90

THE DERIVATION OF THE PATTERN FORMULAE OF TWO-WAY PARTITIONS FROM THOSE OF SIMPLER PATTERNS

By R. A. FISHER *and* J. WISHART.

[Received 28 November, 1930.—Read 11 December, 1930.]

1. *Introductory.*

If the moment generating function of a variable be defined in terms of the frequency element $df(x)$ of a variate x in the form

$$M = \int e^{tx}\, df(x),$$

then, when M can be expanded in a series of powers of t, the cumulative function K, defined by

$$K = \log M,$$

can also be expanded in the form

$$K = \sum_r \frac{\kappa_r t^r}{r\,!}.$$

The coefficients κ_r of this expansion have by different writers been termed semi-invariants and cumulative moment functions. Since both terms are lengthy, and the first somewhat misleading, we propose in what follows to refer to them as the *cumulants* of the distribution.

In a recent paper* it was shown that, if symmetric functions k_r, of the r-th degree, are calculated from the observations of a sample, in such a manner that the mean value of k_r in all possible samples is equal to the r-th cumulant (κ_r) of the infinite population from which the sample has been taken, then the cumulants of the distribution of any k, and of the simultaneous distribution of all such statistics, which are necessarily expressible in terms of the cumulants of the population, may be readily and simply determined by combinatorial methods. The essence of the method is that the coefficient of any single term in the required

---

* R. A. Fisher, *Proc. London Math. Soc.* (2), 30 (1929), 199–238.

formulae is a composite of contributions from one or more two-way partitions, of which one marginal partition represents the formula, while the other specifies the particular term; and for each such partition the numerical coefficient is derived as the number of ways of setting up the partition, while the coefficient in n, the size of the sample, depends on the pattern of the two-way partition, *i.e.* on the number of rows and columns, and the number and distribution of zero entries. A useful list of such pattern formulae has already been given (*loc. cit.*).

The listing of the great number of patterns possible for two-way partitions of larger numbers would, however, be impracticable; the present paper will show how the pattern formulae for such cases may be readily derived from those already listed. Patterns with a large number of rows have necessarily, for formulae of given degree, relatively few entries in each row. The extreme case is that for the terms involving only the variance of the sampled population, in which each row has only two entries; these patterns are of particular interest, since only these occur in the distribution of moment statistics derived from the normal distribution. It will be seen that the method of deriving the pattern function by the addition of a new column is particularly simple in these cases. It should be noted that the pattern functions are the same for multivariate as for univariate problems.

2. *The general method of evaluating a pattern.*

The general procedure for determining the function of n associated with any specified pattern is to consider all the possible ways in which the rows can be separated into separate groups, or separates. Thus with two rows we have only two possible separations, the rows being either amalgamated as in the marginal total, or kept separate; these two separations correspond to the partitions (2) and (1^2) respectively. With three rows we have one separation corresponding to the partition (3), three corresponding to the partition (21), and one corresponding to the partition (1^3). In general the number of separations of r rows corresponding to the partition $(p_1^{\pi_1} p_2^{\pi_2} \ldots)$ is

$$\frac{r!}{\pi_1!\,(p_1!)^{\pi_1}\,.\,\pi_2!\,(p_2!)^{\pi_2}\ldots},$$

and the total number of separations of r rows into s separates is

$$\frac{1}{(s-1)!}\,\Delta^{s-1}(1^{r-1}),$$

the leading $(s-1)$-th divided difference of the series of the $(r-1)$-th powers of the positive integers. Representative numbers are given in Table I.

TABLE I.

Number of separations of r rows into s separates.

Number of rows	Number of separates									Total
	1	2	3	4	5	6	7	8	9	
2	1	1								2
3	1	3	1							5
4	1	7	6	1						15
5	1	15	25	10	1					52
6	1	31	90	65	15	1				203
7	1	63	301	350	140	21	1			877
8	1	127	966	1701	1050	266	28	1		4140
9	1	255	3025	7770	6951	2646	462	36	1	21147

The total number of separations, being [Whitworth, *Choice and Chance* (1886), 95] the coefficient of $x^r/r!$ in the expansion of

$$e^{e^x-1},$$

increases rapidly, and although, usually, large groups of separations making similar contributions may be treated together, it is evidently desirable to shorten the method for cases of more than six or eight rows.

The contribution of each separation to the function required is the product of the factor $n(n-1)\dots(n-s+1)$ with factors for the several columns; these factors are

$$\frac{1}{n},\quad \frac{-1}{n(n-1)},\quad \frac{2!}{n(n-1)(n-2)},\quad \dots,$$

according as the column is represented in 1, 2, 3, ... separates. This process has been already sufficiently exemplified (R. A. Fisher, *loc. cit.*). We shall now show how the pattern-function for a pattern containing a column with only two entries may be obtained from the pattern-functions of simpler patterns, and then consider the parallel procedure to be used when no column contains less than three or more entries.

3. *Expansion of a pattern function in terms of the functions of simpler patterns.*

Consider the pattern

$$\begin{matrix} \cdot & \times & \times & \times \\ \times & \cdot & \times & \times \\ \times & \times & \cdot & \cdot \\ \times & \times & \cdot & \cdot \end{matrix}$$

of which the right-hand column contains only two entries. The fifteen separations which are possible with four rows may be divided into two classes, in one of which (α) the two rows represented in the fourth column lie in the same separate, while in the second class (β) they lie in different separates. In the first class the fourth column contributes the factor n^{-1}, in the second class the factor is $-1/n(n-1)$.

Now in separations of the first class the cofactors of n^{-1} will be the contributions of all its possible separations to the pattern-function of the pattern

$$\begin{matrix} \times & \times & \times \\ \times & \times & . \\ \times & \times & . \end{matrix}$$

in which the fourth column is omitted, and the first two rows of the original pattern are amalgamated. In general we may designate the function of this reduced pattern by A. If now we represent the function of the pattern

$$\begin{matrix} . & \times & \times \\ \times & . & \times \\ \times & \times & . \\ \times & \times & . \end{matrix}$$

in which the rows have not been amalgamated, by B, it appears that the cofactor of $-1/n(n-1)$ in the pattern-function to be evaluated will consist of all the contributions to B which do not occur in A, and the required function is reduced to the form

$$\frac{A}{n}-\frac{B-A}{n(n-1)}\equiv\frac{A}{n-1}-\frac{B}{n(n-1)},$$

which is a general formula for the function of any pattern having a column with only two entries.

In the particular case considered we may at once substitute

$$A=\frac{1}{(n-1)(n-2)},\quad B=\frac{n}{(n-1)^2(n-2)},$$

and obtain
$$\frac{1}{(n-1)^2(n-2)}\left(1-\frac{1}{n-1}\right)\equiv\frac{1}{(n-1)^3},$$

in accordance with the value given in Fisher's list of useful patterns.

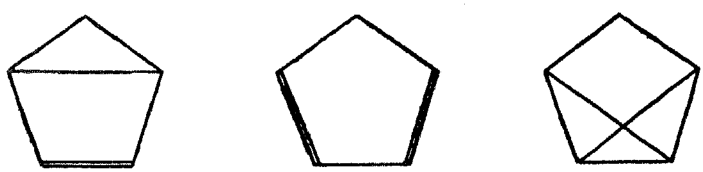
Alternatively, of course, B could have been reduced in turn to functions of two column patterns.

Whenever one or both rows represented in the column to be removed contain only two entries, the pattern-function B vanishes, and we are left simply with

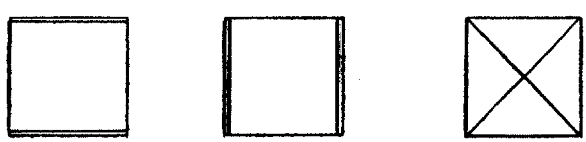
$$A/(n-1);$$

this will always be the case with the pattern of terms involving only variances and covariances, which are the only terms which appear in sampling from normal populations.

For these normal cases use may be made of the symbolical diagram in which each column is represented by a point and each row by a line joining the two points corresponding to the two columns in which it is represented. Thus for the evaluation of $\kappa(3^4 2)$, we have partitions represented by the figures



in each of which one angle at which two lines meet represents the column with only two entries. The reduction formula thus shows that the functions associated with these patterns will be each $1/(n-1)$ of the functions associated with the simpler patterns represented by the three figures below (two of which are equivalent),



that is of those which occur in the evaluation of $\kappa(3^4)$. In respect of the numerical factor, too, it should be noticed that every way of setting up a partition for $\kappa(3^4)$ corresponds to twelve ways of setting up one of the partitions for $\kappa(3^4 2)$, since any one of the six rows may be broken and connected in two ways with the elements of the new column. Thus a correspondence is established for the whole coefficient, and we have for the normal case

$$\kappa(3^4\,2) = \frac{12}{n-1}\,\kappa(3^4)\,\kappa_2,$$

or, in general, since the number of rows in the partition is equal to the power of κ_2 by which the coefficient is multiplied, the addition of a new part 2 is equivalent to the action of the operator

$$\frac{2\kappa_2^2}{n-1}\,\frac{d}{d\kappa_2},$$

an operator by means of which the higher cumulants of simultaneous distributions involving the estimated variance k_2 may be very readily obtained. In the multivariate case the operator for adding a variance or covariance k_{pq} is

$$\sum_{rs} \frac{1}{n-1} (\kappa_{pr}\kappa_{qs}+\kappa_{ps}\kappa_{qr}) \frac{d}{d\kappa_{rs}},$$

where κ_{pq} stands for the covariance of the variates p and q.

4. *Removal of a column of three entries.*

The direct generalization of the method of the last section to partitions containing a column of three entries will evidently express the new function in terms of the functions of five simpler partitions, each obtained by suppressing the column of three entries, namely A, formed by the amalgamation of all three of the rows used in the column suppressed; B_1, B_2, B_3, formed by the amalgamation of two only of these three rows; and C by leaving all three rows distinct. In the new function the cofactor of $1/n$ will evidently be A, that of $-1/n(n-1)$ will be the sum of three quantities $B-A$, and that of $2/n(n-1)(n-2)$ will be

$$C-(B_1-A)-(B_2-A)-(B_3-A)-A = C-B_1-B_2-B_3+2A.$$

Hence the general formula is

$$A\left(\frac{1}{n}+\frac{3}{n(n-1)}+\frac{4}{n(n-1)(n-2)}\right)$$
$$-(B_1+B_2+B_2)\left(\frac{1}{n(n-1)}+\frac{2}{n(n-1)(n-2)}\right)+\frac{2C}{n(n-1)(n-2)}$$
$$\equiv \frac{n}{(n-1)(n-2)} A - \frac{1}{(n-1)(n-2)} (B_1+B_2+B_3) + \frac{2}{n(n-1)(n-2)} C.$$

As an example we may derive the function for the pattern

$$\begin{array}{ccc} \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \\ \times & \times & . \end{array}$$

from those for the simpler patterns

$$\begin{array}{ccc} \begin{array}{cc} \times & \times \\ \times & \times \\ & \\ & \end{array} & \begin{array}{cc} \times & \times \\ \times & \times \\ \times & \times \\ & \end{array} & \begin{array}{cc} \times & \times \\ \times & \times \\ \times & \times \\ \times & \times \end{array} \\ \\ A & B_1 = B_2 = B_3 & C \\ \\ \dfrac{1}{n-1} & \dfrac{n}{(n-1)(n-2)} & \dfrac{n(n+1)}{(n-1)(n-2)(n-3)}. \end{array}$$

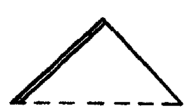
We have then

$$\frac{n}{(n-1)^2(n-2)}-\frac{3n}{(n-1)^2(n-2)^2}+\frac{2(n+1)}{(n-1)^2(n-2)^2(n-3)}$$
$$=\frac{n^3-8n^2+17n+2}{(n-1)^2(n-2)^2(n-3)}.$$

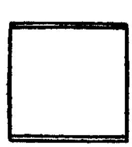
In its application to normal patterns the formula for removing a column of three entries is reduced to the simple formula

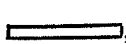
$$\frac{n}{(n-1)(n-2)}A,$$

but it should be noticed that A here is not necessarily the function of a normal pattern, for the other entries of the rows represented in the suppressed column may be in three different columns, in which case A will be the function of a pattern having a row with three entries. When, however, two of the amalgamated rows are duplicates joining the same pair of columns, the function will be expressed in terms of that of a normal pattern. In the representative diagram the effect of this will be to suppress a point, two of the lines from which are duplicates, and to replace the set of three lines meeting there by a single line joining their extremities



as does the broken line in the figure. By two such operations such a diagram as



is reduced to , representing the pattern

$$\begin{matrix}\times & \times\\ \times & \times\end{matrix}$$

and having the function $1/(n-1)$, whence it easily follows that the diagram in question is associated with the function

$$n^2/(n-1)^3(n-2)^2.$$

In the case where three entries appear in the amalgamated row the partition of which A is the function, though not a normal one, is of the kind which must occur in the term in $\kappa_3\kappa_2^{r-1}$, that is in the final term of the corresponding formula. Since any arrangement of a two-way partition having a marginal column represented by the partition $(3\,2^{r-1})$ corresponds to six arrangements of the normal partition of the formula in which a new row of three entries has been added, these coefficients must be in the simple ratio

$$\frac{6n}{(n-1)(n-2)}.$$

For example, the coefficient of the term in $\kappa_3\kappa_2^2$ of the formula for $\kappa(4\,3)$ is

$$\frac{36n}{(n-1)(n-2)},$$

whence it follows that the coefficient of the term in κ_2^5 of the formula for $\kappa(4\,3^2)$ must be

$$\frac{216n^2}{(n-1)^2(n-2)^2},$$

as may be verified from the formula given previously. Equally the corresponding coefficients in the formulae for $\kappa(3\,2^2)$ and $\kappa(3^2 2^2)$ are

$$\frac{48}{(n-1)^2} \quad \text{and} \quad \frac{288n}{(n-1)^3(n-2)}.$$

All the normal terms occurring in the distribution of k_3 and of its simultaneous distribution with other such statistics may then be obtained from the coefficients of simpler formulae.

5. *The removal of a column containing any number of entries.*

In expressing the function of a pattern as a linear function of the patterns formed by deleting a column of r entries, and amalgamating the r corresponding rows into ρ' rows in accordance with the partition $(p_1^{\pi_1} p_2^{\pi_2}\ldots)$, where

$$\Sigma p\pi = r,$$

$$\Sigma \pi = \rho',$$

let $a(p_1^{\pi_1} p_2^{\pi_2}\ldots)$ be the coefficient of the function of any one of the patterns so formed.

The functions consist of the sums of contributions from all possible separations of all the rows; we shall designate by $Q(q_1^{\chi_1} q_2^{\chi_2} \ldots)$ the total contribution of all such separations in which the r rows of the deleted column are separated in any particular way into χ_1 separates containing q_1 of these rows each, χ_2 separates containing q_2 of these rows each, and so on; then, if the partition $(p_1^{\pi_1} p_2^{\pi_2} \ldots)$ can be found by subdividing the parts of the partition $(q_1^{\chi_1} q_2^{\chi_2} \ldots)$ in λ ways, any Q will appear in the expansion with coefficient

$$\Sigma \lambda a (p_1^{\pi_1} p_2^{\pi_2} \ldots),$$

the summation being over all partitions $(p_1^{\pi_1} p_2^{\pi_2} \ldots)$, and this must be equated to its coefficient in the function of the whole pattern, namely

$$(-)^{\rho-1} (\rho-1)! \frac{(n-\rho)!}{n!},$$

where $\Sigma(\chi) = \rho$. We have in this way one such equation for every kind of partition $(q_1^{\chi_1} q_2^{\chi_2} \ldots)$, and these are sufficient to determine the unknown coefficients a.

But if the statistic k_r, defined as having its mean sampling value equal to κ_r, be expanded in the form

$$k_r = \Sigma A (p_1^{\pi_1} p_2^{\pi_2} \ldots) s_{p_1}^{\pi_1} s_{p_2}^{\pi_2} \ldots,$$

where s_p is the sum of the p-th powers of the observations, then the coefficient of $\mu_{q_1}^{\chi_1} \mu_{q_2}^{\chi_2} \ldots$ in the mean value of the expansion, when the mean values of the s-products are expanded in terms of the moments μ, by the general formula previously given (Fisher, *loc. cit.*, p. 207), will be

$$\Sigma \frac{n!}{(n-\rho)!} \mu A (p_1^{\pi_1} p_2^{\pi_2} \ldots),$$

where μ is the number of ways in which the parts of $(p_1^{\pi_1} p_2^{\pi_2} \ldots)$ may be amalgamated to form those of $(q_1^{\chi_1} q_2^{\chi_2} \ldots)$, and this is to be equated to its known coefficient in the expansion of κ_r, namely

$$\frac{(-)^{\rho-1} (\rho-1)!}{\chi_1! \chi_2! \ldots} \frac{r!}{(q_1!)^{\chi_1} (q_2!)^{\chi_2} \ldots}.$$

The relation between λ and μ may be found from the following consideration; the number of ways of dividing r objects into ρ' parts, $p_1^{\pi_1} p_2^{\pi_1} \ldots$, these being grouped into ρ divisions $q_1^{\chi_1} q_2^{\chi_2} \ldots$, may be obtained either by multiplying by λ the number of ways of dividing r objects into parts $q_1^{\chi_1} q_2^{\chi_2} \ldots$, *i.e.*

$$\frac{\lambda}{\chi_1 ! \, \chi_2 ! \ldots} \, \frac{r!}{(q_1 !)^{\chi_1} (q_2 !)^{\chi_2} \ldots},$$

or equally by multiplying by μ the number of ways of dividing r objects into parts $p_1^{\pi_1} p_2^{\pi_2} \ldots$, *i.e.*

$$\frac{\mu}{\pi_1 ! \, \pi_2 ! \ldots} \, \frac{r!}{(p_1 !)^{\pi_1} (p_2 !)^{\pi_2} \ldots}.$$

This relationship must hold for each particular separation of the partition $(p_1^{\pi_1} p_2^{\pi_2} \ldots)$ having *specification* $(q_1^{\chi_1} q_2^{\chi_2} \ldots)$, as defined by MacMahon*, or, as is here required, for all separations taken together. Consequently our second set of equations may be written

$$\Sigma \left\{ \lambda A (p_1^{\pi_1} p_2^{\pi_2} \ldots) \div \frac{r!}{\pi_1 ! \, \pi_2 ! \ldots (p_1 !)^{\pi_1} (p_2 !)^{\pi_2} \ldots} \right\} = (-)^{\rho-1} (\rho - 1)! \, \frac{(n-\rho)!}{n!},$$

showing that we can satisfy the equations for a by putting

$$a(p_1^{\pi_1} p_2^{\pi_2} \ldots) = A (p_1^{\pi_1} p_2^{\pi_2} \ldots) \div \frac{r!}{\pi_1 ! \, \pi_2 ! \ldots (p_1 !)^{\pi_1} (p_2 !)^{\pi_2} \ldots},$$

where the partition functions A have been already given up to partitions of six, that is, as far as is needed for the deletion of a column of six entries, in Fisher's expressions for k_1 to k_6 (*loc. cit.*, 203–4).

Let us illustrate this general method of proof by the case when $r = 4$. The function of a pattern containing a column of four entries is to be expressed in terms of the functions of the fifteen simpler patterns formed by deleting this column, and amalgamating the entries of the four rows in which it is represented in every possible way. We represent by $F(4)$ the function of the patterns found by amalgamating all four rows into one, by $F_1(3\,1)$, $F_2(3\,1)$, $F_3(3\,1)$, $F_4(3\,1)$ the functions of the four patterns which can be found by leaving one row untouched and amalgamating into one the remaining three. Similarly there will be three functions $F(2^2)$, six functions $F(2\,1^2)$, and one function $F(1^4)$. We have to find five coefficients $a(4)$, $a(3\,1)$, $a(2^2)$, $a(2\,1^2)$, and $a(1^4)$, such that the new function shall be

$$a(4)\,F(4) + \sum_1^4 a(3\,1)\,F(3\,1) + \sum_1^3 a(2^2)\,F(2^2) + \sum_1^6 a(2\,1^2)\,F(2\,1^2) + a(1^4)\,F(1^4).$$

---

* P. A. MacMahon, *Combinatory analysis*, 1 (1915), 46.

Now the separations in which all four rows lie in different separates appear only in $F(1^4)$, and the sum of their contributions, $Q(1^4)$, comes into the new function with an additional factor

$$\frac{-6}{n(n-1)(n-2)(n-3)},$$

hence the first of the five equations required is

$$\alpha(1^4) = \frac{-6}{n(n-1)(n-2)(n-3)}.$$

Next any separation in which the four rows lie in three different separates will appear in one of the functions $F(2\,1^2)$ and also in $F(1^4)$, and the sum of the contributions from such separations, $Q(2\,1^2)$, comes into the new function with the additional factor $2/n(n-1)(n-2)$, so that we have the second equation

$$\alpha(2\,1^2)+\alpha(1^4) = \frac{2}{n(n-1)(n-2)}.$$

Thirdly, any separation in which the rows fall two each into two different separates will appear in one of the functions $F(2^2)$, in two of the functions $F(2\,1^2)$, and in the function $F(1^4)$, so that

$$\alpha(2^2)+2\alpha(2\,1^2)+\alpha(1^4) = \frac{-1}{n(n-1)}.$$

It should be noted that the coefficient 2 is the number of ways in which a partition (2^2) can be subdivided into a partition $(2\,1^2)$.

Fourthly, any separation in which three of the rows fall into one separate, and the remaining row into a different one, appears in one function $F(3\,1)$, in three functions $F(2\,1^2)$, and in the function $F(1^4)$, giving the equation

$$\alpha(3\,1)+3\alpha(2\,1^2)+\alpha(1^4) = \frac{-1}{n(n-1)},$$

in which the coefficient 3 is the number of ways in which a partition $(3\,1)$ may be subdivided into partitions $(2\,1^2)$.

Finally, any separation in which all four rows fall in the same separate appears in all the functions, so we have

$$\alpha(4)+4\alpha(3\,1)+3\alpha(2^2)+6\alpha(2\,1^2)+\alpha(1^4) = \frac{1}{n},$$

completing the set of five equations necessary to determine the coefficients α.

The equations may evidently be solved by direct substitution, yielding the solution

$$a(1^4) = \frac{-6}{n(n-1)(n-2)(n-3)},$$

$$a(2\,1^2) = \frac{2}{(n-1)(n-2)(n-3)},$$

$$a(2^2) = \frac{-1}{(n-2)(n-3)},$$

$$a(3\,1) = \frac{-(n+1)}{(n-1)(n-2)(n-3)},$$

$$a(4) = \frac{n(n+1)}{(n-1)(n-2)(n-3)}.$$

The proof of the relationship under discussion depends, however, upon the correspondence of the equations for a with those obtained by putting

$$k_4 = A(4)s_4 + A(3\,1)s_3s_1 + A(2^2)s_2^2 + A(2\,1^2)s_2s_1^2 + A(1^4)s_1^4,$$

and expressing the conditions that the mean value of the statistic so constructed shall be equal to the population parameter κ_4 with its identical expression in terms of the moments

$$\kappa_4 = \mu_4 - 4\mu_3\mu_1 - 3\mu_2^2 + 12\mu_2\mu_1^2 - 6\mu_1^4.$$

Here μ_1^4 will appear in the expansion of the mean value of s_1^4 only, giving

$$n(n-1)(n-2)(n-3)A(1^4) = -6;$$

$\mu_2\mu_1^2$ will appear in the means of $s_2s_1^2$ and of s_1^4, giving

$$n(n-1)(n-2)\{A(2\,1^2)+6A(1^4)\} = 12;$$

μ_2^2 will appear in the means of s_2^2, $s_2s_1^2$, and s_1^4, giving

$$n(n-1)\{A(2^2)+A(2\,1^2)+3A(1^4)\} = -3,$$

while the two remaining equations are

$$n(n-1)\{A(3\,1)+2A(2\,1^2)+4A(1^4)\} = -4$$

and $$n\{A(4)+A(3\,1)+A(2^2)+A(2\,1^2)+A(1^4)\} = 1,$$

the numerical factors in these equations being the number of ways in which the parts of any particular partition may be amalgamated to form those of the partition in the equation of which it occurs.

These equations also may be solved by direct substitution, giving the familiar relations

$$A(1^4) = \frac{-6}{n(n-1)(n-2)(n-3)},$$

$$A(2\,1^2) = \frac{12}{(n-1)(n-2)(n-3)},$$

$$A(2^2) = \frac{-3}{(n-2)(n-3)},$$

$$A(3\,1) = \frac{-4(n+1)}{(n-1)(n-2)(n-3)},$$

$$A(4) = \frac{n(n+1)}{(n-1)(n-2)(n-3)},$$

from which it is evident that we can obtain the corresponding solutions for a by dividing by the number of ways in which four objects may be distributed in the required partition, or in general by

$$\frac{1}{\pi_1!\,\pi_2!\,\ldots}\,\frac{r!}{(p_1!)^{\pi_1}(p_2!)^{\pi_2}\ldots}.$$

6. *Proof of the vanishing of a class of zero functions.*

The fact that the function of a two-way partition can be found by an expression linear in the functions of partitions with one less column, and with the rows represented in the deleted column more or less amalgamated, may be used to prove that the partition function is necessarily zero in a class of cases in which its vanishing has hitherto only been noted empirically. The class in question is that of partitions the rows of which can be divided into two groups, which, whatever may be their internal connections, are only connected with each other by entries in a single column. For example, in the pattern

$$\begin{matrix} \times & \times & \times & \cdot \\ \times & \times & \times & \cdot \\ \cdot & \cdot & \times & \times \\ \cdot & \cdot & \times & \times \end{matrix}$$

the group of columns represented in the two upper rows and the group represented in the two lower rows have in common only a single column, namely the third from the left. Evidently all patterns in which any row has only one entry belong to this class, but, whereas a statistical reason has already been given for patterns with such rows having zero functions, no general demonstration applicable to the whole class has been given.

The demonstration which suggests itself from the methods of this paper is as follows. If any of the columns which are not represented in both groups of rows is removed, all the patterns in terms of which the entire pattern-function is expanded still possess two groups of rows having only one column in common. If all such columns are removed in succession, the expansion contains only the functions of patterns having a single column, all of which have, however, two or more rows.

It is obvious statistically that the function of such patterns must be zero, for the statistics k_r have been defined so that their mean sampling values shall be κ_r and shall contain no product terms in κ's of lower degree; we may, however, give an algebraical proof for a pattern of r rows and one column.

The contribution to the function of any separation into s separates will be

$$n(n-1)\ldots(n-s+1)\,\frac{(-)^{s-1}(s-1)!}{n(n-1)\ldots(n-s+1)}=(-)^{s-1}(s-1)!,$$

and, the number of separations being

$$\frac{1}{(s-1)!}\,\Delta^{s-1}(1^{r-1}),$$

the complete function must be

$$\sum_{s=1}^{r}(-)^{s-1}\Delta^{s-1}(1^{r-1}),$$

which is the finite difference expansion of 0^{r-1}, and is consequently zero when r exceeds unity, as in the class of patterns under consideration.

Summary.

A method is developed of calculating the function of n to be associated with any two-way partition in the evaluation of the cumulants of the sampling distribution of the appropriate moment statistics k, by expanding it in terms of the functions of partitions having simpler patterns. When columns of two or three entries occur the simplification is extremely rapid. The method is, however, generalized for all cases.

A proof is given of the vanishing of the functions corresponding to all patterns in which the rows may be divided into two groups having only a single column in common.