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AN EXAMINATION OF THE DIFFERENT POSSIBLE SOLUTIONS OF A PROBLEM IN INCOMPLETE BLOCKS

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1. Introduction

Recent papers in the *Annals of Eugenics* by Yates (1936) and Bose (1939) have drawn attention to the importance of the combinatorial problem which arises when it is desired to compare a number of "varieties", or experimental treatments, on "blocks" of experimental material, which, for the sake of greater homogeneity contain fewer units than the number of varieties to be used. The practical importance of this type of experimental arrangement has been demonstrated by Yates, who also provides a series of practically valuable solutions. A somewhat larger collection has been since published by Fisher & Yates (1938). Bose, while adding further solutions to those so far discovered, has discussed the intimate connexion of this problem with other branches of mathematics, notably with finite geometries.

Although the greatest practical importance attaches to the first solution of such a problem, it is also of some theoretical interest to discover what other types of solution may exist. The chief purpose of the present paper is to report the results of such an exploration of one of these problems, chosen as being in itself comparatively simple, while at the same time furnishing a multiplicity of solutions.

As a preliminary let us set out the primary arithmetical requirements, and demonstrate an important inequality.

If each block contains a selection of k different varieties out of the number v available, we require a set of b blocks, such that in the whole solution each variety shall occur r times, while each pair of varieties shall occur together λ times. Then the five variable integers are connected by two primary equations:

$$vr = kb, \qquad \text{......(1)}$$

$$(v-1)\lambda = (k-1)r. \qquad \text{......(2)}$$

Sets of numbers fulfilling these two conditions may be thought of as constituting a discontinuous assemblage in three dimensions.

Corresponding to any solution, there will be an infinite series of other solutions obtained by merely repeating the arrangement arrived at n times. In such a series the values of k and v are unchanged, but those of λ, r and b will be multiplied by n. The existence for such a series of problems of solutions corresponding with all solutions of the primary problem is thus assured. Further, if λ, r and b have any common factor, n, there may exist a solution for the same value of v and k and for values λ/n, r/n and b/n, and this will be so if, and only if, a solution of the primary problem exists consisting of sets of n identical blocks. For the practical purpose of obtaining a single solution, therefore, the whole series of problems is solved when it is ascertained that the terminal member has a solution. Any problem with $\lambda = 1$ stands at the head of its series. The types of solution may, however, become much more numerous as the number of replications is increased.

A different type of correspondence is shown by the complementary solution. If, keeping v and b unchanged, we replace each block of k variates by a block of the remaining $v-k$ varieties, then

$$k' = v-k, \quad \text{or} \quad k+k' = v;$$

also, since

$$vr' = k'b = (v-k)b = vb-vr,$$

it follows that

$$r' = b-r, \quad \text{or} \quad r+r' = b.$$

Again, since

$$(v-1)\lambda' = (k'-1)r' = (v-k-1)r'$$

and

$$(v-1)\lambda = (k-1)r,$$

we have

$$(v-1)(\lambda'-\lambda) = vr'-kb-(r'-r)$$
$$= (v-1)(r'-r).$$

Hence

$$\lambda'-\lambda = r'-r, \quad \text{or} \quad r'-\lambda' = r-\lambda.$$

Since to every solution of a problem in which $2k > v$ there thus corresponds a solution of a corresponding problem in which $2k < v$, the number and structure of systems of solutions of the two complementary problems are identical.

If all possible selections of k from v varieties were made we should have

$$b = v!/k!(v-k)!$$
$$r = (v-1)!/(k-1)!(v-k)!$$
$$\lambda = (v-2)!/(k-2)!(v-k)!$$

If H is the highest common factor of these expressions, and if a solution exists for b, r and λ equal each to $1/H$ of the expression above, then all problems associated with the values v and k form a single series. If, however, no solution exists for the highest common factor, all prime multiples will belong to different series. It is important that the smallest proportional set of possible values, b, r, λ may be incapable of giving a solution through having $r < k$. We shall now prove that in no such case is a solution possible.

In relation to any block, let us consider as a variable quantity, x, the number of varieties which any other block has in common with it. There will be $b-1$ such other blocks. The sum of the $b-1$ values of x, which we may write $S(x)$, is easily found; for each of the k varieties in the block appears $r-1$ times in other blocks. Consequently

$$S(x) = k(r-1). \qquad \ldots\ldots(3)$$

The ratio of $S(x^2)$ to $S(x)$ will be the average value of x in blocks chosen to contain one variety in common with the first block. For any chosen variety there will be $r-1$ s blocks, and each of the other $(k-1)$ varieties will occur in these $(\lambda-1)$ times. Hence

$$S(x^2)/S(x) = 1+\frac{(k-1)(\lambda-1)}{r-1}.$$

Then $$S(x^2) = k(r-1)+k(k-1)(\lambda-1). \qquad \text{......(4}$$

But, the sum of the squares of the $\frac{1}{2}(b-1)(b-2)$ differences between two values of x,

$$(b-1)\,S(x^2)-S^2(x)$$

is necessarily positive or zero. Since from (1),

$$b-1 = \frac{vr}{k}-1,$$

we have $$vr(r-1)+vr(k-1)(\lambda-1)-k(r-1)-k(k-1)(\lambda-1)-k^2(r-1)^2.$$

But $$vr(r-1)-vr(k-1) = vr(r-k)$$

and $$vr(k-1)\lambda = r^2(k-1)^2+r(k-1)\lambda, \quad \text{from (2)}$$

$$r^2(k-1)^2-k^2(r-1)^2 = (r+k-2rk)(r-k),$$

$$r(k-1)\lambda-k(k-1)\lambda = (k-1)\lambda(r-k),$$

$$-k(r-1)+k(k-1) = -k(r-k).$$

Hence $$(r-k)\{vr+r-2rk+(k-1)\lambda\}$$
cannot be negative, but

$$(k-1)\lambda = (k-1)r-(v-k)\lambda, \quad \text{from (2)}$$

and $$vr-rk = (v-k)r,$$

hence the chosen expression is factorized in the form

$$(r-k)(v-k)(r-\lambda), \qquad \text{......(5}$$

which, divided by $(b-1)^2$, gives the variance of the $b-1$ values of x corresponding w any block.

Now, in all cases, $v>k$ and $r>\lambda$, hence $r \geqslant k$.

Also, in the limiting case where $r = k$, it follows that x is constant, and evidently is eq to λ. In this limiting type of problem, blocks and varieties are equal in number and fı the same condition.

Observe that when the blocks are complete, the factors $(v-k)$ and $(r-\lambda)$ both vani so that r may be less than k, while still satisfying the requirement that the variance c cannot be negative.

When $\lambda = 1$, equations (3) and (4) reduce to

$$S(x) = S(x^2) = k(r-1),$$

so that x is always 0 or 1, and in fact takes the value unity just $k(r-1)$ times. In other cases more than one distribution of x is possible, and more than one may be realized for different blocks of the same solution.

In the case $r = 8$, $v = 9$, $k = 4$, $b = 18$, $\lambda = 3$ (no. 11 of *Statistical Tables*), it appears that

$$b-1 = 17,$$
$$S(x) = k(v-1) = 28,$$
$$1+(k-1)(\lambda-1)/(r-1) = 1+6/7,$$
$$S(x^2) = 52.$$

There are two possible distributions of 17 values of x such that their sum is 28, and the sum of their squares 52, namely,

x	Frequency distribution (a)	Frequency distribution (b)
0	1	—
1	4	7
2	12	9
3	—	1
Total	17	17

Since in the solution given blocks *abgh* and *cdef* appear, having no letter in common, these must both have the frequency distribution (a), while the blocks *abcd* and *bcdg* having three letters in common must both have the frequency distribution (b).

In general if, in any distribution, four consecutive frequencies can be increased or diminished by the series 1, -3, $+3$, -1, without introducing negative frequencies, the values of $S(x)$ and $S(x^2)$ will be unaltered, and a new solution will be obtained.

2. Standard solutions and sets

Given any solution of a problem of incomplete blocks, we may designate the varieties by letters, supposedly unlimited in number, and arrange the letters in each block in alphabetical order. Since the blocks may themselves now be arranged in alphabetical order, using the same convention as for words in a dictionary, it is obvious that corresponding with any solution there is one and only one solution in standard order. This is called a standard solution. From it a permutation of the blocks will generate a number of solutions, the number being $b!$ if the blocks are all different, as they must be when $\lambda = 1$, but which is a submultiple of $b!$ for solutions containing sets of two or more identical blocks.

Given a standard solution, we may permute the letters, and rearrange the blocks in standard order, so as to obtain either the same or another standard solution. Permutation of the letters, then, will generate a set of $v!$, or some submultiple of $v!$, different standard solutions. Corresponding to any member of a set of less than $v!$ standard solutions there will be a group of permutations of the letters which is inoperative in changing the solution. The number of standard solutions in the set is $v!$ divided by the order of this inoperative permutation group. Any permutation of the letters which gives a new standard solution may be applied to the inoperative group in order to find the inoperative group of the new solution.

We may wish to consider solutions subject to some further restriction. For example, where b is a multiple of r, and therefore v of k, the blocks may be divisible into r divisions, each comprising a complete replication. Such restricted solutions must also have inoperative permutation groups, which must be subgroups (including in that term the two extremes, the identity and the entire group) of the group inoperative for the corresponding unrestricted solution. When the subgroup is a proper subgroup its order must be a factor of the order of the group, and the ratio of these two orders represents the number of ways of subdividing the solution in question into replications of this set. The same unrestricted solution may, of course, be capable of subdivision in two or more ways belonging to different sets, just as a Latin square may have Graeco solutions belonging to different sets of Graeco-Latin squares.

3. Method of specification appropriate to blocks of 3, $\lambda = 1$

In specifying a solution it is useful to determine some character of the individual varieties in which they may be the same or different. With blocks of three this may be done by determining a character of each pair of varieties. Thus we choose two varieties a and b, then when $\lambda = 1$, these uniquely determine a third variety c, with which they constitute a block. The remaining varieties will each occur with a in one block and with b in another. Thus we may find a chain of blocks such as

$$paq \quad qbr \quad ras \quad sbt \quad tau \quad ubv \quad \ldots$$

in which r must differ from p, and t from r, though t may be the same as p. At whatever point recurrence occurs we shall have a pair of closed chains consisting of letters other than a, b and c. The number of varieties must, of course, be odd, since $v-1 = 2r$, and we see that each pair of letters will be associated with some partition of the partible number $\frac{1}{2}(v-3)$, or $(r-1)$, into parts of 2 or more. In the case we shall investigate, $v = 15$, the partible number is 6, and the possible partitions are

$$(6), \quad (4\,2), \quad (3^2) \quad \text{and} \quad (2^3).$$

Since parts of magnitude 2 have a special convenience for generating new sets, our primary interest in the partition lies in the number of these it contains. Thus a partition (2^3) may be marked 3, a partition $(4\,2)$ is marked 1, while partitions 6 and (3^2) which have no part 2 are denoted by the symbols $-$ and $\times$, since there is also a real advantage in distinguishing them.

In order to classify rapidly a number of pairs of varieties appearing in any given solution, it is convenient to write out the $v \times v$ Latin square corresponding with the solution. If the rows, columns and letters of a square are all made to correspond with varieties, then the existence of a block *abc* requires six entries corresponding to a general permutation of the categories, i.e.

Row	*a*	meets column	*b*	at letter	*c*
,,	*a*	,,	*c*	,,	*b*
,,	*b*	,,	*a*	,,	*c*
,,	*b*	,,	*c*	,,	*a*
,,	*c*	,,	*a*	,,	*b*
,,	*c*	,,	*b*	,,	*a*

The square is therefore itself self-adjugate. As an illustration I give the 15×15 square corresponding to Savur's (1939) solution of the problem $v = 15$, $k = 3$, $b = 35$, $r = 7$, $\lambda = 1$.

Table 1. *Self-adjugate* 15×15 *Latin square corresponding to the problem of selecting* 35 *incomplete blocks of* 3 *out of* 15 *varieties* (*Savur's solution*)

a	o	n	m	l	k	j	i	h	g	f	e	d	c	b
o	**b**	m	l	k	j	i	n	g	f	e	d	c	h	a
n	m	**c**	k	j	i	o	l	f	e	d	h	b	a	q
m	l	k	**d**	i	o	n	j	e	h	c	b	a	g	f
l	k	j	i	**e**	n	m	o	d	c	b	a	g	f	h
k	j	i	o	n	**f**	l	m	c	b	a	g	h	e	d
j	i	o	n	m	l	**g**	k	b	a	h	f	e	d	c
i	n	l	j	o	m	k	**h**	a	d	g	c	f	b	e
h	g	f	e	d	c	b	a	**i**	o	n	m	l	k	j
g	f	e	h	c	b	a	d	o	**j**	m	n	k	l	i
f	e	d	c	b	a	h	g	n	m	**k**	o	j	i	h
e	d	h	b	a	g	f	c	m	n	o	**l**	i	j	k
d	c	b	a	g	h	e	f	l	k	j	i	**m**	o	n
c	h	a	g	f	e	d	b	k	l	i	j	o	**n**	m
b	a	g	f	h	d	c	e	j	i	k	h	n	m	**o**

Letters in heavy type on the diagonal indicate with which variety each row or column is taken to correspond. In use this diagonal may be left blank.

From such a square it is easy to read off the entries of the triangular diagram in which each pair of letters is characterized. To do this we fix attention on the two chosen rows and alternate between them following the same column and the same letter alternately. If a cycle of 6, 4 or 3 letters is encountered the partition is determined without further inspection; if a cycle of two, it will be necessary to start with a third letter to determine whether it belongs to a cycle of two (scoring 3), or of four (scoring 1). Thus from rows **a** and **b** of the square above, we may read the cycle ..*nmlkji*..; similarly, from rows **a** and **h** the cycle ..*nlo*.., from **a** and **i** the cycles ..*nf*.. and ..*og*.., while from **a** and **j** the cycle ..*ofmh*.., these being representatives of all four partitions possible.

The triangular table representing the relations of the 105 pairs of varieties is then easily filled. The lines of such a table are read down to the diagonal, and then across horizontally.

Each individual letter is now characterized by the score of each of the fourteen pairs into which it enters. Thus *d*, *f* and *g* each enter into three pairs scoring 1 and one pair scoring $\times$, they may be characterized by the formula ($1^3\times$). The classification of all fifteen letters is then as follows:

$1^3\times$	$3\,1^3\times$	$3\,1^3\times^7$	1^{12}	$3\,1^8$	$3^2\,1^6$	$3^{10}\,1^{90}\times^{14}$
dfg	*abce*	*h*	*lno*	*jkm*	*i*	Total

Table 2. *Triangular diagram showing the character of the double chains associated with each pair of varieties* (*Savur's solution*)

a	—	—	—	—	—	—	×	3	1	1	—	1	—	—
	b	—	—	—	—	—	×	—	3	1	1	—	1	—
		c	—	—	—	—	×	—	—	3	1	1	—	1
			d	—	—	—	×	—	—	—	1	—	1	1
				e	—	—	×	—	1	—	—	3	1	1
					f	—	×	—	—	—	1	—	1	1
						g	×	—	—	—	1	—	1	1
							h	3	—	—	1	—	1	1
								i	1	1	1	1	1	1
									j	1	1	1	1	1
										k	1	1	1	1
											l	1	1	1
												m	1	1
													n	1
														o

Since *h* and *i* are unique, it follows that the same is true of the letter *a* with which they make a block. The twelve other letters fall in four groups of three. It is then easy to find that when *b* is replaced by *c*, *c* by *e* and *e* by *b*, it is necessary, if we are to maintain the same solution, to make the similar cyclic substitutions (*dgf*), (*jkm*), (*lon*), and that no other permutation than this will leave the solution unaltered. The inoperative group is therefore the cyclic group of order 3,

$$(bce)\,(dgf)\,(jkm)\,(lon),$$

and the number of standard solutions in the set is $15!/3$.

4. Interchanges

Any cycle corresponding to a part 2 implies the existence of four blocks, such as

$$pax,\quad xbq,\quad qay,\quad ybp,$$

which may be thought of as a set of four out of the eight possible successions of choices, *a* or *b*, *p* or *q*, *x* or *y*.

When a set of four such blocks occurs in a solution it implies the existence of a part 2 in the partition corresponding with the three pairs of letters *ab*, *pq* and *xy*. It will, therefore, contribute 6 to the total for all the letters. Thus, if taking the total of all the formulae for the fifteen individual letters we have, as in this case, $3^{10}\,1^{90}\times^{14}$, we may obtain the number of such sets of four blocks in the solution by dividing $(3\times 10)+90 = 120$, by 6. There are thus 20 such sets of four blocks in Savur's solution.

Evidently in such a set of four, if we interchange either a and b, or p and q or x and y, we shall obtain a complementary set of four blocks, e.g.

$$apy, \quad aqx, \quad bpx, \quad bqy,$$

in which, as before, the fifteen pairs of these six letters, except the three pairs ab, pq and xy, all occur once. This new set of blocks may therefore replace the old in the complete solution, thus supplying a convenient method of generating new solutions from any given one. Such an interchange is sufficiently designated by the formula

$$\{(ab)\ (pq)\ (xy)\},$$

implying that whichever set of four blocks occurs in the old solution is to be replaced by its alternative. The outer brackets are only required to distinguish an interchange limited to four blocks, from a permutation applied to all.

It is not in general necessary to examine all possible interchanges individually, since the symmetry implied by the existence of any inoperative group of permutations shows that some of them may be equivalent. Thus Savur's solution in the form in which I have taken it from his paper (only replacing numbers by the corresponding letters of the Latin alphabet), contains the four blocks

$$abo, \quad agj, \quad bgi, \quad ijo,$$

and is therefore susceptible to the interchange

$$\{(ai)\ (bj)\ (go)\};$$

applying the inoperative permutation

$$(bce)\ (dgf)\ (jkm)\ (lon),$$

it is clear that the set of three interchanges

$$\{(ai)\ (bj)\ (go)\}, \quad \{(ai)\ (ck)\ (fn)\}, \quad \{(ai)\ (dl)\ (em)\},$$

are all equivalent, and must lead to members of the same set of solutions.

To express the argument more formally, if S stand for Savur's solution, then we have shown that

$$(bce)\ (dgf)\ (jkm)\ (lon)\, S = S.$$

Hence, applying the interchange,

$$\begin{aligned}\{(ai)\ (bj)\ (go)\}\, S &= \{(ai)\ (bj)\ (go)\}.(bce)\ (dgf)\ (jkm)\ (lon)\, S \\ &= (bce)\ (dgf)\ (jkm)\ (lon).\{(ai)\ (dl)\ (em)\}\, S.\end{aligned}$$

In this manner the twenty possible interchanges are reduced to six triplets of equivalent interchanges, and two single interchanges, which are unaltered by the inoperative permutation. The reversal of each of these symmetrical interchanges generates only $15!/3$ standard solutions, so that not more than this number can be different. In consequence, they must lead to sets of at least threefold symmetry. A triplet of equivalent interchanges, on the other hand, might lead to a set without symmetry containing the full number of $15!$

different solutions, and indeed four of the six do so. The two other triplets, which lead to sets of threefold symmetry, necessarily account for similar triplets in the sets to which they lead.

An interchange occasionally leads to a member of the same set as the solution to which it is applied. We may then distinguish between cases in which it re-enters by the same or by a different interchange; both types occur among these solutions.

5. The sets found

In all, seventy-nine distinct sets of solutions were encountered. This was more than I had expected, since the problem with $v = 13$, $k = 3$, $r = 6$, $b = 26$ has only two sets. Classified according to the two simple characteristics (i) the symmetry number, the order of the permutation group for which each solution is invariant, and (ii) the number of possible reversals, they are shown distributed in the following table (Table 3).

Table 3. *Distribution of* 79 *sets in two characters*

Reversals \ Symmetry number	1	2	3	4	5	6	8	12	21	24	32	36	96	168	192	288	2016
2	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
5	5	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
6	3	.	.	2	.	.	.	1	.	.	.	1	.	.	.	.	.
7	4	.	3	.	.	.	.	.	.	.	.	.	.	.	.	.	.
8	3	2	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
9	6	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
10	2	.	1	1	1	1	.	.	.	.	.	.	.	.	.	.	.
11	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
12	4	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
13	2	.	2	.	.	.	.	.	.	.	.	.	.	.	.	.	.
14	1	1	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.
15	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
17	.	.	1	.	.	.	.	1	.	.	.	.	.	.	.	.	.
18	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
19	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
20	1	.	2	.	.	.	.	.	.	.	.	.	.	.	.	.	.
23	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
25	.	.	.	2	.	.	.	.	.	1	.	.	.	.	.	.	.
31	.	2	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
32	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
33	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	1	.
37	.	.	.	1	.	.	.	1	.	1	.	.	.	.	.	.	.
49	.	.	.	.	.	.	1	.	.	.	1	.	.	1	.	.	.
57	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.
73	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.
105	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1
	36	6	12	8	1	1	2	3	1	2	1	1	1	1	1	1	1

In its general character the table resembles that of H. W. Norton for the sets of 7×7 Latin squares, with which it should be compared. The distribution of reversal number is remarkably level with a preference for odd numbers, especially among the larger numbers. As with the Latin square the higher reversal numbers are associated with high symmetry.

It will be noticed that a large number of sets, thirty-six in all, are without symmetry. In these no two varieties are related alike to the others. As these sets contain the full number of 15! standard solutions, while the remaining forty-three sets each contain at most half that number, it appears that the majority of all solutions belong to such complete sets. Actually, of about 60 billion* standard solutions over 47 billion, or 78·12 % belong to sets without symmetry. The distribution of numbers of sets and standard solutions by symmetry number and number in set is shown in the following table:

Table 4. *Distribution of sets in relation to symmetry*

Symmetry no.	No. of solutions in each set	No. of sets	No. of solutions	Percentage	Percentage with as high or higher symmetry
1	1,307674,368000	36	47,076277,248000	78·12	
2	653837,184000	6	3,923023,104000	6·510	21·88
3	435891,456000	12	5,230697,472000	8·682	15·370
4	326918,592000	8	2,615348,736000	4·341	6·688
5	261534,873600	1	261534,873600	0·4340	2·3473
6	217945,728000	1	217945,728000	0·3617	1·9133
8	163459,296000	2	326918,592000	0·5425	1·5516
12	108972,864000	3	326918,592000	0·5425	1·0091
21	62270,208000	1	62270,208000	0·1033	0·4666
24	54486,432000	2	108972,864000	0·1808	0·3633
32	40864,824000	1	40864,824000	0·06781	0·18254
36	36324,288000	1	36324,288000	0·06028	0·11473
96	13621,608000	1	13621,608000	0·02260	0·05445
168	7783,776000	1	7783,776000	0·01292	0·03185
192	6810,804000	1	6810,804000	0·01130	0·01893
288	4540,536000	1	4540,536000	0·007518	0·007626
20160	64,864800	1	64,864800	0·0001076	0·0001076
		79	60,259918,118400		

The cumulative percentages in the right-hand column show that about one solution in 100 has symmetry of 12-fold or more, about one in 1000 of 36-fold, while only one in a million belongs to the set with highest symmetry (20160). The graph (Fig. 1) shows the relationship between the symmetry on a logarithmic scale, and the negative logarithm of the probability of finding symmetry as high or higher.

Table 3 shows also the number of interchanges available in each set. In the thirty-six sets without symmetry these are liable all to lead to different sets. The 363 interchanges of which these sets are capable lead predominantly to sets without symmetry, but a minority lead to smaller sets, in such cases they correspond with a number of equivalent interchanges

* This seems to be the correct English word for 10^{12}. In journalism, and unfortunately even in mathematical works in America, the word is used for 10^9.

in the sets to which they lead. Table 5 shows the distribution of the totality of 1390 interchanges according to the symmetry of the sets from which they originate, and that of the sets to which they lead. The right-hand vertical margin gives the former distribution, and the lower margin the latter.

The entries in the diagonal, representing cases in which the symmetry number is unaltered, include all cases in which an interchange leaves the set unaltered. Thus of the 291

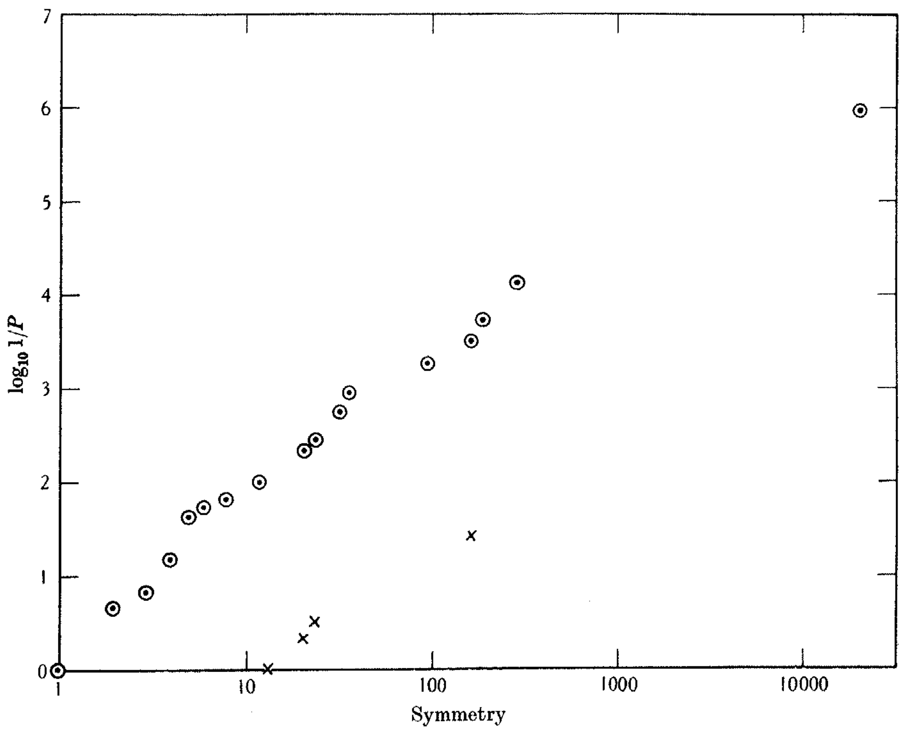

Fig. 1. The relation between the relative frequency, P, of solutions having as high or higher symmetry, and the symmetry number of the sets concerned. Both are plotted logarithmically. ⊙ Solutions of the unrestricted problem. × Solutions of the restricted problem.

cases in which both sets are without symmetry in 25 the sets are identical. In seven of these the interchange also is identical, while the remaining 18 cases represent 9 pairs of reciprocal interchange.

With respect to sets showing symmetry certain variations may be noted in the permutation groups to which they are invariant. In nearly all cases these contain only even permutations. No. 21 is an exception. Of the six sets with twofold symmetry the four with 23 interchanges or less involve six pairs of equivalent varieties, and three unique, while the two sets with 31 interchanges have seven unique varieties, and only four pairs involved in the inoperative permutation.

Ten out of the twelve sets with threefold symmetry have three unique varieties and four triplets which may be cyclically permuted. The remaining two, one with two interchanges and one of those with seven, have five triplets. Of the eight sets with inoperative groups of order four, six have cyclic groups. Those with six interchanges are of the form $(4^3\,2)$, that with eight of the form (4^3), those with 10 and 18 of the form $(4^3\,2)$. One with 25 is of the form $(4^2\,2^2)$, while the other, and that with 37, are non-cyclic.

Table 5

Symmetry of final set

Symmetry of initial set	1	2	3	4	5	6	8	12	21	24	32	36	96	168	192	288	20160	Total
1	291	20	37	11	2	.	1	1	.	.	.	.	.	.	.	.	.	363
2	40	32	10	18	.	2	7	4	.	1	1	.	.	.	.	.	.	115
3	111	15	32	3	.	.	3	1	2	.	.	.	.	.	.	.	.	167
4	44	36	4	32	.	2	6	3	.	5	1	.	1	1	.	.	.	135
5	10	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	10
6	.	6	.	3	.	.	.	.	.	.	.	1	.	.	.	.	.	10
8	8	28	8	12	.	.	10	6	.	4	3	.	1	.	2	.	.	82
12	12	24	4	9	.	.	9	1	.	.	.	.	1	.	.	.	.	60
21	.	.	14	.	.	.	.	.	.	.	.	.	.	.	.	.	.	14
24	.	12	.	30	.	.	12	.	.	3	3	.	.	.	.	2	.	62
32	.	16	.	8	.	.	12	.	.	4	4	.	2	.	2	1	.	49
36	.	.	.	.	.	6	.	.	.	.	.	.	.	.	.	.	.	6
96	.	.	.	24	.	.	12	8	.	.	6	.	.	4	3	.	.	57
168	.	.	.	42	.	.	.	.	.	.	.	.	7	.	.	.	.	49
192	.	.	.	.	.	.	48	.	.	.	12	.	6	.	6	.	1	73
288	.	.	.	.	.	.	.	.	.	24	9	.	.	.	.	.	.	33
20160	.	.	.	.	.	.	.	.	.	.	.	.	.	.	105	.	.	105
	516	189	109	192	2	10	120	24	2	41	39	1	18	5	118	3	1	1390

6. Restricted solutions

Of the 79 sets of solutions of the incomplete block problem, four yield solutions of the more restricted problem in which the 35 blocks are divisible into seven separate replications. These give the sets of solutions of Kirkman's problem, of which a long discussion is given by Rouse Ball. Unless I have overlooked any, there are seven such sets, which accords with Rouse Ball's statement (p. 218) that they are not less than seven nor more than eleven. Another of his remarks, "the number of solutions is said to be $65 \times 13!$, but I do not vouch for the correctness of this result" seems, in fact, to be accurate.

Two different sets may be formed on the solution (No. 79) having the highest symmetry. Thus in

abe	*acd*	*afg*	*ahi*	*ajk*	*alm*	*ano*
chl	*bmo*	*bhj*	*bcf*	*bln*	*bik*	*bdg*
djo	*eij*	*cko*	*dkn*	*ceg*	*cjn*	*cim*
fkm	*fhn*	*dil*	*elo*	*dhm*	*def*	*ehk*
gin	*gkl*	*emn*	*gjm*	*fio*	*gho*	*fjl*

each column is a complete replication. The blocks, and their grouping by replications, are unchanged by the permutations

$$(ah)\ (bl)\ (ce)\ (dk)\ (fo)\ (jm)$$
$$(be)\ (cl)\ (dm)\ (fo)\ (gn)\ (jk)$$
$$(aeb)\ (cdf)\ (bjk)\ (lom)$$
$$(acogjlb)\ (dkhmfne)$$

These generate a group of order 168, so that the same blocks may be subdivided in 120 ways all belonging to the same set of solutions. This group is transitive for all letters save *i*. The number of solutions in the set is $15!/168$ or $13! \times 1\frac{1}{4}$.

A second set of subdivisions of the same thirty-five blocks is represented by

abe	*acd*	*afg*	*ahi*	*ajk*	*alm*	*ano*
chl	*bik*	*bmo*	*bcf*	*bln*	*bdg*	*bhj*
djo	*emn*	*cjn*	*dkn*	*ceg*	*cko*	*cim*
fkm	*fjl*	*dil*	*elo*	*dhm*	*eij*	*def*
gin	*gho*	*ehk*	*gjm*	*fio*	*fhn*	*gkl*

Three of the replications are the same as in the first example, the contents of the remaining four having been redistributed. Here also the inoperative group is of order 168, but is transitive for sets of 7 and 8 letters; it may be generated from

$$(afni)\ (bcje)\ (dlmk)\ (gh)$$
$$(ago)\ (bel)\ (dkm)\ (fhn)$$
$$(agnohfi)\ (blekmcd)$$

The number of restricted solutions from these two sets is thus $13! \times 2\frac{1}{2}$.

Two more sets may be formed on solution no. 70, which has 288-fold symmetry:

abc	*ade*	*afg*	*ahi*	*ajk*	*alm*	*ano*
dhl	*bik*	*bhj*	*bmo*	*bln*	*bdf*	*beg*
ejo	*cmn*	*clo*	*cdg*	*cef*	*cij*	*chk*
fkm	*fho*	*dkn*	*ekl*	*dio*	*ehn*	*djm*
gin	*gjl*	*eim*	*fjn*	*ghm*	*gko*	*fil*

and

abc	*ade*	*afg*	*ahi*	*ajk*	*alm*	*ano*
dhl	*bik*	*bhj*	*bmo*	*bln*	*bdf*	*beg*
ejo	*clo*	*cmn*	*cef*	*cdg*	*cij*	*chk*
fkm	*fjn*	*dio*	*dkn*	*eim*	*ehn*	*djm*
gin	*ghm*	*ekl*	*gjl*	*fho*	*gko*	*fil*

The inoperative groups of these two solutions have in common a subgroup of 8, which is inoperative for both solutions,

$$(ab)\ (dnfo)\ (elgm)\ (hikj)$$
$$(dg)\ (ef)\ (hi)\ (jk)\ (ln)\ (mo)$$
$$(de)\ (fg)\ (hj)\ (ik)\ (lo)\ (mn)$$
$$(ab)\ (dm)\ (eo)\ (fl)\ (gn)\ (hk)$$
$$(ab)\ (dl)\ (en)\ (fm)\ (go)\ (ij)$$

To complete the inoperative group for the first solution it is sufficient to introduce the new element

$$(abc)\ (dlh)\ (enk)\ (foi)\ (gmj),$$

while that for the second solution is completed by

$$(abc)\ (efg)\ (ijk)\ (mno).$$

Both these groups are isomorphic with the complete permutation group of four objects. Each set of solutions therefore provides 12 methods of dividing the same blocks into replications. The common subgroup is transitive for the set of 8 letters *defglmno*, and for the set of 4 letters *hijk*; in the first solution these sets are combined in a set of 12, but in the second solution they remain separate. As each set has 24-fold symmetry it provides $8\frac{3}{4} \times 13!$ distinct solutions. The two sets together give $17\frac{1}{2} \times 13!$, and these added to the two sets based on no. 79, make $20 \times 13!$.

Set no. 52, with only 12-fold symmetry, yields two solutions belonging to different sets; these sets are twins, and may be represented as follows:

aeg	*afj*	*aho*	*aim*	*ejn*	*elm*	*eko*
bfm	*bgk*	*bin*	*beh*	*bcj*	*cdm*	*ack*
cio	*chl*	*cef*	*cgn*	*adn*	*abl*	*bdo*
dhj	*dei*	*dgl*	*dfk*	*gij*	*ghm*	*fgo*
kln	*mno*	*jkm*	*jlo*	*fhn*	*fil*	*hik*

Four replications are common to the two arrangements; the remaining 15 blocks may be divided into three replications by following either the Latin or the Greek letters of the 3×3 Graeco-Latin square

$$\begin{matrix} A\alpha & B\beta & C\gamma \\ B\gamma & C\alpha & A\beta \\ C\beta & A\gamma & B\alpha \end{matrix}$$

These two solutions are both invariant to the same group of permutations as the unrestricted set on which they are formed. They therefore each contribute $15!/12$, or $17\frac{1}{2} \times 13!$, to the number of restricted solutions. The six given so far consequently total $55 \times 13!$.

The last set of restricted solutions is provided by no. 48 having 21-fold symmetry, and is the only subdivision of this solution.

abn	*aco*	*adi*	*aej*	*afk*	*agl*	*ahm*
cgj	*bgm*	*bel*	*bdo*	*bhj*	*bck*	*bfi*
dem	*dhk*	*chn*	*cfm*	*cei*	*dfj*	*cdl*
fhl	*efn*	*fgo*	*ghi*	*dgn*	*eho*	*egk*
iko	*ijl*	*jkm*	*kln*	*lmo*	*imn*	*jno*

The inoperative permutation group is of order 21. The number of solutions provided by this set is therefore 15!/21 or 10 × 13!. The whole number is therefore 65 × 13! as reported so doubtfully by Rouse Ball. More than half of these are contributed by the twin sets based on no. 52. Table 6 shows the actual numbers and their distribution by symmetry number:

Table 6. *Distribution of restricted solutions*

Symmetry no.	No. of solutions in each set	No. of sets	No. of solutions	Percentage	Percentage with as high or higher symmetry
12	108972,864000	2	217945·728000	53·85	—
21	62270,208000	1	62270,208000	15·38	46·15
24	54486,432000	2	108972,864000	26·92	30·77
168	7783·776000	2	15567·552000	3·846	3·846
		7			

The relation between frequency and symmetry is also shown, in comparison with the unrestricted solution in Fig. 1.

Summary

The paper reports the results of an exploration, and is of interest principally as a study in method. The study of the types of configuration meeting specified requirements, must take account of the possibility of a multiplicity of solutions, and it would seem from this example, and from that of the 7 × 7 Latin squares, enumerated in the last volume of this journal, that the greater number of solutions are likely to belong to sets having no symmetrical relationships whatever.

Methods of specifying the invariant characteristics of a solution will vary much with the size of the blocks, and even for blocks of 3 with the frequency (λ) with which each pair of varieties is tested together. An incomplete specification may well suffice to distinguish all the solutions which actually exist, and to determine the size of the set, and other properties of practical value. In addition to such a specification, it is important to possess a ready method, such as that provided by interchange of equivalent sets of blocks, of developing new solutions from any one given.

Seventy-nine sets of solutions have been found for the particular problem of selecting

35 blocks of 3 out of 15 varieties, in such a way that each variety appears seven times and each pair of varieties once in the same block. It is difficult to assess the probability that any further set has escaped detection. Examples of each of these sets, and particulars of symmetry, numbers of simple interchanges, etc., are set out in full in the following table. The inoperative permutation group is indicated by a selection of generators, though no attempt has been made to specify it systematically.

Seven sets also have been found of the more restricted problem, formerly widely known as Kirkman's problem. Only four of the unrestricted solutions lead to solutions of this problem, which are enumerated in this paper.

Key to table of solutions

In the first column, following the serial number of the set, is given the total classification formula for all 15 letters, e.g. $(1^{12} \times^{30})$ in Number 1. Beneath this are the number of interchanges (2) and the symmetry number (3), which is the order of the inoperative permutation group. On the right the classification formulae of the fifteen letters used in the example below are listed in three columns. Below are shown the 35 blocks of an example of the set in question, and finally below that a generator or a selection of the generators, of the inoperative group for this example.

1. $(1^{12} \times^{30})$	*a*	$(\times)$	*f*	$(\times^3)$	*k*	$(1 \times^2)$
	b	.	*g*	(1)	*l*	.
2	*c*	.	*h*	.	*m*	$(1^2 \times^4)$
	d	$(\times^3)$	*i*	.	*n*	.
3	*e*	.	*j*	$(1 \times^2)$	*o*	.

abl	*ahj*	*bfn*	*cfm*	*dil*	*fij*	*him*
ack	*aio*	*bgm*	*cgl*	*djm*	*fhk*	*jko*
adn	*bcj*	*bik*	*chn*	*egj*	*flo*	*jln*
aem	*bdh*	*cdo*	*def*	*ehl*	*gho*	*klm*
afg	*beo*	*cei*	*dgk*	*ekn*	*gin*	*mno*

(*abc*) (*def*) (*ghi*) (*jkl*) (*mno*)

2. $(1^{30} \times^{22})$	*a*	$(\times)$	*f*	$(1^2 \times)$	*k*	$(1^2 \times^3)$
	b	$(\times^3)$	*g*	$(1^2 \times^2)$	*l*	(1^3)
5	*c*	$(1 \times)$	*h*	.	*m*	(1^3)
	d	$(1^2 \times)$	*i*	.	*n*	$(1^3 \times^2)$
1	*e*	.	*j*	.	*o*	$(1^4 \times)$

abk	*ahm*	*bfl*	*cfn*	*dgh*	*fgi*	*hlo*
acj	*ali*	*bgm*	*chk*	*djo*	*fhj*	*ijk*
adn	*bco*	*bhn*	*cim*	*ehi*	*fmo*	*ino*
aef	*bdi*	*cdl*	*dem*	*eko*	*gjl*	*jmn*
ago	*bej*	*ceg*	*dfk*	*eln*	*gkn*	*klm*

3. $(1^{30} \times^{32})$	*a*	$(\times)$	*f*	$(1^2 \times^2)$	*k*	$(1^3 \times)$
	b	$(\times^2)$	*g*	.	*l*	.
5	*c*	$(\times^3)$	*h*	.	*m*	$(1^3 \times^3)$
	d	$(1^2 \times)$	*i*	$(1^2 \times^3)$	*n*	.
1	*e*	.	*j*	.	*o*	$(1^4 \times^4)$

abo	*ahl*	*bfh*	*cfl*	*dhj*	*fgi*	*hik*
acn	*aij*	*bjm*	*cho*	*dno*	*fjk*	*hmn*
adf	*bcg*	*bkn*	*cim*	*efn*	*fmo*	*iln*
aek	*bdi*	*cdk*	*dem*	*egh*	*gjn*	*jlo*
agm	*bel*	*cej*	*dgl*	*eio*	*gko*	*klm*

4. $(1^{30} \times^{32})$	*a*	$(\times^3)$	*f*	(1^2)	*k*	$(1^2 \times^4)$
	b	$(1 \times)$	*g*	$(1^2 \times)$	*l*	$(1^3 \times)$
5	*c*	.	*h*	.	*m*	$(1^3 \times^2)$
	d	$(1 \times^3)$	*i*	$(1^2 \times^2)$	*n*	$(1^3 \times^3)$
1	*e*	.	*j*	$(1^2 \times^4)$	*o*	$(1^5 \times^3)$

abo	*ain*	*bgm*	*cfj*	*dkm*	*eio*	*hjn*
ach	*ajl*	*bhl*	*cgl*	*dln*	*fgi*	*ijm*
ade	*bci*	*bkn*	*cmn*	*efl*	*fhk*	*ikl*
afm	*bdf*	*cdo*	*dgj*	*egn*	*fno*	*jko*
agk	*bej*	*cek*	*dhi*	*ehm*	*gho*	*lmo*

5. $(1^{30} \times^{28})$	*a*	$(\times^4)$	*f*	$(1^2 \times)$	*k*	$(1^2 \times^3)$
	b	(1)	*g*	.	*l*	$(1^2 \times^4)$
5	*c*	$(1 \times)$	*h*	.	*m*	$(1^3 \times)$
	d	$(1 \times^3)$	*i*	$(1^2 \times^2)$	*n*	.
1	*e*	(1^2)	*j*	.	*o*	$(1^5 \times^4)$

abl	*ain*	*bgk*	*cgl*	*djn*	*fhk*	*gjm*
acf	*akm*	*bhj*	*chn*	*dlm*	*fim*	*hil*
ado	*bco*	*bmn*	*cij*	*egn*	*fjo*	*hmo*
aej	*bdi*	*cdk*	*deh*	*eik*	*fln*	*jkl*
agh	*bef*	*cem*	*dfg*	*elo*	*gio*	*kno*

6. $(3^2 1^{24} \times^{22})$	*a*	$(\times^2)$	*f*	(1^2)	*k*	$(1^2 \times^2)$
	b	(1)	*g*	.	*l*	$(1^3 \times)$
5	*c*	$(1 \times^2)$	*h*	$(1^2 \times)$	*m*	$(1^3 \times^2)$
	d	$(1 \times^3)$	*i*	$(1^2 \times^2)$	*n*	$(3 \times^4)$
1	*e*	(1^2)	*j*	.	*o*	$(3\,1 \times)$

abm	*akl*	*bhi*	*cgk*	*djm*	*emo*	*glo*
aci	*ano*	*bjo*	*cho*	*dkn*	*fgm*	*hjk*
ade	*bcf*	*bln*	*cmn*	*efl*	*fin*	*hlm*
afj	*bdg*	*cdl*	*dfh*	*egi*	*fko*	*ijl*
agh	*bek*	*cej*	*dio*	*ehn*	*gjn*	*ikm*

7.

$(1^{36}\times^{18})$	*a*	(1^{2})	*f*	(1^{2})	*k*	$(1^{3}\times^{3})$
	b	.	*g*	.	*l*	.
6	*c*	.	*h*	.	*m*	.
	d	.	*i*	.	*n*	.
36	*e*	.	*j*	$(1^{3}\times^{3})$	*o*	.

abc	*ain*	*bgo*	*cgk*	*dhk*	*ehi*	*gjn*
ado	*ajm*	*bhl*	*chn*	*dim*	*elo*	*hjo*
aek	*bdj*	*bkm*	*cij*	*dln*	*fhm*	*iko*
afl	*ben*	*cde*	*clm*	*cfj*	*fkn*	*jkl*
agh	*bfi*	*cfo*	*dfg*	*egm*	*gil*	*mno*

(*abc*) (*dgf*) (*ehi*) (*jkl*); (*bhcg*) (*dfie*) (*jm*) (*koln*)

8.

$(1^{36}\times^{22})$	*a*	$(\times^{2})$	*f*	$(1^{2}\times^{2})$	*k*	$(1^{3}\times)$
	b	(1)	*g*	$(1^{2}\times^{3})$	*l*	$(1^{3}\times^{2})$
6	*c*	$(1\times)$	*h*	$(1^{2}\times^{5})$	*m*	.
	d	(1^{2})	*i*	(1^{3})	*n*	$(1^{4}\times)$
1	*e*	$(1^{2}\times)$	*j*	.	*o*	$(1^{5}\times^{2})$

abl	*agj*	*bgi*	*cfk*	*dij*	*fhi*	*hko*
acm	*ain*	*bhj*	*cgh*	*dlo*	*fjm*	*hlm*
adh	*bco*	*bmn*	*cjl*	*egl*	*fln*	*ikl*
aek	*bdk*	*cdn*	*dem*	*ehn*	*gkm*	*imo*
afo	*bef*	*cei*	*dfg*	*ejo*	*gno*	*kmn*

9.

$(1^{36}\times^{26})$	*a*	$(1\times^{3})$	*f*	$(1^{2}\times)$	*k*	$(1^{3}\times)$
	b	.	*g*	$(1^{2}\times^{2})$	*l*	.
6	*c*	$(1^{2}\times)$	*h*	.	*m*	.
	d	.	*i*	.	*n*	.
4	*e*	.	*j*	.	*o*	$(1^{6}\times^{4})$

abo	*ahj*	*bfn*	*cfj*	*djk*	*fhm*	*hil*
ack	*aln*	*bgi*	*cin*	*dmn*	*fkl*	*hno*
adi	*bch*	*bkm*	*clm*	*efi*	*ghk*	*ijm*
aem	*bdl*	*cdg*	*deh*	*egl*	*gjn*	*iko*
afg	*bej*	*ceo*	*dfo*	*ekn*	*gmo*	*jlo*

(*ab*) (*cdef*) (*ghij*) (*klmn*)

10

$(1^{36}\times^{28})$	*a*	$(\times^{4})$	*f*	$(1^{2}\times^{2})$	*k*	$(1^{3}\times^{2})$
	b	$(1\times^{2})$	*g*	.	*l*	$(1^{3}\times^{3})$
6	*c*	.	*h*	$(1^{2}\times^{3})$	*m*	$(1^{4}\times)$
	d	$(1^{2}\times)$	*i*	(1^{3})	*n*	.
1	*e*	$(1^{2}\times^{2})$	*j*	$(1^{3}\times)$	*o*	$(1^{4}\times^{2})$

abj	*agm*	*bfh*	*chk*	*djo*	*fgi*	*gkn*
aco	*ahl*	*bkm*	*cil*	*dkl*	*fjk*	*hin*
adi	*bcg*	*blo*	*cjn*	*efl*	*fmo*	*ijm*
aek	*bdn*	*cdf*	*deg*	*ehj*	*gho*	*iko*
afn	*bei*	*cem*	*dhm*	*eno*	*gjl*	*lmn*

11

$(1^{36}\times^{42})$	*a*	$(1\times^{2})$	*f*	$(1^{2}\times^{3})$	*k*	$(1^{3}\times^{5})$
	b	.	*g*	.	*l*	$(1^{4}\times^{3})$
6	*c*	.	*h*	.	*m*	.
	d	.	*i*	.	*n*	.
4	*e*	(1^{2})	*j*	$(1^{3}\times^{5})$	*o*	.

abm	*ahn*	*bhj*	*cgh*	*dhi*	*ein*	*gno*
ace	*ajo*	*bio*	*cik*	*dkn*	*ejk*	*hlo*
adl	*bcn*	*bkl*	*cjm*	*efo*	*fhk*	*ilm*
afi	*bde*	*cdo*	*dfj*	*egl*	*fmn*	*jln*
agk	*bfg*	*cfl*	*dgm*	*ehm*	*gij*	*kmo*

(*abcd*) (*fghi*) (*jk*) (*lmno*)

12.

$(3^{2}\,1^{30}\times^{24})$	*a*	$(1\times)$	*f*	$(1^{2}\times)$	*k*	$(1^{3}\times^{2})$
	b	$(1\times^{2})$	*g*	.	*l*	$(1^{3}\times^{3})$
6	*c*	.	*h*	$(1^{2}\times^{2})$	*m*	$(1^{4}\times)$
	d	(1^{2})	*i*	$(1^{2}\times^{5})$	*n*	$(3\times)$
1	*e*	$(1^{2}\times)$	*j*	(1^{3})	*o*	$(3\,1^{2}\times^{2})$

abj	*aio*	*bfk*	*cgl*	*dhn*	*fgn*	*hlo*
acf	*amn*	*bgo*	*cik*	*dko*	*fhm*	*ijn*
adg	*bcm*	*bhi*	*cno*	*efo*	*fjl*	*ilm*
ael	*bdl*	*cdj*	*dem*	*egi*	*ghj*	*jmo*
ahk	*ben*	*ceh*	*dfi*	*ejk*	*gkm*	*kln*

13.

$(3^{4}\,1^{24}\times^{24})$	*a*	$(1^{2}\times)$	*f*	$(1^{2}\times)$	*k*	$(1^{2}\times)$
	b	.	*g*	.	*l*	.
6	*c*	.	*h*	.	*m*	$(3\times^{6})$
	d	.	*i*	.	*n*	.
12	*e*	.	*j*	.	*o*	(3^{2})

abd	*ahn*	*bho*	*chm*	*din*	*eko*	*ghj*
acl	*ajk*	*bim*	*cio*	*djo*	*eln*	*glm*
aei	*bce*	*bkl*	*cjn*	*dkm*	*fgi*	*hik*
afm	*bfj*	*cdf*	*deg*	*efh*	*fkn*	*ijl*
ago	*bgn*	*cgk*	*dhl*	*ejm*	*flo*	*mno*

(*abcdefghijkl*) (*mn*)

14.

$(1^{42}\times^{14})$	*a*	$(\times^{4})$	*f*	$(1^{2}\times)$	*k*	$(1^{4}\times)$
	b	.	*g*	.	*l*	.
7	*c*	(1^{2})	*h*	.	*m*	(1^{5})
	d	.	*i*	(1^{3})	*n*	.
3	*e*	.	*j*	$(1^{4}\times)$	*o*	.

abi	*agj*	*bfj*	*cgi*	*dhi*	*ejk*	*iko*
acm	*ahk*	*bgk*	*chj*	*djl*	*fgn*	*ilm*
adn	*bcn*	*bhl*	*ckl*	*efi*	*fhm*	*jmo*
aeo	*bdo*	*cde*	*dfk*	*egl*	*gho*	*kmn*
afl	*bem*	*cfo*	*dgm*	*ehn*	*ijn*	*lno*

(*cde*) (*fgh*) (*jkl*) (*mno*)

15.

$(1^{42}\times^{14})$	*a*	$(\times)$	*f*	$(1^{2}\times^{2})$	*k*	(1^{4})
	b	$(1^{2}\times)$	*g*	.	*l*	.
7	*c*	.	*h*	(1^{3})	*m*	$(1^{4}\times)$
	d	.	*i*	$(1^{3}\times)$	*n*	.
3	*e*	$(1^{2}\times^{2})$	*j*	(1^{4})	*o*	.

abk	*agn*	*bgo*	*cgk*	*dgh*	*ein*	*hkn*
acl	*ahi*	*bil*	*cij*	*dik*	*fgl*	*hlo*
adj	*bcd*	*bmn*	*cno*	*dmo*	*fio*	*jko*
aeo	*beh*	*cem*	*del*	*efk*	*gim*	*jln*
afm	*bfj*	*cfh*	*dfn*	*egj*	*hjm*	*klm*

(*bcd*) (*efg*) (*jkl*) (*mno*)

16.

$(1^{42}\times^{28})$	*a*	$(\times)$	*f*	$(1^{3}\times)$	*k*	$(1^{4}\times)$
	b	$(\times^{4})$	*g*	$(1^{3}\times^{2})$	*l*	.
7	*c*	(1^{2})	*h*	$(1^{3}\times^{3})$	*m*	$(1^{4}\times^{2})$
	d	$(1^{2}\times)$	*i*	.	*n*	.
1	*e*	$(1^{2}\times^{3})$	*j*	.	*o*	$(1^{5}\times)$

abn	*agm*	*bfg*	*cgk*	*dgi*	*fhk*	*hio*
aco	*ahl*	*bjo*	*chm*	*dhn*	*fjn*	*ijl*
adk	*bci*	*bkl*	*cln*	*egn*	*flm*	*imn*
aej	*bdm*	*cdj*	*del*	*eik*	*ghj*	*jkm*
afi	*beh*	*cef*	*dfo*	*emo*	*glo*	*kno*

17. $(1^{42} \times^{30})$

	a	$(1\ \times)$	*f*	$(1^2 \times^3)$	*k*	$(1^3 \times^3)$
	b	.	*g*	$(1^3 \times^2)$	*l*	.
7	*c*	.	*h*	.	*m*	$(1^5 \times)$
	d	$(1^2 \times^3)$	*i*	.	*n*	.
3	*e*	.	*j*	$(1^3 \times^3)$	*o*	.

abi	*agm*	*bfj*	*cfn*	*dhm*	*fgo*	*hik*
ach	*ajn*	*bhn*	*cio*	*dij*	*fhl*	*jkm*
ado	*bcg*	*bko*	*clm*	*egk*	*fim*	*jlo*
ael	*bdl*	*cdk*	*def*	*eho*	*ghj*	*kln*
afk	*bem*	*cej*	*dgn*	*ein*	*gil*	*mno*

(abc) (def) (ghi) (jkl) (mno)

18. $(3^2\ 1^{36} \times^{24})$

	a	$(1\ \times)$	*f*	$(1^2 \times^2)$	*k*	(1^4)
	b	.	*g*	$(1^2 \times^4)$	*l*	.
7	*c*	$(1 \times^2)$	*h*	$(1^3 \times)$	*m*	$(1^4 \times^2)$
	d	$(1^2 \times)$	*i*	$(1^3 \times^3)$	*n*	$(3\ 1 \times^3)$
1	*e*	$(1^2 \times^2)$	*j*	(1^4)	*o*	$(3\ 1^2 \times^2)$

abj	*agh*	*bfm*	*cfj*	*dhm*	*fgn*	*hin*
aci	*amo*	*bkl*	*cgk*	*djn*	*fho*	*hjk*
adk	*bch*	*bno*	*cln*	*ehl*	*fik*	*ilm*
aen	*bdi*	*cdo*	*def*	*eij*	*gio*	*jlo*
afl	*beg*	*cem*	*dgl*	*eko*	*gjm*	*kmn*

19. $(3^2\ 1^{36} \times^{28})$

	a	$(1 \times^4)$	*f*	$(1^2 \times^2)$	*k*	$(1^3 \times^2)$
	b	(1^2)	*g*	.	*l*	(1^4)
7	*c*	$(1^2 \times)$	*h*	$(1^2 \times^3)$	*m*	$(1^5 \times^3)$
	d	.	*i*	$(1^3 \times)$	*n*	$(3\ 1 \times^3)$
1	*e*	$(1^2 \times^2)$	*j*	$(1^3 \times^2)$	*o*	$(3\ 1^2 \times^2)$

abk	*agi*	*bfm*	*cfj*	*d*	*fgl*	*hil*
ach	*amn*	*bhn*	*cik*	*dko*	*fin*	*hjk*
adl	*bcl*	*bjo*	*cno*	*efk*	*gho*	*imo*
aej	*bdg*	*cdm*	*den*	*ehm*	*gjn*	*jlm*
afo	*bei*	*ceg*	*dfh*	*elo*	*gkm*	*kln*

20. $(3^2\ 1^{36} \times^{36})$

	a	$(\times^4)$	*f*	$(1^2 \times^3)$	*k*	$(1^3 \times^2)$
	b	$(1^2 \times)$	*g*	.	*l*	$(1^4 \times^3)$
7	*c*	.	*h*	$(1^3 \times)$	*m*	$(1^5 \times^3)$
	d	$(1^2 \times^2)$	*i*	.	*n*	$(3\ 1 \times^3)$
1	*e*	$(1^2 \times^3)$	*j*	$(1^3 \times^2)$	*o*	$(3\ 1^2 \times^4)$

abl	*ajk*	*bgk*	*chk*	*dik*	*ejl*	*gjm*
ace	*ano*	*bhm*	*cil*	*djn*	*eko*	*hjo*
adm	*bcd*	*bij*	*cmn*	*dlo*	*fgl*	*hln*
afi	*ben*	*cfj*	*deg*	*efm*	*fkn*	*imo*
agh	*bfo*	*cgo*	*dfh*	*ehi*	*gin*	*klm*

21. $(1^{48} \times^{14})$

	a	$(\times^2)$	*f*	$(1^3 \times)$	*k*	(1^4)
	b	(1^3)	*g*	.	*l*	.
8	*c*	.	*h*	.	*m*	.
	d	.	*i*	.	*n*	$(1^4 \times^2)$
4	*e*	.	*j*	(1^4)	*o*	$(1^4 \times^6)$

abd	*aik*	*bhm*	*cgh*	*dgn*	*ehn*	*gjk*
ace	*ano*	*bin*	*cij*	*dhi*	*emo*	*hkl*
afl	*bcl*	*bjo*	*cko*	*dlo*	*fho*	*ilm*
agm	*bek*	*cdm*	*dej*	*efi*	*fjm*	*jln*
ahj	*bfg*	*cfn*	*dfk*	*egl*	*gio*	*kmn*

(bcde) (fghi) (jklm)

22. $(3^2\ 1^{42} \times^{14})$

	a	$(\times)$	*f*	(1^3)	*k*	(1^5)
	b	$(1\ \times)$	*g*	$(1^3 \times^2)$	*l*	$(1^5 \times)$
8	*c*	$(1 \times^4)$	*h*	.	*m*	$(1^5 \times^2)$
	d	(1^2)	*i*	(1^5)	*n*	$(3 \times)$
1	*e*	.	*j*	.	*o*	$(3\ 1^2)$

abj	*ahk*	*bgo*	*cgj*	*dhl*	*fho*	*gln*
aci	*ano*	*bhi*	*chn*	*djn*	*fik*	*ijo*
adm	*bcl*	*bkn*	*cmo*	*egk*	*fjl*	*ilm*
ael	*bdf*	*cdk*	*deo*	*ehj*	*fmn*	*jkm*
afg	*bem*	*cef*	*dgi*	*ein*	*ghm*	*klo*

23. $(3^2\ 1^{42} \times^{16})$

	a	(1^2)	*f*	$(1^3 \times)$	*k*	(1^4)
	b	.	*g*	$(1^3 \times^2)$	*l*	$(1^4 \times^2)$
8	*c*	$(1^2 \times)$	*h*	.	*m*	.
	d	.	*i*	.	*n*	$(3 \times^2)$
2	*e*	$(1^3 \times)$	*j*	(1^4)	*o*	$(3\ 1^3)$

abo	*agm*	*bfj*	*cfm*	*dho*	*fil*	*hij*
acj	*aln*	*bhl*	*cgo*	*djm*	*fkn*	*hkm*
adi	*bci*	*bmn*	*ckl*	*efo*	*ghn*	*ino*
aek	*bdk*	*cdn*	*del*	*eim*	*gik*	*jko*
afh	*beg*	*ceh*	*dfg*	*ejn*	*gjl*	*lmo*

(ab) (cd) (ef) (gh) (jk) (lm)

24. $(3^2\ 1^{42} \times^{20})$

	a	$(1\ \times)$	*f*	$(1^3 \times)$	*k*	(1^4)
	b	.	*g*	.	*l*	(1^5)
8	*c*	$(1^2 \times^2)$	*h*	.	*m*	$(1^5 \times)$
	d	$(1^2 \times^3)$	*i*	$(1^3 \times^2)$	*n*	$(3\ 1^2 \times)$
1	*e*	(1^3)	*j*	.	*o*	$(3\ 1^2 \times^4)$

abm	*agj*	*bhi*	*chk*	*dim*	*eko*	*gio*
acl	*ahn*	*bjk*	*cjn*	*dln*	*fgi*	*hlo*
ado	*bcg*	*bno*	*cmo*	*efh*	*fjo*	*ijl*
aei	*bdf*	*cde*	*dgk*	*egn*	*fmn*	*ikn*
afk	*bel*	*cfi*	*dhj*	*ejm*	*ghm*	*klm*

25. $(3^2\ 1^{42} \times^{24})$

	a	$(\times^2)$	*f*	$(1^3 \times^2)$	*k*	$(1^4 \times)$
	b	$(1\ \times)$	*g*	$(1^3 \times^3)$	*l*	$(1^4 \times^3)$
8	*c*	$(1^2 \times)$	*h*	$(1^3 \times^5)$	*m*	$(1^5 \times)$
	d	.	*i*	(1^4)	*n*	$(3\ 1 \times)$
1	*e*	$(1^3 \times^2)$	*j*	.	*o*	$(3\ 1^3 \times)$

abf	*ahj*	*bil*	*cfo*	*dkl*	*fhl*	*gln*
acl	*ano*	*bjo*	*cgj*	*dmo*	*fjk*	*hkm*
adi	*bcm*	*bkn*	*chi*	*efi*	*fmn*	*ijn*
aem	*bdh*	*cdn*	*dej*	*ehn*	*gho*	*iko*
agk	*beg*	*cek*	*dfg*	*elo*	*gim*	*jlm*

26. $(3^2\ 1^{42} \times^{40})$

	a	$(1 \times^2)$	*f*	$(1^3 \times^2)$	*k*	$(1^4 \times^3)$
	b	.	*g*	.	*l*	$(1^5 \times^2)$
8	*c*	$(1 \times^6)$	*h*	.	*m*	.
	d	$(1^2 \times^3)$	*i*	.	*n*	$(3\ 1^2 \times^4)$
2	*e*	.	*j*	$(1^4 \times^3)$	*o*	$(3\ 1^3 \times^2)$

abo	*ahk*	*bfh*	*cfl*	*dhm*	*fgo*	*hio*
acj	*amn*	*bij*	*cgm*	*djo*	*fin*	*hjl*
adl	*bck*	*bln*	*cno*	*egj*	*fjm*	*ikm*
aef	*bdg*	*cdi*	*den*	*eil*	*ghn*	*jkn*
agi	*bem*	*ceh*	*dfk*	*eko*	*gkl*	*lmo*

(ab) (de) (fg) (hi) (jk) (lm)

27. $(1^{54} \times^{24})$

	a	$(1\times)$	*f*	$(1^3\times)$	*k*	$(1^4\times^3)$
	b	$(1^2\times)$	*g*	$(1^3\times^2)$	*l*	$(1^5\times)$
9	*c*	(1^3)	*h*	$(1^3\times^3)$	*m*	$(1^5\times^3)$
	d	$(1^3\times)$	*i*	$(1^4\times)$	*n*	$(1^5\times^4)$
1	*e*	.	*j*	$(1^4\times^2)$	*o*	(1^6)

abi	*agh*	*bhj*	*cgk*	*dho*	*ekl*	*hik*
acl	*ajn*	*bkm*	*cij*	*djk*	*fgn*	*hmn*
adm	*bcf*	*bln*	*cmo*	*efm*	*fhl*	*ilo*
aeo	*bde*	*cdn*	*dfi*	*egj*	*fjo*	*jlm*
afk	*bgo*	*ceh*	*dgl*	*ein*	*gim*	*kno*

28. $(3^2\, 1^{48} \times^{24})$

	a	$(\times^3)$	*f*	$(1^3\times^2)$	*k*	(1^5)
	b	$(1^2\times)$	*g*	.	*l*	$(1^5\times)$
9	*c*	(1^3)	*h*	$(1^3\times^4)$	*m*	$(1^5\times^2)$
	d	$(1^3\times)$	*i*	$(1^4\times)$	*n*	$(3\,1^2)$
1	*e*	$(1^3\times^2)$	*j*	$(1^4\times^2)$	*o*	$(3\,1^3\times^3)$

abg	*ahl*	*bfm*	*cfn*	*dio*	*fgh*	*hik*
acj	*ano*	*bij*	*cgo*	*djk*	*fjl*	*hjo*
adm	*bck*	*blo*	*cim*	*egi*	*fko*	*hmn*
aek	*bdn*	*cdh*	*def*	*ejn*	*gjm*	*iln*
afi	*beh*	*cel*	*dgl*	*emo*	*gkn*	*klm*

29. $(3^2\, 1^{48} \times^{24})$

	a	$(1\times)$	*f*	$(1^3\times^5)$	*k*	$(1^4\times)$
	b	$(1^2\times^2)$	*g*	(1^4)	*l*	$(1^4\times^3)$
9	*c*	.	*h*	$(1^4\times)$	*m*	$(1^5\times)$
	d	$(1^3\times)$	*i*	.	*n*	$(3\,1^2\times^3)$
1	*e*	$(1^3\times^2)$	*j*	.	*o*	$(3\,1^3)$

abg	*aim*	*bfn*	*cgk*	*din*	*elo*	*gjn*
acj	*akl*	*bhk*	*chl*	*djk*	*fgo*	*hio*
ado	*bci*	*bjo*	*cno*	*egh*	*fik*	*hjm*
aef	*bdl*	*cde*	*dfh*	*eij*	*fjl*	*kmo*
ahn	*bem*	*cfm*	*dgm*	*ekn*	*gil*	*lmn*

30. $(3^2\, 1^{48} \times^{30})$

	a	$(\times^3)$	*f*	(1^4)	*k*	$(1^1\times^3)$
	b	$(1\times^5)$	*g*	$(1^4\times)$	*l*	$(1^5\times)$
9	*c*	$(1^2\times)$	*h*	$(1^4\times^2)$	*m*	$(1^5\times^2)$
	d	(1^3)	*i*	.	*n*	$(3\,1^2\times^3)$
1	*e*	$(1^3\times)$	*j*	$(1^4\times^3)$	*o*	$(3\,1^3\times^3)$

abd	*aik*	*bhj*	*cgn*	*dhk*	*eim*	*gkl*
acl	*ano*	*bin*	*chi*	*djo*	*fgj*	*hlm*
aej	*bck*	*blo*	*cjm*	*dmn*	*fio*	*ijl*
afh	*beg*	*cdf*	*del*	*efk*	*fln*	*jkn*
agm	*bfm*	*ceo*	*dgi*	*ehn*	*gho*	*kmo*

31. $(3^4\, 1^{42} \times^{26})$

	a	(1)	*f*	$(1^3\times^3)$	*k*	$(1^4\times)$
	b	$(1^2\times^2)$	*g*	.	*l*	$(3\,1^2\times^3)$
9	*c*	$(1^2\times^3)$	*h*	(1^4)	*m*	$(3\,1^2\times^4)$
	d	$(1^2\times^4)$	*i*	.	*n*	$(3\,1^3\times)$
1	*e*	(1^3)	*j*	.	*o*	$(3\,1^3\times^2)$

abl	*aik*	*bfm*	*cij*	*dhm*	*fgk*	*glm*
acf	*amo*	*bgi*	*ckm*	*dkl*	*fio*	*hil*
adg	*bch*	*bno*	*cln*	*efh*	*fjl*	*hko*
aen	*bdj*	*cdo*	*dei*	*ejm*	*ghn*	*imn*
ahj	*bek*	*ceg*	*dfn*	*elo*	*gjo*	*jkn*

32. $(3^4\, 1^{42} \times^{30})$

	a	$(\times^3)$	*f*	$(1^3\times^2)$	*k*	$(1^4\times^2)$
	b	$(1^2\times^3)$	*g*	$(1^3\times^3)$	*l*	$(3\,1^2\times)$
9	*c*	$(1^2\times^4)$	*h*	(1^4)	*m*	.
	d	$(1^3\times)$	*i*	$(1^4\times)$	*n*	$(3\,1^3\times^2)$
1	*e*	.	*j*	.	*o*	$(3\,1^3\times^5)$

abf	*alo*	*bgo*	*cfi*	*dhi*	*fhn*	*gjl*
ach	*amn*	*bhl*	*cgm*	*dkm*	*fko*	*hjo*
adg	*bck*	*bjm*	*cno*	*efg*	*flm*	*ikl*
aek	*bdn*	*cdl*	*deo*	*ehm*	*ghk*	*imo*
aij	*bei*	*cej*	*dfj*	*eln*	*gin*	*jkn*

33. $(1^{60} \times^{10})$

	a	$(1^3\times^2)$	*f*	(1^4)	*k*	(1^5)
	b	.	*g*	.	*l*	.
10	*c*	.	*h*	.	*m*	.
	d	.	*i*	.	*n*	.
5	*e*	.	*j*	.	*o*	.

abl	*ahj*	*bfi*	*cgj*	*dim*	*fgm*	*hio*
aci	*amn*	*bgk*	*chl*	*dkl*	*fjl*	*hkm*
adg	*bcm*	*bno*	*cko*	*egi*	*fkn*	*ijk*
aek	*bdj*	*cdn*	*deo*	*ejn*	*ghn*	*iln*
afo	*beh*	*cef*	*dfh*	*elm*	*glo*	*jmo*

(*abcde*) (*fghij*) (*klmno*)

34. $(1^{60} \times^{20})$

	a	$(\times^4)$	*f*	$(1^3\times^2)$	*k*	(1^5)
	b	$(1^3\times)$	*g*	.	*l*	$(1^6\times)$
10	*c*	.	*h*	.	*m*	.
	d	.	*i*	(1^5)	*n*	.
3	*e*	.	*j*	.	*o*	$(1^6\times^3)$

abk	*agm*	*bfj*	*cfo*	*dgo*	*fgh*	*imo*
aci	*ahn*	*bgl*	*cgk*	*dhi*	*fik*	*jlm*
adj	*bcn*	*bho*	*chm*	*efm*	*gij*	*jno*
aeo	*bdm*	*cdl*	*dek*	*egn*	*hjk*	*klo*
afl	*bei*	*cej*	*dfn*	*ehl*	*iln*	*kmn*

(*bcd*) (*fgh*) (*ijk*) (*lmn*)

35. $(3^2\, 1^{54} \times^{18})$

	a	$(1\times^2)$	*f*	$(1^3\times)$	*k*	$(1^5\times)$
	b	$(1^2\times)$	*g*	$(1^3\times^2)$	*l*	.
10	*c*	.	*h*	(1^5)	*m*	$(1^7\times)$
	d	$(1^2\times^2)$	*i*	.	*n*	$(3\,1^2\times^6)$
1	*e*	(1^3)	*j*	.	*o*	$(3\,1^4)$

abm	*ago*	*bfi*	*cfo*	*dij*	*fgh*	*hjm*
aci	*akn*	*bgj*	*cgl*	*dno*	*fln*	*hkl*
adh	*bck*	*bhn*	*cjn*	*efm*	*gik*	*ilm*
ael	*bdl*	*cdm*	*deg*	*ein*	*gmn*	*jlo*
afj	*beo*	*ceh*	*dfk*	*ejk*	*hio*	*kmo*

36. $(3^4\, 1^{48} \times^{8})$

	a	(1^3)	*f*	(1^4)	*k*	$(1^4\times)$
	b	.	*g*	.	*l*	.
10	*c*	.	*h*	.	*m*	$(3\,1^2\times^2)$
	d	.	*i*	$(1^4\times)$	*n*	.
4	*e*	(1^4)	*j*	.	*o*	(3^2)

abi	*agm*	*bgk*	*cfg*	*dgh*	*fho*	*him*
aco	*akn*	*bhn*	*chl*	*djm*	*fil*	*hjk*
adl	*bcj*	*blm*	*cin*	*ego*	*fkm*	*iko*
aeh	*bdo*	*cdk*	*dei*	*ejn*	*gij*	*jlo*
afj	*bef*	*cem*	*dfn*	*ekl*	*gln*	*mno*

(*abcd*) (*efgh*) (*ijkl*) (*mn*)

37. $(3^4\ 1^{48} \times^{18})$

	a	$(\times^2)$	*f*	$(1^3 \times^2)$	*k*	$(1^6 \times)$
	b	(1^2)	*g*	$(1^4 \times)$	*l*	$(3\ 1)$
10	*c*	$(1^2 \times^2)$	*h*	(1^5)	*m*	$(3\ 1^2 \times)$
	d	$(1^2 \times^5)$	*i*	.	*n*	$(3\ 1^3 \times^2)$
1	*e*	$(1^3 \times)$	*j*	(1^6)	*o*	$(3\ 1^4 \times)$

abk	*aln*	*bfl*	*cfi*	*djl*	*fgn*	*hio*
acg	*amo*	*bgm*	*chm*	*dmn*	*fkm*	*hjk*
adh	*bcj*	*bhn*	*clo*	*efh*	*ghl*	*ijn*
aei	*bdi*	*cdk*	*deg*	*ejm*	*gik*	*ilm*
afj	*beo*	*cen*	*dfo*	*ekl*	*gjo*	*kno*

38. $(3^6\ 1^{42} \times^{12})$

	a	(1^3)	*f*	(1^3)	*k*	$(3\ 1^2 \times^2)$
	b	.	*g*	$(1^3 \times^2)$	*l*	.
10	*c*	.	*h*	.	*m*	$(3\ 1^3)$
	d	.	*i*	.	*n*	.
6	*e*	.	*j*	$(3\ 1^2 \times^2)$	*o*	.

abi	*agl*	*bfm*	*cfo*	*dgk*	*fij*	*him*
ach	*ajn*	*bhj*	*cik*	*djo*	*fln*	*hkn*
adm	*bcg*	*bko*	*clm*	*efg*	*gho*	*ilo*
aeo	*bdl*	*cdn*	*dei*	*ehl*	*gin*	*jkl*
afk	*ben*	*cej*	*dfh*	*ekm*	*gjm*	*mno*

(ae) (bd) (cf) (gh) (jk) (mn); (abc) (def) (ghi) (jkl) (mno)

39. $(3^4\ 1^{54} \times^{26})$

	a	$(\times^2)$	*f*	$(1^4 \times^2)$	*k*	$(1^7 \times)$
	b	$(1^2 \times^2)$	*g*	(1^5)	*l*	$(3\ 1^2 \times)$
11	*c*	$(1^2 \times^6)$	*h*	$(1^5 \times)$	*m*	$(3\ 1^2 \times^3)$
	d	$(1^3 \times)$	*i*	$(1^5 \times^2)$	*n*	$(3\ 1^3 \times)$
1	*e*	$(1^3 \times^2)$	*j*	(1^6)	*o*	$(3\ 1^5 \times^2)$

abk	*aln*	*bfg*	*cfn*	*dgl*	*fim*	*hil*
acg	*amo*	*bhn*	*chm*	*dmn*	*fjl*	*hko*
adi	*bci*	*bjm*	*clo*	*efo*	*ghj*	*ijk*
aej	*bdo*	*cdj*	*deh*	*egm*	*gio*	*jno*
afh	*bel*	*cek*	*dfk*	*ein*	*gkn*	*klm*

40. $(3^2\ 1^{66} \times^{16})$

	a	$(1^2 \times^2)$	*f*	$(1^4 \times)$	*k*	(1^6)
	b	$(1^3 \times)$	*g*	$(1^4 \times^2)$	*l*	.
12	*c*	$(1^4 \times)$	*h*	$(1^4 \times^3)$	*m*	$(1^7 \times)$
	d	.	*i*	(1^5)	*n*	$(3\ 1^4 \times)$
1	*e*	.	*j*	$(1^5 \times)$	*o*	.

abk	*ahj*	*bfm*	*cgk*	*dgm*	*ekl*	*hlm*
acd	*alo*	*bgi*	*cjm*	*dhn*	*fgo*	*ijo*
aem	*bcl*	*bho*	*cno*	*dil*	*fln*	*imn*
afi	*bdj*	*cei*	*deo*	*efj*	*gjl*	*jkn*
agn	*ben*	*cfh*	*dfk*	*egh*	*hik*	*kmo*

41. $(3^4\ 1^{60} \times^{20})$

	a	$(1^3 \times)$	*f*	$(1^4 \times^2)$	*k*	$(1^6 \times^2)$
	b	.	*g*	$(1^4 \times^3)$	*l*	$(3\ 1)$
12	*c*	$(1^3 \times^3)$	*h*	$(1^5 \times^2)$	*m*	$(3\ 1^3)$
	d	$(1^4 \times)$	*i*	(1^6)	*n*	$(3\ 1^3 \times)$
1	*e*	.	*j*	$(1^6 \times)$	*o*	$(3\ 1^5 \times^2)$

abh	*ajk*	*bfj*	*cgj*	*dhl*	*ejl*	*hio*
acd	*alo*	*bgi*	*cil*	*dij*	*fgl*	*hjm*
aen	*bco*	*bln*	*ckn*	*dmo*	*fhk*	*imn*
afi	*bdk*	*ceh*	*deg*	*efo*	*ghn*	*jno*
agm	*bem*	*cfm*	*dfn*	*eik*	*gko*	*klm*

42. $(3^4\ 1^{60} \times^{20})$

	a	$(1 \times^2)$	*f*	$(1^4 \times^2)$	*k*	(1^7)
	b	$(1^2 \times)$	*g*	$(1^5 \times^2)$	*l*	$(3\ 1 \times^4)$
12	*c*	$(1^2 \times^2)$	*h*	(1^6)	*m*	$(3\ 1^3 \times^2)$
	d	(1^4)	*i*	$(1^6 \times)$	*n*	.
1	*e*	$(1^4 \times)$	*j*	(1^7)	*o*	$(3\ 1^5 \times)$

abh	*agn*	*bfk*	*cfi*	*dkm*	*fgm*	*hik*
ack	*amo*	*bil*	*chm*	*dln*	*fhn*	*hjo*
adi	*bcj*	*bmn*	*clo*	*efo*	*ghl*	*ijn*
aej	*bdo*	*cdg*	*deh*	*eim*	*gio*	*jlm*
afl	*beg*	*cen*	*dfj*	*ekl*	*gjk*	*kno*

43. $(3^8\ 1^{48} \times^{16})$

	a	$(1^2 \times)$	*f*	(1^7)	*k*	$(3\ 1^2 \times)$
	b	$(1^2 \times^2)$	*g*	(1^8)	*l*	$(3\ 1^3)$
12	*c*	$(1^3 \times^2)$	*h*	$(3 \times)$	*m*	.
	d	$(1^4 \times^6)$	*i*	$(3\ 1)$	*n*	$(3\ 1^3 \times)$
1	*e*	(1^5)	*j*	$(3\ 1 \times)$	*o*	$(3\ 1^4 \times)$

abf	*ajm*	*bhi*	*chn*	*dgk*	*ein*	*gmo*
ace	*akl*	*bjo*	*cil*	*dim*	*fgn*	*hjl*
adn	*bcg*	*bmn*	*ckm*	*efo*	*fik*	*hko*
agh	*bdl*	*cdo*	*dej*	*egl*	*flm*	*jkn*
aio	*bek*	*cfj*	*dfh*	*ehm*	*gij*	*lno*

44. $(3^2\ 1^{72} \times^{18})$

	a	$(1^2 \times)$	*f*	(1^5)	*k*	$(1^6 \times^2)$
	b	$(1^2 \times^2)$	*g*	$(1^5 \times)$	*l*	$(1^7 \times)$
13	*c*	$(1^4 \times)$	*h*	$(1^5 \times^2)$	*m*	(1^9)
	d	.	*i*	.	*n*	$(3\ 1^3 \times)$
1	*e*	$(1^4 \times^3)$	*j*	$(1^6 \times)$	*o*	$(3\ 1^5)$

abm	*ahl*	*bfn*	*cgm*	*djl*	*fgi*	*hik*
acf	*ako*	*bhj*	*cil*	*dkn*	*fjo*	*hmn*
adi	*bck*	*bio*	*cno*	*efh*	*fkl*	*ijn*
aen	*bdg*	*cdh*	*deo*	*egk*	*gho*	*jkm*
agj	*bel*	*cej*	*dfm*	*eim*	*gln*	*lmo*

45. $(3^2\ 1^{72} \times^{32})$

	a	$(\times^4)$	*f*	$(1^5 \times)$	*k*	$(1^6 \times^2)$
	b	$(1^4 \times^3)$	*g*	.	*l*	.
13	*c*	.	*h*	$(1^6 \times)$	*m*	.
	d	.	*i*	.	*n*	$(3\ 1^3 \times^4)$
3	*e*	$(1^5 \times)$	*j*	.	*o*	$(3\ 1^6 \times^3)$

abj	*agl*	*bgi*	*cgo*	*dgm*	*eln*	*hjk*
ach	*ano*	*bhn*	*cin*	*djn*	*fgj*	*hmo*
adi	*bcd*	*blm*	*ckm*	*dkl*	*fmn*	*ijm*
aem	*bek*	*cej*	*deo*	*efi*	*gkn*	*iko*
afk	*bfo*	*cfl*	*dfh*	*egh*	*hil*	*jlo*

(bcd) (efg) (hij) (klm)

46. $(3^8\ 1^{54} \times^{14})$

	a	$(1^2 \times)$	*f*	(1^7)	*k*	$(3\ 1^2)$
	b	$(1^2 \times^2)$	*g*	(1^8)	*l*	$(3\ 1^2 \times)$
13	*c*	$(1^3 \times)$	*h*	$(3\ 1)$	*m*	$(3\ 1^4)$
	d	$(1^4 \times^6)$	*i*	$(3\ 1^2)$	*n*	$(3\ 1^4 \times^2)$
1	*e*	(1^6)	*j*	.	*o*	$(3\ 1^5 \times)$

abe	*aik*	*bhm*	*cil*	*dgh*	*emn*	*gkl*
acf	*alm*	*bjn*	*cjm*	*dkn*	*fgn*	*hij*
adn	*bcg*	*bko*	*ckn*	*efl*	*fhk*	*hln*
agj	*bdl*	*cdo*	*dei*	*ego*	*fmo*	*ino*
aho	*bfi*	*ceh*	*dfj*	*ejk*	*gim*	*jlo*

47. $(3^{10}\,1^{48}\times^{20})$

	a	$(1^3\times^2)$	*f*	(1^7)	*k*	$(3\,1^3\times^2)$
	b	.	*g*	$(3\times)$	*l*	.
13	*c*	.	*h*	$(3\,1)$	*m*	.
	d	(1^7)	*i*	.	*n*	$(3\,1^3\times^7)$
3	*e*	.	*j*	.	*o*	$(3^2\,1^3)$

abf	*ajm*	*bhk*	*cho*	*dgi*	*ehn*	*gno*
ace	*akn*	*bjo*	*cil*	*djn*	*emo*	*hij*
adh	*bcd*	*bln*	*cmn*	*dlo*	*fgh*	*hlm*
agl	*bei*	*cfj*	*dek*	*efl*	*fin*	*ikm*
aio	*bgm*	*cgk*	*dfm*	*egj*	*fko*	*jkl*

(abc) (def) (hij) (klm)

48. $(1^{81}\times^{14})$

	a	$(\times^7)$	*f*	$(1^3\times)$	*k*	(1^9)
	b	$(1^3\times)$	*g*	.	*l*	.
14	*c*	.	*h*	.	*m*	.
	d	.	*i*	(1^9)	*n*	.
21	*e*	.	*j*	.	*o*	.

abn	*agl*	*bfi*	*cfm*	*dgn*	*fgo*	*imn*
aco	*ahm*	*bgm*	*cgj*	*dhk*	*fhl*	*jkm*
adi	*bck*	*bhj*	*chn*	*efn*	*ghi*	*jno*
aej	*bdo*	*cdl*	*dem*	*egk*	*ijl*	*kln*
afk	*bel*	*cei*	*dfj*	*eho*	*iko*	*lmo*

(bcdefgh) (ijklmno); (bcg) (ehf) (jmk) (lno)

49. $(3^6\,1^{66}\times^{10})$

	a	(1^3)	*f*	$(1^5\times)$	*k*	$(3\,1^3)$
	b	$(1^4\times)$	*g*	.	*l*	$(3\,1^4)$
14	*c*	.	*h*	$(1^5\times^2)$	*m*	.
	d	(1^5)	*i*	(1^6)	*n*	$(3\,1^4\times)$
1	*e*	$(1^5\times)$	*j*	$(1^6\times)$	*o*	$(3^2\,1^3\times)$

abi	*ajn*	*bfk*	*cfm*	*dlm*	*fgi*	*gkn*
aco	*akm*	*bhm*	*chl*	*dno*	*fhn*	*hij*
adf	*bcn*	*blo*	*cik*	*efo*	*fjl*	*iln*
aeh	*bdj*	*cdg*	*dei*	*ekl*	*gho*	*imo*
agl	*beg*	*cej*	*dhk*	*emn*	*gjm*	*jko*

50. $(3^6\,1^{66}\times^{12})$

	a	$(1^3\times)$	*f*	(1^5)	*k*	$(3\,1^4\times)$
	b	.	*g*	.	*l*	.
14	*c*	$(1^3\times^2)$	*h*	$(1^6\times)$	*m*	$(3\,1^5)$
	d	$(1^4\times)$	*i*	.	*n*	.
2	*e*	.	*j*	$(3\,1^3\times^2)$	*o*	$(3\,1^6)$

abo	*ahl*	*bfm*	*cfi*	*dho*	*fgj*	*hij*
ack	*ajm*	*bik*	*cgh*	*dkn*	*fhn*	*hkm*
adi	*bcl*	*bjn*	*cjo*	*egk*	*fko*	*iln*
aef	*bdg*	*cdm*	*dej*	*eio*	*gim*	*jkl*
agn	*beh*	*cen*	*dfl*	*elm*	*glo*	*mno*

(ab) (de) (fg) (hi) (kl) (mn)

51. $(3^6\,1^{72}\times^{14})$

	a	$(1^3\times^2)$	*f*	$(1^5\times)$	*k*	$(3\,1^4)$
	b	$(1^4\times)$	*g*	.	*l*	$(3\,1^5\times)$
15	*c*	(1^5)	*h*	(1^6)	*m*	.
	d	$(1^5\times)$	*i*	$(1^6\times)$	*n*	$(3\,1^5\times^2)$
1	*e*	.	*j*	$(1^6\times^2)$	*o*	$(3^2\,1^3)$

abj	*ahi*	*bfm*	*cgo*	*dij*	*eno*	*hkl*
ace	*amn*	*bgh*	*cim*	*dlo*	*fgj*	*hmo*
adk	*bck*	*bio*	*cjn*	*egi*	*fhn*	*iln*
afo	*bdn*	*cdh*	*def*	*ehj*	*fik*	*jko*
agl	*bel*	*cfl*	*dgm*	*ekm*	*gkn*	*jlm*

52. $(3^6\,1^{84}\times^8)$

	a	$(1^3\times)$	*f*	$(1\times)$	*k*	$(3\,1^7)$
	b	.	*g*	.	*l*	.
17	*c*	.	*h*	.	*m*	.
	d	.	*i*	.	*n*	.
12	*e*	(1^6)	*j*	$(3\,1^7)$	*o*	.

abl	*aho*	*bfm*	*cgn*	*dgl*	*fgo*	*hik*
ack	*aim*	*bgk*	*chl*	*dhj*	*fhn*	*jkm*
adn	*bcj*	*bin*	*cio*	*ejn*	*fil*	*jlo*
aeg	*bdo*	*cdm*	*dei*	*eko*	*ghm*	*kln*
afj	*beh*	*cef*	*dfk*	*elm*	*gij*	*mno*

(ab) (cd) (fi) (gh) (jn) (ko); (abc) (fgh) (jkl) (mno)

53. $(3^6\,1^{84}\times^{14})$

	a	$(\times)$	*f*	(1^9)	*k*	$(3\,1^3\times)$
	b	$(1^3\times)$	*g*	(1^{10})	*l*	.
17	*c*	.	*h*	.	*m*	$(3\,1^8)$
	d	.	*i*	.	*n*	.
3	*e*	$(1^3\times^7)$	*j*	$(3\,1^3\times)$	*o*	.

abg	*akn*	*bfk*	*cfl*	*dhl*	*fgo*	*gmn*
ach	*alo*	*bij*	*cgk*	*dkm*	*fhm*	*hjk*
adi	*bcm*	*bln*	*cjo*	*ejn*	*fin*	*hno*
aef	*bdo*	*cdn*	*deg*	*eko*	*ghi*	*ikl*
ajm	*beh*	*cei*	*dfj*	*elm*	*gjl*	*imo*

(bcd) (ghi) (jkl) (mno)

54. $(3^8\,1^{84}\times^2)$

	a	(1^6)	*f*	$(1^7\times)$	*k*	$(3\,1^4)$
	b	.	*g*	.	*l*	$(3\,1^6)$
18	*c*	.	*h*	$(3\,1^4)$	*m*	.
	d	.	*i*	.	*n*	.
4	*e*	.	*j*	.	*o*	.

abk	*ahl*	*bgj*	*cfk*	*dko*	*fho*	*gmo*
acg	*ano*	*bim*	*cjn*	*dmn*	*fjm*	*hin*
adj	*bch*	*blo*	*clm*	*efg*	*fln*	*hkm*
aem	*bdf*	*cdi*	*del*	*ehj*	*gil*	*ijo*
afi	*ben*	*ceo*	*dgh*	*eik*	*gkn*	*jkl*

(abcd) (fg) (hijk) (lmno)

55. $(3^8\,1^{84}\times^6)$

	a	(1^5)	*f*	$(1^6\times)$	*k*	$(3\,1^5)$
	b	$(1^5\times)$	*g*	(1^7)	*l*	$(3\,1^6)$
18	*c*	.	*h*	(1^8)	*m*	$(3\,1^6\times)$
	d	(1^6)	*i*	$(3\,1^4)$	*n*	.
1	*e*	$(1^6\times)$	*j*	.	*o*	$(3^2\,1^5)$

abh	*aio*	*bgo*	*cfm*	*djo*	*elm*	*hin*
acl	*amn*	*bim*	*cgk*	*dkm*	*fgh*	*hjk*
adg	*bcj*	*bkn*	*cno*	*efo*	*fik*	*hmo*
aek	*bde*	*cdi*	*dfn*	*egi*	*gjm*	*ijl*
afj	*bfl*	*ceh*	*dhl*	*ejn*	*gln*	*klo*

56. $(3^8\,1^{90}\times^{10})$

	a	$(1^4\times)$	*f*	$(1^6\times^2)$	*k*	$(3\,1^6)$
	b	$(1^5\times)$	*g*	$(1^7\times)$	*l*	.
19	*c*	.	*h*	(1^8)	*m*	$(3\,1^6\times)$
	d	(1^6)	*i*	$(3\,1^5\times)$	*n*	$(3\,1^8)$
1	*e*	$(1^6\times)$	*j*	$(3\,1^6)$	*o*	$(3^2\,1^6\times)$

abl	*agm*	*bgk*	*cfl*	*dhn*	*ekl*	*hik*
aci	*ako*	*bij*	*cgj*	*dlo*	*fio*	*hjo*
adj	*bcn*	*bmo*	*chm*	*efg*	*fjk*	*ilm*
aeh	*bde*	*cdk*	*dfm*	*ein*	*ghl*	*jln*
afn	*bfh*	*ceo*	*dgi*	*ejm*	*gno*	*kmn*

57. $(3^{14}\ 1^{72} \times^{8})$

	a	$(1^{6} \times)$	*f*	$(3\ 1^{3} \times)$	*k*	$(3\ 1^{5})$
	b	$(3\ 1^{3})$	*g*	$(3\ 1^{4} \times)$	*l*	.
19	*c*	$(3\ 1^{3} \times)$	*h*	.	*m*	$(3\ 1^{8})$
	d	.	*i*	.	*n*	.
3	*e*	.	*j*	$(3\ 1^{5})$	*o*	.

abf	*ahk*	*bjo*	*cgn*	*dik*	*ejk*	*gio*
acm	*ail*	*bkm*	*chj*	*djl*	*fgk*	*hin*
adn	*bci*	*bln*	*ckl*	*efn*	*fhl*	*jmn*
aeo	*bdg*	*cde*	*dfm*	*egl*	*fij*	*kno*
agj	*beh*	*cfo*	*dho*	*eim*	*ghm*	*lmo*

(*cde*) (*ghi*) (*jkl*) (*mno*)

58. $(3^{8}\ 1^{96} \times^{8})$

	a	$(1^{3} \times)$	*f*	$(1^{6} \times)$	*k*	$(3\ 1^{7})$
	b	$(1^{6} \times)$	*g*	.	*l*	$(3\ 1^{8})$
20	*c*	.	*h*	$(3\ 1^{3} \times)$	*m*	.
	d	.	*i*	$(3\ 1^{7})$	*n*	.
3	*e*	.	*j*	.	*o*	$(3\ 1^{9})$

abn	*agj*	*bfk*	*cfm*	*dgn*	*fgl*	*ino*
acl	*aho*	*bgo*	*cgi*	*dhl*	*fhj*	*jkn*
adm	*bcj*	*bhm*	*chn*	*efn*	*ghk*	*jlo*
aek	*bdi*	*cdk*	*dej*	*egm*	*ijm*	*kmo*
afi	*bel*	*ceo*	*dfo*	*ehi*	*ikl*	*lmn*

(*bcd*) (*efg*) (*ijk*) (*lmn*)

59. $(3^{10}\ 1^{90} \times^{14})$

	a	$(1^{3} \times)$	*f*	(1^{12})	*k*	$(3\ 1^{3} \times^{7})$
	b	.	*g*	$(3\ 1^{3} \times)$	*l*	$(3\ 1^{8})$
20	*c*	.	*h*	.	*m*	.
	d	(1^{12})	*i*	.	*n*	.
3	*e*	.	*j*	.	*o*	$(3^{2}\ 1^{6})$

abd	*ajm*	*bio*	*chm*	*dgj*	*eik*	*ghn*
acf	*akl*	*bjn*	*cjl*	*dhk*	*emo*	*gim*
aeg	*bce*	*bkm*	*ckn*	*dlo*	*fgk*	*hil*
aho	*bfh*	*cdi*	*den*	*efl*	*fij*	*jko*
ain	*bgl*	*cgo*	*dfm*	*ehj*	*fno*	*lmn*

(*abc*) (*def*) (*ghi*) (*lmn*)

60. $(3^{18}\ 1^{66} \times^{8})$

	a	$(1^{6} \times)$	*f*	$(3\ 1^{4} \times)$	*k*	$(3\ 1^{7})$
	b	$(3\ 1^{3})$	*g*	.	*l*	$(3\ 1^{8})$
20	*c*	$(3\ 1^{3} \times)$	*h*	$(3\ 1^{5})$	*m*	$(3^{2}\ 1^{2} \times)$
	d	.	*i*	$(3\ 1^{5} \times)$	*n*	$(3^{2}\ 1^{3})$
1	*e*	$(3\ 1^{4} \times)$	*j*	$(3\ 1^{6})$	*o*	$(3^{3}\ 1^{3})$

abd	*ahi*	*bhj*	*cfl*	*dim*	*ejm*	*gkm*
ack	*amo*	*bin*	*cij*	*dko*	*fgn*	*hkn*
aen	*bcg*	*blm*	*cmn*	*dln*	*fhm*	*hlo*
afj	*bek*	*cdh*	*def*	*egh*	*fik*	*jkl*
agl	*bfo*	*ceo*	*dgj*	*eil*	*gio*	*jno*

61. $(3^{10}\ 1^{108} \times^{8})$

	a	$(1^{6} \times)$	*f*	$(1^{6} \times)$	*k*	$(3\ 1^{8})$
	b	.	*g*	(1^{10})	*l*	.
23	*c*	.	*h*	$(3\ 1^{6} \times)$	*m*	$(3\ 1^{9})$
	d	.	*i*	.	*n*	$(3^{2}\ 1^{9})$
2	*e*	.	*j*	$(3\ 1^{7})$	*o*	.

abj	*ahm*	*bfn*	*cfk*	*dgh*	*fhl*	*hij*
acn	*ail*	*bhk*	*cgi*	*din*	*fio*	*jkn*
adk	*bcl*	*bim*	*cho*	*efm*	*gjm*	*jlo*
aeo	*bdo*	*cdm*	*del*	*ehn*	*gko*	*klm*
afg	*beg*	*cej*	*dfj*	*eik*	*gln*	*mno*

(*ab*) (*cd*) (*ef*) (*hi*) (*kl*) (*no*)

62. $(3^{22}\ 1^{72} \times^{8})$

	a	$(1^{6} \times)$	*f*	$(3\ 1^{6})$	*k*	$(3^{2}\ 1^{3} \times)$
	b	$(3\ 1^{3} \times)$	*g*	$(3\ 1^{6} \times)$	*l*	.
23	*c*	$(3\ 1^{5})$	*h*	$(3\ 1^{7})$	*m*	$(3^{2}\ 1^{5})$
	d	$(3\ 1^{5} \times)$	*i*	$(3\ 1^{7} \times)$	*n*	$(3^{3}\ 1^{3})$
1	*e*	.	*j*	$(3^{2}\ 1^{3})$	*o*	$(3^{3}\ 1^{5})$

abj	*akn*	*bfk*	*cgm*	*dgl*	*eln*	*hjl*
acd	*alo*	*bgn*	*cin*	*dhn*	*fho*	*hkm*
aem	*bch*	*blm*	*cjo*	*dko*	*fjn*	*ijm*
afi	*bdi*	*cek*	*dej*	*efg*	*gio*	*ikl*
agh	*beo*	*cfl*	*dfm*	*ehi*	*gjk*	*mno*

63. $(3^{22}\ 1^{84})$

	a	$(3\ 1^{5})$	*f*	$(3\ 1^{6})$	*k*	$(3^{2}\ 1^{4})$
	b	.	*g*	.	*l*	$(3^{2}\ 1^{6})$
25	*c*	.	*h*	.	*m*	.
	d	.	*i*	$(3\ 1^{7})$	*n*	$(3^{2}\ 1^{8})$
4	*e*	$(3\ 1^{6})$	*j*	.	*o*	$(3^{4}\ 1^{2})$

abm	*agi*	*bfl*	*cfi*	*dgo*	*fgn*	*imn*
ack	*ahj*	*bgj*	*cgm*	*dhm*	*fhk*	*jln*
adn	*bcn*	*bhi*	*cho*	*efm*	*ghl*	*jmo*
ael	*bdk*	*cdl*	*dei*	*egk*	*ijk*	*klm*
afo	*beo*	*cej*	*dfj*	*ehn*	*ilo*	*kno*

(*ab*) (*cd*) (*ef*) (*gh*); (*ad*) (*bc*) (*eh*) (*fg*) (*ij*) (*lm*)

64. $(3^{22}\ 1^{84})$

	a	$(3\ 1^{5})$	*f*	$(3\ 1^{6})$	*k*	$(3^{2}\ 1^{6})$
	b	.	*g*	.	*l*	.
25	*c*	.	*h*	.	*m*	.
	d	.	*i*	.	*n*	.
4	*e*	.	*j*	$(3\ 1^{9})$	*o*	$(3^{4}\ 1^{2})$

abn	*ago*	*bfk*	*cgl*	*dhk*	*fgm*	*him*
ack	*ail*	*bgj*	*chj*	*dij*	*fhl*	*jkm*
adm	*bcm*	*bho*	*cio*	*ejo*	*fin*	*jln*
aeh	*bdl*	*cdn*	*deg*	*ekl*	*ghn*	*kno*
afj	*bei*	*cef*	*dfo*	*emn*	*gik*	*lmo*

(*abcd*) (*fghi*) (*kl*) (*mn*)

65. $(3^{42}\ 1^{24})$

	a	$(3^{2}\ 1^{4})$	*f*	(3^{3})	*k*	(3^{3})
	b	.	*g*	.	*l*	$(3^{3}\ 1^{3})$
25	*c*	.	*h*	.	*m*	.
	d	(3^{3})	*i*	.	*n*	.
24	*e*	.	*j*	.	*o*	.

abm	*ajk*	*bhk*	*chj*	*djn*	*ekm*	*ghn*
acl	*ano*	*bij*	*cik*	*dkl*	*fho*	*gil*
adg	*bcn*	*blo*	*cmo*	*ehl*	*fim*	*gjm*
aef	*bdf*	*cde*	*dhm*	*ein*	*fjl*	*gko*
ahi	*beg*	*cfg*	*dio*	*ejo*	*fkn*	*lmn*

(*abc*) (*def*) (*hij*) (*lmn*); (*ac*) (*djgh*) (*ekfi*) (*mn*)

66. $(3^{26}\ 1^{108})$

	a	$(3\ 1^{6})$	*f*	$(3\ 1^{8})$	*k*	$(3^{2}\ 1^{6})$
	b	.	*g*	.	*l*	$(3^{2}\ 1^{8})$
31	*c*	$(3\ 1^{7})$	*h*	$(3\ 1^{9})$	*m*	.
	d	$(3\ 1^{8})$	*i*	$(3^{2}\ 1^{6})$	*n*	$(3^{4}\ 1^{6})$
2	*e*	.	*j*	.	*o*	$(3^{4}\ 1^{8})$

abk	*aim*	*bgh*	*chm*	*dgo*	*fgl*	*hkl*
acg	*ajo*	*bio*	*cko*	*dhj*	*fin*	*hno*
adn	*bcf*	*bjm*	*cln*	*efo*	*fjk*	*ijl*
ael	*bdl*	*cdi*	*dek*	*egm*	*gik*	*kmn*
afh	*ben*	*cej*	*dfm*	*ehi*	*gjn*	*lmo*

(*ab*) (*de*) (*fg*) (*ij*)

67.

$(3^{26}\,1^{108})$	*a*	$(3\,1^{6})$	*f*	$(3\,1^{9})$	*k*	$(3^{2}\,1^{6})$
	b	.	*g*	.	*l*	$(3^{2}\,1^{8})$
31	*c*	$(3\,1^{8})$	*h*	$(3^{2}\,1^{5})$	*m*	.
	d	.	*i*	.	*n*	$(3^{3}\,1^{9})$
2	*e*	$(3\,1^{9})$	*j*	$(3^{2}\,1^{6})$	*o*	$(3^{4}\,1^{6})$

abj agi bfl cfn dhk fho him
acm ahn bgh chl dil fij jkn
ado bco bin cik efm gjo jlm
ael bdm cdj den ehj gkl kmo
afk bek ceg dfg eio gmn lno

(ab) (cd) (ef) (hi)

68.

$(3^{26}\,1^{114}\times^{8})$	*a*	$(1^{6}\times)$	*f*	$(3\,1^{9}\times)$	*k*	$(3^{2}\,1^{9})$
	b	$(3\,1^{6}\times)$	*g*	$(3^{2}\,1^{6}\times)$	*l*	.
32	*c*	.	*h*	.	*m*	$(3^{3}\,1^{9})$
	d	.	*i*	.	*n*	.
3	*e*	$(3\,1^{9})$	*j*	$(3^{2}\,1^{9})$	*o*	.

abj ahn bfk cfl dhk fgo hij
ack aio bgn cgj dim fhm jkl
adl bcm bil cho ejo fin jmn
aef bdo cdn deg ekm ghl kno
agm beh cei dfj eln gik lmo

(bcd) (ghi) (jkl) (mno)

69.

$(3^{42}\,1^{72})$	*a*	$(3^{2}\,1^{4})$	*f*	$(3^{3}\,1^{3})$	*k*	$(3^{3}\,1^{3})$
	b	$(3^{2}\,1^{8})$	*g*	.	*l*	.
33	*c*	.	*h*	.	*m*	$(3^{3}\,1^{7})$
	d	$(3^{3}\,1^{3})$	*i*	.	*n*	.
8	*e*	.	*j*	.	*o*	$(3^{3}\,1^{11})$

abn agk bgi cfh dil fgm hin
acm alo bhj cik dko fio hkm
adh bcl bmo cno efn fkl ijm
aei bdf cdj dem eho ghl jkn
afj bek ceg dgn ejl gjo lmn

(bc) (defg) (hijk) (mn); (dj) (ei) (fh) (gk)

70.

$(3^{42}\,1^{72})$	*a*	$(3^{2}\,1^{12})$	*f*	$(3^{3}\,1^{3})$	*k*	$(3^{3}\,1^{3})$
	b	.	*g*	.	*l*	.
33	*c*	.	*h*	.	*m*	.
	d	$(3^{3}\,1^{3})$	*i*	.	*n*	.
288	*e*	.	*j*	.	*o*	.

abc alm bik chk dio ekl fkm
ade ano bln cij djm ejo ghm
afg bdf bmo clo dkn fho gin
ahi beg cdg cmn ehn fil gjl
ajk bhj cef dhl eim fjn gko

(dinfkl) (ehogjm); (ab) (dnel) (fogm) (hk);
(abc) (dimfjl) (ekngho)

71.

$(3^{30}\,1^{132})$	*a*	$(3\,1^{9})$	*f*	$(3\,1^{9})$	*k*	$(3^{2}\,1^{8})$
	b	.	*g*	$(3^{2}\,1^{8})$	*l*	.
37	*c*	.	*h*	.	*m*	$(3^{4}\,1^{10})$
	d	.	*i*	.	*n*	.
24	*e*	.	*j*	.	*o*	.

abc aio bho cfl dgh elo gkn
ade ajk bin cgm dln emn hjl
afm bdj blm chi dmo fjn hkm
agl bek cdk cno efh fko ijm
ahn bfg cej dfi egi gjo ikl

(abc) (dfh) (egi) (jln) (kmo); (bc) (de) (fhgi) (lomn)

72.

$(3^{50}\,1^{72})$	*a*	$(3^{2}\,1^{4})$	*f*	$(3^{3}\,1^{5})$	*k*	$(3^{4}\,1^{2})$
	b	$(3^{2}\,1^{8})$	*g*	$(3^{3}\,1^{6})$	*l*	.
37	*c*	$(3^{3}\,1^{5})$	*h*	.	*m*	$(3^{4}\,1^{6})$
	d	.	*i*	.	*n*	.
4	*e*	.	*j*	.	*o*	(3^{6})

abk alm bij chm dhn ein fjn
acd ano blo cil dik ejm gjo
aef bce bmn cjk djl fgk hio
agi bdf cfo deo egl fhl kln
ahj bgh cgn dgm ehk fim kmo

(ce) (df) (gi) (hj); (cd) (ef) (gh) (ij)

73.

$(3^{50}\,1^{72})$	*a*	$(3^{2}\,1^{8})$	*f*	$(3^{3}\,1^{3})$	*k*	$(3^{4}\,1^{3})$
	b	.	*g*	.	*l*	.
37	*c*	.	*h*	$(3^{3}\,1^{7})$	*m*	.
	d	$(3^{3}\,1^{3})$	*i*	.	*n*	.
12	*e*	.	*j*	.	*o*	$(3^{7}\,1^{3})$

abj akl bio cjo djm eno ghl
aci amn bkm ckn dlo fhn gim
ade bch bln clm ehm fik gjn
afg bdf cdg dhk eil fjl gko
aho beg cef din ejk fmo hij

(de) (fg) (km) (ln); (abc) (efg) (hij) (knm)

74.

$(3^{58}\,1^{120})$	*a*	$(3^{2}\,1^{8})$	*f*	$(3^{3}\,1^{9})$		$(3^{4}\,1^{8})$
	b	.	*g*	$(3^{4}\,1^{6})$		.
49	*c*	$(3^{3}\,1^{9})$	*h*	.	*m*	$(3^{4}\,1^{10})$
	d	.	*i*	$(3^{4}\,1^{8})$	*n*	$(3^{6}\,1^{8})$
8	*e*	.	*j*	.	*o*	$(3^{8}\,1^{6})$

abm aij bho chk djm eim fkm
ace akl bil cjo dlo eko ghm
adf bcf bjk clm efn fgj ikn
ago bde cdn dgk egl fhl jln
ahn bgn cgi dhi ehj fio mno

(ce) (df) (il) (jk); (ab) (cfde) (gh) (ilkj)

75.

$(3^{58}\,1^{120})$	*a*	$(3^{2}\,1^{12})$	*f*	$(3^{4}\,1^{6})$	*k*	$(3^{4}\,1^{6})$
	b	$(3^{3}\,1^{11})$	*g*	.	*l*	.
49	*c*	.	*h*	.	*m*	.
	d	.	*i*	.	*n*	$(3^{6}\,1^{8})$
32	*e*	.	*j*	.	*o*	.

abc alm bgm cfl dgl ehl gio
ade ano bhk cgk dhj eik jlo
afg bdo bil chm dim fho jmn
ahi ben cdn cij efm fin kln
ajk bfj ceo dfk egj ghn kmo

(de) (fmik) (glhj) (no); (fh) (gi) (jk) (lm);
(becd) (fjgk) (hmil) (no)

76.

(3^{98})	*a*	(3^{6})	*f*	(3^{6})	*k*	(3^{7})
	b	.	*g*	.	*l*	.
49	*c*	.	*h*	(3^{7})	*m*	.
	d	.	*i*	.	*n*	.
168	*e*	.	*j*	.	*o*	.

abd ajk bio cin dil ejm fmo
acg aln bjn cjo dko elo ghl
aef bce bkl clm dmn fhi gij
aho bfg cdf deg ehn fjl gkm
aim bhm chk dhj eik fkn gno

(abcdefg) (hijklmn); (agc) (bde) (jnm) (kol)

77.

$(3^{98}\,1^{48})$	*a*	(3^{6})	*f*	$(3^{6}\,1^{8})$	*k*	$(3^{7}\,1^{3})$
	b	.	*g*	.	*l*	.
57	*c*	.	*h*	$(3^{7}\,1^{3})$	*m*	.
	d	.	*i*	.	*n*	.
96	*e*	$(3^{6}\,1^{8})$	*j*	.	*o*	.

abe	*ajn*	*bil*	*cin*	*djl*	*eln*	*fno*
acf	*ako*	*bjo*	*cjm*	*dkm*	*emo*	*ghk*
adg	*bcg*	*bkn*	*ckl*	*efg*	*fhi*	*gij*
ahl	*bdf*	*cde*	*dhn*	*ehj*	*fjk*	*glo*
aim	*bhm*	*cho*	*dio*	*eik*	*flm*	*gmn*

(*bcd*) (*efg*) (*hnkljo*) (*im*); (*ab*) (*cd*) (*hmil*) (*jokn*)

78.

$(3^{114}\,1^{96})$	*a*	$(3^{6}\,1^{8})$	*f*	$(3^{6}\,1^{8})$	*k*	$(3^{8}\,1^{6})$
	b	.	*g*	$(3^{8}\,1^{6})$	*l*	.
73	*c*	.	*h*	.	*m*	.
	d	.	*i*	.	*n*	.
192	*e*	.	*j*	.	*o*	(3^{14})

abo	*akm*	*bhk*	*chl*	*dkn*	*ekl*	*fjl*
ace	*aln*	*bin*	*cim*	*dlm*	*emn*	*gno*
adf	*bcf*	*bjm*	*cjn*	*efo*	*fgm*	*hmo*
agi	*bde*	*cdo*	*dgj*	*egh*	*fhn*	*ilo*
ahj	*bgl*	*cgk*	*dhi*	*eij*	*fik*	*jko*

(*ade*) (*bcf*) (*hij*) (*kml*); (*ab*) (*cfde*) (*glni*) (*jk*)

79.

(3^{210})	*a*	(3^{14})	*f*	(3^{14})	*k*	(3^{14})
	b	.	*g*	.	*l*	.
105	*c*	.	*h*	.	*m*	.
	d	.	*i*	.	*n*	.
20160	*e*	.	*j*	.	*o*	.

abe	*alm*	*bik*	*cim*	*dil*	*elo*	*fkm*
acd	*ano*	*bln*	*cjn*	*djo*	*emn*	*gho*
afg	*bcf*	*bmo*	*cko*	*dkn*	*fhn*	*gin*
ahi	*bdg*	*ceg*	*def*	*ehk*	*fio*	*gjm*
ajk	*bhj*	*chl*	*dhm*	*eij*	*fjl*	*gkl*

(*abe*) (*cdg*) (*ijk*) (*lmo*); (*adge*) (*bc*) (*hlnj*) (*km*)
(*abcdgef*) (*ijlmokn*); (*abdihjofnckmegl*)

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