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THE MOMENTS OF THE DISTRIBUTION FOR NORMAL SAMPLES OF MEASURES OF DEPARTURE FROM NORMALITY

Author's Note (CMS 21.15a) *

In an earlier paper (Paper 74) it had been shown that mean powers and other symmetric functions of the sampling distributions of the k-series of symmetric functions of the observations could be obtained from the combinatorial properties of certain bipartitional functions, thus reducing to direct arithmetic the very heavy algebra by which this type of problem had ordinarily been treated. Equivalent properties are shown in this paper to be possessed by the ratios of such functions to the powers of the same degree of the estimated variance, k_2, in samples from a normal population. We thus find exact formulae in place of the approximate series arrived at in the earlier publication, Section 11 of which is here superseded. The recurrence relation of Section 2 and the use of symbolic operators in Section 5 may well have applications beyond the immediate problem.

The Moments of the Distribution for Normal Samples of Measures of Departure from Normality.

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1. *The Appropriate Symmetric Functions of the Observations.*

If $x_1 \ldots x_n$ are the values of a variate observed in a sample of n, from any population, we may evaluate a series of statistics (k) such that the mean value of k_p will be the pth cumulative moment function of the sampled population; the first three of these are defined by the equations :

$$k_1 = \frac{1}{n} \mathrm{S}(x),$$

$$k_2 = \frac{1}{n-1} \mathrm{S}(x - k_1)^2,$$

$$k_3 = \frac{n}{(n-1)(n-2)} \mathrm{S}(x - k_1)^3;$$

then it has been shown (Fisher, 1929)* that the cumulative moment functions

* R. A. Fisher, "Moments and Product Moments of Sampling Distributions," 'Proc. Lond. Math. Society,' Series 2, vol. 30, pp. 199–238 (1929).

of the simultaneous distribution, in samples, of k_1, k_2, k_3, ..., may be obtained by the direct application of a very simple combination procedure.

The simplest measure of departure from normality will then be

$$\gamma = k_3 k_2^{-3/2},$$

a quantity which is evidently independent of the units of measurement, and in samples from a symmetrical distribution will have a distribution symmetrical about the value zero. In testing the evidence provided by a sample, of departure from normality, the distribution of this quantity in normal samples is required.

Hitherto the exact values of the moments of this distribution have been unknown, though a method of calculating the moments for large samples, in a series of any number of powers of n^{-1}, has been given. It will be shown that the distribution may be investigated by means of a recurrence relation, which yields the moments of the distribution and seems well adapted for the investigation of its other properties.

2. *The Recurrence Relation for* γ.

For values of p from 1 to $n - 1$, let us define ξ_p by the relation

$$\xi_p = (x_p - k_1) + \frac{1}{n-1}(x_n - k_1),$$

or,

$$x - k_1 = \xi - \frac{1}{n-1}(x_n - k_1).$$

Then evidently

$$\overset{n-1}{\underset{1}{\mathrm{S}}}\ (\xi_p) = 0$$

and

$$\overset{n}{\underset{1}{\mathrm{S}}}\,(x_p - k_1)^2 = (x_n - k_1)^2\left(1 + \frac{1}{n-1}\right) + \overset{n-1}{\underset{1}{\mathrm{S}}}\,(\xi^2)$$

while

$$\overset{n}{\underset{1}{\mathrm{S}}}\,(x_p - k_1)^3 = (x_n - k_1)^3\left(1 - \frac{1}{(n-1)^2}\right) - \frac{3}{n-1}(x_p - k_1)\,\mathrm{S}\,(\xi^2) + \mathrm{S}\,(\xi^3),$$

so that, if

$$\frac{n}{n-1}\,(x_n - k_1)^2 = \cot^2\theta\,.\,\mathrm{S}\,(\xi^2),$$

we may express the ratio

$$\gamma_n = \frac{n\sqrt{n-1}}{n-2} \mathrm{S}\,(x-k_1)^3 \div \mathrm{S}^{3/2}\,(x-k_1)^2$$

in terms of the ratio

$$\gamma_{n-1} = \frac{(n-1)\sqrt{n-2}}{n-3} \mathrm{S}\,(\xi^3) \div \mathrm{S}^{3/2}\,(\xi^2)$$

in the recurrence relation

$$\gamma_n = \frac{n\,(n-3)}{(n-1)^{1/2}\,(n-2)^{3/2}} \sin^3\theta \,.\, \gamma_{n-1} - \frac{3\sqrt{n}}{n-2} \cos\theta \sin^2\theta + \sqrt{n}\cos^3\theta, \quad (1)$$

where γ_{n-1} is the value of γ calculated from the sample values excluding x_n, and γ_n is the value calculated from the whole sample of n values.

The value of the recurrence relation in this form lies in the fact that the distribution of θ is independent of that of γ_{n-1}, for whatever may be the values of $\xi_1, \ldots, \xi_{n-1} \div \sqrt{\mathrm{S}\,(\xi^2)}$, if σ be the standard error of the population sampled the distribution of

$$t = (x_n - k_1)\sqrt{n/n-1}$$

will be

$$\frac{1}{\sigma\sqrt{2\pi}} e^{-t^2/2\sigma^2} dt\,;$$

hence if

$$c = t/\sqrt{\mathrm{S}},$$

where S stands for S (ξ^2), since the distribution of S is known to be

$$df = \frac{1}{(2\sigma^2)^{\frac{1}{2}(n-2)} \frac{n-4}{2}!} \mathrm{S}^{\frac{1}{2}(n-4)} e^{-\mathrm{S}/2\sigma^2} d\mathrm{S},$$

the distribution of c will be given by

$$df = \frac{dc}{\sigma\sqrt{2\pi}} \cdot \frac{(2\sigma^2)^{-\frac{1}{2}(n-2)}}{\frac{n-4}{2}!} \int_0^\infty e^{-\frac{\mathrm{S}}{2\sigma^2}(1+c^2)} \mathrm{S}^{\frac{1}{2}(n-3)} d\mathrm{S} = \frac{\frac{n-3}{2}!}{\frac{n-4}{2}!\ \sqrt{\pi}} \frac{dc}{(1+c^2)^{\frac{n-1}{2}}};$$

or, if c is cot θ, the distribution of θ is

$$df = \frac{\frac{n-3}{2}!}{\frac{n-4}{2}!\ \sqrt{\pi}} \sin^{n-3}\theta\, d\theta. \quad (2)$$

independently of the value of γ_{n-1}, as indeed is obvious if the sample is considered geometrically.

3. *The Distribution for Samples of* 3.

The terminal values of γ_n are given by putting $\theta = 0$ and π, when $\gamma_n = \pm\sqrt{n}$ irrespective of the values of γ_{n-1} which is indeed indeterminate at these values. The recurrence relation enables us also by means of a single integration to obtain the distribution of γ_n from that of γ_{n-1}, or alternatively to obtain the moments of the distribution of γ_n in terms of those of the distribution of γ_{n-1} To utilise the recurrence relation in these ways we shall need the distribution of γ for the smallest possible samples, *i.e.*, for $n = 3$.

When $n = 3$, we may represent the 3 deviations of the observations from the mean of the population by

$$x_1 = b + a\cos\phi, \quad x_2 = b + a\cos\left(\phi + \frac{2\pi}{3}\right) x_3 = b + a\cos\left(\phi + \frac{4\pi}{3}\right),$$

then the mean of the sample is b, and the statistics k_2 and k_3 are given by

$$k_2 = \frac{a^2}{2}\left\{\cos^2\phi + \cos^2\left(\phi + \frac{2\pi}{3}\right) + \cos^2\left(\phi + \frac{4\pi}{3}\right)\right\}$$

$$= \tfrac{3}{4}a^2$$

$$k_3 = \frac{3a^3}{2}\left\{\cos^3\phi + \cos^3\left(\phi + \frac{2\pi}{3}\right) + \cos^3\left(\phi + \frac{4\pi}{3}\right)\right\},$$

but

$$\cos^3\phi = \tfrac{1}{4}(\cos 3\phi + 3\cos\phi),$$

hence

$$k_3 = \tfrac{9}{8}a^3\cos 3\phi,$$

and

$$\gamma \equiv k_3 k_2^{-3/2} = \sqrt{3}\cos 3\phi.$$

For the sampling distribution of ϕ, since

$$\frac{\partial(x_1, x_2, x_3)}{\partial(b, a, \phi)} = -\frac{3\sqrt{3}}{2}a,$$

and

$$x_1^2 + x_2^2 + x_3^2 = 3b^2 + \tfrac{3}{2}a^2,$$

we have

$$df = \frac{1}{(\sigma\sqrt{2\pi})^3}\cdot\frac{3\sqrt{3}}{2}\int_{-\infty}^{\infty} e^{-\frac{3b^2}{2\sigma^2}}\,db \int_0^{\infty} ae^{-\frac{3a^2}{4\sigma^2}}\,.\,d\phi,$$

which on integration with respect to b yields

$$df = \frac{1}{(\sigma\sqrt{2\pi})^2}\int_0^{\infty}\frac{3a}{2}e^{-\frac{3a^2}{4\sigma^2}}\,da\,.\,d\phi,$$

and on integration with respect to a, yields simply

$$df = \frac{1}{2\pi}d\phi.$$

Since, we have already found γ as a function of ϕ, we have on substitution

$$df = \frac{d\gamma}{\pi\sqrt{3 - \gamma^2}} \tag{3}$$

as the distribution of γ for the case $n = 3$, since γ takes any particular value six times as ϕ changes from 0 to 2π.

The distribution is, of course, symmetrical, and has the following even moments

$$\mu_2 = 3/2$$
$$\mu_4 = 27/8$$
$$\mu_6 = 135/16$$
$$\dots\dots\dots\dots\dots$$
$$\mu_{2s} = \frac{(s-\frac{1}{2})\,!}{s\,!\,\sqrt{\pi}}\,.\,3^s.$$

4. *The Moments of γ in General.*

The exact distribution for $n > 3$ seems not to be expressible simply in terms of known functions. For the moments about the mean (zero) of the distribution we may proceed as follows: let v_n be the variance of the distribution of γ_n, then squaring both sides of equation (1) and averaging over all possible values we find

$$v_n = \frac{n^2(n-3)^2}{(n-1)(n-2)^3}\sin^6\theta\,.\,v_{n-1} + n\cos^6\theta - \frac{6n}{n-2}\cos^4\theta\sin^2\theta + \frac{9n}{(n-2)^2}\cos^2\theta\sin^4\theta\,;$$

since equation (2) gives the distribution of θ we may now average over all values of θ, by multiplying by the right-hand side of this equation and integrating with respect to θ from 0 to 2π, we then have

$$v_n = \frac{n^2(n-3)^2}{(n-1)(n-2)^3}\cdot\frac{(n-2)\,n\,(n+2)}{(n-1)(n+1)(n+3)}\cdot v_{n-1} + \frac{1}{(n-1)(n+1)(n+3)}\left\{n\,.\,1\,.\,3\,.\,5 - \frac{6n}{n-2}\,.\,1\,.\,3\,(n-2) + \frac{9n}{(n-2)^2}(n-2)\,n\,.\,1\right\}$$

$$= \frac{n^3(n+2)(n-3)^2}{(n-2)^2(n-1)^2(n+1)(n+3)}v_{n-1} + \frac{6n}{(n-2)(n-1)(n+3)}\,.$$

The variance for any particular value of n may now be found by direct substitution; alternatively we may note that if

$$w_n = \frac{(n-2)^2 (n+1)(n+3)}{n^2} v_n,$$

then

$$w_{n-1} = \frac{(n-3)^2 n (n+2)}{(n-1)^2} v_{n-1},$$

and the recurrence relation is reduced to

$$w_n - w_{n-1} = \frac{6(n-2)(n+1)}{n(n-1)} = 6 + 12\left(\frac{1}{n} - \frac{1}{n-1}\right);$$

whence

$$w_n = C + 6n + 12/n,$$

where C is a constant to be determined from

$$w_3 = 4;$$

whence

$$w_n = 6\left(n - 3 + \frac{2}{n}\right) = \frac{6(n-1)(n-2)}{n},$$

and

$$v_n = \frac{6n(n-1)}{(n-2)(n+1)(n+3)},$$

the general formula for the variance of γ_n.

The same process applied to the mean fourth power will give a recurrence formula involving the variance, for which the value found can now be substituted; in this way the mean values of all even powers may be evaluated in succession. Writing v' for the fourth moment, we have the equation

$$\begin{aligned} v_n' = {} & \frac{n^4 (n-3)^4}{(n-1)^2 (n-2)^6} \cdot \frac{(n-2) n (n+2)(n+4)(n+6)(n+8)}{(n-1)(n+1)(n+3)(n+5)(n+7)(n+9)} v_{n-1}' \\ & + \frac{6n^3 (n-3)^2}{(n-1)(n-2)^3} \cdot \frac{6(n^2 - n + 70) n (n+2)}{(n-2)(n-1)(n+1)(n+3)(n+5)(n+7)(n+9)} v_{n-1} \\ & + \frac{108n^2 (31n^3 - 144n^2 + 183n + 70)}{(n-2)^3 (n-1)(n+1)(n+3)(n+5)(n+7)(n+9)}, \end{aligned}$$

or, substituting for v_{n-1},

$$\begin{aligned} v_n' = {} & \frac{n^5 (n-3)^4 (n+2)(n+4)(n+6)(n+8)}{(n-2)^5 (n-1)^3 (n+1)(n+3)(n+5)(n+7)(n+9)} v_{n-1}' \\ & + \frac{108n^2 (2n^4 + 23n^3 + 2n^2 - 237n + 70)}{(n-2)^3 (n-1)(n+1)(n+3)(n+5)(n+7)(n+9)}; \end{aligned}$$

but if

$$w_n' = \frac{(n-2)^4 (n+1)(n+3)(n+5)(n+7)(n+9)}{n^4 (n-1)} v_n',$$

then

$$w_n' - w_{n+1}' = \frac{108\,(n-2)}{n^2 (n-1)^2} (2n^4 + 23n^3 + 2n^2 - 237n + 70)$$

$$= 108 \left\{2n + 23 - \frac{264}{n(n-1)} + 140 \frac{2n-1}{n^2 (n-1)^2}\right\},$$

so that

$$w_n' = 108 \left(n^2 + 24n + C + \frac{264}{n} - \frac{140}{n^2}\right),$$

where C is to be determined from

$$w_3' = 480$$

so that

$$C = -149,$$

$$w_n' = \frac{108\,(n-1)(n-2)}{n^2} (n^2 + 27n - 70),$$

and the fourth moment of γ is given by

$$\mu_4(\gamma) = v_n' = \frac{108n^2 (n-1)^2 (n^2 + 27n - 70)}{(n-2)^3 (n+1)(n+3)(n+5)(n+7)(n+9)}$$

Similarly the sixth moment is found to be

$$\mu_6(\gamma) = \frac{3240n^3 (n-1)^3 (n^4 + 84n^3 + 2695n^2 - 15168n + 20020)}{(n-2)^5 (n+1)(n+3)\ldots(n+15)},$$

and the same method may be applied to determine the higher moments.

From the moments the cumulative moment functions may be determined by the invariable relationships, which for symmetrical distributions become

$$\kappa_2 = \mu_2$$

$$\kappa_4 = \mu_4 - 3\mu_2^2$$

$$\kappa_6 = \mu_4 - 15\mu_2\mu_4 + 30\mu_2^3,$$

which give us the values

$$\kappa_2 = \frac{6n(n-1)}{(n-2)(n+1)(n+3)},$$

$$\kappa_4 = \frac{1296n^2 (n-1)^2 (n-7)(n^2 + 2n - 5)}{(n-2)^3 (n+1)^2 (n+3)^2 (n+5)(n+7)(n+9)},$$

$$\kappa_6 = \frac{466560n^3 (n-1)^3 (7n^6 - 88n^5 - 286n^4 + 3284n^3 + 1667n^2 - 22108n + 20020)}{(n-2)^5 (n+1)^3 (n+3)^3 (n+5)(n+7)(n+9)(n+11)(n+13)(n+15)},$$

from which we may derive the ratios,

$$\frac{1}{4!}\kappa_4\kappa_2^{-2} = \frac{3(n-7)(n^2+2n-5)}{2(n-2)(n+5)(n+7)(n+9)},$$

$$\frac{1}{6!}\kappa_6\kappa_2^{-3} = \frac{3(7n^6-88n^5-286n^4+3284n^3+1667n^2-22108n+20020)}{(n-2)^2(n+5)(n+7)(n+9)(n+11)(n+13)(n+15)},$$

which determine the rate of approach of the distribution of γ to normality as the sample number n is increased. It will be noticed that κ_4 changes from a negative to a positive sign at $n = 7$, and that the corresponding ratio rises to its greatest value about $0\cdot024$ at $n = 22$, while the corresponding ratio for κ_6 starting from positive values has a negative maximum about $-0\cdot0016$ at $n = 8$, is positive again at $n = 13$, and reaches a positive maximum about $+0\cdot0027$ at $n = 32$. Using the reciprocal of n as abscissa the course of these two ratios is shown in figs. 1 and 2.

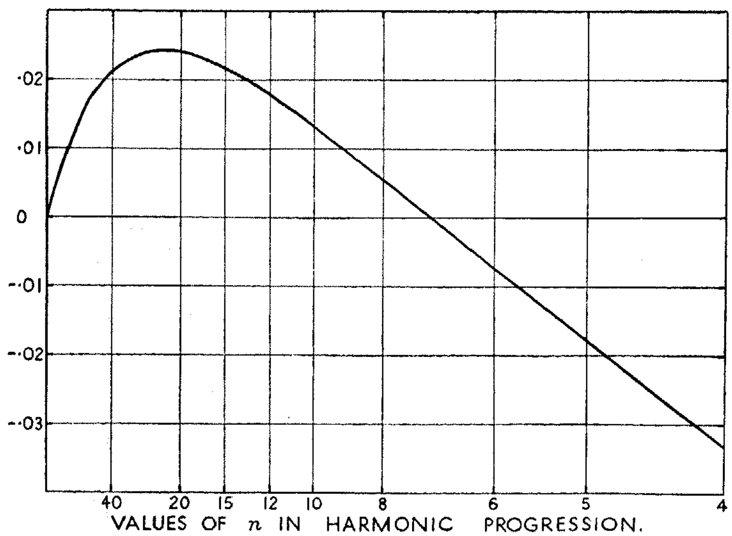

Fig. 1.—Graph of the Ratio $\kappa_4\kappa_2^{-2}/4!$ of the distribution of $\gamma = k_3k_2^{-3/2}$.

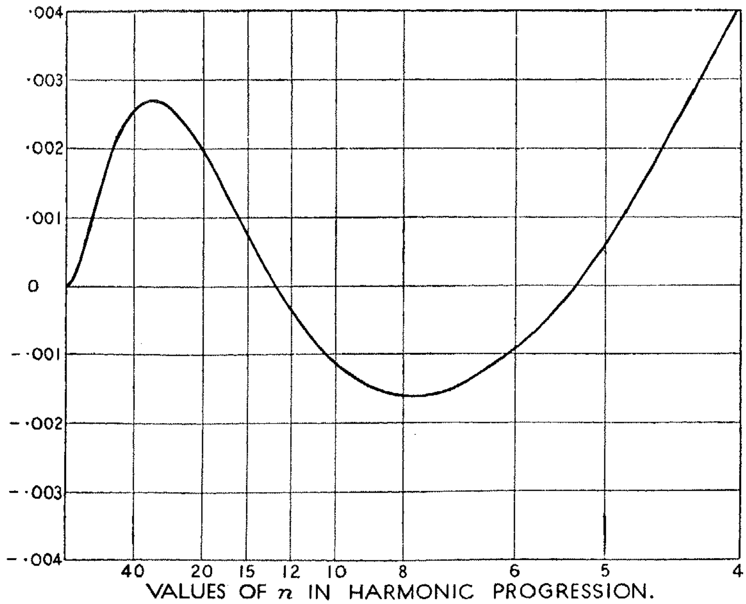

FIG. 2.—Graph of the Ratio $\kappa_6 \kappa_2^{-3}/6\,!$ of the distribution of $\gamma = k_3 k_2^{-3/2}$.

5. *The Moments of the Simultaneous Distribution of Different Measures of Departure from Normality.*

It is obvious that the method of approach adopted in the foregoing sections is applicable to the determination of the moments of the distributions of, or more generally of the simultaneous distribution of, all measures of departure from normality such as

$$\gamma = k_3 k_2^{-3/2}$$
$$\delta = k_4 k_2^{-2}$$
$$\varepsilon = k_5 k_2^{-5/2}$$

and so on.

For δ and ε we find the recurrence relations comparable with that already found for γ, namely

$$\gamma_n = n^{1/2} c^3 - \frac{3n^{1/2}}{n-2} cs^2 + \frac{n(n-3)}{(n-2)^{3/2}(n-1)^{1/2}} s^3 \cdot \gamma_{n-1},$$

$$\delta_n = nc^4 - \frac{6n}{n-2} c^2 s^2 + \frac{3}{n-2} s^4 - \frac{4n^{1/2}(n+1)}{(n-2)^{3/2}(n-1)^{1/2}} cs^3 \gamma_{n-1}$$
$$+ \frac{(n+1)(n-4)}{(n-2)^2} s^4 \cdot \delta_{n-1},$$

$$\varepsilon_n = n^{3/2} c^5 - \frac{10n^{3/2}}{n-2} c^3 s^2 + \frac{15n^{1/2}}{n-2} cs^4 \frac{10n(n+1)}{(n-2)^{3/2}(n-1)^{1/2}}\left(c^2 s^3 - \frac{s^5}{n+4}\right)\gamma_{n-1}$$

$$- \frac{5n^{1/2}(n+5)}{(n-2)^2} cs^4 \delta_{n-1} + \frac{n^2(n^2-25)s^5}{(n-2)^{5/2}(n-1)^{1/2}(n+4)} \varepsilon_{n-1},$$

and by a mere repetition of the algebraic processes employed above, we may obtain a recurrence relation for the mean value of any expression of the form

$$\gamma^a \delta^b \varepsilon^c \ldots\ldots,$$

from which the mean value in question may be derived.

If, in accordance with the notation employed for the designation of the moments of the set of statistics $k_2, k_3, k_4, \ldots$, we represent such a mean product by

$$\mu(\ldots 5^c\, 4^b\, 3^a\, 2^{-r}),$$

where

$$2r = 3a + 4b + 5c + \ldots,$$

so that r is always an integer save for the odd moments which necessarily vanish, we may list the following formulæ :

$$\mu(3^2\, 2^{-3}) = \frac{6n(n-1)}{(n-2)(n+1)(n+3)},$$

$$\mu(4^2\, 2^{-4}) = \frac{24n(n-1)^2}{(n-3)(n-2)(n+3)(n+5)},$$

$$\mu(4\, 3^2 2^{-5}) = \frac{216n^2(n-1)^2}{(n-2)^2(n+1)(n+3)(n+5)(n+7)},$$

$$\mu(5^2\, 2^{-5}) = \frac{120n^2(n+5)(n-1)^3}{(n-4)(n-3)(n-2)(n+1)(n+3)(n+5)(n+7)},$$

$$\mu(3^4\, 2^{-6}) = \frac{108n^2(n-1)^2(n^2+27n-70)}{(n-2)^3(n+1)(n+3)(n+5)(n+7)(n+9)},$$

$$\mu(4^3\, 2^{-6}) = \frac{1728n(n-1)^3(n^2-5n+2)}{(n-3)^2(n-2)^2(n+3)(n+5)(n+7)(n+9)},$$

$$\mu(3^6\, 2^{-9}) = \frac{3240n^3(n-1)^3(n^4+84n^3+2695n^2-15168n+20020)}{(n-2)^5(n+1)(n+3)(n+5)(n+7)(n+9)(n+11)(n+13)(n+15)}.$$

A comparison of these formulæ with those already given (Fisher, 1929), for the cumulative moment functions of k_2, k_3, k_4, which in every case but $\mu(3^4)$ and $\mu(3^6)$ are also the moments of the distribution, shows that

$$\mu\,(3^2) = \frac{6n}{(n-1)\,(n-2)}\,\kappa_2^{\,3},$$

$$\mu\,(4^2) = \frac{24n\,(n+1)}{(n-1)\,(n-2)\,(n-3)}\,\kappa_2^{\,4},$$

$$\mu\,(4\,3^2) = \frac{216n^2}{(n-1)^2\,(n-2)^2}\,\kappa_2^{\,5},$$

$$\mu\,(5^2) = \frac{120n^2\,(n+5)}{(n-1)\,(n-2)\,(n-3)\,(n-4)}\,\kappa_2^{\,5},$$

$$\mu\,(4^3) = \frac{1728n\,(n+1)\,(n^2-5n+2)}{(n-1)^2\,(n-2)^2\,(n-3)^2}\,\kappa_2^{\,6}.$$

Moreover

$$\begin{aligned}\mu\,(3^4) &= \kappa\,(3^4) + 3\kappa^2\,(3^2)\\ &= \left\{\frac{648n^2\,(5n-12)}{(n-1)^3\,(n-2)^3} + \frac{108n^2}{(n-1)^2\,(n-2)^2}\right\}\kappa_2^{\,6}\\ &= \frac{108n^2\,(n^2+27n-70)}{(n-1)^3\,(n-2)^3}\,\kappa_2^{\,6},\end{aligned}$$

and

$$\begin{aligned}\mu\,(3^6) &= \kappa\,(3^6) + 15\kappa\,(3^4)\,\kappa\,(3^2) + 15\kappa^3\,(3^2)\\ &= 3240\,\kappa_2^{\,9}\left\{\frac{144\,(22n^2-111n+142)\,n^3}{(n-1)^5\,(n-2)^5} + \frac{18\,(5n-12)\,n^3}{(n-1)^4\,(n-2)^4} + \frac{n^3}{(n-1)^3\,(n-2)^3}\right\}\\ &= \frac{3240n^3\,(n^4+84n^3+2695n^2-15168n+20020)}{(n-1)^5\,(n-2)^5}\,\kappa_2^{\,9}.\end{aligned}$$

In every case, therefore, the moment of the distribution of γ, δ, ε, ..., is derivable by multiplying by

$$\frac{(n-1)^r}{(n-1)\,(n+1)\,\ldots\,(n+2r-3)\,\kappa_2^{\,r}}$$

the corresponding moment of the distribution of k_3, k_4, k_5, ... Since many of the latter moments may be found relatively expeditiously by means of the combinatorial procedure, this will be the quicker method for the more complex product moments. For moments of high degree, however, such as (3^8) and (3^{10}) it does not seem easy to enumerate with certainty all the combinatorial patterns, and the recurrence method, though necessarily heavy, supplies a valuable check.

An analytical proof of this relationship, or at least analytical grounds for accepting it as general, may be found by the method of transforming the characteristic function previously employed in demonstrating the rules of the combinatorial method. If

$$M(t_1, t_2, \ldots)$$

is the characteristic function of the simultaneous distribution of the variates $x_1, x_2, \ldots$, and if

$$M'(\tau_1, \tau_2, \ldots)$$

is that of variates $\xi_1, \xi_2, \ldots$, defined in terms of $x_1, x_2, \ldots$, by the relations

$$\xi_1 = f_1(x_1, x_2, \ldots),$$
$$\xi_2 = f_2(x_1, x_2, \ldots),$$

then

$$M'(\tau_1, \tau_2, \ldots) = e^{\tau_1 f_1 + \tau_2 f_2 + \cdots} M(t_1, t_2, \ldots)$$

at $t_1 = 0$, $t_2 = 0$, ..., where f_p in the index stands for

$$f_p\left(\frac{d}{dt_1}, \frac{d}{dt_2}, \ldots\right).$$

To apply this theorem to the present case we utilise the fact that in sampling from the normal distribution k_2 is distributed independently of $\gamma, \delta, \varepsilon, \ldots$, in the known distribution

$$df = \frac{1}{\frac{1}{2}(n-3)} \cdot \left(\frac{n-1}{2\kappa_2}\right)^{\frac{1}{2}(n-1)} k_2^{\frac{1}{2}(n-3)} e^{-\frac{1}{2}(n-1)k_2/\kappa_2} dk_2,$$

of which the characteristic function,

$$\int e^{k_2 t_2} df,$$

is

$$\left(1 - \frac{2\kappa_2 t_2}{n-1}\right)^{-\frac{1}{2}(n-1)}$$

Hence the general characteristic function of the simultaneous distribution of $k_2, \gamma, \delta, \ldots$, is of the form

$$\left(1 - \frac{2\kappa_2 t_2}{n-1}\right)^{-\frac{1}{2}(n-1)} M(t_3, t_4, \ldots),$$

where M is the sum of terms such as

$$\mu(\ldots 5^c 4^b 3^a 2^{-r}) \frac{t_3^a}{a!} \frac{t_4^b}{b!} \frac{t_5^c}{c!} \ldots,$$

and from this expression we must be able to derive the function

$$M'(\tau_3, \tau_4, \ldots) = \Sigma\, \mu\,(\ldots 5^c 4^b 3^a)\, \frac{\tau_3{}^a}{a\,!}\,\frac{\tau_4{}^b}{b\,!}\,\frac{\tau_5{}^c}{c\,!}$$

by the action of the operator

$$e^{\tau_3 D_3 D_2{}^{3/2} + \tau_4 D_4 D_2{}^2 + \tau_5 D_5 D_2{}^{5/2} + \ldots}$$

where D_p stands for d/dt_p.

It appears, therefore, without discussing what meaning should be attached to the fractional indices, which find in fact only zero terms on which to operate, that

$$\mu\,(\ldots 5^c\, 4^b\, 3^a) = \mu\,(\ldots 5^c\, 4^b\, 3^a\, 2^{-r})\,.\,\frac{d^r}{dt_2{}^r}\left(1 - \frac{2\kappa_2 t_2}{n-1}\right)^{-\frac{1}{2}(n-1)}$$

at $t_2 = 0$, or that

$$\mu\,(\ldots 5^c\, 4^b\, 3^a) = \mu\,(\ldots 5^c\, 4^b\, 3^a\, 2^{-r})\,.\,\frac{(n-1)\,\ldots\,(n+2r-3)}{(n-1)^r}\,\kappa_2{}^r,$$

which is the relationship required.

Summary.

Two methods are given for discussing the distribution of the ratios of the symmetric functions k_3, k_4, ..., obtained from samples from a normal distribution to the powers of k_2 of the same degree.

The first method consists in the development of recurrence relations expressing the ratios from a sample of n in terms of the corresponding ratios from a sample of $n - 1$ observations, and of a parameter distributed independently in a known distribution. Theoretically all properties of the general distribution could be obtained from these relations in conjunction with a study of samples of 3, 4, 5 ... observations.

The relations are used to derive the exact values of the first three even moments of the simplest ratio γ, and of the simpler non-vanishing moments of the simultaneous distribution of all the ratios. It is observed that these moments are very simply related to the corresponding moments of the distribution of k_3, k_4, ..., given in a previous paper.

The second method is an application of the method of symbolical operators developed by the author, which confirms the generality of the relationship found. The moments of the one distribution may thus be inferred directly from that of the other for which the combinatorial procedure is available.