $\square$

Lemma 5.3.5. *Suppose (E, Φ, ω) is a* Sp_{2n}*-triple, and let (V, ϕ, Q) be the* SO_{2n+1}*-triple obtained by applying* Construction 5. 3. 3. *Then, applying* Construction 5. 3. 1 *recovers (E, ϕ, ω).*

Proof. Let (E', Φ', ω') be the Sp_{2n}-triple obtained by applying Construction 5.3.1 to (V, Φ, Q). Since $\Phi : E \to E$ is an automorphism, the composition

$$\mathrm{pr}_E \circ \phi : E \oplus \mathbb{C} \ni (v, w) \mapsto \Phi(v) \in E$$

has kernel $\{(0, w) \,|\, w \in \mathbb{C}\} \cong \mathbb{C}$. Under this identification it immediately follows that $\Phi = \Phi'$ and $\omega = \omega'$ and thus, $(E, \Phi, \omega) \cong (E', \Phi', \omega')$. □

In summary, we have established the desired one-to-one correspondence.

Theorem 5.3.6. *There is a natural one-to-one correspondence between* SO_{2n+1}*-triples and* Sp_{2n}*-triples. Explicitly, if (V, ϕ, ω) is a* SO_{2n+1}*-triple, then $V_0 = \ker \phi$ is a one-dimensional subspace of V, hence $E = V/V_0$ defines a complex vector space of dimension $2n$. Moreover, the endomorphism $\phi : V \to V$ induces an automorphism $\Phi : E \to E$ and the map $\omega : E \otimes E \to \mathbb{C}$ given by $\omega(v, w) = Q(\phi(\widetilde{v}), \widetilde{w})$ defines a symplectic form on E, which is Φ-compatible, hence (E, Φ, ω) defines a* Sp_{2n}*-triple.*

Conversely, if (E, Φ, ω) is a Sp_{2n}*-triple, then $V = E \oplus \mathbb{C}$ is a complex vector space of dimension $2n + 1$, and the map $\phi : V \to V$ given by $\phi(v, w) = (\Phi(v), 0)$ defines an endomorphism of V. Moreover, the symmetric bilinear form $Q : V \otimes V \to \mathbb{C}$ defined by $Q((v, w), (v, w)) = \omega(v, \Phi(v)) + w^2 \det(\Phi)$ and the polarisation identity is non-degenerate and ϕ-compatible, hence (V, ϕ, Q) defines a* SO_{2n+1}*-triple. Further, these constructions are mutual inverses.*

5.3.2 Correspondence Away From Singularities

Recall that a SO_{2n+1}-Higgs bundle (V, ϕ, Q) has characteristic polynomial of the form

$$\lambda(\lambda^{2n} + a_2\lambda^{2n-2} + \cdots + a_{2n}). \tag{5.16}$$

Moreover, we choose $a_{2i} \in \mathrm{H}^0(C, L^{2i})$ such that

$$p(\lambda) = \lambda^{2n} + a_2\lambda^{2n-2} + \cdots + a_{2n} \tag{5.17}$$

defines a smooth symplectic spectral curve $\pi : S \to C$. Also, the complex analytic subvariety of L defined by $\lambda = 0$ is smooth. Thus, the singularities of the spectral curve defined by (5.16) occur precisely when $\lambda = 0$ and $a_{2n} = 0$. Let $D = (a_{2n})$, then away from D the spectral curve defined by (5.16) is smooth. Now, we will generalise the one-to-one correspondence in Theorem 5.3.6 to SO_{2n+1}-Higgs bundles and Sp_{2n}-Higgs bundles over the open Riemann surface $C_0 := C \setminus D$ with spectral curves defined by (5.16) and (5.17) respectively.

The kernel of the sheaf map $\phi : \mathcal{V} \to \mathcal{O}_C(L \otimes V)$ induces a subbundle $V_0 \subset V$ and since generically the eigenvalue $\lambda = 0$ has multiplicity one, V_0 defines a holomorphic line bundle. Thus, the quotient bundle $E := V/V_0$ defines a rank $2n$ holomorphic vector bundle over C. Now, we define $\omega : E \otimes E \to L$ by

$$\omega(v, w) = Q(\phi(\widetilde{v}), \widetilde{w})$$

where $\widetilde{v}$ and $\widetilde{w}$ are lifts of the sections v and w to V. Note, that E and ω are defined over all of C. Since ϕ is a homomorphism and Q is $\mathcal{O}_C$-bilinear it follows that ω is $\mathcal{O}_C$-bilinear, and the same calculation as the matrix case shows that ω is skew-symmetric. Next, we claim that ω is non-degenerate over C_0. Suppose $y \in C_0$ and $\omega_y(v, w) = 0$ for every $w \in E_y$, i.e., $Q_y(\phi_y(\widetilde{v}), \widetilde{w}) = 0$ for every $\widetilde{w} \in V_y$. Since Q_y is non-degenerate and $\ker(\phi_y) = (V_0)_y$ it follows that $v = 0$, which shows ω is non-degenerate over C_0. Finally, $\phi : V \to L \otimes V$ induces a homomorphism $\Phi : E \to L \otimes E$, which is defined over C, and the same calculation as the matrix case shows ω is Φ-compatible. Therefore, (E, Φ, ω) defines a Sp_{2n}-Higgs bundle over C_0 and by construction the characteristic polynomial of Φ is given by $p(\lambda)$.

Construction 5.3.7. Suppose (V, ϕ, Q) is a generic SO_{2n+1}-Higgs bundle over C whose characteristic polynomial is given by

$$\lambda(\lambda^{2n} + a_2\lambda^{2n-2} + \cdots + a_{2n})$$

where $p(\lambda) = \lambda^{2n} + a_2\lambda^{2n-2} + \cdots + a_{2n}$ defines a smooth symplectic spectral curve $\pi : S \to C$. Then, the kernel of the Higgs field induces a holomorphic subbundle $V_0 \subset V$ that has rank one and $E = V/V_0$ defines a holomorphic vector bundle over C of rank $2n$. Moreover, the map $\omega : E \otimes E \to L$ given by $\omega(v, w) = Q(\phi(\widetilde{v}), \widetilde{w})$ defines a symplectic form on $E|_{C_0}$. Moreover, the Higgs field ϕ induces a homomorphism $\Phi : E \to L \otimes E$ whose characteristic polynomial is given by $p(\lambda)$ and the symplectic form ω is Φ-compatible. Thus, (E, Φ, ω) defines a Sp_{2n}-Higgs bundle over C_0 with smooth spectral curve $S|_{\pi^{-1}(C_0)}$.

Remark 5.3.8. The holomorphic vector bundle E, the Higgs field $\Phi : E \to L \otimes E$, and the symplectic form ω are defined over C. Moreover, ω is $\mathcal{O}_C$-bilinear, skew-symmetric, and Φ-compatible over C. Thus, if ω is non-degenerate over D, then (E, Φ, ω) define a Sp_{2n}-Higgs bundle over C with smooth spectral curve $\pi : S \to C$. We will see in Section 5.3.3 that ω is indeed non-degenerate over D.

Similar to the matrix case to provide the reverse construction we will first construct a canonical section $v_0 \in \mathrm{H}^0(C_0, V_0)$. Let $y \in C_0$ be given. Notice that (V_y, ϕ_y, Q_y) defines a SO_{2n+1}-triple in the matrix sense. Let $e^0, \ldots, e^{2n}$ be an orthonormal frame for V^* over C_0 with respect to Q. Moreover, in this orthonormal frame we may assume without loss of generality that the linear transformation ϕ_y is of the form

$$\phi_y = 0 \oplus \begin{bmatrix} 0 & \lambda_1(y) \\ -\lambda_1(y) & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & \lambda_n(y) \\ -\lambda_n(y) & 0 \end{bmatrix}.$$

Let $\alpha \in \mathrm{H}^0(C_0, L \otimes \wedge^2 V^*)$ be defined by $\alpha(v,w) = Q(\phi(v), w)$. In the orthonormal frame

$$\alpha = i\lambda_1 e^1 \wedge e^2 + \cdots + i\lambda_n e^{2n-1} \wedge e^{2n}$$

and hence,

$$\alpha^n = n! i^n \lambda_1 \cdots \lambda_n \, e^1 \wedge \cdots \wedge e^{2n}.$$

Let v_0 be the section of $L^n \otimes V$ defined by

$$v_0 = i^n \lambda_1 \cdots \lambda_n e_0.$$

Over C_0 the section v_0 is nowhere vanishing. The volume form $\nu = e^0 \wedge \cdots \wedge e^{2n}$ is SO_{2n+1}-invariant and an identical argument to the matrix case shows

$$\alpha^n = n! i_{v_0}(\nu)$$

and

$$\phi(v_0) = 0.$$

The section v_0 is the desired zero eigensection of ϕ over C_0. Now,

$$Q(v_0, v_0) = Q(i^n \lambda_1 \cdots \lambda_n e_0, i^n \lambda_1 \cdots \lambda_n e_0) = (-1)^n \lambda_1^2 \cdots \lambda_n^2 = a_{2n}. \tag{5.18}$$

To motivate the reverse construction further, consider the following short exact sequence of holomorphic vector bundles over C

$$0 \to V_0 \to V \to E \to 0.$$

Dualising the sequence and using Q to identify $V \cong V^*$ we obtain a short exact sequence

$$0 \to E^* \to V \to V_0^* \to 0.$$

Therefore, there is a canonical isomorphism $\det(V) \cong \det(E)^* \otimes V_0^*$. However, $\det(V) \cong \mathcal{O}_C$, and hence, $\det(E) \cong V_0^*$. Finally, since $\omega \in \mathrm{H}^0(C_0, L \otimes \wedge^2 E^*)$ is non-degenerate the section $\omega^n \in \mathrm{H}^0(C_0, L^n \otimes \det(E)^*)$ is nowhere vanishing, hence $\det(E) \cong L^n$ over C_0. Thus, $V_0 \cong L^{-n}$ over C_0. Further, we may use ω to identify $E \cong L \otimes E^*$ over C_0, or, equivalently, $E^* \cong L^{-1}E$ over C_0. Now, we may give the reverse construction.

Suppose (E, Φ, ω) is a Sp_{2n}-Higgs bundle over C where ω is L-valued and Φ has characteristic polynomial $p(\lambda)$. Over C_0 we define the rank $2n+1$ holomorphic vector bundle

$$V := L^{-1}E \oplus L^{-n}.$$

Moreover, we define a symmetric $\mathcal{O}_{C_0}$-bilinear form $Q : V \otimes V \to \mathcal{O}_{C_0}$ by

$$Q((v,w),(v,w)) = \omega(v, \Phi(v)) + w^2 a_{2n}$$

that extends to all sections by the polarisation identity. Moreover, we define a Higgs field $\phi : V \to L \otimes V$ by

$$\phi(v, w) = (\Phi(v), 0).$$

By adapting the arguments from the vector space case it is straightforward to see that (V, ϕ, Q) defines a SO_{2n+1}-Higgs bundle over C_0 whose spectral curve is defined by

$$\lambda(\lambda^{2n} + a_2\lambda^{2n-2} + \cdots + a_{2n}).$$

Note, since C_0 is open the condition $\det(V) \cong \mathcal{O}_{C_0}$ is vacuous.

Construction 5.3.9. Suppose (E, Φ, ω) is a Sp_{2n}-Higgs bundle over C with characteristic polynomial $p(\lambda) = \lambda^{2n} + a_2\lambda^{2n-2} + \cdots + a_{2n}$ that defines a smooth spectral curve $\pi : S \to C$. Let $D := (a_{2n})$ and set $C_0 := C \setminus D$. Over C_0 we define a rank $2n + 1$ holomorphic vector bundle $V := L^{-1}E \oplus L^{-n}$ and a symmetric, $\mathcal{O}_{C_0}$-bilinear form $Q : V \otimes V \to \mathcal{O}_{C_0}$ by $Q((v, w), (v, w)) = \omega(v, \Phi(v)) + w^2 a_{2n}$ and the polarisation identity. Moreover, we define a Higgs field $\phi : V \to L \otimes V$ by $\phi(v, w) = (\Phi(v), 0)$ and the triple (V, ϕ, Q) defines a SO_{2n+1}-Higgs bundle over C_0 whose characteristic polynomial is given by $\lambda p(\lambda)$.

The same arguments are the matrix case shows Construction 5.3.7 and Construction 5.3.9 are mutual inverses establishing the desired correspondence.

Theorem 5.3.10. *Consider the polynomial*

$$\lambda(\lambda^{2n} + a_2\lambda^{2n-2} + \cdots + a_{2n})$$

where $a_{2i} \in \mathrm{H}^0(C, L^{2i})$ *is chosen so that* $p(\lambda) = \lambda^{2n} + a_2\lambda^{2n-2} + \cdots + a_{2n}$ *defines a smooth symplectic spectral curve. Let* $D := (a_{2n})$ *and set* $C_0 := C \setminus D$*. Then there is a natural one-to-one correspondence between* SO_{2n+1}*-Higgs bundles over* C_0 *whose characteristic polynomial is defined by* $\lambda p(\lambda)$*, and* Sp_{2n}*-Higgs bundles over* C_0 *whose characteristic polynomial is defined by* $p(\lambda)$*. Explicitly, if* (V, ϕ, Q) *is a* SO_{2n+1}*-Higgs bundle with characteristic polynomial* $\lambda p(\lambda)$*, then the kernel of the Higgs field induces a holomorphic line subbundle* $V_0 \subset V$ *and* $E = V/V_0$ *defines a holomorphic vector bundle of rank* $2n$ *and the Higgs field* ϕ *induces a homomorphism* $\Phi : V \to L \otimes V$*. Moreover,* $\omega : E \otimes E \to L$ *given by* $\omega(v, w) = Q(\phi(\widetilde{v}), \widetilde{w})$ *defines a* Φ*-compatible symplectic form over* C_0 *and thus,* (E, Φ, ω) *defines a* Sp_{2n}*-Higgs bundle over* C_0 *with characteristic polynomial* $p(\lambda)$*.*

Conversely, suppose (E, Φ, ω) *is a* Sp_{2n}*-Higgs bundle whose characteristic polynomial is* $p(\lambda)$*. Over* C_0 *we define a rank* $2n+1$ *holomorphic vector bundle by* $V := L^{-1}E \oplus L^{-n}$ *and a Higgs field* $\phi : V \to L \otimes V$ *by* $\phi(v, w) = (\Phi(v), 0)$*. Next, we define a symmetric,* $\mathcal{O}_{C_0}$*-bilinear form* $Q : V \otimes V \to \mathcal{O}_{C_0}$ *by* $Q((v, w), (v, w)) = \omega(v, \Phi(v)) + w^2 a_{2n}$ *and the polarisation identity. The bilinear form is non-degenerate and* ϕ*-compatible, hence* (V, ϕ, Q) *defines a* SO_{2n+1}*-Higgs bundle over* C_0 *with characteristic polynomial* $\lambda p(\lambda)$*. Further, these constructions are mutual inverses.*

5.3.3 Extending Correspondence Over Singularities

Suppose (V, ϕ, ω) is a generic SO_{2n+1}-Higgs bundle, i.e., the characteristic polynomial is given by $\lambda p(\lambda)$ where $p(\lambda) = \lambda^{2n} + a_2\lambda^{2n-2} + \cdots + a_{2n}$ defines a smooth symplectic spectral curve $\pi : S \to C$. Let $D = (a_{2n})$ and set $C_0 = C \setminus D$. Before extending the correspondence over the singularities we will make one assumption.

Assumption 5.3.11. We will assume that $a_{2n-2} \in \mathrm{H}^0(C, L^{2n-2})$ and $a_{2n} \in \mathrm{H}^0(C, L^{2n})$ have no zeros in common.

Remark 5.3.12. The assumption is valid since L is basepoint-free. Recall that basepoint-free holomorphic line bundles admit sections that share no zeros in common. Thus, take $s_1, s_2 \in \mathrm{H}^0(C, L)$ that share no zeros, then $s_1^{2n-2} \in \mathrm{H}^0(C, L^{2n-2})$ and $s_2^{2n} \in \mathrm{H}^0(C, L^{2n})$ share no zeros.

Lemma 5.3.13. *The zeros of $a_{2n} \in \mathrm{H}^0(C, L^{2n})$ are simple.*

Proof. Let x be a zero of a_{2n}. Choose a local coordinate z centred at x, and by shrinking the chart if necessary we may assume that L is trivialised in this neighbourhood. In this neighbourhood the tautological section is a holomorphic function and locally about $0 \in L_x$ the spectral curve S is defined as the zero locus of

$$p(\lambda, z) = \lambda^{2n} + a_2(z)\lambda^{2n-2} + \cdots + a_{2n}(z).$$

Differentiating with respect to λ we see

$$\partial_\lambda p(\lambda, z) = 2n\lambda^{2n-1} + (2n-2)a_2(z)\lambda^{2n-3} + \cdots + 2a_{2n-2}(z)\lambda$$

hence $\partial_\lambda p(\lambda, z)|_{(0,0)} = 0$. Since S is smooth, $\partial_z p(\lambda, z)|_{(0,0)} \neq 0$ and since

$$\partial_z p(\lambda, z) = a_2'(z)\lambda^{2n-2} + \cdots + a_{2n}'(z)$$

it follows that $a_{2n}'(0) \neq 0$. □

Let (E, Φ, ω) be the Sp_{2n}-Higgs bundle over C_0 obtained by applying Construction 5.3.7. The triple (E, ϕ, ω) is defined over C and to show (E, ϕ, ω) defines a Sp_{2n}-Higgs bundle over C it suffices to show ω is non-degenerate over D. To show this we require the following lemma.

Lemma 5.3.14. *Let x be a zero of* $\det(\Phi)$*. If* $\dim(\ker(\Phi_x)) > 1$*, then* $\mathrm{ord}_x(\det(\Phi)) > 1$.

Proof. Suppose $\dim(\ker(\Phi_x)) > 1$. Choose two linearly independent vectors e_1, e_2 in $\ker(\Phi_x)$ and extend to a basis $e_1, e_2, \ldots, e_{2n}$ for E_x. Let z be a local coordinate centred at x trivialising E and extend the basis to a local frame $e_1(z), \ldots, e_{2n}(z)$ for E. Of course, $\Phi(z)e_1(z)$ and $\Phi(z)e_2(z)$ vanish at $z = 0$. Now,

$$\Phi(z)e_1(z) \wedge \Phi(z)e_2(z) \wedge \cdots \wedge \Phi(z)e_{2n}(z) = \det(\Phi)(z)e_1(z) \wedge \cdots \wedge e_{2n}(z).$$

The left-hand-side vanishes to at least second order, hence $\mathrm{ord}_x(\det(\Phi)) > 1$. □

Corollary 5.3.15. *Suppose $x \in C$ is a simple zero of* $\det(\Phi)$*, then* $\dim(\ker(\Phi_x)) = 1$.

Using Corollary 5.3.15 we can show $\omega : E \otimes E \to \mathcal{O}_C$ is non-degenerate over D. Let $x \in D$, i.e., $a_{2n}(x) = 0$. Since $a_{2n-2}(x) \neq 0$ it follows that the zero generalised eigenspace of ϕ_x, which we denote by A is 3-dimensional. Let B be the direct sum of the remaining generalised eigenspaces. By the primary decomposition theorem

$$V_x = A \oplus B.$$

Since the generalised eigenspaces are ϕ_x-invariant, and Q pairs opposite eigenvalues, the decomposition is ϕ_x and Q_x-invariant. Hence, let $\alpha : A \to L_x \otimes A$ be the restriction of ϕ_x. Notice that α is nilpotent. By fixing a basis of A we may identify $\mathfrak{so}(A) \cong \mathfrak{so}_3(\mathbb{C})$ then since α is nilpotent α belongs to one of three conjugacy classes. The three conjugacy classes may be represented by Jordan canonical form, i.e.,

$$J_1 := 0_3, \quad J_2 := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad J_3 := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We claim that α belongs to the J_2 conjugacy class.

Suppose on the contrary that α belongs to the J_1 conjugacy class, i.e., $\alpha = 0$. Then $\phi_x = 0_3 \oplus M$ where $M : B \to L_x \otimes B$ is the restriction of ϕ_x. We may assume without loss of generality that the first basis element spans $(V_0)_x$ and hence, $\Phi_x = 0_2 \oplus M$. However, this implies $\dim(\ker(\Phi_x)) > 1$, contradicting Corollary 5.3.15.

Suppose instead that α belongs to the J_3 conjugacy class. By a suitable choice of basis e_1, e_2, e_3 we may assume $\alpha = J_3$. Then, $\alpha(e_2) = e_1$ and $\alpha(e_1) = \alpha(e_3) = 0$. Now,

$$Q_x(e_1, e_2) = Q_x(\alpha(e_2), e_2) = -Q_x(e_2, \alpha(e_2)) = -Q_x(e_2, e_1)$$

hence, $Q_x(e_1, e_2) = 0$. Moreover,

$$Q_x(e_1, e_1) = Q_x(\alpha(e_2), e_1) = -Q_x(e_2, 0) = 0$$

and

$$Q_x(e_1, e_3) = Q_x(\alpha(e_2), e_3) = -Q_x(e_2, 0) = 0.$$

Therefore, $Q_x(e_1, v) = 0$ for every $v \in A$, which contradicts non-degeneracy. Therefore, α belongs to the conjugacy class J_2 and thus, $\dim(\ker(\phi_x)) = 1$, so $(V_0)_x = \ker(\phi_x)$.

Lemma 5.3.16. *The L-valued 2-form $\omega : E \otimes E \to L$ defined by $\omega(v, w) = Q(\phi(\widetilde{v}), \widetilde{w})$ is non-degenerate over C.*

Proof. We have already seen that ω is non-degenerate over $C \setminus D$. Let $x \in D$ be given and suppose $\omega_x(v, w) = 0$ for every $w \in E_x$. Then, $Q_x(\phi_x(\widetilde{v}), \widetilde{w}) = 0$ for every $\widetilde{w} \in V_x$. Since Q is non-degenerate, $\phi_x(\widetilde{v}) = 0$, i.e., $\widetilde{v} \in \ker(\phi_x)$. However, $(V_0)_x = \ker(\phi_x)$ and thus, $v = 0$ in E_x. $\square$

Therefore, the triple (E, Φ, ω) defines a Sp_{2n}-Higgs bundle over C and thus, Construction 5.3.7 defines a global construction.

Now, we will extend the reverse construction over D. Construction 5.3.9 does not canonically extend over D since $Q(v_0, v_0) = a_{2n}$ and hence, Q_x is degenerate for each $x \in D$. To overcome this obstruction, we will characterise $\mathcal{V}$ as a subsheaf of $\mathcal{M}_C(L^{-1}E \oplus L^{-n})$, i.e., we will allow certain poles so that the Higgs field ϕ and Q extends over D such that Q is ϕ-compatible and non-degenerate. In what proceeds let $U_1 = C_0$ and $V_1 = E$. Consider the short exact sequence of holomorphic vector bundles over C

$$0 \to V_0 \xrightarrow{\iota} V \xrightarrow{j} V_1 \to 0$$

where $\iota : V_0 \to V$ denotes the canonical inclusion and $j : V \to V_1$ denotes the natural projection map. Dualising the sequence and using Q to identify $V \cong V^*$ gives the short exact sequence

$$0 \to V_1^* \xrightarrow{j^*} V \xrightarrow{\iota^*} V_0^* \to 0.$$

Let U_2 be the disjoint union of sufficiently small open discs around each $x \in D$. By local triviality we can choose a splitting $\tau : V_0^*|_{U_2} \to V|_{U_2}$, i.e., $V|_{U_2} \cong (V_1^* \oplus V_0^*)|_{U_2}$ with isomorphism

$$\psi_2 : (V_1^* \oplus V_0^*)|_{U_2} \ni (a, b) \mapsto j^*(a) + \tau(b) \in V|_{U_2}.$$

Over U_1 notice that $Q_0 := Q|_{V_0}$ is non-degenerate and defines an isomorphism $V_0 \cong V_0^*$. Moreover, by non-degeneracy, $V_0 \cap V_0^\perp = \{0\}$ and by identifying $V_0^\perp \cong \mathrm{Ann}(V_0) \cong V_1^*$ over U_1 there is an isomorphism $V|_{U_1} \cong (V_1^* \oplus V_0)|_{U_1}$ given by

$$\psi_1 : (V_1^* \oplus V_0)|_{U_1} \ni (a, b) \mapsto j^*(a) + \iota(b) \in V|_{U_1}.$$

We wish to determine the difference in the decomposition over $U_{12} := U_1 \cap U_2$, i.e., we will compute the transition map

$$\psi_{21} := \psi_2^{-1} \circ \psi_1 : (V_1^* \oplus V_0)|_{U_{12}} \to (V_1^* \oplus V_0^*)|_{U_{12}}.$$

Suppose $(a, 0) \in \mathrm{H}^0(U_{12}, V_1^* \oplus V_0)$, then since $\psi_1(a, 0) = \psi_2(a, 0)$ it is straightforward to see $\psi_{21}(a, 0) = (a, 0)$. Now, let $(0, b) \in \mathrm{H}^0(U_{12}, V_1^* \oplus V_0)$, then $\psi_{21}(0, b) = \psi_2^{-1}(\iota(b))$. To understand $\psi_2^{-1}(\iota(b))$ we let $u : V_0 \to V_1^*$ be the homomorphism defined by $u(b) = \mathrm{pr}_1(\psi_2^{-1}(\iota(b)))$ where $\mathrm{pr}_1 : V_1^* \oplus V_0^* \to V_1^*$ denotes the projection map. Since $\iota^* \circ \iota = Q_0$ it follows that $\psi_{21}(0, b) = (u(b), Q_0(b))$ and thus,

$$\psi_{21}(a, b) = (a + u(b), Q_0(b)).$$

A straightforward calculation shows that the inverse map

$$\psi_{12} : (V_1^* \oplus V_0^*)|_{U_{12}} \to (V_1^* \oplus V_0)|_{U_{12}}$$

is given by

$$\psi_{12}(a,b) = (a - u(Q_0^{-1}(b)), Q_0^{-1}(b)).$$

Now, we extend the map ψ_{12} and ψ_{21} over D by allowing poles. Recall that $(Q_0)_x = 0$ for each $x \in D$ and hence, $(V_0)_x \subset (V_0^{\perp})_x \cong (V_1^*)_x$. Moreover, for every $b \in (V_0)_x$

$$(\psi_{21})_x(0,b) = (u_x(b), 0),$$

which shows $\iota_x = j_x^* \circ u_x$ and thus, $u_x : (V_0)_x \to (V_1^*)_x$ is injective for each $x \in D$. This provides enough machinery to characterise the sheaf of sections of V over C as a subsheaf of $\mathcal{M}_C(V_1^* \oplus V_0)$.

Proposition 5.3.17. *The sheaf $\mathcal{V}$ can be realised as the subsheaf of $\mathcal{M}_C(V_1^* \oplus V_0)$ that has simple poles along D with residues valued in*

$$\Gamma_x = \{(-i_x(w), w) \,|\, w \in (V_0)_x\}$$

where $i_x : (V_0)_x \to (V_1^*)_x$ is injective.

Proof. Recall that $V \cong V_1^* \oplus V_0$ away from D, however since $V_0 \subset V$ is a degenerate subbundle with respect to Q the isomorphism does not extend over D. Using ψ_1^{-1} we may realise $\mathcal{V}$ as a subsheaf of $\mathcal{M}_C(V_1^* \oplus V_0)$ and since ψ_1 is an isomorphism away from D it is clear that any poles will necessarily lie along D. Since $\psi_1^{-1} = \psi_{12} \circ \psi_2^{-1}$, and over U_2 the map $\psi_2^{-1} : V|_{U_2} \to (V_1^* \oplus V_0^*)|_{U_2}$ is an isomorphism it suffices to characterise the subsheaf

$$\psi_{12}\, \mathcal{O}_C(V_1^* \oplus V_0^*) \subset \mathcal{M}_C(V_1^* \oplus V_0).$$

Fix $x \in D$ and choose local holomorphic coordinate z centred at x that trivialises L. Since a_{2n} has a simple zero at x we may assume without loss of generality that $a_{2n} = z$. Moreover, recall that there is a canonical section v_0 of $L^n \otimes V_0$ such that $Q(v_0, v_0) = z$. The section v_0 defines an isomorphism $V_0 \cong L^{-n}$ away from D and it follows that in this trivialisation $Q_0 = \frac{1}{z}$. Therefore, given a local holomorphic section $(a(z), b(z))$ of $V_1^* \oplus V_0$ we see

$$\psi_{12}(a(z), b(z)) = \left(a(z) - \frac{u(z)b(z)}{z}, \frac{b(z)}{z}\right).$$

Since $u(0) = u_x$ is injective the result follows. $\square$

Therefore, if (V, ϕ, Q) is a SO_{2n+1}-Higgs bundle with characteristic polynomial $\lambda p(\lambda)$, and (E, Φ, ω) is the associated Sp_{2n}-Higgs bundle with characteristic polynomial $p(\lambda)$, then $\mathcal{V}$ is the subsheaf of $\mathcal{M}_C(L^{-1}E \oplus L^{-n})$ comprised of sections holomorphic away from $D = (\det(\Phi))$ with simple poles along D whose residues are valued in $\Gamma_x = \{(-i_x(w), w) \,|\, w \in L_x^{-n}\}$ for each $x \in D$ where $i_x : L_x^{-n} \to L_x^{-1}E_x$ is injective. Note, the symplectic form $\omega : E \otimes E \to L$ defines an isomorphism $E^* \cong L^{-1}E$.

We can now extend the construction over the singularities. Let (E, Φ, ω) be a Sp_{2n}-Higgs bundle whose characteristic polynomial is given by $p(\lambda)$. Set $D = (a_{2n})$ and define a rank $2n+1$ holomorphic vector bundle $V \to C$ whose sheaf of sections $\mathcal{V}$ is the subsheaf of $\mathcal{M}_C(L^{-1}E \oplus L^{-n})$ comprised of section holomorphic away from D that have simple poles along D with residues valued in

$$\Gamma_x = \{(-i_x(w), w) \,|\, w \in L_x^{-n}\}$$

for each $x \in D$ where $i_x : L_x^{-n} \to L_x^{-1}E_x$ is injective. Away from D we define a Higgs field $\phi : V \to L \otimes V$ by

$$\phi(v, w) = (\Phi(v), 0)$$

and a symmetric $\mathcal{O}_C$-bilinear form $Q : V \otimes V \to \mathcal{O}_C$ by

$$Q((v, w), (v, w)) = \omega(v, \Phi(v)) + w^2 a_{2n}$$

that extends to every section by the polarisation identity. Recall that the triple (V, ϕ, Q) defines a SO_{2n+1}-Higgs bundle over $C_0 = C \setminus D$. To extend the construction over D we require ϕ and Q to extend holomorphically over D such that Q is ϕ-compatible and non-degenerate. The only flexibility in defining V is choosing $i_x : L_x^{-n} \to L_x^{-1}E_x$ for each $x \in D$. Thus, we will choose i_x that give our desired extensions. Of course, we need V to have trivial determinant. Note, since the rank of V is odd there is no ambiguity in choosing an orientation.

Lemma 5.3.18. *The holomorphic vector bundle $V \to C$ characterised by the subsheaf $\mathcal{V} \subset \mathcal{M}_C(L^{-1}E \oplus L^{-n})$ of sections that are holomorphic away from D with simple poles along D whose residues are valued in $\Gamma_x = \{(-i_x(w), w) \,|\, w \in L_x^{-n}\}$ has trivial determinant.*

Proof. By construction the vector bundle V fits into a short exact sequence

$$0 \to L^{-n} \to V \to E \to 0.$$

There is a canonical isomorphism $\det(V) \cong L^{-n} \otimes \det(E)$. Since the symplectic form $\omega : E \otimes E \to L$ is non-degenerate, ω^n defines a nowhere vanishing section of $L^n \otimes \det(E)^*$ and thus, $\det(E) \cong L^n$, which implies $\det(V) \cong \mathcal{O}_C$. □

Lemma 5.3.19. *The Higgs field $\phi : V \to L \otimes V$ extends holomorphically over D if and only if $i_x(L_x^{-n}) = \ker(\Phi_x)$.*

Proof. Fix $x \in D$ and choose a local holomorphic coordinate z centred at x. In the coordinate neighbourhood we may extend the homomorphism $i_x : L_x^{-n} \to L_x^{-1}E_x$ to a local section $i(z)$ of $\mathrm{Hom}(L^{-n}, L^{-1}E)$. Suppose $w(z)$ is a local section of L^{-n}, then

$$\phi(z)\left(-\frac{i(z)w(z)}{z}, \frac{w(z)}{z}\right) = \left(-\frac{1}{z}\Phi(z)i(z)w(z), 0\right).$$

Thus, ϕ extends holomorphically over x if and only if $\Phi_x(i_x(w)) = 0$ where $w = w(0)$, or, equivalently, $i_x(L_x^{-n}) = \ker(\Phi_x)$. □

Remark 5.3.20. The condition $i_x(L_x^{-n}) = \ker(\Phi_x)$ determines i_x up to scale.

Thus, we choose $i_x : L_x^{-n} \to L_x^{-1}E_x$ such that $i_x(L_x^{-n}) = \ker(\Phi_x)$. Now, we will appropriately scale i_x for each $x \in D$ so that Q extends holomorphically over D. By construction if both entries in Q are holomorphic, then the output is holomorphic. Hence, we only need to consider the cases where one entry is holomorphic and one is meromorphic, and when both entries are meromorphic. Fix $x \in D$ choose local coordinate z centred at x. In the coordinate neighbourhood we may extend i_x to a local section $i(z)$ for $\mathrm{Hom}(L^{-n}, L^{-1}E)$. Suppose $w(z)$ is a local section of L^{-n}. Consider

$$Q\left(\left(-\frac{i(z)w(z)}{z}, \frac{w(z)}{z}\right), (-i(z)w(z), w(z))\right) = \frac{1}{z}\omega(i(z)w(z), \Phi(z)i(z)w(z)) + \frac{w(z)^2}{z}a_{2n}(z). \tag{5.19}$$

By Lemma 5.3.19 note that $\Phi_x(i_x(w)) = 0$, hence $\frac{1}{z}\omega(i(z)w(z), \Phi(z)i(z)w(z))$ extends holomorphically over $z = 0$. Moreover, since $a_{2n}(0) = 0$ it follows that $\frac{w(z)^2}{z}a_{2n}(z)$ extends holomorphically over $z = 0$. Therefore, the expression in (5.19) is holomorphic at $z = 0$. Thus, we have reduced to only considering when both entries are meromorphic. Hence, consider

$$Q\left(\left(-\frac{i(z)w(z)}{z}, \frac{w(z)}{z}\right), \left(-\frac{i(z)w(z)}{z}, \frac{w(z)}{z}\right)\right) = \frac{1}{z^2}\omega(i(z)w(z), \Phi(z)i(z)w(z)) + \frac{w(z)^2}{z^2}a_{2n}(z). \tag{5.20}$$

Writing $a_{2n}(z) = za_{2n}'(0) + z^2A(z)$ where $A(z)$ is some holomorphic function, and $\Phi(z) = \Phi_x + z\Phi_x'(0) + z^2B(z)$ where $B(z)$ is some holomorphic function, then since $\Phi_x(i_x(w)) = 0$ it follows that (5.20) simplifies to

$$\frac{1}{z}\left(\omega(i_x(w), \Phi_x'(i_x(w))) + w^2a_{2n}'(0)\right) \mod \mathcal{O}_x.$$

Therefore, Q extends holomorphically over D if and only if $\omega(i_x(w), \Phi_x'(i_x(w))) = -w^2a_{2n}'(0)$ for each $x \in D$.

Proposition 5.3.21. *We may choose i_x such that $\omega(i_x(w), \Phi_x'(i_x(w))) = -w^2a_{2n}'(0)$ for each $x \in D$.*

Proof. Fix $x \in D$ and choose a local coordinate (U, z) centred at x such that U is biholomorphic to a disc. Choose a trivialising section $\ell := \ell(z)$ of L over U. Locally, $\Phi(z)$ has eigenvalues $\pm\lambda_1(z)\ell, \ldots, \pm\lambda_n(z)\ell$. Since $a_{2n-2}(0) \neq 0$ the zero generalised eigenspace of Φ_x is two-dimensional so we may assume without loss of generality that $\lambda_i(0) = 0$ if and only if $i = 1$. Then, the characteristic polynomial factors as

$$p(\lambda, z) = (\lambda^2 + \alpha)\,\widehat{p}(\lambda, z)$$

where $\alpha = -\lambda_1(z)^2$ and $\widehat{p}(\lambda, z) = (\lambda^2 - \lambda_2(z)^2)\cdots(\lambda^2 - \lambda_{n-1}(z)^2)$. Note that if $\beta = (-1)^{n-1}\lambda_2(z)^2 \cdots \lambda_{n-1}(z)^2$, then $\beta(0) = a_{2n-2}(0)$ and $\alpha\beta = a_{2n}$, and it follows that

$$\alpha'(0) = \frac{a'_{2n}(0)}{a_{2n-2}(0)}.$$

Let $E_0 \subset E|_U$ be the subbundle induced by the kernel of $\Phi(z)^2 + \alpha(z)\ell^2$. Then $(E_0)_x$ is the zero generalised eigenspace of Φ_x, hence $(E_0)_x$ is two-dimensional, which implies E_0 has rank 2. Now, choose a basis e_1, e_2 for $(E_0)_x$ such that $\Phi_x(e_1) = 0$ and $\Phi_x(e_2) = e_1\ell_x$. Extend e_2 to a local section $e_2(z)$ of E over U such that $\Phi(z)e_2(z) \neq 0$, and set $e_1(z) := \ell^{-1}\Phi(z)e_2(z)$. The sections $e_1(z), e_2(z)$ form a frame for E_0 and $\Phi(z)e_2(z) = e_1(z)\ell$. Hence, $\Phi^2(z)e_2(z)\ell^{-1} = \Phi(z)e_1(z)$ and by definition of E_0, $\Phi^2(z) = -\alpha(z)\ell^2$, thus $\Phi(z)e_1(z) = -\alpha(z)e_2(z)\ell$. Therefore, in the local frame $e_1(z), e_2(z)$ for E_0

$$\Phi(z) = \begin{bmatrix} 0 & \ell \\ -\alpha(z)\ell & 0 \end{bmatrix}.$$

Now, since $\Phi_x(i_x(\ell_x^{-n})) = 0$ it follows that $i_x(\ell_x^{-n}) = ue_1\ell_x^{-1}$ for some $u \in \mathbb{C}^*$. Thus, we may extend the homomorphism i_x to a section of $L^{n-1}E$ by defining $i(z) = ue_1(z)\ell^{n-1}$, i.e., $i(z)\ell^{-n} = ue_1(z)\ell^{-1}$. Then,

$$\Phi(z)i(z)\ell^{-n} = \begin{bmatrix} 0 & \ell \\ -\alpha(z)\ell & 0 \end{bmatrix} \begin{bmatrix} u\ell^{-1} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -u\alpha(z) \end{bmatrix}.$$

Differentiating and evaluating at $z = 0$ it follows that $\Phi'_x(i_x\ell_x^{-n}) = -u\frac{a'_{2n}(0)}{a_{2n-2}(0)}e_2$. Therefore, $\omega(i_x(w), \Phi'_x(i_x(w))) = -w^2 a'_{2n}(0)$ if and only if

$$u^2\omega_x(e_1, e_2) = a_{2n-2}(0)\ell_x. \tag{5.21}$$

Since ω_x is non-degenerate, $\omega_x(e_1, e_2) \neq 0$, and since $a_{2n-2}(0)\ell_x \neq 0$ we may choose u up to sign so that (5.21) holds. □

Remark 5.3.22. The condition $u^2\omega_x(e_1, e_2) = a_{2n-2}(0)\ell_x$ may be extended to a local section by decreeing $u^2\omega(e_1(z), e_2(z)) = a_{2n-2}(z)\ell$. Note, this also shows that the choice of i_x is unique up to sign.

We are left to show that Q is non-degenerate over D, i.e., $Q_x : V_x \otimes V_x \to \mathbb{C}$ is non-degenerate for each $x \in D$. Note, it is clear that the extension of Q over D is ϕ-compatible. Recall that in the coordinate patch (U, z) from the proof of Proposition 5.3.21 the characteristic polynomial factors by $p(\lambda, z) = (\lambda^2 + \alpha)\,\widehat{p}(\lambda, z)$, and the kernel of $\Phi(z)^2 + \alpha$ defines a rank 2 subbundle $E_0 \subset E|_U$. Now, let $E_1 \subset E|_U$ be the subbundle induced by the kernel of $\widehat{p}(\Phi(z), z)$. Note that $(E_1)_x$ is the sum of the non-zero generalised eigenspaces and by the primary decomposition theorem $E_x = (E_0)_x \oplus (E_1)_x$. Thus, $\dim((E_1)_x) = 2n-2$

so the holomorphic vector bundle $E_1 \to U$ has rank $2n-2$. By the Cayley-Hamilton theorem $E|_U = E_0 \oplus E_1$. Moreover, since ω pairs opposite eigenvalues, the decomposition is orthogonal with respect to ω, and since each generalised eigenspace is Φ-invariant it follows that each E_i is Φ-invariant. Now, consider the decomposition

$$(L^{-1}E \oplus L^{-n})|_U \cong (L^{-1}|_U\, E_0 \oplus L^{-n}|_U) \oplus L^{-1}|_U\, E_1$$

then since $i_x(L_x^{-n}) = \ker(\Phi_x) \subset (E_0)_x$ we see $\Gamma_x \subset L_x^{-1}(E_0)_x \oplus L_x^{-n}$. Hence, $V|_U = V_0 \oplus V_1$ where $\mathcal{V}_0 \subset \mathcal{M}_U(L^{-1}E_0 \oplus L^{-n})$ is the subsheaf of sections holomorphic away from $z=0$ with a simple pole at $z=0$ whose residue is valued in Γ_x, and $V_1 = L^{-1}|_U\, E_1$. We claim that the decomposition $V|_U = V_0 \oplus V_1$ is orthogonal with respect to Q. To see this, let $y \in U$ be distinct to x, then it suffices to prove that the decomposition $V_y = (V_0)_y \oplus (V_1)_y$ is orthogonal with respect to Q_y since the result follows by continuity. Away from x there is an isomorphism $V_0 \cong L^{-1}E_0 \oplus L^{-n}$ and so $Q_y : (V_0)_y \otimes (V_1)_y \to \mathbb{C}$ is given by the restriction of Q_y to $(V_0)_y \otimes (V_1)_y$. Suppose $(u,v) \in (V_0)_y$ and $w \in (V_1)_y$, then it follows by the polarisation identity that

$$Q_y((u,v),(w,0)) = \frac{1}{2}\left(\omega_y(u,\Phi_y(w)) + \omega_y(w,\Phi_y(u))\right).$$

However, $(V_0)_y$ and $(V_1)_y$ are orthogonal with respect to ω_y and each vector space is Φ_y-invariant so it follows that $Q_y((u,v),(w,0)) = 0$, which proves the claim. Therefore, $Q|_U = Q_0 \oplus Q_1$ where $Q_i : V_i \otimes V_i \to \mathcal{O}_U$ denotes the restriction of $Q|_U$ to V_i. To show Q_x is non-degenerate it suffices to show both $(Q_0)_x$ and $(Q_1)_x$ are non-degenerate.

To see that $(Q_1)_x$ is non-degenerate suppose $(Q_1)_x((u,0),(v,0)) = 0$ for every $u \in (V_1)_x$, i.e., $\omega_x(u,\Phi_x(v)) = 0$ for every $u \in L_x^{-1}(E_1)_x$. Since ω_x is non-degenerate, $\Phi_x(v) = 0$. However, Φ_x is invertible over E_1 and thus, $v = 0$.

Finally, to show that $(Q_0)_x$ is non-degenerate we will use a local frame. Let $e_1(z), e_2(z)$ be the frame constructed for E_0 from the proof of Proposition 5.3.21. We replace $e_1(z), e_2(z)$ by $ue_1(z), ue_2(z)$ so that the condition for Q to extend holomorphically over x becomes

$$\omega(e_1(z), e_2(z)) = a_{2n-2}(z)\ell$$

where $\ell = \ell(z)$ is a section locally trivialising L. Let $e_3(z) := \ell^{-n}$ and we decree $\Phi(z)e_3(z) = 0$. Then, $\Gamma_x = \langle(-e_1\ell_x^{-1}, e_3)\rangle$ and thus,

$$f_1 = \left(-\frac{e_1(z)\ell^{-1}}{z}, \frac{e_3(z)}{z}\right), \quad f_2 = (e_1(z)\ell^{-1}, e_3(z)), \quad f_3 = (e_2(z)\ell^{-1}, 0)$$

defines a frame for V_0. Now,

$$\begin{aligned}
Q(f_1, f_1) &= Q\left(\left(-\frac{e_1(z)\ell^{-1}}{z}, \frac{e_3(z)}{z}\right), \left(-\frac{e_1(z)\ell^{-1}}{z}, \frac{e_3(z)}{z}\right)\right) \\
&= \frac{\ell^{-2}}{z^2}\omega(e_1(z), \Phi(z)e_1(z)) + \frac{a_{2n}(z)}{z^2} \\
&= -\frac{\ell^{-1}\alpha(z)}{z^2}\omega(e_1(z), e_2(z)) + \frac{a_{2n}(z)}{z^2} \\
&= \frac{-\alpha(z)a_{2n-2}(z) + a_{2n}(z)}{z^2}.
\end{aligned}$$

Notice that $\alpha(z)a_{2n-2}(z) = a_{2n}(z) + \alpha(z)\Gamma(z)$ where $\Gamma(z)$ is a holomorphic function such that $\Gamma(0) = 0$. Hence, $\alpha(z)a_{2n-2}(z) - a_{2n}(z) = \alpha(z)\Gamma(z)$, and the right-hand-side has a zero of at least order three at $z = 0$. Thus, evaluating at $z = 0$

$$Q_x(f_1, f_1) = 0.$$

Next, by the polarisation identity

$$\begin{aligned}
Q(f_2, f_3) &= Q((-e_1(z)\ell^{-1}, e_3(z)), (e_2(z)\ell^{-1}, 0)) \\
&= \ell^{-2}\omega(e_1(z), \Phi(z)e_2(z)) \\
&= \ell^{-1}\omega(e_1(z), e_1(z)) \\
&= 0.
\end{aligned}$$

Similar calculations show $Q(f_2, f_2) = 0$ and $Q(f_1, f_3) = 0$ too. Also,

$$Q(f_3, f_3) = \ell^{-2}\omega(e_2(z), \Phi(z)e_2(z)) = -\ell^{-1}\omega(e_1(z), e_2(z)) = -a_{2n-2}(z).$$

Finally,

$$Q(f_1, f_2) = -\frac{\ell^{-2}}{z}\omega(e_1(z), \Phi(z)e_1(z)) + \frac{a_{2n}(z)}{z} = \frac{2a_{2n}(z) + \alpha(z)\Gamma(z)}{z}.$$

Since $\alpha(z)$ vanishes to second order at $z = 0$ but $a_{2n}(z)$ has a simple zero, evaluating at $z = 0$ gives

$$Q_x(f_1, f_2) = 2a'_{2n}(0).$$

Therefore, with respect to the frame f_1, f_2, f_3 for V_0 the bilinear form $(Q_0)_x$ is given by

$$(Q_0)_x = \begin{bmatrix} 0 & 2a'_{2n}(0) & 0 \\ 2a'_{2n}(0) & 0 & 0 \\ 0 & 0 & -a_{2n-2}(0) \end{bmatrix}.$$

Thus, $\det((Q_0)_x) = 4a'_{2n}(0)^2 a_{2n-2}(0) \neq 0$, and hence, $Q : V \otimes V \to \mathcal{O}_C$ is non-degenerate, which establishes the desired construction.

Construction 5.3.23. Suppose (E, Φ, ω) is a Sp_{2n}-Higgs bundle over C with characteristic polynomial $p(\lambda)$ and set $D = (a_{2n})$. Let (V, ϕ, Q) be the SO_{2n+1}-Higgs bundle defined over $C \backslash D$ given by applying Construction 5.3.9. Then we can extend (V, ϕ, Q) holomorphically over D by identifying the sheaf $\mathcal{V}$ with the subsheaf of $\mathcal{M}_C(L^{-1}E \oplus L^{-n})$ comprised of sections that are holomorphic away from D with simple poles along D whose residues are valued in $\Gamma_x = \{(-i_x(w), w) \,|\, w \in L_x^{-n}\}$ for each $x \in D$ where $i_x : L_x^{-n} \to L_x^{-1}E_x$ is injective. Moreover, by choosing i_x such that $i_x(L_x^{-n}) = \ker(\Phi_x)$ the Higgs field extends holomorphically over D. In fact, by choosing i_x such that

$$\omega_x(i_x(w), \Phi'_x(i_x(w))) = -w^2 a'_{2n}(0)$$

for every $w \in L_x^{-n}$, the bilinear form Q extends holomorphically over D, and the resulting bilinear form is non-degenerate. Further, Q is ϕ-compatible and thus, (V, ϕ, Q) defines a SO_{2n+1}-Higgs bundle over C. Also, ϕ has characteristic polynomial $\lambda p(\lambda)$.

Finally, we will determine when different choices of i_x define isomorphic SO_{2n+1}-Higgs bundles. First we recall from Proposition 4.3.16 that an automorphism of a Higgs bundle with smooth spectral curve is a constant multiple of the identity.

Proposition 5.3.24. *Let (E, Φ, ω) be a generic Sp_{2n}-Higgs bundle where ω is L-valued. Suppose (V, ϕ, Q) and (V', ϕ', Q') are SO_{2n+1}-Higgs bundles defined by $\{i_x\}_{x\in D}$ and $\{i'_x\}_{x\in D}$ respectively. Then $(V, \phi, Q) \cong (V', \phi', Q')$ if and only if $i'_x = -i_x$ for each $x \in D$.*

Proof. Suppose $\psi : (V, \phi, Q) \to (V', \phi', Q')$ is an isomorphism. Since ψ commutes with the Higgs fields, ψ induces an automorphism $\psi_2 : L^{-n} \to L^{-n}$. Hence, $\psi_2 = c_2$ is a constant multiple of the identity since ψ_2 is an automorphism of a holomorphic line bundle over a compact Riemann surface. Moreover, ψ induces an isomorphism on the quotient bundle, i.e., $\psi_1 : L^{-1}E \to L^{-1}E$. In fact ψ_1 defines an automorphism of the Higgs bundle $(L^{-1}E, \Phi)$ whose spectral curve is smooth, so $\psi_1 = c_1$ is a constant multiple of the identity. Viewing $\psi : \mathcal{V} \to \mathcal{V}'$ as a sheaf map

$$\psi = \begin{bmatrix} c_1 & * \\ 0 & c_2 \end{bmatrix}.$$

However, away from D there is an isomorphism $V \cong V'$ and thus, $* = 0$. By identifying $\mathcal{V}$ and $\mathcal{V}'$ as appropriate subsheaves of $\mathcal{M}_C(L^{-1}E \oplus L^{-n})$ it follows by continuity that $* = 0$ everywhere, and hence,

$$\psi = \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}.$$

Since ψ is an isomorphism, ψ preserves the residues, i.e., $\psi(\Gamma_x) = \Gamma'_x$ for each $x \in D$. Let $(-i_x(w), w) \in \Gamma_x$ and consider

$$\psi(-i_x(w), w) = \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix} \begin{bmatrix} -i_x(w) \\ w \end{bmatrix} = \begin{bmatrix} -c_1 i_x(w) \\ c_2 w \end{bmatrix} = \begin{bmatrix} -\frac{c_1}{c_2} i_x\,(c_2 w) \\ c_2 w \end{bmatrix} = \begin{bmatrix} -i'_x(c_2 w) \\ c_2 w \end{bmatrix}.$$

Recall that the choice of homomorphisms i_x are unique up to sign, hence $i'_x = t_x i_x$ where $t_x = \pm 1$. However, we have shown $t_x = \frac{c_1}{c_2}$, which is constant, so t_x is independent of $x \in D$. Therefore, either $c_1 = c_2$, which holds if and only if $i_x = i'_x$ for each $x \in D$, or $c_1 = -c_2$, which is true if and only if $i_x = -i'_x$ for each $x \in D$. □

We conclude this section by giving a coordinate-free description for Q to extend holomorphically over D. The map $i_x : L_x^{-n} \to L_x^{-1} E_x$ is equivalent to a vector $i_x \in L_x^{n-1} E_x$. From $i_x(\ell_x^{-n}) = \ell_x^{-1} e_1$ we see $i_x = \ell_x^{n-1} e_1$ and $\Phi_x(i_x) = 0$. We claim that there exists $j_x \in L_x^{n-2} E_x$ such that $\Phi_x(j_x) = i_x$. Consider $j_x = \ell_x^{n-2} e_2$, then

$$\Phi_x(j_x) = \ell_x^{n-2} \Phi_x(e_2) = \ell_x^{n-1} e_1 = i_x.$$

Moreover,

$$\omega_x(i_x, j_x) = \omega_x(\ell_x^{n-1} e_1, \ell_x^{n-2} e_2) = \ell_x^{2n-3} \omega_x(e_1, e_2) = a_{2n-2}(x).$$

Since $\Phi_x(j_x) \neq 0$ and $\Phi_x^2(j_x) = 0$ we see $j_x \in L_x^{n-2} \otimes (\ker(\Phi_x^2)/\ker(\Phi_x))$. Hence, if instead we chose $j'_x = j_x + c i_x$, then

$$\omega_x(i_x, j'_x) = \omega_x(i_x, j_x) + c\omega_x(i_x, i_x) = \omega_x(i_x, j_x).$$

Therefore, the condition $\omega_x(i_x, j_x) = a_{2n-2}(x)$ is independent of choice of $j_x \in L_x^{n-2} \otimes (\ker(\Phi_x^2)/\ker(\Phi_x))$.

5.4 SO_{2n+1}-Spectral Data

Let (V, ϕ, Q) be a generic SO_{2n+1}-Higgs bundle with corresponding Sp_{2n}-Higgs bundle (E, Φ, ω) where $\omega : E \otimes E \to L$ is valued in L and the Higgs field ϕ has characteristic polynomial

$$p(\lambda) := \lambda^{2n} + a_2 \lambda^{2n-2} + \cdots + a_{2n},$$

which defines a smooth symplectic spectral curve $\pi : S \to C$. The spectral curve S possesses a canonical involution defined by $\sigma(\lambda) = -\lambda$. Recall that (V, ϕ, Q) is characterised by a choice of homomorphism $i_x : L_x^{-n} \to L_x^{-1} E_x$, unique up to sign, for each $x \in D := (a_{2n})$. Hence, to compute the spectral data for (V, ϕ, Q) we will compute the spectral data for (E, ϕ, ω) and then deduce further structure endowed by the family of homomorphisms $\{i_x\}_{x \in D}$. Note, since the symplectic form is L-valued we must modify the computation in the Sp_{2n}-case.

Suppose that under the spectral curve correspondence the Higgs bundle (E, Φ) corresponds to $N \in \mathrm{Pic}(S)$, i.e., $E \cong \pi_*(N)$ and $\Phi : E \to L \otimes E$ is induced by the tautological section $\lambda : N \to \pi^*(L) \otimes N$. Recall that $(E, -\Phi)$ corresponds to $\sigma^*(N) \in \mathrm{Pic}(S)$ and (E^*, Φ^*) corresponds to $N^*(R) \in \mathrm{Pic}(S)$ where $R := R_\pi$ denotes the ramification divisor associated to $\pi : S \to C$. Then, by the projection formula, the Higgs bundle (LE^*, Φ^*) corresponds to $\pi^*(L) \otimes N^*(R)$. The symplectic form ω induces an isomorphism $E \cong LE^*$, and since ω is Φ compatible it follows that ω induces an isomorphism of Higgs bundles $(E, -\Phi) \cong (LE^*, \Phi^*)$. Therefore, under the spectral curve correspondence there is an induced isomorphism $\theta : \sigma^*(N) \to \pi^*(L)N^*(R)$. Therefore, the spectral data for (E, Φ, ω) is given by $N \in \mathrm{Pic}(S)$ and a nowhere-vanishing section $\theta \in \mathrm{H}^0(S, \sigma^*(N^*)\pi^*(L)N^*(R))$. Now, we will encode the data of $\{i_x\}_{x\in D}$ into the section $\theta \in \mathrm{H}^0(S, \sigma^*(N^*)\pi^*(L)N^*(R))$.

Identifying $x \in D$ with $\lambda = 0$ in S canonically defines a divisor D_S in S comprised of the fixed points of σ, i.e., $y \in D_S$ if and only if $\sigma(y) = y$. Note also that $\pi(y) = x \in D$, and hence, $\theta_y \in N_y^{-2}L_xR_y$. By choosing a local holomorphic coordinate (U, z) centred at $x \in C$ trivialising L recall that the characteristic polynomial locally factors

$$p(\lambda, z) = (\lambda^2 + \alpha)\widehat{p}(\lambda, z).$$

such that the kernel of the sheaf map $\lambda^2 + \alpha$ induces a rank 2 subbundle $F \subset E|_U$. Moreover, the element $y \in D_S$ such that $\pi(y) = x$ belongs to the restriction of $N \in \mathrm{Pic}(S)$ to the spectral curve defined by $\lambda^2 + \alpha$, which we will again denote by $\pi : S \to C$ and denote the restriction of N by N. Thus, $F \cong \pi_*(N)$ and we have distilled the problem to studying the isomorphism in a rank 2 bundle. Let $\ell := \ell(z)$ be a trivialising section of L over U. Now, we will construct a coordinate w centred y such that $z = w^2$. Indeed, by changing z if necessary we may assume that $\alpha = -z\ell^2$, and hence, S is defined by $\lambda^2 = z\ell^2$. By setting $\lambda = w\ell$ it follows that $z = w^2$. Of course, we need to determine the fibre of F at x, i.e., F_x, which can be obtained by the sheaf $\mathcal{F}$, namely

$$F_x \cong \mathcal{F} \otimes \mathcal{O}_x/\mathfrak{m}_x$$

where $\mathfrak{m}_x \subset \mathcal{O}_x$ is the unique maximal ideal. Since $F \cong \pi_*(N)$ and the map π is given by $z = w^2$ it follows that

$$F_x \cong \mathcal{N} \otimes \mathcal{O}_y/\mathfrak{m}_y^2.$$

Hence, if $n \in N_y$ defines a basis for N_y, then $n, nw \mod w^2$ defines a basis for F_x. For brevity we set $e_1 = nw$ and $e_2 = n$. Since $\theta_y \in N_y^{-2}L_xR_y$ we may write $\theta_y = \frac{an^{-2}\ell_x}{w}$ for some $a \in \mathbb{C}^*$. Now, the vector bundle V is determined by a family of homomorphisms $\{i_x\}_{x\in D}$ satisfying the quadratic condition

$$\omega_x(i_x, j_x) = a_{2n-2}(x)$$

which is independent of $j_x \in L_x^{n-2} \otimes \ker(\Phi_x^2)/\ker(\Phi_x)$ such that $\Phi_x(j_x) = i_x$. Since $\Phi_x(i_x) = 0$ we write $i_x = ue_1\ell_x^{n-1}$, then we set $j_x = ue_2\ell_x^{n-2}$. Note that the family $\{i_x\}_{x\in D}$ is completely determined by the family $\{j_x\}_{x\in D}$. Now,

$$\omega(i_x, j_x) = u^2\ell_x^{2n-3}\omega(e_1, e_2). \tag{5.22}$$

Recall from Proposition 5.1.10 that the symplectic form $\omega : F \otimes F \to L|_U$ is defined by

$$\omega(a, b) = \mathrm{Tr}_{S/U}(a \otimes \theta\sigma^*(b)).$$

Hence, since $\sigma(y) = y$

$$\omega(e_1, e_2) = \mathrm{Tr}_{S/U}\left(nw\frac{an^{-2}\ell_x}{w}n\right) = a\ell_x \,\mathrm{Tr}_{S/U}(1) = 2a\ell_x \tag{5.23}$$

where $\mathrm{Tr}_{S/U}(1) = 2$ since S has degree 2. Therefore, combining (5.22) and (5.23) gives

$$2au^2 = a_{2n-2}(0). \tag{5.24}$$

Now, recall by the adjunction formula that $\mathcal{O}_S(R) \in \pi^*(L^{2n-1})$, which is locally determined by

$$\frac{1}{w} \mapsto \frac{\partial_\lambda p(\lambda, z)}{w} \mod w^2.$$

Since $\frac{\partial_\lambda p(\lambda,z)}{w} = 2a_{2n-2}(z)\ell \mod w^2$ it follows that

$$\theta_y = 2aa_{2n-2}(0)n^{-2}\ell_x^{2n}.$$

Finally, notice that $\theta_y j_x^2 \in L_x^{4n-4}$, and

$$\theta_y j_x^2 = 2au^2 a_{2n-2}(0)\ell_x^{4n-4}.$$

By (5.24) we see $\theta_y j_x^2 = a_{2n-2}(x)^2$, or, equivalently,

$$\theta_y = \left(\frac{a_{2n-2}(x)}{j_x}\right)^2.$$

Therefore, we have shown that the family of homomorphisms $\{j_x\}_{x\in D}$, which determines the family of homomorphisms $\{i_x\}_{x\in D}$ defines a square root of the homomorphisms θ_y.

Proposition 5.4.1. *Let (V, ϕ, Q) be a generic SO_{2n+1}-Higgs bundle with corresponding Sp_{2n}-Higgs bundle (E, Φ, ω) that has characteristic polynomial $p(\lambda) \in \mathrm{H}^0(C, \pi^*(L^{2n}))$ defining a smooth symplectic spectral curve $\pi : S \to C$. Suppose that, under the spectral curve correspondence, the Sp_{2n}-Higgs bundle (E, Φ, ω) has spectral data (N, θ) where $N \in \mathrm{Pic}(S)$ such that $\pi_*(N) \cong E$ and $\theta \in \mathrm{H}^0(S, \sigma^*(N^*)\pi^*(L^{2n})N^*(R))$ is a nowhere vanishing section. Let D_S be denote the set of fixed points of the canonical involution $\sigma : S \to S$. Then, the family of homomorphisms $\{i_x\}_{x\in D}$ defining the SO_{2n+1}-Higgs bundle (V, ϕ, Q) is equivalent to a choice of square root of θ_y for each $y \in D_S$.*

Finally, we will reformulate the spectral data in the language of torsors of abelian varieties. First, we let $T = S/\sigma$, then we may factor $\pi : S \to C$ by

$$\begin{tikzcd} S \arrow[r, "p"] \arrow[dr, "\pi"'] & T \arrow[d, "q"] \\ & C \end{tikzcd}$$

where $p : S \to T$ denoted the canonical projection map, which is a branched double cover. Since p is ramified recall that the pullback map p^* is injective and the Prym variety $\mathrm{Prym}(S,T)$ is connected, which implies $U \in \mathrm{Prym}(S,T)$ if and only if $p^*(\mathrm{Nm}_{S/T}(U)) \cong \mathcal{O}_S$. By Lemma 5.1.14 it follows that $U \in \mathrm{Prym}(S,T)$ if and only if $\sigma^*(U) \cong U^*$.

For each $m \in \mathbb{Z}$ we define the $\mathrm{Prym}(S,T)$-torsor

$$\mathcal{T}_m = \{U \in \mathrm{Pic}(S) \mid \mathrm{Nm}_{S/T}(U) \cong q^*(L^m)\}.$$

The condition $\mathrm{Nm}_{S/T}(U) \cong q^*(L^m)$ is equivalent to $U\sigma^*(U) \cong \pi^*(L^m)$, which is equivalent to a nowhere vanishing section $\chi \in \mathrm{H}^0(S, \sigma^*(U^*)U^*\pi^*(L^m))$ that is unique up to scale. Hence,

$$\mathcal{T}_m = \{(U,[\chi]) \mid U \in \mathrm{Pic}(S),\ \chi \in \mathrm{H}^0(S, \sigma^*(U^*)U^*\pi^*(L^m)) \text{ nowhere vanishing}\}.$$

Thus, the spectral data for Sp_{2n}-Higgs bundles with L-valued symplectic form is given by the $\mathrm{Prym}(S,T)$-torsor $\mathcal{T}_{2n}$. Moreover, the spectral data for generic SO_{2n+1}-Higgs bundles is given by

$$\widetilde{\mathcal{T}}_{2n} := \mathcal{T}_{2n} \times \{\{j_x\}/_{\pm 1} \mid \Phi_x(j_x) = i_x,\ \omega(i_x, j_x) = a_{2n-2}(x)\}.$$

Generic SO_{2n+1}-Higgs bundles defining Sp_{2n}-Higgs bundles is described by the projection map

$$\mathrm{pr} : \widetilde{\mathcal{T}}_{2n} \to \mathcal{T}_{2n}.$$

To compute the degree of the projection map recall that $a_{2n} \in \mathrm{H}^0(C, L^{2n})$, so the zero divisor D has cardinality $|D| = \deg(L^{2n}) = 2n\deg(L)$ since a_{2n} only has simple zeros. Moreover, there is a choice of two homomorphisms i_x for each $x \in D$, i.e., there are $2^{2n\deg(L)}$ different family of $\{i_x\}_{x\in D}$ that define (possibly isomorphic) SO_{2n+1}-Higgs bundles whose associated Sp_{2n}-Higgs bundle is the same. However, by Proposition 5.3.24 the families $\{i_x\}_{x\in D}$ and $\{-i_x\}_{x\in D}$ define isomorphic Higgs bundles so by passing to isomorphism classes it follows that $\mathrm{pr} : \widetilde{\mathcal{T}}_{2n} \to \mathcal{T}_{2n}$ has degree $2^{2n\deg(L)-1}$.

We now endeavour to compute the number of connected components of $\widetilde{\mathcal{T}}_{2n}$ and describe them exactly. In particular, we will show that the components are torsors of the dual Prym variety, $\mathrm{Prym}(S,T)^\vee$. First consider the following lemma.

Lemma 5.4.2. *There is a canonical isomorphism*

$$\mathrm{Prym}(S,T)^\vee \cong \mathrm{Prym}(S,T)/p^*\,\mathrm{Jac}(T)[2].$$

Proof. Since $\mathrm{Prym}(S,T) = \ker(\mathrm{Nm}_{S/T})$ consider the short exact sequence

$$0 \to \mathrm{Prym}(S,T) \to \mathrm{Jac}(S) \xrightarrow{\mathrm{Nm}_{S/T}} \mathrm{Jac}(T) \to 0.$$

By dualising the sequence and identifying $\mathrm{Nm}_{S/T}^\vee = p^*$ it follows that $\mathrm{Prym}(S,T)^\vee \cong \mathrm{Jac}(S)/p^*\,\mathrm{Jac}(T)$. Restricting to $\mathrm{Prym}(S,T)$ defines an isomorphism

$$\mathrm{Prym}(S,T)^\vee \cong \mathrm{Prym}(S,T)/(p^*\,\mathrm{Jac}(T) \cap \mathrm{Prym}(S,T)).$$

Thus, it suffices to show $p^* \operatorname{Jac}(T) \cap \operatorname{Prym}(S,T) \cong p^* \operatorname{Jac}(T)[2]$. Indeed, let $\ell := u - v \in \operatorname{Jac}(T)$. Then, $\operatorname{Nm}_{S/T}(p^*(\ell)) = p(p^{-1}(\ell)) = 2\ell$ and thus, $p^*(\ell) \in \operatorname{Prym}(S,T)$ if and only if $2\ell = 0$, i.e., $\ell \in \operatorname{Jac}(T)[2]$, and the result follows. □

Since $\operatorname{Prym}(S,T)$ defines a complex torus the squaring map $s : \mathcal{T}_n \to \mathcal{T}_{2n}$ is surjective. Note, $s(U_1) = s(U_2)$, i.e., $U_1^2 \cong U_2^2$ if and only if $U_1 \cong U_2 \otimes A$ for some $A \in \operatorname{Prym}(S,T)[2]$. Notice that $\sigma^*(A) \cong A$, and hence, the involution $\sigma : S \to S$ lifts to an isomorphism $\widetilde{\sigma} : A \to A$, i.e., $\widetilde{\sigma} = c\,\mathrm{id}_A$ for some $c \in \mathbb{C}^*$. However, we may assume without loss of generality that $c = 1$, so σ has two involutive lifts $\pm\widetilde{\sigma}$. We may to extend this assignment to $\widetilde{\mathcal{T}}_{2n}$ by

$$\mathcal{T}_n \ni (U, [\chi]) \mapsto (U^2, [\chi^2], [\chi|_{D_S}]/_{\pm 1}) \in \widetilde{\mathcal{T}}_{2n}$$

where we recall that by Proposition 5.4.1 the family $\{j_x\}_{x \in D}$ is entirely equivalent to a choice of square root of χ_y^2 for each $y \in D_S$.

Suppose $(U_1^2, [\chi_1^2], [\chi_1|_{D_S}]) \cong (U_2^2, [\chi_2^2], [\chi_2|_{D_S}])$. Then $U_1 \cong U_2 \otimes A$ for some some $A \in \operatorname{Prym}(S,T)[2]$ and since $[\chi_1^2] = [\chi_2^2]$, we see $\chi_1 = \chi_2 \otimes \widetilde{\sigma}$. Moreover, from $[\chi_1|_{D_S}]/_{\pm 1} = [\chi_2|_{D_S}]/_{\pm 1}$ it follows that $\widetilde{\sigma}|_{D_S} = +1$, and hence, $A/_{\widetilde{\sigma}}$ defines a holomorphic line bundle over T. Therefore, $A \cong p^*(B)$ for some $B \in \operatorname{Jac}(T)[2]$, and we have shown that the induced assignment

$$\mathcal{T}_n / p^* \operatorname{Jac}(T)[2] \to \widetilde{\mathcal{T}}_{2n}$$

defines an injective map.

Proposition 5.4.3. *The map*

$$\mathcal{T}_n / p^* \operatorname{Jac}(T)[2] \to \mathcal{T}_{2n}$$

that is induced from the squaring map has degree $2^{2n\deg(L)-2}$.

Proof. The dimension of $\operatorname{Jac}(S)$ and $\operatorname{Jac}(T)$ are given by $g(S)$ and $g(T)$ respectively. Moreover, the dimension of $\operatorname{Prym}(S,T)$ is given by $g(S) - g(T)$, and since $p : S \to T$ is a branched double cover it follows that the degree of the map is given by $2^{2(g(S)-2g(T))}$. The ramification points of the map $p : S \to T$ are precisely the fixed points of the involution σ of which there are $2n\deg(L)$ points. Hence, by the Riemann-Hurwitz formula

$$2g(S) - 2 = 2(2g(T) - 2) + 2n(\deg(L)).$$

Solving for $g(S) - 2g(T)$ gives

$$g(S) - 2g(T) = n\deg(L) - 1.$$

Thus, the map of interest has degree

$$2^{2(g(S)-2g(T))} = 2^{2n\deg(L)-2}.$$

□

Since the assignment $\mathrm{pr} : \widetilde{\mathcal{T}}_{2n} \to \mathcal{T}_{2n}$ has degree $2^{2n\deg(L)-1}$ and the diagram

$$\begin{tikzcd} \mathcal{T}_n/p^*\operatorname{Jac}(T)[2] \arrow[r] \arrow[dr, "s"'] & \widetilde{\mathcal{T}}_{2n} \arrow[d, "\mathrm{pr}"] \\ & \mathcal{T}_{2n} \end{tikzcd}$$

commutes, it follows that $\widetilde{\mathcal{T}}_{2n}$ has two connected components that are isomorphic to $\mathcal{T}_n/p^*\operatorname{Jac}(T)[2]$. However, $\mathcal{T}_n/p^*\operatorname{Jac}(T)[2]$ defines a torsor of $\operatorname{Prym}(S,T)/p^*\operatorname{Jac}(T)[2]$, which is isomorphic $\operatorname{Prym}(S,T)^\vee$ by Lemma 5.4.2. Therefore, we have shown that isomorphism classes of SO_{2n+1}-Higgs bundles with generic spectral curve S is in one-to-one correspondence with two distinct torsors of $\operatorname{Prym}(S,T)^\vee$.

5.5 SO_{2n+1}-Hitchin Fibration

Consider a characteristic polynomial associated to a SO_{2n+1}-Higgs bundle, i.e.,

$$p(\lambda) = \lambda(\lambda^{2n} + a_2\lambda^{2n-2} + \cdots + a_{2n}).$$

It is well known that the coefficients $a_2, \ldots, a_{2n}$ form a homogenous basis for the invariant polynomials on $\mathfrak{so}_{2n+1}$. Thus, the SO_{2n+1}-Hitchin fibration is the map

$$h : \mathcal{M}_{\mathrm{SO}_{2n+1}} \to \bigoplus_{i=1}^{n} \mathrm{H}^0(C, L^{2i})$$

that sends

$$h(V,\phi,Q) = (a_2, \ldots, a_{2n}).$$

Notice that the SO_{2n+1} and Sp_{2n} Hitchin bases are the same so a computation of its dimension can be found in (5.12). Now, a generic point $(a_2, \ldots, a_{2n}) \in \bigoplus_{i=1}^{n} \mathrm{H}^0(C, L^{2i})$ defines a generic SO_{2n+1}-spectral curve. Recall that a generic SO_{2n+1}-spectral curve canonically defines a smooth Sp_{2n}-spectral curve $\pi : S \to C$. In (5.4) we established that the set of isomorphism classes of SO_{2n+1}-Higgs bundles that correspond to S is in one-to-one correspondence with two torsors of $\operatorname{Prym}(S,T)^\vee$. Hence, we have computed the generic fibres of the SO_{2n+1}-Hitchin fibration.

Theorem 5.5.1. *Suppose $(a_2, \ldots, a_{2n})$ is a generic point in the SO_{2n+1}-Hitchin base that defines a smooth Sp_{2n}-spectral curve $\pi : S \to C$. Then the fibre $h^{-1}(a_2, \ldots, a_{2n})$ consists of two connected components each of which are torsors of $\operatorname{Prym}(S,T)^\vee$.*

Therefore, each connected component of a generic fibre of the SO_{2n+1}-Hitchin fibration is isomorphic as algebraic varieties to the dual Prym variety defining the corresponding generic fibre of the Sp_{2n}-Hitchin fibration. This establishes Langlands duality between Sp_{2n} and SO_{2n+1} Hitchin fibrations since ${}^L\mathrm{Sp}_{2n} \cong \mathrm{SO}_{2n+1}$.

Chapter 6

Type D_n-Higgs Bundles

In this final chapter, we compute the generic fibres of the SO_{2n}-Hitchin fibration, where we extend Hitchin's work in [Hit87b] to the case we replace the canonical bundle with a basepoint-free positive degree holomorphic line bundle. Similar to the SO_{2n+1} case, generic spectral curves are not smooth due to the *Pfaffian*. For $n > 1$, the singularities are generically ordinary double points, which are mild singularities, and the spectral curves are irreducible. Although there is no canonical reduction to a smooth spectral curve as in the SO_{2n+1} case, we resort to *normalising* the spectral curve. Normalising involves removing the singularities to obtain an open Riemann surface, then completing the curve in the sense of compactifying. Instead of resorting to the complex algebraic method, we give an explicit complex analytic normalisation, which develops more machinery we then use to establish the SO_{2n} spectral curve correspondence, which is in terms of the normalisation. The generic fibres are torsors of a Prym variety corresponding to a double étale covering, which we showed to be self-dual in Chapter 3. Since ${}^L\mathrm{SO}_{2n} \cong \mathrm{SO}_{2n}$ this realises Langlands duality in the SO_{2n} Hitchin fibration. For proof ${}^L\mathrm{SO}_{2n} \cong \mathrm{SO}_{2n}$ consult Appendix A.

6.1 SO_{2n}-Higgs Bundles and generic SO_{2n}-Spectral Curves

Principal SO_{2n}-bundles over C correspond to rank $2n$-holomorphic vector bundles over C equipped with a non-degenerate, bilinear, symmetric 2-form $Q : E \otimes E \to \mathcal{O}_C$. Hence, SO_{2n}-*Higgs bundles* are triples (E, ϕ, Q) where $E \to C$ is a rank $2n$-holomorphic vector bundle, $\phi : E \to L \otimes E$ is a holomorphic vector bundle homomorphism, and $Q : E \otimes E \to \mathcal{O}_C$ is a symplectic form such that

$$Q(\phi v, w) + Q(v, \phi w) = 0. \tag{6.1}$$

Fix a SO_{2n}-Higgs bundle (E, ϕ, Q). The eigenvalues of the Higgs field are generically distinct. Hence, to understand the characteristic polynomial of (E, ϕ, Q), let $A \in \mathfrak{so}_{2n}(\mathbb{C})$

have distinct eigenvalues λ_i with corresponding eigenvectors $v_i \in \mathbb{C}^{2n}$. Notice that,

$$\lambda_i Q(v_i, v_j) = Q(Av_i, v_j) = -Q(v_i, Av_j) = -\lambda_j Q(v_i, v_j).$$

Hence,

$$(\lambda_i + \lambda_j)Q(v_i, v_j) = 0. \tag{6.2}$$

Thus, by (6.2) we see $Q(v_i, v_i) = 0$ whenever $\lambda_i \neq 0$. Since Q is non-degenerate it follows that eigenvalues occur in opposite pairs. Therefore, the characteristic polynomial of A is even, i.e.

$$\det(x - A) = x^{2n} + a_2 x^{2n-2} + \cdots + a_{2n}. \tag{6.3}$$

However, the polynomial $a_{2n} = \det(A)$ is the square of a polynomial p_n called the *Pfaffian*. Moreover, the polynomials $a_2, \ldots, a_{2n-2}, p_n$ in (6.3) form a homogeneous basis for the invariant polynomials of $\mathfrak{so}_{2n}(\mathbb{C})$. Thus, the characteristic polynomial for (E, ϕ, Q) is of the form

$$\det(\lambda - \phi) = \lambda^{2n} + a_2 \lambda^{2n-2} + \cdots + p_n^2 \tag{6.4}$$

where $a_{2i} \in \mathrm{H}^0(C, L^{2i})$ for $i = 1, \ldots, n-1$ and $p_n \in \mathrm{H}^0(C, L^n)$. Hence, the spectral curve S of (E, ϕ, Q) possesses a canonical involution $\sigma(\lambda) = -\lambda$. Moreover, the fixed points of σ are precisely where $\lambda = 0$ and $p_n = 0$.

Now, since L is basepoint-free, L^n is basepoint-free [1]. Therefore, by Corollary 2.2.12 we may fix $p_n \in \mathrm{H}^0(C, L^n)$ whose zeros are simple. Now, we compute generic SO_{2n}-spectral curves with respect to the fixed section p_n.

Lemma 6.1.1. *The singular set of a generic* SO_{2n}*-spectral curve is contained in the set of fixed points of the involution* $\sigma(\lambda) = -\lambda$.

Proof. Consider sections of L^{2n} of the form

$$s = \lambda^{2n} + a_2 \lambda^{2n-2} + \cdots + p_n^2$$

where $a_{2i} \in \mathrm{H}^0(C, L^{2i})$ for $i = 1, \ldots, n-1$. Allowing the a_i to vary, the zero set of s defines a linear system of divisors over Y. To compute the base locus consider the sections

$$s_1 = \lambda^{2n} + p_n^2 \quad \text{and} \quad s_2 = \lambda^{2n} + a_2 \lambda^{2n-2} + p_n^2$$

where $a_2 \in \mathrm{H}^0(C, L^2)$ is not identically zero. Then, when s_1 and s_2 mutually vanish we see $a_2 \lambda^{2n-2} = 0$. Since L is basepoint-free we may choose $a_2 \neq 0$, hence $\lambda = 0$, and thus, $p_n = 0$. Clearly, the points where $\lambda = 0$ and $p_n = 0$ belong to the base locus of the linear system. Therefore, the base locus of the system are precisely the fixed points of the involution σ, and by Bertini's theorem the result follows. $\square$

[1] Let $p \in C$. Since L is basepoint free we may choose a section $s : C \to L$ such that $s(p) \neq 0_p$. Then $s^n : C \to L^n$ is a holomorphic section of L^n and $s^n(p) \neq 0_p$.

Of course, Bertini's theorem only ensures smoothness away from the base locus and does not determine smoothness at the base locus. Suppose now (E, ϕ, Q) has a generic spectral curve $\pi : S \to C$ that is defined by

$$p(\lambda) = \lambda^{2n} + a_2\lambda^{2n-2} + \cdots + p_n^2 \tag{6.5}$$

where p_n has simple zeros. We make one assumption namely, that a_{2n-2} and p_n share no common zeros. To see that this assumption is valid, by Lemma 2.2.10 we may choose sections $s, t \in \mathrm{H}^0(C, L)$ such that s and t have no zeros in common. Thus, $s^n \in \mathrm{H}^0(C, L^n)$ and $t^{2n-2} \in \mathrm{H}^0(C, L^{2n-2})$ have no zeros in common. Throughout the rest of this chapter, generic spectral curves satisfy this further assumption. Under these assumptions the fixed points of the involution σ are ordinary double points of S.

Lemma 6.1.2. *Suppose $\pi : S \to C$ is a spectral curve defined by* (6.5)*. Then the fixed points of the involution $\sigma(\lambda) = -\lambda$ are ordinary double points of S.*

Proof. Let $x \in C$ be a zero of p_n and choose a local coordinate z centred at x. By shrinking the coordinate neighbourhood if necessary, we may trivialise L and locally view the tautological section λ as a holomorphic function. Then, (λ, z) defined local coordinates on Y centred at $0 \in L_x$. In this coordinate system, $(0,0) \in S$ and the spectral curve is locally defined about $(0,0)$ by $p(\lambda, z) = 0$. Since the zeros of p_n are simple we may write

$$p_n(z) = az + \text{higher order terms}.$$

where $a \neq 0$. Since a_{2n-2} and p_n share no zeros we may write

$$a_{2n-2}(z) = -b^2 + \text{higher order terms}$$

where $b \neq 0$. It follows that the equation defining S has a local Taylor series expansion

$$p(\lambda, z) = (az + b\lambda)(az - b\lambda) + \text{higher order terms}.$$

Therefore, $(0,0)$ is an ordinary double point of S. □

Combining Lemma 6.1.1 and Lemma 6.1.2 we obtain the following immediate corollary.

Corollary 6.1.3. *The singular points of S are precisely the fixed points of $\sigma(\lambda) = -\lambda$, i.e., the set of points where $\lambda = 0$ and $p_n = 0$. Hence, the spectral curve is not smooth since* $\deg(L) > 0$.

Although generic SO_{2n}-spectral curves are not smooth, generic SO_{2n}-spectral curves as defined in (6.5) are irreducible when $n > 1$.

Proposition 6.1.4. *If $n > 1$, then a generic SO_{2n}-spectral curve $\pi : S \to C$ as defined in* (6.5) *is irreducible as a scheme.*

Proof. Suppose $S = S_1 \cup S_2$ where S_1 and S_2 are spectral curves of degree d_1 and d_2 respectively such that $d_1 + d_2 = 2n$. By Corollary 6.1.3 the singularities of S are fixed points of the involution σ, of which there are $\deg(L^n) = n\deg(L)$ such points. The points of intersection of S_1 and S_2 are singularities of S, hence S_1 and S_2 can intersect in at most $n\deg(L)$ points. Moreover, since each singularity is an ordinary double point, each point of intersection has multiplicity 1, i.e., a transverse intersection of smooth points. However, taking the resultant, S_1 and S_2 intersect with multiplicity $\deg(L^{d_1 d_2}) = d_1 d_2 \deg(L)$, and each zero of the resultant must be simple since the intersection multiplicities all equal 1. Thus,

$$d_1 d_2 \deg(L) \leq \frac{d_1 + d_2}{2} \deg(L).$$

Since $\deg(L) \neq 0$ this is only the case when $d_1 = d_2 = 1$. Therefore, if $n > 1$, then S is irreducible. $\square$

6.2 Normalisation of Generic Spectral Curve

The process of *normalisation* canonically associates a compact Riemann surface to a singular complex analytic variety such that away from the singularities, the curves are isomorphic. Consider the formal definition and a uniqueness result, namely that when a normalisation exists, it is unique up to isomorphism.

Definition 6.2.1. Let P be a singular complex analytic variety with singular set P_{sing}. A *normalisation of* P is a pair $(\widetilde{P}, \nu)$ where $\nu : \widetilde{P} \to P$ is a holomorphic map such that

(i) ν is surjective;

(ii) $\nu^{-1}(P_{\text{sing}})$ is finite;

(iii) $\nu : \widetilde{P} \setminus \nu^{-1}(P_{\text{sing}}) \to P \setminus P_{\text{sing}}$ is injective.

Lemma 6.2.2. *Suppose $(\widetilde{P}, \nu)$ and $(\widetilde{Q}, \theta)$ are two normalisations of a singular complex analytic variety P with singular set P_{sing}. Then there exists a biholomorphism $f : \widetilde{P} \to \widetilde{Q}$ such that $\nu = \theta \circ f$.*

Proof. Consider the biholomorphic mapping

$$\widetilde{P} \setminus \nu^{-1}(P_{\text{sing}}) \xrightarrow{\ \nu\ } P \setminus \Pi \xrightarrow{\ \theta^{-1}\ } \widetilde{Q} \setminus \theta^{-1}(P_{\text{sing}}).$$

This mapping extends continuously to all of $\widetilde{P}$. The extended map is a surjective holomorphic map, which is injective on a dense open subset of $\widetilde{P}$. Hence, the map is a biholomorphism. Denote this map by $f : \widetilde{P} \to \widetilde{Q}$ then, immediately, $\nu = \theta \circ f$. $\square$

Thus, we dedicate the rest of this section to constructing the normalisation of a generic spectral curve $\pi : S \to C$ since by Lemma 6.2.2 the normalisation is unique. The normalisation of a complex analytic variety is isomorphic away from the singular points. Hence, we only need to describe the normalisation about singularities.

Let $y \in S$ be a singular point and suppose that locally in a deleted neighbourhood of y, the spectral curve is the disjoint union of m irreducible analytic curves

$$S = S_1 \cup \cdots \cup S_m.$$

By Corollary 4.2.7 the polynomial $p(\lambda)$ defining S factors into irreducible polynomials each defining S_i, i.e. $p = p_1 \cdots p_m$. By Proposition 4.2.6 the irreducible factors correspond to the orbits of monodromy. Thus, we have reduced to normalising each irreducible component. Choose local coordinate (U, z) centred at $x = \pi(y)$ such that U is biholomorphic to the unit disc Δ. Fix a local irreducible component V defined by $q(\lambda, z)$, and suppose the corresponding zeros are $\lambda_1(z), \ldots, \lambda_k(z)$. Denote the induced monodromy by ρ, which has order t. Let $\widetilde{\Delta}$ be another copy of the unit disc, and consider the t-fold map

$$\widetilde{\Delta}^* \ni w \mapsto w^t \in \Delta^*. \tag{6.6}$$

The map in (6.6) is surjective and holomorphic. Moreover, under this map, one loop around the origin in $\widetilde{\Delta}^*$ induces t-loops around the origin in Δ^*. Since the monodromy ρ has order t, the induced monodromy from pulling back $q(\lambda, z)$ under (6.6) is trivial. Therefore, there exists holomorphic functions $\lambda_1(w), \ldots, \lambda_k(w)$ on $\widetilde{\Delta}^*$ such that

$$q(\lambda_i(w^t), w^t) = 0$$

for $i = 1, \ldots, k$. An identical argument to the proof of Proposition 4.2.6 show that each $\lambda_i(w)$ extends to a holomorphic function over $\widetilde{\Delta}$, which we denote by $\lambda_i(w)$.

Proposition 6.2.3. *The irreducible analytic curve component V is biholomorphic to the punctured unit disc away from the singularity.*

Proof. Adopting the notation of the foregoing fix a $\lambda_i(w)$ and consider the holomorphic map $g : \widetilde{\Delta} \to V$ defined by

$$g(w) = (\lambda_i(w^t), w^t).$$

To see that this function is injective suppose $g(w) = g(u)$, i.e.

$$(\lambda_i(w^t), w^t) = (\lambda_i(u^t), u^t).$$

Then, $u = \exp(2\pi i l/t)w$ for some $l \in \mathbb{Z}$ and thus,

$$\lambda_i(w^t) = \lambda_i(\exp(2\pi i l)w^t)$$

where $\exp(2\pi il)w^t$ denotes the value of w^t looping around the origin l times. Since the order of monodromy is t for $\lambda_i(w^t)$ to remain unchanged it follows that $l = tr$ for some $r \in \mathbb{Z}$. Thus,

$$u = \exp(2\pi itr/t)w = \exp(2\pi ir)w = w.$$

Therefore, the holomorphic function g is injective. Since the t-fold map in (6.6) is surjective and since the polynomial $q(\lambda, z)$ is irreducible, the monodromy acts transitively so as w varies in $\widetilde{\Delta}$ one sees that $\lambda_i(w^t)$ assumes all values of $\lambda_1(z), \ldots, \lambda_k(z)$ where $z \in \Delta$. Therefore, g maps $\widetilde{\Delta}$ onto V. Finally, from the implicit function theorem $V \setminus \{(0,0)\}$ defines a Riemann surface with holomorphic coordinate z, and the map induced map $g : \widetilde{\Delta}^* \to V \setminus \{(0,0)\}$ is holomorphic since its local representation is $z = w^t$. It follows easily that g is a biholomorphism. $\square$

Set $S_{\text{reg}} := S \setminus S_{\text{sing}}$, i.e. S_{reg} denotes the set of regular points of the spectral curve S. An immediately consequence of irreducibility of S is that S_{reg} is connected. For simplicity we first consider the case S has one singular point $y \in S_{\text{sing}}$. Suppose locally about y the spectral curve S has m irreducible curve components. From Proposition 6.2.3 we obtain m unit discs Δ_j and m holomorphic functions $g_j : \Delta_j^* \to S_{\text{reg}}$, that are biholomorphic onto their image sets. Define the set

$$\widetilde{S} := S_{\text{reg}} \bigcup_{g_1} \Delta_1 \cdots \bigcup_{g_m} \Delta_m,$$

where $S_{\text{reg}} \bigcup_{g_1} \Delta_1$ is defined by

$$S_{\text{reg}} \bigcup_{g_1} \Delta_1 := (S_{\text{reg}} \sqcup \Delta_1)/_{\sim}$$

where $\sim$ is the equivalence relation generated by $p \sim g_1(p)$. Since $g_1 : \Delta_1^* \to S_{\text{reg}}$ is biholomorphic onto its image set, $S_{\text{reg}} \bigcup_{g_1} \Delta_1$ can be endowed with holomorphic coordinates, thus inheriting the structure of a Riemann surface. Repeating this successively births the desired Riemann surface

$$\widetilde{S} := S_{\text{reg}} \bigcup_{g_1} \Delta_1 \bigcup_{g_2} \Delta_2 \cdots \bigcup_{g_m} \Delta_m.$$

The compactness is clear. The desired normalisation $\nu : \widetilde{S} \to S$ is given by

$$\nu(p) = \begin{cases} p & \text{for } p \in S_{\text{reg}} \\ g_i(p) & \text{for } p \in \Delta_i \end{cases}.$$

To see that ν is a normalisation, notice that away from the singular point y, the map is given by $\nu(p) = p$, which is injective. The set of points in $\widetilde{S}$ that map to y under ν are precisely the origin of each disc, so $|\nu^{-1}(y)| = m$, and the surjective condition is clear.

Suppose instead the spectral curve has several singularities $S_{\text{sing}} := \{y_1, \ldots, y_l\}$ then the construction generalises in the obvious manner. Namely, for each y_i we obtain m_i discs δ_{i_j} and m_i holomorphic functions $g_{i_j} : \Delta^*_{i_j} \to S_{\text{reg}}$ that are biholomorphic onto their image sets. Then we define

$$\begin{aligned}\widetilde{S} := S_{\text{reg}} \bigcup_{g_{1_1}} \Delta_{1_1} \bigcup_{g_{1_2}} \Delta_{1_2} \cdots \bigcup_{g_{1_{m_1}}} \Delta_{1_{m_1}} \\ \bigcup_{g_{2_1}} \Delta_{2_1} \bigcup_{g_{2_2}} \Delta_{2_2} \cdots \bigcup_{g_{2_{m_2}}} \Delta_{2_{m_2}} \\ \vdots \\ \bigcup_{g_{l_1}} \Delta_{l_1} \bigcup_{g_{l_2}} \Delta_{l_2} \cdots \bigcup_{g_{l_{m_l}}} \Delta_{l_{m_l}}.\end{aligned}$$

We define $\nu : \widetilde{S} \to S_{\text{reg}}$ by

$$\nu(p) = \begin{cases} p & \text{for } p \in S_{\text{reg}} \\ g_{i_j}(p) & \text{for } p \in \Delta_{i_j} \end{cases}.$$

Throughout the rest of the chapter, we let $\nu : \widetilde{S} \to S$ denote the normalisation of the generic spectral curve $\pi : S \to C$.

6.3 SO_{2n}-Spectral Curve Correspondence

In general, the normalisation of a spectral curve is not a spectral curve. Hence, we modify arguments from the spectral curve correspondence to obtain the SO_{2n}-spectral curve correspondence. Consider the branched cover $\widetilde{\pi} : \widetilde{S} \to C$ defined by the diagram

$$\begin{tikzcd} \widetilde{S} \arrow[r, "\nu"] \arrow[dr, "\widetilde{\pi}"'] & S \arrow[d, "\pi"] \\ & C \end{tikzcd}.$$

By the universal property of the normalisation, the involution $\sigma : S \to S$ has a unique lift $\widetilde{\sigma} : \widetilde{S} \to \widetilde{S}$, which is an involution. In fact, the lift $\widetilde{\sigma}$ acts freely on $\widetilde{S}$.

Lemma 6.3.1. *The involution $\widetilde{\sigma} : \widetilde{S} \to \widetilde{S}$ has no fixed points.*

Proof. Away from the fixed points of the involution σ, the map $\nu : \widetilde{S} \to S$ is a biholomorphism. Hence, it suffices to show that the points in $\widetilde{S}$ that map to fixed points of σ under ν are not fixed points of $\widetilde{\sigma}$. Let $y \in S$ such that $\sigma(y) = y$. Recall that there are two eigenvalues passing through y, say λ_1 and $-\lambda_1$. Hence, in a sufficiently small open neighbourhood V of y, the pre-image $\nu^{-1}(V)$ has two connected components U^+ and U^-. Thus, $\nu^{-1}(y) = \{y^+, y^-\}$ where $y^\pm \in U^\pm$. Hence, it suffices to prove $\widetilde{\sigma}(y^\pm) = y^\mp$. However, this is clear since $\sigma(\lambda_1) = -\lambda_1$, hence $\widetilde{\sigma}$ interchanges U^+ and U^-. $\square$

The automorphism group generated by $\widetilde{\sigma}$ is isomorphic to $\mathbb{Z}_2$, which vacuously acts properly discontinuously on $\widetilde{S}$, so $T := \widetilde{S}/\widetilde{\sigma}$ defines a compact Riemann surface. Moreover, the canonical map $p : \widetilde{S} \to T$ is an unramified double cover since by Lemma 6.3.1 the involution $\widetilde{\sigma}$ has no fixed points. Therefore, by Proposition 3.5.9 the pullback map $p^* : \mathrm{Jac}(T) \to \mathrm{Jac}(S)$ is not injective.

Since the spectral curve S is not a smooth scheme, the Higgs field $\phi : E \to L \otimes E$ is not necessarily regular, so $\ker(\lambda - \pi^*\phi)$ does not necessarily define a holomorphic line bundle over S. However, using the fact that $\widetilde{S}$ is a Riemann surface, we may construct a holomorphic line bundle over $\widetilde{S}$.

By Proposition 2.1.1 the sheaf map

$$\nu^*(\lambda) - \widetilde{\pi}^*(\phi) : \widetilde{\pi}^*(\mathcal{E}) \to \mathcal{O}_{\widetilde{S}}(\widetilde{\pi}^*(L \otimes E))$$

induces a holomorphic vector bundle $A \to \widetilde{S}$. Since the eigenvalues of ϕ are generically distinct it follows that $A \to \widetilde{S}$ defines a holomorphic line bundle. Let $R := R_{\widetilde{\pi}}$ denote the ramification divisor of $\widetilde{\pi} : \widetilde{S} \to C$. Then the holomorphic line bundle $N := A(R)$ is the analogous line bundle to the one constructed in the general linear spectral curve correspondence. Thus, we claim that $E \cong \widetilde{\pi}_*(N)$ as vector bundles over C with the isomorphism given by

$$\begin{tikzcd} \widetilde{\pi}_*(\mathcal{N}) \arrow[r] \arrow[dr, "\psi"'] & \widetilde{\pi}_*((\widetilde{\pi}^*(\mathcal{E}))(R)) \arrow[d, "\mathrm{Tr}_{\widetilde{S}/C}"] \\ & \mathcal{E} \end{tikzcd} \tag{6.7}$$

Away from the zeros of $p_n \in \mathrm{H}^0(C, L^n)$, the normalisation $\nu : \widetilde{S} \to S$ restricts to a biholomorphism. Thus, by the GL_{2n}-spectral curve correspondence, (6.7) induces an isomorphism $E \cong \widetilde{\pi}_*(N)$ away from the zeros of p_n. Therefore, it suffices to show (6.7) is an isomorphism near the zeros of p_n. First, we need a result from linear algebra.

Lemma 6.3.2. *Let $B : \mathbb{C}^m \to \mathbb{C}^m$ be an endomorphism with corresponding characteristic polynomial*

$$\det(x - B) = (x - x_1)^{m_1} \cdots (x - x_k)^{m_k}$$

where $x_1, \ldots, x_k$ are the eigenvalues. Let $I' \subset \{1, \ldots, k\}$ and let V_i denote the generalised eigenspaces of x_i. Suppose $p'(x) = \prod_{j \in I'} (x - x_j)^{e_j}$ where $e_j \geq 0$, then

$$\ker(p'(B)) \subset \bigoplus_{j \in I'} V_j.$$

Proof. By the primary decomposition theorem $\mathbb{C}^m = \bigoplus_{i=1}^k V_i$. For brevity set $H := \ker(p'(B))$. Let $v \in H$ be given. Writing $v = \sum_{i=1}^k v_i$ where $v_i \in V_i$, then

$$p'(B)v = \sum_{i=1}^{k} p'(B)v_i = \sum_{i=1}^{k} \left\{ \prod_{j \in I'} (B - x_j)^{e_j} v_i \right\} = 0.$$

Hence,

$$\prod_{j \in I'} (B - x_j)^{e_j} v_i = 0$$

and if $i \neq j$ then $(B - x_j)^{e_j}$ is invertible on V_i so $\prod_{j \in I'} (B - x_j)^{e_j} v_i = 0$ if and only if $v_i = 0$ whenever $i \notin I'$. Therefore, $H \subset \bigoplus_{j \in I'} V_j$. □

Proposition 6.3.3. *Suppose (E, ϕ) is a Higgs bundle over C that has corresponding spectral curve $\pi : S \to C$. Suppose at $x \in C$ there is an open neighbourhood U of x such that $S|_{\pi^{-1}(U)} = S' \cup S''$ is a disjoint union. Then over U the Higgs bundle (E, ϕ) decomposes as the sum of two Higgs bundles over U where one Higgs bundle has spectral curve S' and the other has spectral curve S'', i.e., $(E, \phi)|_U \cong (E', \phi') \oplus (E'', \phi'')$ where (E', ϕ') and (E'', ϕ'') has spectral curves S' and S'' respectively.*

Proof. Let $p(\lambda)$ denote the polynomial defining S. Then by Corollary 4.2.7 the polynomial factors $p(\lambda) = p'(\lambda)p''(\lambda)$ where $p'(\lambda)$ and $p''(\lambda)$ are the polynomials defining S' and S'' respectively. It follows from Proposition 2.1.1 that $E' = \ker(p'(\phi))$ and $E'' = \ker(p''(\phi))$ define holomorphic vector bundles over U. Moreover, E', E'' are holomorphic subbundles of E, and from the definition it is clear that E', E'' are ϕ-invariant. Generically the eigenvalues of ϕ are distinct so it is clear that $\mathrm{rank}(E') = \deg(p')$ and $\mathrm{rank}(E'') = \deg(p'')$. Over U let z be local coordinates centred at x, then it follows by Lemma 6.3.2 that E'_x is contained in the generalised eigenspaces of $\phi(x)$ corresponding to zeros of $p'(\lambda, 0)$. Similarly, E''_x is contained in the generalised eigenspaces of $\phi(x)$ corresponding to zeros of $p''(\lambda, 0)$. However, since S' and S'' are disjoint, $p'(\lambda, 0)$ and $p''(\lambda, 0)$ have distinct zeros, hence $E'_x \cap E''_x = \{0\}$. Also, $\dim(E'_x) + \dim(E''_x) = \dim(E_x)$ and thus, $E_x \cong E'_x \oplus E''_x$. Moreover, for $y \in U \setminus \{x\}$ it is clear that $E_y \cong E'_y \oplus E''_y$. Therefore, $E|_U \cong E' \oplus E''$ and since the subbundles E' and E'' are ϕ-invariant we see $\phi|_U = \phi' \oplus \phi''$ and thus, $(E, \phi)|_U \cong (E', \phi') \oplus (E'', \phi'')$ where (E', ϕ') and (E'', ϕ'') has spectral curves S' and S'' respectively. □

Suppose $x \in C$ is a zero of $p_n \in \mathrm{H}^0(C, L^n)$. Then we can choose an open neighbourhood $U \subset C$ of x not containing any other zeros of p_n such that $S|_{\pi^{-1}(U)} = S' \cup S''$ is a disjoint union where S' has degree 2 and contains the singularity of S, and S'' is smooth. By Proposition 6.3.3 we see

$$(E, \phi)|_U \cong (E', \phi') \oplus (E'', \phi'') \tag{6.8}$$

where (E', ϕ') and (E'', ϕ'') has spectral curve S' and S'' respectively. For each $y \in U \setminus \{x\}$ the Higgs field has distinct eigenvalues at y so clearly E'_y and E''_y are orthogonal with respect to Q. Thus, by continuity E'_x and E''_x are orthogonal with respect to Q. Therefore, the decomposition in (6.8) is orthogonal with respect to Q.

Now, since S'' is smooth, $S|_{\pi^{-1}(U)}$ has local normalisation $\widetilde{S}' \cup S''$ where $\widetilde{S}'$ is the normalisation of S'. Let A', A'' and N', N'' be the restrictions of A and N to $\widetilde{S}'$ and S'' respectively. Also, let $\widetilde{\pi}'$ and $\widetilde{\pi}''$ denote the restriction of $\widetilde{\pi}$ to $\widetilde{S}'$ and $\widetilde{S}''$ respectively. Locally,

$$\widetilde{\pi}_*(N) \cong \widetilde{\pi}'_*(N') \oplus \widetilde{\pi}''_*(N'')$$

and over U the sheaf map ψ in (6.7) decomposes as $\psi = \psi' \oplus \psi''$ where

$$\psi' : \widetilde{\pi}'_*(\mathcal{N}') \to \mathcal{E}'|_U$$

and

$$\psi'' : \widetilde{\pi}''_*(\mathcal{N}'') \to \mathcal{E}''|_U.$$

Recall that by the GL_{2n}-spectral curve correspondence, $\psi'' : \widetilde{\pi}''_*(\mathcal{N}'') \to \mathcal{E}''|_U$ is an isomorphism. Therefore, we have reduced to showing $\psi' : \widetilde{\pi}'_*(\mathcal{N}') \to \mathcal{E}'|_U$ is an isomorphism. That is, we have reduced to proving the rank 2 case. As remarked earlier, $E|_U \cong E' \oplus E''$ is orthogonal with respect to Q and the induced form $Q' : E' \otimes E' \to \mathcal{O}_U$ is symmetric, bilinear, and non-degenerate. Therefore, (E', ϕ', Q') defines an O_2-Higgs bundle. In fact, shrinking U if necessary we may trivialise $\wedge^2 E' \cong \mathcal{O}_U$ and choose an orientation so that (E', ϕ', Q') becomes a SO_2-Higgs bundle. For brevity, we will now omit the primes from the notation and consider (E, ϕ, Q) to be a SO_2-Higgs bundle with spectral curve $\pi : S \to U$. Now, the spectral curve S is defined by

$$\lambda^2 + p^2 = 0$$

and setting $\mu = ip$ we see S is defined by $(\lambda - \mu)(\lambda + \mu) = 0$. Hence, $\lambda = \mu$ or $\lambda = -\mu$ and the two eigenvalues $\mu, -\mu$ corresponds to the two sheets of the normalisation $\widetilde{S}$. Recall that p has a simple zero at x, so μ has a simple zero at x. It follows that $\phi(x)$ has eigenvalue 0 with multiplicity 2, hence $\phi(x)$ is nilpotent. However, the only nilpotent element of $\mathfrak{so}_2(\mathbb{C})$ is 0, thus $\phi(x) = 0$. Therefore, $\phi = \mu\beta$ for some matrix β with eigenvalues $1, -1$. Hence, in an appropriate local frame

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Moreover, in this frame

$$\phi = \begin{pmatrix} \mu & 0 \\ 0 & -\mu \end{pmatrix}.$$

Now, we will use this local matrix representation to verify ψ is an isomorphism. Note, $\widetilde{\pi}$ is unramified locally, so $A = N$. Let U^+ and U^- denote the two sheets of $\widetilde{S}$, then $\lambda|_{U^+} = \mu I$ and $\lambda|_{U^-} = -\mu I$. Hence,

$$\lambda - \phi = \begin{cases} \begin{pmatrix} 0 & 0 \\ 0 & 2\mu \end{pmatrix} & \text{on } U^+ \\ \\ \begin{pmatrix} -2\mu & 0 \\ 0 & 0 \end{pmatrix} & \text{on } U^- \end{cases}$$

Thus,

$$N = \ker(\lambda - \phi) = \begin{cases} \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle & \text{on } U^+ \\ \\ \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle & \text{on } U^- \end{cases}$$

Therefore, the map ψ is given by

$$\begin{pmatrix} f \\ g \end{pmatrix} \mapsto \begin{cases} \left\langle \begin{pmatrix} f \\ 0 \end{pmatrix} \right\rangle & \text{on } U^+ \\ \\ \left\langle \begin{pmatrix} 0 \\ g \end{pmatrix} \right\rangle & \text{on } U^- \end{cases} \mapsto \begin{pmatrix} f \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ g \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}.$$

which is an isomorphism. Thus, we have established the following proposition.

Proposition 6.3.4. *Let (E, ϕ, Q) is a SO_{2n}-Higgs bundle with generic spectral curve $\pi : S \to C$. Suppose $\nu : \widetilde{S} \to S$ is the normalisation of $\widetilde{S}$ and set $\widetilde{\pi} = \pi\nu$ and $R = R_{\widetilde{\pi}}$. Then $A = \ker(\nu^*(\lambda) - \widetilde{\pi}^*)$ defines a holomorphic line bundle on $\widetilde{S}$ and the sheaf map*

$$\begin{tikzcd} \widetilde{\pi}_*(\mathcal{N}) \arrow[r, "i"] \arrow[dr, "\psi"'] & \widetilde{\pi}_*(\widetilde{\pi}^*(\mathcal{E})(R)) \arrow[d, "\mathrm{Tr}_{\widetilde{S}/C}"] \\ & \mathcal{E} \end{tikzcd}$$

defines an isomorphism $\widetilde{\pi}_*(A(R)) \cong E$.

The same argument as the general linear case shows that given $M \in \mathrm{Pic}(\widetilde{S})$, then $(\widetilde{\pi}_*(M), \widetilde{\pi}_*(\nu^*\lambda))$ defines a Higgs bundle on C, and this assignment is mutual inverse to Proposition 6.3.4. Hence, the Higgs bundle (E, ϕ) corresponds to a holomorphic line bundle $N \to \widetilde{S}$, and we will use Q to deduce further properties of N. We will adapt arguments from the Sp_{2n}-spectral curve correspondence to obtain the desired result.

Lemma 6.3.5. *Suppose the Higgs bundle (E, ϕ) corresponds to $N \in \mathrm{Pic}(\widetilde{S})$. Then the Higgs bundle $(E, -\phi)$ corresponds to $\widetilde{\sigma}^*(N) \in \mathrm{Pic}(\widetilde{S})$.*

Proof. The Higgs bundle (E', ϕ') over C where $E' = \widetilde{\pi}_*(\widetilde{\sigma}^*(N))$ and ϕ' is induced by $\nu^*(\lambda) : \widetilde{\sigma}^*(N) \to \widetilde{\pi}^*(L) \otimes \widetilde{\sigma}^*(N)$ corresponds to $\sigma^*(N) \in \mathrm{Pic}(\widetilde{S})$, thus it suffices to prove $(E', \phi') \cong (E, -\phi)$ as Higgs bundles. Let $U \subset C$ be a given open subset. Then $\widetilde{\pi}_*(\mathcal{N})(U) = \mathcal{N}(\widetilde{\pi}^{-1}(U))$ and since $\widetilde{\sigma}$ preserves $\widetilde{\pi}^{-1}(U)$ there is an isomorphism of sections

$$\mathcal{N}(\widetilde{\pi}^{-1}(U)) \ni s \mapsto \widetilde{\sigma}^* s \in \widetilde{\sigma}^*(\mathcal{N})(\widetilde{\pi}^{-1}(U)) \tag{6.9}$$

and since $\sigma(\lambda) = -\lambda$ it easily follows that

$$\widetilde{\sigma}^*(\nu^*\lambda)s = -(\nu^*\lambda)\widetilde{\sigma}^* s \tag{6.10}$$

for every $s \in \mathcal{N}(\widetilde{\pi}^{-1}(U))$. Equation (6.9) gives an isomorphism $E \cong E'$ and (6.10) shows that this isomorphism preserves the Higgs fields. Therefore, $(E, -\phi) \cong (E', \phi')$. □

Lemma 6.3.6. *Suppose the Higgs bundle (E, ϕ) corresponds to $N \in \mathrm{Pic}(\widetilde{S})$. Then the dual Higgs bundle (E^*, ϕ^*) corresponds to $N^*(R) \in \mathrm{Pic}(\widetilde{S})$.*

Proof. The Higgs bundle (E', ϕ') over C where $E' = \widetilde{\pi}_*(N^*(R))$ and ϕ' is induced by $\nu^*(\lambda) : N^*(R) \to \widetilde{\pi}^*(L) \otimes N^*(R)$ corresponds to $N^*(R) \in \mathrm{Pic}(S)$. It suffices to show $(E', \phi') \cong (E^*, \phi^*)$. Recall from relative duality we have the non-degenerate pairing

$$E \otimes E' \ni (a, b) \mapsto \mathrm{Tr}_{\widetilde{S}/C}(a \otimes b) \in \mathcal{O}_C.$$

This induces an isomorphism $E' \cong E^*$, so it suffices to show $(\phi a, b) = (a, \phi' b)$ for every $a \in \mathcal{E}(U)$ and $b \in \mathcal{E}'(U)$ where $U \subset C$ is a given open subset of C. Recall that $E \cong \widetilde{\pi}_*(N)$ so let $\widetilde{a} \in \mathcal{N}(\widetilde{\pi}^{-1}(U))$ be the section corresponding to a under the isomorphism. Notice that $\phi a = \nu^*(\lambda)\widetilde{a}$ and $\phi' b = \nu^*(\lambda) b$ and thus,

$$(\phi a, b) = \mathrm{Tr}_{\widetilde{S}/C}(\nu^*(\lambda) a \otimes b) = \mathrm{Tr}_{\widetilde{S}/C}(a \otimes \nu^*(\lambda) b) = (a, \phi' b),$$

hence $(E', \phi') \cong (E^*, \phi^*)$. □

Now, the non-degenerate symmetric bilinear form $Q : E \otimes E \to \mathcal{O}_C$ induces a canonical isomorphism $E \cong E'$. Moreover, since $Q(\phi x, y) = -Q(x, \phi y)$ the induced isomorphism preserves $-\phi$ and ϕ^*. Thus, $(E, -\phi) \cong (E^*, \phi^*)$ as Higgs bundles. Then, by Lemma 6.3.5 and Lemma 6.3.6 there is an induced isomorphism $\theta : \widetilde{\sigma}^*(N) \to N^*(R)$ of line bundles, which establishes one direction in the desired correspondence.

Proposition 6.3.7. *Suppose (E,ϕ,Q) is a SO_{2n}-Higgs bundles with generic spectral curve $\pi : S \to C$. Let $\nu : \widetilde{S} \to S$ be the normalisation of S. Suppose (E,ϕ,Q) corresponds to $N \in \mathrm{Pic}(\widetilde{S})$. Then Q gives rise to an isomorphism $\theta : \widetilde{\sigma}^*(N) \to N^*(R)$. Equivalently, Q gives rise to a non-vanishing section θ of $\widetilde{\sigma}^*(N^*) \otimes N^*(R)$.*

Remark 6.3.8. Since $\widetilde{S}$ is a compact Riemann surface, the isomorphism $\theta : \widetilde{\sigma}^*(N) \to N^*(R)$ is unique up to scale, and rescaling the symmetric form amounts to rescaling the isomorphism.

Proposition 6.3.7 provides one assignment in the desired correspondence. We will establish the other assignment to establish the correspondence. Fix $M \in \mathrm{Pic}(\widetilde{S})$ such that $\widetilde{\sigma}^*(M) \cong M^*(R)$ and fix an isomorphism $\tau : \widetilde{\sigma}^*(M) \to M^*(R)$. The Higgs bundle (V,ξ) over C where $V = \widetilde{\pi}_*(M)$ and $\xi : E \to L \otimes E$ is induced by $\nu^*(\lambda) : N \to \widetilde{\pi}^*(L) \otimes N$ corresponds to $M \in \mathrm{Pic}(\widetilde{S})$. Using the isomorphism $\tau : \widetilde{\sigma}^*(M) \to M^*(R)$ we will construct a symmetric, $\mathcal{O}_C$-bilinear non-degenerate form $\mu : V \otimes V \to \mathcal{O}_C$. Let $U \subseteq C$ be a given open subset and let $a, b \in \mathcal{V}(U) = \mathcal{O}_{\widetilde{S}}(M)(\widetilde{\pi}^{-1}(U))$. Notice that $\tau\widetilde{\sigma}^* b \in \mathcal{O}_{\widetilde{S}}(M^*(R))(\widetilde{\pi}^{-1}(U))$, hence using the pairing from relative duality define

$$\mu(a,b) = \mathrm{Tr}_{\widetilde{S}/C}(a \otimes \tau\widetilde{\sigma}^* b). \tag{6.11}$$

This pairing is $\mathcal{O}_C$-bilinear. Moreover, since the pairing from relative duality is non-degenerate, and τ is non-vanishing, it follows that μ is non-degenerate.

Lemma 6.3.9. *The $\mathcal{O}_C$-bilinear pairing $\mu : V \otimes V \to \mathcal{O}_C$ from (6.11) is compatible with the Higgs field $\xi : V \to L \otimes V$.*

Proof. Since the Higgs field ξ is induced by $\nu^*(\lambda) : M \to \widetilde{\pi}^*(L) \otimes M$ it follows that $\xi b = \nu^*(\lambda) b$. Then,

$$\mu(a,\xi b) = \mathrm{Tr}_{\widetilde{S}/C}(a \otimes \tau\widetilde{\sigma}^*(\nu^*\lambda) b) = -\mathrm{Tr}_{\widetilde{S}/C}((\nu^*\lambda) a \otimes \tau\widetilde{\sigma}^* b) = -\mu(\xi a, b)$$

where we used the fact $\nu^*\lambda$ and $\widetilde{\sigma}^*$ anti-commute. $\square$

Similar to the symplectic case there is a canonical lift of $\widetilde{\sigma}$ to $\widetilde{\sigma}^*(M^*) \otimes M^*(R)$. Now, it is clear that $\mathrm{Tr}_{\widetilde{S}/C}(\widetilde{\sigma}^* s) = \mathrm{Tr}_{\widetilde{S}/C}(s)$ so to determine a necessary and sufficient condition for $\mu : V \otimes V \to \mathcal{O}_C$ to be symmetric notice that

$$\xi(b,a) = \mathrm{Tr}_{\widetilde{S}/C}(b \otimes \tau\widetilde{\sigma}^* a) = \mathrm{Tr}_{\widetilde{S}/C} \widetilde{\sigma}^*(b \otimes \tau\widetilde{\sigma}^* a) = \mathrm{Tr}_{\widetilde{S}/C}(a \otimes (\widetilde{\sigma}^*\tau)\widetilde{\sigma}^* b)$$

and thus, $\xi : V \otimes V \to \mathcal{O}_C$ is symmetric if and only if $\widetilde{\sigma}^*\tau = \tau$. Compared to the symplectic case the condition $\widetilde{\sigma}^*\tau = \tau$ is not automatic.

Recall we have the canonical unramified map $p : \widetilde{S} \to T$, so we may factor $\widetilde{\pi} : \widetilde{S} \to C$ via.

$$\begin{tikzcd} \widetilde{S} \arrow[r, "p"] \arrow[dr, "\widetilde{\pi}"'] & T \arrow[d, "q"] \\ & C \end{tikzcd}$$

Lemma 6.3.10. *Let $A \in \mathrm{Pic}(\widetilde{S})$ be given. Then $p^*(\mathrm{Nm}_{\widetilde{S}/T}(A)) \cong A\widetilde{\sigma}^*(A)$.*

Proof. Since $\widetilde{S}$ is a compact Riemann surface it suffices to use divisors, and since point divisors generate the class divisor group modulo linear equivalence it suffices to prove the equality with point divisors. Let $x \in \widetilde{S}$ be a given point divisor. The map $\widetilde{\sigma} : \widetilde{S} \to \widetilde{S}$ is a biholomorphism, thus the degree is preserved under pulling back by $\widetilde{\sigma}$ and since $\widetilde{\sigma}$ is an involution it is clear that $\widetilde{\sigma}^*(x) = \widetilde{\sigma}(x)$. Hence, $x + \widetilde{\sigma}^*(x) = x + \widetilde{\sigma}(x)$. Now, by definition $\mathrm{Nm}_{\widetilde{S}/T}(x) = p(x)$ so we wish to compute $p^*(p(x))$. Recall that $\widetilde{\sigma}$ has no fixed points, then since $p(x) = p(\widetilde{\sigma}(x))$ and p is a double cover it follows that $p^*(p(x)) = x + \widetilde{\sigma}(x)$. Therefore, $x + \widetilde{\sigma}^*(x) = p^*(\mathrm{Nm}_{\widetilde{S}/T}(x))$. □

By Lemma 6.3.10 the isomorphism $\tau : \widetilde{\sigma}^*(M) \to M^*(R)$, which is equivalent to an isomorphism $\tau : M\widetilde{\sigma}^*(M) \to \mathcal{O}_{\widetilde{S}}(R)$, and can be viewed an isomorphism $\tau : p^*(\mathrm{Nm}_{\widetilde{S}/T}(M)) \to \mathcal{O}_{\widetilde{S}}(R)$. Since p is unramified there is an isomorphism $\mathcal{O}_{\widetilde{S}}(R) \cong p^*(\mathcal{O}_T R_q)$. Therefore, τ may be viewed as an isomorphism

$$\tau : p^*(\mathrm{Nm}_{\widetilde{S}/T}(M)) \to p^*(\mathcal{O}_T R_q). \tag{6.12}$$

Now, it is clear that $\widetilde{\sigma}^*\tau = \tau$ if and only if τ descends to an isomorphism $\mathrm{Nm}_{\widetilde{S}/T}(M) \cong \mathcal{O}_T(R_q)$. Therefore, $\mu : V \otimes V \to \mathcal{O}_C$ is symmetric if and only if τ descends to an isomorphism $\mathrm{Nm}_{\widetilde{S}/T}(M) \cong \mathcal{O}_T(R_q)$. In other words, (V, ξ, μ) defines a O_{2n}-Higgs bundle if and only if τ descends to an isomorphism $\mathrm{Nm}_{\widetilde{S}/T}(M) \cong \mathcal{O}_T(R_q)$. Of course, we want (V, ξ, μ) to be a SO_{2n}-Higgs bundle for which we have to show $\det(V) \cong \mathcal{O}_C$. Assume now that τ descends to an isomorphism $\mathrm{Nm}_{\widetilde{S}/T}(M) \cong \mathcal{O}_T R_q$. To show $\det(V)$ is trivial we require a few results.

Lemma 6.3.11. *There is a short exact sequence of sheaves*

$$0 \to \mathcal{O}_S \to \nu_*(\mathcal{O}_{\widetilde{S}}) \to \bigoplus_{\sigma(p)=p} \mathbb{C}_p \to 0$$

where $\mathbb{C}_p$ denotes the skyscraper sheaf at $p \in S$.

Proof. It suffices to verify exactness at each stalk, so fix $p \in S$ such that $\sigma(p) = p$. Choose local coordinates z centred at $p \in S$. Now, $\mathcal{O}_{S,p} \cong \mathcal{O}_x[\lambda]/(\lambda^2 - z^2)$ and since $\mathcal{O}_x[\lambda]$ is a free $\mathcal{O}_x[\lambda^2]$-module of rank 2 with basis $1, \lambda$ it follows that

$$\mathcal{O}_x[\lambda]/(\lambda^2 - z^2) \cong \mathcal{O}_x \oplus \lambda\mathcal{O}_x.$$

Also,

$$(\nu_*\mathcal{O}_{\widetilde{S}})_p \cong \mathcal{O}_x[\lambda]/(\lambda - z) \oplus \mathcal{O}_x[\lambda]/(\lambda + z) \cong \mathcal{O}_x \oplus \mathcal{O}_x.$$

Thus, we define

$$\mathcal{O}_{S,p} \ni f(z) + \lambda g(z) \mapsto (f(z) + zg(z), f(z) - zg(z)) \in (\nu_*\mathcal{O}_{\widetilde{S}})_p \tag{6.13}$$

and

$$(\nu_*\mathcal{O}_{\widetilde{S}})_p \ni (a(z), b(z)) \mapsto a(0) - b(0) \in \mathbb{C}. \tag{6.14}$$

Each map is clearly injective and surjective respectively and moreover, the image of (6.13) clearly lies in the kernel of map (6.14). To see the opposite inclusion note that since $a(0) - b(0) = 0$ we may write $a(z) - b(z) = zh(z)$ where $h(z) \in \mathcal{O}_x$. Then one can easily verify that

$$\frac{a(z) + b(z)}{2} + \lambda\frac{h(z)}{2} \mapsto (a(z), b(z)).$$

Therefore, the sequence

$$0 \to \mathcal{O}_{S,p} \to (\nu_*\mathcal{O}_{\widetilde{S}})_p \to \mathbb{C} \to 0$$

is exact, and we obtain an exact sequence of sheaves

$$0 \to \mathcal{O}_S \to \nu_*(\mathcal{O}_{\widetilde{S}}) \to \bigoplus_{\sigma(p)=p} \mathbb{C}_p \to 0. \tag{6.15}$$

□

Lemma 6.3.12. *There is an isomorphism of holomorphic line bundles*

$$\det(\widetilde{\pi}_*(\mathcal{O}_{\widetilde{S}})) \cong L^{2n-2n^2}.$$

Proof. Applying the direct image functor π_* to the short exact sequence in (6.15) yields

$$0 \to \pi_*(\mathcal{O}_S) \to \widetilde{\pi}_*(\mathcal{O}_{\widetilde{S}}) \to \bigoplus_{p_n(y)=0} \mathbb{C}_y \to R^1\pi_*(\mathcal{O}_S) \to \cdots$$

Let $x \in C$. Then $\pi^{-1}(x)$ is finite and thus, $\mathrm{H}^1(\pi^{-1}(x), \mathcal{O}_S) = 0$. Hence, by Grauert's base change theorem $R^1\pi_*(\mathcal{O}_S) = 0$. Thus, we have a short exact sequence of sheaves

$$0 \to \pi_*(\mathcal{O}_S) \to \widetilde{\pi}_*(\mathcal{O}_{\widetilde{S}}) \to \bigoplus_{p_n(y)=0} \mathbb{C}_y \to 0. \tag{6.16}$$

Let $j : \pi_*(\mathcal{O}_S) \to \widetilde{\pi}_*(\mathcal{O}_{\widetilde{S}})$ be the sheaf map in (6.16). Fix a zero $y \in C$ of p_n, and choose local coordinate z centred at y. Then $\pi_*(\mathcal{O}_S)_y$ are $\widetilde{\pi}_*(\mathcal{O}_{\widetilde{S}})_y$ and free $\mathcal{O}_y$-modules of rank $2n$ so we may choose a basis for each $\mathcal{O}_y$-module such that $j_y : \pi_*(\mathcal{O}_S)_y \to \widetilde{\pi}_*(\mathcal{O}_{\widetilde{S}})_y$ is in smith normal form, i.e. $j_y = \mathrm{diag}(z^{e_1}, \dots, z^{e_{2n}})$ where $0 \leq e_1 \leq \dots \leq e_{2n}$. Since $\mathrm{coker}(j_y)$ has dimension 1 it follows that $j_y = \mathrm{diag}(1, \dots, 1, z)$, hence $\det(j_y) = z$. Therefore, $\det(j)$ has a simple zero at each zero of p_n and hence,

$$\det(\widetilde{\pi}_*\mathcal{O}_{\widetilde{S}}) \otimes \det(\pi_*\mathcal{O}_S)^* \cong \mathcal{O}_C((p_n)) \cong L^n.$$

Recall from Proposition 4.2.3 that $\pi_*\mathcal{O}_S \cong \mathcal{O}_C \oplus L^{-1} \oplus \cdots \oplus L^{-(2n-1)}$. Thus, $\det(\pi_*\mathcal{O}_S) \cong L^{-n(2n-1)}$ and it follows that

$$\det(\widetilde{\pi}_*\mathcal{O}_{\widetilde{S}}) \cong L^{2n-2n^2}.$$

□

Lemma 6.3.13. *The determinant of V is trivial, i.e.,* $\det(V) \cong \mathcal{O}_C$.

Proof. Recall that $V = \widetilde{\pi}_*(M)$ and $\mathrm{Nm}_{\widetilde{S}/T}(M) \cong \mathcal{O}_T(R_q)$. By Proposition 4.6.7 there is an isomorphism

$$\det(V) \cong \mathrm{Nm}_{\widetilde{S}/C}(M) \otimes \det(\widetilde{\pi}_*\mathcal{O}_{\widetilde{S}}).$$

Notice that $\mathrm{Nm}_{\widetilde{S}/C} = \mathrm{Nm}_{T/C} \circ \mathrm{Nm}_{\widetilde{S}/T}$, hence $\mathrm{Nm}_{\widetilde{S}/C}(M) \cong \mathrm{Nm}_{T/C}(\mathcal{O}_T R_q)$ and by Lemma 6.3.12 we see

$$\det(V) \cong \mathrm{Nm}_{T/C}(\mathcal{O}_T R_q) \otimes L^{2n-2n^2}.$$

Thus, it suffices to show $\mathrm{Nm}_{T/C}(\mathcal{O}_T R_q) \cong L^{2n^2-2n}$. By Proposition 4.6.7 there is an isomorphism

$$\mathrm{Nm}_{T/C}(\mathcal{O}_T R_q) \cong \det(q_*\mathcal{O}_T R_q) \otimes \det(q_*\mathcal{O}_T)^*.$$

From Relative Duality it follows that $\det(q_*\mathcal{O}_T R_q) \cong \det(q_*\mathcal{O}_T)^*$ and thus,

$$\mathrm{Nm}_{T/C}(\mathcal{O}_T R_q) \cong \det(q_*\mathcal{O}_T)^{-2}.$$

Since $\det(q_*\mathcal{O}_T) \cong \mathcal{O}_C \oplus L^{-2} \oplus \cdots L^{-2(n-1)}$ it follows that $\det(q_*\mathcal{O}_T) \cong L^{n-n^2}$. Therefore,

$$\mathrm{Nm}_{T/C}(\mathcal{O}_T R_q) \cong L^{2n^2-2n}.$$

$\square$

Therefore, (V, ξ, μ) defines an O_{2n}-Higgs bundle with $\det(V) \cong \mathcal{O}_C$. To define a SO_{2n}-Higgs bundle, we need to choose an orientation for V. There are two choices of orientation. One preserves the Pfaffian, and the other changes the sign of the Pfaffian. Therefore, choosing the orientation for V that preserves the Pfaffian establishes (V, ξ, μ) as a SO_{2n}-Higgs bundle. Thus, we have established the following proposition.

Proposition 6.3.14. *Let $M \to \widetilde{S}$ be a holomorphic line bundle equipped with an isomorphism $\tau : \widetilde{\sigma}^*(M) \to M^*(R)$ that descends to an isomorphism $\mathrm{Nm}_{\widetilde{S}/T}(M) \cong \mathcal{O}_T(R_q)$. Then the Higgs bundle (V, ξ) where $\xi : V \to L \otimes V$ is induced by $\nu^*(\lambda) : M \to \widetilde{\pi}^*(L) \otimes M$ corresponds to M. Moreover, the isomorphism $\tau : \widetilde{\sigma}^*(M) \to M^*(R)$ defines a non-degenerate symmetric $\mathcal{O}_C$-bilinear form $\mu : V \otimes V \to \mathcal{O}_C$ compatible with ξ. Also, $\det(V) \cong \mathcal{O}_C$ and can be endowed with an orientation preserving the Pfaffian so (V, ξ, μ) defines a SO_{2n}-Higgs bundle.*

Rescaling the isomorphism defines the same Higgs bundle up to isomorphism, and an identical argument to the general linear case shows that the two constructed assignments are mutual inverses. Therefore, we have established the following correspondence.

Theorem 6.3.15. *There is a one-to-one correspondence between isomorphism classes of SO_{2n}-Higgs bundles (E, ϕ, Q) with generic spectral curve S, and holomorphic line bundles N over the normalisation $\widetilde{S}$ such that $\mathrm{Nm}_{\widetilde{S}/T}(N) \cong \mathcal{O}_T(R_q)$ where $T = \widetilde{S}/\widetilde{\sigma}$ and $\widetilde{\sigma}$ denotes the lift of the involution $\sigma(\lambda) = -\lambda$.*

Since the norm map $\mathrm{Nm}_{\widetilde{S}/T} : \mathrm{Pic}(\widetilde{S}) \to \mathrm{Pic}(T)$ is surjective we may choose a holomorphic line bundle $N_0 \to \widetilde{S}$ such that $\mathrm{Nm}_{\widetilde{S}/T}(N_0) = \mathcal{O}_T(R_q)$. Then, writing $N \cong N_0 \otimes U$ it follows that $\mathrm{Nm}_{\widetilde{S}/T}(U) \cong \mathcal{O}_T$. Therefore, the set of holomorphic line bundles $N \to \widetilde{S}$ such that $\mathrm{Nm}_{\widetilde{S}/T}(N) \cong \mathcal{O}_T(R_q)$ defines a torsor of $\ker(\mathrm{Nm}_{\widetilde{S}/T})$ and we have arrived at the SO_{2n}-spectral curve correspondence.

Theorem 6.3.16 (SO_{2n}-Spectral Curve Correspondence)**.** *The isomorphism classes of* SO_{2n}*-Higgs bundle* (E, ϕ, Q) *with generic spectral curve* $\pi : S \to C$ *is in one-to-one correspondence with a torsor of* $\ker(\mathrm{Nm}_{\widetilde{S}/T}) \subset \mathrm{Pic}(\widetilde{S})$ *where* $\widetilde{S}$ *is the normalisation of* S *and* $T := \widetilde{S}/\widetilde{\sigma}$ *where* $\widetilde{\sigma}$ *is the lift of the involution* $\sigma(\lambda) = -\lambda$.

Now, we will compute the number of connected components of $\mathrm{Nm}_{\widetilde{S}/T}$. Recall that the number of connected components of $\mathrm{Nm}_{\widetilde{S}/T}$ is equal to the cardinality of the cokernel of $p_* : \pi_1(\mathrm{Jac}(\widetilde{S})) \to \pi_1(\mathrm{Jac}(T))$, which is precisely the index $(\pi_1(\mathrm{Jac}(T)) : p_*(\pi_1(\mathrm{Jac}(\widetilde{S}))))$. The fundamental group of a complex tori is canonically isomorphic to its lattice, hence $\pi_1(\mathrm{Jac}(\widetilde{S})) \cong \mathrm{H}^1(\widetilde{S}, \mathbb{Z})$, and $\pi_1(\mathrm{Jac}(T)) \cong \mathrm{H}^1(T, \mathbb{Z})$. Under this identification it is clear that the induced map of the first cohomology is the *wrong way map*, i.e., $p_! : \mathrm{H}^1(\widetilde{S}, \mathbb{Z}) \to \mathrm{H}^1(T, \mathbb{Z})$. Moreover the index $(\mathrm{H}^1(T, \mathbb{Z}) : p_!(\mathrm{H}^1(\widetilde{S}, \mathbb{Z})))$ is equal to $(\mathrm{H}_1(T, \mathbb{Z}) : p_*\mathrm{H}_1(\widetilde{S}, \mathbb{Z}))$

Suppose $p : \widetilde{S} \to T$ is determined by the kernel of homomorphism $r : \pi_1(T) \to \mathbb{Z}_2$, i.e., $p_*(\pi_1(\widetilde{S})) = \ker(r)$. Consider now the following commutative diagram

$$\begin{tikzcd} \pi_1(\widetilde{S}) \arrow[r, "p_*"] \arrow[d, "\mathrm{pr}"] & \pi_1(T) \arrow[r, "r"] \arrow[d, "\mathrm{pr}"] & \mathbb{Z}_2 \arrow[d, equal] \\ \mathrm{H}_1(\widetilde{S}, \mathbb{Z}) \arrow[r, "p_*"] & \mathrm{H}_1(T, \mathbb{Z}) \arrow[r, "r"] & \mathbb{Z}_2 \end{tikzcd}$$

The top row is exact, which in fact implies that the bottom row is exact. Indeed, from the commutativity it is clear that $p_*\mathrm{H}_1(\widetilde{S}, \mathbb{Z}) \subset \ker(r)$. On the other hand, if $r(\gamma) = 0$ where $\gamma \in \mathrm{H}_1(T, \mathbb{Z}_2)$, then we may choose a lift $\widehat{\gamma} \in \pi_1(T)$. Then, $r(\widehat{\gamma}) = 0$ and since the top row is exact we may choose $\widehat{\tau} \in \pi_1(\widetilde{S})$ such that $p_*(\widehat{\tau}) = \widehat{\gamma}$. It follows that $p_*(\tau) = \gamma$ where $\tau := \mathrm{pr}(\widehat{\tau})$, and thus, $p_*\mathrm{H}_1(\widetilde{S}, \mathbb{Z}) = \ker(r)$. Therefore, $(\mathrm{H}_1(T, \mathbb{Z}) : p_*\mathrm{H}_1(\widetilde{S}, \mathbb{Z})) = 2$, shows the number of connected components of $\mathrm{Nm}_{\widetilde{S}/T}$ is equal to 2. Moreover, one connected component is $\mathrm{Prym}(\widetilde{S}, T)$ and it is clear that the other connected component is a torsor of $\mathrm{Prym}(\widetilde{S}, T)$, which establishes the following proposition.

Proposition 6.3.17. *The kernel of the norm map* $\ker(\mathrm{Nm}_{\widetilde{S}/T})$ *associated to the étale double cover* $p : \widetilde{S} \to T$ *consists of two connected components where the connected connected component of the identity is the Prym variety* $\mathrm{Prym}(\widetilde{S}, T)$ *and the other connected component is a torsor of* $\mathrm{Prym}(\widetilde{S}, T)$.

6.4 SO_{2n}-Hitchin Fibration

Recall that the characteristic polynomial of a generic SO_{2n}-Higgs bundle is of the form

$$p(\lambda) = \lambda^{2n} + a_2\lambda^{2n-2} + \cdots + p_n^2 \tag{6.17}$$

where $a_{2i} \in \mathrm{H}^0(C, L^{2i})$ for $i = 1, 2, \ldots, n-1$ and $p_n \in \mathrm{H}^0(C, L^n)$. The coefficients $a_2, a_4, \ldots, a_{2n-2}, p_n$ form a homogeneous basis for the ring of invariant polynomials of $\mathfrak{so}_{2n}(\mathbb{C})$. Hence the SO_{2n}-Hitchin fibration is given by

$$h : \mathcal{M}_{\mathrm{SO}_{2n}} \to \bigoplus_{i=1}^{n-1} \mathrm{H}^0(C, L^{2i}) \oplus \mathrm{H}^0(C, L^n)$$

that sends

$$h(E, \phi, Q) = (a_2, \ldots, a_{2n-2}, p_n)$$

where the characteristic polynomial for (E, ϕ, Q) is of the form (6.17).

6.4.1 Dimension of SO_{2n}-Hitchin Base

Again we will assume $\deg(L) \geq 2g$ so that L is basepoint-free and $\mathrm{H}^1(C, L^i) = 0$ for every $i \geq 1$. Then, by the Riemann-Roch theorem

$$h^0(C, L^{2i}) = 2i\deg(L) + 1 - g$$

for $i = 1, \ldots, n-1$. Moreover,

$$h^0(C, L^n) = n\deg(L) + 1 - g$$

Thus,

$$\sum_{i=1}^{n-1} h^0(C, L^{2i}) = n(n-1)\deg(L) + (n-1)(1-g).$$

Therefore,

$$\dim\left(\bigoplus_{i=1}^{n-1} \mathrm{H}^0(C, L^{2i}) \oplus \mathrm{H}^0(C, L^n)\right) = n^2\deg(L) + n(1-g). \tag{6.18}$$

To compute the Hitchin base for $L = K_C$ notice that since $\deg(K_C^i) > 2g-2$ for $i > 1$ it suffices to substitute $\deg(L) = 2g-2$ into (6.18), which gives

$$\dim\left(\bigoplus_{i=1}^{n-1} \mathrm{H}^0(C, K_C^{2i}) \oplus \mathrm{H}^0(C, K_C^n)\right) = n(2n-1)(g-1).$$

6.4.2 Generic Fibres of SO_{2n}-Hitchin Fibration

A generic point in the hitchin base $(a_2, \ldots, p_n)$ canonically defines a generic SO_{2n} spectral curve $\pi : S \to C$, and thus, we may use the SO_{2n}-spectral curve correspondence to classify the generic fibres of the SO_{2n}-Hitchin fibration.

Theorem 6.4.1. *Let $(a_2, \ldots, p_n)$ be a generic point in the SO_{2n}-Hitchin base that defines a generic SO_{2n}-spectral curve $\pi : S \to C$ with canonical involution σ. Denote the normalisation of S by $\widetilde{S}$ and denote the unique lift of σ by $\widetilde{\sigma}$. Then, the generic fibre $h^{-1}(a_2, \ldots, p_n)$ is comprised of two connected components each of which are torsors of the Prym variety $\mathrm{Prym}(\widetilde{S}, T)$ where $T := \widetilde{S}/\widetilde{\sigma}$.*

Proof. The preimage $h^{-1}(a_2, \ldots, p_n)$ corresponds to isomorphism classes of SO_{2n}-Higgs bundles whose associated spectral curve is $\pi : S \to C$. By the SO_{2n}-spectral curve correspondence $h^{-1}(a_2, \ldots, p_n)$ corresponds to a torsor of $\ker(\mathrm{Nm}_{\widetilde{S}/T})$. Moreover, by Proposition 6.3.17, $\ker(\mathrm{Nm}_{\widetilde{S}/T})$ consists of two connected components that are torsors of $\mathrm{Prym}(\widetilde{S}, T)$. Therefore, $h^{-1}(a_2, \ldots, p_n)$ is comprised of two connected components each of which are torsors of the Prym variety $\mathrm{Prym}(\widetilde{S}, T)$. □

6.4.3 Dimension of Generic Fibres

Suppose $\pi : S \to C$ is a generic SO_{2n}-spectral curve defined by the polynomial

$$p(\lambda) = \lambda^{2n} + a_2\lambda^{2n-2} + \cdots + p_n^2.$$

Let $\nu : \widetilde{S} \to S$ denote the corresponding normalisation and $\widetilde{\pi} : \widetilde{S} \to C$ the induced map. We will to compute $\deg(R_{\widetilde{\pi}})$, however contrary to the other cases $\widetilde{S}$ is not a spectral curve in general. Away from the ordinary double points the curves S and $\widetilde{S}$ are isomorphic. Moreover, away from the singularities the holomorphic line bundle associated to the ramification divisor of $\pi : S \to C$ is $L^{2n(2n-1)}$ by the adjunction formula. Hence, we will study the discriminant of S at the ordinary doubles points to determine the correction in the degree of the ramification divisor of $\widetilde{\pi} : \widetilde{S} \to C$. Recall that the ordinary double points of S correspond to zeros of the Pfaffian. Let $x \in C$ be a zero of the Pfaffian. We claim that $(0, x)$ corresponds to a double zero of the discriminant. Locally the spectral curve may be written as the disjoint union $S = S_1 \cup S_2$ of spectral curves where S_1 has degree 2 and contains the singularity, and S_2 is smooth with degree $2n - 2$. The polynomial factors by $p(\lambda) = p_1(\lambda)p_2(\lambda)$ where p_i defines S_i. Now, the discriminant of S is given by

$$\mathrm{Res}(p(\lambda), \partial_\lambda p(\lambda)) = \mathrm{Res}(p_1(\lambda), \partial_\lambda p(\lambda))\, \mathrm{Res}(p_2(\lambda), \partial_\lambda p(\lambda)).$$

Notice that $p_1(0, x) = 0$ and $p_2(0, x) \neq 0$. Hence, we only need to consider $\mathrm{Res}(p_1(\lambda), \partial_\lambda p(\lambda))$. Since $\partial_\lambda p(\lambda) = p_2(\lambda)\partial_\lambda p_1(\lambda) + p_1(\lambda)\partial_\lambda p_2(\lambda)$ it follows that we only need to study the discriminant of S_1. We may choose a local coordinate z centred at x such that

$$p_1(\lambda, z) = \lambda^2 + z^2.$$

Therefore, the discriminant of S_1 is locally given by $-4z^2$, which proves the claim. Since the Pfaffian has $n\deg(L)$ simple zeros, which correspond to double zeros of the discriminant divisor the ramification divisor of the induced map $\widetilde{\pi} : \widetilde{S} \to C$ has degree

$$\deg(R_{\widetilde{\pi}}) = 2n(2n-1)\deg(L) - 2n\deg(L) = 2n(2n-2)\deg(L).$$

Thus, by Riemann-Hurwitz

$$2g(\widetilde{S}) - 2 = 2n(2g-2) + 2n(2n-2)\deg(L).$$

Solving for $g(\widetilde{S})$ gives

$$g(\widetilde{S}) = 2n(g-1) + 2n(n-1)\deg(L) + 1.$$

Let $T = \widetilde{S}/\widetilde{\sigma}$, then since the natural projection map $\widetilde{S} \to T$ is a double étale cover it follows from Riemann-Hurwitz that

$$g(T) = \frac{1}{2}g(\widetilde{S}) + \frac{1}{2}.$$

Hence, the dimension of the generic fibre $\mathrm{Prym}(\widetilde{S}, T)$ is given by

$$\dim(\mathrm{Prym}(\widetilde{S}, T)) = n(g-1) + n(n-1)\deg(L). \tag{6.19}$$

Now, if $L = K_C$, then it follows that

$$\dim(\mathrm{Prym}(\widetilde{S}, T)) = n(2n-1)(g-1). \tag{6.20}$$

6.4.4 Self-Duality in Generic Fibres

In (3.5.4) we showed that the Prym variety associated to an étale double cover is principally polarised, i.e., admits a principal polarisation Ξ satisfying $i_0^* c_1(\Theta) = 2\Xi$ where $i_0 : \mathrm{Prym}(\widetilde{S}, T) \to \mathrm{Jac}(\widetilde{S})$ denotes inclusion and Θ is a theta divisor on $\mathrm{Jac}(\widetilde{S})$, which implies the Prym variety is self-dual. Since the double cover $p : \widetilde{S} \to T$ is étale we see $\mathrm{Prym}(S, T)^\vee \cong \mathrm{Prym}(S, T)$. In fact, this implies that the other connected component of $\ker(\mathrm{Nm}_{\widetilde{S}/T})$ is self-dual as a torsor of an abelian variety. To see this, let P_1 denote the other connected component of $\ker(\mathrm{Nm}_{\widetilde{S}/T})$ and let $i_1 : P_1 \to \mathrm{Jac}(\widetilde{S})$ denote inclusion. Suppose $a \in P_1$, then the translation map $t_a : \mathrm{Prym}(\widetilde{S}, T) \to P_1$ defines an isomorphism of algebraic varieties, and moreover, the following square commutes

$$\begin{tikzcd}
\mathrm{Prym}(\widetilde{S}, T) \arrow[r, "i_0"] \arrow[d, "t_a"'] & \mathrm{Jac}(\widetilde{S}) \arrow[d, "t_a"'] \\
P_1 \arrow[r, "i_1"] & \mathrm{Jac}(\widetilde{S})
\end{tikzcd}$$

Passing to the second integral cohomology gives the commutative square

$$\begin{tikzcd}
\mathrm{H}^2(\mathrm{Prym}(\widetilde{S},T),\mathbb{Z}) & \mathrm{H}^2(\mathrm{Jac}(\widetilde{S}),\mathbb{Z}) \arrow[l, "i_0^*"'] \\
\mathrm{H}^2(P_1,\mathbb{Z}) \arrow[u, "t_a^*"] & \mathrm{H}^2(\mathrm{Jac}(\widetilde{S}),\mathbb{Z}) \arrow[l, "i_1^*"'] \arrow[u, "t_a^*"']
\end{tikzcd}$$

that is, $i_0^* t_a^* = t_a^* i_1^*$. Now, the first Chern class satisfies naturality, so $c_1(t_a^*\Theta) = t_a^* c_1(\Theta)$. It follows that

$$t_a^* i_1^* c_1(\Theta) = i_0^* c_1(t_a^*\Theta) = i_0^* c_1(\Theta) = 2\Xi.$$

This proves the duality in P_1. Therefore, the generic fibres of the SO_{2n}-Hitchin fibration are self-dual, which demonstrates Langlands duality in the Hitchin fibration since ${}^L\mathrm{SO}_{2n} \cong \mathrm{SO}_{2n}$.

Appendix A

Langlands Duality for Classical Simple Lie Groups

A.1 Defining the Langlands Dual Group

Fix a complex connected semisimple Lie group G, and let $\mathfrak{g}$ denote the Lie algebra of G. An algebraic subalgebra $\mathfrak{t} \subset \mathfrak{g}$ is abelian with each element semisimple if and only if $\mathfrak{t}$ is the tangent algebra of a torus $T \subset G$. Moreover, the subalgebra is maximal, hence Cartan if and only if $T \subset G$ is maximal. Thus, fix a maximal torus $T \subset G$ with corresponding Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$.

A.1.1 Dual Root Systems

Let $(E, (\, , \,))$ be a Euclidean space equipped with an inner product. There is another inner product $\langle \, , \, \rangle$ on E defined by $\langle \alpha, \beta \rangle := \frac{2(\alpha,\beta)}{(\beta,\beta)}$. We wish to endow a Euclidean structure on the dual space E^*. To endow the desired structure note that we may regard E as the dual space of E^* under the canonical isomorphism $E \cong (E^*)^*$. Consider the isomorphism $E \ni \lambda \mapsto u_\lambda \in E^*$ where u_λ is defined by

$$(\lambda, \mu) = \mu(u_\lambda) \tag{A.1}$$

for every $\mu \in E$. Under this isomorphism, there is a canonical Euclidean structure given by $(u_\lambda, u_\mu) := (\lambda, \mu)$. Hence, the Euclidean structure on E naturally induces a Euclidean structure $(E^*, (\, , \,))$, and under this correspondence, to every root system $\Delta \subset (E, (\, , \,))$ there is a naturally associated root system $\Delta^\vee \subset (E^*, (\, , \,))$ called the dual root system, which we will now construct.

Let $\alpha \in \Delta$ be a root. The *coroot* $\alpha^\vee \in \Delta^\vee$ associated to α is defined by $\alpha^\vee := \frac{2u_\alpha}{(\alpha,\alpha)}$. Notice that

$$\mu(\alpha^\vee) = \frac{2(\mu, \alpha)}{(\alpha, \alpha)} = \langle \mu, \alpha \rangle \tag{A.2}$$

for every $\mu \in E$.

Lemma A.1.1. *The set of coroots* $\Delta^\vee := \{\alpha^\vee \in E^* \,|\, \alpha \in \Delta\}$ *defines a root system in* $(E^*, (\ ,\ ))$ *called the dual root system.*

Proof. Since Δ is finite and does not contain 0 it easily follows that $\Delta^\vee$ is finite and does not contain 0. Moreover, since Δ spans E it follows from the isomorphism $E \ni \lambda \mapsto u_\lambda \in E^*$ that Δ^* spans E^*. A straightforward calculation shows that $c\alpha^\vee = (\frac{1}{c}\alpha)^\vee$. Thus, $c\alpha^\vee \in \Delta^\vee$ if and only if $\frac{1}{c}\alpha \in \Delta$. However, $\frac{1}{c}\alpha \in \Delta$ if and only if $c = \pm 1$. It is easy to show $\langle \alpha^\vee, \beta^\vee \rangle = \langle \beta, \alpha \rangle$ and the integrality condition immediately follows. Finally, to see that the reflection $r_{\alpha^\vee}$ permutes $\Delta^\vee$ notice that $\alpha^\vee$ is a rescaling of u_α, hence $r_{\alpha^\vee} = r_{u_\alpha}$. Let $\beta \in \Delta$ be a root and suppose $r_\alpha(\beta) = \gamma$, i.e., $\beta - \langle \beta, \alpha \rangle \alpha = \gamma$. Then, it follows that

$$r_{u_\alpha}(u_\beta) = u_\beta - \langle \beta, \alpha \rangle u_\alpha.$$

Let $\lambda \in E$, then by (A.1) we see

$$\lambda(u_\beta - \langle \beta, \alpha \rangle u_\alpha) = (\beta - \langle \beta, \alpha \rangle \alpha, \lambda) = (\gamma, \lambda) = \lambda(u_\gamma).$$

Since $\lambda \in E$ was arbitrary it follows that $r_{u_\alpha}(u_\beta) = u_\gamma$. Moreover, since reflections are isometries we see $(\gamma, \gamma) = (\beta, \beta)$ and it follows that $r_{\alpha^\vee}(\beta^\vee) = \gamma^\vee$. Therefore, $r_{\alpha^\vee}$ permutes $\Delta^\vee$ and thus, $\Delta^\vee$ defines a root system. $\square$

A.1.2 Root Lattice and Weight Lattice

Fix a root system $\Delta \subset (E, (\ ,\ ))$ with corresponding coroot system $\Delta^\vee \subset (E^*, (\ ,\ ))$. The integral span of the root system $\Delta \subset E$ defines a lattice in E called the *root lattice* denoted Λ_{rt}. To see that Λ_{rt} defines a lattice recall that Δ has a base Π, which is a vector space basis of E such that each $\alpha \in \Delta$ may be written as $\alpha = \sum_{\beta \in \Pi} k_\beta \beta$ where $k_\beta \in \mathbb{Z}$ and each non-zero k_β have the same sign. Thus, the integral span of Δ equals the integral span of Π, which is certainly a lattice. Similarly, the integral span of the $\Delta^\vee \subset E^*$ defines a lattice in E^* called the *coroot lattice* denoted by Λ_{cort}. Now, we will introduce two additional lattices.

Definition A.1.2. The *weight lattice* of Δ is defined by

$$\Lambda_{\mathrm{wt}} := \{\mu \in E \,|\, \langle \mu, \alpha \rangle \in \mathbb{Z} \text{ for every } \alpha \in \Delta\}.$$

Also, the *coweight lattice* of Δ denoted by Λ_{cowt} is defined to be the weight lattice of $\Delta^\vee$, i.e.,

$$\Lambda_{\mathrm{cowt}} := \{w \in E^* \,|\, \langle w, \beta \rangle \in \mathbb{Z} \text{ for every } \beta \in \Delta^\vee\}.$$

Remark A.1.3. The integrality axiom of Δ implies $\Lambda_{\mathrm{rt}} \subseteq \Lambda_{\mathrm{wt}}$, likewise $\Lambda_{\mathrm{cort}} \subseteq \Lambda_{\mathrm{cowt}}$. Moreover, since the weight root lattices have the same rank the inclusion $\Lambda_{\mathrm{rt}} \subseteq \Lambda_{\mathrm{wt}}$ has finite index, and similarly $\Lambda_{\mathrm{cort}} \subseteq \Lambda_{\mathrm{cowt}}$ has finite index.

It is not immediately clear that Λ_{wt} and Λ_{cowt} define lattices. However, this follows since Λ_{wt} is the dual lattice for Λ_{cort}, and Λ_{cowt} is the dual lattice of Λ_{rt}.

Lemma A.1.4. *The weight lattice is dual to the coroot lattice, i.e., $\Lambda_{\mathrm{wt}} = \Lambda^*_{\mathrm{cort}}$, and the coweight lattice is dual to the root lattice, i.e., $\Lambda_{\mathrm{cowt}} = \Lambda^*_{\mathrm{rt}}$.*

Proof. Let $\mu \in \Lambda_{\mathrm{wt}}$, i.e., $\langle \mu, \alpha \rangle \in \mathbb{Z}$ for every $\alpha \in \Delta$. By (A.2) we see $\langle \lambda, \alpha \rangle = \mu(\alpha^\vee)$ and hence

$$\Lambda_{\mathrm{wt}} = \{\mu \in E \mid \mu(\alpha^\vee) \in \mathbb{Z} \text{ for every } \alpha^\vee \in \Delta^\vee\}.$$

Therefore, $\Lambda_{\mathrm{wt}} = \Lambda^*_{\mathrm{cort}}$, and the proof $\Lambda_{\mathrm{cowt}} = \Lambda^*_{\mathrm{rt}}$ is similar. $\square$

A.1.3 Character and Cocharacter Lattices

For the rest of the section $\Delta \subset (\mathfrak{t}^*, (\ ,\ ))$ denotes the unique root system determined by the connected complex semisimple Lie group G and the maximal torus T. In fact, Δ defines a natural real form of the complex vector space $\mathfrak{t}^*$, which we will now construct. To construct the desired real form we will follow [EW06, Section 10.6].

Lemma A.1.5. *The following statements are true*

(i) *If $t \in \mathfrak{t}$ and $t \neq 0$, then there exists a root $\alpha \in \Delta$ such that $\alpha(t) \neq 0$.*

(ii) *The root system Δ spans $\mathfrak{t}^*$.*

Proof. To prove (i) let $t \in \mathfrak{t}$ and suppose $\alpha(t) = 0$ for every $\alpha \in \Delta$. Let $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$ be the root space decomposition. Notice that for every $x \in \mathfrak{g}_\alpha$ we have $[t, x] = \alpha(t)x = 0$ and thus, $t \in Z(\mathfrak{g})$. However, since $\mathfrak{g}$ is semisimple, $Z(\mathfrak{g}) = 0$, hence $t = 0$. Now (ii) is a reformulation of (i). Indeed, let $W \subset \mathfrak{t}^*$ denote the span of Δ and suppose W is a proper subspace. Then the annihilator $W^\circ = \{t \in \mathfrak{t}^* \mid \alpha(t) = 0 \text{ for every } \alpha \in \Delta\}$ has positive dimension, which is a contradiction. $\square$

By Lemma A.1.5 (ii) let $\alpha_1, \ldots, \alpha_n$ be a vector space basis for $\mathfrak{t}^*$ such that $\alpha_i \in \Delta$ for $i = 1, \ldots, n$.

Lemma A.1.6. *Every $\beta \in \Delta$ is a linear combination of the α_i with coefficients in $\mathbb{Q}$.*

Proof. Since $\alpha_1, \ldots, \alpha_n$ is a vector space basis for $\mathfrak{t}^*$ we may write $\beta = \sum_{i=1}^n c_i\alpha_i$ where $c_i \in \mathbb{C}$. Notice that for each $j = 1, \ldots, n$

$$(\beta, \alpha_j) = \sum_{i=1}^n c_i(\alpha_i, \alpha_j).$$

In terms of matrices

$$\begin{bmatrix}(\beta,\alpha_1)\\ \vdots \\ (\beta,\alpha_n)\end{bmatrix} = \begin{bmatrix}(\alpha_1,\alpha_1) & \cdots & (\alpha_n,\alpha_1)\\ \vdots & \ddots & \vdots \\ (\alpha_1,\alpha_n) & \cdots & (\alpha_n,\alpha_n)\end{bmatrix}\begin{bmatrix}c_1\\ \vdots \\ c_n\end{bmatrix}$$

The $n \times n$ matrix is the matrix representing the non-degenerate bilinear form $(\,,\,)$ with respect to the chosen basis. Each entry is rational and hence the inverse matrix has rational entries. Moreover, $(\beta, \alpha_j) \in \mathbb{Q}$ for $j = 1, \ldots, n$ and it follows that $c_1, \ldots, c_n$ are rational. □

By Lemma A.1.6 the root system $\Delta \subset \mathfrak{t}^*$ belongs to the real subspace spanned by the roots $\alpha_1, \ldots, \alpha_n$, which we will denote by $\mathfrak{t}^*_{\mathbb{R}}$. Since the roots span $\mathfrak{t}^*$ it is easy to see that $\mathfrak{t}^*_{\mathbb{R}}$ defines a real form of $\mathfrak{t}^*$. The inner product $(\,,\,)$ on $\mathfrak{t}^*$ restricts to a real-valued inner product on $\mathfrak{t}^*_{\mathbb{R}}$. Therefore, we may regard $\Delta \subset (\mathfrak{t}^*_{\mathbb{R}}, (\,,\,))$.

The last notion required is that of character and cocharacter lattices. Recall that a character of a Lie group is a 1-dimensional representation, so a character of T is a representation $\chi : T \to \mathbb{G}_m$. We define the *character group* of T by $X^*(T) := \mathrm{Hom}(T, \mathbb{G}_m)$. To compute the character group of T recall that $T \cong (\mathbb{G}_m)^n$ where n is the rank of the Lie group G.

Proposition A.1.7. *A character $\chi : \mathbb{G}_m^n \to \mathbb{G}_m$ is necessarily of the form*

$$\chi(x_1, \ldots, x_n) = x_1^{t_1} \cdots x_n^{t_n}$$

where $t_1, \ldots, t_n \in \mathbb{Z}$.

Proof. Let $\chi : \mathbb{G}_m^n \to \mathbb{G}_m$ be a given character. Notice that $\chi \circ \iota_j : \mathbb{G}_m \to \mathbb{G}_m$ is a character of $\mathbb{G}_m$ for $j = 1, \ldots, n$ where $\iota_j : \mathbb{G}_m \to \mathbb{G}_m^n$ is defined by $x \mapsto (1, \ldots, 1, x, 1, \ldots, 1)$. Moreover, χ is determined by $\chi \circ \iota_j$ for $j = 1, \ldots, n$. Thus, it suffices to compute the characters of $\mathbb{G}_m$. Suppose $\xi : \mathbb{G}_m \to \mathbb{G}_m$ is a character of $\mathbb{G}_m$. The character ξ canonically defines a character $\xi : U(1) \to \mathbb{G}_m$ of $U(1)$. Since $U(1) \subset \mathbb{G}_m$ is compact it follows that any character of $U(1)$ is an endomorphism of $U(1)$. The endomorphisms of $U(1)$ are of the form $U(1) \ni z \mapsto z^a \in U(1)$ where $a \in \mathbb{Z}$ and it follows that $\xi(x) = x^a$ for every $x \in \mathbb{G}_m$. □

By Proposition A.1.7 the character group $X^*(T)$ of the torus T defines a free abelian group of rank n, i.e., $X^*(T) \cong \mathbb{Z}^n$. Moreover, given a character $\chi : T \to \mathbb{G}_m$ the differential at the unit has purely imaginary image $\mathrm{d}\chi : \mathfrak{t} \to \mathbb{C}$, and hence, $\frac{\mathrm{d}\chi}{2\pi i} : \mathfrak{t} \to \mathbb{R}$. In fact, this defines an injective group homomorphism $X^*(T) \to \mathfrak{t}^*_{\mathbb{R}}$. Therefore, the injection sends an integral basis of $X^*(T)$ to a real basis of $\mathfrak{t}^*_{\mathbb{R}}$. Thus, $X^*(T)$ embeds as a lattice in $\mathfrak{t}^*_{\mathbb{R}}$. See [OV90, pp 113] for more details. This allows us to define the character and cocharacter lattice.

Definition A.1.8. The *character lattice* of (G,T) is the character group $X^*(T)$, and the *cocharacter lattice* of (G,T), denoted $X_*(T)$ is the dual lattice to $X^*(T)$.

In [OV90, pp 174] it is shown that the character lattice is an intermediate lattice for the root and weight lattices, i.e., $\Lambda_{\text{rt}} \subseteq X^*(T) \subseteq \Lambda_{\text{wt}}$. Hence, passing to the dual lattices it follows that $\Lambda_{\text{cort}} \subseteq X_*(T) \subseteq \Lambda_{\text{cowt}}$, i.e., the cocharacter lattice is an intermediate lattice for the coroot and coweight lattices. Now, consider the definition of root datum.

Definition A.1.9. The *root datum* of G with maximal torus T is defined to be the quadruple $(X^*(T), \Delta, X_*(T), \Delta^\vee)$.

Before defining the Langlands dual group we will reference two theorems that together give the classification of (complex) connected semisimple Lie groups.

Theorem A.1.10 ([OV90, Theorem 9, pp 192]). *A connected semisimple Lie group G is determined uniquely up to isomorphism by its Dynkin diagram and the character lattice $X^*(T)$ of a maximal torus $T \subset G$. More precisely, if G_1, G_2 are two connected semisimple Lie groups, $T_i \subset G_i$ their maximal torus, Π_i the corresponding system of simple roots then for any isomorphism $\psi : \Pi_1 \to \Pi_2$ which maps $X^*(T_1)$ onto $X^*(T_2)$ there exists an isomorphism $\Phi : G_1 \to G_2$ mapping T_1 onto T_2 and inducing ψ.*

Theorem A.1.11 ([OV90, Theorem 10, pp 192]). *Let $\Delta \subset E$ be a reduced root system with root and weight lattices Λ_{rt} and Λ_{wt} respectively. For any intermediate lattice $\Lambda_{\text{rt}} \subseteq \Lambda \subseteq \Lambda_{\text{wt}}$ there exists a connected semisimple Lie group G with maximal torus T such that there is a root system isomorphism $\Delta_G \to \Delta$ mapping $X^*(T)$ into Λ.*

By Theorem A.1.10, the complex connected semisimple Lie group G with maximal torus T is completely determined by $X^*(T)$ and Δ. By Theorem A.1.11 any intermediate lattice $\Lambda_{\text{rt}} \subseteq \Lambda \subseteq \Lambda_{\text{wt}}$ is the character lattice of a maximal torus of some complex connected semisimple Lie group. Therefore, every complex connected semisimple Lie group is completely determined by its root datum and moreover, a quadruple of the form $(\Lambda, \Delta, \Lambda^*, \Delta^\vee)$ where $\Lambda_{\text{rt}} \subseteq \Lambda \subseteq \Lambda_{\text{wt}}$ is an intermediate lattice determines a complex connected semisimple Lie group. In particular, the study of complex connected semisimple Lie groups is equivalent to the study of root datum, which naturally leads to the Langlands dual group.

Definition A.1.12. Suppose G with maximal torus T has root datum $(X^*(T), \Delta, X_*(T), \Delta^\vee)$. Then the *Langlands dual group* of G denoted ${}^LG$ is defined to be the connected complex semisimple Lie group whose root datum is given by $(X_*(T), \Delta^\vee, X^*(T), \Delta)$.

A.1.4 Properties of the Langlands Dual Group

Computing the Langlands dual group can be tedious so to aid its computation we will show how to compute the centre and fundamental group from the root datum. First, we need the following proposition.

Proposition A.1.13 ([Bou05, Proposition 11, pp 314])**.** *The group G is simply connected if and only if $X^*(T) = \Lambda_{\mathrm{wt}}$, or, equivalently, $X_*(T) = \Lambda_{\mathrm{cort}}$*

Before computing the centre and fundamental group from the root datum we require one more lemma.

Lemma A.1.14. *There is a canonical isomorphism*

$$T \cong \mathfrak{t}/X_*(T).$$

Proof. Let $\chi : T \to \mathbb{G}_m$ be a given character. Let $\exp : \mathfrak{t} \to T$ denote the standard exponential map, which is surjective. Then for every $t \in \mathfrak{t}$ one has $\chi(\exp(t)) = e^{2\pi i\langle \chi, t\rangle}$ where the latter χ is viewed as a linear form on $\mathfrak{t}$. Thus, $\ker(\exp)$ is precisely the set of $t \in \mathfrak{t}$ such that $\langle \chi, t\rangle \in \mathbb{Z}$, which is precisely the cocharacter lattice. Therefore, by the first isomorphism theorem $T \cong \mathfrak{t}/X_*(T)$. □

Proposition A.1.15. *There is a canonical isomorphism*

$$Z(G) \cong \Lambda_{\mathrm{cowt}}/X_*(T).$$

Proof. The conjugate action of G on itself induces the adjoint action $G \to \mathrm{GL}(\mathfrak{g})$, which is a linear representation of G on its Lie algebra $\mathfrak{g}$. Since G is connected, an element $g \in G$ is central whenever the adjoint action is trivial. Considering the root space decomposition $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$, and let $\Delta' \subset X^*(T)$ denote the set of characters that correspond to the roots, i.e., elements in Δ. Then, it follows that $Z(G) = \cap_{\alpha'\in\Delta'} \ker(\alpha')$. Under the identification $T \cong \mathfrak{t}/X_*(T)$ from Lemma A.1.14 the subset $\ker(\alpha') \subset T$ corresponds to the set of $t \in T$ such that $\langle \alpha', t\rangle \in \mathbb{Z}$ and it immediately follows that $Z(G) \cong \Lambda_{\mathrm{cowt}}/X_*(T)$. □

Proposition A.1.16. *There is a canonical isomorphism*

$$\pi_1(G) \cong X_*(T)/\Lambda_{\mathrm{cort}}.$$

Proof. Let $\rho : \widetilde{G} \to G$ denote the universal cover of G and let $\widetilde{T}$ denote its maximal torus. Recall that there is a canonical isomorphism $\pi_1(G) \cong \ker\rho$. Moreover, since $\rho^{-1}Z(G) = Z(\widetilde{G})$ it follows that $\ker(\rho) \subseteq Z(\widetilde{G})$. Hence, $\pi_1(G)$ can naturally be viewed as a subgroup of $Z(\widetilde{G})$. Since $\widetilde{G}$ is simply connected it follows from Lemma A.1.13 and Proposition A.1.15 that $Z(\widetilde{G}) \cong \Lambda_{\mathrm{cowt}}/\Lambda_{\mathrm{cort}}$. Therefore, $\pi_1(G)$ is identified as the kernel of the canonical map $\Lambda_{\mathrm{cowt}}/\Lambda_{\mathrm{cort}} \to \Lambda_{\mathrm{cowt}}/X_*(T)$, which is precisely $X_*(T)/\Lambda_{\mathrm{cort}}$, i.e., $\pi_1(G) \cong X_*(T)/\Lambda_{\mathrm{cort}}$. □

Remark A.1.17. Since both containments $\Lambda_{\mathrm{cort}} \subseteq X_*(T) \subseteq \Lambda_{\mathrm{cowt}}$ have finite index we see $Z(G)$ and $\pi_1(G)$ are finite abelian groups.

It immediately follows from Proposition A.1.15 and Proposition A.1.16 that $\widehat{Z(G)} \cong X^*(T)/\Lambda_{\mathrm{rt}}$ and $\widehat{\pi_1(G)} \cong \Lambda_{\mathrm{wt}}/X^*(T)$ where $\widehat{A}$ denotes the Pontryagin dual of the group A. Moreover, the root datum defining ${}^LG$ is given by $(X_*(T), \Delta^\vee, X^*(T), \Delta)$, which is now used to compute the centre and fundamental group of ${}^LG$.

Lemma A.1.18. *There are canonical isomorphisms*

$$Z({}^LG) \cong \widehat{\pi_1(G)}$$

and

$$\pi_1({}^LG) \cong \widehat{Z(G)}.$$

Proof. Langlands duality interchanges the root and coroot lattices, weight and coweight lattices, and character and cocharacter lattices. Therefore, by Proposition A.1.15 we see $Z({}^LG) \cong X^*(T)/\Lambda_{\mathrm{rt}}$, which is canonically isomorphic to $\widehat{\pi_1(G)}$. By Proposition A.1.16 there is a canonical isomorphism $\pi_1({}^LG) \cong \Lambda_{\mathrm{wt}}/X^*(T)$, which is canonically isomorphic to $\widehat{Z(G)}$. □

By the fundamental theorem of finitely generated abelian groups every finite abelian group is the product of finitely many cyclic groups. Although not canonical there is an isomorphism of groups $\widehat{\mathbb{Z}_n} \cong \mathbb{Z}_n$ [1] which establishes the following corollary.

Corollary A.1.19. *Langlands duality interchanges the centre and fundamental group, i.e., $Z({}^LG) \cong \pi_1(G)$ and $\pi_1({}^LG) \cong Z(G)$.*

A.2 Computing Weight Lattice Modulo Root Lattice

To compute the Langlands dual groups notice that the number of intermediate lattices $\Lambda_{\mathrm{rt}} \subseteq \Lambda \subseteq \Lambda_{\mathrm{wt}}$ is in bijection with subgroups of the quotient group $\Lambda_{\mathrm{wt}}/\Lambda_{\mathrm{rt}}$. We will compute the Langlands dual group for the classical simple Lie groups, hence in type A_n, B_n, C_n, and D_n we will compute the weight lattice modulo the root lattice. In each case $\mathfrak{t}$ will denote the standard Cartan subalgebra.

A.2.1 Type A_n

The root system A_n corresponds to $\mathfrak{sl}_{n+1}(\mathbb{C})$. Let $e_1, \ldots, e_{n+1}$ denote the standard basis for E. The Cartan subalgebra in this case is given by trace-free diagonal matrices, hence consider the canonical identification

$$\mathfrak{t} \cong \{(t_1, \ldots, t_{n+1}) \in \mathbb{C}^{n+1} \,|\, t_1 + \cdots + t_{n+1} = 0\}.$$

[1] The element 1 generates $\mathbb{Z}_n$ and under Pontryagin duality is identified to a n-th root of unity, which generates a cyclic group of order n.

Now, $\Delta = \{e_i - e_j \mid 1 \le i, j \le n+1;\ i \ne j\}$ and Δ has simple roots $\alpha_i = e_i - e_{i+1}$ where $1 \le i \le n$. Moreover, A_n is simply-laced and $(\alpha_i, \alpha_i) = 2$ so notice that $(\ ,\ ) = \langle\ ,\ \rangle$. Also,

$$\langle \alpha_i, \alpha_j \rangle = \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } |i-j| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Lemma A.2.1. *The root lattice is given by*

$$\Lambda_{\mathrm{rt}} = \left\{ \sum_{i=1}^{n+1} a_i e_i \mid a_i \in \mathbb{Z};\ \sum_{i=1}^{n+1} a_i = 0 \right\} =: \mathcal{S}.$$

Proof. Suppose $w \in \Lambda_{\mathrm{rt}}$, i.e. $w = a_1\alpha_1 + \cdots + a_n\alpha_n$ where $a_i \in \mathbb{Z}$. Then,

$$w = a_1 e_1 + (a_2 - a_1)e_2 + \cdots + (a_n - a_{n-1})e_n - a_n e_{n+1}$$

and

$$a_1 + (a_2 - a_1) + \cdots + (a_n - a_{n-1}) - a_n = 0,$$

hence $w \in \mathcal{S}$.

Conversely suppose $v \in \mathcal{S}$, i.e., $v = \sum_{i=1}^{n+1} a_i e_i$ where $a_i \in \mathbb{Z}$ and $\sum_{i=1}^{n+1} a_i = 0$. Then,

$$v = a_1(e_1 - e_2) + a_2(e_2 - e_3) + \cdots + a_{n+1}(e_n - e_{n+1}),$$

that is,

$$v = a_1\alpha_1 + (a_1 + a_2)\alpha_2 + \cdots + (a_1 + \cdots + a_n)\alpha_n,$$

thus $v \in \Lambda_{\mathrm{rt}}$. $\square$

Recall that $\Lambda_{\mathrm{wt}} = \{w \in \mathfrak{t}^*_{\mathbb{R}} \mid (w, \alpha_i) \in \mathbb{Z} \text{ for } 1 \le i \le n+1\}$. Let $w \in \Lambda_{\mathrm{wt}}$, then $(w, \alpha_i) = w_i - w_{i+1} \in \mathbb{Z}$ and it follows that $w_i \equiv w_j \mod \mathbb{Z}$. Thus, $w_i = m_i + r$ where $m_i \in \mathbb{Z}$ and $0 \le r < 1$ for $1 \le i \le n+1$. However, $w \in \mathfrak{t}^*_{\mathbb{R}} \subset \mathfrak{t}$, hence $w_1 + \cdots + w_{n+1} = 0$ and so

$$(m_1 + \cdots + m_{n+1}) + (n+1)r = 0$$

thus, $r \in \frac{1}{n+1}\mathbb{Z}$.

Consider the homomorphism

$$\Lambda_{\mathrm{wt}} \ni (w_1, \ldots, w_{n+1}) \mapsto (n+1)w_1 \mod (n+1) \in \mathbb{Z}_{n+1},$$

which is surjective since $\left(\frac{1}{n+1}, \ldots, \frac{1}{n+1}, -\frac{n}{n+1}\right) \mapsto 1$. The kernel is precisely the set of $w \in \Lambda_{\mathrm{wt}}$ with $w_i \in \mathbb{Z}$ for each $i = 1, \ldots, n+1$, i.e., the kernel is Λ_{rt}. Therefore,

$$\Lambda_{\mathrm{wt}}/\Lambda_{\mathrm{rt}} \cong \mathbb{Z}_{n+1}.$$

A.2.2 Type B_n

The root system B_n corresponds to $\mathfrak{so}_{2n+1}(\mathbb{C})$, and the cartan subalgebra $\mathfrak{t}$ consists of diagonal matrices of the form

$$\operatorname{diag}(0, t_1, \ldots, t_n, -t_1, \ldots, -t_n).$$

Hence, there is a canonical identification $\mathfrak{t} \cong \{(t_1, \ldots, t_n) \in \mathbb{C}^n\}$. Let $e_1, \ldots, e_n$ denote the standard basis of E. Now, $\Delta = \{\pm e_i \pm e_j, \pm e_i \,|\, 1 \leq i < j \leq n\}$ and the simple roots are given by $\alpha_i = e_i - e_{i+1}$ where $1 \leq i \leq n-1$ and $\alpha_n = e_n$. Similar to the A_n case

$$\langle \alpha_i, \alpha_j \rangle = \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } |i-j| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

for $1 \leq i, j \leq n-1$. Also, $\langle \alpha_n, \alpha_n \rangle = 2$ and $\langle \alpha_n, \alpha_i \rangle = -1$ if $i = n-1$ and 0 otherwise. Finally, $\langle \alpha_i, \alpha_n \rangle = -2$ for $i = n-1$ and 0 otherwise.

Lemma A.2.2. *The root lattice is given by*

$$\Lambda_{\mathrm{rt}} = \left\{ \sum_{i=1}^{n} a_i e_i \,|\, a_i \in \mathbb{Z} \right\} = \mathbb{Z}^n.$$

Proof. It is clear that $\Lambda_{\mathrm{rt}} \subseteq \mathbb{Z}^n$. Suppose $v \in \mathbb{Z}^n$, i.e., $v = a_1 e_1 + \ldots + a_n e_n$ where $a_i \in \mathbb{Z}$. Notice that

$$v = a_1(e_1 - e_2) + (a_1 + a_2)(e_2 - e_3) + \cdots + (a_1 + \cdots a_{n-1})(e_{n-1} - e_n) + (a_1 + \cdots + a_n)e_n$$

that is,

$$v = a_1\alpha_1 + (a_1 + a_2)\alpha_2 + \cdots + (a_1 + \cdots + a_n)\alpha_n.$$

Hence, $v \in \Lambda_{\mathrm{rt}}$ and thus, $\Lambda_{\mathrm{rt}} = \mathbb{Z}^n$. $\square$

The weight lattice is given by $\Lambda_{\mathrm{wt}} = \{w \in \mathfrak{t}^*_{\mathbb{R}} \,|\, \langle w, \alpha_i \rangle \in \mathbb{Z} \text{ for } 1 \leq i \leq n\}$. Suppose $1 \leq i \leq n-1$, then it follows that $w_i - w_{i+1} \in \mathbb{Z}$ and thus, $w_i \equiv w_j \mod \mathbb{Z}$, for $w_j = m_j + r$ where $m_j \in \mathbb{Z}$ and $0 \leq r < 1$. From $\langle w, \alpha_n \rangle = 2w_n \in \mathbb{Z}$ we see $r \in \frac{1}{2}\mathbb{Z}$, hence either $r = 0$, or $r = \frac{1}{2}$.

Consider the homomorphism

$$\Lambda_{\mathrm{wt}} \ni (m_1 + r, \ldots, m_n + r) \mapsto 2r \in \mathbb{Z}_2,$$

which is clearly surjective. Moreover, an element belongs to the kernel exactly when $r = 0$, i.e., the kernel is Λ_{rt}. Therefore,

$$\Lambda_{\mathrm{wt}}/\Lambda_{\mathrm{rt}} \cong \mathbb{Z}_2.$$

A.2.3 Type C_n

The root system C_n corresponds to $\mathfrak{sp}_{2n}(\mathbb{C})$, and the Cartan subalgebra $\mathfrak{t}$ consists of diagonal matrices

$$\operatorname{diag}(t_1, \ldots, t_n, -t_1, \ldots, t_n).$$

Hence, there is a canonical identification $\mathfrak{t} \cong \{(t_1, \ldots, t_n) \in \mathbb{C}^n\}$. Let $e_1, \ldots, e_n$ denote the standard basis of E. Now, $\Delta = \{\pm e_i \pm e_j, \pm 2e_i \,|\, 1 \leq i < j \leq n\}$ and the simple roots are given by $\alpha_i = e_i - e_{i+1}$ where $1 \leq i \leq n-1$ and $\alpha_n = 2e_n$. Similar to the B_n case

$$\langle \alpha_i, \alpha_j \rangle = \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } |i-j| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

for $1 \leq i, j \leq n-1$. Also, $\langle \alpha_n, \alpha_n \rangle = 2$ and $\langle \alpha_n, \alpha_i \rangle = -2$ if $i = n-1$ and 0 otherwise. Finally, $\langle \alpha_i, \alpha_n \rangle = -1$ if $i = n-1$ and 0 otherwise.

Lemma A.2.3. *The root lattice is given by*

$$\Lambda_{\mathrm{rt}} = \left\{ \sum_{i=1}^{n} a_i e_i \,|\, a_i \in \mathbb{Z};\ \sum_{i=1}^{n} a_i \in 2\mathbb{Z} \right\} =: \mathcal{S}.$$

Proof. Suppose $v = a_1\alpha_1 + \cdots + a_n\alpha_n$ where $a_i \in \mathbb{Z}$. Then,

$$v = a_1e_1 + (a_2 - a_1)e_2 + \cdots + (a_{n-1} - a_{n-2})e_{n-1} + (2a_n - a_{n-1})e_n.$$

Notice that

$$a_1 + (a_2 - a_1) + \cdots + (a_{n-1} - a_{n-2}) + (2a_n - a_{n-1}) = 2a_n,$$

which is even, hence $\Lambda_{\mathrm{rt}} \subseteq \mathcal{S}$.

Conversely, suppose $w = \sum_{i=1}^n a_ie_i$ where $a_i \in \mathbb{Z}$ and $\sum_{i=1}^n a_i \in 2\mathbb{Z}$. Then,

$$w = a_1(e_1 - e_2) + (a_1 + a_2)(e_2 - e_3) + \cdots + \frac{1}{2}(a_1 + \cdots + a_n)2e_n,$$

i.e.,

$$w = a_1\alpha_1 + (a_1 + a_2)\alpha_2 + \cdots + \frac{1}{2}(a_1 + \cdots + a_n)\alpha_n$$

and since $a_1 + \cdots + a_n \in 2\mathbb{Z}$ it follows that $w \in \Lambda_{\mathrm{rt}}$, so $\Lambda_{\mathrm{rt}} = \mathcal{S}$. $\square$

The weight lattice is given by $\Lambda_{\mathrm{wt}} = \{w \in \mathfrak{t}^*_{\mathbb{R}} \,|\, \langle w, \alpha_i \rangle \in \mathbb{Z} \text{ for } 1 \leq i \leq n\}$, hence $w_i \equiv w_j \mod \mathbb{Z}$. Notice that $\langle w, \alpha_n \rangle = w_n \in \mathbb{Z}$ and thus, $w_i \in \mathbb{Z}$ for $1 \leq i \leq n$.

Consider the homomorphism

$$\Lambda_{\mathrm{wt}} \ni (w_1, \ldots, w_n) \mapsto \sum_{i=1}^{n} w_i \mod 2 \in \mathbb{Z}_2,$$

which is clearly surjective. Moreover, the kernel is precisely Λ_{rt}, and thus

$$\Lambda_{\mathrm{wt}}/\Lambda_{\mathrm{rt}} \cong \mathbb{Z}_2.$$

A.2.4 Type D_n

The root system D_n corresponds $\mathfrak{so}_{2n}(\mathbb{C})$ so the Cartan subalgebra $\mathfrak{t}$ consists of matrices of the form

$$\operatorname{diag}(t_1, \ldots, t_n, -t_1, \ldots, -t_n).$$

Hence, there is a canonical identification $\mathfrak{t} \cong \{(t_1, \ldots, t_n) \in \mathbb{C}^n\}$. Let $e_1, \ldots, e_n$ denote the standard basis for E. The root system is given by $\Delta = \{\pm e_i \pm e_j \,|\, 1 \leq i < j \leq n\}$, and the simple roots are given by $\alpha_i = e_i - e_{i+1}$ for $1 \leq i \leq n-1$ and $\alpha_n = e_{n-1} + e_n$. Similar to the C_n case

$$\langle \alpha_i, \alpha_j \rangle = \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } |i-j| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

for $1 \leq i, j \leq n-1$. Also, $\langle \alpha_n, \alpha_n \rangle = 2$ and for $1 \leq i \leq n-1$ we see $\langle \alpha_n, \alpha_i \rangle = -1 = \langle \alpha_i, \alpha_n \rangle$ if $i = n-1$ and 0 otherwise.

Lemma A.2.4. *The root lattice is given by*

$$\Lambda_{\mathrm{rt}} = \left\{ \sum_{i=1}^{n} a_i e_i \,|\, a_i \in \mathbb{Z};\ \sum_{i=1}^{n} a_i \in 2\mathbb{Z} \right\} =: \mathcal{S}.$$

Proof. Suppose $v \in \Lambda_{\mathrm{rt}}$, i.e., $v = a_1\alpha_1 + \cdots + a_n\alpha_n$ where $a_i \in \mathbb{Z}$. Then,

$$v = a_1 e_1 + (a_2 - a_1)e_2 + \cdots + (a_{n-1} - a_{n-2} + a_n)e_{n-1} + (a_n - a_{n-1})e_n$$

and hence,

$$a_1 + (a_2 - a_1) + \cdots (a_{n-1} - a_{n-2} + a_n) + (a_n - a_{n-1}) = 2a_n,$$

which is even. Thus, $v \in \mathcal{S}$ and so $\Lambda_{\mathrm{rt}} \subseteq \mathcal{S}$.

Conversely, suppose $w = a_1 e_1 + \cdots + a_n e_n$ where $a_i \in \mathbb{Z}$ and $a_1 + \cdots + a_n \in 2\mathbb{Z}$. Then,

$$v = a_1\alpha_1 + (a_1 + a_2)\alpha_2 + \cdots + \frac{1}{2}(a_1 + \cdots + a_{n-1} - a_n)\alpha_{n-1} + \frac{1}{2}(a_1 + \cdots + a_n)\alpha_n.$$

Notice that $\frac{1}{2}(a_1 + \cdots + a_n) \in \mathbb{Z}$ and moreover,

$$\frac{1}{2}(a_1 + \cdots + a_{n-1} - a_n) = (a_1 + \cdots + a_{n-1}) - \frac{1}{2}(a_1 + \cdots + a_n) \in \mathbb{Z}.$$

Therefore, $w \in \Lambda_{\mathrm{rt}}$ and thus, $\Lambda_{\mathrm{rt}} = \mathcal{S}$. □

Since D_n is simply-laced the weight lattice is given by

$$\Lambda_{\text{wt}} = \{w \in \mathfrak{t}^*_{\mathbb{R}} \mid (w, \alpha_i) \in \mathbb{Z} \text{ for } 1 \leq i \leq n\}$$

and it follows that $w_i \equiv w_j \mod \mathbb{Z}$ and $w_{n+1} + w_n \in \mathbb{Z}$. From this description it is clear that $w_i = m_i + r$ where $m_i \in \mathbb{Z}$ and $r = 0$ or $r = \frac{1}{2}$.

We will compute $\Lambda_{\text{wt}}/\Lambda_{\text{rt}}$ by computing the cosets explicitly. Notice that if $w \in \Lambda_{\text{wt}}$, then $4w \in \Lambda_{\text{rt}}$ and thus $[\Lambda_{\text{wt}} : \Lambda_{\text{rt}}] \leq 4$. Consequently, given four distinct cosets the quotient group can be computed by studying the cosets' order.

Let $v = \left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$ and $w = (1, \ldots, 0)$. Suppose first that n is odd. Notice that $v \notin \Lambda_{\text{rt}}$ and $3v \notin \Lambda_{\text{rt}}$. Moreover, $2v \notin \Lambda_{\text{rt}}$ since $\sum_{i=1}^{n} 1$ is odd. Therefore,

$$\Lambda_{\text{wt}}/\Lambda_{\text{rt}} = \{[0], [v], [2v], [3v]\} \cong \mathbb{Z}_4.$$

Suppose now that n is even. Again $v \notin \Lambda_{\text{rt}}$, and $w \notin \Lambda_{\text{rt}}$. Moreover, $2w \in \Lambda_{\text{rt}}$ and $2v \in \Lambda_{\text{rt}}$ since n is even. Now, $v + w \notin \Lambda_{\text{rt}}$ and it is clear that $[v + w]$ is distinct to $[v]$ and $[w]$ and it follows that

$$\Lambda_{\text{wt}}/\Lambda_{\text{rt}} = \{[0], [v], [w], [v + w]\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2.$$

Therefore, the weight lattice modulo the root lattice depends on the parity of n, in particular

$$\Lambda_{\text{wt}}/\Lambda_{\text{rt}} \cong \begin{cases} \mathbb{Z}_4 & \text{if } n \text{ is odd} \\ \mathbb{Z}_2 \times \mathbb{Z}_2 & \text{if } n \text{ is even.} \end{cases}$$

A.3 Computing Langlands Dual Groups

Using the computation of the weight lattice modulo the root lattice in the root systems A_n, B_n, C_n and D_n we will compute the Langlands dual groups of the classical simple matrix Lie groups by studying the centre and fundamental groups. In each case, T denotes the maximal torus that corresponds to the Cartan subalgebra $\mathfrak{t}$.

A.3.1 Langlands Duality for $\mathrm{SL}_n(\mathbb{C})$ and $\mathrm{PGL}_n(\mathbb{C})$

The root system A_{n-1} where $n \geq 2$ is simply laced, hence self dual. Moreover, A_{n-1} corresponds to $\mathfrak{sl}_n(\mathbb{C})$. In (A.2.1) it was shown that $\Lambda_{\text{wt}}/\Lambda_{\text{rt}} \cong \mathbb{Z}_n$. The subgroups of $\mathbb{Z}_n$ are precisely $\mathbb{Z}_m$ where m divides n. Hence, every distinct subgroup of $\mathbb{Z}_n$ has unique order and thus, every distinct intermediate lattice $\Lambda_{\text{rt}} \subseteq \Lambda \subseteq \Lambda_{\text{wt}}$ has unique index. Therefore, there is no ambiguity computing the Langlands dual group by interchanging the centre and fundamental group.

The centre of $\mathrm{SL}_n(\mathbb{C})$ naturally corresponds to n-th root of unity, hence $Z(\mathrm{SL}_n(\mathbb{C})) \cong \mathbb{Z}_n$. Recall that the centre is isomorphic to $X^*(T)/\Lambda_{\text{rt}}$ hence, $X^*(T)/\Lambda_{\text{rt}} \cong \mathbb{Z}_n$. However,

$X^*(T)/\Lambda_{\text{rt}} \subset \Lambda_{\text{wt}}/\Lambda_{\text{rt}} \cong \mathbb{Z}_n$ and it follows that $X^*(T) \cong \Lambda_{\text{wt}}$. Therefore, by Proposition A.1.13 we see $\pi_1(\mathrm{SL}_n(\mathbb{C})) = 1$, i.e., $\mathrm{SL}_n(\mathbb{C})$ is simply connected.

Thus, the Langlands dual group of $\mathrm{SL}_n(\mathbb{C})$ is the complex connected semisimple Lie group with root system A_{n-1} that has trivial centre and fundamental group isomorphic to $\mathbb{Z}_n$. We claim $\mathrm{PGL}_n(\mathbb{C})$ is the Langlands dual group of $\mathrm{SL}_n(\mathbb{C})$. Since every complex number admits a n-th root it follows that $\mathrm{PGL}_n(\mathbb{C}) = \mathrm{PSL}_n(\mathbb{C})$, hence $\mathrm{SL}_n(\mathbb{C})$ is the universal cover of $\mathrm{PGL}_n(\mathbb{C})$. Since universal covers are local diffeomorphisms it follows that the Lie algebra $\mathfrak{pgl}_n(\mathbb{C})$ is isomorphic to $\mathfrak{sl}_n(\mathbb{C})$ and thus, $\mathrm{PGL}_n(\mathbb{C})$ has root system A_{n-1}.

Lemma A.3.1. *The fundamental group of* $\mathrm{PGL}_n(\mathbb{C})$ *is a cyclic group of order n, i.e.,* $\pi_1(\mathrm{PGL}_n(\mathbb{C})) \cong \mathbb{Z}_n$.

Proof. Consider the fibration

$$\mathbb{Z}_n \to \mathrm{SL}_n(\mathbb{C}) \to \mathrm{PGL}_n(\mathbb{C})$$

then passing to the long exact sequence in homotopy gives an exact sequence

$$1 \to \pi_1(\mathrm{PGL}_n(\mathbb{C})) \to \mathbb{Z}_n \to 1.$$

Therefore, $\pi_1(\mathrm{PGL}_n(\mathbb{C})) \cong \mathbb{Z}_n$. □

Since the fundamental group of $\mathrm{PGL}_n(\mathbb{C})$ is isomorphic to $\Lambda_{\text{wt}}/X^*(T) \subseteq \Lambda_{\text{wt}}/\Lambda_{\text{rt}} \cong \mathbb{Z}_n$ it follows that $X^*(T) \cong \Lambda_{\text{rt}}$, hence $Z(\mathrm{PGL}_n(\mathbb{C})) = 1$. Therefore, $\pi_1(\mathrm{SL}_n(\mathbb{C})) \cong Z(\mathrm{PGL}_n(\mathbb{C}))$ and $Z(\mathrm{SL}_n(\mathbb{C})) \cong \pi_1(\mathrm{PGL}_n(\mathbb{C}))$ and $\mathfrak{sl}_n(\mathbb{C}) \cong \mathfrak{pgl}_n(\mathbb{C})$ and thus,

$${}^L\mathrm{SL}_n(\mathbb{C}) \cong \mathrm{PGL}_n(\mathbb{C}).$$

A.3.2 Langlands Duality for Sp_{2n} and $\mathrm{SO}_{2n+1}(\mathbb{C})$

The root system B_n corresponds to $\mathfrak{so}_{2n+1}(\mathbb{C})$ and C_n corresponds to $\mathfrak{sp}_{2n}(\mathbb{C})$. Also, the root systems B_n and C_n are dual.[2] Moreover, in (A.2.2) and (A.2.3) it was shown that the weight lattice modulo the root lattice in both cases is isomorphic to $\mathbb{Z}_2$. The only subgroups of $\mathbb{Z}_2$ are itself and the trivial group, hence the character lattice is either the weight lattice or root lattice in both cases. We will prove that Sp_{2n} and $\mathrm{SO}_{2n+1}(\mathbb{C})$ are Langlands dual, and by the discussion it suffices to compute their centre and fundamental groups.

Lemma A.3.2. *The centre of* $\mathrm{SO}_m(\mathbb{C})$ *depends on the parity of m. If m is even, then* $Z(\mathrm{SO}_m(\mathbb{C})) \cong \mathbb{Z}_2$*, and if m is odd, then* $Z(\mathrm{SO}_m(\mathbb{C})) \cong 1$.

[2] Dualsing the root system amounts to reversing the arrows in the Dynkin diagram, so it is clear that $B_n^\vee \cong C_n$ since B_n has Dynkin diagram and C_n has Dynkin diagram

Proof. Recall that the centre of $\mathrm{GL}_m(\mathbb{C})$ is given by $\{c1_m \mid c \in \mathbb{C}^*\}$, and $c1_m \in \mathrm{SO}_m(\mathbb{C})$ if and only if $c^m = 1$ and $c^2 = 1$. If m is even, $c = \pm 1$ and $Z(\mathrm{SO}_m(\mathbb{C})) \cong \mathbb{Z}_2$. However, if m is odd, $c = 1$ and $Z(\mathrm{SO}_m(\mathbb{C})) \cong 1$. $\square$

Lemma A.3.3. *The fundamental group of* $\mathrm{SO}_m(\mathbb{C})$ *where* $m \geq 3$ *is a cyclic group of order* 2*, i.e.,* $\pi_1(\mathrm{SO}_m(\mathbb{C})) \cong \mathbb{Z}_2$.

Proof. The complex Lie group $\mathrm{SO}_m(\mathbb{C})$ is the complexification of $\mathrm{SO}_m(\mathbb{R})$, and $\mathrm{SO}_m(\mathbb{R}) \subset \mathrm{SO}_m(\mathbb{C})$ is a maximal compact Lie subgroup. By considering polar decomposition there is a deformation retraction from $\mathrm{SO}_m(\mathbb{C}) \to \mathrm{SO}_m(\mathbb{R})$, hence $\pi_1(\mathrm{SO}_m(\mathbb{C})) \cong \pi_1(\mathrm{SO}_m(\mathbb{R}))$. Thus, it suffices to compute $\pi_1(\mathrm{SO}_m(\mathbb{R}))$. Now, $\mathrm{SO}_m(\mathbb{R})$ acts on $\mathbb{R}^n$ preserving the standard inner product, which induces a natural transitive action on the unit sphere with stabiliser $\mathrm{SO}_{m-1}(\mathbb{R})$. Hence, consider the fibration

$$\mathrm{SO}_m(\mathbb{R}) \to \mathrm{SO}_{m+1}(\mathbb{R}) \to S^m$$

then passing to the long exact sequence in homotopy gives

$$\cdots \to \pi_2(S^m) \to \pi_1(\mathrm{SO}_m(\mathbb{R})) \to \pi_1(\mathrm{SO}_{m+1}(\mathbb{R})) \to \pi_1(S^m) \to 1.$$

However, $\pi_2(S^m)$ and $\pi_1(S^m)$ are trivial since $m \geq 3$ and it follows that $\pi_1(\mathrm{SO}_m(\mathbb{R})) \cong \pi_1(\mathrm{SO}_{m+1}(\mathbb{R}))$. In particular, $\pi_1(\mathrm{SO}_m(\mathbb{R})) \cong \pi_1(\mathrm{SO}_3(\mathbb{R}))$ for each $m \geq 3$. Therefore, it suffices to compute $\pi_1(\mathrm{SO}_3(\mathbb{R}))$, and since $\mathrm{SO}_3(\mathbb{R})$ is homeomorphic to $\mathbb{RP}^3$ we see $\pi_1(\mathrm{SO}_3(\mathbb{R})) \cong \pi_1(\mathbb{RP}^3) \cong \mathbb{Z}_2$. $\square$

To compute the centre of Sp_{2n} recall that a $2n \times 2n$ matrix M belongs to Sp_{2n} if and only if $M^\top \Omega M = \Omega$ where $\Omega = \begin{bmatrix} 0 & 1_n \\ -1_n & 0 \end{bmatrix}$. Hence, $c1_{2n} \in \mathrm{Sp}_{2n}$ if and only if $c^2 = 1$, i.e., $c = \pm 1$, hence $Z(\mathrm{Sp}_{2n}) \cong \mathbb{Z}_2$. Recall that the centre is isomorphic to $X^*(T)/\Lambda^{\mathrm{rt}} \subset \Lambda_{\mathrm{wt}}/\Lambda_{\mathrm{rt}} \cong \mathbb{Z}_2$ and it follows that $\Lambda_{\mathrm{wt}} = X^*(T)$. In particular, Sp_{2n} is simply connected, i.e., $\pi_1(Sp_{2n}) \cong 1$.

Therefore, $Z(\mathrm{SO}_{2m+1}(\mathbb{C})) \cong \pi_1(\mathrm{Sp}_{2n})$ and $Z(\mathrm{Sp}_{2n}) \cong \pi_1(\mathrm{SO}_{2m+1}(\mathbb{C}))$, and the Lie algebras $\mathfrak{sp}_{2n}(\mathbb{C})$ and $\mathfrak{so}_{2n+1}(\mathbb{C})$ are dual, which shows

$${}^L\mathrm{Sp}_{2n} \cong \mathrm{SO}_{2n+1}(\mathbb{C}).$$

A.3.3 Langlands Duality for $\mathrm{SO}_{2n}(\mathbb{C})$

The root system D_n is simply laced, hence self-dual and corresponds to $\mathfrak{so}_{2n}(\mathbb{C})$. In (A.2.4) it was shown $\Lambda_{\mathrm{wt}}/\Lambda_{\mathrm{rt}}$ depends on the parity of n, namely $\Lambda_{\mathrm{wt}}/\Lambda_{\mathrm{rt}} \cong \mathbb{Z}_4$ if n is odd, and $\Lambda_{\mathrm{wt}}/\Lambda_{\mathrm{rt}} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and n is even.

Suppose first that n is odd. Then the distinct subgroups of $\Lambda_{\mathrm{wt}}/\Lambda_{\mathrm{rt}} \cong \mathbb{Z}_4$ have unique order, so the distinct intermediate lattices $\Lambda_{\mathrm{rt}} \subseteq \Lambda \subseteq \Lambda_{\mathrm{wt}}$ has distinct index so it suffices to determine the Langlands dual groups by computing the centre and fundamental group. By Lemma A.3.2 we see $Z(\mathrm{SO}_{2n}(\mathbb{C})) \cong \mathbb{Z}_2$, and Lemma A.3.3 we see $\pi_1(\mathrm{SO}_{2n}(\mathbb{C})) \cong \mathbb{Z}_2$. Hence, $Z(\mathrm{SO}_{2n}(\mathbb{C})) \cong \pi_1(\mathrm{SO}_{2n}(\mathbb{C}))$ and since $\mathfrak{so}_{2n}(\mathbb{C})$ is self-dual we have shown

$$^L\mathrm{SO}_{2n}(\mathbb{C}) \cong \mathrm{SO}_{2n}(\mathbb{C}) \quad \text{where } n \text{ is odd.}$$

Suppose now that n is even. We will prove $^L\mathrm{SO}_{2n}(\mathbb{C}) \cong \mathrm{SO}_{2n}(\mathbb{C})$. Since $\Lambda_{\mathrm{wt}}/\Lambda_{\mathrm{rt}} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ there are three distinct subgroups of order 2, hence there are three distinct intermediate lattices of index 2. Thus, it is no longer sufficient to compute the duality from the centre and fundamental group alone. Recall from (A.2.4) the root lattice is given by

$$\Lambda_{\mathrm{rt}} = \{(a_1, \dots, a_n) \in \mathbb{Z}^n \mid a_1 + \cdots + a_n \in 2\mathbb{Z}\},$$

and the weight lattice is given by

$$\Lambda_{\mathrm{wt}} = \left\{(w_1, \dots, w_n) \in \mathbb{R}^n \mid w_i = m_i + r, \text{where } m_i \in \mathbb{Z} \text{ and } r = 0 \text{ or } r = \frac{1}{2}\right\}$$

Moreover, $\Lambda_{\mathrm{wt}}/\Lambda_{\mathrm{rt}} = \{[0], [v], [w], [v+w]\}$ where $v = \frac{1}{2}(1, \dots, 1)$ and $w = (1, \dots, 0)$. Hence, the subgroup lattice of $\Lambda_{\mathrm{wt}}/\Lambda_{\mathrm{rt}}$ is given by

$$\begin{tikzcd}[arrows=dash]
 & \mathbb{Z}_2 \times \mathbb{Z}_2 \arrow[dl] \arrow[d] \arrow[dr] & \\
\langle v \rangle \arrow[dr] & \langle w \rangle \arrow[d] & \langle v + w \rangle \arrow[dl] \\
 & 1 &
\end{tikzcd}$$

and the corresponding lattice of lattices is given by

$$\begin{tikzcd}[arrows=dash]
 & \Lambda_{\mathrm{wt}} \arrow[dl] \arrow[d] \arrow[dr] & \\
\Lambda_{\mathrm{rt}} + \langle v \rangle \arrow[dr] & \Lambda_{\mathrm{rt}} + \langle w \rangle \arrow[d] & \Lambda_{\mathrm{rt}} + \langle v + w \rangle \arrow[dl] \\
 & \Lambda_{\mathrm{rt}} &
\end{tikzcd} .$$

It is clear that $\Lambda_{\mathrm{rt}} + \langle w \rangle = \mathbb{Z}^n$, and the dual lattice of the standard lattice is isomorphic to the standard lattice, hence $(\Lambda_{\mathrm{rt}} + \langle w \rangle)^* \cong \Lambda_{\mathrm{rt}} + \langle w \rangle$. Now, $\Lambda_{\mathrm{rt}} + \langle v \rangle \subseteq \Lambda_{\mathrm{wt}}$ has index 2, so $\Lambda_{\mathrm{wt}}^* \subseteq (\Lambda_{\mathrm{rt}} + \langle v \rangle)^*$ has index 2. However, $\Lambda_{\mathrm{wt}}^* = \Lambda_{\mathrm{cort}}$ and since the root system is self dual it follows that $\Lambda_{\mathrm{rt}} \subseteq (\Lambda_{\mathrm{rt}} + \langle v \rangle)^*$ has index 2. To compute $(\Lambda_{\mathrm{rt}} + \langle v \rangle)^*$ it suffices to check which coset belongs to the lattice.

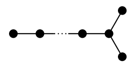
Now, $(v, v) = \frac{n}{4}$ so if 4 divides n, then $(v, v) \in \mathbb{Z}$ and if 4 does not divide n, then $(v, v) \notin \mathbb{Z}$. Similarly, $(v + w, v) = \frac{n}{4} + \frac{1}{2}$ and it follows that $(v + w, v) \in \mathbb{Z}$ if 4 does not divide n, and $(v + w, v) \notin \mathbb{Z}$ if 4 divides n. Thus,

$$(\Lambda_{\mathrm{rt}} + \langle v\rangle)^* \cong \begin{cases} \Lambda_{\mathrm{rt}} + \langle v\rangle & \text{if 4 divides } n \\ \Lambda_{\mathrm{rt}} + \langle v + w\rangle & \text{if 4 does not divide } n. \end{cases}$$

Likewise,

$$(\Lambda_{\mathrm{rt}} + \langle v + w\rangle)^* \cong \begin{cases} \Lambda_{\mathrm{rt}} + \langle v + w\rangle & \text{if 4 divides } n \\ \Lambda_{\mathrm{rt}} + \langle v\rangle & \text{if 4 does not divide } n. \end{cases}$$

Therefore, the only intermediate lattice preserved under dualising for each n is $\Lambda_{\mathrm{rt}} + \langle w\rangle$. Thus, to prove ${}^L\mathrm{SO}_{2n}(\mathbb{C}) \cong \mathrm{SO}_{2n}(\mathbb{C})$ we will show the character lattice is $\Lambda_{\mathrm{rt}} + \langle w\rangle$.

The root system D_n is simple and [FH91, Proposition D.40] shows that the outer automorphisms of a simple Lie algebra are precisely the group of graph automorphisms of the associated Dynkin diagram. Considering the Dynkin diagram for D_n : 
there is a canonical diagram automorphism, namely $\phi : \mathfrak{t}^*_{\mathbb{R}} \to \mathfrak{t}^*_{\mathbb{R}}$ defined by $\phi(e_i) = e_i$ for $i \neq n$ and $\phi(e_n) = -e_n$ and notice that $\phi^2 = \mathrm{id}$. Hence, ϕ is an outer automorphism of $\mathfrak{so}_{2n}(\mathbb{C})$. Moreover, since ϕ acts by conjugating by a determinant -1 orthogonal matrix it immediately follows that $\phi(\mathrm{SO}_{2n}(\mathbb{C})) = \mathrm{SO}_{2n}(\mathbb{C})$. Therefore, to prove the character lattice of $\mathrm{SO}_{2n}(\mathbb{C})$ is given by $\Lambda_{\mathrm{rt}} + \langle w\rangle$ it suffices to prove $\Lambda_{\mathrm{rt}} + \langle w\rangle$ is the only intermediate lattice invariant under ϕ.

Observe that $\phi(v) = \frac{1}{2}(1, \ldots, 1, -1) = v - e_n$ and since $w + e_n \in \Lambda_{\mathrm{rt}}$ notice that $w \equiv -e_n \mod \Lambda_{\mathrm{rt}}$ so $\phi(v) = v + w \mod \Lambda_{\mathrm{rt}}$ and thus, $\phi(\Lambda_{\mathrm{wt}} + \langle v\rangle) = \Lambda_{\mathrm{rt}} + \langle v + w\rangle$. Since $\phi(w) = w$ it follows immediately that $\phi(\Lambda_{\mathrm{wt}} + \langle w\rangle) = \Lambda_{\mathrm{wt}} + \langle w\rangle$. Therefore, $\Lambda_{\mathrm{wt}} + \langle w\rangle$ is the only lattice to be preserved by ϕ and thus,

$${}^L\mathrm{SO}_{2n}(\mathbb{C}) \cong \mathrm{SO}_{2n}(\mathbb{C}).$$

A.3.4 Exceptional Lie Algebras

Each exceptional Lie algebra is self-dual so computing Langlands duality distills down to knowing the index of the root lattice in the weight lattice. Of course, if the order is not prime or not one we cannot immediately conclude duality for every group without more information, however it turns out that in each case the index is either prime or one. The reader may consult [Car05, Appendix A] for a statement of the index in each exceptional case.

Appendix B

Positive Line Bundles on Complex Tori

This appendix is self-contained and states the necessary machinery to prove that a positive line bundle on a complex torus is precisely a positive definite line bundle. We will give a reference for every lemma stated, however, we will only prove the desired result. In what proceeds we assume that the reader is familiar with the definition of a hermitian line bundle and has knowledge of some Hodge theory, namely knowledge of harmonic forms, particularly on complex tori.

Let X be a complex manifold. Every holomorphic line bundle $\mathcal{L} \to X$ is equipped with a canonical operator $\overline{\partial}_{\mathcal{L}} : \Gamma(X, \mathcal{L}) \to \Omega^{0,1}(X, \mathcal{L})$. We may describe the operator in terms of transition functions and an open cover. Suppose $\{U_i\}$ is an open cover of X that trivialises $\mathcal{L}$ over each U_i, and let g_{ij} denote the corresponding transition functions. Then, for a smooth section s of $\mathcal{L}$ we set $s_i := s|_{U_i}$ and we define $(\overline{\partial}_{\mathcal{L}} s)_i := \overline{\partial} s_i$. To see that this is well-defined we write $s_i = g_{ij} s_j$, then by the product rule

$$\overline{\partial} s_i = \overline{\partial}(g_{ij} s_j) = g_{ij} \overline{\partial} s_j,$$

which proves that the operator is well-defined.

Theorem B.0.1 ([Wel08, Theorem 3.2.1])**.** *Let X be a complex manifold and suppose $(\mathcal{L}, h)$ is a hermitian line bundle on X. Then, there exists a unique connection ∇ called the Chern connection that satisfies*

(i) $\nabla^{0,1} = \overline{\partial}_{\mathcal{L}}$;

(ii) ∇ *is h-compatible, i.e.,*

$$\mathrm{d}h(s,t) = h(\nabla s, t) + h(s, \nabla t).$$

Proposition B.0.2 ([Wel08, Proposition 6.2.2])**.** *Suppose that X is compact. A line bundle $\mathcal{L} \to X$ is positive if and only if $\mathcal{L}$ admits a hermitian metric h such that the curvature of the associated Chern connection ∇ is a positive $(1,1)$-form.*

Now, let $A = V/\Gamma$ be a complex torus. We identify the Néron-Severi group $\mathrm{NS}(A)$ with the group of hermitian forms $H : V \times V \to \mathbb{C}$ with $\Im H(\Gamma, \Gamma) \subseteq \mathbb{Z}$.

Definition B.0.3. A *semicharacter* for a hermitian form $H : V \times V \to \mathbb{C}$ is a map $\chi : \Gamma \to U(1)$ satisfying

$$\chi(\lambda + \mu) = \chi(\lambda)\chi(\mu)\exp(\pi i H(\lambda, \mu)).$$

Let $H : V \times V \to \mathbb{C}$ be a hermitian form on V with corresponding semicharacter χ. Define $\alpha : \Gamma \times V \to \mathbb{C}^*$ by

$$\alpha(\lambda, v) = \chi(\lambda)\exp\left(\pi H(v, \lambda) + \frac{\pi}{2}H(\lambda, \lambda)\right).$$

The pair defines a holomorphic line bundle $L(H, \chi) \to A$ where the total space is the complex manifold defined by

$$L(H, \chi) = (V \times \mathbb{C})/\Gamma$$

where Γ acts on $V \times \mathbb{C}$ by $(\lambda, (v, t)) \mapsto (v + \lambda, \alpha(\lambda, v)t)$.

Lemma B.0.4 ([BL04, Lemma 2.2.1])**.** *The first Chern class of $L(H, \chi)$ corresponds to the hermitian form H.*

Theorem B.0.5 (Appell-Humbert Theorem [BL04, Theorem 2.2.3])**.** *For every holomorphic line bundle $\mathcal{L} \to A$ there exists a unique hermitian form H and corresponding semicharacter χ such that $\mathcal{L} \cong L(H, \chi)$.*

Before proving the desired result we will define the average of a closed 2-form on A. Let $\alpha \in \Omega^k(A)$ be a closed 2-form. There is a canonical representation $A \ni a \mapsto t_a^* \in \mathrm{GL}(\Omega^k(A))$ for each k. A k-form is a constant if it is invariant under translation, and constant k-forms on A are precisely harmonic k-forms. We define the *average* of a k-form α to be the k-form

$$\mathrm{avg}(\alpha) = \frac{1}{\mathrm{vol}(A)} \int_A t_a^*(\alpha)\,\mathrm{d}a$$

where $\mathrm{d}a$ is a Haar measure. It is clear that the average is constant, and hence, harmonic. Moreover, by the dominated convergence theorem, the mean value theorem, and the fact that $d \circ t_a^* = t_a^* \circ \mathrm{d}$ it follows that $\mathrm{d}\mathrm{avg}(\alpha) = \mathrm{avg}(\mathrm{d}\alpha)$. This in fact implies that the average is the unique harmonic representative of the de Rham cohomology class $[\alpha] \in \mathrm{H}^k_{dR}(A)$. Indeed, if β is the unique harmonic representative, then $\alpha = \beta + \mathrm{d}\eta$ for some $(k-1)$-form η. Then, since β is constant, $\mathrm{avg}(\beta) = \beta$ and it follows that

$$\mathrm{avg}(\alpha) = \beta + \mathrm{d}\mathrm{avg}(\eta) = \beta.$$

We can now prove our desired result.

Proposition B.0.6. *A line bundle $\mathcal{L} \to A$ is positive definite if and only if it is positive.*

Proof. Suppose that $\mathcal{L} \cong L(H, \chi)$ is positive definite. To prove that $\mathcal{L}$ is positive we will construct a hermitian metric on $\mathcal{L}$ such that the curvature of the associated Chern connection naturally corresponds to H. Consider the hermitian form $h : (V \times \mathbb{C}) \times (V \times \mathbb{C}) \to \mathbb{C}$ defined by

$$h((v,t),(v,s)) = \bar{t}\exp(-\pi H(v,v))s.$$

It is straightforward to show that h is invariant under the action of the lattice, and hence, descends to a hermitian metric on $\mathcal{L}$. Choose a vector space basis $e_1, \ldots, e_n$ for V and set $H_{ij} := H(e_i, e_j)$. For coordinates $v = (v^1, \ldots, v^n)$ we write $v = v^i e_i$. Then,

$$\frac{i}{2\pi}\overline{\partial}\partial \log(\exp(-\pi H(v,v))) = \frac{i}{2} H_{ij}\,\mathrm{d}v^i \wedge \mathrm{d}\overline{v}^j,$$

which shows iF_∇ is a positive $(1,1)$-form where ∇ is the associated Chern connection to $(\mathcal{L}, h)$.

Conversely, suppose $\mathcal{L}$ is a positive line bundle, i.e., there exists a hermitian metric h on $\mathcal{L}$ such that iF_∇ is a positive (1,1)-form where ∇ is the Chern connection. Recall that under the de Rham isomorphism theorem $\frac{i}{2\pi}F_\nabla \in \Omega^2(A)$ represents $c_1(\mathcal{L})$. From the previous discussion, the unique harmonic representative associated $\frac{i}{2\pi}F_\nabla$ is given by

$$\frac{1}{\mathrm{vol}(A)}\int_A t_a^*\left(\frac{i}{2\pi}F_\nabla\right)\mathrm{d}a.$$

Since iF_∇ is everywhere positive, the average must be positive and it follows that the hermitian form associated to $c_1(\mathcal{L})$ must be positive definite. $\square$

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