The Asymptotic Approach to Behren's Integral, with Further Tables for the d Test of Significance

Fisher, Ronald Aylmer, Sir, 1890-1962

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Integrating with respect to y now gives the leading term

$$s^4 d^3;$$

adding three times the coefficient of d^3 to the coefficient of y^2, we obtain

$$(s^4 + 4s^2c^2)\,d;$$

and adding once the coefficient of d to the term absolute in y, we obtain the check indicated above. The coefficient of n_1^{-1} in S is therefore

$$\tfrac{1}{4}\{s^4d^2 + (s^4 + 4s^2c^2)\},$$

or, as is more convenient for subsequent work,

$$\tfrac{1}{4}\{s^4(d^2+1) + 4s^2c^2\}.$$

The check occurring for all powers of s and c at the last stage of the work is of great service for the heavier polynomials. The term involving s only in the coefficient of n^{-r} is of course the same as in the expansion for t. This term can only occur when n_2 is not involved. Equally there is above no term in c^4, which could only occur if n_1 were not involved.

The expressions obtained are most conveniently recorded in tabular form for the contributions of the second, third and fourth degrees. In Table 1, the expression is set out for the probability of exceeding any given value of d. The columns give the coefficients of the various polynomials in d^2, from which common factors have been removed, and included at the head of each column with the appropriate powers of $\sin\theta$ and $\cos\theta$. Each subdivision of the table has a divisor consisting of powers of n_1 and n_2 with a corresponding numerical coefficient. Since interchanging n_1 and n_2 is equivalent to interchanging $\cos\theta$ and $\sin\theta$ symmetrically related terms have not been repeated.

Table 1. $\dfrac{1}{4n_1}\{s^4(d^2+1)+4s^2c^2\} + \dfrac{1}{4n_2}\{4s^2c^2 + c^4(d^2+1)\}$

		8	48	192
	s^8	s^6c^2	s^4c^4	s^2c^6
d^6	3	.	.	.
d^4	−7	7	.	.
d^2	−5	−10	5	.
d^0	−3	−3	−3	1
		$\div 6(16n_1^2)$		

	4		4
	s^6c^2	s^4c^4	s^2c^6
d^6	.	1	.
d^4	1	−13	1
d^2	−6	57	−6
d^0	3	−81	3
		$\div 16n_1n_2$	

		4	32	1536	384	1536
	s^{12}	$s^{10}c^2$	s^8c^4	s^6c^6	s^4c^8	s^2c^{10}
d^{10}	1	.	.	.	.	.
d^8	−11	11	.	.	.	.
d^6	14	−92	19	.	.	.
d^4	6	78	−111	2	.	.
d^2	−3	12	51	−7	13	.
d^0	−15	−21	−3	1	−19	1
			$\div 6(64n_1^3)$			

Table 1 (*continued*)

	4 $s^{10}c^2$	s^8c^4	8 s^6c^6	48 s^4c^8	192 s^2c^{10}
d^{10}	.	3	.	.	.
d^8	3	−97	7	.	.
d^6	−52	1210	−164	5	.
d^4	198	−6462	1230	−73	1
d^2	−108	10887	−3324	269	−6
d^0	−21	−2301	2127	−237	3

$\div 6(64n_1^2 n_2)$

	s^{16}	16 $s^{14}c^2$	32 $s^{12}c^4$	384 $s^{10}c^6$	3840 s^8c^8	30720 s^6c^{10}	184320 s^4c^{12}	737280 s^2c^{14}
d^{14}	15	.	.	.	.	.	.	.
d^{12}	−375	75	.	.	.	.	.	.
d^{10}	2225	−1550	1075	.	.	.	.	.
d^8	−2141	7247	−17689	1151	.	.	.	.
d^6	−939	−5052	61434	−14278	679	.	.	.
d^4	−213	−1299	27822	33518	−5819	211	.	.
d^2	915	210	−2985	−8060	7775	−1066	29	.
d^0	945	585	1035	−105	−615	537	−59	1

$\div 360(256n_1^4)$

	4 $s^{14}c^2$	$s^{12}c^4$	4 $s^{10}c^6$	32 s^8c^8	1536 s^6c^{10}	384 s^4c^{12}	1536 s^2c^{14}
d^{14}	.	1	.	.	.	.	.
d^{12}	1	−57	11	.	.	.	.
d^{10}	−34	1311	−510	19	.	.	.
d^8	333	−14547	8583	−681	2	.	.
d^6	−996	71451	−63804	8178	−51	13	.
d^4	519	−120699	195165	−39270	390	−201	1
d^2	126	33309	−180990	66735	−1005	741	−6
d^0	27	2043	18945	23805	630	−573	3

$\div 6(256n_1^3 n_2)$

	192 $s^{14}c^2$	48 $s^{12}c^4$	8 $s^{10}c^6$	s^8c^8	8 s^6c^{10}	48 s^4c^{12}	192 s^2c^{14}
d^{14}	.	.	.	9	.	.	.
d^{12}	.	.	21	−609	21	.	.
d^{10}	.	15	−1066	18359	−1066	15	.
d^8	3	−509	20521	−297323	20521	−509	3
d^6	−52	5570	−185748	2,554419	−185748	5570	−52
d^4	198	−22758	779715	−10,804275	779715	−22748	198
d^2	−108	28803	−1,288170	19,150245	−1,288170	28803	−108
d^0	−21	−4809	523215	−9,418185	523215	−4809	−21

$\div 36(256n_1^2 n_2^2)$

As explained in § 3, using the method of Cornish & Fisher (1937), from these first few terms of the expansion of the probability integral we may pass to the corresponding terms of the expression of d in terms of x, where x is the normal deviate at a given level of significance. These are set out in Table 2 in an arrangement similar to that used in Table 1. It will be noticed that the highest powers of x^2 are now the same as the corresponding powers of n, and that the coefficients are generally not so large.

An unforeseen check on the accuracy of the table may be noticed at this stage. If we let $\cos^2\theta = -\sin^2\theta$, as with the 'circular points at infinity', the terms of the second degree become

$$(-9x^4+72x^2-87)\div 32n_1^2$$

and

$$(-9x^4+72x^2-87)\div 16n_1n_2,$$

or, in all

$$-3(3x^4-24x^2+29)\left(\frac{1}{n_1}+\frac{1}{n_2}\right)^2\div 32.$$

Table 2. *Expansion of d in terms of x*

$$\frac{d-x}{x}=\frac{1}{4n_1}\{s^4(x^2+1)+4c^2s^2\}+\dots$$

		8	48	192
	s^8	s^6c^2	s^4c^4	s^2c^6
x^4	5	4	.	.
x^2	16	−1	4	.
x^0	3	3	−1	1

$\div 6(16n_1^2)$

	4		4
	s^6c^2	s^4c^4	s^2c^6
x^4	.	−9	.
x^2	−3	48	−3
x^0	5	−47	5

$\div 16n_1n_2$

		4	16	64	384	1536
	s^{12}	$s^{10}c^2$	s^8c^4	s^6c^6	s^4c^8	s^2c^{10}
x^6	3	.	15	.	.	.
x^4	19	61	−79	32	.	.
x^2	17	−16	88	−101	11	.
x^0	−15	−21	−18	18	−15	1

$\div 6(64n_1^3)$

	4		8	48	192
	$s^{10}c^2$	s^8c^4	s^6c^6	s^4c^8	s^2c^{10}
x^6	.	249	−60	.	.
x^4	29	−3191	697	−36	.
x^2	−128	8003	−2116	219	−3
x^0	27	−2301	1671	−193	5

$\div 6(64n_1^2n_2)$

		16	32	384	3840	30720	184320	737280
	s^{16}	$s^{14}c^2$	$s^{12}c^4$	$s^{10}c^6$	s^8c^8	s^6c^{10}	s^4c^{12}	s^2c^{14}
x^8	79	92	−1354	336	.	.	.	.
x^6	776	−407	15664	−4986	384	.	.	.
x^4	1482	2841	−23007	16963	−3410	160	.	.
x^2	−1920	−1485	5040	−8380	5228	−808	26	.
x^0	−945	−585	−1215	15	−633	429	−53	1

$\div 360(256n_1^4)$

Table 2 (*continued*)

	12		12	48	64	1152	1536
	$s^{14}c^2$	$s^{12}c^4$	$s^{10}c^6$	s^8c^8	s^6c^{10}	s^4c^{12}	s^2c^{14}
x^8	.	−783	480	−105	.	.	.
x^6	−11	15576	−8097	2422	−480	.	.
x^4	93	−67930	36103	−15769	6089	−33	.
x^2	−105	49696	−45643	31546	−18437	210	−3
x^0	−1	−2559	6813	−13206	12642	−181	5

$\div 6(256 n_1^3 n_2)$

	192	144	8		8	144	192
	$s^{14}c^2$	$s^{12}c^4$	$s^{10}c^6$	s^8c^8	s^6c^{10}	s^4c^{12}	s^2c^{14}
x^8	.	.	2740	−60407	2740	.	.
x^6	.	332	−69745	1,127096	−69745	332	.
x^4	29	−3847	474831	−6,465786	474831	−3847	29
x^2	−128	8532	−967803	13,863168	−967803	8532	−128
x^0	27	−2145	455553	−7,774263	455553	−2145	27

$\div 36(256 n_1^2 n_2^2)$

Corresponding relations are found in the third and fourth degrees, so that in this imaginary direction the percentile point depends only on the harmonic mean of n_1 and n_2. I have not investigated the analytical basis of these relationships, which, apart from offering a valuable check, go far to simplify the expansions for d in terms of multiples of the modular angle.

Just as the expansion of Table 1 may be used directly, though laboriously, to obtain the probability value for any observed value of d, so with less, but still considerable, labour the expansion in Table 2 will give the value of d corresponding with any chosen probability. The advantage of this latter approach is, however, only fully realized when a few more steps are taken: (i) we may rewrite the terms in inverse powers of n_1 and n_2 as terms in powers of σ and δ, where

$$\frac{1}{n_1} + \frac{1}{n_2} = \sigma, \quad \frac{1}{n_2} - \frac{1}{n_1} = \delta;$$

(ii) the powers of $\sin^2\theta$ and $\cos^2\theta$ may be expressed in terms of $\cos 2\theta$, $\cos 4\theta$, $\cos 6\theta$, etc.; (iii) numerical values may be substituted for x corresponding with different chosen levels of significance.

The two first of these operations are obviously best taken together. For the first degree we then have

$$x(3x^2+7) \qquad \sigma/32,$$

$$x(x^2-3)\cos 4\theta \qquad \sigma/32,$$

$$x(x^2+1)\cos 2\theta \qquad \delta/8.$$

Naturally terms with even powers of δ involve even multiples of 2θ, while those with odd powers of δ have only odd multiples of 2θ.

In the second degree we have, for the coefficient of σ^2,

$$x(127x^4+584x^2+609) \quad \div 3 \,.\, 2^{12},$$
$$x(5x^4-24x^2-21)\cos 4\theta \quad \div 2^{10},$$
$$3x(-3x^4+24x^2-29)\cos 8\theta \quad \div 2^{12};$$

for the coefficient of $2\sigma\delta$,

$$x(17x^4+36x^2-21)\cos 2\theta \quad \div 2^{10},$$
$$x(-11x^4+20x^2+87)\cos 6\theta \quad \div 3 \,.\, 2^{10};$$

and for the coefficient of δ^2,

$$x(13x^4+32x^2+27) \quad \div 3 \,.\, 2^{8},$$
$$x(-x^4-7)\cos 4\theta \quad \div 2^{8}.$$

The absence of any term in $\delta^2 \cos 8\theta$ is a consequence of the property alluded to above. It will be observed that the highest multiple of θ appearing is four times the power of σ plus twice the power of δ in the term concerned.

For the third degree, the coefficient of σ^3 is

$$x(429x^6+1697x^4+7951x^2-309) \quad \div 3 \,.\, 2^{17},$$
$$x(31x^6-437x^4-2827x^2+3033)\cos 4\theta \quad \div 2^{18},$$
$$x(-71x^6+1053x^4-2541x^2-609)\cos 8\theta \quad \div 2^{17},$$
$$9x(9x^6-131x^4+451x^2-321)\cos 12\theta \quad \div 2^{18};$$

the coefficient of $3\sigma^2\delta$ is

$$x(465x^6+1369x^4-1669x^2-3837)\cos 2\theta \quad \div 9 \,.\, 2^{15},$$
$$x(-603x^6+2605x^4+9239x^2-3873)\cos 6\theta \quad \div 9 \,.\, 2^{16},$$
$$x(83x^6-565x^4-879x^2+2889)\cos 10\theta \quad \div 3 \,.\, 2^{16};$$

that of $3\sigma\delta^2$ is

$$x(243x^6+1151x^4+217x^2-1155) \quad \div 9 \,.\, 2^{14},$$
$$x(-21x^6-17x^4+149x^2+369)\cos 4\theta \quad \div 9 \,.\, 2^{11},$$
$$x(69x^6-103x^4-593x^2-2517)\cos 8\theta \quad \div 9 \,.\, 2^{14};$$

while the coefficient of δ^3 is

$$x(9x^6+89x^4+19x^2-237)\cos 2\theta \quad \div 3 \,.\, 2^{12},$$
$$x(3x^6-13x^4+49x^2+177)\cos 6\theta \quad \div 3 \,.\, 2^{12}.$$

In the fourth degree the terms in σ^4 are

$$x(283277x^8-157592x^6+12{,}162126x^4+7{,}227120x^2-17{,}859555) \quad \div 45 \,.\, 2^{25},$$
$$x(-2059x^8+20536x^6-116850x^4+227232x^2+91125)\cos 4\theta \quad \div 3 \,.\, 2^{22},$$
$$x(-131x^8+20008x^6-118994x^4-113488x^2+213453)\cos 8\theta \quad \div 2^{23},$$
$$3x(299x^8-7608x^6+47794x^4-65824x^2-6741)\cos 12\theta \quad \div 2^{22},$$
$$9x(-429x^8+9624x^6-62478x^4+131088x^2-64573)\cos 16\theta \quad \div 2^{25};$$

the coefficients of $4\sigma^3\delta$ are

$x(19999x^8+102431x^6-235543x^4-841235x^2+1{,}234260)\cos 2\theta \quad \div\ 15.2^{23},$

$x(-95497x^8+993427x^6+2{,}133069x^4-7{,}814295x^2-8{,}934840)\cos 6\theta \quad \div\ 45.2^{23},$

$x(43631x^8-546116x^6+44238x^4+5{,}685180x^2-1{,}323045)\cos 10\theta \quad \div\ 15.2^{23},$

$x(-1221x^8+16332x^6-17786x^4-171124x^2+193719)\cos 14\theta \quad \div\ 2^{23};$

the coefficients of $6\sigma^2\delta^2$ are

$x(33893x^8+227872x^6-65946x^4-1{,}276200x^2-1{,}818675) \quad \div\ 135.2^{20},$

$x(-7459x^8+3520x^6+192022x^4+346232x^2+170469)\cos 4\theta \quad \div\ 9.2^{21},$

$x(3989x^8-16352x^6-106746x^4-144552x^2+289917)\cos 8\theta \quad \div\ 9.2^{20},$

$x(-9047x^8+56000x^6+204606x^4+191640x^2-1{,}596015)\cos 12\theta \quad \div\ 27.2^{21};$

the coefficients of $4\sigma\delta^3$ are

$x(-8936x^8+124481x^6+458137x^4+605915x^2+1{,}806075)\cos 2\theta \quad \div\ 15.2^{23},$

$x(117128x^8-80003x^6-871851x^4-6{,}027345x^2-17{,}113185)\cos 6\theta \quad \div\ 45.2^{23},$

$x(-1039x^8+2164x^6+5338x^4+67220x^2+233565)\cos 10\theta \quad \div\ 15.2^{19};$

while the coefficients of δ^4 are

$x(797x^8+11128x^6+16446x^4-65760x^2-87075) \quad \div\ 45.2^{17},$

$x(x^8-136x^6+22x^4+1664x^2+3585)\cos 4\theta \quad \div\ 3.2^{15},$

$x(-15x^8+216x^6-394x^4-3296x^2-9039)\cos 8\theta \quad \div\ 3.2^{17}.$

The diagrams show radially the coefficients of σ, σ^2 and σ^3 in the expansions for the six chosen percentile points. As might have been anticipated from the extreme non-normality of 'Student's' distribution for one degree of freedom, the departure from circularity of these contours is quite pronounced. Their change of shape as we pass from one level of significance to another shows how little reliance should be placed on qualitative conjectures as to the form of the distribution in which the level of significance is not taken into account.

5. Numerical values at chosen levels of significance

The numerical values at the six chosen levels of significance from 10 to 0·2 %, these being the double probabilities, appropriate to a deviation which may be judged significant whether positive or negative, begin with Table 3 (p. 162).

Here are shown the numerical values of the polynomials in x which have been given in the last section, for six special values chosen from those previously tabulated by Cornish & Fisher, namely,

Level of significance	Deviate x
%	
10	1·6448,53627
5	1·9599,63985
2	2·3263,47874
1	2·5758,29303
0·5	2·8070,33768
0·2	3·0902,23231

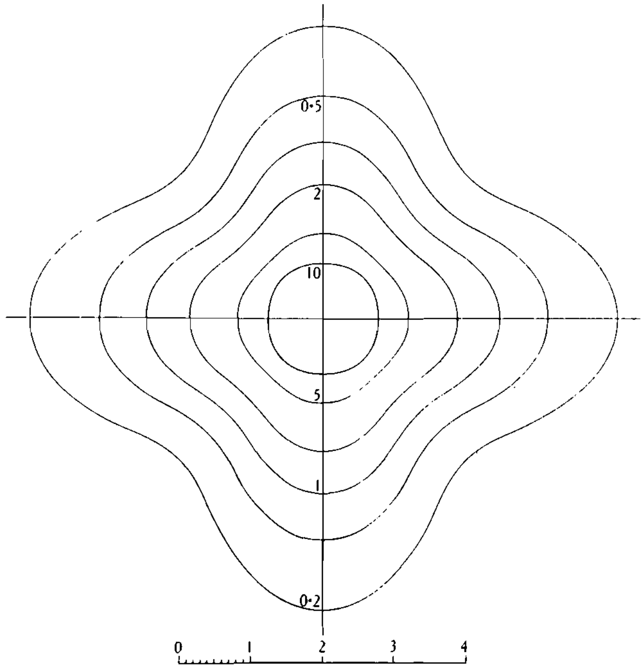

Fig. 1. Contours showing variation in the coefficient of σ for changing angle. The sign of the term in $\cos 4\theta$ changes between 10 and 5 %.

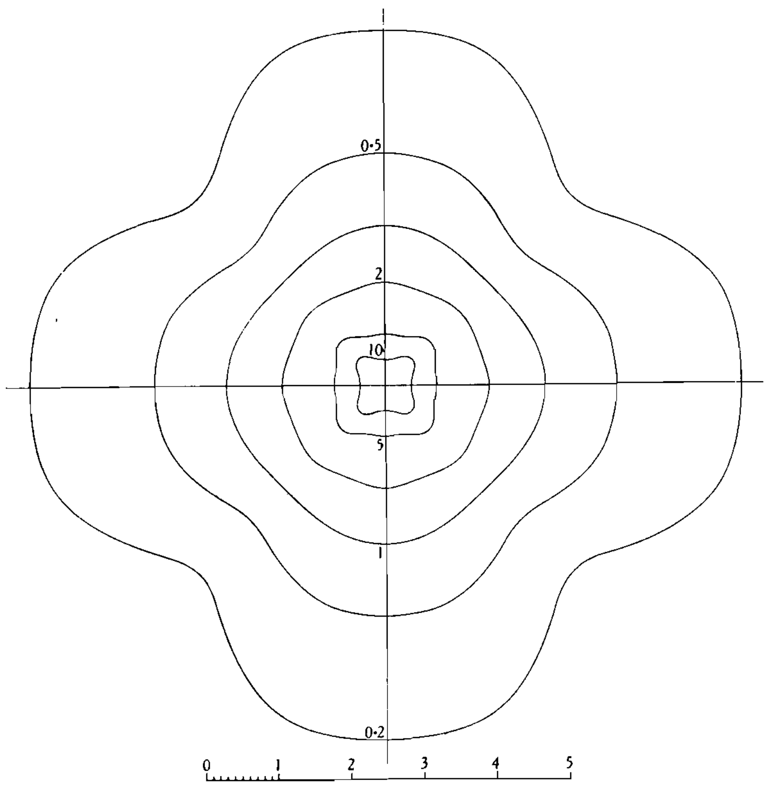

Fig. 2. The coefficients of σ^2 for varying angle. Note the transitions in form due to the coefficients of $\cos 4\theta$ and $\cos 8\theta$ changing sign between 1 and 2 %.

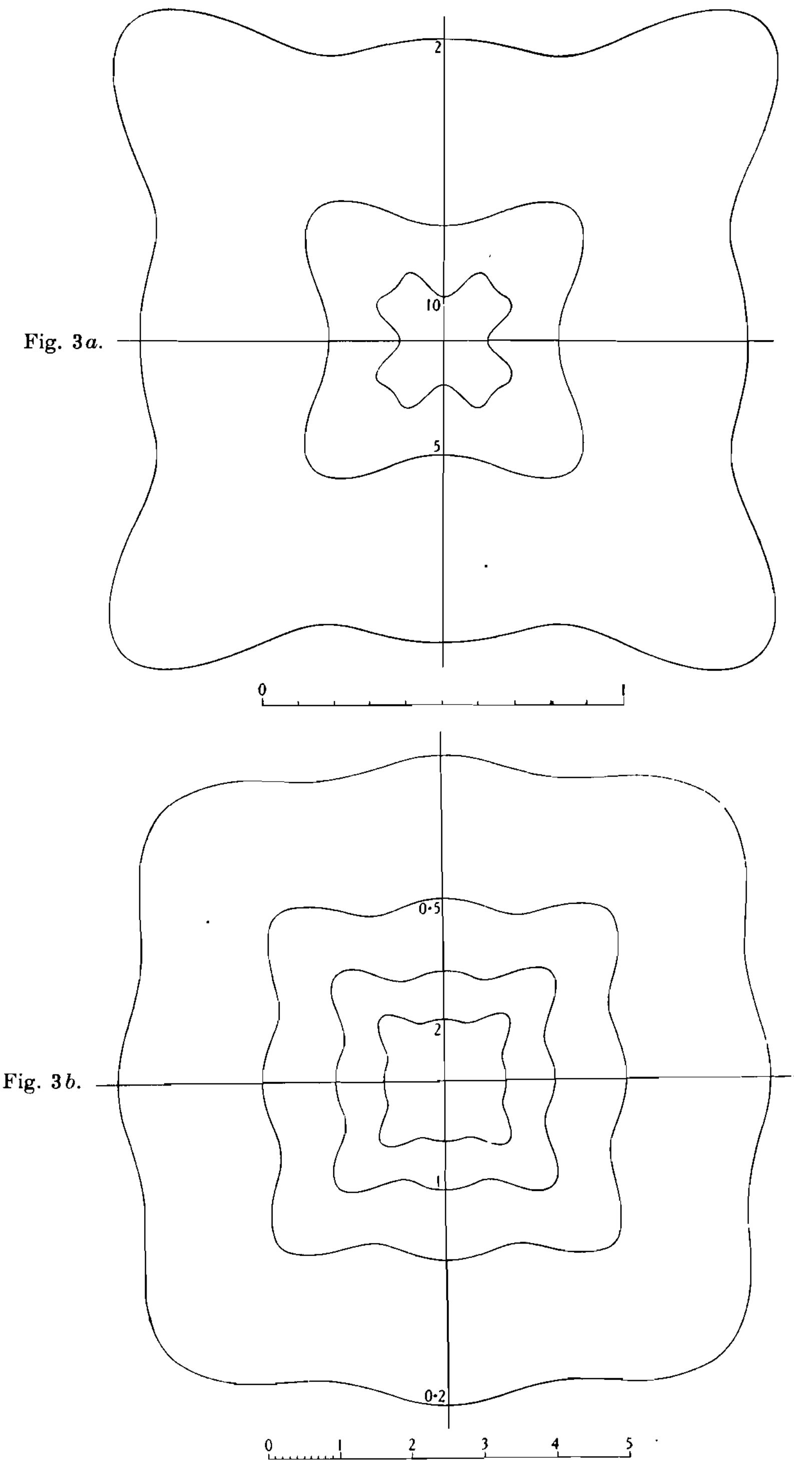

Fig. 3. Coefficients of σ^3 for varying angle. The signs of terms in cos 8θ and cos 12θ change between 10 and 5 % (see Table 3). 3*a* shows the three inner contours on a larger scale; 3*b* the four outer contours on a smaller scale. The 2 % contour is shown in both figures.

As the order of magnitude of the coefficients is uniform throughout, it has been thought sufficient to give them to seven decimal places. They thus afford a basis of higher accuracy, should this be desired, to the later tables involving special angles.

The difference between values at 0 or 90° and those at 45° may be picked out at a glance from the coefficients of $\cos 4\theta$, for the first and second degree, and from the sum of the coefficients of $\cos 4\theta$ and $\cos 12\theta$ for the third and fourth degrees. It is obvious that the 5 % point, discussed by Bartlett, is near a zero, but positive, in the coefficient of σ, but is negative in the coefficient of σ^2. Evidently a change of sign is to be expected at some rather small value of n, if n_1 and n_2 have a common value n.

Table 4 (p. 164) gives the numerical values at the same levels of significance of coefficients of σ, δ, and their powers and products up to the fourth degree, for values of θ from 0 to 45° at 15° intervals. The values from 45° to 90° are the same as those for the complementary angles, with the sign changed when δ is involved to an odd power.

At any of these chosen angles, the required percentile point can be easily found for known values of n_1 and n_2. Using these first to calculate σ and δ, it is convenient to complete the calculation in two stages. Starting with the full double expansion

$$\Sigma\Sigma a_{ij}\sigma^i\delta^j,$$

we may first remove δ by calculating

$$b_i = a_{i0}+\delta(a_{i1}+\delta(a_{i2}+\delta(a_{i3}+\ )))$$

for values of i from 0 to 4. Lastly, we substitute for σ in the form

$$b_0+\sigma(b_1+\sigma(b_2+\sigma(b_3+\sigma b_4))).$$

The percentile values required should be calculated for two adjacent angles at least, and the results interpolated.

Example. The 1 % point is required when $n_1 = 13$, $n_2 = 10$, $\theta = 21°$.

We have first
$$\sigma = \tfrac{1}{13}+\tfrac{1}{10} = 0\cdot176923,$$
$$\delta = \tfrac{1}{10}-\tfrac{1}{13} = 0\cdot023077.$$

Taking, $\theta = 15°$, we find

$$b_0 = 2\cdot575829+\delta(2\cdot12893+\delta(2\cdot46538+\delta(\ldots)))$$
$$= 2\cdot62629,$$
$$b_1 = 2\cdot41181,$$
$$b_2 = 2\cdot26458,$$
$$b_3 = 1\cdot55343,$$
$$b_4 = 0\cdot82473.$$

Using next the numerical value of σ, we find the percentile point at 15° to be

$$3\cdot13339;$$

as the last adjustment is 0·00081, we should not expect the fourth decimal digit to be correct.

Table 3. *Harmonic coefficients for six chosen levels of significance*

First degree:

%		$\cos 4\theta$
10	0·7770201	−0·0151356
5	1·1345972	+0·0515384
2	1·6891963	0·1753408
1	2·1656840	0·2925898
0·5	2·6875871	0·4280234
0·2	3·4425511	0·6324798

σ

%	$\cos 2\theta$
10	0·7618846
5	1·1861356
2	1·8645371
1	2·4582738
0·5	3·1156105
0·2	4·0750309

δ

Second degree:

%		$\cos 4\theta$	$\cos 8\theta$
10	0·4174610	−0·0792441	0·0168338
5	0·7538917	−0·0754337	0·0271666
2	1·4178452	−0·0100925	+0·0221838
1	2·1118436	+0·1002957	−0·0034488
0·5	2·9914869	0·2750050	−0·0537629
0·2	4·4681870	0·6209879	−0·1661074

σ^2

%	$\cos 2\theta$	$\cos 6\theta$
10	0·3226081	0·0324426
5	0·7046649	+0·0009598
2	1·5260624	−0·0961260
1	2·4305057	−0·2218152
0·5	3·6132725	−0·4005435
0·2	5·6524975	−0·7294301

$2\sigma\delta$

%		$\cos 4\theta$
10	0·4470595	−0·0920088
5	0·8721971	−0·1665725
2	1·7597018	−0·3297653
1	2·7220639	−0·5133734
0·5	3·9702503	−0·7575214
0·2	6·1083680	−1·1853005

δ^2

Third degree:

%		$\cos 4\theta$	$\cos 8\theta$	$\cos 12\theta$
10	0·1761950	−0·0451801	−0·0148330	+0·0066933
5	0·3967407	−0·0935945	+0·0171058	−0·0007708
2	0·9490968	−0·1788337	0·0927595	−0·0232004
1	1·6537347	−0·2545615	0·1601448	−0·0412811
0·5	2·6953145	−0·3341788	0·2144227	−0·0479726
0·2	4·7464565	−0 4340256	0·2197755	−0·0130323

σ^3

%	$\cos 2\theta$	$\cos 6\theta$	$\cos 10\theta$
10	0·0606690	0·0787811	−0·0165749
5	0·2413365	0·1192182	−0·0410733
2	0·7961822	+0·1058792	−0·0622391
1	1·5824039	−0·0175592	−0·0468078
0·5	2·8124878	−0·3061493	+0·0212472
0·2	5·3441065	−1·0646437	0·2397113

$3\sigma^2\delta$

Table 3 (*continued*)

Third degree (*continued*):

$3\sigma\delta^2$

%		cos 4θ	cos 8θ
10	0·1413309	+0·0206849	−0·0391406
5	0·4045879	−0·0531599	−0·0319468
2	1·1398202	−0·3346116	+0·0346136
1	2·1299202	−0·7720307	0·1601475
0·5	3·6338755	−1·4902625	0·3839727
0·2	6·6537031	−3·0255204	0·8909915

δ^3

%	cos 2θ	cos 6θ
10	0·0862213	0·0366539
5	0·2646992	0·0547820
2	0·7381673	0·1016549
1	1·3490679	0·1689690
0·5	2·2480752	0·2795105
0·2	3·9981150	0·5210591

Fourth degree:

σ^4

%		cos 4θ	cos 8θ	cos 12θ	cos 16θ
10	0·0559800	+0·0096049	−0·0564124	0·0178640	+0·0000808
5	0·1571470	−0·0034690	−0·1019629	0·0557365	−0·0081058
2	0·4587268	−0·0566340	−0·1146548	0·0727262	−0·0086008
1	0·9113202	−0·1573378	−0·0288312	+0·0163927	+0·0118826
0·5	1·6814960	−0·3640233	+0·2031786	−0·1278718	0·0541224
0·2	3·4574499	−0·9374118	0·8496253	−0·4594351	0·1256138

$4\sigma^3\delta$

%	cos 2θ	cos 6θ	cos 10θ	cos 14θ
10	0·0021848	0·0002062	+0·0385926	−0·0138664
5	0·0365146	0·0728027	−0·0022404	−0·0077310
2	0·2142264	0·2674470	−0·1700471	+0·0399371
1	0·5522866	0·4715614	−0·3747321	0·1043107
0·5	1·1954975	0·6685248	−0·5884851	0·1713648
0·2	2·7903274	0·7446222	−0·6844539	0·1853463

$6\sigma^2\delta^2$

%		cos 4θ	cos 8θ	cos 12θ
10	0·0033391	0·0951151	−0·0865075	0·0151706
5	0·0872547	0·1510588	−0·1971870	0·0582193
2	0·4481251	+0·1126387	−0·3437410	0·1345405
1	1·0831337	−0·1710000	−0·3294510	0·1707438
0·5	2·2422380	−0·9078900	−0·0148395	+0·1273935
0·2	5·0239614	−3·1088703	+1·3274791	−0·2067282

$4\sigma\delta^3$

%	cos 2θ	cos 6θ	cos 10θ
10	0·0574221	−0·0764955	0·0461907
5	0·1246494	−0·0834939	0·0581902
2	0·2825822	+0·0385975	+0·0303837
1	0·4609573	0·3689301	−0·0764607
0·5	0·6857037	1·0916619	−0·3304636
0·2	1·0248625	3·0706744	−1·0596949

δ^4

%		cos 4θ	cos 8θ
10	0·0165206	0·0469194	−0·0363226
5	0·1175297	+0·0280134	−0·0461972
2	0·4903092	−0·0883215	−0·0504244
1	1·0907730	−0·2907328	−0·0466136
0·5	2·1253613	−0·6369711	−0·0414882
0·2	4·4804178	−1·3931654	−0·0514103

Table 4. *Coefficients of σ, δ, etc. for chosen values both of the level of significance and of θ*

First degree:

%	0°	15°	30°	45°	
10	0·76188	0·76945	0·78459	0·79215	
5	1·18614	1·16037	1·10883	1·08306	
2	1·86454	1·77687	1·60153	1·51386	
1	2·45827	2·31198	2·01939	1·87309	σ
0·5	3·11561	2·90160	2·47358	2·25956	
0·2	4·07503	3·75879	3·12631	2·81007	
10	0·76188	0·65981	0·38094	0	
5	1·18614	1·02722	0·59307	0	
2	1·86454	1·61474	0·93227	0	
1	2·45827	2·12893	1·22914	0	δ
0·5	3·11561	2·69820	1·55781	0	
0·2	4·07503	3·52908	2·03751	0	

Second degree:

%	0°	15°	30°	45°	
10	0·35505	0·36942	0·44867	0·51354	
5	0·70562	0·70259	0·77802	0·85649	
2	1·42994	1·40171	1·41180	1·45012	
1	2·20869	2·16372	2·06342	2·00810	σ^2
0·5	3·21273	3·15587	2·88087	2·66272	
0·2	4·92307	4·86173	4·24075	3·68109	
10	0·35505	0·27939	0·12886	0	
5	0·70562	0·61026	0·35137	0	
2	1·42994	1·32161	0·85916	0	
1	2·20869	2·10488	1·43707	0	$2\sigma\delta$
0·5	3·21273	3·12919	2·20718	0	
0·2	4·92307	4·89521	3·55568	0	
10	0·35505	0·40106	0·49306	0·53907	
5	0·70562	0·78891	0·95549	1·03877	
2	1·42994	1·59482	1·92458	2·08947	
1	2·20869	2·46538	2·97876	3·23545	δ^2
0·5	3·21273	3·59149	4·34901	4·72777	
0·2	4·92307	5·51572	6·70102	7·29367	

Third degree:

%	0°	15°	30°	45°	
10	0·12288	0·15433	0·21290	0·19985	
5	0·31948	0·34216	0·43421	0·50821	
2	0·83982	0·83650	0·96893	1·24389	
1	1·51804	1·48766	1·65966	2·10972	σ^3
0·5	2·52759	2·46899	2·70722	3·29189	
0·2	4·51917	4·43259	4·84055	5·41329	
10	0·12288	0·06690	−0·05673	0	
5	0·31948	0·24457	−0·01909	0	
2	0·83982	0·74341	+0·26109	0	
1	1·51804	1·41094	0·78536	0	$3\sigma^2\delta$
0·5	2·52759	2·41729	1·72302	0	
0·2	4·51917	4·42054	3·85655	0	

Table 4 (*continued*)

Third degree (*continued*):

%	0°	15°	30°	45°	
10	0·12288	0·17124	0·15056	0·08151	
5	0·31948	0·39398	0·44714	0·42580	
2	0·83982	0·95521	1·28982	1·50905	$3\sigma\delta^2$
1	1·51804	1·66383	2·43586	3·06210	
0·5	2·52759	2·69676	4·18702	5·50811	
0·2	4·51917	4·69545	7·72097	10·57022	
10	0·12288	0·07467	0·00646	0	
5	0·31948	0·22924	0·07757	0	
2	0·83982	0·63927	0·26743	0	δ^3
1	1·51804	1·16833	0·50556	0	
0·5	2·52759	1·94689	0·84452	0	
0·2	4·51917	3·46247	1·47800	0	

Fourth degree:

%	0°	15°	30°	45°	
10	0·02712	0·07108	0·09721	−0·02782	
5	0·09935	0·15471	0·26965	−0·00519	
2	0·35156	0·41931	0·62140	+0·31938	σ^4
1	0·75343	0·82473	1·01486	1·03532	
0·5	1·44690	1·49871	1·60699	2·43069	
0·2	3·03584	2·96056	2·97910	5·82954	
10	0·02712	−0·01952	+0·01325	0	
5	0·09935	+0·04026	−0·05953	0	
2	0·35156	0·29820	−0·22539	0	$4\sigma^3\delta$
1	0·75343	0·71249	−0·33063	0	
0·5	1·44690	1·39657	−0·27934	0	
0·2	3·03584	2·84873	+0·40099	0	
10	0·02712	0·07898	0·01421	−0·19345	
5	0·09935	0·20316	0·16854	−0·31921	
2	0·35156	0·54177	0·69822	−0·14280	$6\sigma^2\delta^2$
1	0·75343	0·99162	1·50410	+0·75394	
0·5	1·44690	1·66832	2·83100	3·00790	
0·2	3·03584	3·01251	5·70793	9·66704	
10	0·02712	0·00972	0·12830	0	
5	0·09935	0·05756	0·17491	0	
2	0·35156	0·21841	+0·11789	0	$4\sigma\delta^3$
1	0·75343	0·46542	−0·17668	0	
0·5	1·44690	0·88003	−0·91404	0	
0·2	3·03584	1·80528	−3·08809	0	
10	0·02712	0·05814	0·01122	−0·06672	
5	0·09935	0·15464	0·12662	+0·04332	
2	0·35156	0·47136	0·55968	0·52821	δ^4
1	0·75343	0·96871	1·25945	1·33489	
0·5	1·44690	1·82762	2·46459	2·72084	
0·2	3·03584	3·80954	5·20271	5·82217	

At 30°, similarly, we have

$$b_0 = 3{\cdot}60579, \quad b_1 = 2{\cdot}08960, \quad b_2 = 2{\cdot}12260,$$
$$b_3 = 1{\cdot}62914, \quad b_4 = 1{\cdot}01486,$$

giving the 1 % point

$$3{\cdot}05195.$$

Direct interpolation of the angles now gives

$$3{\cdot}101.$$

'Student's' t for ten degrees of freedom, corresponding with 0°, is 3·169. Using 15° interval in angle, the second difference at 15° is therefore $-0{\cdot}045$. Linear interpolation of the angle may therefore have given an underestimate of the percentile point to the extent cf about 0·005. Only exceptionally, therefore, will any more accurate process be required.

Sukhatmé has given values at 1 %, and we may compare the above with those obtainable by interpolation in his table. For $n_1 = 13$ interpolation linear in $1/n$ will be sufficient, but for n_2 we shall have to use four-point interpolation.

Since

$$\tfrac{24}{13} = 2 - \tfrac{2}{13},$$

we take

$$d_{13} = d_{12} - \tfrac{2}{13}(d_{12} - d_{24}).$$

Thus at 15° we find

$$n_2 = 6, \quad d = 3{\cdot}638 - \tfrac{2}{13}(0{\cdot}005) = 3{\cdot}63723,$$
$$n_2 = 8, \quad d = 3{\cdot}306 - \tfrac{2}{13}(0{\cdot}006) = 3{\cdot}30508,$$
$$n_2 = 12, \quad d = 3{\cdot}030 - \tfrac{2}{13}(0{\cdot}007) = 3{\cdot}02892,$$
$$n_2 = 24, \quad d = 2{\cdot}795 - \tfrac{2}{13}(0{\cdot}009) = 2{\cdot}79362.$$

Since

$$\tfrac{24}{10} = 2{\cdot}4,$$

interpolation for $n_2 = 10$, between the values 8 and 12, is given by

$$d_{10} = d_{12} + 0{\cdot}4(d_8 - d_{12}),$$

while from the values for 6 and 24 we have

$$d_{10} = d_6 + \frac{1\,.\,4}{3}(d_6 - d_{24}).$$

These two linear interpolations are

$$3{\cdot}02892 + 0{\cdot}11046 = 3{\cdot}13938,$$
$$2{\cdot}79362 + 0{\cdot}39368 = 3{\cdot}18730.$$

The difference is 0·04792, and this, multiplied by

$$\tfrac{1}{2}(0{\cdot}4)(0{\cdot}6) = 0{\cdot}12,$$

is subtracted from the first value to give the four-point interpolate, i.e.

$$3{\cdot}13938 - 0{\cdot}00575 = 3{\cdot}13363,$$

agreeing satisfactorily with 3·13339 obtained from the tables given in this paper.

Tables 3 and 4 supply the test of significance for sufficiently large values of n_1 and n_2 by means of calculations which are rapid when $n_1 = n_2$, and consequently $\delta = 0$, but are necessarily somewhat laborious for unequal samples. The case where $n_1 = \infty$ however, for which

$\delta = \sigma$, is of sufficient importance to justify special treatment. In this case the discrepancy between the two mean values is compounded of two parts, one of which is normally distributed with a variance which is known with exactitude, being estimated from n_1 degrees of freedom. The other source of error, however, which may be of comparable magnitude, has a variance estimated from only a finite number $n_2 = n$ degrees of freedom; it is therefore fiducially distributed in 'Student's' distribution, for n degrees of freedom. The difference between the means thus has a distribution compounded of a Gaussian and a 'Student's' distribution with an estimated ratio between the parameters, σ/s; this ratio is equated to $\tan\theta$. The tabulated value d for any level of significance is multiplied by $\sqrt{(s^2+\sigma^2)}$ to give the corresponding allowable discrepancy between the means.

Table 5 (p. 168) gives the coefficients of n^{-1}, and its powers up to n^{-5}, at intervals of 5° in θ. Since convergence is not so rapid as for t ($\theta = 0$), and since the coefficients are not of constant sign, the inclusion of the fifth powers gives a useful increase of precision for accurate work.

Example. Let us check Sukhatmé's values for the 5 % points at $n = 24$.

For the five values of θ we have

	15°	30°	45°	60°	75°
Normal deviate	1·95996	1·95996	1·95996	1·95996	1·95996
Term in n^{-1}	0·09115	0·07091	0·04513	0·02149	0·00555
,, n^{-2}	0·00471	0·00423	0·00329	0·00179	0·00047
,, n^{-3}	0·00018	0·00013	0·00013	0·00013	0·00004
,, n^{-4}	0·00001	0·00001	−0·00001	0·00000	0·00000
,, n^{-5}	0·00000	0·00000	0·00000	0·00000	0·00000
	2·05601	2·03524	2·00850	1·98337	1·96602
Sukhatmé	2·057	2·036	2·009	1·984	1·966

There is good agreement with Sukhatmé's values, though it would appear that the latter have a slight positive bias.

The corresponding values at 1 % are

	15°	30°	45°	60°	75°
Normal deviate	2·57583	2·57583	2·57583	2·57583	2·57583
Term in n^{-1}	0·18504	0·13536	0·07805	0·03293	0·00763
,, n^{-2}	0·01535	0·01374	0·00910	0·00376	0·00073
,, n^{-3}	0·00086	0·00086	0·00082	0·00044	0·00008
,, n^{-4}	0·00004	0·00003	0·00002	0·00004	0·00001
,, n^{-5}	0·00000	0·00000	0·00000	0·00000	0·00000
	2·77712	2·72582	2·66382	2·61300	2·58428
Sukhatmé	2·779	2·727	2·665	2·615	2·586

The values confirm the small positive bias in Sukhatmé's figures, which in consequence make the test of significance a trifle too strict.

Finally, in Table 6 (p. 170) I give the significant values appropriate to an error compounded of components having respectively Normal and 'Student's' distributions. These are given at intervals of 10° of the angle

$$\tan^{-1}\sigma/s,$$

where σ is the standard deviation of the normal component, which will be derived either from

Table 5. *Error compounded of 'Student' and Normal distributions. Coefficients of expansion for chosen percentile points*

θ	10 %	5 %	2 %	1 %	0·5 %	0·2 %	
0°	1·523769	2·372271	3·729074	4·916548	6·231221	8·150062	
5°	1·513107	2·351143	3·690173	4·861556	6·158075	8·050010	
10°	1·481362	2·288681	3·575607	4·699842	5·943188	7·756335	
15°	1·429264	2·187590	3·391603	4·440906	5·599797	7·287871	
20°	1·358029	2·052179	3·147962	4·099639	5·148609	6·674035	
25°	1·269378	1·888081	2·857250	3·695024	4·615937	5·952101	
30°	1·165530	1·701896	2·533794	3·248526	4·031381	5·163827	
35°	1·049194	1·500799	2·192587	2·782326	3·425304	4·351786	
40°	0·923543	1·292137	1·848203	2·317614	2·826397	3·555836	
45°	0·792156	1·083059	1·513856	1·873094	2·259564	2·810071	1/n
50°	0·658943	0·880197	1·200656	1·463865	1·744357	2·140593	
55°	0·528035	0·689434	0·917168	1·100768	1·294101	1·564301	
60°	0·403646	0·515760	0·669257	0·790252	0·915770	1·088796	
65°	0·289918	0·363214	0·460247	0·534728	0·610586	0·713343	
70°	0·190754	0·234914	0·291326	0·333345	0·375217	0·430725	
75°	0·109641	0·133143	0·162130	0·183051	0·203401	0·229711	
80°	0·049488	0·059475	0·071423	0·079799	0·087756	0·097782	
85°	0·012487	0·014912	0·017752	0·019701	0·021520	0·023765	
0°	1·42020	2·82250	5·71975	8·83476	12·85090	19·69227	
5°	1·40810	2·80907	5·71444	8·84607	12·89013	19·78887	
10°	1·37500	2·77072	5·69299	8·86291	12·97296	20·00921	
15°	1·32925	2·71202	5·63974	8·83885	13·00573	20·16787	
20°	1·28079	2·63719	5·53187	8·71098	12·86487	20·02421	
25°	1·23698	2·54682	5·34408	8·41767	12·43494	19·36076	
30°	1·19945	2·43625	5·05470	7·91631	11·64424	18·05312	
35°	1·16312	2·29655	4·65213	7·19650	10·48742	16·10994	
40°	1·11794	2·11808	4·14002	6·28535	9·02939	13·67211	
45°	1·05261	1·89526	3·53959	5·24354	7·39049	10·97476	1/n²
50°	0·95874	1·63054	2·88778	4·15351	5·71855	8·28707	
55°	0·83415	1·33584	2·23136	3·10299	4·15665	5·85005	
60°	0·68401	1·03076	1·61807	2·16803	2·81552	3·83041	
65°	0·51989	0·73835	1·08735	1·40009	1·75715	2·30052	
70°	0·35714	0·47989	0·66350	0·81985	0·99307	1·24509	
75°	0·21170	0·27099	0·35331	0·41933	0·48899	0·58704	
80°	0·09751	0·12012	0·14912	0·17083	0·19259	0·22163	
85°	0·02488	0·02991	0·03592	0·04013	0·04414	0·04919	
0°	0·98300	2·55585	6·71858	12·14430	20·22069	36·15339	
5°	0·98669	2·56366	6·71853	12·11206	20·11528	35·85360	
10°	0·98399	2·56005	6·68854	12·01730	19·88796	35·29201	
15°	0·94341	2·48706	6·57164	11·88030	19·75801	35·24301	
20°	0·83975	2·30790	6·35106	11·75962	19·94778	36·34916	
25°	0·67788	2·04854	6·08814	11·73322	20·52345	38·57838	
30°	0·50083	1·79595	5·88910	11·82888	21·28186	41·05111	1/n³
35°	0·37399	1·64994	5·81947	11·94938	21·76181	42·36270	
40°	0·35166	1·65946	5·83238	11·86337	21·40662	41·23266	
45°	0·44436	1·78561	5·77103	11·29602	19·81622	37·12393	
50°	0·60768	1·91731	5·45184	10·07759	16·95897	30·50151	
55°	0·76129	1·93068	4·77467	8·25728	13·22026	22·61511	

Table 5 (*continued*)

θ	10 %	5 %	2 %	1 %	0·5 %	0·2 %	
60°	0·82832	1·75533	3·78769	6·10561	9·25470	14·95580	
65°	0·77188	1·40756	2·66703	4·00136	5·72332	8·69012	
70°	0·60835	0·97396	1·62903	2·27196	3·05745	4·33942	
75°	0·39270	0·56115	0·83261	1·07801	1·36051	1·79485	$1/n^3$
80°	0·18962	0·24665	0·32810	0·39498	0·46679	0·56973	
85°	0·04946	0·06033	0·07391	0·08384	0·09360	0·10639	
0°	0·4339	1·5893	5·6250	12·0548	23·1504	48·5735	
5°	0·4373	1·5854	5·5815	11·9517	22·9638	48·2627	
10°	0·4673	1·6545	5·6799	11·9671	22·6687	46·8752	
15°	0·5639	1·9195	6·2078	12·4548	22·4426	43·4612	
20°	0·7289	2·3017	6·8395	12·8542	21·6126	37·8938	
25°	0·8599	2·4283	6·6378	11·9366	19·3431	32·4463	
30°	0·7599	1·8690	4·9404	9·2697	16·2840	31·6810	
35°	0·2756	0·5619	2·2590	6·2121	14·89561	39·3657	
40°	−0·5140	−0·9665	+0·1460	5·0439	17·3929	54·3629	
45°	−1·2553	−1·8771	−0·0092	6·8938	23·1989	69·6539	$1/n^4$
50°	−1·5258	−1·6341	+1·8361	10·5475	28·7402	76·4062	
55°	−1·1553	−0·4337	4·3257	13·3539	30·0644	70·0542	
60°	−0·3725	+0·9460	5·8004	13·3282	25·8311	53·1778	
65°	+0·3661	1·7395	5·5066	10·4930	17·9998	33·0280	
70°	0·7150	1·7026	3·9137	6·4698	9·9784	16·4049	
75°	0·6423	1·1370	2·0749	3·0315	4·2299	6·2291	
80°	0·3572	0·5160	0·7716	1·0004	1·2595	1·6461	
85°	0·0982	0·1232	0·1560	0·1806	0·2045	0·2336	
0°	0·1934	0·7329	3·3142	8·4560	18·9070	47·1226	
5°	0·1869	0·7007	3·2070	8·2703	18·6618	46·9632	
10°	0·1575	0·5002	2·3071	6·3896	15·6622	43·7995	
15°	0·0606	0·1007	0·8875	3·7196	11·8334	40·9903	
20°	−0·0643	0·1546	2·3177	7·5429	18·9708	50·9807	
25°	+0·1748	1·9606	9·8659	22·2925	40·4488	66·6234	
30°	1·3587	5·8660	20·2117	37·4438	54·5896	51·5741	
35°	3·3343	9·4695	23·2590	33·9403	34·7425	−11·8377	
40°	4·5853	8·7434	12·0233	6·7152	−13·3944	−79·5722	
45°	3·3225	2·2734	−7·8525	−25·2064	−50·3299	−83·4155	$1/n^5$
50°	−0·3675	−6·1308	−21·8406	−36·7544	−44·5208	−10·1326	
55°	−4·0134	−10·4875	−20·6743	−22·2177	−5·2770	+81·6597	
60°	−5·0279	−8·4399	−8·5958	+1·8217	31·6422	125·4594	
65°	−3·2551	−3·0722	+2·9446	16·1918	42·2440	107·6484	
70°	−0·7649	+0·9681	6·9988	16·0039	30·5982	61·8609	
75°	+0·5706	1·9001	5·0495	8·8187	14·1136	24·0389	
80°	0·5985	1·0721	1·9571	2·8473	3·9524	5·7830	
85°	0·1907	0·2548	0·3495	0·4292	0·5161	0·6426	

Table 6. *Significant values at six levels for error compounded of components distributed in Normal and 'Student's' distribution*

	Degrees of freedom							
Angle	10	12	15	20	30	60	∞	
0° (t)	1·812	1·782	1·753	1·725	1·697	1·671	1·645	
10°	1·808	1·778	1·750	1·722	1·696	1·670	1·645	
20°	1·794	1·767	1·741	1·716	1·692	1·668	1·645	
30°	1·774	1·751	1·728	1·706	1·685	1·665	1·645	
40°	1·749	1·730	1·711	1·694	1·677	1·661	1·645	
50°	1·721	1·707	1·693	1·680	1·668	1·656	1·645	10 %
60°	1·693	1·684	1·675	1·667	1·659	1·652	1·645	
70°	1·668	1·664	1·659	1·656	1·652	1·648	1·645	
80°	1·651	1·650	1·649	1·648	1·647	1·646	1·645	
90° (x)	1·645	1·645	1·645	1·645	1·645	1·645	1·645	
0° (t)	2·228	2·179	2·131	2·086	2·042	2·000	1·960	
10°	2·219	2·171	2·126	2·082	2·039	1·999	1·960	
20°	2·194	2·151	2·109	2·069	2·031	1·995	1·960	
30°	2·157	2·120	2·085	2·051	2·019	1·989	1·960	
40°	2·112	2·083	2·056	2·030	2·005	1·982	1·960	
50°	2·066	2·046	2·026	2·008	1·991	1·975	1·960	5 %
60°	2·024	2·011	1·999	1·989	1·978	1·969	1·960	
70°	1·989	1·984	1·978	1·973	1·968	1·964	1·960	
80°	1·967	1·966	1·965	1·963	1·962	1·961	1·960	
90° (x)	1·960	1·960	1·960	1·960	1·960	1·960	1·960	
0° (t)	2·764	2·681	2·602	2·528	2·457	2·390	2·326	
10°	2·748	2·668	2·592	2·520	2·452	2·388	2·326	
20°	2·704	2·631	2·563	2·498	2·438	2·380	2·326	
30°	2·637	2·576	2·520	2·466	2·417	2·370	2·326	
40°	2·559	2·513	2·470	2·430	2·393	2·358	2·326	
50°	2·481	2·450	2·421	2·394	2·370	2·347	2·326	2 %
60°	2·414	2·396	2·379	2·364	2·351	2·338	2·326	
70°	2·364	2·356	2·349	2·343	2·337	2·331	2·326	
80°	2·335	2·334	2·332	2·330	2·329	2·328	2·326	
90° (x)	2·326	2·326	2·326	2·326	2·326	2·326	2·326	
0° (t)	3·169	3·055	2·947	2·845	2·750	2·660	2·576	
10°	3·148	3·037	2·932	2·835	2·743	2·657	2·576	
20°	3·086	2·985	2·892	2·804	2·723	2·647	2·576	
30°	2·993	2·909	2·831	2·760	2·693	2·632	2·576	
40°	2·883	2·820	2·762	2·709	2·661	2·616	2·576	
50°	2·775	2·733	2·695	2·661	2·630	2·601	2·576	1 %
60°	2·684	2·661	2·640	2·622	2·605	2·590	2·576	
70°	2·620	2·611	2·603	2·595	2·588	2·582	2·576	
80°	2·586	2·584	2·582	2·580	2·579	2·577	2·576	
90° (x)	2·576	2·576	2·576	2·576	2·576	2·576	2·576	
0° (t)	3·581	3·429	3·286	3·153	3·030	2·915	2·807	
10°	3·553	3·405	3·267	3·139	3·020	2·910	2·807	
20°	3·473	3·338	3·214	3·099	2·994	2·897	2·807	0·5 %
30°	3·350	3·237	3·134	3·040	2·955	2·878	2·807	
40°	3·203	3·119	3·042	2·974	2·912	2·857	2·807	

Table 6 (*continued*)

Angle	Degrees of freedom 10	12	15	20	30	60	∞	
50°	3·058	3·003	2·954	2·911	2·872	2·838	2·807	
60°	2·939	2·910	2·884	2·861	2·841	2·823	2·807	
70°	2·859	2·848	2·838	2·829	2·821	2·814	2·807	0·5 %
80°	2·818	2·816	2·814	2·812	2·810	2·809	2·807	
90° (x)	2·807	2·807	2·807	2·807	2·807	2·807	2·807	
0° (t)	4·144	3·930	3·733	3·552	3·386	3·232	3·090	
10°	4·106	3·898	3·708	3·533	3·372	3·225	3·090	
20°	3·999	3·809	3·636	3·479	3·336	3·207	3·090	
30°	3·832	3·671	3·528	3·399	3·284	3·181	3·090	
40°	3·630	3·508	3·401	3·308	3·226	3·153	3·090	
50°	3·425	3·347	3·280	3·222	3·172	3·128	3·090	0·2 %
60°	3·259	3·219	3·185	3·156	3·131	3·110	3·090	
70°	3·152	3·138	3·126	3·116	3·106	3·098	3·090	
80°	3·103	3·100	3·098	3·096	3·094	3·092	3·090	
90° (x)	3·090	3·090	3·090	3·090	3·090	3·090	3·090	

theory, or from a large number of degrees of freedom available for estimation, while s is an unbiased estimate of the standard error of the second component, based on n degrees of freedom.

The values chosen for n are the harmonic series 10, 12, 15, 20, 30, 60, so that for any value exceeding 10 interpolation is easy, using the argument $60/n$. For full accuracy four-point interpolation is usually necessary for the angle; for direct tests of significance the tabular values will usually be sufficient.

Summary

A certain amount of confusion and controversy has arisen in respect of the test of significance, first given by Behrens, for the difference between the means of two samples not supposedly drawn from equally variable populations, or from populations having a known variance ratio. In such cases not only a single hypothetical variance, but the ratio of two such variances, requires to be 'Studentized', or eliminated, by means of its fiducial distribution. The first two sections of the present paper are therefore given to a somewhat fuller examination of the logic of this and analogous inferences than has previously appeared.

The analytic difficulties of Behrens's solution make it desirable to use in this case the method of asymptotic approach, which I had previously published conjointly with 'Student' in 1926. By this means expressions for the relevant probability are obtained in terms of powers and products of reciprocals of the two degrees of freedom involved (Table 1).

These are then inverted to give, in Table 2, expressions for any chosen percentile point of Behrens's distribution in terms of the corresponding normal deviate. These are alternatively expressed in terms of the cosines of multiples of the angle, the tangent of which is the ratio of the estimated standard deviations of the two means.

Table 3 gives the numerical coefficients of these cosines for six chosen levels of significance, as far as the fourth degree. Table 4 gives the corresponding values at angular intervals of 15 degrees.

Tables 5 and 6 are devoted to presenting fuller material for the important case in which one of the means to be compared, being based on a large mass of relatively inaccurate observations, may be taken to be normally distributed with known variance ($n_1 = \infty$). Table 5 gives the coefficients of inverse powers of n_2 for six levels of significance at angular intervals of 5 degrees, while Table 6 gives the significant values at the same six levels at intervals of 10 degrees for values of n_2 in the harmonic progression 10, 12, 15, etc.

Comparison with the previously published tables of Sukhatmé, obtained by direct numerical integration, show that the latter are accurate very nearly to the third decimal place, but show, on the whole, a slight positive bias, making the test of significance, but only to a very slight degree, too stringent.

It may be noted that the entries of Table 6 for 0°, which are simply values of 'Student's' t, supply values at two higher levels of significance than have been previously published.

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