THE UNIVERSITY OF ADELAIDE
Department of Mechanical Engineering

DESIGN AND STRUCTURAL MODIFICATIONS OF VIBRATORY SYSTEMS TO ACHIEVE PRESCRIBED MODAL SPECTRA

PhD Dissertation

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Abstract

This thesis concerns with problems associated with design and structural modification of vibratory systems. The aim is to meet prescribed modal and spectral requirement. Several common problems encountered in practical engineering applications are described, and novel strategies for solving these problems are then proposed. The mathematical formulations of these problems have been generated, and solution methods are developed.

The first problem concerns with developing a systematic approach for design of conservative vibratory systems with prescribed natural frequencies. Since, in general, this problem has more design parameters (namely the independent elements of the mass and stiffness matrices) than number of design constraints (i.e. the number of natural frequencies to be assigned), it has a family of solutions. We are only interested in these solutions which are physically realisable, i.e. solutions which can be physically constructed. We thus assume that a physically realisable stiffness matrix of a system is known, and then calculate a realisable mass matrix, so that the desired natural frequencies are obtained.

A second problem concerns with a case where in addition to the prescribed natural frequencies, corresponding mass-normalised mode shapes are also specified. This problem is analysed for situations where all of the system's natural frequencies and mode shapes are specified, and also for the case when these frequencies and their associated mode shapes are only partially prescribed. When all of the natural frequencies and mass-normalised mode shapes are prescribed, the problem is overdetermined, i.e there are more constraints than

there are independent design parameters. In general, there are no physically realisable solutions for this case. Therefore, we formulate and solve an optimisation problem leading to an approximate solution which is optimal in a specified sense. A partial specification of natural frequencies and mode shapes may result in a problem which has no realisable solutions, a unique solution, or a family of solution. This depends on the ratio between the number of prescribed and a total number of natural modes for the system. We present a solution method which can cope with any of these cases.

The two remaining problems concern with determining the necessary structural modifications to an existing system, based on measured modal analysis data. Problem 3 deals with assignment of natural frequencies only, while in Problem 4 we assume that both the natural frequencies and the corresponding mass-normalised mode shapes are specified. In both of these problems, our aim is to determine the necessary modifications based on the measured test data only. Because modal analysis data only partially describe the system, we are unable to obtain exact solutions to either problem. We overcome the difficulties of the inherent truncation of the modal data by formulating and solving optimisation problems giving approximate solutions which are optimal in a specified sense.

The developed algorithms were numerically tested on arbitrarily chosen examples, and a simple experiment was designed and carried out to test the suitability of the generated theory in a practical application.

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D.D.Sivan

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Glossary of Principal Symbols

n	-	number of degrees-of-freedom
i	-	index
j	-	index
ω_i	-	i^{th} natural frequency
$\mathbf{M}$	-	mass matrix of a system
$\mathbf{K}$	-	stiffness matrix of a system
$\mathbf{\Lambda}$	-	eigenvalues matrix of a system
$\mathbf{\Phi}$	-	corresponding mass-normalised eigenvectors matrix
λ_i	-	ii^{th} diagonal element of $\mathbf{\Lambda}$, $\lambda_i=\omega_i^2$
ϕ_i	-	corresponding i^{th} column vector of $\mathbf{\Phi}$
$\mathbf{\Lambda}^*$	-	desired eigenvalues matrix
$\mathbf{\Phi}^*$	-	desired mass-normalised eigenvectors matrix
λ_i^*	-	ii^{th} diagonal element of $\mathbf{\Lambda}^*$
ϕ_i^*	-	corresponding i^{th} column vector of $\mathbf{\Phi}^*$
m_i	-	mass of i^{th} element
k_{ij}	-	ij^{th} element of $\mathbf{K}$
s_{ij}	-	the stiffness constant of a spring connecting mass i to mass j
m	-	number of truncated modes ($m<n$ always)
$\mathbf{\Lambda}_1$	-	eigenvalue matrix containing first m eigenvalues of $\mathbf{\Lambda}$
$\mathbf{\Phi}_1$	-	matrix containing first m corresponding column vectors of $\mathbf{\Phi}$
$\mathbf{\Lambda}_1^*$	-	eigenvalue matrix containing first m eigenvalues of $\mathbf{\Lambda}^*$
$\mathbf{\Phi}_1^*$	-	matrix containing first m corresponding column vectors of $\mathbf{\Phi}^*$
I_n	-	Identity matrix of size n (suffix n can be any value)
$\mathbf{\Lambda}_2$	-	eigenvalue matrix containing last n- m eigenvalues of $\mathbf{\Lambda}$
$\mathbf{\Phi}_2$	-	matrix containing corresponding n-m column vectors of $\mathbf{\Phi}$
$\Delta\mathbf{M}$	-	mass modification matrix
$\Delta\mathbf{K}$	-	stiffness modification matrix

δm_i	-	modification to an i^{th} mass element
δs_i	-	modification to an i^{th} stiffness element
$\mathbf{M}_{mod}$	-	a mass matrix of a modified system
$\mathbf{K}_{mod}$	-	a stiffness matrix of a modified system
$\mathbf{B}_{ij}^{(K)}$	-	a connectivity matrix describing the connections of spring s_{ij}
$\mathbf{y}_K$	-	an augmented vector for stiffness sensitivity
$\mathbf{F}_K$	-	a stiffness sensitivity matrix
$\mathbf{s}$	-	a vector consisting of all independent spring constants s_{ij}
$\mathbf{B}_i^{(M)}$	-	a mapping matrix for m_i onto $\mathbf{M}$
$\mathbf{y}_M$	-	an augmented matrix for mass sensitivity
$\mathbf{m}$	-	a vector consisting of all independent mass elements
$\mathbf{F}_M$	-	a mass sensitivity matrix
$\mathbf{k}_i$	-	i^{th} column vector of $\mathbf{K}$
$\mathbf{X}$	-	an arbitrary symmetric matrix
$\mathbf{Y}$	-	an arbitrary symmetric matrix
$\mathbf{D}$	-	a diagonal matrix
$\mathbf{R}$	-	in sections 2, 8 and 9, a residual matrix defined by (2.15)
	-	in section 4, a matrix defined by (4.27)
	-	in section 5, a matrix defined by (5.6)
	-	in section 7, a matrix defined by (7.19)
$\mathbf{\Psi}$	-	in sections 2, 8 and 9 is defined in (2.18) and (2.19)
	-	in section 3, a modal matrix of a modified system
	-	in section 5, defined by (5.39)
$\bar{\mathbf{\Lambda}}$	-	eigenvalues matrix which is an approximation to $\mathbf{\Lambda}^*$
$\bar{\mathbf{\Phi}}$	-	in section 8, a corresponding mass-normalised eigenvectors matrix
	-	in section 9, an approximation to $\mathbf{\Phi}^*$
$\mathbf{A}$	-	in section 3 and 5, any real symmetric matrix
	-	in section 5, a matrix defined by (5.23)
	-	in section 6, a matrix defined by (6.23)
$\boldsymbol{a}_i$	-	i^{th} column vector of $\mathbf{A}$
$\mathbf{\Omega}$	-	in section 3, an eigenvalue matrix of a modified system
	-	in section 8, an eigenvalue matrix of (8.3)
α	-	in section 4, a scalar constant defined by (4.3)

$G_{i,j}$	-	scalar constants defined by (4.23)
F_i	-	scalar constants defined by (4.24)
t	-	an iteration index
$\mathbf{U}$	-	an orthonormal matrix
ε	-	a residual error function
E_t	-	a residual error function defined by (4.35)
$\mathbf{Q}$	-	an orthonormal matrix
$\bar{v}_1, \bar{v}_2$	-	two arbitrary vectors in example of Figure 5.1
$\bar{v}_1^*, \bar{v}_2^*$	-	two orthogonal vectors in example of Figure 5.1
$\bar{A}_1, \bar{A}_2$	-	the square of the lengths of the projection vectors in Figure 5.1
$\mathbf{G}_{(i)}$	-	a matrix defined by (5.30) and (5.43)
$\mathbf{z}_i$	-	a vector defined by (5.31) and (5.44)
$\mathbf{E}_{(i)}$	-	a matrix defined by (5.32) and (5.45)
$\mathbf{\Phi}_{(i)}$	-	a modal matrix defined by (5.39)
$\mathbf{E_X}$	-	a known component of the IP Decomposition of $\mathbf{X}$
$\mathbf{D_X}$	-	a diagonal component of the IP Decomposition of $\mathbf{X}$
β_{ij}	-	known coefficients in $\mathbf{D_X}$ defined in (7.1)
$\mathbf{F}$	-	in section 7, a matrix defined by (7.23)
	-	in section 8, a matrix defined by (8.4)
$\mathbf{E}$	-	a matrix defined by (7.22)
$\mathbf{G}$	-	in section 7,a matrix defined by (7.33)
	-	in section 8, a matrix defined by (8.5)
$\mathbf{H}_i$	-	a mapping matrix defined by (7.36)
$\mathbf{g}_i$	-	an i^{th} column vector of $\mathbf{G}$
$\mathbf{P}$	-	in section 7, a sensitivity matrix defined by (7.42)
	-	in section 9, a matrix defined by (9.3)
$\mathbf{T}$	-	a matrix defined by (9.4)
$\mathbf{H}$	-	a matrix defined by (9.5)
σ_i	-	i^{th} singular value of any matrix
$\mathbf{S}$	-	a diagonal matrix of singular values

In addition to above principal symbols, some symbols for local applications are introduced and defined in the appropriate sections of the text.

Section 1 *INTRODUCTION*

Design of mechanical systems for a specified range of static and dynamic requirements is a basic problem in Mechanical Engineering. However, while design of static systems is well established, design to meet the dynamic requirements is not yet fully developed. This thesis deals with several problems encountered in the design and analysis of dynamic vibratory systems. The common aim among these problems is to develop algorithms which would allow a systematic approach for the design of vibratory systems with prescribed natural frequencies and mode shapes.

A classical problem in vibration analysis is to determine spectral properties (namely the natural frequencies and mode shapes) of a system with known physical parameters (i.e. mass, stiffness and damping space functions), under specified excitation forces and for a given set of the initial and boundary conditions. Solution technique requires precise knowledge of all the above data as well as a set of the partial differential equations which describe the motion of the system. In practice, however, these precise data and the description of motion is only available for a limited range of relatively simple systems.

For complex systems, where this information is not easily available, it is customary to develop a discrete analytical model which approximates the behaviour of a real system. In problems analysed in this thesis we only consider conservative analytical models (i.e Mass-Spring and Finite Element with no damping). This was done in order to simplify the analysis. We realise that some amount of damping is always present in physical systems, however for many mechanical systems its magnitude is relatively small and may be ignored.

Once a suitable model is selected, its physical parameters can be determined, and then the spectral properties can be evaluated. The requirements that the system has prescribed natural frequencies and mode shapes is ensured by the adjustment of the physical parameters in the model until the desired spectral properties are obtained. This is a "trial-and-error" process and it requires repeated computations of the spectral properties for each modification of the model. Furthermore, if the spectral requirements consist of multiple constraints, then currently there is no systematic method for determining the necessary modifications to physical parameters which improve one or more dynamic property without detrimental effects on any of the others. In this thesis we have developed systematic methods for obtaining the physical parameters of the system from the prescribed spectral data. Thus we have solved the classical vibration problem in "reverse", and consequently the class of problems that we have analysed is known as *inverse vibration problems.*

When the necessary physical parameters for the selected model are determined, the design stage is completed. The next stage is the development process where a prototype of the system is built and is then tested experimentally. The measured spectral properties are then checked for compliance with the design specifications. A common method for measuring the dynamic behaviour of the mechanical systems is the *modal analysis testing*. If the results of modal analysis tests show that the system does not meet the design requirements, then some structural modifications are needed. At present the most common method for determining these modifications is a "trial-and-error" experimental process. The limitation of this approach is that it is costly, time consuming and does not cope well if multiple constraints are placed on the spectral properties. In this thesis we have developed a systematic methods for determining the necessary structural modifications based on the results of modal analysis tests. In these problems we determine the necessary structural modifications using only the physically measured spectral properties, and assume that the analytical model is not available. Therefore the error due to discrepancy between the dynamic behaviour of the selected analytical model and the actual physical system is removed from the calculations.

Using the measured modal analysis data presents several problems. Firstly, since some amount of damping is always present in the physical system, the measured modes will be complex. However, the survey of the relevant literature have shown that there have been several published papers dealing with the methods of extracting the real modes from the measured complex ones. Therefore, we assume that any one of the accepted methods (which are described in the section 3 of this thesis) may be applied prior to the application of our algorithms. Secondly, since modal analysis data do not contain a complete description of the system, there is insufficient information to find the exact modified parameters which yield the desired spectrum. The difficulties arising from the inherently truncated data provided by modal analysis were overcome by formulating suitable optimisation problems, which could then be solved.

A common requirement imposed on the solution to all problems studied in this thesis was that all determined parameters of the system should be *physically realisable*. This requirement resulted in the following two *physical realisability constraints*:

1) all determined masses and stiffnesses must be real and non-negative,
2) the shape of the obtained mass and stiffness matrices must comply with the requirements of the selected analytical model.

Failure to comply with the physical realisability constraint (1) would result in a system which can not be physically reconstructed. And failure to meet the constraint (2) would prevent the translation of the obtained mathematical solution into the real physical system. These constraints were dealt with separately for each problem studied, depending on the chosen analytical models and assumed known data.

Two different kinds of spectral requirements were examined. Initially we have examined cases where only natural frequencies assignments were sought. Then assignment of both the natural frequencies and the corresponding mode shapes was investigated. Application of the two types of spectral constraints to both the design and development stages as described above resulted in the formulation of four distinct problems. Description and definition of these four problems is presented in section 2. Section 3 contains a review of background knowledge and literature survey. Analysis of these four problems is described in sections 4, 5, 8 and 9. In sections 6 and 7 we present solutions to some different variations of the problem described in section 5. Developed algorithms for the solution to the above problems were extensively tested on numerical examples, and the results from some selected few of these examples are presented in appropriate subsections for each problem. The practical application of the developed theory to a real physical structure was confirmed by designing and carrying out a simple experimental program, results of which are presented in section 10. Conclusions and summary of this work is given section 11. The relevant references are listed in section 12, and the raw measured data from the experiments are shown in the Appendices.

Section 2 PROBLEM DESCRIPTIONS AND DEFINITIONS

In this section we describe some common engineering design problems where spectral requirements are included in the design specifications. The current standard design procedures are discussed and some changes to these procedures are suggested. These changes result in the formulation of four distinct inverse vibration problems.

2.1 Problem 1: Design to Achieve Desired Natural Frequencies

Consider the shaft-pulley system shown in Figure 1. Design of such systems for a specified range of loads and power transmission requirements is one of the basic tasks in mechanical engineering.

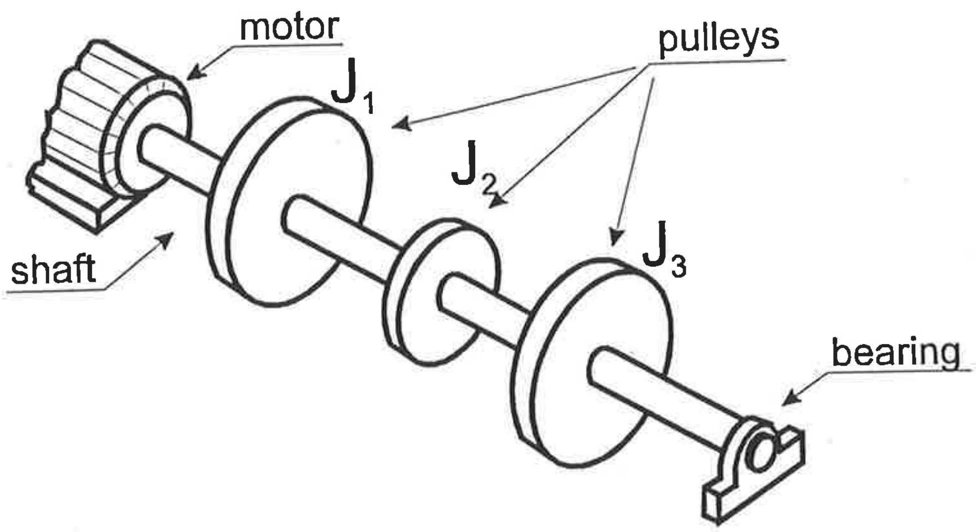


Figure 2.1: A Shaft-Pulley Assembly

Suppose that design specifications state the range of expected rotary speeds of the shaft, and hence restrict the allowed natural frequencies of the system. The current standard design approach for such a system is as follows:

Current Design Process

Input: Total number of pulleys and their position on the shaft; Loads on the system and power transmission requirements; Bands of restricted frequencies.

Design Procedure:

1. Select materials and sizes for all pulleys (Note: this may be predetermined), and hence obtain an estimate of each pulley's mass.
2. Select material for the shaft, and determine the required size(s) of the shaft to operate under the required loading conditions.
3. Select a suitable analytical model for the vibrational analysis of the system, and using information from steps 1 and 2 construct the mass and stiffness matrices of the system.
4. Using the mass and stiffness matrices obtained in step 3 calculate the natural frequencies for the torsional and transverse modes of vibration, and check that these frequencies do not fall within the restricted frequency ranges.
5. If natural frequencies do not fall into the restricted ranges, then design process is finished. If they do, then modify the mass and stiffness matrices and repeat from step 4.

Output (*In theory*): A shaft-pulley system which meets all design specifications.

It is clear that step 5 constitutes an open-ended iterative loop in the above procedure. If the natural frequency restrictions are relatively simple (e.g. that all natural frequencies must be above some specified value), then the above procedure would perform satisfactorily. However, if the design constraints are more complex (e.g. if it is necessary to interlace some or all of the natural frequencies in-between the restricted frequency bands), then the above procedure would not be adequate. This is because there exists no well established method for modifying the mass and stiffness matrices to produce the exactly desired adjustments in the natural frequencies. Therefore, obtaining the desired natural frequencies by mass and stiffness modification would resort to a "trial-and-error" process with no guarantee of ultimate success. We note that the determination of the required mass and stiffness modifications to produce prescribed changes in the natural frequencies constitutes an inverse vibration problem. Thus we propose to formulate a suitable inverse vibration problem, which could be solved, and include the solution for this problem in the current design procedure. This modification to the current design procedure would remove the open-ended loop described in step 5, and ensure availability of solution which meets all design specifications.

We begin the formulation of our problem by selecting a discrete mass-spring system as a model for the shaft-pulley assembly. This choice is made because the mass-spring system is a simplest system to analyse, but which also provides a good model for the dynamic behaviour of such assemblies (especially in the torsional modes of vibration). In this model pulleys will be represented as discrete masses and shaft segments between the pulleys as springs. Therefore, modifications to the mass matrix will only affect the pulleys, while changes to the stiffness matrix will only affect the shaft.

The chosen mass-spring analytical model for a shaft-pulley assembly dictates that the mass and stiffness matrices must comply with the following physical realisability requirements:

1) The mass matrix must be real, positive and diagonal.
2) The stiffness matrix must be real and symmetric, with positive dominant main diagonal and negative elsewhere. (Note:" dominant main diagonal" implies that the sum of all elements in each row (or column) is greater than or equal to zero.)

Also, the choice of the mass-spring analytical model restricts the maximum number of natural frequencies that can be assigned to n, where n is the number of pulleys on the shaft. The natural frequencies of a mass-spring system are then the solutions to the following equation:

$$det\ (\mathbf{K} - \varpi_i^2\,\mathbf{M}) = 0, \quad i = 1,2,...,n. \tag{2.1}$$

where $\mathbf{K}$ is a real, symmetric stiffness matrix (nxn),

$\mathbf{M}$ is a real, positive and diagonal mass matrix (nxn),

ϖ_i is an i-th natural frequency of the system.

To obtain the desired adjustments in the natural frequencies, a designer has a choice of modifying either the shaft, or the pulleys, or both. Since there are more design parameters (i.e non-zero elements in **K** and **M**) than constraints (at most n desired natural frequencies), the equation (2.1) has a family of solutions. We restrict our interest to only those solutions which are physically realisable, i.e solutions which correspond to realistic physical systems. We are also aware that as well as meeting the spectral specifications, an adequate solution must also satisfy other constraints. Those constraints are that the shaft should be able to operate under the specified loads and that it retains a practical geometry and composition. The practical geometry and composition considerations include the following requirements:

a) The manufacture of the shaft must be from a single bar stock, i.e uniform material and "in one piece".

b) The shaft should preferably be of a circular cross-section and its diameter should be kept as uniform as possible, i.e avoid unnecessary "stepping" to minimise manufacture costs and reduce stress concentrations.

c) There should be no obstructions for the assembly of pulleys onto the shaft, i.e no "troughs" in the middle, etc.

d) The shaft-pulley assembly must resemble a simply connected mass-spring system, i.e no cross-connection between non-adjacent pulleys. This imposes the restriction that the stiffness matrix must be tri-diagonal.

e) The shaft-pulley assembly must operate satisfactorily under the specified loads, i.e minimum strength requirements must be met which dictate minimum size limit.

f) To maintain a good correlation with the chosen analytical mass-spring model, the mass of the shaft should be considerably less than the mass of the pulleys, i.e maximum size limit.

Clearly, the majority of the above requirements concern only design of a shaft. The only constraint on the design of the pulleys is that they must meet minimum strength requirements. Since current design procedures ensure that the shaft satisfy all of the above requirements, we propose to restrict the modifications for natural frequency adjustment to pulleys only.

We also note that currently the spectral requirements of the design specifications are only used in checking the obtained solution. They are not used directly in any of the design calculations. Thus we believe that a vital piece of design information is not being fully utilised. An alternative approach that we propose is as follows:

1) Select a desired set of natural frequencies which are well separated from the restricted frequency bands.
2) Determine the required masses of the pulleys which in combination with the stiffness properties of the shaft produce the desired natural frequency spectrum.

The proposed design procedure is then as follows:

Proposed Design Procedure

Input: Total number of pulleys and their position on the shaft; Loads on the system and power transmission requirements; Bands of restricted frequencies.

New Design Procedure:

1. Estimate mass limits for each pulley based on strength requirements.
2. Select material for the shaft, and determine the required size(s) of the shaft to operate under the required loading conditions.
3. Using the information from step 2 construct the stiffness matrix under the assumption of a mass-spring model.

4. Select a set of desired natural frequencies which are well separated from the restricted frequency bands.
5. Using the information from steps 3 and 4 calculate the necessary mass matrix for the system to have the desired natural frequencies.
6. Check that the obtained masses are within the acceptable limits.

Output: A shaft-pulley system which meets all design specifications.

We note that step 5 in the new procedure involves solving an inverse vibration problem. The mathematical formulation of the inverse problem to be solved is then as follows:

Problem 1: Mass matrix evaluation to achieve desired natural frequencies.

Given the real, symmetric stiffness matrix **K** and a set of desired natural frequencies $\{\varpi_1^*, \varpi_2^*, \ldots, \varpi_n^*\}$, find a real, positive and diagonal mass matrix **M**, such that the roots of the characteristic polynomial equation

$$det\,(\mathbf{K} - \lambda\,\mathbf{M}) = 0$$

are $\lambda = \varpi_i^{*2}$ (i=1,2,..,n).

The analysis of this problem is presented in section 4. Although the above problem was formulated specifically in relation to the design of shaft-pulley assemblies, the solution may be applied to the assignment of natural frequencies of any vibratory system which can be represented by a mass-spring model. This fact is demonstrated by the results of the experimental tests on the model of a building which are presented in section 10.

2.2 Problem 2: Design to Achieve Desired Natural Frequencies and Mode Shapes.

Inverse vibration problems associated with the construction of vibratory systems from the known set of desired natural frequencies and mode shapes have many applications in engineering. These include system reconstruction, modification and design. In this thesis we are concerned with the design problem of constructing a physically realisable mass-spring system with prescribed natural frequencies and mode shapes. This problem arises when controlling the maximal deflection of a harmonically excited system.

The natural frequencies and mode shapes of an undamped vibratory system are characterised by the solutions to the following equation:

$$\mathbf{K\Phi} = \mathbf{M\Phi\Lambda}\ , \tag{2.2}$$

where **K** is a positive semi-definite symmetric *stiffness matrix*,

M is a positive definite symmetric *mass matrix*,

$\mathbf{\Phi}$ is a *modal matrix* (which describes the mode shapes of the system),

$\mathbf{\Lambda} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, is a *spectral matrix* (which describes the natural frequencies ϖ_i of the system by relation $\lambda_i = \varpi_i^2$),

n is the number of *degrees of freedom*.

If we stipulate that the modal matrix $\mathbf{\Phi}$ must be *mass-normalised*, then it is well known that the following bi-orthogonality relations hold:

$$\mathbf{\Phi}^T\mathbf{M\Phi} = \mathbf{I}_n \tag{2.3}$$

$$\mathbf{\Phi}^T\mathbf{K\Phi} = \mathbf{\Lambda}. \tag{2.4}$$

For a multiple connected mass-spring system, the mass matrix **M** is real, positive and diagonal. Denote

$$\mathbf{M} = \mathrm{diag}(m_1, m_2, ..., m_n),\ m_i > 0,\ m_i \in \mathbb{R};\ \ i=1,2,...,n. \tag{2.5}$$

The stiffness matrix $\mathbf{K} = [\ k_{ij}\ ]$ is symmetric, and has the following properties:

$$\begin{aligned}
&\text{(a)} \quad k_{ii} > 0,\ i=1,2,...n \\
&\text{(b)} \quad k_{ij} \leq 0,\ i \neq j;\ i=1,2,...,n;\ j=2,3,...,n; \\
&\text{(c)} \quad \sum_{j=1}^{n} k_{ij} \geq 0,\ i=1,2,...,n
\end{aligned} \tag{2.6}$$

In words, **K** has positive diagonal elements, non-positive off-diagonal elements, and it is weakly diagonally dominant.

Suppose we want to determine a physically realisable mass-spring system which has a prescribed eigenvalue matrix $\mathbf{\Lambda}$ with corresponding mode shape matrix $\mathbf{\Phi}$. If we use the orthogonality equations (2.3) and (2.4), we have

$$\mathbf{M} = \mathbf{\Phi}^{-T}\mathbf{\Phi}^{-1} \tag{2.7}$$

$$\mathbf{K} = \mathbf{\Phi}^{-T}\mathbf{\Lambda}\mathbf{\Phi}^{-1}\ . \tag{2.8}$$

However, this solution in general would not be physically realisable. This is demonstrated by the following example:

Example 2.2.1:

Suppose the desired dynamic properties, $\mathbf{\Lambda}^*$ and $\mathbf{\Phi}^*$, for a five degrees-of-freedom mass-spring system are :

$$\mathbf{\Lambda}^* = \text{diag}\ \{\ 50,\ 100,\ 200,\ 400,\ 800\ \}$$

and

$$\mathbf{\Phi}^* = \begin{bmatrix} 0.1 & -0.1 & 0.2 & -0.4 & 0.1 \\ 0.1 & 0.1 & 0.2 & 0.1 & 0.3 \\ 0.1 & -0.1 & 0.3 & 0.2 & -0.4 \\ 0.1 & -0.3 & -0.1 & -0.1 & -0.1 \\ 0.3 & 0.2 & -0.1 & 0.1 & 0.1 \end{bmatrix}$$

We wish to determine physically realisable **M** and **K** which have dynamic characteristics as close as possible to the above desired properties.

There is no exact solution for these data since if we use equations (2.7) and (2.8), we obtain

$$M = \mathbf{\Phi}^{*-T}\mathbf{\Phi}^{*-1} = \begin{bmatrix} 6.6406 & -4.5515 & 1.0830 & -4.7646 & 2.7310 \\ -4.5515 & 13.5005 & -0.3195 & 8.6019 & -4.9737 \\ 1.0830 & -0.3195 & 3.5646 & -1.1215 & 1.0213 \\ -4.7646 & 8.6019 & -1.1215 & 14.5886 & -1.5877 \\ 2.7310 & -4.9737 & 1.0213 & -1.5877 & 8.9672 \end{bmatrix}$$

$$K = \mathbf{\Phi}^{*-T}\mathbf{\Lambda}^*\mathbf{\Phi}^{*-1} = \begin{bmatrix} 2216.1 & -2100.0 & 230.0 & -1446.8 & 476.9 \\ -2100.0 & 6358.5 & -1.6181 & 2763.1 & -1762.7 \\ 230.0 & -1618.1 & 1559.2 & -915.4 & 352.6 \\ -1446.8 & 2763.1 & -915.4 & 2324.0 & -699.3 \\ 476.9 & -1762.7 & 352.6 & -699.3 & 945.7 \end{bmatrix}$$

Clearly, both **M** and **K** do not have the form required for a mass-spring system, and therefore are not physically realisable.

Since equations (2.7) and (2.8) represent the unique solution to equations (2.3) and (2.4) we conclude that generally there is no exact physically realisable solution to this problem. However, we may obtain a physically realisable system with spectral properties that are close to the required data, by solving the following optimisation problem:

Problem 2 **:** Determination of a Physically Realisable System

Given sets of desired eigenvalues $\{\lambda_1^*, \lambda_2^*, ..., \lambda_n^*\}$ with corresponding mass-normalised eigenvectors $\{\phi_1^*, \phi_2^*, ..., \phi_n^*\}$. Denote by

$$\Phi^* = [\ \phi_1^* |\ \phi_2^* |\ ...|\ \phi_n^*\ ] \tag{2.9}$$

the column partitioning of Φ^*, and let

$$\Lambda^* = \mathrm{diag}(\ \lambda_1^*, \lambda_2^*, ..., \lambda_n^*\ ). \tag{2.10}$$

Determine physically realisable **K** and **M** corresponding to a discrete mass-spring system, with spectral properties Φ and Λ satisfying eq. (2.2), such that the norms $\|\Phi^* - \Phi\|$ and $\|\Lambda^* - \Lambda\|$ are minimised.

The analysis of this problem is given in section 5. A related problem associated with the reconstruction of physically realisable systems from the incomplete prescribed modal and spectral data is considered in section 6. In section 7 we define a special form for the mass and stiffness matrices, and then show how the method developed in section 5 can be extended to reconstructing matrices of this form.

2.3 Modifications of the Existing Structures to Obtain Desired Spectral Properties.

Analytical models simulate the behaviour of real physical structures. Application of such models is necessary because we are unable to define the differential equations governing the motion for most practical mechanical systems. In order to define these equations some assumptions about the properties of the real structure, which simplify the analysis, are usually made. The most common technique used in simplifying the analysis of a complex structure is to model this structure as a *lumped parameter* or *discrete* system. In a lumped parameter system the structure is divided into a finite number of discrete elements of known mass, and which are connected to each other by springs and dampers with known stiffness and damping constants. A designer is then able to estimate the forces acting on each element and, thus, obtain the differential equations of motion, which then may be solved.

However, the equations derived in this way may still be too complicated to solve in many cases, and therefore to obtain a solution some other simplifications may have to be made. A commonly used assumption is absence of damping in the structure. Systems without damping are called *conservative* systems, because the energy of the system is not dissipated through damping and thus conserved. Assumption of conservative system greatly simplifies the dynamic response analysis of the system, and in many cases allows a solution to be obtained which would not be possible otherwise. This is especially true in applications with inverse problems.

The number and severity of these assumptions vary greatly depending on the choice of a particular model, and also on the level of sophistication to which this model is developed. However, all assumptions which simplify the analysis of the system also introduce some uncertainty in how accurately does chosen model would simulate the behaviour of a real physical structure. It is well known that the behaviour of all analytical models will differ to some degree from the behaviour of the actual structure. Because of this difference, once the analytical stage is completed, the design process usually requires to build a prototype of the structure and then test this prototype experimentally.

A common method for measuring the dynamic behaviour of vibratory systems is modal analysis testing. The natural frequencies and mode shapes which are measured by modal analysis represent the actual physical dynamic behaviour of the system. Therefore the modal analysis data is free from the inaccuracies due to analytical assumptions. If modal analysis tests show that the dynamic behaviour does not meet design specifications, then some modifications to the system would be necessary. The usual approach at present is to adjust the dynamic behaviour by an experimental trial-and-error process. This process has several disadvantages. The main drawback is that currently there is no systematic method of obtaining modifications which produce exactly the desired changes in the spectral properties. Consequently, if the sought adjustments are relatively complicated, the above process is ill-suited for the task. We want to develop a systematic approach which would allow a designer to calculate the necessary modifications to the structure so that the desired adjustments in the spectral properties are achieved.

In developing such systematic approach, there are two choices. The first approach is to use what is known as a *model updating* technique. In this approach the original analytical model is modified in such a way that its spectral properties closely correlate with the measured modal behaviour. This process is carried out by mathematical manipulations using all of the known data, including the physical and spectral properties of the original analytical model, measured modal analysis data and the desired spectral characteristics. Then based on the assumption that the new updated model is now representing the "true" model of the structure, a designer determines the "corrections" which must be applied to the physical parameters of the original model. The original system is then redesigned for the desired dynamic spectrum incorporating these "corrections". The necessary structural modifications to the prototype are then determined by comparing the redesigned system with the original one. The main weakness of this approach is that it assumes that the calculated "corrections" to the physical properties of the analytical model are constant parameters that can be superimposed from one system to another. This may or may not be true, and can vary from one design problem to another. If the modifications to the prototype determined this way still do not produce the desired spectral characteristics when implemented, then it is difficult to see what should be the next step. Repeating the above model updating process would not guarantee any better results, and due to cost and time limitations the procedure can not be carried out indefinitely.

An alternative approach is to determine the necessary modifications to the structure directly from the modal analysis data only, without using any of the data from the analytical model. In this way any inaccuracies due to the analytical model assumptions are eliminated, since only the measured data is used. The modification matrices must still have the form demanded by the analytical model so that they can be translated into actual physical changes in the structure, but the overall effect of analytical assumptions is minimised.

However, using only modal analysis data creates an additional problem. This problem arises from the difficulty in measuring a "complete" set of modal data for many practical structures. Real physical structures have infinite number of natural frequencies, but due to time and equipment limitations only a finite number of these frequencies can be measured. For example, the maximum sampling rate of the available equipment determines the maximum natural frequency that can be measured. Also, in general, the mode shapes can only be measured at a finite number of points and not continuously along the structure.

Fortunately, in most practical engineering problems the design requirements for spectral properties are restricted to a specified frequency range. This frequency range is usually from zero to some maximum stated value. The spectral properties outside this frequency range are of no interest to the designer, and therefore no restrictions on them are imposed. For the problems considered in this thesis we assume that the number of spectral pairs (i.e natural frequencies and corresponding mode shapes) that are measured is equal to the number of spectral pairs in the design constraints. The difficulties arising from the inherently truncated data provided by modal analysis are overcome by formulating optimal modification problems, which are then solved.

This thesis deals with conservative systems which may be modelled analytically by the generalised eigenvalue problem

$$\mathbf{K\Phi} = \mathbf{M\Phi\Lambda} \ , \ \mathbf{\Phi}^T\mathbf{M\Phi} = \mathbf{I}_n \tag{2.11}$$

where **K**, **M**, $\mathbf{\Phi}$, $\mathbf{\Lambda}$ and n are as defined in equation (2.2). We also stipulate an additional requirement for the diagonal elements of $\mathbf{\Lambda}$, which is that $\lambda_1 < \lambda_2 < \ldots < \lambda_n$.

Partitioning $\mathbf{\Phi}$ and $\mathbf{\Lambda}$ in the form :

$$\mathbf{\Phi} = [\ \mathbf{\Phi}_1 \mid \mathbf{\Phi}_2\ ]\ , \quad \mathbf{\Phi}_1 \ nxm \text{ real matrix, } m < n \tag{2.12}$$

and

$$\mathbf{\Lambda} = \left[\begin{array}{c|c} \mathbf{\Lambda}_1 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{\Lambda}_2 \end{array}\right] \ , \ \mathbf{\Lambda}_1 = \text{diag}(\lambda_1, \ldots, \lambda_m), \tag{2.13}$$

we assume that $\mathbf{\Phi}_1$, measured at n points, and $\mathbf{\Lambda}_1$ are known from modal analysis tests, while the submatrices $\mathbf{\Phi}_2$ and $\mathbf{\Lambda}_2$ cannot be obtained by measurements and remain unknown. Any actual structure will be damped and the measured modes will therefore be complex. We assume that the real modes $\mathbf{\Phi}_1$ have been extracted from these complex modes by one of the accepted procedures (see section 3). We also assume that for the low frequency range, the behaviour of the system is adequately modelled by equation (2.11).

Suppose that a design specification states that the smallest m eigenvalues should be λ_1^*, λ_2^*, ..., λ_m^*. Then we may ask the following question: If $\mathbf{\Phi}_1$ and $\mathbf{\Lambda}_1$ are known, is it possible to determine physically realisable matrices $\mathbf{\Delta M}$ and $\mathbf{\Delta K}$, such that $\mathbf{\Lambda}^*$=diag $(\lambda_1^*,\lambda_2^*,\ldots,\lambda_m^*)$ together with the corresponding modal matrix $\mathbf{\Phi}^*$ satisfy the following equations

$$(\mathbf{K}+\mathbf{\Delta K})\mathbf{\Phi}^* = (\mathbf{M}+\mathbf{\Delta M})\mathbf{\Phi}^*\mathbf{\Lambda}^* \quad \text{and} \quad \mathbf{\Phi}^{*T}(\mathbf{M}+\mathbf{\Delta M})\mathbf{\Phi}^* = \mathbf{I}_m \ ? \tag{2.14}$$

Since the truncated modal data $\mathbf{\Phi}_1$ and $\mathbf{\Lambda}_1$ do not determine **K** and **M** uniquely, it is clear that (2.14) cannot be satisfied generally. We overcome this difficulty by formulating a following related optimisation problem, which gives an optimal approximation, in some particular sense, to the solution of (2.14).

Similar to Ram and Braun [46], let us denote the residual matrix **R** by:

$$R = [(K + \Delta K)\bar{\Phi} - (M + \Delta M)\bar{\Phi}\bar{\Lambda}] \tag{2.15}$$

where $\bar{\Lambda}$ and $\bar{\Phi}$ are some approximations to the desired Λ^* and the corresponding Φ^* respectively. Then we formulate the following problem:

Problem 3: Optimal Modification for Natural Frequencies

Given Φ_1, Λ_1 and Λ^*. Find physically realisable incremental matrices **ΔK** and **ΔM** such that:

$$\| (\mathbf{M}+\Delta\mathbf{M})^{-\frac{1}{2}}\mathbf{R} \|_F \text{ is minimised} \tag{2.16}$$

Subject to $\bar{\Phi} \in \text{span}(\Phi_1)$.

The requirement that $\bar{\Phi}$ belongs to the column range of Φ_1, is needed to make the problem solvable.

If $\| \mathbf{R} \| = 0$, then so is $\| (\mathbf{M}+\Delta\mathbf{M})^{-\frac{1}{2}}\mathbf{R} \|$. In this case $\bar{\Phi} = \Phi^*$ and $\bar{\Lambda}=\Lambda^*$, i.e the solution of Problem 3 is also the solution of (2.14). When $\| \mathbf{R} \|$ is small, the solution of Problem 3 approximates the solution of (2.14). Hence by minimising the norm (2.16) we obtain an *optimal approximation.* In fact, the residual matrix **R** is weighted by $(\mathbf{M}+\Delta\mathbf{M})^{-\frac{1}{2}}$, this is for convenience purposes only. The analysis of this problem is presented in section 8. The solution methods are discussed for several commonly used analytical models (including mass-spring and finite elements). The important case where the mass and the stiffness matrices are interrelated is also studied. This analysis may be applicable to the problems of modifying the axially vibrating rod and the transversely vibrating beam.

When modal matrix Φ^* is also included in the design specifications, the corresponding optimisation problem becomes:

Problem 4: Optimal Modification for Natural Frequencies and Mode Shapes

Given Φ_1, Λ_1, Φ^* and Λ^*. Find physically realisable incremental matrices **ΔK** and **ΔM** such that:

$$\| (\mathbf{M}+\Delta\mathbf{M})^{-1/2}\mathbf{R} \|_F \text{ is minimised} \tag{2.17}$$

Subject to $\bar{\Phi} \in \text{span}(\Phi_1)$.

Problem 4 was studied by Ram and Braun in [46]. The authors have shown that there is a family of solution to this problem, and that it is characterised by the following equations:

$$\Delta M = \Phi_1^{T\dagger}\left(\Psi^{-T}\Psi^{-1} - I_m\right)\Phi_1^{\dagger} + Y - \Phi_1^{T\dagger}\Phi_1^{T}Y\Phi_1\Phi_1^{\dagger} \tag{2.18}$$

$$\Delta K = \Phi_1^{T\dagger}\left(\Psi^{-T}\Lambda^{*}\Psi^{-1} - \Lambda_1\right)\Phi_1^{\dagger} + X - \Phi_1^{T\dagger}\Phi_1^{T}X\Phi_1\Phi_1^{\dagger} \tag{2.19}$$

where $\mathbf{\Phi}_1^{\dagger}$ denotes the Moore-Penrose pseudoinverse of $\mathbf{\Phi}_1$, $\mathbf{\Psi}= \mathbf{\Phi}_1^{\dagger}\mathbf{\Phi}^*$, and $\mathbf{X}$ and $\mathbf{Y}$ are arbitrary *m*x*m* real symmetric matrices.

The family of solutions described by equations (2.18) and (2.19) contains all possible solutions to Problem 4. However, we are only interested in these solutions which are physically realisable. In [46] Ram and Braun have discussed the requirements for physical realisability, but no method was developed for extracting realisable solutions from the general family described by (2.18) and (2.19). In this thesis our aim was to develop such method for extracting realisable solutions, and the its derivation is given in section 9. It was found that the physically realisable solutions for $\mathbf{\Delta M}$ and $\mathbf{\Delta K}$ are, in fact, independent of the arbitrary matrices $\mathbf{X}$ and $\mathbf{Y}$.

Section 3 *LITERATURE SURVEY*

The scope of this thesis falls within the field of *inverse vibration problems*, and a closely related field in linear algebra known as *inverse eigenvalue problems*. The inverse vibration problems have applications, among others, in the areas of *design*, *model reconstruction* (also referred to as *system identification* problems), *structural modifications* and *model updating* of vibratory systems. The *design* and *structural modifications* problems aim to control the vibrations of the systems in a desired fashion. The *model reconstruction* and *model updating* problems aim at obtaining an optimal analytical model, which approximates as close as possible the vibrations of the actual system. The book of Gladwell [1] and two review papers by the same author [2,3] give an excellent introduction and overview of the current state of knowledge in the general field of inverse vibration problems, and in the area of *model reconstruction* in particular. We draw extensively on the comprehensive material from these sources.

3.1 Inverse Problems

The term *inverse* is used to distinguish these problems from the classical problems in vibration and in linear algebra, which are commonly known as *direct* problems. In the classical direct problems the aim is to determine the *behaviour* of a system (such as natural frequencies and/or mode shapes) from known *physical properties* of the system (e.g. mass, density, elastic constants, size etc). Inverse problems are concerned with the determination or estimation of such physical properties from the desired and/or experimentally measured behaviour. The *design* problems aim to develop methodical algorithms for determining the physical properties of a system from the prescribed desired behaviour. In the *structural modifications* problems the aim is to develop algorithms which use measured modal data (usually from experimental modal analysis testing of a prototype) to determine the necessary modifications to the physical properties which produce the prescribed or desired behaviour. The *model reconstruction* problems concern with the methods of obtaining the physical properties of the system from the measured experimental data, while in the *model updating* problems the aim is to minimise the difference between the measured behaviour and the theoretical behaviour of the existing analytical model.

In the above definition of the inverse problems, the term *determination* refers to problems where the sought properties can be computed exactly from the given behaviour. In these idealised, essentially mathematical problems the fundamental assumption is that all of the required data is *exact* and *complete*, meaning that it is sufficient to determine the system uniquely. The vibration analysis of most engineering problems usually contains a significant amount of uncertainty. This uncertainty arises from a lack of detailed knowledge about the vibratory system itself (namely the shape of its mass, stiffness and damping matrices), the desired behaviour (which is usually specified only approximately in terms of permitted or restricted ranges), and the experimentally measured data (which always contains some errors due to noise, equipment limits etc). Therefore, in most engineering problems the necessary data is inaccurate and incomplete, thus, at best, permitting only an *estimation* of the sought properties.

Most published work in the field of inverse vibration problems (including the material presented in this thesis) may be classified as a combination of mainly the *determination* approach, with some elements of the *estimation* approach. Thus, in order to make the problems solvable and depending on a type of a particular problem, most of the necessary data (e.g. the shapes of the mass and stiffness matrices, the prescribed natural frequencies and/or mode shapes, etc) are assumed to be known, exact and complete. At the same time, the essential physical limitations (such as a requirement that the values of the mass and stiffness elements must be real and positive, and the inherent perturbations and truncations in the measured modal analysis data) are taken into consideration.

In [1-3] the inverse vibration problems are categorised according to the type of the mechanical systems (namely *continuous* or *discrete*, *damped* or *undamped*), and the type of the prescribed behaviour (namely *spectral* or *modal* (or both), *complete* or *truncated*, *nodal* or *isospectral*). *Nodal* inverse problems concern with the reconstruction of physical properties of the system from data relating to the position of the nodes. The term *isospectral* refers to the studies of distinct vibratory systems that have the same eigenvalues. Some discussion was also given of applications of the developed system identification techniques to *fault detection* problems.

The paper by Chu [4] gives a very good, up-to-date review of the closely related field of inverse eigenvalue problems. Although this paper is not specifically directed towards the subject of vibration analysis, it contains a substantial amount of information on the latest developments in this field, especially in the area of *design* and control.

3.2 Inverse Multiplicative Eigenvalue Problem

Of particular importance to the scope of this thesis, is the so called *inverse multiplicative eigenvalue problem (IMEP)*. This problem was first formulated by Downing and Householder [5] in 1956. Their definition of the IMEP was as follows:

> Given a real and symmetric matrix **A**, determine a real, positive and diagonal matrix **D**, such that the equation
>
> $$\det(\mathbf{A} - \lambda\mathbf{D}) = 0 \qquad (3.1)$$
>
> has prescribed roots.

Comparison of the above definition of IMEP with the definition of Problem 1 given in section 2, shows that the two problems are identical. In [5], an iterative algorithm for a solution to IMEP was presented, and this algorithm was shown to have a local quadratic convergence. When tested on the numerical examples, this local convergence characteristics of the algorithm was found to be inadequate for a robust solution to Problem 1. A further discussion of this algorithm and its performance is given in section 4, where we develop a solution to Problem 1.

Over the following years since its first formulation, the IMEP has attracted interest from several researchers, and a number of papers were published dealing with various aspects of the problem. The formulation of IMEP was expanded and generalised to include complex and non-symmetric matrices **A** and complex diagonal matrices **D**. Several papers were also published dealing with the necessary and sufficient conditions criteria for the existence of a solution to a given IMEP.

Hadeler [6] has defined and proved some sufficient conditions for existence of a solution to an IMEP. An alternative algorithm for the solution was also presented, along with the statement of convergence criterion. From the numerical example given in [6], it appears that the developed algorithm works, but it seems to be more complicated than the method of [5].

Kublanovskaja [7] suggested a general approach to the solution of the so called *generalised inverse eigenvalue problem*, of which IMEP was a specific case. However, due to the more general nature of the problem, the presented algorithm appears to be quite complex. This algorithm is broadly based on the method of Hadeler [6].

De Oliveira [8] expanded the results of Hadeler [6] for the sufficient conditions for a solution to IMEP. The results in [8] allow for the application of the conditions of [8] and the algorithm of [6] to cases where matrix **A** is non-symmetric. It was shown that the results are also valid for the solution to a so called *inverse multiplicative permanent root problem* (i.e the case when $\mathbf{D} = \mathbf{I}_n$). Although the results of [8] are not directly applicable to Problem 1, this paper provides a good explanation to the results of [6].

A major contribution to the analysis of IMEP was made by Friedland [9,10]. In [9], the author presented a proof of the existence of a solution to an IMEP when both **A** (which is not necessarily symmetric) and **D** are complex-valued matrices, and when all principal minors of **A** are distinct from zero. The paper also contains the mathematical proof that the number of different matrices **D** (i.e. the maximum number of distinct solutions to an IMEP) is at most *n!*. This last result is very important for our analysis of Problem 1, and its significance is further discussed in section 4. In [10] the same author suggested a solution for an IMEP when **A** and **D** are both complex-valued. Since in our Problem 1 the requirement is for both **A** and **D** to be real, this solution is not applicable to our analysis.

Dias da Silva [11] extended the results of Friedland [9,10] for the sufficient conditions to IMEP for complex-valued matrices, and He Xuchu and Dai Hua [12] continued the work of Hadeler [6] and De Oliveira [8] to find better sufficient conditions for the solution to IMEP. The method of proof in [12] is similar to Hadeler's. The presented numerical example illustrates that there exists a solution to IMEP according to the conditions established in [12], even though conditions given in [6] and [8] are not satisfied. Based on this example, the authors conclude that their conditions are better than those of [6] and [8]. However, in general, the sufficient conditions established in [6] are simpler to calculate.

Biegler-König [13] defined the sufficient conditions for the solution of a generalised inverse eigenvalue problem, of which IMEP is a special case. Formulation and a numerical solution method for this problem was proposed by Kublanovskaja in [7]. While Friedland in [9,10] has proved some conditions for a complex-valued IMEP, in [13] the author is investigating conditions for obtaining a real-valued solution. The method of proof is similar to Hadeler's in [6], and some results of [6] are special cases of the theorems in [13]. The *generalised inverse eigenvalue problem* in this paper is defined as follows:

> Let $\mathbf{A}_i$ be real matrices (i=0,1,2,...,n), and a set of prescribed real eigenvalues $\{\Lambda_1^*, \Lambda_2^*, ..., \Lambda_n^*\}$. Find a set of real parameters $\{c_1, c_2, ..., c_n\}$, such that a matrix
>
> $$A(c) = A_0 + \sum_{i=1}^{n} c_i A_i \tag{3.2}$$
>
> has eigenvalues $\{\Lambda_1^*, \Lambda_2^*, ..., \Lambda_n^*\}$.

The results in [13] were given for general real matrices $\mathbf{A}_i$ and a special case when matrices $\mathbf{A}_i$ are both real and symmetric. The IMEP may be expressed in the form required by equation (3.2), by setting $\mathbf{A}_0 = 0$ and $\mathbf{A}_i = \boldsymbol{a}_i \boldsymbol{e}_i^T$ (i=1,2,...,n), where $\boldsymbol{a}_i$ is the ith row vector of $\mathbf{A}_i$ and $\boldsymbol{e}_i$ is ith element unit vector (i.e the ith element in $\boldsymbol{e}_i$ is equal to 1, and all other elements are zeros). However, the resultant matrices $\mathbf{A}_i$ will not, in general, be symmetric. Therefore, it was specifically stated in [13] that a general IMEP described by (3.1) may not be expressed in the form of equation (3.2) if matrices $\mathbf{A}_i$ are to be symmetric. In the review paper by Chu [4] (see above) eigenvalue problems of the form (3.2) were classified as *parametrized inverse eigenvalue problems* (in reference to parameters $\{c_1, c_2, \ldots, c_n\}$), thus dropping the use of the term *generalised.*

Nocedal and Overton [14] gave a summary of the numerical methods for solving eigenvalue problems of the form (3.2), and in a subsequent paper, Friedland, Nocedal and Overton [15] presented a general overview and comparison between the various solution approaches to this problem. A total of four different methods of solution were presented in [15], three of which were published previously and one new. The analysis was given for the problems where the desired eigenvalues were all distinct, and also for the case when there are multiple identical eigenvalues. A general convergence analysis was carried out for both the distinct and multiple eigenvalues. The comparison between the four methods was presented, and a numerical example was given which illustrated the results.

Although the results of [15] were not directly applicable to Problem 1, Joseph [16] has used one of the methods presented in [15] as a basis for an algorithm to solve what he called *inverse eigenvalue problem in structural design.* The author have assumed that the mass and the stiffness matrices of a structure, **M** and **K**, are functions of n independent structural parameters $\{c_1, c_2, \ldots, c_n\}$, and comply with the following forms

$$M(c) = M_0 + \sum_{i=1}^{n} c_i M_i \tag{3.3}$$

$$K(c) = \sum_{i=1}^{n} c_i K_i \tag{3.4}$$

Some known results from the *eigenvalue sensitivity* theory were then applied to determine $\{c_1, c_2, ..., c_n\}$ such that the resulting eigenvalues of the system described by **M** and **K** were equal to the prescribed set of desired eigenvalues $\{\lambda_1^*, \lambda_2^*, ..., \lambda_m^*\}$ $(m \leq n)$. The method of [16] is directly applicable to solve Problem 1. Numerical testing have shown that the performance of this method has the same problematic local convergence characteristics as the algorithm of Downing and Householder in [5]. However, one major advantage of this method is that it permits assignment of a partial set of desired eigenvalues (i.e when $m < n$), thus allowing application of the method to problems with truncated modal data. Because of this property, the algorithm was used as a foundation for our solution to Problem 3. Further discussions of the Joseph's algorithm and its application to Problems 1 and 3 are given in the appropriate parts of sections 4 and 8.

3.3 Sensitivity Methods

The *eigenvalue sensitivity* theory, referred to above, and the associated subject of *eigenvalue derivatives* was used extensively in this thesis. In its simplest form, the principle of matrix derivatives is demonstrated by a following example. Suppose that we have matrix **A** which is a function of n independent parameters $\{c_1, c_2, ..., c_n\}$ and can be expressed in the form of equation (3.2). Then it is clear that

$$\frac{\partial A(c)}{\partial (c_i)} = A_i \quad , \quad i=1,2,...,n \ . \tag{3.5}$$

Matrices $\mathbf{A}_i$ are commonly called *connectivity* or *mapping* matrices, describing the connections of each individual parameter c_i or simply defining the grid position of this parameter within the global matrix $\mathbf{A}(c)$. Determining matrix derivatives of $\mathbf{A}(c)$ with respect to parameters c_i is a simple process because (3.2) is a linear and uncoupled equation in c_i. In the eigenvalue sensitivity analysis, the aim is to obtain the relations describing the effect of each of the elements of a matrix on the eigenvalues of that matrix. However, determining the eigenvalues of a matrix is not a linear process. Therefore, there is no exact way of describing the effect of a particular element on the eigenvalues. Consequently, most of the published work on this subject concerned with determining an approximation to these relations, which are optimal in some way. It is clear that obtaining an accurate relations describing the effect of an individual structural parameter (i.e mass, stiffness or damping element) on the natural frequencies and mode shapes (i.e eigenvalues and eigenvectors) is the key to a solution of most problems in inverse vibrations applications to *design* and *structural modifications.*

In a review paper on structural modifications problems, Baldwin and Hutton [17] give a good summary of the literature on eigenvalue sensitivity methods. This paper is primarily concerned with what may be described as *direct structural modification problems.* These are problems in which the aim is to determine the dynamic behaviour of a system based on known changes in the structural parameters and the measured behaviour of the original unmodified structure. The *inverse structural modifications,* where the aim is to obtain the structural modifications necessary to meet specified constraints on natural frequencies and mode shapes, are only briefly discussed. However, most of the theory on sensitivity analysis is equally applicable to inverse problems. The material presented mainly deals with conservative (i.e undamped) systems, and is classified into three main categories. Namely, the techniques based on the *assumption of small modifications*, these for *localised modifications*, and finally, techniques using the *modal approximation* approach.

The techniques for *localised modifications* deal with determining the dynamic behaviour of the modified system based on the precise knowledge of the location of the structural changes in addition to their magnitude. An interesting characteristic of these problems is that, apparently, the modified behaviour can be determined exactly, although only by an iterative procedure. In all other categories only approximate solutions may be obtained. In applications involving inverse problems a major problematic area is how to satisfy the physical realisability constraints on the necessary modifications based only on the knowledge of the desired and measured behaviour of the system. Thus, specifying additional constraints on the exact location of modifications would add an extra layer of difficulty to what is already a complicated problem. Therefore, in this thesis we have not attempted to specifically formulate or to solve any of the *inverse localised* problems. However, some limited control over the location of structural modifications is possible under some circumstances in the analysis of sections 6 and 9. The reader is referred to these sections for further details.

The category based on the assumption of *small modifications* dealt exclusively with the sensitivity methods. This category was further subdivided into three separate approaches, namely methods based on *Rayleigh's principle*, methods based *on eigenvalue derivatives*, and finally, methods based on *modal perturbation theory*. All three approaches lead to very similar formulations and are discussed below.

3.3.1 Rayleigh's method

The Rayleigh's approach is based on the equation

$$\mu_i = \frac{\psi_i^T(K + \Delta K)\psi_i}{\psi_i^T(M + \Delta M)\psi_i} \tag{3.6}$$

where **M** and **K** are the mass and stiffness matrices of the original system, **ΔM** and **ΔK** are the matrices denoting respectively modifications to the mass and stiffness, and μ_i and ψ_i are the ith eigenvalue and the corresponding eigenvector of the modified system.

It is assumed that, for small modifications, the mode shapes do not change appreciably and therefore we may substitute $\psi_i = \phi_i$ into (3.6). Also, applying the orthogonality properties $\phi_i^T \mathbf{K} \phi_i = \lambda_i$ and $\phi_i^T \mathbf{M} \phi_i = 1$ (where λ_i and ϕ_i are the ith eigenvalue and eigenvector of the original system), equation (3.6) becomes

$$\mu_i = \frac{\lambda_i + \phi_i^T \Delta K \phi_i}{1 + \phi_i^T \Delta M \phi_i} \tag{3.7}$$

Thus, if **ΔM** and **ΔK**, and λ_i and ϕ_i, are known, then μ_i can be calculated. Clearly, if the modifications are not small then the assumption that $\psi_i = \phi_i$ is not valid, and consequently, the method should not be applied.

3.3.2 Eigenvalue derivative method

The *eigenvalue derivative* approach considers the structural modification problem in terms of a rate of change of an eigenvalue with respect to a structural parameter change. It was shown in [17] that

$$\frac{\partial \lambda_i}{\partial c_j} = \phi_i^T \left[\frac{\partial K}{\partial c_j} - \lambda_i \frac{\partial M}{\partial c_j} \right] \phi_i \qquad (3.8)$$

where c_j is some j^{th} structural parameter.

This relation may then be used in a Taylor series expansion to give a first or second order approximation to the natural frequencies of the modified structure. Early work in this area was conducted by McCalley [18] and Wittrick [19], and, in general, the mathematical foundations have been discussed by Lancaster [20]. A major treatment of the entire problem, including the calculation of the mode shapes of a modified structure, was presented by Fox and Kapoor [21]. In [21] a first-order solution was considered. Two methods were derived to calculate the mode shape derivatives. The first method expressed the eigenvector derivatives in terms of a series expansion in the unmodified eigenvectors and, hence, required the knowledge of the full modal characteristics of the original structure (although truncation is possible). A second method expressed the eigenvector derivative only in terms of the corresponding frequency and eigenvector of the original structure. However, although the second method was potentially more attractive, some numerical difficulties were encountered which prevented its successful implementation. These difficulties were eventually solved by Nelson [22].

Van Belle [23] presented a theory of adjoint structures to calculate the differential sensitivities of mechanical structures. This work has been expanded by Van Honacker [24] to derive expressions for the differential, finite difference and frequency response sensitivities for natural frequencies and mode shapes of a viscously damped vibratory system. Second-order terms of Taylor expansion are included in the analysis to obtain expressions for "large-change" sensitivities.

Second- and higher-order solutions have also been investigated by Rudisil [25, 26], Muira and Schmit [27], Van Belle [28] and Rizai and Bernard [29]. Wang, Heylen and Sas [30] summarised the developed procedures, but found that the methods based on truncated Taylor expansions are limited in their applications to small modifications, and that inclusion of higher-order terms does not always ensure a more accurate solution.

To and Ewins [31] used the closed-form properties of the Rayleigh's method and the theoretical basis of the eigenvalue and eigenvector derivatives analysis, to develop a powerful iterative algorithm, which is not restricted to just small modifications. Instead of assuming that the eigenvectors of a modified system remain unchanged, as in the Rayleigh's method above, the authors express modified modes as linear combinations of the original modes. The coefficients in these linear combinations of original modes were termed *mode participation factors.* The method involves (starting initially from equation of the form (3.7)) obtaining an estimate for the modified eigenvalues of a system, and then using these estimates to calculate the mode participation factors. The mode participation factors are then in turn used to calculate the better approximations to the modified eigenvalues. The procedure is then repeated until convergence is achieved. The authors claim that this procedure has 'superconvergence characteristics', and that the exact modal properties of the modified structure may be determined. The effects of modal truncation and sensitivity to input data perturbations are also presented.

Zimoch [32] used the first-order Taylor expansion to determine the sensitivity matrices for the eigenvalues and mode shapes. The effects of the structural modifications on the dynamic behaviour of the system could then be estimated in a computationally efficient way. The method could be applied to damped as well as conservative systems. A more accurate solution may be achieved by inclusion of a second-order term of the Taylor's expansion. This inclusion does not require any alterations to the procedure, but the penalty is the necessity to carry out much more involved calculations. In [33] Zimoch applied the method of [32] to solve an inverse problem. The formulation of his problem is very similar to our Problem 2, except that no physical realisability constraints were imposed on either mass or stiffness matrices. Without these constraints a solution can be obtained by a trivial process described in section 2.2 (see equations (2.7) and (2.8)). However, the motivation of Zimoch was to develop a more computationally efficient method for determining the changes in the physical parameters of a system to achieve the desired eigenvalues and eigenvectors.

Joseph [16] (see above) used equation (3.8) to develop an iterative algorithm which solves an inverse problem, and, in theory, is not restricted to small modifications. The method required an estimate of the initial values for the structural parameters $\{c_1, c_2, ..., c_n\}$. Then the mass and stiffness matrices, **M** and **K**, may be constructed via equations (3.3) and (3.4).

Using equation (3.8) and an example of (3.5), the eigenvalue derivatives were calculated for the system described by **M** and **K**. Application of a Newton-Raphson method using these eigenvalue derivatives, allowed calculations of better estimates for $\{c_1, c_2, \ldots, c_n\}$. This procedure could then be repeated indefinitely, until convergence was achieved.

3.3.3 Perturbations methods

In the perturbation approach it is assumed that the mass and stiffness matrices of the modified system , $\mathbf{M}_{\mathbf{mod}}$ and $\mathbf{K}_{\mathbf{mod}}$, and the corresponding eigenvalue and eigenvector matrices, $\boldsymbol{\Omega}$ and $\boldsymbol{\Psi}$, are related to the properties of the original system by

$$\begin{aligned} \mathbf{M}_{\mathbf{mod}} &= \mathbf{M} + \Delta\mathbf{M} & \mathbf{K}_{\mathbf{mod}} &= \mathbf{K} + \Delta\mathbf{K} \\ \boldsymbol{\Omega} &= \boldsymbol{\Lambda} + \Delta\boldsymbol{\Lambda} & \boldsymbol{\Psi} &= \boldsymbol{\Phi} + \Delta\boldsymbol{\Phi} \end{aligned} \tag{3.9}$$

It is also assumed that for small modifications, all $\boldsymbol{\Delta}$ terms are sufficiently small. Thus, substituting properties (3.9) into an equation of the form (2.2) governing the motion of the modified system, and neglecting all terms of $\boldsymbol{\Delta}^2$ and higher, it is possible to obtain an approximate relation between the structural modification $\Delta\mathbf{M}$ and $\Delta\mathbf{K}$ and $\boldsymbol{\Omega}$ and $\boldsymbol{\Psi}$. The precise form of such relationship is dependent on the assumptions and conditions of a particular problem, and thus, are too numerous to be given here.

The first treatment of the structural modification problem was by Rayleigh [34], who used this type of approach and derived an approximate solution in terms of modal coordinates from energy expressions. Jones [35] extended this work to include general perturbations and derived expressions for natural frequency and mode shape changes. Romstad, Hutchinson and Runge [36] investigated a variety of more general perturbation formulations using a power series approach.

Stetson and Harrison [37] have extended the modal perturbation approach to treat the inverse problem of determining the structural modifications necessary to meet the specified constraints on natural frequencies and mode shapes. The method of [37], uses the results from NASTRAN finite element analysis software to determine the analytical model of the original structure. It then processes these data, taking account of the physical realisability constraints, to obtain the necessary changes in the thicknesses of the structural elements. The aim of the method is to minimise the necessary structural changes, while obtaining the desired dynamic properties. Sandstrom and Anderson [38] extended this work by directly relating the physical changes in the natural frequencies and mode shapes to changes in structural parameters.

Sandstrom, Anderson and others [39,40] reported that perturbation approach based on the linear energy formulation gives good accuracy for natural frequency goals, but is often not accurate for significant mode shape changes. In [39] the authors have extended the linear perturbation approach of [37], including all of the non-linear terms in the perturbation equation. This was done because, as was demonstrated by a numerical example of a typical problem, the second-order terms can be as large as the first-order terms. Thus neglecting second-order terms may lead to large errors, particularly in redesign of mode shapes. In [40] the authors have developed a non-linear, iterative algorithm, which was partitioned into two stages, namely the 'predictor' and the 'corrector' stages. The 'predictor' phase is essentially an improved version of the algorithm in [37], which gives a first-order approximation for the required structural changes. In the 'corrector' phase, these approximations are used to calculate a first-order estimate for the desired eigenvectors. These eigenvectors are then used in general perturbation equations to find the corrections for the structural changes. The process is then repeated as many times as necessary to achieve the acceptable dynamic behaviour. The algorithm of [40] requires precise knowledge of the physical properties of the original structure (i.e its finite element model) as an input to the problem.

Zhang, Wang, Allemang and Brown [41] used the perturbation approach to find an approximate solution to a problem which is identical to our Problem 3. The method uses power expansion of a perturbation equation to find the necessary mass modifications so that the desired natural frequencies are achieved. Results are presented for both the first and the second order approaches. The input to the problem was assumed to consist of only the specified desired natural frequencies, and a truncated set of measured modal analysis data. The method of [41] is applicable to any damped or conservative system, whose mass matrix is diagonal, and where the mass and stiffness matrices are independent of each other (e.g. mass-spring system). The algorithm also allows to control the location of mass modifications, and an optimisation procedure for the best locations and magnitude of mass modifications is given. However, the method is based on the perturbation approach and its performance is acceptable only if the sought changes in the natural frequencies are relatively small. Numerical simulations have shown that this method performs inadequately in a general case, when desired changes are not sufficiently small. Therefore, our aim was to develop an alternative method which would not have this limitation.

3.4 Modal Approximation Methods

The *modal approximation* methods are based on the assumption that the eigenvectors of the original unmodified structure form a complete vector basis to describe the motion of any modified structure. Mathematically, this assumption implies that the eigenvectors of the modified system belong to the space spanned by the eigenvectors of the original structure. We have also used this assumption in our formulation of Problems 3 and 4. This approach allows us to obtain an approximate solution to these problems, which is optimal in a Rayleigh-Ritz sense. Our solutions are based on the theory developed by Parlett [42], and which is described in detail at the beginning of section 8.

The results presented in sections 8 and 9 of this thesis, are the extension of the work done by Ram and Braun [43 - 46] in this field. In [43 - 46] the authors dealt with problems arising specifically when the necessary structural modifications are determined based on the modal analysis data, and assuming no knowledge of any other information about the structure. It was shown by Berman [47], that even under the most favourable laboratory conditions, there are severe limitations on obtaining modal analysis data which is complete, i.e. which completely describes all modes of the system. Therefore, the use of modal analysis results alone (i.e without any additional information from other sources) inevitably introduce the problems of *modal truncation errors*. The presence of *modal truncation errors* imply that there is insufficient information to find the exact values for the physical parameters of a system from the measured modal analysis data. Thus, the difficulties arising from the inherently truncated data provided by modal analysis may only be overcome by formulating problems for approximate solutions which are optimal in some specified sense.

Another problem with using modal analysis data, arise when an assumption is made that a system under consideration is conservative. This assumption of a conservative system was made in [43 - 46] and also in our formulation of Problems 3 and 4. The eigenvalues and eigenvectors of a conservative system are real-valued, whereas eigenvalues and eigenvectors of a damped system are complex-valued. Since any actual structure will always have some degree of damping, the measured modal data is always complex-valued. In the analysis of [43 - 46] and in our analysis of sections 8 and 9, we assume that the real-valued modes may be extracted from these measured complex modes. There are a number of available methods for such extraction, ranging from complicated mathematical procedures to a very simple process of truncating the imaginary part. Zhang and Lallement [48] present a summary of three extraction methods, and describe the comparison of their relative performance. The application of the these methods to a test structure showed that the results are sufficiently accurate. However, due to the requirements placed on the input data in two of the methods, they were judged to be of mathematical interest only. The third method was considered suitable for engineering applications. In a recent paper by Ahmadian, Gladwell and Ismail [49] it was shown analytically that a real mode most correlated with a complex measured mode is the real part of the same complex mode when it is rotated so that the norm of its real part is maximised. Clearly, an assumption of a conservative system is suitable only for the lightly damped structures. If damping in the structure is not negligible, then the real modes extracted by any of the above methods are completely different from the mode shapes of a conservative analytical model. Consequently, large errors will result from any attempt to correlate this fundamentally different data.

In [43] Ram and Braun used the result of Parlett [42] to formulate and to solve a direct structural modification problem. The developed algorithm yields an approximate solution which is optimal in a Rayleigh-Ritz sense. In [44] same authors derived the upper and lower bounds on eigenvalues (i.e natural frequencies) of a modified structure based on truncated modal testing results. In [43] it was shown that a solution which is optimal in a Rayleigh-Ritz sense provides an upper bound for the predicted natural frequencies of a modified structure. In [44] a method for obtaining the lower bounds for the natural frequencies was developed, and a procedure for predicting a modal truncation error was presented. In [45] the authors obtained bounds on the eigenvectors due to structural modification. In [46] a method developed in [43] was applied to an inverse modification problem. This inverse problem is identical to our Problem 4. The authors were able to characterise all possible family of solutions to this problem (see section 9 for the equations characterising those solutions for $\mathbf{\Delta M}$ and $\mathbf{\Delta K}$). However, although constraints for physically realisable solutions were stated and an optimisation problem for determining such solutions formulated, this optimisation problem was not solved and consequently physical realisability constraints were not enforced. In section 9 we give an alternative formulation for the problem of extracting realisable solutions from the family characterised in [46]. This alternative problem is then solved, thus complementing the results of [46].

Tsuei and Yee [50] presented a solution for a single parameter modification (either a mass or stiffness element) of a conservative system. The method is based on the force response of the original system, and allows to shift one natural frequency to a prescribed value. Since the method does not require iterations, it is computationally efficient. However, due to the coupling between the modes, a prescribed shift in one natural frequency causes other natural frequencies to shift as well, and these "secondary" shifts are not controllable. Thus, this method can not be used for assigning multiple (i.e. more than one) natural frequencies. In [51] same authors extended the results of [50] to applications with damped systems. The new algorithm does require iterations, but it is claimed that it converges very fast. Ram [52] considered the problem of how to enlarge a spectral gap of some vibrating continuous and discrete systems (including taut spring, non-uniform beam and a mass-spring system) by introducing two appropriate oscillators at the proper locations..

A different approach to a structural modification problem was presented by Coppolino [53]. The author uses the measured truncated modal data and the stiffness matrix from the finite element model of the original structure to determine the so called *residual modal matrix.* The residual modal matrix is obtained by substituting unit load vectors at the location and instead of the required structural modifications. The measured modal matrix, augmented by the residual matrix, then describe an exact static response characterisation of the original structure due to application of unit loads. This is, in effect, equivalent to obtaining a static response of the modified structure. The numerical example given in [53] for a 1416 degrees-of-freedom system demonstrated the application of the developed method.

An equivalent vector form of equation (2.2) for a conservative system is

$$\mathbf{K}\phi_i = \varpi_i^2\mathbf{M}\phi_i \tag{3.10}$$

or, alternatively

$$\mathbf{M}^{-1}\mathbf{K}\phi_i = \varpi_i^2\phi_i \tag{3.11}$$

where ϖ_i and ϕ_i are the i^{th} natural frequency and the corresponding mode shape of a system. There are well established methods for calculating ϖ_i and ϕ_i from the measured experimental data. Because both sides of equation (3.11) are multiplied from the right side by ϕ_i, ϕ_i is sometimes referred to as the *right modal vector.*

Zhang, Allemang and Brown [54] have shown that the same information which is used to extract ϕ_i from the frequency response function of a system, may also be used to extract the so called *left modal vector*, ξ_i, which is defined by

$$\xi_i^T\mathbf{M}^{-1}\mathbf{K} = \varpi_i^2\ \xi_i^T \tag{3.12}$$

The left modal vector is related to the right modal vector via

$$\xi_i = \mathbf{M}\phi_i \tag{3.13}$$

and, assuming mass-normalisation, it can be immediately shown that

$$\xi_i^T\phi_j = \begin{cases} 1\ , & i=j \\ 0\ , & i\neq j \end{cases} . \tag{3.14}$$

Thus, it appears that more than the usual information can be extracted from the experimental modal analysis results. This additional information can then be used to negate the inherent incompleteness of the measured data, and thus allow to circumvent entirely the effects of the modal truncation errors. However, the extraction of the left modal vectors appears to be very sensitive to noise, and hence it is not clear whether it can be done with sufficient accuracy in practical applications. Also, unlike the well established procedures for extracting right modal vectors, extraction of the left modal vectors requires to solve a set of ill-posed equations of deficient rank, which may lead to additional large errors. If, on the other hand, the left modal vectors are available and accurate, then they can be immediately applied to solve problems in structural modifications.

Based on this assumption of the availability of left modal vectors, Bucher and Braun [55] developed an exact solution to an inverse structural modification problem. Their method allows an exact assignment of the natural frequencies and mode shapes based on incomplete modal analysis data, provided that the prescribed mode shapes belong to the space spanned by the original measured modal vectors. If the prescribed mode shapes do not belong to the space spanned by the measured vectors, a method for approximate assignment is also developed, which gives an optimal solution in a least squares sense.

In two recent papers Bucher and Braun [56,57] have developed a computationally efficient optimisation procedure for minimising the vibratory response of a system by structural modifications. This procedure may be applied in cases where only a truncated set of measured modal data is available, or when a complete set of analytical data is assumed to be known. In [56] the authors develop the theoretical basis for their method, and in a "companion" paper [57] they provide detailed examples of application of the derived theory.

3.5 Model Reconstruction and Model Updating Problems

The work of Gladwell [1-3] (see above) gives a comprehensive introduction to the problems of *model reconstruction*, which are also often referred to as problems of *system identification*. The book [58] and the recent paper [59] by Friswell and Mottershead give a review of the current state of knowledge in the subject of *model updating*. In this section we only present few results which are of most relevance to the scope of this thesis.

Boley and Golub [60] have reviewed various algorithms for reconstruction of Jacobi matrices from the knowledge of their eigenvalues, eigenvalues of their principal sub-matrices, and/or knowledge of some specified elements of the normalised eigenvectors. Ram and Coldwell [61] found a solution for reconstructing a free-free, multi-connected mass-spring system (i.e system where none of its springs or masses are attached to the ground, but where each mass may be connected via a spring element to any other masses) from known sets of the natural frequencies of the system. The required sets of natural frequencies included the original system, and the frequencies of the systems when each of the masses, in turn, was pinned to the ground, thus restricting its movement. Gladwell and Movahhedy [62] have obtained the set of the necessary and sufficient conditions to ensure positive mass and stiffness parameters for the three-degree-of-freedom case. Movahhedy, Ismail and Gladwell [63] have examined the problems associated with reconstruction of such systems from the experimentally measured data. Ram and Gladwell [64] solved the problem of reconstructing the finite element model of an axially vibrating rod from the knowledge of some of its eigenvalues and eigenvectors. The minimum requirement for a closed-form solution is one eigenvalue and two eigenvectors. However, with this minimal data the algorithm is very sensitive to perturbations. This sensitivity of the method is decreased if overdetermined data (i.e more data than is minimally necessary) is used, in which case a solution is obtained by a least squares approach. Ram [65,66] then extended the method of [64] to find the equivalent solutions for reconstruction of a longitudinally vibrating continuous rod and a discrete model of a transversely vibrating beam.

Starek and Inman [67 - 71] have studied the problems associated with the reconstruction of non-conservative systems. In [67] the authors have developed a method of solution which ensured that the mass, stiffness and damping matrices are real-valued, provided that all of the eigenvalues of a system are complex. In [68,69] the method was improved to ensure that the matrices are also symmetric, thus enhancing the physical realisability properties of the solution. In [70] an alternative approach to the method of [69] was presented, which further improved the realisability properties of a solution by ensuring that the obtained matrices, in addition to being real and symmetric, are also positive definite. In [71] the method has been further developed to include systems with real-valued eigenvalues associated with overdamped modes.

3.6 General References

The book of Golub and Van Loan [72] was an invaluable reference for understanding concepts of linear algebra and matrix analysis. In particular, the algorithm for solving an *orthogonal Procrustes problem* [72, p.582] formed a foundation for the solution to Problem 2 described in section 5, and a more general Problem 2(b) presented in section 7. A detailed description of the orthogonal Procrustes problem and the procedure for its solution are given in section 5.

The physical realisability constraints for the mass and stiffness elements were a major focus in this thesis. The principal demand for the realisable mass and stiffness elements is that they must be real and non-negative. Thus, the method for solving a *non-negative least squares problem* given in the book of Lawson and Hanson [73, p. 161] was used as a primary tool in our analysis throughout this thesis. The algorithm for solving a *non-negative least squares problem* is also available as a standard function **nnls** in MATLAB.

Chu [74] has discussed the effect of the rate of convergency of the two methods for an *inverse singular value* problem, which is closely related to the inverse eigenvalue problems. He found that a quadratically converging algorithm converges fast but locally, while a linearly converging algorithm converges globally but at a slower rate. Thus it is possible that the convergence characteristics of an algorithm may be improved by reducing its rate of convergence. We have successfuly applied this principle in our solution to Problem 1.

Section 4 PROBLEM 1: DESIGN FOR NATURAL FREQUENCIES[1]

In this section we analyse a solution to Problem 1, which is formulated in section 2.1. In this problem we assume that the vibratory behaviour of a system can be adequately approximated by the behaviour of a conservative mass-spring analytical model. It is also assumed that a physically realisable stiffness matrix **K** and a set of desired natural frequencies $\{\varpi_1^*, \varpi_2^*, \ldots, \varpi_n^*\}$ are known.

Denoting:

$$\mathbf{\Lambda}^* = \text{diag}\,(\lambda^*_1, \lambda^*_2, \ldots, \lambda^*_n)\ ;\ \lambda^*_i = \omega_i^{*2}\ ;\ 0 > \lambda^*_1 > \lambda^*_2 > \ldots > \lambda^*_n \tag{4.1}$$

we wish to find a real, positive and diagonal mass matrix **M** such that the roots of the characteristic polynomial

$$\det\,(\,\mathbf{K}-\lambda\mathbf{M}) = 0 \tag{4.2}$$

are the prescribed diagonal elements of $\mathbf{\Lambda}^*$.

[1] Material presented in this section has been published in [75].

This problem is similar to the *inverse multiplicative eigenvalue problem* which was first formulated by Downing and Householder [5]. Many authors (eg. [6 - 15]) have presented alternative methods of solution for this, and similar, problems as well as some partial conditions for the existence of real solutions. Most notable work was done by Friedland [9, 10] and Friedland, Nocedal and Overton in [15]. Recently, Joseph [16] has developed a related method which solves a similar problem to the one studied here. Two of the algorithms ([5] and [16]) can be applied directly to solve the problem under consideration.

It should be noted that the existing methods of solution are based on iterative procedures. The problem, however, can be expressed as a system of *n* equations with *n* unknowns. The possibility of finding a closed-form solution is investigated in section 4.1, and a closed-form solutions are obtained there for two and three degrees-of-freedom systems. It appears however that this method cannot be effective for high order systems due to the complexity of the non-linear equations involved. In section 4.2 we describe the requirements for a practical method of solution and discuss the need of a new algorithm. The new algorithm is presented in section 4.3. Some numerical simulations are given in section 4.4, and the conclusions are summarised in section 4.5.

4.1 A Closed-Form Solution

Equation (4.2) can be written in the following form:

$$\det\,(\mathbf{K}\text{-}\lambda\mathbf{M}) = \alpha(\lambda-\lambda^*_1)(\lambda-\lambda^*_2)(\lambda-\lambda^*_3)\ldots(\lambda-\lambda^*_n) \tag{4.3}$$

where λ^*_i (i=1,2,...,n) are the given eigenvalues, and α is some constant.

We wish to find positive diagonal **M** such that (4.3) is satisfied. Both sides of (4.3) can be expressed as polynomials in λ of order n. Equating the coefficients of the two polynomials will produce n equations with n unknowns. This is demonstrated by the following examples:

4.1.1 A two degree-of-freedom system.

A general two degrees-of-freedom system has the following stiffness and mass matrices:

$$K = \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix}, \quad M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \tag{4.4}$$

Suppose **K** and λ_1^*, λ_2^* are given. The problem under consideration is to find m_1 and m_2 such that (4.3) holds, i.e

$$\det \begin{pmatrix} k_{11}-\lambda m_1 & k_{12} \\ k_{12} & k_{22}-\lambda m_2 \end{pmatrix} = \alpha(\lambda-\lambda_1^*)(\lambda-\lambda_2^*) \tag{4.5}$$

Expanding both sides we have

$$\begin{aligned} m_1 m_2 \lambda^2 - (k_{11}m_2 + k_{22}m_1)\lambda + (k_{11}k_{22} - k_{12}^2) \\ = \alpha\lambda^2 - \alpha(\lambda_1^* + \lambda_2^*)\lambda + \alpha\lambda_1^*\lambda_2^* \end{aligned} \tag{4.6}$$

which yields:

$$m_1 m_2 = \alpha \tag{4.7}$$

$$k_{11}m_2 + k_{22}m_1 = \alpha(\lambda_1^* + \lambda_2^*) \tag{4.8}$$

$$k_{11}k_{22} - k_{12}^2 = \alpha\lambda_1^*\lambda_2^* \tag{4.9}$$

From (4.9) α can be expressed in terms of the given data as:

$$\alpha = \frac{k_{11}k_{22} - k_{12}^2}{\lambda_1^* \lambda_2^*} \quad . \tag{4.10}$$

and (4.7) and (4.8) can then be used to produce a quadratic equation in m_2 :

$$k_{11}m_2^2 - \alpha(\lambda_1^* + \lambda_2^*)m_2 + k_{22}\alpha = 0 \tag{4.11}$$

The solutions then are

$$m_2 = \frac{\alpha(\lambda_1^* + \lambda_2^*) \pm \sqrt{\alpha^2(\lambda_1^* + \lambda_2^*)^2 - 4k_{11}k_{22}\alpha}}{2k_{11}} \tag{4.12}$$

$$m_1 = \frac{\alpha}{m_2} \tag{4.13}$$

We note that provided the necessary condition $k_{11}k_{22} > k_{12}^2$ is satisfied, there are two physically realisable solutions if

$$\alpha(\lambda_1^* + \lambda_2^*)^2 > 4k_{11}k_{22} \tag{4.14a}$$

one physically realisable solution if

$$\alpha(\lambda_1^* + \lambda_2^*)^2 = 4k_{11}k_{22} \tag{4.14b}$$

and no real solution otherwise.

4.1.2 A three degree-of-freedom system

Suppose

$$\boldsymbol{K} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{12} & k_{22} & k_{23} \\ k_{13} & k_{23} & k_{33} \end{bmatrix}, \quad \boldsymbol{M} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \tag{4.15}$$

then similar analysis gives:

$$A_1 = m_1 m_2 m_3 \tag{4.16}$$

$$A_2 = B_1 m_2 m_3 + B_2 m_1 m_3 + B_3 m_1 m_2 \tag{4.17}$$

$$A_3 = C_1 m_1 + C_2 m_2 + C_3 m_3 \tag{4.18}$$

where

$$\begin{aligned}
A_1 &= \frac{k_{11}k_{22}k_{33} + 2k_{12}k_{13}k_{23} - k_{11}k_{23}^2 - k_{22}k_{13}^2 - k_{33}k_{12}^2}{\lambda_1^* \lambda_2^* \lambda_3^*} ; \\
A_2 &= A_1(\lambda_1^* + \lambda_2^* + \lambda_3^*) \ ; \ A_3 = -A_1(\lambda_1^*\lambda_2^* + \lambda_1^*\lambda_3^* + \lambda_2^*\lambda_3^*) ; \\
B_1 &= k_{11} \ ; \ B_2 = k_{22} \ ; \ B_3 = k_{33} \ ; \ C_1 = k_{23}^2 - k_{22}k_{33} ; \\
C_2 &= k_{13}^2 - k_{11}k_{33} \ ; \ C_3 = k_{12}^2 - k_{11}k_{22} .
\end{aligned} \tag{4.19}$$

Solving for m_2 and m_1 as functions of m_3, we obtain:

$$m_2 = \frac{C_1\left[A_2 - \dfrac{B_3 A_1}{m_3}\right] - B_2 m_3(A_3 - C_3 m_3)}{(C_1 B_1 - C_2 B_2)m_3} \tag{4.20}$$

and

$$m_1 = \frac{A_1}{m_2 m_3} \tag{4.21}$$

Substitution of (4.20) and (4.21) into (4.18) results in the following 6th order polynomial in m_3 :

$$\begin{aligned}
&B_2B_1C_3^2m_3^6 - 2B_2B_1A_3C_3m_3^5 + (B_2A_2C_2C_3 + B_2B_1A_3^2 + A_2B_1C_1C_3)m_3^4 + \\
&(B_2^2A_1C_2^2 - 2B_2A_1B_1C_1C_2 + A_1B_1^2C_1^2 - B_3B_2A_1C_2C_3 - B_2A_2A_3C_2 - \\
&B_3A_1B_1C_1C_3 - A_2B_1A_3C_1)m_3^3 + (B_3B_2A_1A_3C_2 + B_3A_1B_1A_3C_1 + A_2^2C_1C_2)m_3^2 - \\
&2B_3A_1A_2C_1C_2m_3 + B_3^2A_1^2C_1C_2 = 0
\end{aligned} \tag{4.22}$$

There are generally six roots to the polynomial (4.22). We may substitute each root into (4.20) and find corresponding m_2. Then (4.21) gives us m_1. There are therefore at most six possible solutions in this case. However, in general, the existence of a physically realisable solution is not guaranteed.

4.1.3 The general case

Let
$$\binom{n}{p}_q \,, \quad q = 1,\ldots,\frac{n!}{p!(n-p)!}$$

denote the *combinations of elements* $\{m_1, m_2, \ldots, m_n\}$ *taken p elements at a time without repetitions*, and let

$$\prod \binom{n}{p}_q \,, \; q = 1,\ldots,\frac{n!}{p!(n-p)!}$$

denote the *products of the elements in these combinations.*

Then for a general nxn system (4.2) can be written as follows

$$\det(K - \lambda M) = \lambda^n \left[\prod \binom{n}{n}\right] + \lambda^{n-1} \left[\sum_{q=1}^{n} \left[G_{n-1,q} \prod \binom{n}{n-1}_q\right]\right] + \ldots\ldots \quad \ldots + \lambda \left[\sum_{q=1}^{n} \left[G_{1,q} \prod \binom{n}{1}_q\right]\right] + G_0 = 0 \tag{4.23}$$

where $G_{i,q}$ (i=1,...,n-1) are constants and are explicitly determined from the elements of **K**.

Similarly we may write

$$\alpha(\lambda - \lambda_1^*)(\lambda - \lambda_2^*)\ldots.(\lambda - \lambda_n^*) = \alpha\lambda^n + \alpha F_1\lambda^{n-1} + \alpha F_2\lambda^{n-2} + \ldots \\ \ldots + \alpha F_{n-1}\lambda + \alpha F_n = 0 \tag{4.24}$$

where the coefficients F_i (i=1,...,n) are constants and can be determined from λ_1^*, ..., λ_n^*.

Therefore equating the coefficients of (4.23) and (4.24), we have following system of n equations in n unknowns (m_1 , ...,m_n):

$$\begin{aligned}
\alpha &= \frac{G_0}{F_n} = \prod\binom{n}{n} \\
\alpha F_1 &= \sum_{q=1}^{n}\left[G_{n-1,q}\prod\binom{n}{n-1}_q\right] \\
&\vdots \\
\alpha F_p &= \sum_{q=1}^{\frac{n!}{p!(n-p)!}}\left[G_{n-p,q}\prod\binom{n}{n-p}_q\right] \\
&\vdots \\
\alpha F_{n-1} &= \sum_{q=1}^{n}\left[G_{1,q}\prod\binom{n}{1}_q\right]
\end{aligned} \tag{4.25}$$

It is difficult to determine a solution to these equations when n is large (we were unable to find a solution for the 4x4 case using a symbolic manipulator!). Friedland in [9] found that there are at most *n!* different solutions for the general problem. However, the existence of a physically realisable solution is not guaranteed. Also, in practical engineering designs, it is likely that the desired natural frequencies would be permitted to have some finite tolerance ranges. Thus any solution that would fall within these tolerances, would be suitable. The method presented in section 4.2 requires that the values for the desired natural frequencies be specified precisely, and then allows a finite number of possible solutions to be obtained. Approximate solutions cannot be obtained by this method.

4.2 Existing Methods

Two of the methods of solution for the inverse multiplicative eigenvalue problem, namely Downing and Householder (from now on D&H) [5] and Joseph [16], can be applied directly to the problem investigated here. Both methods are iterative, and have a local quadratic convergency. Thus, they do converge when the starting point is sufficiently close to a local solution. If they converge, the solution is obtained with a small number of iterations. However, if the starting point is not sufficiently close to a local solution the iterative algorithms may diverge or oscillate about the true solution.

Chu in [74] has discussed the effect of the rate of convergency of the two methods for a closely related *inverse singular value* problem. He found that a quadratically converging algorithm converges fast but locally, the linearly converging algorithm converges globally but at a slower rate. Thus it is possible that the convergence characteristics of an algorithm may be improved by reducing its rate of convergence.

A further consideration about the suitability of the two currently available methods was made based on the physical realisability criteria for the solution. In order to satisfy the physical realisability criteria, any obtained solution (i.e mass matrix) must be real, positive and diagonal. Although both methods always satisfy the diagonality requirement, the algorithm of Joseph may converge into a complex solution. The D&H algorithm, on the other hand, if converge, would always converge into a real and positive solution. Thus, the D&H algorithm was chosen as a basis for a new algorithm, and it is summarised below:

Algorithm 4.1: Downing and Householder Method

Input: Stiffness matrix **K** and eigenvalues matrix Λ^* .

Algorithm:

1) Choose an initial guess for a diagonal, positive-definite mass matrix $\mathbf{M}_0$.
2) Set iteration index t=0.
3) Calculate the spectral decomposition:

$$\mathbf{M}_t^{-1/2}\mathbf{K}\,\mathbf{M}_t^{-1/2} = \mathbf{U}_t\Lambda_t\mathbf{U}_t^T$$

   where $\Lambda_t = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, $\lambda_1>\lambda_2>\ldots>\lambda_n$ is an eigenvalue matrix, $\mathbf{U}_t$ is an nxn orthonormal matrix (i.e $\mathbf{U}_t\mathbf{U}_t^T=\mathbf{I}_n$) and $\mathbf{I}_n$ is the nxn identity matrix.

4) Calculate the real diagonal matrix $\mathbf{Z}_t$, satisfying: $\Lambda_t=\Lambda^*(\mathbf{I}_n+\mathbf{Z}_t)$

   If the elements of $\mathbf{Z}_t$ are sufficiently small, then stop.

5) Solve a system of linear equations:

$$\sum_{j=1}^{n} |u_{ji}|^2\, d_{jj} = z_{ii} \qquad (i= 1,\ \ldots,\ n)$$

   where u_{ji} and z_{ii} are elements of $\mathbf{U}_t$ and $\mathbf{Z}_t$ respectively.

6) Set $\mathbf{D}_t=\mathrm{diag}(d_{11}, \ldots, d_{nn})$.
7) Calculate the next iteration for the mass matrix :

$$\mathbf{M}_{t+1}=(\mathbf{I}_n- 0.5\mathbf{D}_t)^{-2}\mathbf{M}_t$$

8) Set iteration index t = t+1, and repeat from step 3.

Output (if converges)**:** Real, positive and diagonal mass matrix **M**.

4.3 The New Algorithm

Suppose that steps 1 to 3 of the D&H algorithm were carried out. Then, the spectral decomposition

$$\mathbf{M}_t^{-1/2}\mathbf{K}\mathbf{M}_t^{-1/2} = \mathbf{U}_t \,\Lambda_t \mathbf{U}_t^{T} \tag{4.26}$$

is known.

Define **R** as follows:

$$\mathbf{R} = \mathbf{U}_t \,\Lambda^{*1/2}\Lambda_t^{-1/2}\mathbf{U}_t^{T} \;, \tag{4.27}$$

then

$$R\; U_t \,\Lambda_t U_t^T R \;=\; U_t \,\Lambda^*\; U_t^T \tag{4.28}$$

which means that the eigenvalues of $\mathbf{R}\mathbf{M}_t^{-1/2}\mathbf{K}\mathbf{M}_t^{-1/2}\mathbf{R}$ are the diagonal elements of Λ^*. Thus, if **R** is real and diagonal, then an exact solution for the mass matrix $\mathbf{M}=\mathbf{M}_{t+1}$ is given by:

$$\mathbf{M}_{t+1} = \mathbf{R}^{-1}\,\mathbf{M}_t\,\mathbf{R}^{-1} \;. \tag{4.29}$$

But in general **R** is not diagonal, and consequently $\mathbf{M}_{t+1}$ would not be diagonal either. However, if we could find a real diagonal matrix, which is close to **R** in some sense, it may obtain an approximation for $\mathbf{M}_{t+1}$. Therefore, we define the following optimisation problem.

Given **R** *as in* (4.27). *Find a real diagonal matrix* **D**, *such that the residual error:*

$$\varepsilon \;=\; \left\| R \;-\; D \right\|_F^2 \tag{4.30}$$

is minimised.

Applying the equality:

$$\| R - D \|_F^2 = trace(R^T R) + trace(D^T D) - 2\, trace(D^T R) \tag{4.31}$$

then for any matrix **R** of (4.27) and using the fact that **D** is diagonal, ε is given by:

$$\varepsilon = trace(R^T R) + \sum_{i=1}^{n} \left[d_{ii}^2 - 2 d_{ii} r_{ii} \right] \tag{4.32}$$

$$= trace(R^T R) + \sum_{i=1}^{n} \left[d_{ii} - r_{ii} \right]^2 - \sum_{i=1}^{n} r_{ii}^2 \tag{4.33}$$

where d_{ii} and r_{ii} are the diagonal elements of **D** and **R** respectively. Then from (4.33) it is clear that ε is minimised when

$$d_{ii} = r_{ii} \,. \tag{4.34}$$

Thus, the residual error ε is minimised when the diagonal elements of **D** are equal to the diagonal elements of **R**. Applying this result, the approximation for the mass matrix $\mathbf{M}_{t+1}$ is equal to $\mathbf{D}^{-2}\mathbf{M}_t$. The following iterative algorithm is then proposed:

Algorithm 4.2: The New Method

Input: Stiffness matrix **K** and eigenvalues matrix Λ^*.

Algorithm:

1) Choose an initial guess for a diagonal, positive-definite mass matrix $\mathbf{M}_0$.
2) Set iteration index t = 0.

3) Calculate the spectral decomposition:

$$\mathbf{M}_t^{-\frac{1}{2}}\mathbf{K}\mathbf{M}_t^{-\frac{1}{2}} = \mathbf{U}_t\,\Lambda_t\mathbf{U}_t^T$$

4) If the norm $\| \Lambda^* - \Lambda_t \|_F$ is sufficiently small, then stop.

5) Calculate $\mathbf{R} = \mathbf{U}_t\,\Lambda^{*\frac{1}{2}}\Lambda_t^{-\frac{1}{2}}\mathbf{U}_t^T$

6) Form a real diagonal matrix **D** from the diagonal elements of **R**.

7) Calculate the next iteration of mass $\mathbf{M}_{t+1}=\mathbf{D}^{-2}\mathbf{M}_t$

8) Set iteration index t = t+1 and repeat from step 3.

Output: Real, positive and diagonal mass matrix **M**.

This algorithm and the two existing methods were tested on some numerical examples, and their performances were compared. It appears that the new algorithm is linearly convergent, and thus, in general, a significantly larger number of iterations is necessary than with the two existing methods. However, this disadvantage is balanced by a better global behaviour, in the sense that it usually converges to a solution even if the initial guess is not close to an actual solution. This relation between the rate of convergence of an algorithm and its convergence characteristics is similar to the one described by Chu in [74].

In all numerical examples tested, the new algorithm has converged into an optimal local solution. Furthermore, the diagonal elements of **D** are always real, and as a result **M** is always real and positive.

4.4 Numerical Examples

The algorithms were tested with various combinations of **K** and Λ^*. As expected, the parameters that have the most significant influence on the performance of the algorithms are the initial guess for the mass matrix. We present here the results of the numerical tests for one combination of **K** and Λ^*, for several different initial guesses.

Consider the 10 degree-of-freedom system, with stiffness matrix

$$K = \begin{bmatrix} 200 & -10 & -20 & -5 & -5 & -10 & 0 & 0 & -50 & -50 \\ -10 & 100 & 0 & 0 & 0 & 0 & -20 & -10 & -20 & -10 \\ -20 & 0 & 300 & -40 & -30 & -60 & -10 & 0 & -20 & -10 \\ -5 & 0 & -40 & 400 & -30 & -40 & -50 & -20 & -10 & -70 \\ -5 & 0 & -30 & -30 & 150 & -10 & -5 & -5 & -20 & 0 \\ -10 & 0 & -60 & -40 & -10 & 250 & 0 & 0 & 0 & -80 \\ 0 & -20 & -10 & -50 & -5 & 0 & 120 & -5 & 0 & -10 \\ 0 & -10 & 0 & -20 & -5 & 0 & -5 & 250 & 0 & -100 \\ -50 & -20 & -20 & -10 & -20 & -0 & 0 & 0 & 350 & -40 \\ -50 & -10 & -10 & -70 & 0 & -80 & -10 & -100 & -40 & 400 \end{bmatrix}$$

We wish to find a positive-definite and diagonal matrix **M**, such that the eigenvalues of the system are the diagonal elements of :

$$\Lambda^* = \mathrm{diag}\ (500, 450, 400, 350, 300, 250, 200, 150, 100, 50).$$

Define

$$\mathbf{E}_t = \| \Lambda^* - \Lambda_t \|_2 \tag{4.35}$$

where t is an iteration index. $\mathbf{E}_t$ is then equal to the maximum difference between any two corresponding elements of Λ^* and Λ_t, and thus measures the accuracy of the solution. The results of six tests are presented here. The initial conditions are shown in Table 4.1.

Test No.	Diagonal Elements of the Initial Guess, $\mathbf{M}_0$	Diagonal Elements of the Λ_0	Residual $\mathbf{E}_0$
1	1, 1, 1, 1, 1, 1, 1, 1, 1, 1	507.2, 415.4, 364.6, 335.4, 231.2, 206.3, 160.6, 129.1, 108.4, 61.8	68.8
2	1, 2, 1, 3, 1, 4, 1, 5, 1, 6	369.5, 313.0, 191.1, 165.1, 139.5, 97.5, 83.5, 52.6, 48.0, 22.6	208.9
3	1, 1, 1, 2, 1, 1, 1, 1, 4, 4	345.3, 271.8, 245.6, 211.5, 201.4, 145.2, 116.6, 93.0, 87.8, 39.3	178.2
4	4, 1, 10, 3, 2, 2, 2, 2, 4, 10	151.8, 130.5, 124.1, 107.3, 88.9, 73.6, 52.8, 49.7, 31.4, 15.6	348.2
5	1 , 1 , 1, 1, 1, 1, 1, 1, 4, 4	428.1, 344.2, 267.2, 226.7, 202.7, 145.5, 120.6, 93.2, 88.6, 40.7	132.8
6	1.5, 0.5, 3.5, 1.0, 0.5, 1.0, 0.5, 0.5, 1.5, 3.5	522.9, 443.0, 312.5, 266.4, 245.2, 234.6, 171.9, 138.6, 94.1, 46.5	87.5

Table 4.1: The Initial Conditions

Figure 4.1 displays the results from the first twelve iterations of each algorithm. The logarithmic scale in this figure shows that the new algorithm converges linearly. Table 4.2 shows the obtained solutions for the various algorithms. The criteria used to terminate iterations was either when $\mathbf{E}_t > 10^4$ (i.e algorithm diverged), or when $\mathbf{E}_{t+1} - \mathbf{E}_t < 0.01$ (i.e algorithm converged).

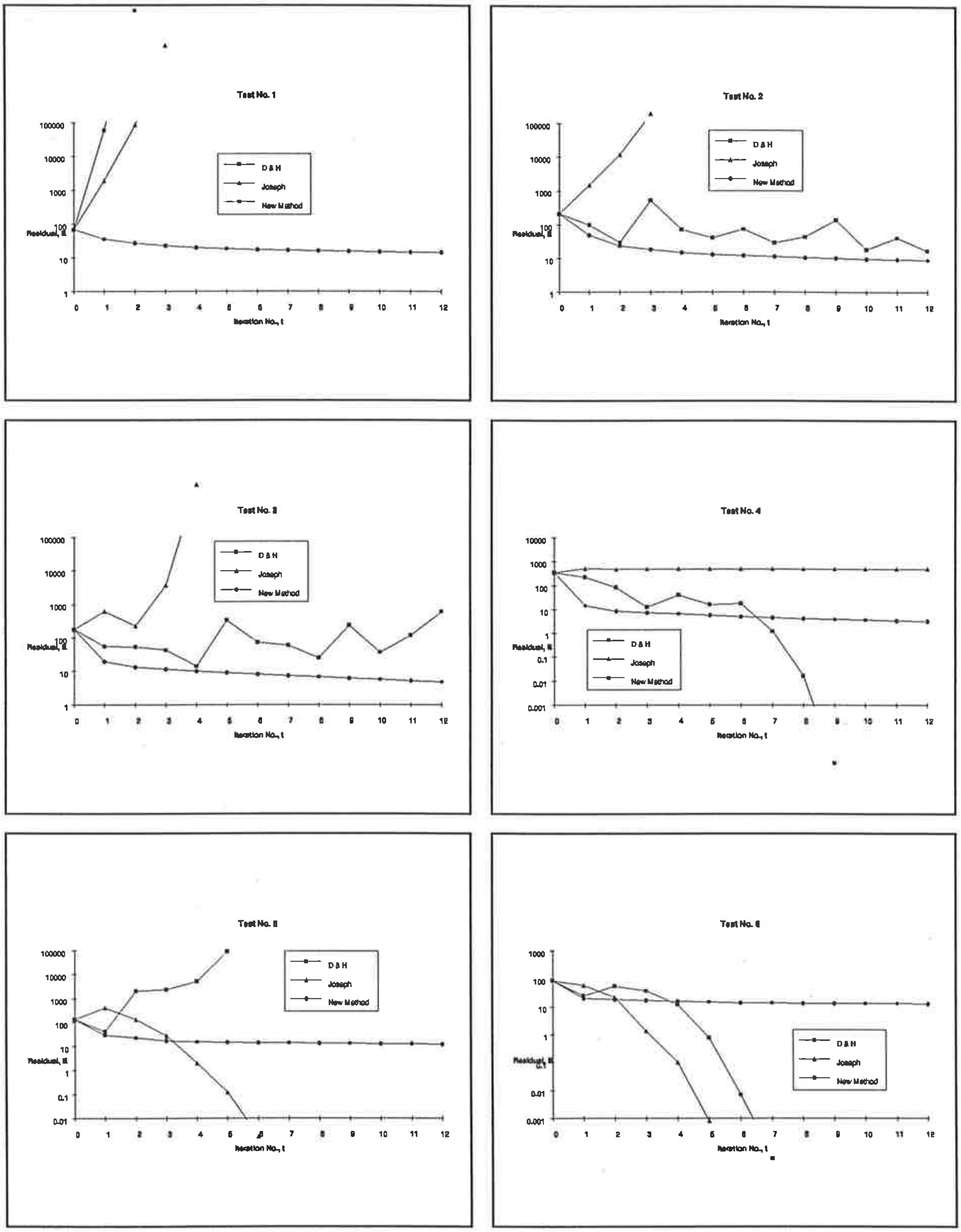


Figure 4.1 : Residual (E_t) vs Iteration Number (t) plot for the first 12 iterations

Test No.	Method	Iterations to Convergence (t)	Diagonal Elements of the Obtained Mass Matrix, $\mathbf{M}_t$ (given to 2 decimal places only)	Diagonal Elements of Λ_t (given to 1 decimal place only)	Residual $\mathbf{E}_t$
1	D & H	Diverge	-	-	-
	Joseph	Diverge	-	-	-
	New Method	425	0.60, 1.60, 0.96, 0.95, 0.75, 1.10, 0.83, 0.68, 1.03, 1.16	504.8, 451.4, 401.0, 353.1, 294.7, 254.9, 201.4, 151.2, 94.8, 50.7	5.3
2	D & H	No Convergence	-	-	-
	Joseph	Diverge	-	-	-
	New Method	113	0.56, 0.99, 0.74, 1.27, 0.46, 1.18, 0.375, 1.80, 0.77, 3.27	500.4, 450.6, 399.9, 351.6, 297.8, 251.3, 197.5, 149.9, 99.9, 50.5	2.5
3	D & H	No Convergence	-	-	-
	Joseph	No Convergence	-	-	-
	New Method	1050	0.70, 0.54, 0.71, 1.15, 0.50, 0.65, 0.62, 0.585, 3.64, 3.58	500.0, 450.0, 400.0, 350.0, 300.0, 250.0, 200.0, 150.0, 100.0, 50.0	0.0012
4	D & H	9	1.36, 0.26, 3.24, 0.90, 0.49, 0.75, 0.51, 0.58, 1.43, 2.97	500.0, 450.0, 400.0, 350.0, 300.0, 250.0, 200.0, 150.0, 100.0, 50.0	3.9×10^{-6}
	Joseph	4	Physically Unrealisable	All zeros	500
	New Method	396	1.36, 0.33, 3.13, 0.93, 0.60, 0.63, 0.49, 0.57, 1.13, 3.16	500.1, 449.9, 400.0, 351.8, 298.3, 249.9, 200.1, 150.0, 100.0, 50.0	1.8
5	D & H	Diverge	-	-	-
	Joseph	6	0.83, 0.34, 1.73, 1.29, 0.31, 0.61, 0.50, 0.65, 3.71, 3.36	500.0, 450.0, 400.0, 350.0, 300.0, 250.0, 200.0, 150.0, 100.0, 50.0	0.0017
	New Method	2356	0.58, 0.62, 0.74, 0.89, 0.52, 0.93, 0.50, 0.65, 3.62, 3.57	500.0, 450.1, 400.0, 350.0, 300.1, 250.1, 199.8, 150.0, 100.0, 50.0	0.2
6	D & H	7	1.15, 0.27, 3.36, 0.94, 0.43, 0.86, 0.38, 0.59, 2.16, 2.79	500.0, 450.0, 400.0, 350.0, 300.0, 250.0, 200.0, 150.0, 100.0, 50.0	3.7×10^{-5}
	Joseph	5	1.36, 0.35, 3.23, 1.06, 0.38, 0.71, 0.48, 0.53, 1.44, 2.99	500.0, 450.0, 400.0, 350.0, 300.0, 250.0, 200.0, 150.0, 100.0, 50.0	0.0008
	New Method	2137	1.45, 0.43, 3.62, 1.43, 0.35, 0.67, 0.42, 0.54, 1.2, 2.35	500.0, 450.0, 400.0, 350.0, 300.0, 250.0, 200.0, 150.0, 100.0, 50.0	0.0103

Table 4.2: Obtained Solutions

In test no.1 the D&H and Joseph's algorithms both diverged from first iteration onwards. The new algorithm achieved first iteration with $\mathbf{E}_1$=35.7 and converged to an approximate solution with $\mathbf{E}_{425}$=5.3 after 425 iterations. In test no.2 the Joseph's algorithm diverged from first iteration onwards. The D&H algorithm has neither converged nor diverged. It exhibited oscillatory behaviour about the solution. The closest point reached by the algorithm was at $\mathbf{E}_{12}$ =17.0 after 12 iterations. The new algorithm achieved $\mathbf{E}_1$=49.8 , $\mathbf{E}_2$=23.5 and converged at $\mathbf{E}_{113}$=2.5. In test no.3 both D&H and Joseph's algorithms have shown an oscillatory behaviour for the first few iterations and then began to diverge from t=2 and t=10 respectively. The new algorithm achieved $\mathbf{E}_1$=19.1, $\mathbf{E}_2$=13.3 and converged at $\mathbf{E}_{1050}$=0.0012. In test no.4 the D&H algorithm converged quadratically at t=9. The Joseph's algorithm has also converged, but not in the desired direction (refer Table 4.2 and Figure 4.2). The new algorithm achieved $\mathbf{E}_1$=14.4, $\mathbf{E}_2$=8.5 and converged at $\mathbf{E}_{396}$=1.8. In test no.5 the D&H algorithm diverged from the second iteration onwards, the Joseph's algorithm converged quadratically after 6 iterations, and the new algorithm achieved $\mathbf{E}_1$=29.1, $\mathbf{E}_2$=22.7 and converged at $\mathbf{E}_{2356}$=0.2. In test no.6 both the D&H and Joseph's algorithms have converged quadratically after 7 and 5 iterations respectively, and the new algorithm achieved $\mathbf{E}_1$=20.1, $\mathbf{E}_2$=18.3 and converged at $\mathbf{E}_{2137}$=0.01 .

Thus we note that in some of the above examples the new method performs better, in terms of global convergency, than the existing algorithms.

4.5 Conclusions

In this section we considered the problem of selecting the masses of a mass-spring system to achieve the desired natural frequencies. A closed-form solution for a two and three degree-of-freedom systems was given, but it appears impractical to obtain similar solutions for high order systems. Two existing iterative methods were then examined numerically, and found to have a local quadratic convergency. A new iterative method was then suggested, and numerical simulations show that it has better global convergency, but at a slower rate. Similar behaviour has been observed by Chu in [74] for a closely related inverse singular value problem. Perhaps the best strategy is to begin iterations with our method, and then switch to a quadratic method once within a proximity of a solution.

Although D&H method was selected as a basis for our algorithm, the method of Joseph is better suitable for problems where the desired natural frequency spectrum is incomplete (i.e less than n desired natural frequencies are specified) or for systems where mass and stiffness matrices are not independent of each other. Therefore, in section 6, where we analyse a problem of structural modifications to achieve prescribed natural frequencies, it was more convenient to base our solution algorithm on the method of Joseph.

Section 5 *PROBLEM 2:*

DESIGN FOR NATURAL FREQUENCIES AND MODE SHAPES[2]

In this section we present a solution to Problem 2, which was formulated in section 2.2. In Problem 2 we wish to determine mass and stiffness matrices $\mathbf{M}$ and $\mathbf{K}$ corresponding to a physically realisable mass-spring system, such that its modal and spectral properties, described by the modal matrix $\mathbf{\Phi}$ and the spectral matrix $\mathbf{\Lambda}$, are as close as possible to the prescribed modal and spectral matrices $\mathbf{\Phi}^*$ and $\mathbf{\Lambda}^*$.

We realise that the two problems of determining $\mathbf{\Phi}$ and $\mathbf{\Lambda}$ corresponding to a realisable system can be solved separately. Also note that satisfying equations (2.3) and (2.4) is a sufficient condition for equation (2.2) to hold.

[2] Material presented in this section has been accepted for publication in the Journal of Sound and Vibration, reference [76].

Consider now the problem of determining the optimal mode shape matrix $\mathbf{\Phi}$,

Problem 2.1: Determination of Mode Shapes

Given $\mathbf{\Phi}^*$, determine $\mathbf{\Phi}$ such that $\mathbf{M} = \mathbf{\Phi}^{-T}\mathbf{\Phi}^{-1}$ is a diagonal positive definite matrix, and which minimises the norm $\|\mathbf{\Phi}^* - \mathbf{\Phi}\|$.

We analyse this problem in section 5.1. Once the solution $\mathbf{\Phi}$ is found, we solve the following problem

Problem 2.2: Determination of Eigenvalue Matrix

Given $\mathbf{\Lambda}^*$ and $\mathbf{\Phi}$, determine $\mathbf{\Lambda}$ which minimises the norm $\|\mathbf{\Lambda}^* - \mathbf{\Lambda}\|$, such that $\mathbf{K} = \mathbf{\Phi}^{-T}\mathbf{\Lambda}\mathbf{\Phi}^{-1}$ satisfies the properties given by (2.6).

We present the global optimal solution to this problem in section 5.2. Determining the global optimal solution is computationally expensive. We therefore present another, local optimal approximation in section 5.3. A numerical example demonstrating the algorithms is presented in section 5.4, and conclusions are drawn in section 5.5.

5.1 Mode Shape Optimisation.

Let $\mathbf{D}$ = diag $(d_1, d_2, ..., d_n)$, $d_i \neq 0$, and let $\mathbf{Q}$ be an orthonormal matrix, that is, $\mathbf{QQ}^T = \mathbf{I}_n$. If $\mathbf{\Phi} = \mathbf{DQ}$, then the mass matrix $\mathbf{M}$ obtained by equation (6) is physically realisable, since

$$\mathbf{M} = \mathbf{\Phi}^{-T}\mathbf{\Phi}^{-1} = (\mathbf{D}^{-1}\mathbf{Q})(\mathbf{Q}^T\mathbf{D}^{-1}) = \mathbf{D}^{-2} \tag{5.1}$$

is a positive definite diagonal matrix.

Thus a solution to Problem 2.1 can be obtained by determining a diagonal matrix $\mathbf{D}$ and an orthonormal matrix $\mathbf{Q}$, such that

$$\min_{\mathbf{D},\mathbf{Q}} \| \Phi^* - \mathbf{DQ} \| \ . \tag{5.2}$$

In solving this problem we will make use of the following result. Given two *nxn* matrices $\mathbf{A}$ and $\mathbf{B}$, the well known *orthogonal Procrustes problem*, is to determine an orthonormal matrix $\mathbf{Q}$, such that

$$\min_{\mathbf{Q}} \| \mathbf{A} - \mathbf{BQ} \|_F. \tag{5.3}$$

An algorithm for solving this problem is given below (see e.g. Golub and van Loan [72, p.582]).

Algorithm 5.1: Orthogonal Procrustes Problem

Input: Two nxn matrices $\mathbf{A}$ and $\mathbf{B}$.

Algorithm:

1) Set $\mathbf{C} = \mathbf{B}^T\mathbf{A}$.
2) Compute the singular value decomposition $\mathbf{C}=\mathbf{U\Sigma V}^T$.
3) Evaluate $\mathbf{Q}=\mathbf{UV}^T$.

Output: Orthonormal $\mathbf{Q}$, which solves (5.3).

Thus we may choose a diagonal matrix $\mathbf{D}_0$ as an initial guess and obtain an orthonormal $\mathbf{Q}_0$ which minimises $\| \Phi^* - \mathbf{D}_0\mathbf{Q}_0 \|_F$, by using Algorithm 5.1. We now show how to obtain a matrix $\mathbf{D}_1$ such that

$$\| \Phi^* - \mathbf{D}_1\mathbf{Q}_0 \|_F \leq \| \Phi^* - \mathbf{D}_0\mathbf{Q}_0 \|_F. \tag{5.4}$$

The Frobenius norm is invariant under orthonormal multiplication. Hence

$$\| \mathbf{\Phi}^* - \mathbf{D}_1\mathbf{Q}_0 \|_F = \| \mathbf{\Phi}^*\mathbf{Q}_0^T - \mathbf{D}_1 \|_F, \tag{5.5}$$

Define
$$\mathbf{R} = \mathbf{\Phi}^*\mathbf{Q}_0^T, \tag{5.6}$$

and denote
$$\epsilon = \| R - D_1 \|_F^2 \tag{5.7}$$

Using the equality

$$\|R - D_1\|_F^2 = trace(R^TR) + trace(D_1^TD_1) - 2\ trace(D_1^TR) \tag{5.8}$$

we find that

$$\epsilon = trace\ (R^TR) + \sum_{i=1}^{n} \left[d_{ii}^2 - 2d_{ii}r_{ii} \right] \tag{5.9}$$

$$= trace(R^TR) + \sum_{i=1}^{n} \left[d_{ii} - r_{ii} \right]^2 - \sum_{i=1}^{n} r_{ii}^2 \tag{5.10}$$

where $\mathbf{D}_1$ = diag (d_{ii}) and $\mathbf{R} = [\ r_{ij}\ ]$. Then from (5.10) it is clear that ϵ is minimised when

$$d_{ii} = r_{ii}\ . \tag{5.11}$$

Thus, the residual error ϵ is minimised when the diagonal elements of $\mathbf{D}_1$ are equal to the diagonal elements of $\mathbf{R}$. Having determined a diagonal matrix $\mathbf{D}_1$ satisfying (5.4), we can reapply the Algorithm 5.1 with $\mathbf{\Phi}^*$ and $\mathbf{D}_1$ as an input and find an orthonormal matrix $\mathbf{Q}_1$ such that

$$\| \mathbf{\Phi}^* - \mathbf{D}_1\mathbf{Q}_1 \|_F \leq \| \mathbf{\Phi}^* - \mathbf{D}_1\mathbf{Q}_0 \|_F\ . \tag{5.12}$$

Continuing in this manner iteratively, we obtain an approximation to the Problem 2.1. The following algorithm summarises this result.

Algorithm 5.2: Approximate Solution to Problem 2.1

Input: An nxn modal matrix $\mathbf{\Phi}^*$.

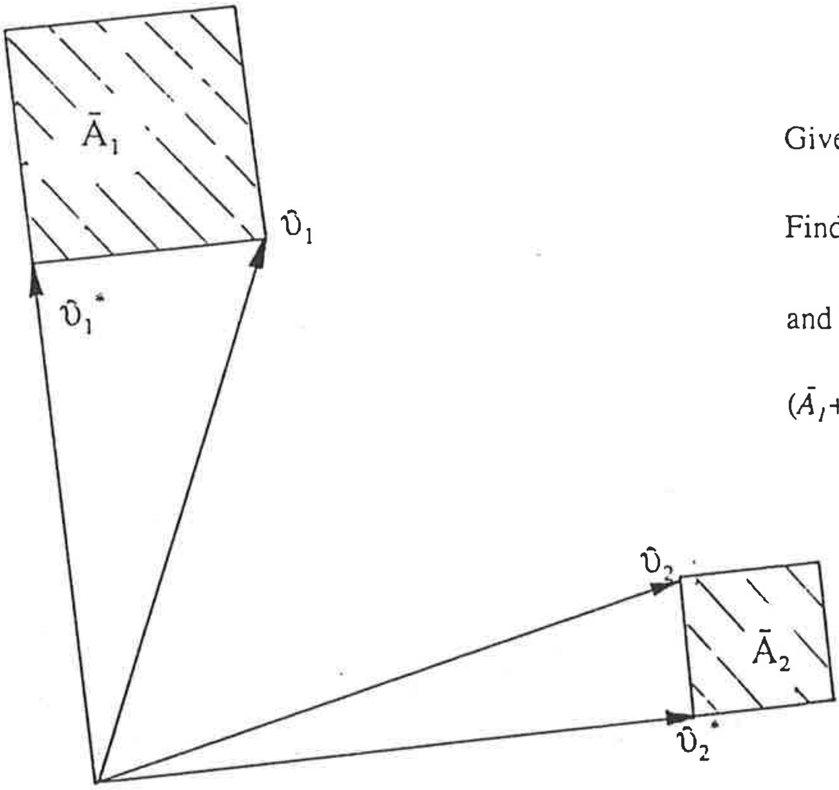
Algorithm:

1) Set initial guess $\mathbf{D}_0$ and a tolerance for convergence ϵ.
2) For i=0, 1, 2, ...
   a) Evaluate $\mathbf{C} = \mathbf{D}_i^T\mathbf{\Phi}^*$.
   b) Compute the singular value decomposition $\mathbf{C}=\mathbf{U\Sigma V}^T$.
   c) Evaluate $\mathbf{Q}_i = \mathbf{UV}^T$.
   d) Obtain $\mathbf{R}=\mathbf{\Phi}^*\mathbf{Q}_i^T$.
   e) $\mathbf{D}_{i+1}$ = diag $(r_{11}, r_{22}, \ldots, r_{nn})$.
   f) Test convergence
      (i) Set $N_1 = \| \mathbf{\Phi}^* - \mathbf{D}_i\mathbf{Q}_i \|_F$, $N_2 = \| \mathbf{\Phi}^* - \mathbf{D}_{i+1}\mathbf{Q}_i \|_F$.
      (ii) If $(N_1 - N_2) \leq \epsilon$, go to 3.
3) $\mathbf{D} = \mathbf{D}_{i+1}$, $\mathbf{Q} = \mathbf{Q}_i$.

Output: A diagonal matrix $\mathbf{D}$ and an orthonormal matrix $\mathbf{Q}$ which approximate the solution of (5.2).

It follows from (5.4) and (5.12) that $\| \mathbf{\Phi}^* - \mathbf{D}_i\mathbf{Q}_i \|_F$ is a monotonic non-increasing function of an iteration index i. The Algorithm 5.2 thus necessarily converge.

The geometrical interpretation of the Algorithm 5.2 is also clear. The stage of calculating $\mathbf{Q}_i$ is equivalent to finding an optimal *rotation* of the matrix $\mathbf{D}_i$ into $\mathbf{\Phi}^*$. The subsequent stage of determining $\mathbf{D}_{i+1}$ is equivalent to *projecting* the column vectors of $\mathbf{\Phi}^*$ onto the axis defined by the column vectors of $\mathbf{Q}_i$. This *rotation-projection* procedure is then carried out until an optimal combination of **D** and **Q**, which solves (5.2), is achieved. Note that problem (5.2) itself can be described geometrically for a two- and three-dimensional space. Figure 5.1 shows an equivalent geometrical problem to (5.2) in a two-dimensional space.



Given two vectors $\hat{\upsilon}_1$ and $\hat{\upsilon}_2$.

Find two orthogonal vectors $\hat{\upsilon}_1^*$ and $\hat{\upsilon}_2^*$, such that the sum $(\bar{A}_1+\bar{A}_2)$ is minimised.

Figure 5.1: Geometric problem equivalent to (5.2) in a two-dimensional space.

5.2 Global Optimisation for Eigenvalues

Using the method described in section 5.1, we obtain a matrix $\mathbf{\Phi} = \mathbf{DQ}$, which satisfies the physical realisability criteria for **M** while minimising $\| \mathbf{\Phi}^* - \mathbf{DQ} \|_F$. In this section we will use this result to obtain a physically realisable **K** which satisfies equation (2.4) while minimising $\| \mathbf{\Lambda}^* - \mathbf{\Lambda} \|$.

The physical realisability criteria for the connectivity of **K**, as described in (2.6), arise from the requirement that the stiffness of all the springs in mass-spring systems must be non-negative. Thus, if we ensure that all the springs have non-negative stiffness, then we necessarily satisfy the conditions of (2.6).

The stiffness matrix **K**, may be written in the following form

$$K = \sum_{p=0}^{n-1} \sum_{q=p+1}^{n} s_{pq} B_{pq}^{(K)} \tag{5.13}$$

where s_{pq} is the stiffness of the spring connecting mass p to mass q, s_{op} represents the stiffness of the spring which connects mass p to the ground, and $\mathbf{B}_{pq}^{(K)}$ is the matrix describing the spring connection between mass p and mass q,

$$B_{pq}^{(K)} = \left[b_{ij}^{(K)}\right] = \begin{cases} b_{pp}^{(K)} = b_{qq}^{(K)} = 1 \\ b_{pq}^{(K)} = b_{qp}^{(K)} = -1 \\ b_{ij}^{(K)} = 0 \ elsewhere \end{cases}, \ (p \neq q). \tag{5.14}$$

Substituting equation (5.13) into equation (2.4), we obtain

$$\Lambda = \sum_{p=0}^{n-1} \sum_{q=p+1}^{n} s_{pq} (\Phi^T B_{pq}^{(K)} \Phi) \ . \tag{5.15}$$

Each of the ij-th element of Λ is thus given by

$$\lambda_{ij} = \sum_{p=0}^{n-1} \sum_{q=p+1}^{n} s_{pq} (\phi_i^T B_{pq}^{(K)} \phi_j) \ . \tag{5.16}$$

Let $N = \frac{1}{2}(n^2 + n)$ and construct the vectors

$$\mathbf{y}_K = (y_1^{(K)}, y_2^{(K)}, \ldots, y_N^{(K)})^T = (\lambda_{11}, \lambda_{12}, \lambda_{13}, \ldots, \lambda_{1n}, \lambda_{22}, \lambda_{23}, \ldots, \lambda_{2n}, \lambda_{33}, \ldots, \lambda_{nn})^T \qquad (5.17)$$

and

$$\mathbf{x} = (x_1, x_2, x_3, \ldots, x_N)^T = (s_{01}, s_{12}, s_{13}, \ldots, s_{1n}, s_{02}, s_{23}, \ldots, s_{2n}, s_{03}, \ldots, s_{nn})^T. \qquad (5.18)$$

Denote

$$\boldsymbol{F}_K = \left[f_{ij}^{(K)}\right] = \frac{\partial y_i^{(K)}}{\partial x_j}, \qquad (i,j = 1, 2,\ldots, N) \qquad (5.19)$$

then all the elements of $\mathbf{F}_K$ can be evaluated using equation (5.16). Equation (5.15) can be written in a vector form

$$\mathbf{F}_K \mathbf{x} = \mathbf{y}_K. \qquad (5.20)$$

In order to satisfy the physical realisability criteria we require all the elements of $\mathbf{x}$ to be non-negative. Setting $\mathbf{\Lambda}=\mathbf{\Lambda}^*$ we may determine the vector $\mathbf{y}_K$ and solve the following *non-negative least squares problem*

$$\min_{\mathbf{x}} \; \| \mathbf{F}_K \mathbf{x} - \mathbf{y}_K \|_2, \text{ subject to } \mathbf{x} \geq 0. \qquad (5.21)$$

An algorithm for the solution of this problem is given in [73, p.161]. (The standard MATLAB function **nnls** solves this problem). Thus the stiffnesses s_{pq} can be obtained from the solution $\mathbf{x}$ of (5.21), via equation (5.18), which in turn determines the matrix $\mathbf{K}$ by (5.13).

The above process gives an optimal solution to the eigenvalue matrix optimisation problem, because it is the best positive solution in a least square sense. We note that in order to obtain a solution for the n degrees-of-freedom system, we need to solve an augmented system (5.21) of dimension N. This is a computational barrier, and an alternative approach is presented in the next section.

5.3 Local Optimisation for Eigenvalues.

Alternatively, the stiffness matrix **K** may be obtained by a local optimisation procedure. Setting $\mathbf{\Lambda} = \mathbf{\Lambda}^*$ and multiplying both sides of equation (2.4) by $\mathbf{\Phi}^{-1}$, we have

$$\mathbf{\Phi}^T\mathbf{K} = \mathbf{\Lambda}^*\mathbf{\Phi}^{-1} . \tag{5.22}$$

Denote

$$\mathbf{A} = \mathbf{\Lambda}^*\mathbf{\Phi}^{-1} \tag{5.23}$$

and partition **A** and **K** as follows

$$\mathbf{A} = [\ \boldsymbol{a}_1 \mid \boldsymbol{a}_2 \mid \boldsymbol{a}_3 \mid .... \mid \boldsymbol{a}_n\ ], \tag{5.24}$$

$$\mathbf{K} = [\mathbf{k}_1 \mid \mathbf{k}_2 \mid \mathbf{k}_3 \mid .....\mid \mathbf{k}_n\ ]. \tag{5.25}$$

Then from equation (5.22), each column of **A** is given by

$$\mathbf{\Phi}^T\mathbf{k}_j = \boldsymbol{a}_j \quad (j = 1, ...,n) . \tag{5.26}$$

We now show how to solve equation (5.26) column by column sequentially. The stiffness matrix **K** for a general mass-spring system of order n has the following form

$$K = \begin{bmatrix} k_{11} & -k_{12} & -k_{13} & -k_{14} & \cdots & -k_{1n} \\ -k_{21} & k_{22} & -k_{23} & -k_{24} & \cdots & -k_{2n} \\ -k_{31} & -k_{32} & k_{33} & -k_{34} & \cdots & -k_{3n} \\ -k_{41} & -k_{42} & -k_{43} & k_{44} & \cdots & -k_{4n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -k_{n1} & -k_{n2} & -k_{n3} & -k_{n4} & \cdots & k_{nn} \end{bmatrix} \tag{5.27}$$

and physical realisability requires that

$$\text{(a)} \qquad k_{ij} = k_{ji} \geq 0 \text{ , for all } 1\leq i,j\leq n, \text{ and} \qquad (5.28)$$

$$\text{(b)} \qquad k_{jj} - \sum_{\substack{i=1 \\ i\neq j}}^{n} k_{ji} \geq 0 \qquad (j=1,2,...,n)$$

The physical parameters appearing in the first column of K can be approximated by solving

$$\min_{\mathbf{k}_1} \|\Phi^T\mathbf{k}_1 - a_1\| \text{ , subject to } \mathbf{G}_{(1)}\mathbf{k}_1 \geq 0 \qquad (5.29)$$

$$\text{where} \quad G_{(1)} = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 \end{bmatrix} \text{ ; } G_{(1)} \in \mathbb{R}^{n\,x\,n}. \qquad (5.30)$$

Then setting

$$\mathbf{z}_1 = \mathbf{G}_{(1)}\mathbf{k}_1 \qquad (5.31)$$

and

$$E_{(1)} = \Phi^T G_{(1)}^{-1} \qquad (5.32)$$

we find that (5.29) can be transformed to the standard non-negative least squares form.

$$\min_{\mathbf{z}_1} \|\mathbf{E}_{(1)}\mathbf{z}_1 - a_1\|, \text{ subject to } \mathbf{z}_1 \geq 0. \qquad (5.33)$$

The solution $\mathbf{z}_1$ of (5.33) then determines the physical stiffnesses in $\mathbf{k}_1$, as shown

$$\mathbf{k}_1 = \mathbf{G}_{(1)}^{-1}\mathbf{z}_1. \qquad (5.34)$$

In a similar manner the physical parameters appearing in $\mathbf{k}_j$, the j^{th} column of $\mathbf{K}$, can be approximated. By the symmetry of $\mathbf{K}$ the first (j-1) elements in the j^{th} step have been already determined in the previous steps. Hence denoting

$$\hat{k}_j = \left[-k_{1j}, \dots, -k_{j-1\,j}, \sum_{i=1}^{j-1} k_{ij}, 0, 0, \dots, 0 \right]^T \tag{5.35}$$

$$\bar{k}_j = \left[0, \dots, 0, \bar{k}_{jj}, -k_{j+1\,j}, \dots, -k_{nj} \right]^T \tag{5.36}$$

we may write

$$k_j = \hat{k}_j + \bar{k}_j \quad , \tag{5.37}$$

where $\hat{\mathbf{k}}_j$ is known and $\bar{\mathbf{k}}_j$ is to be determined.

Substituting (5.37) into equation (5.26) gives

$$\Phi^T \hat{k}_j + \Phi^T \bar{k}_j = a_j \tag{5.38}$$

Let Φ be partitioned in the form

$$\Phi = \left[\frac{\Psi}{\Phi_{(j)}} \right], \quad \Phi_{(j)} \in \mathbb{R}^{(n-j+1)\,x\,n} \quad . \tag{5.39}$$

Define

$$a_j^* = a_j - \Phi^T \hat{k}_j \quad , \tag{5.40}$$

and by truncating the zero elements of the vector $\bar{\mathbf{k}}_j$ in (5.36), set

$$k_j^* = \left[\bar{k}_{jj}, -k_{j+1\,j}, \dots, -k_{nj} \right] \quad . \tag{5.41}$$

Then a non-negative $\mathbf{k}_j^*$ which approximates the solution of equation (5.38) in least square sense, can be obtained by solving

$$\min_{\mathbf{k}_j^*} \| \Phi^T_{(j)} \mathbf{k}_j^* - a_j^* \|, \text{ subject to } \mathbf{G}_{(j)} \mathbf{k}_j^* \geq 0 \tag{5.42}$$

$$\text{where} \quad G_{(j)} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \end{bmatrix} \quad ; \quad G_{(j)} \in \mathbb{R}^{(n-j+1)\,x\,(n-j+1)} \tag{5.43}$$

Denote

$$\mathbf{z}_j = \mathbf{G}_{(j)}\mathbf{k}_j^* \tag{5.44}$$

and

$$E_{(j)} = \Phi_{(j)}^T G_{(j)}^{-1} \tag{5.45}$$

then the standard non-negative least square form of (5.42) is given by

$$\min_{\mathbf{z}_j} \ \| \mathbf{E}_{(j)}\mathbf{z}_j - a_j^* \|, \text{ subject to } \mathbf{z}_j \geq 0. \tag{5.46}$$

Solving (5.46) for $\mathbf{z}_j$, then $\mathbf{k}_j^*$ can be obtained by

$$\mathbf{k}_j^* = \mathbf{G}_{(j)}^{-1}\mathbf{z}_j \ . \tag{5.47}$$

This determines the unknown stiffnesses in the j^{th} column of **K**. Applying this process for j=2, ...,n evaluates the complete matrix **K** in a physically realisable form. The following algorithm summarises the above process.

Algorithm 5.3: Approximate Solution to Problem 2.2

Input: A modal matrix Φ (obtained in section 5.1), and a desired spectral matrix Λ^*.

Algorithm: 1) Calculate $\mathbf{A}$ using equation (5.23) and partition $\mathbf{A}$ as in (5.24). This determines the vectors a_j , j=1,2,...,n.

2) Construct the matrix $\mathbf{G}_{(1)}$ as in (5.30).
3) Determine the matrix $\mathbf{E}_{(1)}$ using (5.32).
4) Determine the vector $\mathbf{z}_1$ by solving the non-negative least square problem (5.33).
5) Obtain $\mathbf{k}_1$ from (5.34). This determines the first row and column of $\mathbf{K}=[k_{ij}]$.
6) for j = 2, 3, ..., n
   (a) Set the vector $\hat{\mathbf{k}}_j$ using (5.35).
   (b) Obtain $\mathbf{\Phi}_{(j)}$ by partitioning $\mathbf{\Phi}$ as in (5.39).
   (c) Determine a_j^* from equation (5.40).
   (d) Construct $\mathbf{G}_{(j)}$ as in (5.43) and calculate $\mathbf{E}_{(j)}$ by (5.45).
   (e) Determine $\mathbf{z}_j$ by solving the non-negative least square problem (5.46).
   (f) Calculate $\mathbf{k}_j^*$ from equation (5.47).
   (g) Construct vector $\bar{\mathbf{k}}_j$ by augmenting $\mathbf{k}_j^*$ with zero elements as shown in (5.36) and (5.41).
   (h) Obtain $\mathbf{k}_j$ from equation (5.37). This determines the j-th row and column of $\mathbf{K}$, without destroying the symmetry of its first (j-1) rows and columns.

Output: A physically realisable stiffness matrix $\mathbf{K}$ which approximates the solution of Problem 2.2 in the local optimisation sense.

The computational expense of this process is approximately equal to solving n times a non-negative least square problem of dimensions $n, (n-1), \ldots, 1$. This is more efficient then solving an augmented system of dimension N.

5.4 Numerical Example

The local optimisation solution obtained by Algorithm 5.3 is not the optimal solution in the global sense, such as described in section 5.1. It is shown in this section by means of a numerical example that the quality of solution is not greatly affected.

Consider a solution to the problem described in Example 2.2.1. The desired dynamic properties, $\mathbf{\Lambda}^*$ and $\mathbf{\Phi}^*$, for a five degrees-of-freedom mass-spring system are:

$$\mathbf{\Lambda}^* = \text{diag}\ (\ 50,\ 100,\ 200,\ 400,\ 800\ )$$

and

$$\mathbf{\Phi}^* = \begin{bmatrix} 0.1 & -0.1 & 0.2 & -0.4 & 0.1 \\ 0.1 & 0.1 & 0.2 & 0.1 & 0.3 \\ 0.1 & -0.1 & 0.3 & 0.2 & -0.4 \\ 0.1 & -0.3 & -0.1 & -0.1 & -0.1 \\ 0.3 & 0.2 & -0.1 & 0.1 & 0.1 \end{bmatrix}$$

and we wish to determine physically realisable **M** and **K** which have dynamic characteristics as close as possible to the above data.

It was shown that there is no exact solution for these data since **M** and **K** obtained by equations (2.7) and (2.8) are not physically realisable solutions for a mass-spring system. We now show how to determine an optimal solution.

Applying Algorithm 5.2, we obtain a diagonal matrix **D** and an orthonormal matrix **Q**, such that $\mathbf{\Phi} = \mathbf{DQ}$ is given by

$$\Phi = \begin{bmatrix} 0.1232 & -0.0333 & 0.2107 & -0.3988 & 0.0414 \\ 0.0578 & -0.0363 & 0.1922 & 0.1503 & 0.2680 \\ 0.1337 & -0.0707 & 0.3292 & 0.1820 & -0.3767 \\ 0.1186 & -0.2856 & -0.1045 & 0.0061 & 0.0073 \\ 0.3233 & 0.1725 & -0.1021 & 0.0324 & 0.0088 \end{bmatrix}$$

and $\| \Phi^{*} - \Phi \|_F$ is minimised.

Substituting Φ in equations (2.7) and (2.8), we obtain

$$M = diag(\ 4.5152,\ 7.3516,\ 3.2650,\ 9.3757,\ 6.8583)$$

and

$$K = \begin{bmatrix} 1523.9 & -216.2 & -392.0 & -146.0 & -240.6 \\ -216.2 & 4009.4 & -1356.4 & -48.7 & 10.5 \\ -392.0 & -1356.4 & 1597.1 & -178.5 & -135.7 \\ -146.0 & -48.7 & -178.5 & 976.0 & -47.9 \\ -240.6 & 10.5 & -135.7 & -47.9 & 506.5 \end{bmatrix}$$

The mass matrix **M** is now physically realisable, whereas the stiffness matrix **K** is not realisable. Therefore, setting $\Lambda = \Lambda^{*}$ and applying the global optimisation procedure described in section 5.2, we obtain the following realisable stiffness matrix

$$K' = \begin{bmatrix} 1512.0 & -227.2 & -337.7 & -144.3 & -245.4 \\ -227.2 & 4012.4 & -1277.9 & -41.7 & 0 \\ -337.7 & -1277.9 & 1690.1 & -37.6 & -36.9 \\ -144.3 & -41.7 & -37.6 & 939.8 & -69.1 \\ -245.4 & 0 & -36.9 & -69.1 & 454.2 \end{bmatrix}$$

A mass-spring system corresponding to the mass matrix **M** and the stiffness matrix $\mathbf{K}'$ is shown in Figure 5.2.

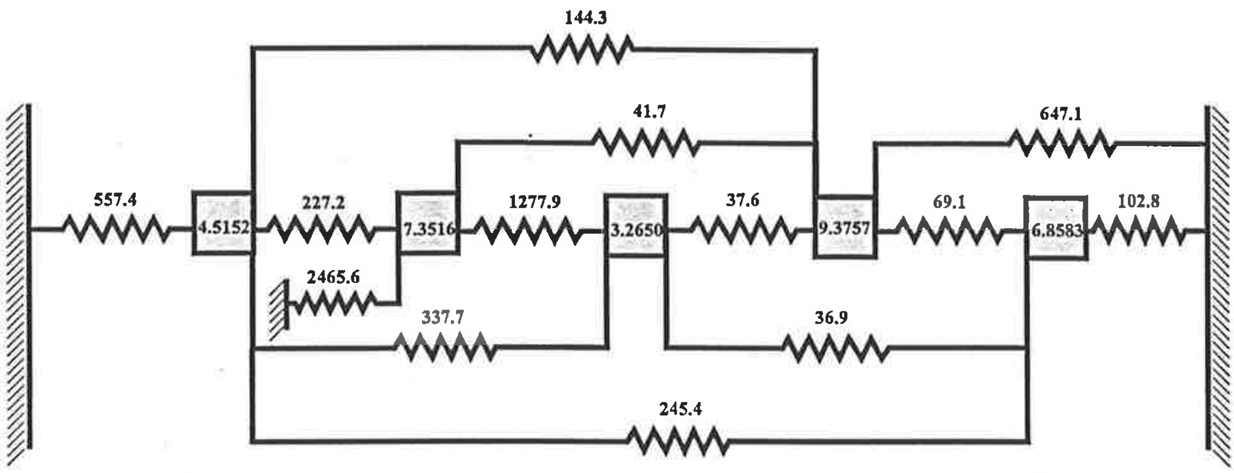


Figure 5.2: A mass-spring system corresponding to **M** and $\mathbf{K}'$.

This realisable mass-spring system has the following modal data

$$\mathbf{\Lambda}' = diag(\ 52.8,\ 101.2,\ 214.3,\ 401.3,\ 795.2)$$

$$\mathbf{\Phi}' = \begin{bmatrix} 0.0982 & -0.0265 & 0.2488 & -0.3848 & 0.0335 \\ 0.0267 & -0.0165 & 0.1995 & 0.1600 & 0.2639 \\ 0.0553 & -0.0273 & 0.3379 & 0.2010 & -0.3846 \\ 0.0939 & -0.3087 & -0.0488 & 0.0137 & -0.0002 \\ 0.3538 & 0.1203 & -0.0691 & 0.0375 & 0.0012 \end{bmatrix}$$

This compares reasonably well with the desired properties $\mathbf{\Lambda}^*$ and $\mathbf{\Phi}^*$.

However, the above solution is computationally expensive. Applying Algorithm 5.3 to the above example, we obtain a physically realisable stiffness matrix

$$K'' = \begin{bmatrix} 1523.9 & -216.2 & -392.0 & -146.0 & -240.6 \\ -216.2 & 4009.3 & -1356.4 & -48.7 & 0 \\ -392.0 & -1356.4 & 1748.4 & 0 & 0 \\ -146.0 & -48.7 & 0 & 976.0 & -47.9 \\ -240.6 & 0 & 0 & -47.9 & 506.5 \end{bmatrix}$$

Figure 5.3 shows a mass-spring system corresponding to the mass matrix **M** and the stiffness matrix $\mathbf{K}''$.

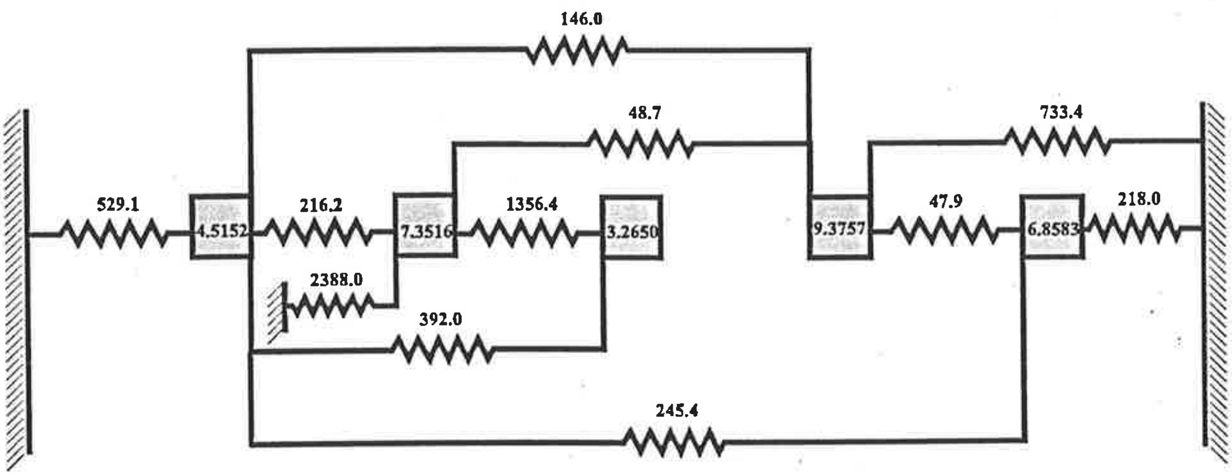


Figure 5.3: A mass-spring system corresponding to **M** and $\mathbf{K}''$.

This mass-spring system has the following modal properties

$$\Lambda'' = diag(\ 62.6,\ 104.0,\ 200.5,\ 407.6,\ 821.7)$$

$$\Phi'' = \begin{bmatrix} 0.0985 & -0.0314 & 0.2479 & -0.3838 & 0.0454 \\ 0.0251 & -0.0174 & 0.2013 & 0.1683 & 0.2574 \\ 0.0471 & -0.0255 & 0.3384 & 0.1863 & -0.3927 \\ 0.0841 & -0.3116 & -0.0474 & 0.0161 & -0.0028 \\ 0.3578 & 0.1087 & -0.0661 & 0.0400 & -0.0021 \end{bmatrix}$$

We note that the global optimal solution is slightly better than the local one. They both, however, lead to essentially similar systems. Table 5.1 shows the cosines of the angles between the desired mode shapes and the modes of the physical systems which have been obtained. Let θ be the angle between two eigenvectors. Then cos(θ)=1 indicates identical eigenvectors.

Mode No., j	1	2	3	4	5
$\mathrm{Cos}\measuredangle(\phi^*_j,\phi_j)$	0.9885	0.9210	0.9988	0.9586	0.9580
$\mathrm{Cos}\measuredangle(\phi^*_j,\phi'_j)$	0.9648	0.9015	0.9852	0.9541	0.9558
$\mathrm{Cos}\measuredangle(\phi^*_j,\phi''_j)$	0.9587	0.8948	0.9845	0.9506	0.9571

Table 5.1: Cosines of angles between the desired mode shapes and their approximations

We asked for mass-normalised eigenvectors. Hence the amplitude ratio between the desired mode shapes and their approximation is of interest as well. These amplitude ratios are given in Table 5.2.

Mode No., j	1	2	3	4	5
$\Vert \phi_j \Vert / \Vert \phi^*_j \Vert$	1.0918	0.8617	1.0541	0.9688	0.8773
$\Vert \phi'_j \Vert / \Vert \phi^*_j \Vert$	1.0649	0.8346	1.0835	0.9683	0.8838
$\Vert \phi''_j \Vert / \Vert \phi^*_j \Vert$	1.0657	0.8323	1.0835	0.9605	0.8915

Table 5.2: Amplitude ratios between the desired mode shapes and their approximations

The results in Tables 5.1 and 5.2 present a good agreement between the desired mode shapes and the modes obtained.

5.5 Conclusions

The problem of constructing a mass-spring system with prescribed eigenvalues and mode shapes has been addressed. This is a non-linear approximation problem since the number of constraints, the eigendata, is larger than the number of free parameters, the number of masses and springs in the system.

It is shown that the problems of determining the mass and stiffness matrices can be solved separately. First, an optimal set of mode shapes associated with a physically realisable mass matrix is obtained. This is done by a convergent iterative algorithm. Then a physically realisable stiffness matrix is determined using the optimal mode shapes obtained in the previous stage.

Two methods of obtaining a physically realisable stiffness matrix have been suggested. One method determines a global optimal solution in a least square sense. This method involves non-linear optimisation of large matrices of order N for a problem with n degrees of freedom. The other method breaks the problem into n sub-problems of small dimensions and determines a local optimal solution for each sub-problem. The result is a computationally economical method of solution. It is shown through a numerical example that both methods lead to similar solutions.

We note that in the given numerical example, the modal shapes of Φ, Φ' and Φ'' do not have the same number of changes of sign among the elements of their columns as in the corresponding columns of Φ^*. Based on this criteria, we may say that the mode shapes of Φ, Φ' and Φ'' are *qualitatively* different from the corresponding mode shapes of Φ^*. In the analysis we used to solve Problem 2, we have optimised the modal vectors to be as close as possible to the prescribed modal vectors in the Frobenius norm sense, without imposing the additional constraints of sign changes. Thus, extending the above method to allow for such sign change control would constitute a significant improvement to the method.

The results presented in this paper may be applied in designing physically realisable systems with prescribed spectral constrains, and in identifying realisable systems from modal test data.

Section 6 *PROBLEM 2(a):*

RECONSTRUCTION FROM TRUNCATED MODAL AND SPECTRAL DATA

In section 5 we have considered a problem associated with reconstruction of a physically realisable mass-spring system from the prescribed set of modal and spectral data. The main assumption on which we based our analysis is that the prescribed modal and spectral data are such that the resultant matrices $\mathbf{\Phi}^*$ and $\mathbf{\Lambda}^*$ are full matrices of size nxn. Thus in section 5 we have developed a solution to the problem of reconstruction from a *complete* set of data. In many (perhaps most) practical applications the desired modal and spectral properties are usually specified for only the first few lower modes, thus leading to a *truncated* set of prescribed modal and spectral data. In this section we show how to reconstruct various models of vibratory systems from such truncated sets of data. Since in all other respects, apart from the truncation of $\mathbf{\Phi}^*$ and $\mathbf{\Lambda}^*$, this problem is identical to Problem 2, we designate it as Problem 2(a).

Partitioning Φ^* and Λ^* in the form :

$$\Phi^* = [\ \Phi_1^* \mid \Phi_2^*\ ],\quad \Phi_1^* \text{ is an } n\text{x}l \text{ real matrix, } l < n \tag{6.1}$$

and

$$\Lambda^* = \left[\begin{array}{c|c} \Lambda_1^* & \mathbf{0} \\ \hline \mathbf{0} & \Lambda_2^* \end{array}\right],\quad \Lambda_1 = \text{diag}(\lambda_1^*, \ldots, \lambda_l^*), \tag{6.2}$$

we assume that Φ_1^* and Λ_1^* are specified, while the submatrices Φ_2^* and Λ_2^* remain unknown. Then the formulation of a problem we want to solve is as follows.

Problem 2(a): *Reconstruction from truncated modal and spectral data.*

Given Φ_1^* and Λ_1^*, determine physically realisable mass and stiffness matrices **M** and **K**, such that the vibratory system contains modes which are as close as possible to the prescribed data.

We note that the orthogonality relations

$$\Phi_1^{*T}\mathbf{M}\Phi_1^* = \mathbf{I}_l \tag{6.3}$$

and

$$\Phi_1^{*T}\mathbf{K}\Phi_1^* = \Lambda_1^* \tag{6.4}$$

are still valid. However, due to the effects of truncation, obtaining **M** and **K** which satisfy (6.3) and (6.4) respectively is not necessarily a solution to Problem 2(a). In fact, there may exist many different combinations of **M** and **K** that satisfy (6.3) and (6.4), but which are not solutions to Problem 2(a). Thus, in order for us to have a solution, **M** and **K** should satisfy

$$\mathbf{K}\Phi_1^* = \mathbf{M}\Phi_1^*\Lambda_1^*, \tag{6.5}$$

and, to maintain the mass-normalisation properties, **M** should also satisfy (6.3).

As in the method presented in section 5, we realise that the solutions for **M** and **K** may be obtained separately. First, we find **M** satisfying (6.3), and then determine **K** such that (6.5) holds.

6.1 Reconstruction of a Mass Matrix

The mass matrix **M**, may be written in the following form

$$M = \sum_{j=1}^{n} m_j B_j^{(M)} \tag{6.6}$$

where m_j is the mass of the j^{th} element, and $\mathbf{B}_j^{(\mathbf{M})}$ is the mapping matrix. For a mass-spring model $\mathbf{B}_j^{(\mathbf{M})}$ is as follows

$$B_j^{(M)} = \left[b_{pq}^{(M)}\right] = \begin{cases} b_{jj}^{(M)} = 1 \\ b_{pq}^{(M)} = 0 \quad elsewhere, \quad p \neq q \end{cases} \tag{6.7}$$

Substituting equation (6.7) into equation (6.3), we obtain

$$I_l = \sum_{j=1}^{n} m_j \Phi_1^{*T} B_j^{(M)} \Phi_1^* \tag{6.8}$$

Each element of $\mathbf{I}_l$ is given by

$$I_l = [\delta_{pq}] = \begin{cases} \delta_{pq} = 1, \ p = q \\ \delta_{pq} = 0, \ p \neq q \end{cases} \tag{6.9}$$

Partitioning $\mathbf{\Phi}_1^*$ into column vectors as shown

$$\Phi_1^* = \left[\phi_1^* \mid \phi_2^* \mid \ldots \mid \phi_l^* \right] \tag{6.10}$$

then from (6.8) each element δ_{pq} must be equal to

$$\delta_{pq} = \sum_{j=1}^{n} m_j \, \phi_p^{*\,T} B_j^{(M)} \phi_q^{*} \tag{6.11}$$

Let N= ½(l^2+ l) and construct the vectors

$$\mathbf{y_M} = (y_1^{(M)}, y_2^{(M)}, \ldots, y_N^{(M)})^T = (\delta_{11}, \delta_{12}, \delta_{13}, \ldots, \delta_{1l}, \delta_{22}, \delta_{23}, \ldots, \delta_{2l}, \delta_{33}, \ldots, \delta_{ll})^T \tag{6.12}$$

and

$$\mathbf{m} = (m_1, m_2, \ldots, m_n)^T \tag{6.13}$$

Denote

$$F_M = \left[f_{ij}^{(M)}\right] = \frac{\partial y_i^{(M)}}{\partial m_j}, \quad (i = 1, 2, \ldots, N;\ j = 1, 2, \ldots, n) \tag{6.14}$$

then all the elements of $\mathbf{F_M}$ can be evaluated using equation (6.11). Equation (6.8) can be written in a vector form

$$\mathbf{F_M}\,\mathbf{m} = \mathbf{y_M}. \tag{6.15}$$

Since $\mathbf{F_M}$ and $\mathbf{y_M}$ are known, (6.15) can be solved for **m**, and the mass matrix **M** can then be determined from the elements of vector **m** by equation (6.6).

We note that in order to obtain a solution for the system of size nxl ($l<n$), we need to solve an augmented system (6.15) of size N$\times n$ (N= ½(l^2+ l)). However, in this case augmentation is based on the smaller dimension l, whereas number of independent parameters available for optimisation is fixed at n. Therefore depending on the value of l there are three possibilities for the solution to (6.15).

Set $r = n$-N, then if r>0 there will be a family of solutions for **m**. This family of solutions is characterised by the following equation

$$\mathbf{m} = \mathbf{F_M}^{\dagger}\mathbf{y_M} + \mathbf{V_r b} \tag{6.16}$$

where $\mathbf{F_M}^{\dagger}$ is the Moore-Penrose pseudoinverse of $\mathbf{F_M}$, **b** is an arbitrary vector of dimension rx1, and $\mathbf{V_r}$ is a matrix of dimension n xr which is obtained by a following procedure

Calculate singular value decomposition $\mathbf{F_M} = \mathbf{USV^T}$, *and partition*

the nxn matrix $\mathbf{V} = [\mathbf{V_N} \mid \mathbf{V_r}]$, *where* $\mathbf{V_N}$ *is n*xN, *and* $\mathbf{V_r}$ *is nxr.* (6.17)

If $r = 0$, then $\mathbf{F_M}$ is a full square matrix, and there will be one unique solution for **m**. This unique solution is

$$\mathbf{m} = \mathbf{F_M}^{-1}\mathbf{y_M} \,. \tag{6.18}$$

And finally, if r<0, then there are no solutions for **m**, and only an approximate solution (which is optimal in a least squares sense) can be obtained by

$$\mathbf{m} = \mathbf{F_M}^{\dagger}\mathbf{y_M} \,. \tag{6.19}$$

However, we also note that in order to satisfy the physical realisability criteria we require all the elements of **m** to be positive. Therefore, if solutions of (6.16), (6.18) and (6.19) do not yield positive **m**, it may have to be determined by solving the following *non-negative least squares problem*

$$\min_{\mathbf{m}} \; \| \mathbf{F_M}\, \mathbf{m} - \mathbf{y_M} \|_2 \,, \text{ subject to } \mathbf{m} \geq 0. \tag{6.20}$$

This will produce an optimal non-negative solution to the vector **m** in a least square sense.

The above procedure is summarised by the following algorithm.

Algorithm 6.1: Determination of a Mass Matrix

Input: Desired modal data $\Phi_1^*(n\text{x}l)$.

Algorithm:

1) Column partition Φ_1^* as in (6.10).
2) Set N= ½(l^2+ l).
3) Construct vectors $\mathbf{y}_M$ as in (6.12), via (6.9).
4) Form vector **m** of dimension *nx1* as in (6.13).
5) Construct matrix $\mathbf{F}_M$ using (6.14) and (6.11).
6) (a) If n > N, then determine **m** by equation (6.16),
   (b) if n = N, then determine **m** by (6.18),
   (c) if n < N, then determine **m** by (6.19).
7) If **m** obtained in step 6 is not non-negative, then determine **m** by solving (6.20).
8) Construct **M** from the elements of **m** using (6.6).

Output: Physically realisable mass matrix **M**.

We note that Algorithm 6.1 can be used to determine **M** corresponding to any chosen analytical model, not just for a mass-spring system. All that is required is to use an appropriate mapping matrix $\mathbf{B}_j^{(M)}$ in equation (6.6).

6.2 Reconstruction of a Stiffness Matrix

Setting $$\mathbf{A}^T = \mathbf{M}\Phi_1^* \Lambda_1^*, \tag{6.21}$$

equation (6.5) becomes $$\mathbf{K}\Phi_1^* = \mathbf{A}^T. \tag{6.22}$$

Since Φ_1^* and $\mathbf{A}$ are known, the stiffness matrix $\mathbf{K}$ can be determined. As in section 5, we may determine both the global and local optimal solutions to equation (6.22).

6.2.1 Local optimal solution

Taking the transpose of (6.22) and using the fact that $\mathbf{K}$ is symmetric, we obtain

$$\Phi_1^{*T}\mathbf{K} = \mathbf{A}. \tag{6.23}$$

Partition $\mathbf{A}$ and $\mathbf{K}$ as follows

$$\mathbf{A} = [\ \boldsymbol{a}_1 \mid \boldsymbol{a}_2 \mid \boldsymbol{a}_3 \mid \ldots. \mid \boldsymbol{a}_n\ ], \tag{6.24}$$

$$\mathbf{K} = [\mathbf{k}_1 \mid \mathbf{k}_2 \mid \mathbf{k}_3 \mid \ldots..\mid \mathbf{k}_n\ ]. \tag{6.25}$$

Then from equation (6.23), each column of $\mathbf{A}$ is given by

$$\Phi_1^{*T}\mathbf{k}_j = \boldsymbol{a}_j \quad (j = 1, \ldots, n)\ . \tag{6.26}$$

Equation (6.26) is identical to equation (5.26), except that matrix Φ_1^* has the dimensions of nxl. The same procedure as in section 5.3 can then be used to find a solution for $\mathbf{K}$, sequentially column by column. The resulting solution would be a local approximation for **K**. However, a general multi-connected mass-spring system of order n has ½(n^2+n) independent spring elements, while the total number of constraints that need to be satisfied in equation (6.23) is n times l (and where $l < n$). Thus, if l is less than ½(n+1), then there will exist a family of exact solutions for **K**. By selecting a local optimisation method for calculating elements of **K**, we can only obtain approximate solutions to equation (6.23). Therefore, in this particular problem, a global approach for the solutions of **K** appears to be more suitable.

6.2.2 Global solution for stiffness matrix

The stiffness matrix **K** for a system with J independent spring elements, may be written in the following form

$$K = \sum_{q=1}^{J} s_q \, B_q^{(K)} \tag{6.27}$$

where s_q is the stiffness of the q^{th} spring, and $\mathbf{B}_q^{(K)}$ is the mapping matrix corresponding to the chosen analytical model for the system. For example, equations (5.13) and (5.14) describe the **K** and $\mathbf{B}_q^{(K)}$ for a multi-connected mass-spring system of order n, and for such system J =½(n^2 + n).

Substituting (6.27) into (6.23), we obtain

$$\sum_{q=1}^{J} s_q \, \Phi_1^{*T} B_q^{(K)} = A \quad . \tag{6.28}$$

Setting

$$B^{(q)} = \left[b_{ij}^{(q)}\right] = \Phi_1^{*T} B_q^{(K)} , \tag{6.29}$$

then each ij^{th} element of **A**, a_{ij}, is given by

$$a_{ij} = \sum_{q=1}^{J} s_q \, b_{ij}^{(q)} . \tag{6.30}$$

Since there are $n \times l$ independent elements in **A**, in order to obtain a global solution for **K** we need to solve simultaneously a system of $n \times l$ linear equations of the form (6.30). We proceed in a similar fashion to the analysis described by equations (6.12) - (6.20).

Let $\bar{N} = n \times l$ and construct the vectors

$$\mathbf{y}_K = (y_1^{(K)}, y_2^{(K)}, \ldots, y_{\bar{N}}^{(K)})^T = (a_{11}, a_{12}, \ldots, a_{1n}, a_{21}, \ldots, a_{2n}, a_{31}, \ldots, a_{ln})^T \tag{6.31}$$

and

$$\mathbf{s} = (s_1, s_2, \ldots, s_J)^T. \tag{6.32}$$

Denote

$$F_K = \left[f_{ij}^{(K)}\right] = \frac{\partial y_i^{(K)}}{\partial s_j} , \quad (i = 1, 2, \ldots, \bar{N};\ j = 1, 2, \ldots, J) \tag{6.33}$$

then all the elements of $\mathbf{F_K}$ can be evaluated using equation (6.30). Equation (6.22) can be written in a vector form

$$\mathbf{F_K}\,\mathbf{s} = \mathbf{y_K}. \tag{6.34}$$

Since $\mathbf{F_K}$ and $\mathbf{y_K}$ are known, (6.34) can be solved for **s**, and the stiffness matrix **K** can then be determined from the elements of vector **s** via equation (6.27).

As in a case of solution to (6.15), there are three possibilities for the solution to (6.34). Setting $r = J - \bar{N}$, then if $r > 0$ there will be a family of solutions for $\mathbf{s}$. This family of solutions is characterised by the following equation

$$\mathbf{s} = \mathbf{F_K}^{\dagger}\mathbf{y_K} + \mathbf{V_r b} \tag{6.35}$$

where $\mathbf{F_K}^{\dagger}$ is the Moore-Penrose pseudoinverse of $\mathbf{F_K}$, $\mathbf{b}$ is an arbitrary vector of dimension rx1, and $\mathbf{V_r}$ is a matrix of dimension J$\mathrm{x}r$ which is obtained by the following procedure

Calculate singular value decomposition $\mathbf{F_K} = \mathbf{USV^T}$, *and partition the JxJ matrix* $\mathbf{V} = [\mathbf{V}_{\bar{N}} \mid \mathbf{V_r}]$, *where* $\mathbf{V}_{\bar{N}}$ *is* $J\mathrm{x}\bar{N}$, *and* $\mathbf{V_r}$ *is Jxr.* (6.36)

If $r = 0$, then $\mathbf{F_K}$ is a full square matrix, and there will be one unique solution for $\mathbf{s}$. This unique solution is

$$\mathbf{s} = \mathbf{F_K}^{-1}\mathbf{y_K} \, . \tag{6.37}$$

And finally, if $r<0$, then there are no solutions for $\mathbf{s}$, and only an approximate solution (which is optimal in a least squares sense) can be obtained by

$$\mathbf{s} = \mathbf{F_K}^{\dagger}\mathbf{y_K} \, . \tag{6.38}$$

However, we also note that in order to satisfy the physical realisability criteria we require all the elements of $\mathbf{s}$ to be positive. Therefore, if solutions of (6.35), (6.37) and (6.38) do not yield positive $\mathbf{s}$, it may have to be determined by solving the following *non-negative least squares problem*

$$\min_{\mathbf{s}} \; \| \mathbf{F_K s} - \mathbf{y_K} \|_2 , \text{ subject to } \mathbf{s} \geq 0. \tag{6.39}$$

This will produce an optimal non-negative solution to the vector $\mathbf{s}$ in a least square sense.

The above process is summarised by the following algorithm.

Algorithm 6.2: Determination of a Stiffness Matrix

Input: Desired modal and spectral data $\Phi_1^*$and Λ_1^*, and mass matrix **M** obtained by Algorithm 6.1.

Algorithm:

1) Calculate the matrix **A** by equation (6.21).
2) Set Ñ= $n \times l$.
3) Construct vectors $\mathbf{y}_K$ as in (6.31).
4) Form a vector **s** of dimension $Jx1$ as in (6.32).
5) Construct a matrix $\mathbf{F}_K$ using (6.33) and (6.30).
6) (a) If $J >$ Ñ, then determine **s** by equation (6.35),

   (b) if $J =$ Ñ, then determine **s** by (6.37),

   (c) if $J <$ Ñ, then determine **s** by (6.38).
7) If **s** obtained in step 6 is not non-negative, then determine **s** by solving (6.39).
8) Construct **K** from the elements of **s** using (6.27).

Output: Physically realisable mass matrix **K**.

Algorithm 6.2 can be used to determine **K** corresponding to any chosen analytical model by substituting an appropriate mapping matrix $\mathbf{B}_q^{(K)}$ in equation (6.27).

6.3 Numerical Example

Suppose that the desired dynamic properties, Λ_1^* and Φ_1^*, for a five degrees-of-freedom mass-spring system are:

$$\Lambda_1^* = \text{diag}\ (\ 50,\ 100,\ 200\ )$$

and

$$\Phi_1^* = \begin{bmatrix} 0.1 & -0.1 & 0.2 \\ 0.1 & 0.1 & 0.2 \\ 0.1 & -0.1 & 0.3 \\ 0.1 & -0.3 & -0.1 \\ 0.3 & 0.2 & -0.1 \end{bmatrix}$$

and we wish to determine physically realisable **M** and **K** which have dynamic characteristics as close as possible to the above data.

In the above data l=3 and n=5. First calculating the mass matrix **M**, we note that parameter $N=\frac{1}{2}(l^2+l)= 6$, and since n<N, there is no exact solution for **M**. The approximate solution for **M** is obtained by Algorithm 6.1 (using step 6(c)), and this solution is

$$\mathbf{M} = \text{diag}\ (\ 7.3458,\ 5.2048,\ 3.6097,\ 7.2876,\ 7.0552)\ .$$

Calculating **K**, we note that $J = \frac{1}{2}(n^2+n)= 15$ and $\bar{N} = n\times l=15$, and since $J= \bar{N}$, there is one exact solution for **K**. Applying Algorithm 6.2 (with step 6(b)), this solution is

$$K = \begin{bmatrix} 1062.6 & 45.5 & 85.3 & -222.9 & -184.4 \\ 45.5 & 540.9 & 219.4 & -128.9 & -105.0 \\ 85.3 & 219.4 & 436.5 & -45.7 & -142.3 \\ -222.9 & -128.9 & -45.7 & 753.2 & -35.6 \\ -184.4 & -105.0 & -142.3 & -35.6 & 513.1 \end{bmatrix} .$$

However, since some of the off-diagonal elements of the above **K** are positive, this stiffness matrix is not physically realisable. To obtain the physically realisable **K** we perform step 7 of the Algorithm 6.2 and obtain

$$K = \begin{bmatrix} 1217.7 & 0 & 0 & -254.4 & -180.9 \\ 0 & 925.6 & 0 & 0 & -203.1 \\ 0 & 0 & 634.9 & -110.7 & -82.9 \\ -254.4 & 0 & -110.7 & 805.5 & -69.7 \\ -180.9 & -203.1 & -82.9 & -69.7 & 537.9 \end{bmatrix}$$

This **K** is physically realisable, and the mass-spring system corresponding to the obtained mass and stiffness matrices, **M** and **K**, is shown in Figure 6.1.

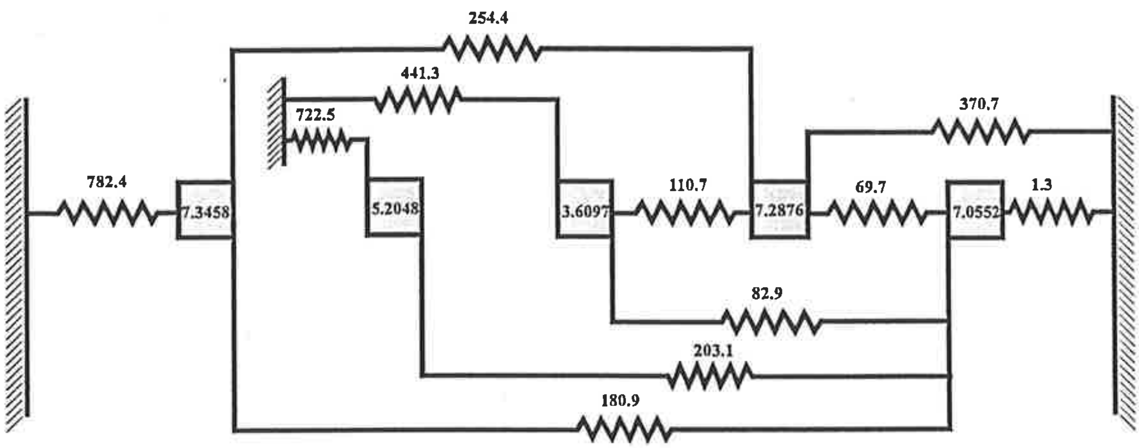


Figure 6.1: Mass-spring system corresponding to **M** and **K**.

The eigenvalues and the corresponding mass-normalised eigenvectors of this system are

$$\Lambda = \text{diag}\ (\ 50.8213,\ 100.6356,\ 172.2435,\ 184.2177,\ 198.3586)$$

and

$$\Phi = \begin{bmatrix} 0.1087 & -0.0997 & -0.2450 & 0.1048 & 0.2083 \\ 0.0966 & 0.0835 & 0.0309 & -0.3626 & 0.2084 \\ 0.0914 & -0.0739 & 0.3870 & 0.1972 & 0.2730 \\ 0.1371 & -0.3051 & 0.0426 & -0.0980 & -0.1181 \\ 0.3144 & 0.1653 & 0.0044 & 0.0593 & -0.1096 \end{bmatrix}$$

We note that the first, second and fifth modes in the $\mathbf{\Lambda}$ and $\mathbf{\Phi}$ above compare very well with the desired properties $\mathbf{\Lambda}_1^*$ and $\mathbf{\Phi}_1^*$. For a good correlation it is required that the eigenvalue ratio, the amplitude ratio of the eigenvectors and the values of cosines between the two eigenvectors are all as close as possible to 1. The values of these ratios and cosines are presented in Table 6.1.

Desired Mode, i	Corresponding Obtained Mode, j	Eigenvalue Ratio, λ_j/λ_i^*	Amplitude Ratio of Eigenvectors, $\|\phi_j\|/\|\phi_i^*\|$	Cosine of an angle between the two eigenvectors, $\mathrm{Cos}(\angle\phi_j\phi_i^*)$
1	1	1.0164	0.9399	0.9956
2	2	1.0064	1.0587	0.9944
3	5	0.9918	1.0072	0.9966

Table 6.1 : Comparison between the desired and the obtained modes.

The results summarised in Table 6.1 demonstrate that a very good correlation is achieved between the desired and the obtained modes.

6.4 Conclusions

In this section we have developed a method for reconstructing an analytical model of the vibratory system from a truncated set of desired modal properties. This method is general and is not restricted to any particular form of the mass and the stiffness matrices. Mass and stiffness matrices corresponding to any chosen analytical model can be reconstructed by this method.

It was shown that depending on the dimensions of the known desired data, we may obtain a family of solutions, a unique solution or an approximate solution which is optimal in a specified sense.

The presented numerical example demonstrated the application of the algorithm to a five degrees-of-freedom mass-spring system, and the obtained results showed a very good correlation between the desired and the obtained modal data.

Section 7 *PROBLEM 2(b): INDEPENDENT PARAMETER DECOMPOSITION*

In this section we define a special form for the mass and stiffness matrices, **M** and **K**, which are more general than these corresponding to a mass-spring system. It is demonstrated by examples that matrices of this form may correspond to various analytical models of vibratory systems, including a Finite Element model. We then show how to reconstruct these matrices from the prescribed modal and spectral data, $\mathbf{\Phi}^*$ and $\mathbf{\Lambda}^*$. The problem we solve is identical to Problem 2, with the exception that the shapes of **M** and **K** are not necessarily correspond to a discrete mass-spring model. Thus, we designate this problem as Problem 2(b). The definition of the new matrix type for **M** and **K** is given below.

Definition 7.1: *Independent Parameter Decomposition*

Suppose that a symmetric mass matrix $\mathbf{X}=[x_{ij}]$ of size *nxn* is such that all of its elements x_{ij} can be expressed as prescribed linear functions of *n* unknown independent parameters $\{\varkappa_1, \varkappa_2, ..., \varkappa_n\}$. Then we define this matrix to be an *independent parameter decomposable* if it can be described by a following product

$$\mathbf{X} = \mathbf{E_X}\, \mathbf{D_X}\, \mathbf{E_X}^T, \tag{7.1}$$

where the numerical values of all elements of $\mathbf{E_X}$ (nxn) are known, and $\mathbf{D_X} = \text{diag}\{d_1, d_2, ..., d_n\}$, $d_i \neq 0$, with each diagonal element d_i equal to some known linear function of $\{\varkappa_1, \varkappa_2, ..., \varkappa_n\}$, i.e

$$d_i = \sum_{j=1}^{n} \beta_{ij} \chi_j\,, \quad (i = 1, 2, ..., n) \tag{7.2}$$

and β_{ij} are known.

The above definition is demonstrated by the following examples.

<u>*Example 7.1: A simply-connected mass-spring system*</u>

Consider a simply-connected mass-spring system shown in Figure 7.1

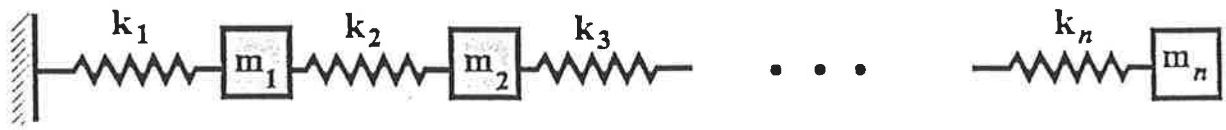


Figure 7.1: A simply-connected mass-spring system.

The mass and stiffness matrices corresponding to system have the following form

$$\mathbf{M} = \text{diag}\ (m_1, m_2, ..., m_n\ ),$$

and

$$K = \begin{bmatrix} k_1+k_2 & -k_2 & 0 & 0 & \cdots & 0 & 0 \\ -k_2 & k_2+k_3 & -k_3 & 0 & \cdots & 0 & 0 \\ 0 & -k_3 & k_3+k_4 & -k_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & -k_{n-2} & k_{n-2}+k_{n-1} & -k_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & -k_{n-1} & k_{n-1}+k_n & -k_n \\ 0 & 0 & 0 & \cdots & 0 & -k_n & k_n \end{bmatrix}$$

Both matrices are IP decomposable because they can be expressed as products

$$\mathbf{M} = \mathbf{E_M D_M E_M}^T \tag{7.3}$$

$$\mathbf{K} = \mathbf{E_K D_K E_K}^T \tag{7.4}$$

where $\mathbf{D_M} = \mathbf{M} = \text{diag}\ (m_1, m_2, ..., m_n\ )$ (7.5)

$\mathbf{E_M} = \mathbf{I}_n = \text{diag}\ (1,1,...,1),$ (7.6)

$\mathbf{D_K} = \text{diag}\ (k_1, k_2, ..., k_n)\ ,$ (7.7)

and

$$E_K = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix} . \tag{7.8}$$

Example 7.2: A 2 d.o.f. finite element model of an axially vibrating rod

The mass and stiffness matrices of a 2 d.o.f. finite element model for a longitudinally vibrating rod are

$$M = \begin{bmatrix} \dfrac{m_1 + m_2}{3} & \dfrac{m_2}{6} \\ \dfrac{m_2}{6} & \dfrac{m_2}{3} \end{bmatrix}$$

and

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} ,$$

where m_1, m_2, k_1 and k_2 are the masses and the stiffnesses of the two elements.

Both of these matrices are IP decomposable since they can be described by products of the form (7.3) and (7.4), respectively, with

$$D_M = \begin{bmatrix} \dfrac{m_1}{3} + \dfrac{m_2}{4} & 0 \\ 0 & \dfrac{m_2}{6} \end{bmatrix} , \qquad E_M = \begin{bmatrix} 1 & \dfrac{1}{\sqrt{2}} \\ 0 & \sqrt{2} \end{bmatrix} ,$$

and

$$D_K = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} , \qquad E_K = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} .$$

7.1 Reconstruction of IP Decomposable Mass Matrices

In Problem 2 we want to determine the mass and stiffness matrices **M** and **K** corresponding to a physically realisable mass-spring system, such that its modal and spectral properties, described by modal matrix $\mathbf{\Phi}$ and spectral matrix $\mathbf{\Lambda}$, are as close as possible to the prescribed modal and spectral matrices $\mathbf{\Phi}^*$ and $\mathbf{\Lambda}^*$. In this section we generalise the solution of Problem 2 for the IP decomposable matrices **M** and **K**.

As in section 5, the two problems of determining $\mathbf{\Phi}$ and $\mathbf{\Lambda}$ corresponding to a realisable system can be solved separately. Consider now the problem of determining the optimal mode shape matrix $\mathbf{\Phi}$.

If **M** is an IP decomposable mass matrix, then by definition 7.1 it may be expressed by a product of the form of equation (7.3), where $\mathbf{E_M}$ is a known matrix of size *n*x*n*, and

$$\boldsymbol{D}_M = diag\left(d_1^{(M)}, d_2^{(M)}, \ldots, d_n^{(M)}\right), \; d_i^{(M)} \neq 0, \tag{7.9}$$

is a diagonal matrix such that each element $d_i^{(M)}$ (i=1,...,n) is equal to a known linear function of mass elements $\{m_1, m_2, \ldots, m_n\}$, i.e

$$d_i^{(M)} = \sum_{j=1}^{n} \beta_{ij}^{(M)} m_j \, , \quad (i = 1, 2, \ldots, n) \tag{7.10}$$

where $\beta_{ij}^{(M)}$ are known.

The mass matrix **M** must also satisfy the orthogonality equation

$$\mathbf{M} = \mathbf{\Phi}^{-T}\mathbf{\Phi}^{-1}. \tag{7.11}$$

Let **Q** be an orthonormal matrix, that is, $\mathbf{QQ}^T = \mathbf{I}_n$, and set matrix

$$D = D_M^{-1/2} = diag\left\{ \frac{1}{\sqrt{d_1^{(M)}}}, \frac{1}{\sqrt{d_2^{(M)}}}, \ldots, \frac{1}{\sqrt{d_n^{(M)}}} \right\} , \tag{7.12}$$

then equations (7.11) and (7.3) will both be satisfied if the modal matrix $\boldsymbol{\Phi}$ is equal to

$$\Phi = E_M^{-T} D\, Q \quad . \tag{7.13}$$

Thus a solution to our problem can be obtained by determining a diagonal matrix **D** and an orthonormal matrix **Q**, such that

$$\min_{\mathbf{D},\mathbf{Q}} \| \boldsymbol{\Phi}^* - \mathbf{E}_M^{-T}\mathbf{DQ} \| \; . \tag{7.14}$$

In solving this problem we will make use of the algorithm 5.1 for solving an *orthogonal Procrustes problem*, which is described in section 5.

Thus setting

$$\mathbf{A} = \boldsymbol{\Phi}^* \tag{7.15}$$

$$\mathbf{B}_t = \mathbf{E}_M^{-T}\mathbf{D}_t, \text{ (where t designates an iteration index)}, \tag{7.16}$$

we may choose a diagonal matrix $\mathbf{D}_0$ as an initial guess and obtain an orthonormal $\mathbf{Q}_0$ which minimises $\| \mathbf{A} - \mathbf{B}_0\mathbf{Q}_0 \|_F$, by using Algorithm 5.1. We now show how to obtain a matrix $\mathbf{D}_1$ such that

$$\| \boldsymbol{\Phi}^* - \mathbf{E}_M^{-T}\mathbf{D}_1\mathbf{Q}_0 \|_F \leq \| \boldsymbol{\Phi}^* - \mathbf{E}_M^{-T}\mathbf{D}_0\mathbf{Q}_0 \|_F. \tag{7.17}$$

The Frobenius norm is invariant under orthonormal multiplication. Hence

$$\| \mathbf{\Phi}^* - \mathbf{E}_M^{-T}\, \mathbf{D}_1\mathbf{Q}_0 \|_F = \| \mathbf{\Phi}^*\mathbf{Q}_0^T - \mathbf{E}_M^{-T}\mathbf{D}_1 \|_F, \tag{7.18}$$

Define
$$\mathbf{R} = \mathbf{\Phi}^*\mathbf{Q}_0^T, \tag{7.19}$$

and denote
$$\epsilon = \| R - E_M^{-T}D_1 \|_F^2 \tag{7.20}$$

Using the equality

$$\|R - E_M^{-T}D_1\|_F^2 = trace(R^TR) + trace(D_1^TE_M^{-1}E_M^{-T}D_1) - 2\ trace(D_1^TE_M^{-1}R) \tag{7.21}$$

and setting

$$E = E_M^{-1}E_M^{-T} \tag{7.22}$$

$$F = E_M^{-1}\ R \quad , \tag{7.23}$$

equation (7.21) may be written as

$$\|R - E_M^{-T}D_1\|_F^2 = trace(R^TR) + trace(D_1^TED_1) - 2\ trace(D_1^TF) \quad . \tag{7.24}$$

Therefore, applying the knowledge that $\mathbf{D}_1$ is a diagonal matrix and its diagonal elements are defined in equation (7.12), we find that

$$\epsilon = trace\ (R^TR) + \sum_{i=1}^{n}\left[\frac{e_{ii}}{d_i^{(M)}} - \frac{2f_{ii}}{\sqrt{d_i^{(M)}}}\right] \tag{7.25}$$

where e_{ii} and f_{ii} are the diagonal elements of matrices **E** and **F** respectively . Differentiating ε with respect to $d_i^{(M)}$ and equating to zero to obtain the minimisation criteria, we obtain

$$\frac{\partial\varepsilon}{\partial d_i} = -\frac{e_{ii}}{d_i^{(M)^2}} + \frac{f_{ii}}{d_i^{(M)^{1.5}}} = 0 \tag{7.26}$$

Then from (7.26) it is clear that ϵ is minimised when

$$\frac{1}{\sqrt{d_i^{(M)}}} = \frac{f_{ii}}{e_{ii}} . \tag{7.27}$$

Thus, the residual error ϵ is minimised when the diagonal elements of $\mathbf{D}_1$ are equal to the ratios of the diagonal elements of $\mathbf{F}$ and the corresponding diagonal elements of $\mathbf{E}$. Having determined a diagonal matrix $\mathbf{D}_1$ satisfying (7.17), we can reapply the Algorithm 5.1 with $\mathbf{\Phi}^*$ and $\mathbf{D}_1$ as an input and find an orthonormal matrix $\mathbf{Q}_1$ such that

$$\| \mathbf{\Phi}^* - \mathbf{E}_M^{-T} \mathbf{D}_1\mathbf{Q}_1 \|_F \leq \| \mathbf{\Phi}^* - \mathbf{E}_M^{-T}\mathbf{D}_1\mathbf{Q}_0 \|_F . \tag{7.28}$$

Continuing in this manner iteratively, we obtain an approximate solution for the optimal modal matrix $\mathbf{\Phi}$. The mass elements $\{m_1, m_2, ..., m_n\}$ can then be determined by solving the system of linear equations (7.10) after substitution of the obtained values for $\{d_1^{(M)}, d_2^{(M)}, ..., d_n^{(M)}\}$. The following algorithm summarises this result.

***Algorithm 7.1**:* Approximate solution for the modal matrix $\mathbf{\Phi}$ and the mass elements

Input: An nxn modal matrix $\mathbf{\Phi}^*$, an nxn matrix $\mathbf{E}_M$ and coefficients $\beta_{ij}^{(M)}$ (i,j=1,...,n).

Algorithm:

1) Set initial guess $\mathbf{D}_0$ and a tolerance for convergence ϵ.
2) For t=0, 1, 2, ...
   a) Evaluate $\mathbf{C} = \mathbf{D}_t^T\mathbf{E}_M^{-1}\mathbf{\Phi}^*$.
   b) Compute the singular value decomposition $\mathbf{C}=\mathbf{U\Sigma V}^T$.
   c) Evaluate $\mathbf{Q}_t = \mathbf{UV}^T$.

d) Obtain $\mathbf{F} = \mathbf{E}_M^{-1}\mathbf{\Phi}^*\mathbf{Q}_t^T$.

e) Set $\mathbf{E} = \mathbf{E}_M^{-1}\mathbf{E}_M^{-T}$

f) Form $D_{t+1} = diag\left\{\frac{f_{11}}{e_{11}}, \frac{f_{22}}{e_{22}}, \ldots, \frac{f_{nn}}{e_{nn}}\right\}$

g) Test convergence

(i) Set $N_1 = \|\mathbf{\Phi}^* - \mathbf{E}_M^{-T}\mathbf{D}_t\mathbf{Q}_t\|_F$, $N_2 = \|\mathbf{\Phi}^* - \mathbf{E}_M^{-T}\mathbf{D}_{t+1}\mathbf{Q}_t\|_F$.

(ii) If $(N_1 - N_2) \leq \epsilon$, go to 3.

3) $\mathbf{\Phi} = \mathbf{E}_M^{-T}\mathbf{D}_{t+1}\mathbf{Q}_t$

4) Calculate

$$d_i^{(M)} = \left[\frac{e_{ii}}{f_{ii}}\right]^2, \quad i=1, \ldots, n.$$

5) Solve the following system of linear equations for m_j (j=1, ...,n).

$$\sum_{j=1}^{n} \beta_{ij}^{(M)} m_j = d_i^{(M)} \ ; \quad i=1, \ldots, n.$$

Output: An optimal modal matrix $\mathbf{\Phi}$ and a set of mass elements $\{m_1, m_2, \ldots, m_n\}$.

It follows from (7.17) and (7.28) that the norm $\|\mathbf{\Phi}^* - \mathbf{E}_M^{-T}\mathbf{D}_t\mathbf{Q}_t\|_F$ is a monotonic non-increasing function of an iteration index t. The Algorithm 7.1 thus necessarily converge.

7.2 Reconstruction of IP Decomposable Stiffness Matrices

In this section we will use the results obtained from Algorithm 7.1 to reconstruct an IP decomposable stiffness matrix **K**, such that the norm $\|\mathbf{\Lambda}^{*}-\mathbf{\Lambda}\|$ (which represents the difference between the desired and the obtained eigenvalues) is minimised.

If **K** is an IP decomposable stiffness matrix, then by definition 7.1 it may be expressed by a product of the form of equation (7.4), where $\mathbf{E_K}$ is a known matrix of size *n*x*n*, and

$$\boldsymbol{D}_K = diag\left(d_1^{(K)},\ d_2^{(K)},\ \ldots,\ d_n^{(K)}\right),\ d_i^{(K)} \neq 0, \tag{7.29}$$

is a diagonal matrix such that each element $d_i^{(K)}$ (i=1,...,n) is equal to a known linear function of stiffness elements $\{k_1, k_2, \ldots, k_n\}$, i.e

$$d_i^{(K)} = \sum_{j=1}^{n} \beta_{ij}^{(K)} k_j\ , \quad (i = 1,\ 2,\ \ldots,\ n) \tag{7.30}$$

where $\beta_{ij}^{(K)}$ are known.

The orthogonality equation for the stiffness matrix is

$$\mathbf{\Phi}^T\mathbf{K}\mathbf{\Phi} = \mathbf{\Lambda}. \tag{7.31}$$

Thus substituting (7.4) into (7.31), we obtain

$$\mathbf{\Phi}^T\ \mathbf{E_K}\mathbf{D_K}\mathbf{E_K}^T\mathbf{\Phi} = \mathbf{\Lambda}. \tag{7.32}$$

Let

$$\mathbf{G} = \mathbf{E_K}^T\mathbf{\Phi}\ , \tag{7.33}$$

then equation (7.32) becomes

$$\mathbf{G}^T\mathbf{D_K}\mathbf{G} = \mathbf{\Lambda}\ . \tag{7.34}$$

A diagonal matrix $\mathbf{D_K}$ may be expressed in terms of its elements as

$$D_K = \sum_{i=1}^{n} d_i^{(K)} H_i \tag{7.35}$$

where H_i is a nxn mapping matrix and it is equal to

$$H_i = \left[h_{pq}\right] = \begin{cases} h_{ii} = 1 \\ h_{pq} = 0 \quad elsewhere \end{cases} \tag{7.36}$$

Substituting (7.35) into (7.34) we obtain

$$\Lambda = \sum_{i=1}^{n} d_i^{(K)} \left(G^T H_i G\right) \ . \tag{7.37}$$

Partitioning **G** into column vectors

$$G = [\ g_1 \mid g_2 \mid .... \mid g_n\ ]\ , \tag{7.38}$$

then each pq[th] element of **Λ** is given by

$$\lambda_{pq} = \sum_{i=1}^{n} d_i^{(K)} \left(g_p^T H g_q\right) \ . \tag{7.39}$$

Let N= ½(n^2+ n) and construct the vectors

$$\mathbf{y} = (y_1, y_2, y_3, ..., y_N)^T = (\lambda_{11}, \lambda_{12}, ..., \lambda_{1n}, \lambda_{22}, \lambda_{23}, ..., \lambda_{2n}, \lambda_{33}, ..., \lambda_{nn})^T \tag{7.40}$$

and

$$\mathbf{d} = (d_1^{(K)}, d_2^{(K)}, ..., d_n^{(K)})^T \ . \tag{7.41}$$

Denote

$$P = \left[p_{ij}\right] = \frac{\partial y_i}{\partial d_j^{(K)}}\ , \quad (i = 1, 2,..., N;\ j = 1,..., n) \tag{7.42}$$

then all the elements of **P** can be evaluated using equation (7.39).

Equation (7.37) can be written in a vector form

$$\mathbf{P}\ \mathbf{d} = \mathbf{y}. \tag{7.43}$$

Setting $\mathbf{\Lambda}=\mathbf{\Lambda}^*$, we may determine the vector $\mathbf{y}$ and solve equation (7.43) for $\mathbf{d}$. However, since N>n for any n>1, there is, in general, no solution to (7.43). The approximate solution for $\mathbf{d}$, which is optimal in a least squares sense, can be obtained by

$$\mathbf{d} = \mathbf{P}^{\dagger}\mathbf{y} \tag{7.44}$$

where $\mathbf{P}^{\dagger}$ is the Moore-Penrose pseudo-inverse of $\mathbf{P}$.

The stiffness elements $\{k_1, k_2, ..., k_n\}$ can then be obtained by solving the system of linear equations defined by (7.30). This process is summarised by the following algorithm.

***Algorithm 7.2**:* Approximate solution for the stiffness elements

Input: Modal matrix Φ (nxn), desired eigenvalues matrix Λ^*(nxn), matrix $\mathbf{E}_K$ (nxn) and coefficients $\beta_{ij}^{(K)}$ (i,j=1,...,n).

Algorithm:

1) Calculate $\mathbf{G} = \mathbf{E}_K^T\Phi$ and partition it as in equation (7.38).
2) Set $\Lambda = \Lambda^*$ and form vector $\mathbf{y}$ as in (7.40)
3) Construct matrix $\mathbf{P}$ using equations (7.42),(7.49) and (7.36).
4) Calculate vector $\mathbf{d}$, as defined by (7.41), using (7.44).
5) Solve the following system of linear equations for k_j (j=1, ...,n).

$$\sum_{j=1}^{n} \beta_{ij}^{(K)} m_j = d_i^{(K)} \ ; \quad i=1, \ ..., \ n.$$

Output: A set of stiffness elements $\{k_1, k_2, ..., k_n\}$, corresponding to an optimal $\mathbf{K}$, which minimises the norm $\|\Lambda^* - \Lambda\|$.

We note that Algorithm 7.2 requires augmentation of the system from size n to size N. This is a computational barrier. In section 5 we presented a computationally less expensive method for obtaining a local optimal solution for **K**, but there we had to satisfy a system of N equations with N variable parameters. In this section we are required to satisfy a system of N equations with only n variables, therefore we are forced to seek a global optimal solution in order to achieve the best possible quality.

7.3 Numerical Example

Consider a 2 d.o.f. finite element model for a longitudinally vibrating rod of Example 7.2. The mass and stiffness matrices for this model are of the following form

$$M = \begin{bmatrix} \dfrac{m_1 + m_2}{3} & \dfrac{m_2}{6} \\ \dfrac{m_2}{6} & \dfrac{m_2}{3} \end{bmatrix}$$

and

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

where m_1, m_2, k_1 and k_2 are the masses and the stiffnesses of the two elements in the model. The components of the independent parameter decomposition for these matrices were shown to be

$$D_M = \begin{bmatrix} \dfrac{m_1}{3} + \dfrac{m_2}{4} & 0 \\ 0 & \dfrac{m_2}{6} \end{bmatrix}, \qquad E_M = \begin{bmatrix} 1 & \dfrac{1}{\sqrt{2}} \\ 0 & \sqrt{2} \end{bmatrix},$$

and

$$\boldsymbol{D}_K = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}, \qquad \boldsymbol{E}_K = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} .$$

Suppose that we wish to determine **M** and **K**, corresponding to the above form, such that the modal and spectral properties of the system be as close as possible to the desired properties described by

$$\boldsymbol{\Phi}^* = \begin{bmatrix} 0.1 & 0.2 \\ 0.2 & -0.3 \end{bmatrix}, \quad \boldsymbol{\Lambda}^* = \begin{bmatrix} 10 & 0 \\ 0 & 100 \end{bmatrix} .$$

Applying Algorithm 7.1, we obtain an optimal modal matrix

$$\boldsymbol{\Phi} = \begin{bmatrix} 0.1300 & 0.1920 \\ 0.2114 & -0.2831 \end{bmatrix}$$

and the mass elements m_1= 35.6119 and m_2=26.9242, which correspond to a mass matrix

$$\boldsymbol{M} = \begin{bmatrix} 20.8454 & 4.4874 \\ 4.4874 & 8.9747 \end{bmatrix}$$

which satisfies the requirements of the prescribed form and the orthogonality properties of equation (7.11).

Applying Algorithm 7.2 with the obtained modal matrix **Φ**, we determine the stiffness elements k_1= 526.7766 and k_2= 356.3036, which correspond to a stiffness matrix

$$K = \begin{bmatrix} 883.0803 & -356.3036 \\ -356.3036 & 356.3036 \end{bmatrix}$$

which is of the required form.

The modal and spectral properties of the system corresponding to the above **M** and **K** are

$$\mathbf{\Phi} = \begin{bmatrix} 0.1314 & 0.1911 \\ 0.2094 & -0.2847 \end{bmatrix}, \quad \mathbf{\Lambda} = \begin{bmatrix} 11.26 & 0 \\ 0 & 99.86 \end{bmatrix}$$

which correlate very well with the desired properties $\mathbf{\Phi}^*$ and $\mathbf{\Lambda}^*$.

7.4 Conclusions

In this section we have defined a special form for mass and stiffness matrices, which are more general than those corresponding to a mass-spring system. Methods for constructing such matrices to suit the prescribed modal and spectral properties were then developed.

The Algorithm 7.1 for reconstruction of the mass matrices is an extension of the theory developed in section 5, and is in fact a generalisation of that theory. The Algorithm 7.2 for reconstruction of the stiffness matrices is based on a matrix sensitivity analysis, and requires augmentation of the system in order to obtain a global optimal solution. This augmentation carries a significant computational penalty. However, due to an inherent deficiency in the number of variable parameters available for optimisation, the quality of a solution is problematic, and thus requires a global approach rather than a more computationally efficient local optimisation approach.

The numerical example based on a 2 degrees-of-freedom finite element model of a longitudinally vibrating rod was presented. This example has demonstrated the application of the developed methods, and the results obtained correlated well with the prescribed modal and spectral properties.

Section 8 *PROBLEM 3: MODIFICATIONS FOR DESIRED NATURAL FREQUENCIES*[3]

In this section we present the analysis for the solution of Problem 3 formulated in section 2.3. In this problem the exact mass and stiffness matrices of the system, **M** and **K**, are assumed to be unknown. The only information which is assumed to be known about the system are the measured modal analysis data contained in the matrices $\mathbf{\Lambda}_1$ and $\mathbf{\Phi}_1$. We then want to determine physically realisable modifications to the mass and stiffness (i.e $\Delta\mathbf{M}$ and $\Delta\mathbf{K}$), based only on $\mathbf{\Lambda}_1$ and $\mathbf{\Phi}_1$, so that the modified system would have spectral properties as close as possible to the desired spectrum described by $\mathbf{\Lambda}^*$.

It was shown in section 2.3 that an approximate solution to this problem may be obtained by solving the following norm minimisation problem

[3] Material presented in this section has been published in [77].

$$\| (\mathbf{M} + \Delta\mathbf{M})^{-1/2}\mathbf{R} \|_F, \text{ subject to } \bar{\mathbf{\Phi}} \in \text{span}(\mathbf{\Phi}_1) \tag{8.1}$$

where the residual matrix **R** is given by

$$R = [(K + \Delta K)\bar{\Phi} - (M + \Delta M)\bar{\Phi}\bar{\Lambda}] \tag{8.2}$$

and $\bar{\mathbf{\Lambda}}$ and $\bar{\mathbf{\Phi}}$ are some approximations to the desired $\mathbf{\Lambda}^*$ and the corresponding $\mathbf{\Phi}^*$ respectively.

It is shown in Parlett [42,pp.321-323] that if we determine the eigensolution, $\mathbf{\Psi}$ and $\mathbf{\Omega}$, of

$$\mathbf{F\Psi} - \mathbf{G\Psi\Omega} = 0 \tag{8.3}$$

where

$$\mathbf{F} = \mathbf{\Phi}_1^T(\mathbf{K}+\Delta\mathbf{K})\mathbf{\Phi}_1 \tag{8.4}$$

$$\mathbf{G} = \mathbf{\Phi}_1^T(\mathbf{M}+\Delta\mathbf{M})\mathbf{\Phi}_1, \tag{8.5}$$

then $\bar{\mathbf{\Lambda}} = \mathbf{\Omega}$ and $\bar{\mathbf{\Phi}} = \mathbf{\Phi}_1\mathbf{\Psi}$ minimise (8.1) under the required constraint that $\bar{\mathbf{\Phi}} \in \text{span}(\mathbf{\Phi}_1)$.

The matrices **K** and **M** are not given, and cannot be determined. However, using the orthogonality properties

$$\mathbf{\Phi}_1^T\mathbf{K}\mathbf{\Phi}_1 = \mathbf{\Lambda}_1 \tag{8.6}$$

and

$$\mathbf{\Phi}_1^T\mathbf{M}\mathbf{\Phi}_1 = \mathbf{I}_m, \tag{8.7}$$

we have

$$\mathbf{F} = \mathbf{\Lambda}_1 + \mathbf{\Phi}_1^T\Delta\mathbf{K}\mathbf{\Phi}_1 \tag{8.8}$$

and

$$\mathbf{G} = \mathbf{I}_m + \mathbf{\Phi}_1^T\Delta\mathbf{M}\mathbf{\Phi}_1. \tag{8.9}$$

Hence, if $\mathbf{\Delta K}$ and $\mathbf{\Delta M}$ are known, then $\mathbf{F}$ and $\mathbf{G}$ can be obtained and eigenproblem (8.3) can be solved for $\mathbf{\Psi}$ and $\mathbf{\Omega}$. Our goal is to solve the *inverse eigenvalue problem* of determining $\mathbf{\Delta K}$ and $\mathbf{\Delta M}$ where $\mathbf{\Omega} = \mathbf{\Lambda}^*$ is prescribed.

As shown in [42], the obtained solution is the Rayleigh-Ritz approximation of (2.14) from the subspace which is spanned by $\mathbf{\Phi}_1$. Thus, it follows that the desired eigenvalues are upper bounds on the eigenvalues of the actual modified system (2.14).

First we consider a simpler case of modifications to a discrete mass-spring system, and then extend these results for more complex models. An important case where mass and stiffness matrices are interrelated is also considered.

8.1 Modifying a Mass-Spring System.

A mass-spring system is the simplest model to analyse. In this model the mass and stiffness matrices are independent. Therefore, it is possible to change one without introducing changes to the other.

The global mass modifications matrix, $\mathbf{\Delta M}$, can be written in terms of its elements as follows

$$\Delta M = \sum_{i=1}^{n} \delta m_i B_i^{(M)} \tag{8.10}$$

where δm_i represents the change in the i^{th} mass element, and $\mathbf{B}_i^{(\mathbf{M})}$ is the mapping matrix

$$B_i^{(M)} = \left[b_{pq}^{(M)}\right] = \begin{cases} b_{ii}^{(M)} = 1 \\ b_{pq}^{(M)} = 0 \quad elsewhere \end{cases} \tag{8.11}$$

The incremental stiffness matrix may be written in the following form

$$\Delta K = \sum_{i=0}^{n-1} \sum_{j=i+1}^{n} \delta s_{ij} B_{ij}^{(K)} \tag{8.12}$$

where δs_{ij} is the change in the stiffness of the spring connecting mass i to mass j, and δs_{oi} represents the change in the stiffness of the spring which connects mass i to the ground, and $\mathbf{B}_{ij}^{(K)}$ is the matrix describing the spring connection between mass i and mass j

$$B_{ij}^{(K)} = \left[b_{pq}^{(K)}\right] = \begin{cases} b_{ii}^{(K)} = b_{jj}^{(K)} = 1 \\ b_{ij}^{(K)} = b_{ji}^{(K)} = -1 \qquad , \qquad i \neq j \\ b_{pq}^{(K)} = 0 \quad elsewhere \end{cases} \tag{8.13}$$

It follows, therefore, that

$$G = I_m + \sum_{i=1}^{n} \delta m_i M_i \tag{8.14}$$

and

$$F = \Lambda_1 + \sum_{i=0}^{n-1} \sum_{j=i+1}^{n} \delta s_{ij} K_{ij} \tag{8.15}$$

where

$$\mathbf{K}_{ij} = \mathbf{\Phi}_1^{\mathbf{T}} \mathbf{B}_{ij}^{(\mathbf{K})} \mathbf{\Phi}_1 \tag{8.16}$$

and

$$\mathbf{M}_i = \mathbf{\Phi}_1^{\mathbf{T}} \mathbf{B}_i^{(\mathbf{M})} \mathbf{\Phi}_1 \ . \tag{8.17}$$

Hence

$$\frac{\partial G}{\partial(\delta m_i)} = M_i \quad , \quad \frac{\partial G}{\partial(\delta s_{ij})} = \mathbf{0} \tag{8.18}$$

$$\frac{\partial F}{\partial(\delta s_{ij})} = K_{ij} \quad , \quad \frac{\partial F}{\partial(\delta m_i)} = 0 \tag{8.19}$$

and, as in [16], we have

$$\frac{\partial \lambda_i^*}{\partial a_j} = \psi_i^T(a^t) \left[\frac{\partial F}{\partial a_j} - \lambda_i(a^t) \frac{\partial G}{\partial a_j} \right] \psi_i(a^t) \tag{8.20}$$

where

$$a^t = (a_1, a_2, ..., a_N) = (\delta s_{01}, \delta s_{02}, ..., \delta s_{0n}, \delta s_{12}, ..., \delta s_{1n}, ..., \delta s_{n-1,n}, \delta m_1, ..., \delta m_n) \ , \tag{8.21}$$

N = ½(n^2+3n), t is an iteration index, $\lambda_i(a^t)$ is the i^{th} eigenvalue and $\psi_i(a^t)$ is the corresponding eigenvector, both obtained in the t^{th} iteration by using an iterate vector a^t.

The modification matrices **ΔM** and **ΔK** may then be found by an algorithm similar to Joseph [16]. This algorithm uses the Newton-Raphson method to find a new approximation for the structural modification parameters from some arbitrarily selected initial guess. The new approximation is calculated by determining the matrix derivatives and eigenvalue sensitivities as shown in (8.18), (8.19) and (8.20) above. These values determine a Jacobian matrix **J**, from which the eigenvalue sensitivity elements, calculated in (8.20), are found. Then a set of linear equations is solved to find the changes for improving the initially chosen structural modification parameters. The algorithm is summarised below, and its full derivation can be found in [16].

Algorithm 8.1: *Approximate solution to Problem 3*

Input: Measured modal analysis data Λ_1 and Φ_1

Algorithm:

1) Set iteration index t = 0 and choose an initial guess for structural modification parameters a^t.
2) Form matrices $\Delta\mathbf{K}$ and $\Delta\mathbf{M}$, by using structural modification parameters contained in the elements of a^t using (8.21).
3) Calculate **F** and **G**, by using (8.8) and (8.9).
4) By solving equation $\mathbf{F}\Psi(a^t) = \mathbf{G}\Psi(a^t)\Lambda(a^t)$ compute the smallest m eigenvalues $\lambda_i(a^t)$ (i=1,...,m), $\lambda_1 < \lambda_2 < ... < \lambda_m$, and the corresponding eigenvectors $\psi_i(a^t)$ normalised with respect to **G**.
5) Perform a convergence test, if $\| \lambda_i(a^t) - \lambda_i^* \|$ is sufficiently small, stop.
6) Compute the Jacobian matrix, $\mathbf{J}=[J_{ij}]$, i = 1,..., m; j=1,...,N {N = ½(n^2+3n)}; where the $(i,j)^{th}$ element of **J** is given by

$$J_{ij} = \frac{\partial \lambda_i^*}{\partial a_j} = \psi_i^T(a^t) \left[\frac{\partial F}{\partial a_j} - \lambda_i(a^t) \frac{\partial G}{\partial a_j} \right] \psi_i(a^t)$$

7) Calculate the singular value decomposition $\mathbf{J}=\mathbf{USV}^T$, where $\mathbf{UU}^T=\mathbf{I}_m$, $\mathbf{VV}^T=\mathbf{I}_N$ and **S** is the matrix containing r (r $\leq$ m) singular values of **J**.

8) Form vector **Δ** from the diagonal elements of the matrix $(\Lambda(\boldsymbol{a}^t) - \Lambda^*)$,

   i.e $\mathbf{\Delta} = \text{diag}\,(\Lambda(\boldsymbol{a}^t) - \Lambda^*)$.

9) Solve a set of linear equations $\mathbf{Jd} = \mathbf{\Delta}$ and obtain a family of possible solutions for **d**

$$\mathbf{d} = \mathbf{J}^{\dagger}\mathbf{\Delta} + \mathbf{V}^{*}\boldsymbol{b} \ ,$$

   where $\mathbf{J}^{\dagger}$ is a Moore-Penrose pseudoinverse of **J**, $\mathbf{V}^{*}$ is a matrix consisting of the last (N-r) columns of **V**, and $\boldsymbol{b}$ is an arbitrary (N-r)x1 vector.

10) Calculate the next approximation for the structural modification parameters

$$\boldsymbol{a}^{t+1} = \boldsymbol{a}^{t} - \mathbf{d}$$

11) Set $t = t+1$ and repeat from step 2.

Output: Vector $\boldsymbol{a}^{t+1}$ containing the required structural modifications parameters.

In order to ensure the physical realisablility of the solution, the elements of $\boldsymbol{a}^{t+1}$ must be such that the mass and stiffness elements of a modified structure are real and positive. Since the original matrices **M** and **K** are assumed to be unknown, precise limits for permissible reduction of the mass and stiffness elements are also unknown. However, if all elements of $\boldsymbol{a}^{t+1}$ are made non-negative, then the obtained modifications will not require reduction in any structural parameter, thus avoiding the problem described above. To make the elements of $\boldsymbol{a}^{t+1}$ non-negative it is required to select a vector $\boldsymbol{b}$, such that the elements of **d** obtained in step 9 are less than or equal to the corresponding elements of $\boldsymbol{a}^{t}$. This may be achieved by inserting the following procedure between steps 9 and 10 of Algorithm 8.1:

Procedure 8.1:

i) Calculate $\boldsymbol{d} = \boldsymbol{J}^{\dagger}\boldsymbol{\Delta}$

ii) Calculate $\boldsymbol{\beta} = \boldsymbol{a}^{t} - \boldsymbol{d}$

iii) Set

$$\gamma(i) = \begin{cases} \beta(i) & , \beta(i) \geq 0 \\ 0 & , \beta(i) < 0 \end{cases} \quad ; i = 1,2,\ldots,N$$

vi) Set $\boldsymbol{\alpha} = \boldsymbol{\beta} - \boldsymbol{\gamma}$

vii) Calculate $\boldsymbol{b} = V^{*+}\,\boldsymbol{\alpha}$ *, where* V^{*+} *is a Moore-Penrose pseudoinverse of* V^{*}

viii) If $\| \boldsymbol{b} \|$ *is sufficiently small, stop.*

ix) Set $\boldsymbol{d} = \boldsymbol{d} + V^{*}\boldsymbol{b}$*, and repeat from step (ii) .*

Algorithm 8.1 allows to determine the vector $\boldsymbol{a}^{t+1}$ which contains structural modification parameters. The mass and stiffness modification matrices $\mathbf{\Delta M}$ and $\mathbf{\Delta K}$ can then be determined from the elements of $\boldsymbol{a}^{t+1}$ by (8.21). The obtained solution is optimal in a Rayleigh-Ritz sense, and the residual (8.2) is minimised for all possible systems with truncated modal matrix $\boldsymbol{\Phi}$ taken from the subspace spanned by $\boldsymbol{\Phi}_1$ and where $\boldsymbol{\Lambda} \approx \boldsymbol{\Lambda}^{*}$.

8.2 Special Case I: Modifications to mass only

Consider now the case where only the mass matrix is subject to modification (i.e $\mathbf{\Delta K}=0$). This problem consists of m equations with n unknowns, $\delta m_1, \delta m_2, \ldots, \delta m_n$.

Here we have

$$\mathbf{F} = \mathbf{\Lambda}_1 \tag{8.22}$$

and

$$G = I_m + \sum_{i=1}^{n} \delta m_i M_i \tag{8.23}$$

and the problem can be solved by Algorithm 8.1. Example 8.1 demonstrates this procedure.

8.3 Special Case II: Interrelated System

Frequently, the mass and the stiffness matrices are interrelated. For example, the longitudinally vibrating rod may be modelled by finite difference model, with

$$m_i = \rho A_i l_i \tag{8.24}$$

$$s_i = \frac{EA_i}{l_i} \tag{8.25}$$

where: m_i and s_i are the mass and stiffness of the i^{th} element, respectively,

E and ρ are the Young's Modulus and density of the rod, and

A_i and l_i are, respectively, the cross-sectional area and length of the i^{th} element.

Here, the mass elements are interrelated to the stiffnesses via A_i and l_i . Hence, a change in **M** (or δm) causes a respective change in **K** (or δs).

If we wish to modify the natural frequencies of a rod by changing only the cross-sectional area (i.e A_i), then the corresponding changes in the mass and stiffness elements are

$$\delta m_i = \rho l_i \delta A_i \tag{8.26}$$

$$\delta s_i = \frac{E\ \delta A_i}{l_i} \quad , \tag{8.27}$$

and we may write

$$\delta m_i = C_i^{(M)}\ \delta A_i \tag{8.28}$$

$$\delta s_i = C_i^{(K)}\ \delta A_i \tag{8.29}$$

where $C_i^{(M)}$ and $C_i^{(K)}$ are constants.

Thus, the global modification matrices are

$$\boldsymbol{\Delta K} = \sum_{i=1}^{n} \delta s_i\ \boldsymbol{B}_i^{(K)} \tag{8.30}$$

$$\boldsymbol{\Delta M} = \sum_{i=1}^{n} \delta m_i\ \boldsymbol{B}_i^{(M)} \tag{8.31}$$

where $\mathbf{B}_i^{(M)}$ is the same as in (8.11), and

$$\boldsymbol{B}_i^{(K)} = \left[b_{pq}^{(K)}\right] = \begin{cases} b_{i-1\ i-1}^{(K)} = b_{ii}^{(K)} = 1 \\ b_{i-1\ i}^{(K)} = b_{i\ i-1}^{(K)} = -1 \\ b_{pq}^{(K)} = 0 \quad elsewhere \end{cases} \quad . \tag{8.32}$$

Substituting (8.28),(8.31) and (8.29),(8.30) into (8.10) and (8.9) respectively, we obtain

$$\boldsymbol{F} = \boldsymbol{\Lambda}_1 + \boldsymbol{\Phi}_1^T \left[\sum_{i=1}^{n} \delta A_i\ C_i^{(K)}\ \boldsymbol{B}_i^{(K)}\right] \boldsymbol{\Phi}_1 \tag{8.33}$$

and

$$\boldsymbol{G} = \boldsymbol{I}_m + \boldsymbol{\Phi}_1^T \left[\sum_{i=1}^{n} \delta A_i\ C_i^{(M)}\ \boldsymbol{B}_i^{(M)}\right] \boldsymbol{\Phi}_1 \tag{8.34}$$

or equivalently

$$F = \Lambda_1 + \sum_{i=1}^{n} \delta A_i \, K_i \tag{8.35}$$

and

$$G = I_m + \sum_{i=1}^{n} \delta A_i \, M_i \tag{8.36}$$

where

$$\mathbf{K}_i = \mathbf{\Phi}_1^{\mathbf{T}} \mathrm{C}_i^{(K)} \mathbf{B}_i^{(\mathbf{K})} \mathbf{\Phi}_1 \tag{8.37}$$

$$\mathbf{M}_i = \mathbf{\Phi}_1^{\mathbf{T}} \mathrm{C}_i^{(M)} \mathbf{B}_i^{(\mathbf{M})} \mathbf{\Phi}_1 \quad . \tag{8.38}$$

Thus, the problem may be solved by Algorithm 8.1.

A similar analysis may be applied when using finite element modelling. For the finite element model of the rod, $\mathbf{B}_i^{(\mathbf{M})}$ is given by:

$$B_i^{(M)} = \left[b_{pq}^{(M)} \right] = \begin{cases} b_{i-1\, i-1}^{(M)} = b_{ii}^{(M)} = \dfrac{1}{3} \\ b_{i-1\, i}^{(M)} = b_{i\, i-1}^{(M)} = \dfrac{1}{6} \\ b_{pq}^{(M)} = 0 \qquad elsewhere \end{cases} \tag{8.39}$$

and $\mathbf{B}_i^{(\mathbf{K})}$ is the same as in (8.32).

We note that the method presented is flexible, and can be applied to other models of vibratory systems. The only change which is needed, is to use the appropriate mapping matrices associated with the chosen system.

8.4 Numerical Examples

Example 8.1: Mass only modifications

Consider a 10 degrees of freedom mass-spring system with

$$K = \begin{bmatrix} 200 & -10 & -20 & -5 & -5 & -10 & 0 & 0 & -50 & -50 \\ -10 & 100 & 0 & 0 & 0 & 0 & -20 & -10 & -20 & -10 \\ -20 & 0 & 300 & -40 & -30 & -60 & -10 & 0 & -20 & -10 \\ -5 & 0 & -40 & 400 & -30 & -40 & -50 & -20 & -10 & -70 \\ -5 & 0 & -30 & -30 & 150 & -10 & -5 & -5 & -20 & 0 \\ -10 & 0 & -60 & -40 & -10 & 250 & 0 & 0 & 0 & -80 \\ 0 & -20 & -10 & -50 & -5 & 0 & 120 & -5 & 0 & -10 \\ 0 & -10 & 0 & -20 & -5 & 0 & -5 & 250 & 0 & -100 \\ -50 & -20 & -20 & -10 & -20 & 0 & 0 & 0 & 350 & -40 \\ -50 & -10 & -10 & -70 & 0 & -80 & -10 & -100 & -40 & 400 \end{bmatrix}$$

and

$$\mathbf{M} = \mathrm{diag}\ (1, 1, 1, 1, 1, 1, 1, 1, 1, 1)\ .$$

The smallest three eigenvalues and the corresponding mass-normalised eigenvectors of the system are

$$\mathbf{\Lambda}_1 = \mathrm{diag}\ (61.8300,\ 108.3525,\ 129.1425)$$

and

$$\mathbf{\Phi}_1 = \begin{bmatrix} 0.2605 & -0.1625 & 0.3016 \\ 0.5453 & 0.7597 & 0.2873 \\ 0.2050 & -0.2277 & 0.0955 \\ 0.2301 & -0.1751 & -0.0392 \\ 0.2694 & -0.4256 & 0.1967 \\ 0.2558 & -0.2886 & 0.1673 \\ 0.5081 & -0.0279 & -0.8355 \\ 0.2159 & -0.1136 & 0.1054 \\ 0.1610 & -0.0617 & 0.1441 \\ 0.2668 & -0.1792 & 0.1385 \end{bmatrix}$$

Assume that only $\mathbf{\Phi}_1$ and $\mathbf{\Lambda}_1$ are given, and that no other information is known. We wish to find modifications to the mass matrix only, such that the first three eigenvalues of the modified system are

$$\Lambda^* = diag(\ 1,\ 2,\ 3) \quad .$$

Applying Algorithm 1 with $\boldsymbol{a}^0 = \mathbf{0}$ we obtain

$$\Delta M = diag(41.139,\ 82.580,\ 16.728,\ 7.127,\ 47.813,\ 32.122,\ 54.471,\ 14.350,\ 11.193,\ 24.213) \quad .$$

Setting new mass matrix $\mathbf{M}_{\mathbf{mod}} = \mathbf{M} + \Delta\mathbf{M}$, we find that the eigenvalues of the modified system are

$$\mathbf{\Lambda}_{\mathbf{mod}} = diag(\ 0.99,\ 1.96,\ 2.72,\ 3.92,\ 6.10,\ 11.58,\ 17.33,\ 21.14,\ 29.48,\ 50.50) \quad .$$

The small discrepancy between the desired eigenvalues of $\mathbf{\Lambda}^*$ and the smallest three eigenvalues of $\mathbf{\Lambda}_{\mathbf{mod}}$ is due to the truncation error, which is unavoidable.

Example 8.2 : Longitudinally Vibrating Rod

Consider a uniform axially vibrating rod, fixed at x=0 and free to oscillate at x=L, with uniform properties ρ=E=L=A=1. The first three eigenpairs of the rod, obtained for a 10 degrees of freedom finite element model, are shown in Table 8.1.

Position from the Fixed End of the Rod	Mode Shapes		
	1^{st} mode	2^{nd} mode	3^{rd} mode
0.1	0.2217	-0.6540	1.0527
0.2	0.4379	-1.1655	1.4888
0.3	0.6434	-1.4229	1.0527
0.4	0.8330	-1.3701	0.0000
0.5	1.0021	-1.0187	-1.0527
0.6	1.1465	-0.4452	-1.4888
0.7	1.2627	0.2254	-1.0527
0.8	1.3478	0.8468	0.0000
0.9	1.3997	1.2836	1.0527
1.0	1.4171	1.4406	1.4888
Eigenvalues	2.4725	22.6205	64.9165

Table 8.1 : First Three Modes of a Uniform Cantilever Rod of Example 2

We wish to change the cross-sectional area of the rod, so that the eigenvalues of the modified rod will be

$$\Lambda^* = diag(\ 1,\ 15,\ 100\ ) \quad .$$

Applying Algorithm 8.1, we obtain the following area modifications for the finite elements model

$$\delta A = diag(-0.6698, -0.8567, -0.3661,\ 0.4642,\ 0.3919, -0.3530, -0.3213,\ 0.5501,\ 0.9244,\ 0.5870) \quad .$$

When these changes are implemented, the lowest three eigenvalues of the finite element model of the modified rod are:

$$\Lambda_{mod} = diag\ (\ 0.7,\ 12.7,\ 96.6) \quad .$$

Note that as predicted before, the desired eigenvalues λ_i^* are higher than their corresponding true eigenvalues $\lambda_{mod\ i}$ of the modified system.

Example 8.3: <u>Sensitivity Test</u>

The sensitivity of Algorithm 8.1 to perturbations is now demonstrated.

Let $$\Lambda_1 = \text{diag}\ (\ 60,\ 105,\ 130\ ),$$

and

$$\Phi_1 = \begin{bmatrix} 0.3 & -0.2 & 0.3 \\ 0.5 & 0.8 & 0.3 \\ 0.2 & -0.2 & 0.1 \\ 0.2 & -0.2 & 0.0 \\ 0.3 & -0.4 & 0.2 \\ 0.3 & -0.3 & 0.2 \\ 0.5 & 0.0 & -0.8 \\ 0.2 & -0.1 & 0.1 \\ 0.2 & -0.1 & 0.1 \\ 0.3 & -0.2 & 0.1 \end{bmatrix} \quad .$$

which may be obtained by "rounding off" the elements of the eigenpairs in Example 8.1.

Repeating Example 8.1 with these values, we obtain

$$\Delta M = diag(\ 36.478,\ 50.249,\ 15.308,\ 14.211,\ 15.418,\ 32.902,\ 92.94,\ 24.081,\ 24.081,\ 50.464)$$

which is quite different from the modifications obtained in Example 8.1. However, we find that the eigenvalues of the modified system are

$$\Lambda_{mod} = diag(\ 0.97,\ 1.96,\ 3.09,\ 5.79,\ 7.64,\ 8.83,\ 12.06,\ 14.57,\ 19.03,\ 27.53)$$

and the three smallest eigenvalues of Λ_{mod} represent a good estimate to the desired eigenvalues. Thus the introduction of perturbations to the given data, caused convergence to a different possible solution.

8.5 Conclusions

In this section we have defined an optimisation problem, which allows us to overcome the effect of truncation. It was shown that this optimisation problem may be solved by applying the algorithm of Joseph [16] (with some minor alterations) to obtain a physically realisable solution. The obtained solutions are optimal in a Rayleigh-Ritz sense. The desired eigenvalues are thus higher than the eigenvalues of the actual modified system.

Using this approach we may also modify vibratory systems with interrelated mass and stiffness matrices. Some examples were given and the sensitivity of the problem to perturbation has been numerically demonstrated.

Section 9 *PROBLEM 4:*

MODIFICATIONS FOR DESIRED NATURAL FREQUENCIES AND MODE SHAPES

In this section we present the analysis of Problem 4, which was formulated in section 2.3. Our Problem 4 is identical to the problem investigated by Ram and Braun in [46]. They have shown that a family of optimal solutions (in a Rayleigh-Ritz sense) to this problem is characterised by the following equations

$$\Delta M = \Phi_1^{T\dagger}\left(\Psi^{-T}\Psi^{-1} - I_m\right)\Phi_1^{\dagger} + Y - \Phi_1^{T\dagger}\Phi_1^{T}Y\Phi_1\Phi_1^{\dagger} \tag{9.1}$$

$$\Delta K = \Phi_1^{T\dagger}\left(\Psi^{-T}\Lambda^{*}\Psi^{-1} - \Lambda_1\right)\Phi_1^{\dagger} + X - \Phi_1^{T\dagger}\Phi_1^{T}X\Phi_1\Phi_1^{\dagger} \tag{9.2}$$

where $\Phi_1^{\dagger}$ denotes the Moore-Penrose pseudoinverse of Φ_1, $\Psi = \Phi_1^{\dagger}\Phi^{*}$, and **X** and **Y** are arbitrary *m*x*m* real symmetric matrices.

We note that all elements in the equations (9.1) and (9.2) are known, with the exception of matrices **X** and **Y**. Since **X** and **Y** can be arbitrarily assigned, a solution for **ΔM** and **ΔK** can be evaluated. However, in general, such arbitrarily selected **X** and **Y** do not result in a physically realisable solution. In other words, the obtained mathematical solutions do not give us a hint on how to change the geometry or the material properties of the structure in order to get the required modifications. Therefore, here our main aim is to develop a method for extracting a *physically realisable* solutions for **ΔM** and **ΔK** from the general family of solutions defined by equations (9.1) and (9.2).

Setting
$$P = \Phi_1^{T\dagger}\left(\Psi^{-T}\Psi^{-1} - I_m\right)\Phi_1^{\dagger} \tag{9.3}$$

$$T = \Phi_1^{T\dagger}\left(\Psi^{-T}\Lambda^{*}\,\Psi^{-1} - \Lambda_1\right)\Phi_1^{\dagger} \tag{9.4}$$

$$H = \Phi_1\Phi_1^{\dagger} \tag{9.5}$$

equations (9.1) and (9.2) become

$$\mathbf{\Delta M = P + Y - H^T Y\, H} \tag{9.6}$$

$$\mathbf{\Delta K = T + X - H^T X\, H}\ . \tag{9.7}$$

We note that equations (9.6) and (9.7) have identical form, and also that this form is very similar to the well known *Discrete Lyapunov Equation (DLP)*. The solution for DLP is available in the Control Toolbox of MATLAB under the function name **dlyap**. The **dlyap** algorithm of MATLAB allows to determine **Y**, such that equation (9.6) holds for any given matrices **ΔM**, **P** and **H** (or determine **X** for any given **ΔK**, **T** and **H** in a case of equation

(9.7)). Our problem is different, since neither $\mathbf{\Delta M}$ nor $\mathbf{\Delta K}$ are known. However, it appears that we should be able to arbitrarily choose any $\mathbf{\Delta M}$ and $\mathbf{\Delta K}$, and then calculate $\mathbf{Y}$ and $\mathbf{X}$ which satisfy (9.6) and (9.7). The implication of this is that we may choose $\mathbf{\Delta M}$ and $\mathbf{\Delta K}$ to be zero matrices, and then obtain $\mathbf{Y}$ and $\mathbf{X}$ which still, supposedly, give a Rayleigh-Ritz approximation to the desired natural frequencies and mode shapes. Clearly, this is not physically possible. A detailed investigation of this apparent contradiction has shown that both (9.6) and (9.7) are ill-conditioned to be solved by **dlyap** algorithm, and therefore the contradiction does not really exist. And interestingly, the reasons behind this ill-conditioning have also provided a key element in deriving a solution to our problem. The following analysis describes this solution.

The matrix $\mathbf{H}$, defined by (9.5), is a product of a matrix by its pseudoinverse. Calculating the singular values decomposition of $\mathbf{H}$, we obtain

$$\mathbf{H} = \mathbf{U}\,\mathbf{Z}\,\mathbf{U}^T \tag{9.8}$$

where $\mathbf{U}\mathbf{U}^T = \mathbf{U}^T\mathbf{U} = \mathbf{I}_n$, and the partitioned singular values matrix $\mathbf{Z}$ is as follows

$$Z = \left[\begin{array}{c|c} I_m & O \\ \hline O & O \end{array}\right] \tag{9.9}$$

and where $\boldsymbol{O}$ represents submatrices with all elements equal to zero.

Substituting (9.8) into (9.6) and (9.7), we obtain

$$\mathbf{\Delta M} = \mathbf{P} + \mathbf{Y} - \mathbf{U}\mathbf{Z}\mathbf{U}^T\mathbf{Y}\,\mathbf{U}\mathbf{Z}\mathbf{U}^T \tag{9.10}$$

$$\mathbf{\Delta K} = \mathbf{T} + \mathbf{X} - \mathbf{U}\mathbf{Z}\mathbf{U}^T\mathbf{X}\,\mathbf{U}\mathbf{Z}\mathbf{U}^T\ . \tag{9.11}$$

Multiplying both sides of (9.10) and (9.11) by $\mathbf{U}^T$ and $\mathbf{U}$, we get

$$\mathbf{U}^T\Delta\mathbf{M}\ \mathbf{U} = \mathbf{U}^T\ \mathbf{P}\ \mathbf{U} + \mathbf{U}^T\mathbf{Y}\ \mathbf{U} - \mathbf{Z}\mathbf{U}^T\mathbf{Y}\ \mathbf{U}\mathbf{Z} \tag{9.12}$$

$$\mathbf{U}^T\Delta\mathbf{K}\ \mathbf{U} = \mathbf{U}^T\ \mathbf{T}\ \mathbf{U} + \mathbf{U}^T\mathbf{X}\ \mathbf{U} - \mathbf{Z}\mathbf{U}^T\mathbf{X}\ \mathbf{U}\mathbf{Z}\ . \tag{9.13}$$

Setting
$$\mathbf{P}^* = \mathbf{U}^T\ \mathbf{P}\ \mathbf{U} \tag{9.14}$$

$$\mathbf{T}^* = \mathbf{U}^T\ \mathbf{T}\ \mathbf{U} \tag{9.15}$$

$$\mathbf{Y}^* = \mathbf{U}^T\mathbf{Y}\ \mathbf{U} \tag{9.16}$$

$$\mathbf{X}^* = \mathbf{U}^T\mathbf{X}\ \mathbf{U}, \tag{9.17}$$

then equations (9.12) and (9.13) become

$$\mathbf{U}^T\Delta\mathbf{M}\ \mathbf{U} = \mathbf{P}^* + \mathbf{Y}^* - \mathbf{Z}\,\mathbf{Y}^*\mathbf{Z} \tag{9.18}$$

$$\mathbf{U}^T\Delta\mathbf{K}\ \mathbf{U} = \mathbf{T}^* + \mathbf{X}^* - \mathbf{Z}\,\mathbf{X}^*\mathbf{Z}\ . \tag{9.19}$$

Partitioning $\mathbf{U}$, $\mathbf{P}^*$, $\mathbf{T}^*$, $\mathbf{Y}^*$ and $\mathbf{X}^*$ as follows

$$\mathbf{U} = [\ \mathbf{U}_1 \mid \mathbf{U}_2\ ]\ ,\ \mathbf{U}_1 \text{ is } nxm \text{ real matrix } m<n \tag{9.20}$$

$$P^* = \left[\begin{array}{c|c} P_1^* & P_2^* \\ \hline P_2^{*T} & P_3^* \end{array}\right]\ ,\ P_1^* \ \ is\ \ mxm \tag{9.21}$$

$$T^* = \left[\begin{array}{c|c} T_1^* & T_2^* \\ \hline T_2^{*T} & T_3^* \end{array}\right]\ ,\ T_1^* \ \ is\ \ mxm \tag{9.22}$$

$$Y^* = \left[\begin{array}{c|c} Y_1^* & Y_2^* \\ \hline Y_2^{*T} & Y_3^* \end{array}\right]\ ,\ Y_1^* \ \ is\ \ mxm \tag{9.23}$$

$$X^* = \left[\begin{array}{c|c} X_1^* & X_2^* \\ \hline X_2^{*T} & X_3^* \end{array}\right] \quad , X_1^* \ is \ mxm \tag{9.24}$$

then, also using (9.9), equations (9.18) and (9.19) may be written as follows:

$$\begin{aligned}\left[\begin{array}{c|c} U_1^T \Delta M U_1 & U_1^T \Delta M U_2 \\ \hline U_2^T \Delta M U_1 & U_2^T \Delta M U_2 \end{array}\right] = \left[\begin{array}{c|c} P_1^* & P_2^* \\ \hline P_2^{*T} & P_3^* \end{array}\right] + \left[\begin{array}{c|c} Y_1^* & Y_2^* \\ \hline Y_2^{*T} & Y_3^* \end{array}\right] - \\ \left[\begin{array}{c|c} I_m & O \\ \hline O & O \end{array}\right] \left[\begin{array}{c|c} Y_1^* & Y_2^* \\ \hline Y_2^{*T} & Y_3^* \end{array}\right] \left[\begin{array}{c|c} I_m & O \\ \hline O & O \end{array}\right]\end{aligned} \tag{9.25}$$

and

$$\begin{aligned}\left[\begin{array}{c|c} U_1^T \Delta K U_1 & U_1^T \Delta K U_2 \\ \hline U_2^T \Delta K U_1 & U_2^T \Delta K U_2 \end{array}\right] = \left[\begin{array}{c|c} T_1^* & T_2^* \\ \hline T_2^{*T} & T_3^* \end{array}\right] + \left[\begin{array}{c|c} X_1^* & X_2^* \\ \hline X_2^{*T} & X_3^* \end{array}\right] - \\ \left[\begin{array}{c|c} I_m & O \\ \hline O & O \end{array}\right] \left[\begin{array}{c|c} X_1^* & X_2^* \\ \hline X_2^{*T} & X_3^* \end{array}\right] \left[\begin{array}{c|c} I_m & O \\ \hline O & O \end{array}\right] .\end{aligned} \tag{9.26}$$

Therefore we obtain

$$\left[\begin{array}{c|c} U_1^T \Delta M U_1 & U_1^T \Delta M U_2 \\ \hline U_2^T \Delta M U_1 & U_2^T \Delta M U_2 \end{array}\right] = \left[\begin{array}{c|c} P_1^* & P_2^* \\ \hline P_2^{*T} & P_3^* \end{array}\right] + \left[\begin{array}{c|c} Y_1^* & Y_2^* \\ \hline Y_2^{*T} & Y_3^* \end{array}\right] - \left[\begin{array}{c|c} Y_1^* & O \\ \hline O & O \end{array}\right] \tag{9.27}$$

and

$$\left[\begin{array}{c|c} U_1^T \Delta K U_1 & U_1^T \Delta K U_2 \\ \hline U_2^T \Delta K U_1 & U_2^T \Delta K U_2 \end{array}\right] = \left[\begin{array}{c|c} T_1^* & T_2^* \\ \hline T_2^{*T} & T_3^* \end{array}\right] + \left[\begin{array}{c|c} X_1^* & X_2^* \\ \hline X_2^{*T} & X_3^* \end{array}\right] - \left[\begin{array}{c|c} X_1^* & O \\ \hline O & O \end{array}\right] . \tag{9.28}$$

Simplifying (9.27) and (9.28) further we obtain

$$\left[\begin{array}{c|c} U_1^T \Delta M U_1 & U_1^T \Delta M U_2 \\ \hline U_2^T \Delta M U_1 & U_2^T \Delta M U_2 \end{array}\right] = \left[\begin{array}{c|c} P_1^* & P_2^* + Y_2^* \\ \hline P_2^{*T} + Y_2^{*T} & P_3^* + Y_3^* \end{array}\right] \tag{9.29}$$

and

$$\left[\begin{array}{c|c} U_1^T \Delta K U_1 & U_1^T \Delta K U_2 \\ \hline U_2^T \Delta K U_1 & U_2^T \Delta K U_2 \end{array}\right] = \left[\begin{array}{c|c} T_1^* & T_2^* + X_2^* \\ \hline T_2^{*T} + X_2^{*T} & T_3^* + X_3^* \end{array}\right] \tag{9.30}$$

Thus, the reasons behind the ill-conditioning of the equations (9.6) and (9.7) for the function **dlyap** of MATLAB now become clear. Separating the first elements from the partitioned matrices in (9.29) and (9.30), we note that the following relations must be satisfied

$$\mathbf{U_1^T \Delta M\, U_1 = P_1^*} \tag{9.31}$$

and

$$\mathbf{U_1^T \Delta K\, U_1 = T_1^*}. \tag{9.32}$$

Equations (9.31) and (9.32) are independent of **Y** and **X**. Since **P**, **T** and **H** are known, $\mathbf{P_1^*}$, $\mathbf{T_1^*}$ and $\mathbf{U_1}$ are also predetermined. Thus, selecting arbitrary **ΔM** and **ΔK**, would not, in general, satisfy (9.31) and (9.32), and therefore, the fundamental condition for a successful application of the **dlyap** algorithm is violated.

We note that in equations (9.31) and (9.32) the only unknowns are **ΔM** and **ΔK**, and therefore these matrices can be calculated. It also follows from the **dlyap** algorithm, that for any so obtained **ΔM** and **ΔK**, all other elements of the equations (9.29) and (9.30) (except for these described in (9.31) and (9.32)) may, in general, be satisfied by some particular matrices **Y** and **X**. Thus, it is a sufficient condition for determining a solution to our problem, if we obtain physically realisable **ΔM** and **ΔK** which satisfy (9.31) and (9.32). The following analysis shows the necessary procedures for achieving this aim.

9.1 Mass Modifications

In general, for a *n* degrees-of-freedom system, the mass matrix **M** would contain *n* independent parameters corresponding to the masses of each of the elements which are part of the system. However, when evaluating the necessary modifications to the system's mass (i.e **ΔM**), a designer may wish to restrict any such modification to only *l* (*l* < *n*) elements.

The global mass modification matrix **ΔM** can then be expressed as

$$\Delta M = \sum_{q=1}^{l} \delta m_q \, B_q^{(M)} \tag{9.33}$$

where δm_q is a modification to the mass of the q[th] element, and $\mathbf{B}_q^{(M)}$ is the *nxn* mapping matrix corresponding to a chosen analytical model.

Substituting equation (9.33) into equation (9.31), we obtain

$$P_1^* = \sum_{q=1}^{l} \delta m_q \, U_1^T B_q^{(M)} U_1 \quad . \tag{9.34}$$

Also from (9.14), (9.20) and (9.21), $\mathbf{P}_1^*$ is equal to

$$P_1^* = \left[p_{ij}^*\right] = U_1^T \, P \, U_1 \quad . \tag{9.35}$$

Partitioning $\mathbf{U}_1$ into column vectors as shown

$$U_1 = \left[\, u_1 \,|\, u_2 \,|\, \ldots \,|\, u_m \right] , \tag{9.36}$$

then each element p_{ij}^* must be equal to

$$p_{ij}^* = \sum_{q=1}^{l} \delta m_q \, u_i^T B_q^{(M)} u_j \quad . \tag{9.37}$$

Let N= $\frac{1}{2}(m^2+ m)$ and construct the vectors

$$\mathbf{y_M} = (y_1^{(M)}, y_2^{(M)}, \ldots, y_N^{(M)})^T = (p_{11}^*, p_{12}^*, p_{13}^*, \ldots, p_{1m}^*, p_{22}^*, \ldots, p_{mm}^*)^T \tag{9.38}$$

and

$$\boldsymbol{\delta}\mathbf{m} = (\delta m_1, \delta m_2, \ldots, \delta m_l)^T. \tag{9.39}$$

Denote

$$F_M = \left[f_{ij}^{(M)}\right] = \frac{\partial y_i^{(M)}}{\partial(\delta m_j)} , \quad (i = 1, 2, \ldots, N; \, j = 1, 2, \ldots, l) \tag{9.40}$$

then all the elements of $\mathbf{F_M}$ can be evaluated using equation (9.37). Equation (9.31) can be written in a vector form

$$\mathbf{F_M} \, \boldsymbol{\delta}\mathbf{m} = \mathbf{y_M}. \tag{9.41}$$

Since $\mathbf{F_M}$ and $\mathbf{y_M}$ are known, (9.41) can be solved for $\boldsymbol{\delta}\mathbf{m}$, and the mass modification matrix $\boldsymbol{\Delta}\mathbf{M}$ can then be determined from the elements of vector $\boldsymbol{\delta}\mathbf{m}$ by equation (9.33).

We note that in order to obtain a solution for the system of size nxm ($m<n$), we need to solve an augmented system (9.41) of size N$\times l$ (N= ½(m^2+ m)). However, in this case augmentation is based on the smaller dimension m, whereas number of independent parameters available for optimisation is fixed at l. Therefore depending on the values of l and m there are three possibilities for the solution to (9.41).

Set $r = l$-N, then if r>0 there will be a family of solutions for $\boldsymbol{\delta}\mathbf{m}$. This family of solutions is characterised by the following equation

$$\boldsymbol{\delta}\mathbf{m} = \mathbf{F_M}^{\dagger}\mathbf{y_M} + \mathbf{V_r b} \tag{9.42}$$

where $\mathbf{F_M}^{\dagger}$ is the Moore-Penrose pseudoinverse of $\mathbf{F_M}$, $\mathbf{b}$ is an arbitrary vector of dimension rx1, and $\mathbf{V_r}$ is a matrix of dimension l$\times r$ which is obtained by a following procedure

Calculate singular value decomposition $\mathbf{F_M}=\mathbf{WSV}^{\mathbf{T}}$, *and partition the lxl matrix* $\mathbf{V} = [\mathbf{V_N} \mid \mathbf{V_r}]$, *where* $\mathbf{V_N}$ *is* lxN, *and* $\mathbf{V_r}$ *is* l$\times r$. (9.43)

If r = 0, then $\mathbf{F_M}$ is a full square matrix, and there will be one unique solution for $\boldsymbol{\delta}\mathbf{m}$. This unique solution is

$$\boldsymbol{\delta}\mathbf{m} = \mathbf{F_M}^{-1}\mathbf{y_M} \; . \tag{9.44}$$

And finally, if r<0, then there are no solutions for $\boldsymbol{\delta}\mathbf{m}$, and only an approximate solution (which is optimal in a least squares sense) can be obtained by

$$\boldsymbol{\delta}\mathbf{m} = \mathbf{F_M}^{\dagger}\mathbf{y_M} \; . \tag{9.45}$$

If it is desired that all the elements of $\boldsymbol{\delta}\mathbf{m}$ to be positive, and if solutions of (9.42), (9.44) and (9.44) do not yield positive $\boldsymbol{\delta}\mathbf{m}$, than it may be obtained by solving the following *non-*

negative least squares problem

$$\min_{\boldsymbol{\delta m}} \quad \| \mathbf{F_M}\boldsymbol{\delta m} - \mathbf{y_M} \|_2 \text{, subject to } \boldsymbol{\delta m} \geq 0. \tag{9.46}$$

This will produce an optimal non-negative solution to the vector $\boldsymbol{\delta m}$ in a least square sense. It should be noted, however, that $\boldsymbol{\Delta M}$ which is a solution of (9.1) is itself only a Rayleigh-Ritz approximation to the solution of the modification problem. Therefore, an approximate solution to (9.1), obtained by (9.45) and (9.46), is in reality "an approximation to an approximation", which may not be acceptable in applications based on possible poor quality of the solutions. Thus, from practical considerations, it appears that it may be best to restrict the application of this method to systems where $\boldsymbol{\delta m}$ can be determined by either (9.42) or (9.44), which requires that $l \geq N$.

The above procedure ensures that the form of the obtained mass modification matrix $\boldsymbol{\Delta M}$ corresponds to a physically realisable system via equation (9.33). The procedure is also independent of an arbitrary choice for the matrix $\mathbf{Y}$, and it is summarised by the following algorithm.

Algorithm 9.1: Determination of a Mass Modification Matrix

Input: Modal test data $\boldsymbol{\Phi}_1(nxm)$ and $\boldsymbol{\Lambda}_1(mxm)$, and desired modal data $\boldsymbol{\Phi}^*(nxm)$ and $\boldsymbol{\Lambda}^*(mxm)$.

Algorithm:

1) Calculate **P** and **H** using (9.3) and (9.5).

2) Obtain the singular value decomposition $\mathbf{H} = \mathbf{UZU^T}$.

3) Column partition $\mathbf{U} = [\ \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m, \ldots, \mathbf{u}_n]$.

4) Set $\mathbf{U}_1 = [\ \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m]$.

5) Calculate $\mathbf{P}_1^* = \mathbf{U}_1^T\mathbf{P}\,\mathbf{U}_1$.

6) Set N= ½(m^2+ m).

7) Construct vectors $\mathbf{y}_M$ as in (9.38).

8) Form vector $\boldsymbol{\delta}\mathbf{m}$ of dimension $lx1$ as in (9.39).

9) Construct matrix $\mathbf{F}_M$ using (9.40) and (9.37).

10) (a) If l > N, then determine $\boldsymbol{\delta}\mathbf{m}$ by equation (9.42),

(b) if l = N, then determine $\boldsymbol{\delta}\mathbf{m}$ by (9.44),

(c) if l < N, then determine $\boldsymbol{\delta}\mathbf{m}$ by (9.45).

11) If desire non-negative $\boldsymbol{\delta}\mathbf{m}$ and the one obtained in step 10 is not, then determine $\boldsymbol{\delta}\mathbf{m}$ by solving (9.46).

12) Construct $\Delta\mathbf{M}$ from the elements of $\boldsymbol{\delta}\mathbf{m}$ using (9.33).

Output: Physically realisable mass modification matrix $\Delta\mathbf{M}$.

9.2 Stiffness Modifications

The number of independent spring elements in a n degrees-of-freedom mass-spring model may vary from (n-1) in a case of a *free-free simply-connected* system to ½(n^2+n) for a *multi-connected* system. Thus assuming that J (where $J \le$ ½(n^2+n)) of the spring elements are available for modifications, the global stiffness modification matrix $\Delta\mathbf{K}$ can then be expressed as

$$\Delta K = \sum_{q=1}^{J} \delta s_q B_q^{(K)} \tag{9.47}$$

where δs_q is a modification to the stiffness of the q^{th} spring element, and $\mathbf{B}_q^{(K)}$ is the nxn mapping matrix corresponding to a chosen analytical model.

We note that the form of equations (9.31) and (9.32), and also of (9.33) and (9.47), are identical. Therefore, by substituting matrix **T**, defined by (9.4), for matrix **P**, and also substituting vector

$$\boldsymbol{\delta}\mathbf{s} = (\delta s_1, \delta s_2, ..., \delta s_J) \tag{9.48}$$

for vector $\boldsymbol{\delta}\mathbf{m}$, we may use the same procedure for evaluating $\boldsymbol{\Delta}\mathbf{K}$ as was used for calculating $\boldsymbol{\Delta}\mathbf{M}$. The required procedure is described by the following algorithm.

Algorithm 9.2: Determination of a Stiffness Modification Matrix

Input: Modal test data $\Phi_1(nxm)$ and $\Lambda_1(mxm)$, and desired modal data $\Phi^*(nxm)$ and $\Lambda^*(mxm)$.

Algorithm:

1) Calculate **T** and **H** using (9.4) and (9.5).
2) Obtain the singular value decomposition $\mathbf{H} = \mathbf{UZU}^T$.
3) Column partition $\mathbf{U} = [\, \mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_m, ..., \mathbf{u}_n]$.
4) Set $\mathbf{U}_1 = [\, \mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_m]$.
5) Calculate

$$T_1^* = \left[t_{ij}^*\right] = U_1^T \; T \; U_1 \quad .$$

6) Set $N = ½(m^2 + m)$.

7) Construct vector

$$\mathbf{y}_K = (y_1^{(K)}, y_2^{(K)}, \ldots, y_N^{(K)})^T = (t_{11}^*, t_{12}^*, t_{13}^*, \ldots, t_{1m}^*, t_{22}^*, \ldots, t_{mm}^*)^T .$$

8) Form vector $\boldsymbol{\delta}\mathbf{s}$ of dimension Jx1 as in (9.48).

9) Construct matrix

$$\boldsymbol{F}_K = \left[f_{ij}^{(K)}\right] = \frac{\partial y_i^{(K)}}{\partial(\delta s_j)}, \quad (i = 1, 2, \ldots, N;\ j = 1, 2, \ldots, J) \quad ,$$

using equation

$$t_{ij}^* = \sum_{q=1}^{J} \delta s_q\, \boldsymbol{u}_i^T \boldsymbol{B}_q^{(K)} \boldsymbol{u}_j \quad .$$

10) (a) If $J > N$, then $\boldsymbol{\delta}\mathbf{s} = \mathbf{F}_K^{\dagger}\mathbf{y}_K + \mathbf{V}_r\mathbf{b}$,

($\mathbf{V}_r$ is obtained by a procedure similar to (9.43), $\mathbf{b}$ is an arbitrary vector).

(b) if $J = N$, then $\boldsymbol{\delta}\mathbf{s} = \mathbf{F}_K^{-1}\mathbf{y}_K$

(c) if $J < N$, then $\boldsymbol{\delta}\mathbf{s} = \mathbf{F}_K^{\dagger}\mathbf{y}_K$

11) If desire non-negative $\boldsymbol{\delta}\mathbf{s}$ and the one obtained in step 10 is not, then determine $\boldsymbol{\delta}\mathbf{s}$ by solving

$$\min_{\delta\mathbf{s}} \ \| \mathbf{F}_K\delta\mathbf{s} - \mathbf{y}_K \|_2 , \text{ subject to } \delta\mathbf{s} \geq 0$$

12) Construct $\mathbf{\Delta K}$ from the elements of $\boldsymbol{\delta}\mathbf{s}$ using (9.47).

Output: Physically realisable stiffness modification matrix $\mathbf{\Delta K}$.

9.3 Numerical Examples

Example 9.1

Consider a mass spring system shown in Figure 9.1.

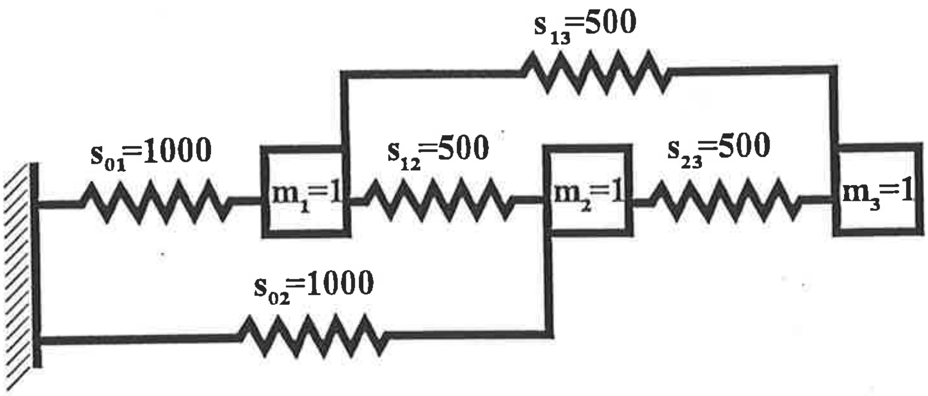


Figure 9.1: A three-degree-of-freedom mass-spring system

The mass and stiffness matrices of this system are as follows

$$\mathbf{M} = \text{diag}\ (1,1,1)$$

and

$$K = \begin{bmatrix} 2000 & -500 & -500 \\ -500 & 2000 & -500 \\ -500 & -500 & 1000 \end{bmatrix}.$$

The spectral and modal properties associated with this system are as follows:

$$\Lambda = \text{diag}\ (500,\ 2000,\ 2500)$$

and

$$\mathbf{\Phi} = \begin{bmatrix} 0.4082 & -0.5774 & -0.7071 \\ 0.4082 & -0.5774 & 0.7071 \\ 0.8165 & 0.5774 & 0.0000 \end{bmatrix} .$$

Now, assume that the physical properties of this system, namely its mass and stiffness matrices, are not known, and also assume that the only available information about the system are the first two of its modes, i.e

$$\mathbf{\Lambda}_1 = \text{diag}\ (500,\ 2000),$$

and

$$\mathbf{\Phi}_1 = \begin{bmatrix} 0.4082 & -0.5774 \\ 0.4082 & -0.5774 \\ 0.8165 & 0.5774 \end{bmatrix} .$$

Suppose that we want to modify the system so that all elements of $\mathbf{\Phi}_1$ are not larger than 0.5, but we also want to achieve this without increasing the magnitude of the existing elements. Under these constraints the desired modal matrix, $\mathbf{\Phi}^*$, is

$$\mathbf{\Phi}^* = \begin{bmatrix} 0.4 & -0.5 \\ 0.4 & -0.5 \\ 0.5 & 0.5 \end{bmatrix} .$$

We also want to modify the spectral properties of the system so that the desired eigenvalues of the system are

$$\mathbf{\Lambda}^* = \text{diag}\ (\ 500,\ 1500)\ .$$

Based on the dimensions of $\mathbf{\Phi}_1$ and $\mathbf{\Lambda}_1$, we realise that in the above system there are a maximum of three independent mass elements and a maximum of six independent spring elements available for modification. The number of constraints to be satisfied by the solutions to (9.31) and (9.32) is equal to three for both mass and stiffness modifications. Therefore, we expect that there exists one unique solution for $\boldsymbol{\delta}\mathbf{m}$, and a family of solutions for $\boldsymbol{\delta}\mathbf{s}$.

Applying Algorithm 9.1, using step 10(b), we obtain the following unique solution for the mass modification matrix, $\mathbf{\Delta M}$, corresponding to a mass-spring analytical model

$$\mathbf{\Delta M} = \text{diag}\ (0.2346,\ 0.2346,\ 1.0247).$$

Since there exists a family of solutions for $\boldsymbol{\delta}\mathbf{s}$, we choose a minimal norm solution (determined by using step 10(a) in the Algorithm 9.2 with $\boldsymbol{b}$ being zero vector), and obtain the following stiffness modification matrix $\mathbf{\Delta K}$

$$\boldsymbol{\Delta K} = \begin{bmatrix} -197.5 & 0.0 & -67.9 \\ 0.0 & -197.5 & -67.9 \\ -67.9 & -67.9 & 938.3 \end{bmatrix}.$$

The modified mass and stiffness matrices for the system are then as follows

$$\mathbf{M}_{\mathbf{mod}} = \mathbf{M} + \mathbf{\Delta M} = \text{diag}\ (1.2346,\ 1.2346,\ 2.0247)$$

and

$$\boldsymbol{K}_{\mathbf{mod}} = \boldsymbol{K} + \boldsymbol{\Delta K} = \begin{bmatrix} 1802.5 & -500.0 & -567.9 \\ -500.0 & 1802.5 & -567.9 \\ -567.9 & -567.9 & 1938.3 \end{bmatrix}.$$

The mass-spring system corresponding to $\mathbf{M}_{\mathbf{mod}}$ and $\mathbf{K}_{\mathbf{mod}}$ is shown in Figure 9.2.

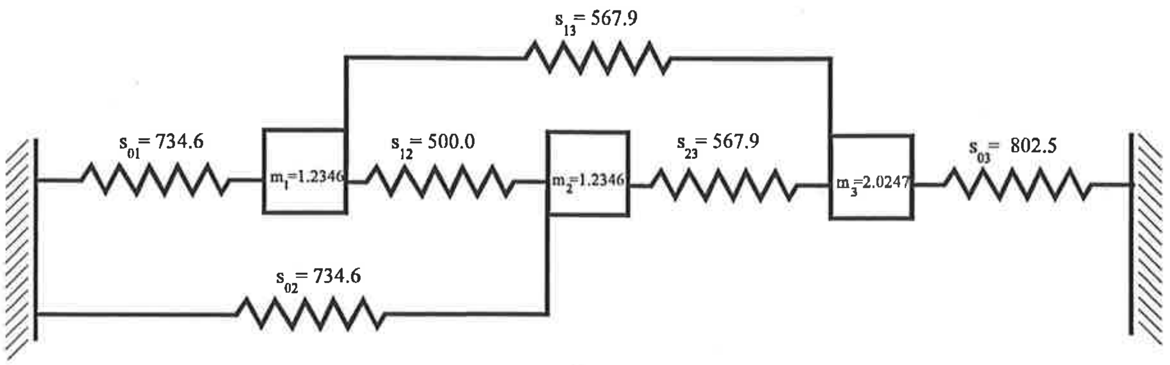


Figure 9.2: A modified mass-spring system

The eigenvalues and mass-normalised eigenvectors corresponding to this modified system are

$$\mathbf{\Lambda}_{\mathbf{mod}} = \text{diag}\,(495.8,\ 1516.5,\ 1865.0\ )$$

and

$$\mathbf{\Phi}_{\mathbf{mod}} = \begin{bmatrix} 0.4279 & -0.4710 & -0.6364 \\ 0.4279 & -0.4710 & 0.6364 \\ 0.5202 & 0.4726 & 0.0000 \end{bmatrix}.$$

The visual comparison between the two desired modes and the first two modes of the modified system show good correlation. However, a good correlation requires that the eigenvalue ratio, the amplitude ratio of the eigenvectors and the values of cosines between the two eigenvectors are all as close as possible to 1. The values of these ratios and cosines are presented in Table 9.1.

Desired Mode, i	Corresponding Obtained Mode, j	Eigenvalue Ratio, λ_j/λ_i^*	Amplitude Ratio of Eigenvectors, $\Vert\phi_j\Vert/\Vert\phi_i^*\Vert$	Cosine of an angle between the two eigenvectors, $\cos(\angle\phi_j\phi_i^*)$
1	1	0.9916	1.0570	0.9999
2	2	1.0110	0.9431	1.0000

Table 9.1 : Comparison between the desired and the obtained modes.

Results in Table 9.1 demonstrate that a very good correlation is achieved between the desired and the obtained modes.

Example 9.2: Sensitivity test

In this example we examine the sensitivity of the developed method to small perturbations in the measured data. Suppose that the matrices $\mathbf{\Lambda}_1$ and $\mathbf{\Phi}_1$ of Example 9.1 were measured with some perturbations, and are as follows

$$\mathbf{\Lambda}_1 = \text{diag}\ (450,\ 2050)$$

and

$$\mathbf{\Phi}_1 = \begin{bmatrix} 0.4 & -0.6 \\ 0.4 & -0.6 \\ 0.8 & 0.6 \end{bmatrix} .$$

Repeating Example 9.1 with the new Λ_1 and Φ_1, we obtain the following mass and stiffness modification matrices

$$\Delta\mathbf{M} = \text{diag}\ (0.2701,\ 0.2701,\ 1.0216)$$

and

$$\Delta K = \begin{bmatrix} -142.7 & 0.0 & -44.4 \\ 0.0 & -142.7 & -44.4 \\ -44.4 & -44.4 & 946.0 \end{bmatrix}.$$

The modified mass and stiffness matrices for the system then are

$$\mathbf{M}_{\mathbf{mod}} = \mathbf{M} + \Delta\mathbf{M} = \text{diag}\ (1.2701,\ 1.2701,\ 2.0216)$$

and

$$K_{\mathbf{mod}} = K + \Delta K = \begin{bmatrix} 1857.3 & -500.0 & -544.4 \\ -500.0 & 1857.3 & -544.4 \\ -544.4 & -544.4 & 1946.0 \end{bmatrix}.$$

The mass-spring system corresponding to the above $\mathbf{M}_{\mathbf{mod}}$ and $\mathbf{K}_{\mathbf{mod}}$ is shown in Figure 9.3.

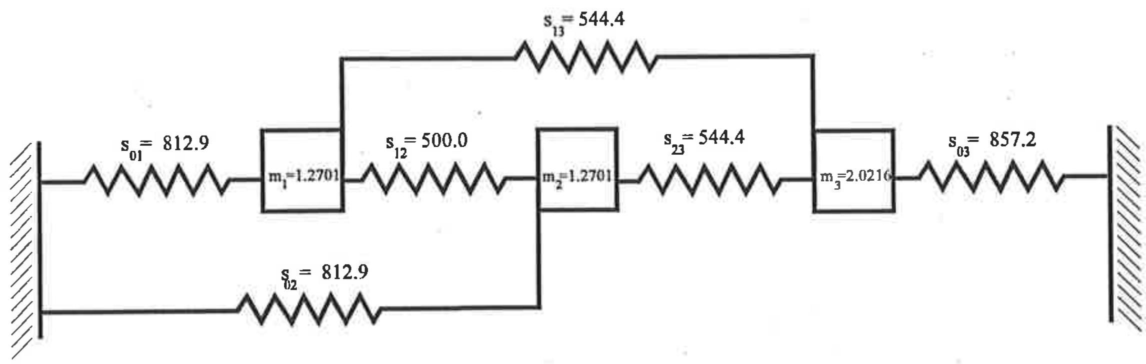


Figure 9.3: A modified mass-spring system from sensitivity test

The eigenvalues and mass-normalised eigenvectors corresponding to this modified system are

$$\Lambda_{mod} = \text{diag}\,(532.3,\ 1499.0,\ 1856.0\ ),$$

and

$$\Phi_{mod} = \begin{bmatrix} 0.4186 & -0.4674 & -0.6274 \\ 0.4186 & -0.4674 & 0.6274 \\ 0.5239 & 0.4692 & 0.0000 \end{bmatrix}.$$

We note that the resulting solution differs marginally from the solution of Example 9.1. However, the differences are small, and the correlation between the desired and the obtained modes is very good. Table 9.2 shows the eigenvalue ratio, the amplitude ratio of the eigenvectors and the values of cosines between the two eigenvectors.

Desired Mode, i	Corresponding Obtained Mode, j	Eigenvalue Ratio, λ_j/λ_i^*	Amplitude Ratio of Eigenvectors, $\lVert \phi_j \rVert / \lVert \phi_i^* \rVert$	Cosine of an angle between the two eigenvectors, $\text{Cos}(\angle\phi_j\phi_i^*)$
1	1	1.0646	1.0471	1.0000
2	2	0.9993	0.9360	1.0000

Table 9.2 : Comparison between the desired and the obtained modes.

Results in Table 9.2 show that despite the introduction of perturbations into the measured data, the quality of the obtained solution is not greatly affected. Therefore we conclude that the developed method is sufficiently robust to perform adequately when perturbations are relatively small.

9.4 Conclusions

A method for determining physically realisable mass and stiffness modifications has been developed. The method is broadly based on the results of Ram and Braun in [30], and it allows determination of the mass and stiffness modification matrices corresponding to any chosen analytical model (i.e the method is general, and is not restricted to any specific analytical model).

Depending on the dimensions of the measured modal data contained in $\mathbf{\Lambda}_1$ and $\mathbf{\Phi}_1$, the method allows to obtain a family of solutions, an unique solution, or an optimal approximate solution for the mass modification matrix $\mathbf{\Delta M}$, and the stiffness modification matrix $\mathbf{\Delta K}$. However, since $\mathbf{\Delta M}$ and $\mathbf{\Delta K}$ themselves constitute only an approximation to the desired solution, it is recommended that the method is applied only in situations where exact solutions for $\mathbf{\Delta M}$ and $\mathbf{\Delta K}$ are available.

The method was tested on a numerical example, and a solution obtained showed a good correlation between the desired and the obtained modal properties. The sensitivity of the method to small perturbations was also performed, and the method was found to be sufficiently robust to cope adequately with introduced perturbations without noticeable deterioration in its performance.

Section 10 *EXPERIMENT*

The main aim of the experimental work described in this section was to check if the developed theoretical results can be used in practical design applications. It is well known that the dynamic behaviour of a discrete system is fundamentally different from the behaviour of a continuous system. In practice, all measured modal analysis data is obtained from a real structure, which behaves like a continuous system. Thus, there is an obvious possibility that the measured modal data may be incompatible with the chosen analytical model of a test structure. In general, a finite element model gives a good correlation with the behaviour of a continuous system for approximately a third of its modes. A discrete mass-spring system would probably give a reasonable correlation for even less number of modes. These "well-correlated" modes correspond to the lower natural frequency end of the spectrum, and the lower the natural frequency of a mode, the better is the correlation.

Provided the measured modal data and the desired modes are within this range of good correlation, the performance of the algorithms should, in principle, be acceptable. However, this condition may prove to be too restrictive for many practical cases.

Most of the theoretical results described in the previous sections were based on the assumption that a vibratory system may be modelled as a conservative discrete mass-spring system. It was the prime objective of this experiment to test whether such assumption may be successfully applied to a practical engineering structure. A simple "desk-top" test rig, which could be used for testing and demonstrating the developed theory, was deemed sufficient to achieve our objectives.

Because our aim was to test a practical engineering structure, we specifically did not want to use an experimental model which consisted of lumped masses connected by light springs. To use such model is equivalent to testing a physical mass-spring system, which is not representative of any obvious engineering application. At the same time, we wanted to use an experimental model which would give a good correlation with the behaviour of a mass-spring system. To do otherwise would have created a large uncertainty in testing the performance of the algorithms.

The two systems considered appropriate for our test model were the torsional shaft-pulley system (see figure 2.1) and a "building" model which is shown in figure 10.1. Both of these test models may represent a large number of real engineering structures. The torsional system is clearly representative of any rotational machinery power transmission trains or gear box assemblies. The "building" model may represent any cantilever structure, such as buildings, aeroplane wing, and many other.

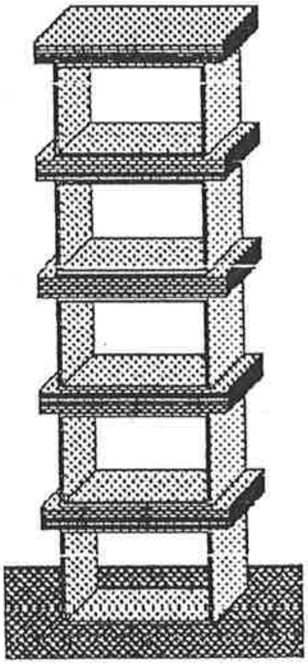


Figure 10.1: Example of a "building" model

The "building" model was chosen as best suitable because of cost, simplicity and safety considerations. To measure the torsional modes would have required the use of a more sophisticated equipment and a more complicated test set-up. Also, to demonstrate the resonance of this torsional system, it had to be driven at high rotational speeds (while resonating) which was considered too unsafe.

The Algorithm 4.2 for solving Problem 1 (see section 5) was then chosen as most suitable for the experimental assessment. This algorithm was selected because, unlike algorithms for Problems 2 and 4 (see sections 6 and 9), it only involved changing the masses of the structure, thus greatly simplifying the design of the test model. Also, unlike algorithm for Problem 3 (see section 8), it is not sensitive to perturbations in the measured modal analysis data, thus giving a more stable platform for the experimental assessment of its performance.

10.1 Test Model Description

The test model consisted of nine extruded aluminium box sections, which represented the "walls" of a building, and a large number of steel plates of various thickness (and hence mass), which were sandwiched between the adjacent box sections to obtain the necessary "floor" mass at each location. The aluminium box sections had a uniform thickness of 3mm throughout, and its dimensions were 160mm(long) x 100mm(wide) x 100mm(high). The steel plates had dimensions of 215mm(long) x 100mm(wide) and were made in various thicknesses to allow for different mass configurations. The adjacent box sections with plates in between were joined together by mild steel, M6x1.0, hexagonal head bolts of appropriate lengths.

The overall, general layout of the assembled model is shown in Figure 10.2. Figure 10.3 shows details of a joint connection between the two adjacent box sections with steel plates in-between.

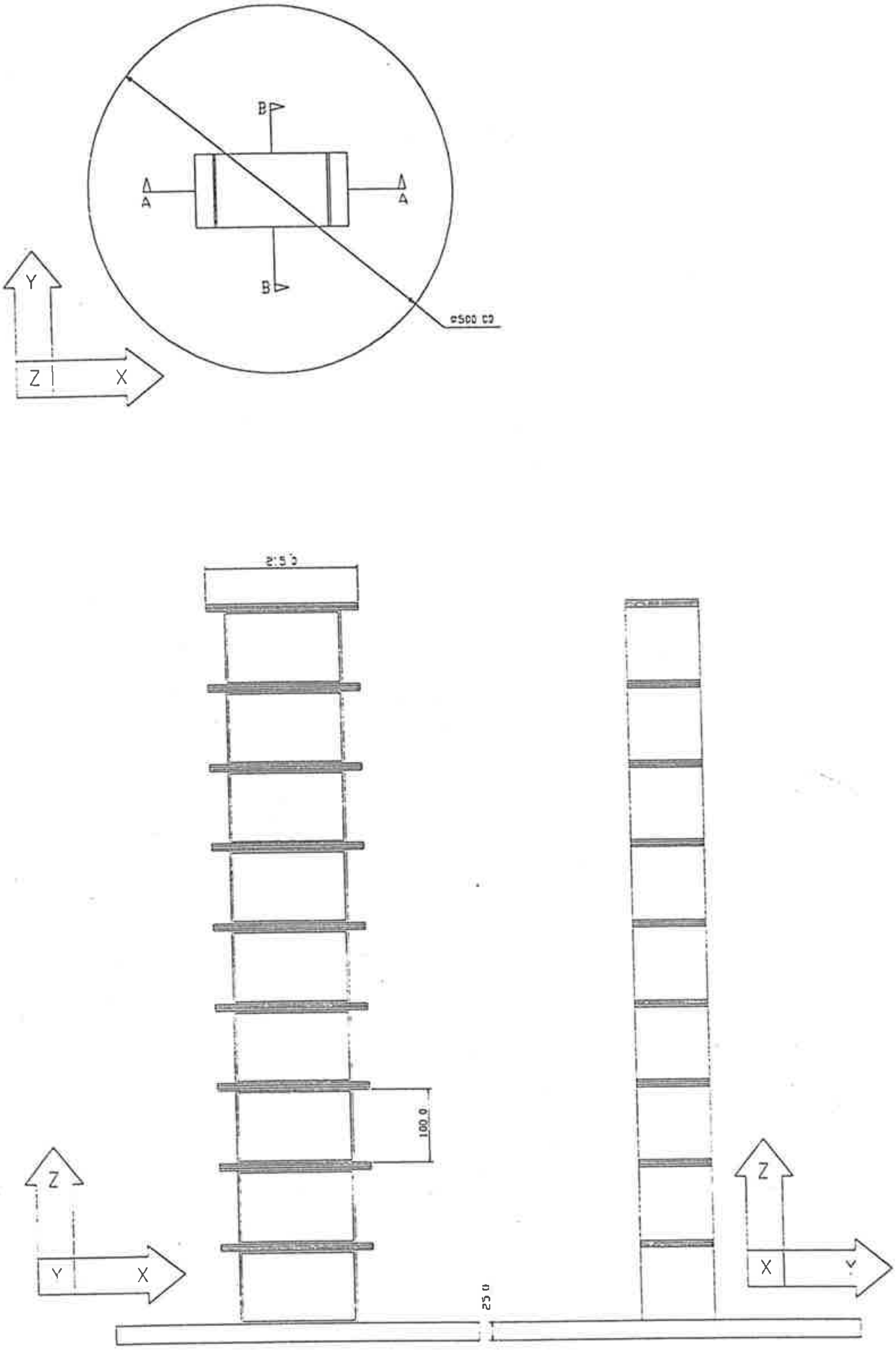


Figure 10.2 : The overall general layout of the test structure.

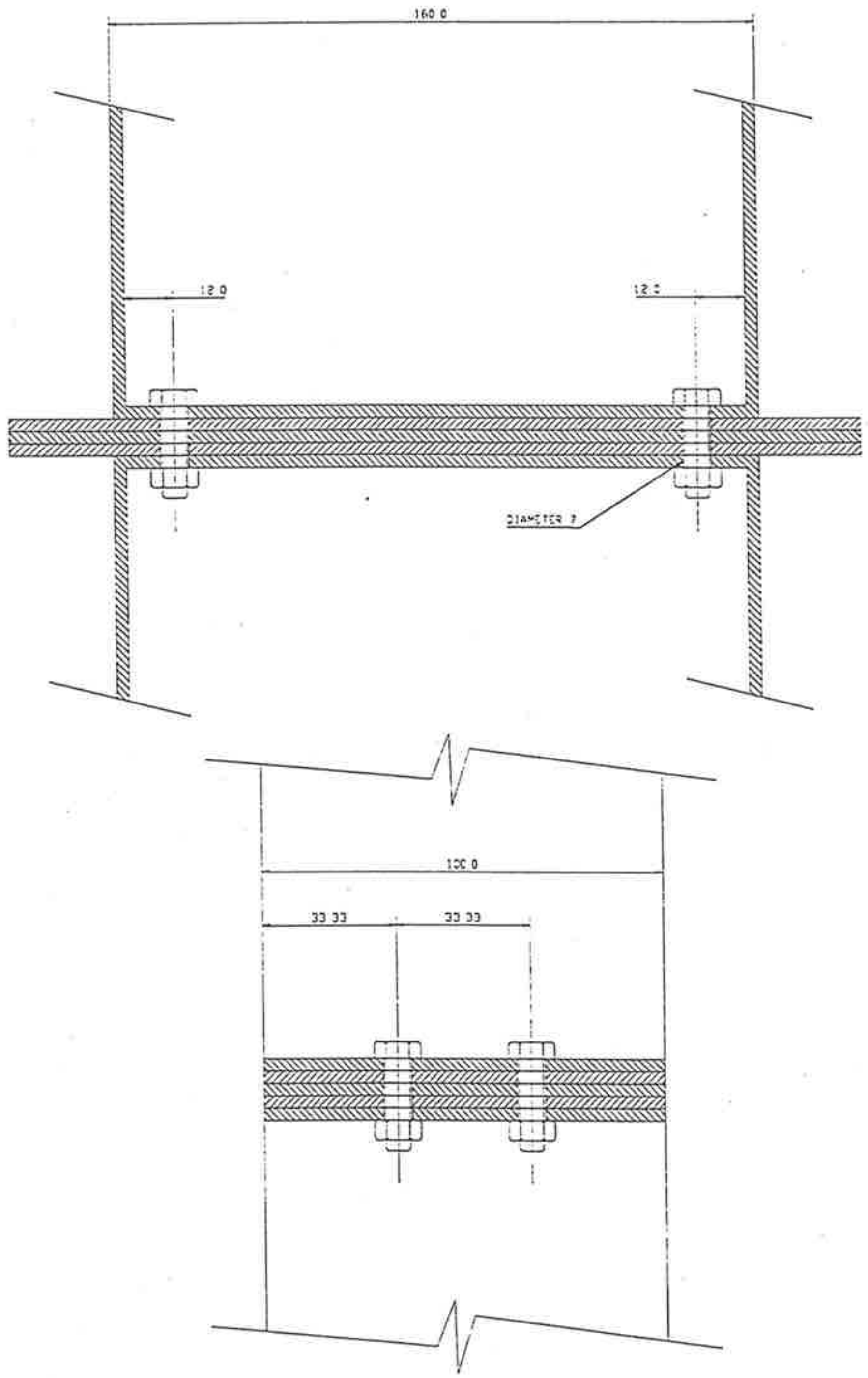


Figure 10.3 : Details of a joint connection between adjacent box sections.

10.2 Mass Elements of the Test Model

Figure 10.4 shows a schematic layout between two adjacent "floors" of the test structure.

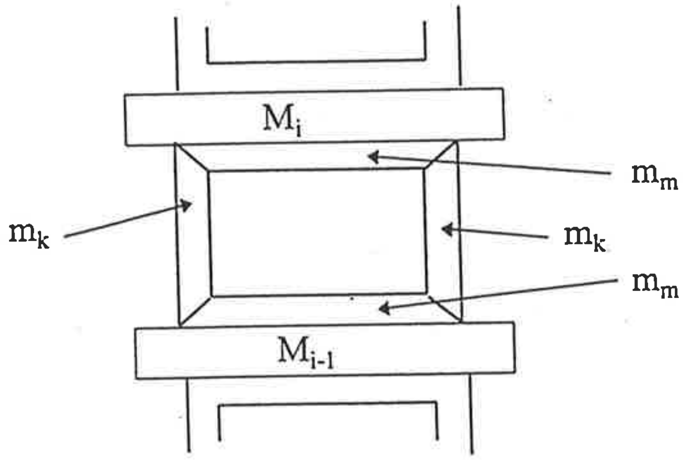


Figure 10.4 : A schematic layout between two adjacent "floors".

In the theoretical mass-spring system, the springs, which connect each mass element to others, are themself have no mass. In a real physical structure this is clearly not the case. Each aluminium box section had a finite mass, although, in general, this mass was very small relative to the mass of the steel plates at each "floor". A choice had to be made whether to ignore the mass of the aluminium box sections, or to include it in the calculations of the "floor" mass. To maintain the accuracy of the test model, it was decided to include the mass of the box sections in our calculations. The effective mass of each "floor" was calculated based on the well known Rayleigh's method (see e.g. Thomson [78], pp. 24-25). The resulting mass matrix for our test structure was then assumed to have the following form

$$M = \begin{bmatrix} M_1 + 2(m_m + \frac{m_k}{3}) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & M_2 + 2(m_m + \frac{m_k}{3}) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & M_3 + 2(m_m + \frac{m_k}{3}) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & M_4 + 2(m_m + \frac{m_k}{3}) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & M_5 + 2(m_m + \frac{m_k}{3}) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & M_6 + 2(m_m + \frac{m_k}{3}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & M_7 + 2(m_m + \frac{m_k}{3}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & M_8 + 2(m_m + \frac{m_k}{3}) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M_9 + (m_m + \frac{m_k}{3}) \end{bmatrix} \quad (10.1)$$

where M_i (i=1,2,...,9) was the mass of the steel plates and connecting bolts at the i^{th} "floor", m_m is the mass of a horizontal segment of a box, and m_k is the mass of a vertical segment of a box.

From measurement it was found that the mass of each box section was approximately equal to 540 grams, and also, from the dimensions of the box section, we know that

$$m_m = 1.6 m_k \quad . \qquad (10.2)$$

The mass of each box section is equal to

$$2(m_m + m_k) = 540. \qquad (10.3)$$

Thus from (10.2) and (10.3) we find that

$$m_m \approx 166 \text{ grams} \tag{10.4}$$

and $$m_k \approx 104 \text{ grams.} \tag{10.5}$$

And, consequently, the sum

$$m_m + \frac{m_k}{3} = 201 \; grams \; . \tag{10.6}$$

This value was then used to determine the additional mass from steel plates which would give us the desired total mass at each "floor". In general, by carefully manipulating with the plates of different masses, we were able to achieve the mass at each "floor" which was nominally within ±2 grams of the desired value.

10.3 Determination of the Stiffness

The stiffness matrix corresponding to our test model, was assumed to have the following form

$$K = \begin{bmatrix} 2k & -k & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -k & 2k & -k & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -k & 2k & -k & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -k & 2k & -k & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -k & 2k & -k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -k & 2k & -k & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -k & 2k & -k & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -k & 2k & -k \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -k & k \end{bmatrix} \tag{10.7}$$

where k was the stiffness constant of each box section.

The value of k was estimated theoretically using the listed properties of aluminium and the information about the dimensions and the shape of the box section. Using Thomson [78, p.178] the stiffness constant k is given by

$$k = 24\frac{EI}{l^3} \tag{10.8}$$

where E = Young's Modulus of Elasticity of aluminium

I = moment of inertia of a box section

l = height of the box section.

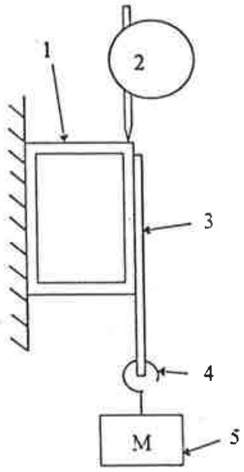
It was not clear which value had to be used for height l of the box section, i.e 94mm or 100mm. Therefore, both values were used to determine the upper and lower bounds for k. Substituting the listed values for E = 70 - 75 GPa, and using other dimensions of a box section to calculate I, the theoretical value of k was found to be

$$k = 378 \text{ kN/m} - 488 \text{ kN/m}. \tag{10.9}$$

Such a large uncertainty (over 20%) was considered too great for our experiment, and therefore a simple procedure was carried out to measure the value of k experimentally.

10.3.1 Stiffness determination experiment

Figure 10.5 shows the schematic layout of the experimental set-up for stiffness measurement.



APPARATUS

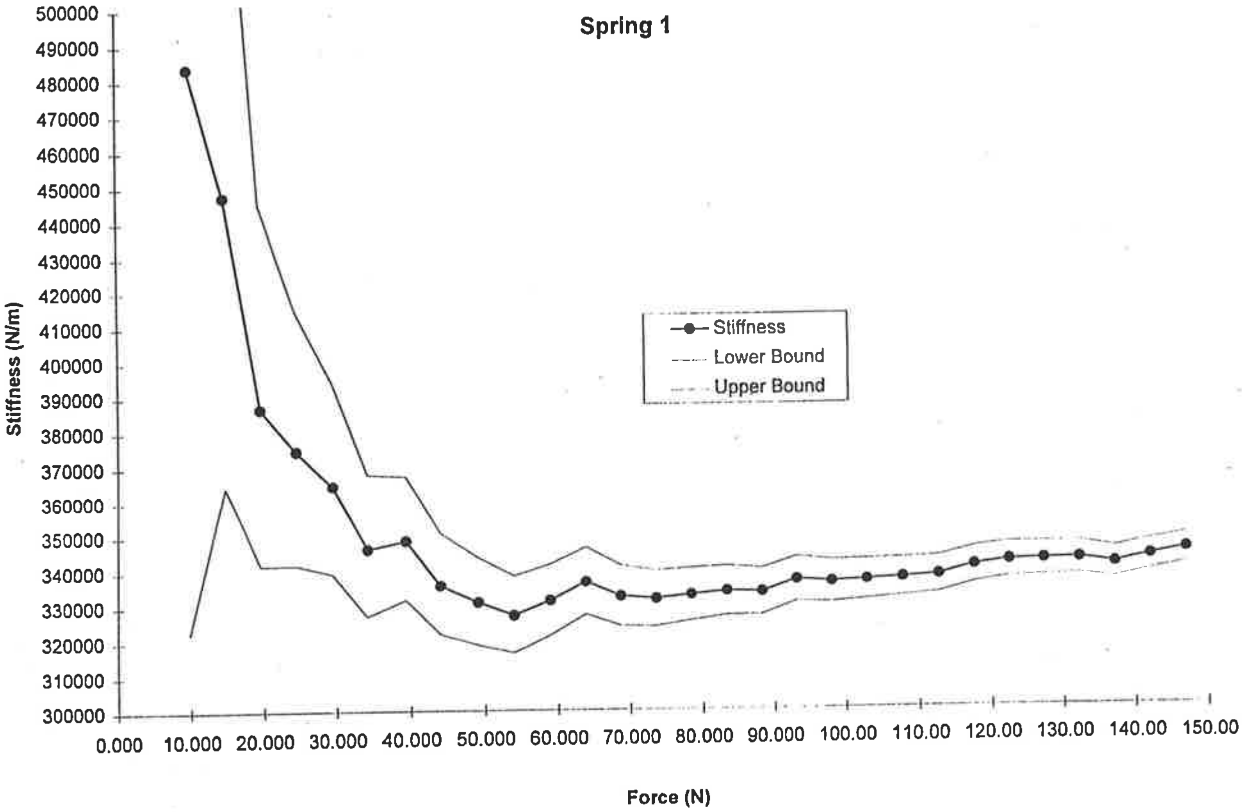
1. Aluminium Box Section
2. Dial Indicator
3. Mounting Plate
4. Hook
5. Load Masses

Figure 10.5: Schematic layout of the stiffness measurement experiment

The method was:

1. Mount the box section onto a solid, straight vertical surface as shown in Figure 10.5.
2. Attach the mounting plate to the other side of the box section as shown, and place the hook through the hole in the mounting plate.
3. Position the dial indicator so that its tip is touching the front edge of the box section, and set the dial reading to zero.
4. Add 0.5kg masses, one at a time, and measure the deflections from the dial indicator.

This process was repeated for three randomly selected box sections, and the results from this experiment are shown in Figure 10.6. The raw data from this experiment is given in the Appendix A. We note the value of the stiffness is significantly higher under small loads (and hence small deflections). However, experimental uncertainty in measured values of small deflections was very large, mainly due to dial resolution limitations. At higher loads the value of the stiffness seemed to "settle" around the approximately 350 kN/m mark.



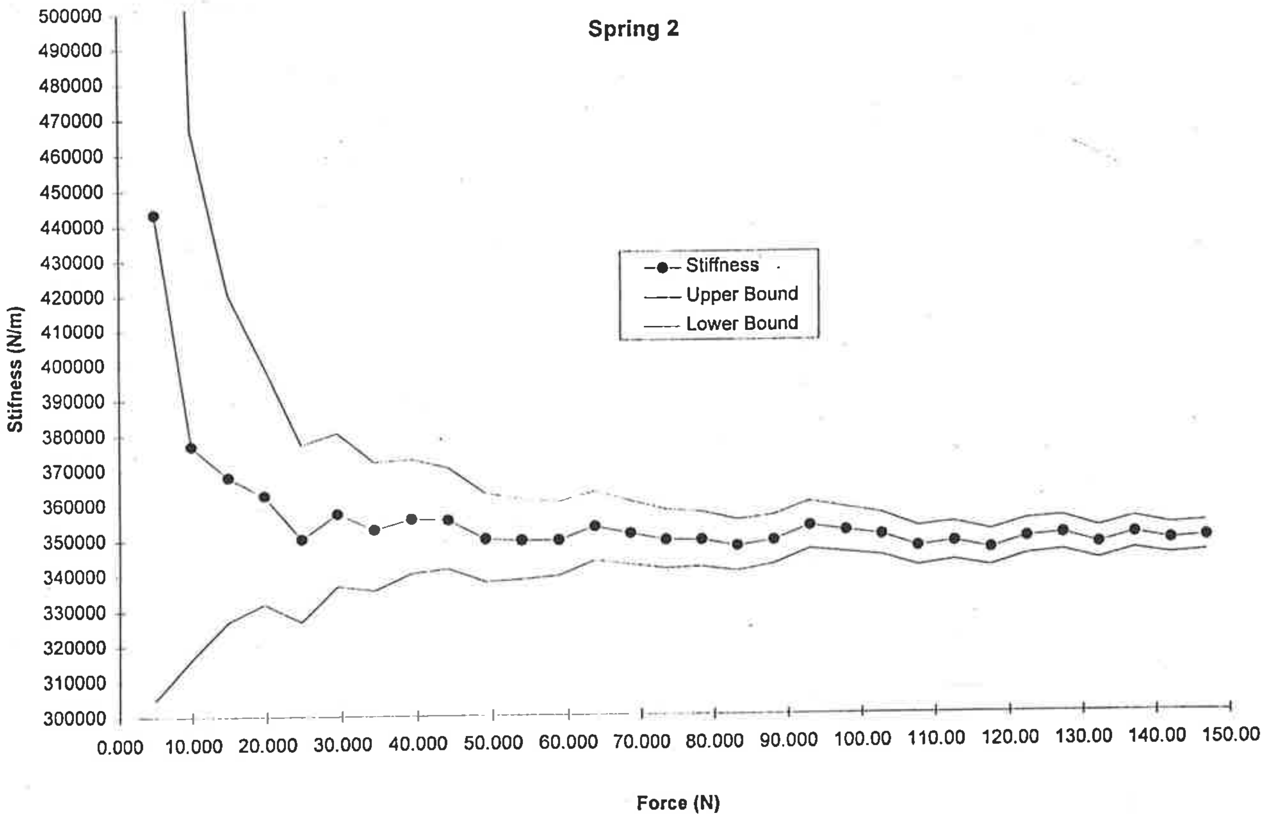


Figure 10.6: Measured stiffness values

10.3.2 Optimal value for the stiffness, k

It was anticipated that during the modal analysis testing of the "building" model the deflections of the "walls" would be relatively small. Therefore, a large uncertainty in what value should be used for stiffness k still existed, despite the availability of the measured data. In the end, the optimal value for the stiffness k was determined by performing a Chi-squared test, which is defined as

$$\chi_0^2 = \sum_{i=1}^{9} \frac{(a_i - b_i)^2}{a_i} \tag{10.10}$$

where χ_0^2 = Chi-squared value
a_i = i^{th} measured experimental natural frequency
b_i = i^{th} analytically determined natural frequency.

Clearly, in the equation (10.10) if $a_i = b_i$ (for i=1,...,9), then $\chi_0^2 = 0$. Thus, the objective was to find the stiffness value, k, which produced analytical natural frequencies b_i (i=1,...,9), such that the magnitude of the χ_0^2 were minimised. The procedure was then as follows. The modal analysis tests were performed on several configurations of a test structure (i.e different mass configurations at each floor), and the measured natural frequencies of each configuration recorded. The analytical values for the natural frequencies of the model with the same mass configurations were calculated using a number of different values for the stiffness k. Then, setting a_i to be the measured natural frequencies and b_i to be the corresponding analytically determined natural frequencies, the values of the χ_0^2 were calculated using equation (10.10). The value of the stiffness k, which consistently gave the lowest $\varkappa_0^2$ value was

$$k = 378 \text{ kN/m} \tag{10.11}$$

and this value was then used in all subsequent experiments. The raw data from these Chi-squared tests is given in the Appendix B.

10.4 Experimental Testing of the Algorithm 4.2

The schematic layout of the experimental set-up is shown in Figure 10.7. The equipment used for these experiments are listed below.

EQUIPMENT USED

1. The "Building" Model.
2. Brüel and Kjær Accelerometer - model 9040.
3. Brüel and Kjær Signal Analyser - model 2032.
4. Brüel and Kjær Charge Amplifier - model 5666.
5. Brüel and Kjær Impulse Hammer - model 1234.
6. IBM Compatible Personal Computer.

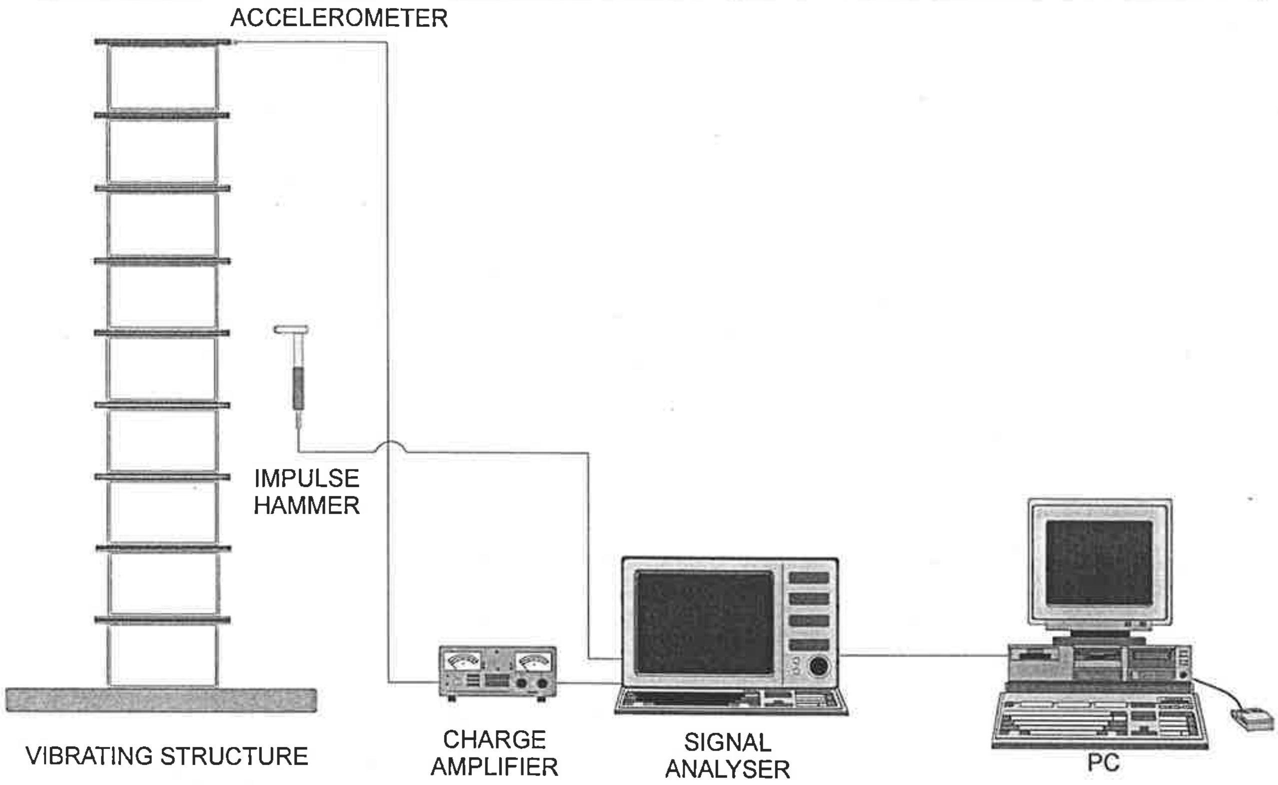


Figure 10.7 : Schematic layout of the Algorithm Testing Experiment.

The experimental procedure was then as follows:

1. Nine different natural frequencies were arbitrarily chosen.
2. Using the stiffness k =378 kN/m, the stiffness matrix of the "building" model was constructed via equation (10.7).
3. Algorithm 4.2 for solving Problem 1 (see section 4) was then applied to determine the necessary mass matrix. (Note: Sometimes several repetitions of the algorithm (with different starting values for the initial guess of the mass matrix) were required, in order to obtain the natural frequencies which were adequately close to the desired frequencies.)

4. Using the obtained mass matrix from step (3) and equations (10.1) and (10.6), the masses of the steel plates to be added at each "floor" were determined.
5. The physical test structure was then assembled with the determined amount of plates at each "floor".
6. The structure was lightly struck by the impulse hammer, consecutively at each "floor", each time recording the natural frequencies of the structure.
7. The measured natural frequencies at each "floor" were averaged, and compared with the desired natural frequencies and the natural frequencies of the analytical system determined by the Algorithm 4.2.

The raw data from these experiments is given in the Appendix C. In Figure 10.8 we present the graphical comparison between the measured, the desired and the analytically determined (by Algorithm 4.2) natural frequencies.

In all results shown in Figure 10.8, the "Frequency" axis is set between the same limits (from 0-220Hz), thus allowing easy visual comparison of the data from all tests. We also want to emphasise the following point. Although in theory Algorithm 4.2 should permit unrestricted assignment of arbitrarily chosen natural frequencies, in our experiment the achievable natural frequency range was approximately 5Hz to 220Hz. This limitation was a direct consequence of the physical constraints on the smallest and the largest mass that we could have at the "floors". Clearly, the smallest mass was simply the mass of the aluminium box sections with no steel plates added. The maximum obtainable mass was governed by the available supply of the steel plates (which was approximately 70kg).

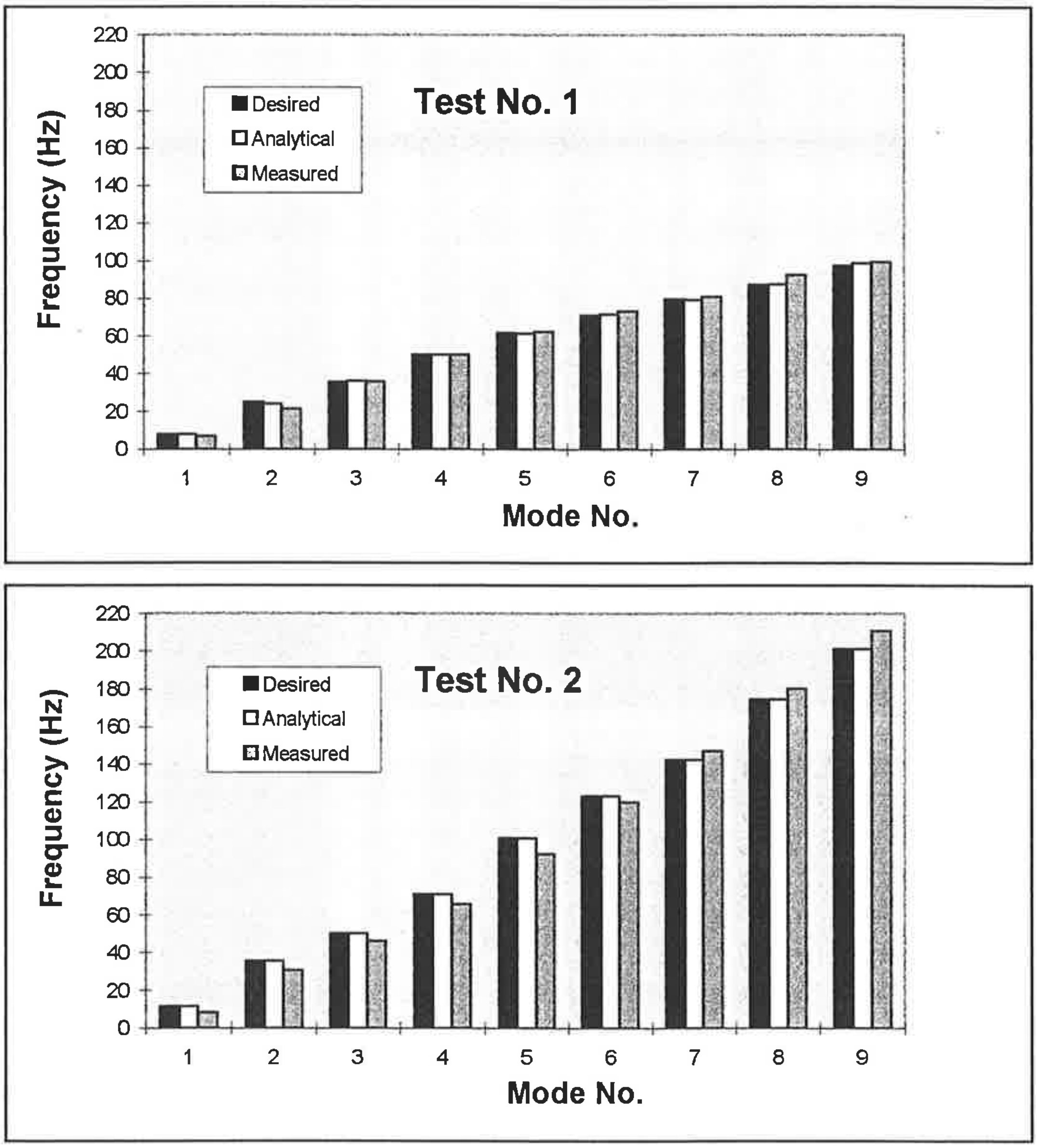
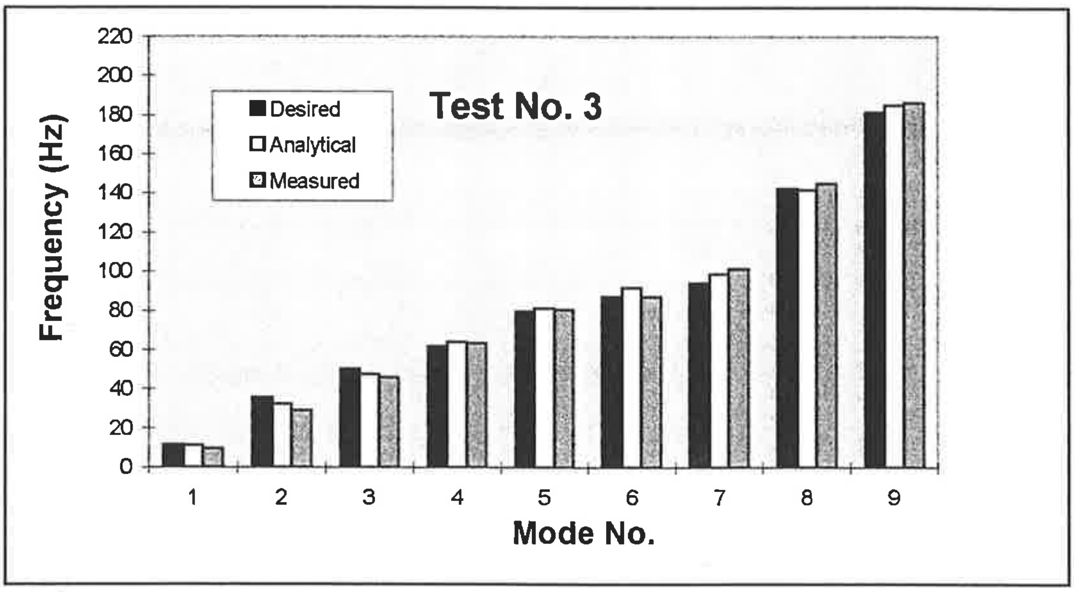


Figure 10.8 : Comparison between the desired, measured and analytical (using Algorithm 4.2) natural frequencies



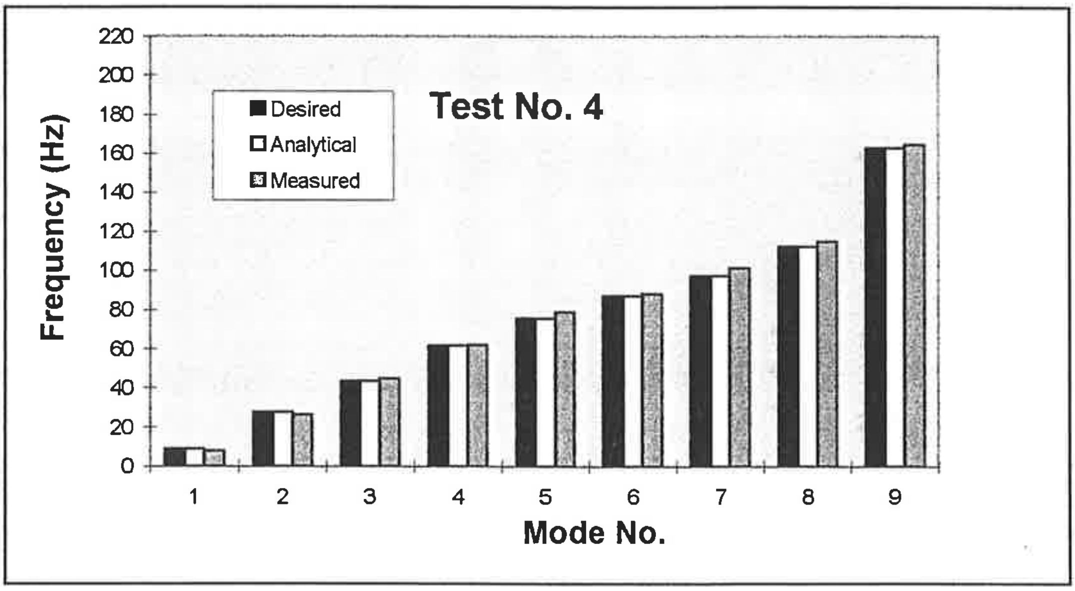
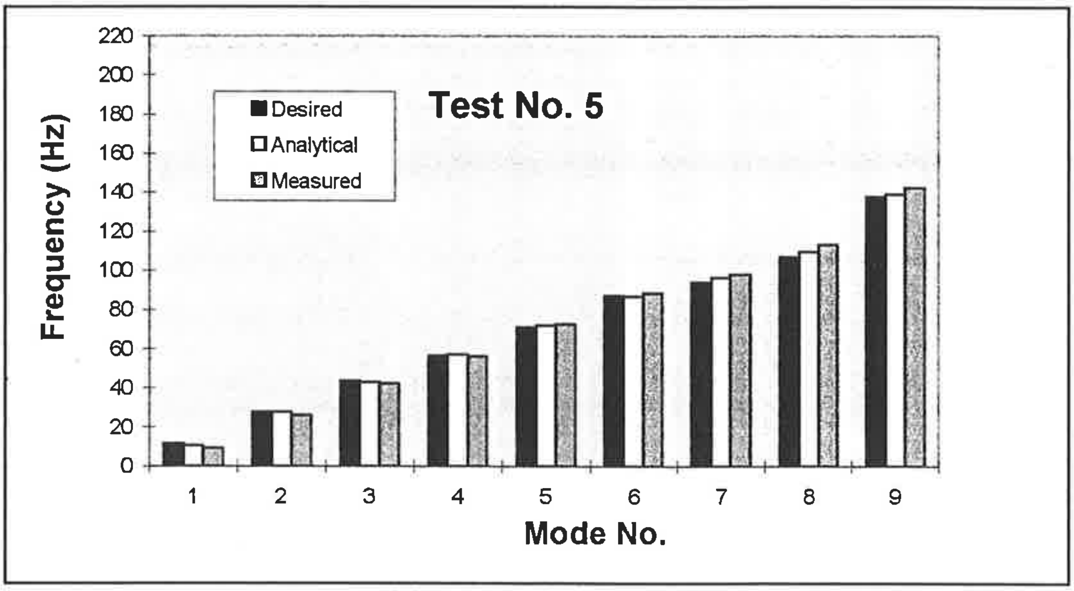


Figure 10.8 (cont'd): Comparison between the desired, measured and analytical (using Algorithm 4.2) natural frequencies.



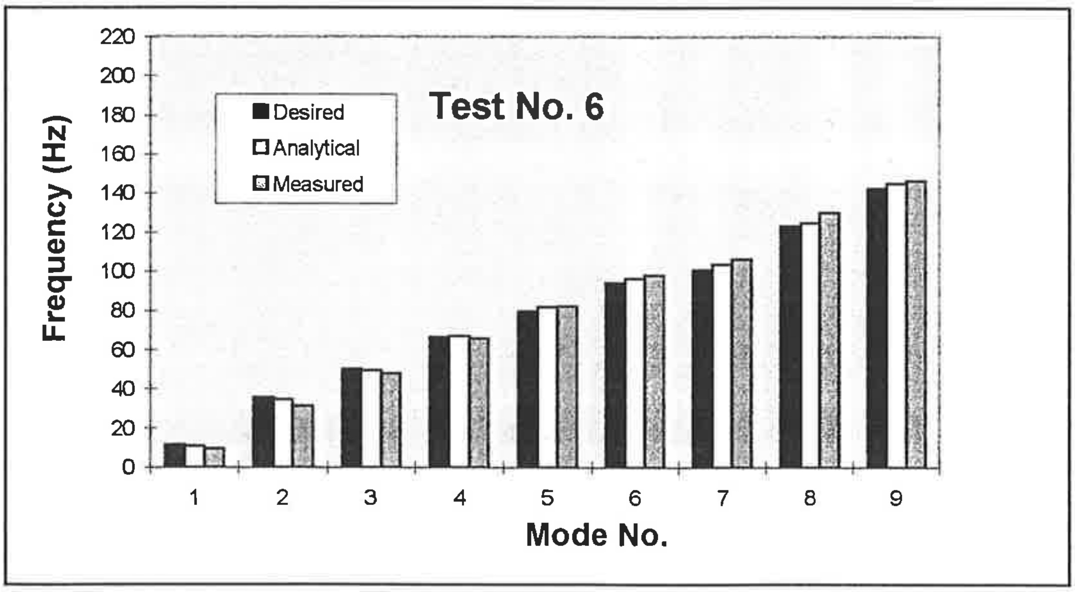
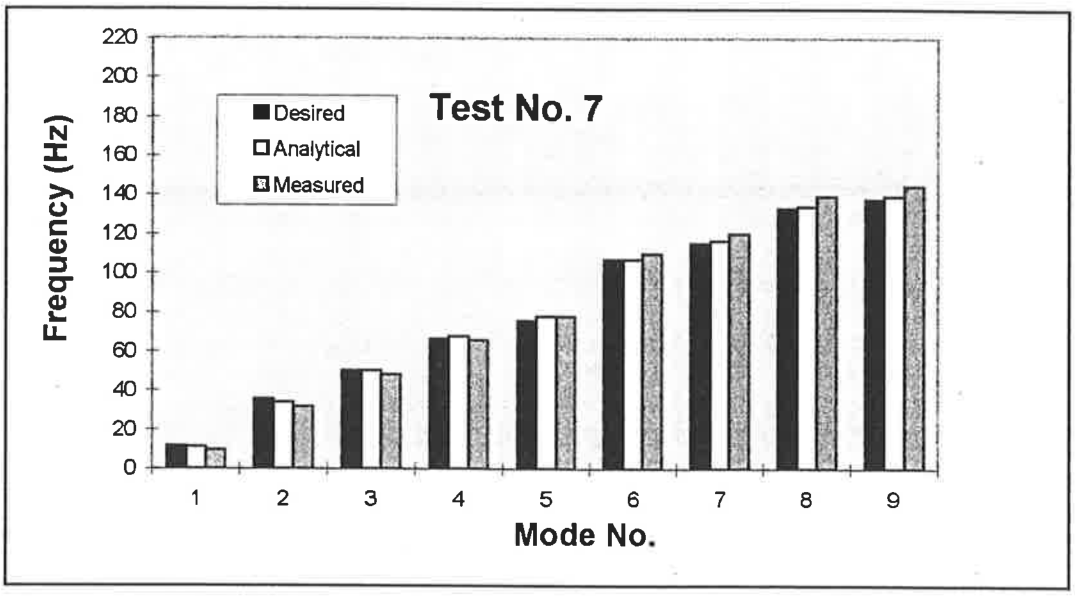


Figure 10.8 (cont'd): Comparison between the desired, measured and analytical (using Algorithm 4.2) natural frequencies.



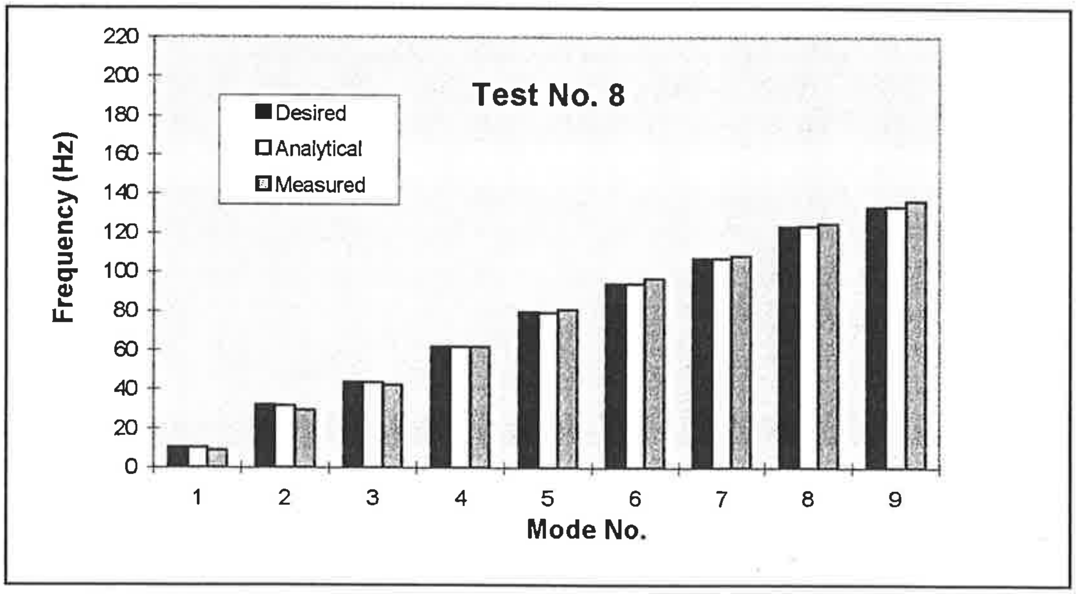
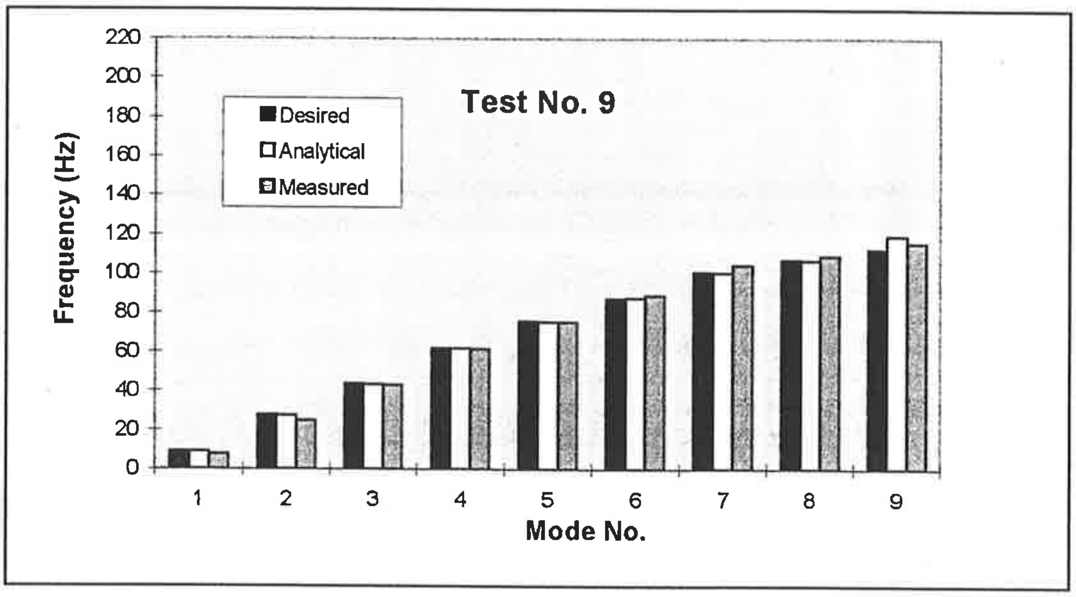


Figure 10.8 (cont'd): Comparison between the desired, measured and analytical (using Algorithm 4.2) natural frequencies.



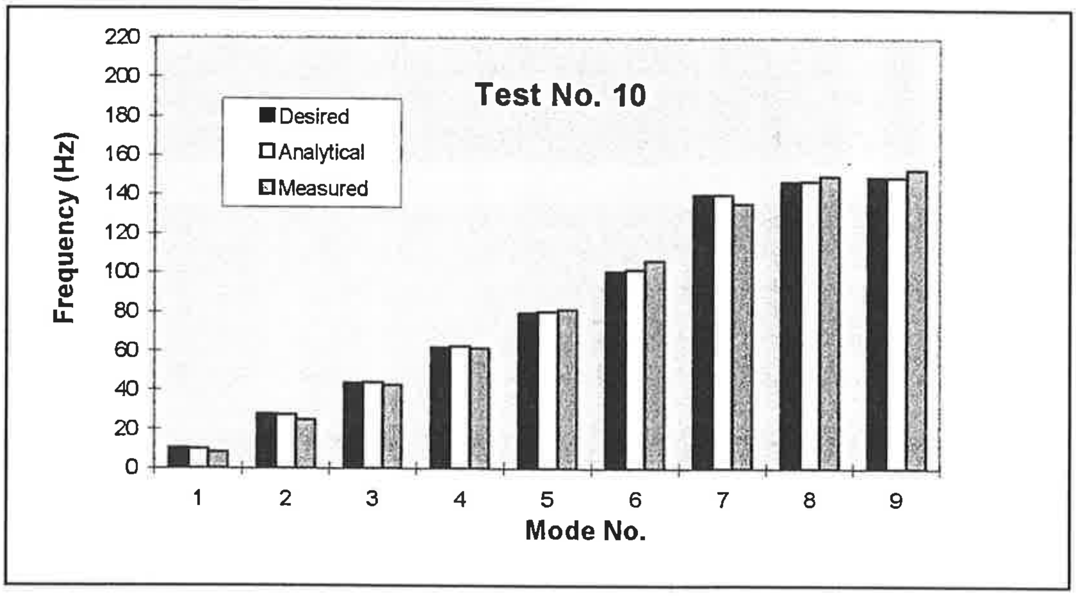


Figure 10.8 (cont'd): Comparison between the desired, measured and analytical (using Algorithm 4.2) natural frequencies.

10.5 Conclusions

The results presented in Figure 10.8 show a good correlation between the desired natural frequencies and the natural frequencies both of the test structure and of the analytical model determined by Algorithm 4.2. Therefore, the main aim of this experimental program, which was to test whether the assumption of conservative mass-spring system is acceptable in the practical engineering applications, was achieved and the answer is positive. However, we are fully aware that the chosen test structure was highly "tailored" and optimised for conformance with such analytical model, and that most "real-life" structures would not be so successful. The experimental test model was, however, well suited to the stated scope of our experiment, and it is representative of some useful engineering structures. The Algorithm 4.2 was found to work well in applications to a real physical structure, and it has a potential of being a very useful tool for the design of vibratory systems to suit natural frequency requirements.

Section 11 *CONCLUSIONS*

The conclusions pertaining specifically to the particular problems investigated, were given at the end of the appropriate sections dealing with those problems. Here we present the general conclusion, which are applicable to all results given in this thesis.

The material presented in this thesis contains a logically complete set of solutions to practical problems dealing with the design and structural modifications of structures, which may be adequately modelled by a mass-spring analytical system. In section 4 we developed a method which allows determination of a mass matrix when the stiffness matrix and a complete set of the desired natural frequencies of a system are known. A derivative of a method by Joseph [16] allows a similar procedure to be followed when the desired natural frequency set is truncated, i.e when not every natural frequency is precisely specified. In section 5 we presented a method for optimal reconstruction of a mass-spring system from a complete set of prescribed spectral and modal data. The analysis given in section 6 then allows us to do the same when the prescribed spectral and modal data are incomplete. In section 7 we have extended the solution method of section 5 to a more general class of the mass and stiffness matrices (i.e. which not necessarily correspond only to a mass-spring system). In section 8 we used the method of Joseph [16] as a basis for developing a new algorithm for obtaining the necessary mass and stiffness modifications to an existing structure, so that the natural frequencies of a modified system are as close as possible to the prescribed values. And finally, in section 9 we developed a method for extracting the physically realisable set of solutions for a problem of structural modifications where both spectral and modal constraints are present. A family of solutions to this problem was originally characterised by Ram and Braun [46], but no method of obtaining a physically realisable solution was developed. Our result thus complements and completes the solution given in [46].

The physical realisability of a solution was the main criteria that had to be satisfied in all of the methods developed in this thesis. All of the presented methods aim at being useful in practical engineering applications, rather than just being of purely mathematical interest. The author hopes that the main contribution of this work would be to make available a useful practical set of design tools which may be applied to "real-life" problems. To some extent this contribution was recognised by publication and the feedback from the three refereed papers [75,76,77], which deal respectively with the material of section 4, section 5 and section 8. Two pending papers [79,80], containing the material developed in sections 6 and 9, will also soon be submitted for a journal publication. The practical application of the method developed in section 4 has also been demonstrated by the experimental results, which are given in section 10.

The author also believes that the work developed in this thesis has filled a small void in the knowledge of inverse vibration problems, particularly in applications with conservative mass-spring systems. However, several of the developed methods (e.g. algorithms of sections 6, 7, 8 and 9) do have an "in-build" ability to cope with the mass and stiffness matrices corresponding to vibratory systems other than the mass-spring model. For example, they may be applied to the mass and stiffness matrices corresponding to a finite element model. Some open problems concerned with improving the developed methods were identified (for example: How to control the sign changes in the obtained modal vectors in Problem 2?), but we leave those problems for later investigations.

Last, but not least, it should be emphasised that engineering solutions must not only be *physically realisable* but *practical* as well. This means that additional constrains may need to be taken into account (e.g. the maximal allowed mass, geometrical and spatial restrictions, etc.). Hence, as expected, the design process involves a combination of experience, intuition and science. In this thesis we have focused on the latter only. A great improvement to the developed methods would be to enable direct prescriptions of *practical* solutions. However, we also leave this important task for later study.

Section 12 *REFERENCES*

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APPENDIX A:

Raw Data from Stiffness Measurement Tests

Spring 1					Error	
Mass (Kg)	Force (N)	Dist. (mm)	Dist. (m)	Stiffness	Lower Bound	Upper Bound
0.493	4.836	0.01	1.00E-05	483633	967266	322422
1.003	9.839	0.022	2.20E-05	447247	578790	364423
1.498	14.695	0.038	3.80E-05	386721	445315	341753
1.986	19.483	0.052	5.20E-05	374667	414525	341801
2.49	24.427	0.067	6.70E-05	364581	393982	339263
3	29.430	0.085	8.50E-05	346235	367875	327000
3.482	34.158	0.098	9.80E-05	348555	367295	331635
4.004	39.279	0.117	1.17E-04	335720	350708	321961
4.485	43.998	0.133	1.33E-04	330811	343733	318825
5.001	49.060	0.15	1.50E-04	327065	338344	316515
5.503	53.984	0.163	1.63E-04	331193	341674	321336
6.003	58.889	0.175	1.75E-04	336511	346408	327164
6.504	63.804	0.192	1.92E-04	332314	341199	323879
6.996	68.631	0.207	2.07E-04	331550	339756	323730
7.493	73.506	0.221	2.21E-04	332608	340307	325249
7.991	78.392	0.235	2.35E-04	333582	340834	326632
8.493	83.316	0.25	2.50E-04	333265	340067	326731
8.992	88.212	0.262	2.62E-04	336685	343235	330380
9.488	93.077	0.277	2.77E-04	336019	342196	330061
9.982	97.923	0.291	2.91E-04	336507	342390	330822
10.478	102.789	0.305	3.05E-04	337014	342631	331578
10.979	107.704	0.319	3.19E-04	337630	343006	332420
11.48	112.619	0.331	3.31E-04	340238	345456	335175
11.977	117.494	0.344	3.44E-04	341553	346591	336660
12.472	122.350	0.358	3.58E-04	341761	346601	337053
12.968	127.216	0.372	3.72E-04	341979	346638	337443
13.47	132.141	0.388	3.88E-04	340569	345015	336236
13.972	137.065	0.4	4.00E-04	342663	347001	338433
14.467	141.921	0.412	4.12E-04	344459	348701	340339
14.964	146.797	0.424	4.24E-04	346219	350350	342184

Spring 2					Error	
Mass (Kg)	Force	Dist. (mm)	Dist. (m)	Stiffness	Lower Bound	Upper Bound
0.497	4.876	0.011	1.10E-05	443234	812595	304723
0.999	9.800	0.026	2.60E-05	376930	466676	316135
1.5	14.715	0.04	4.00E-05	367875	420429	327000
1.996	19.581	0.054	5.40E-05	362607	399607	331877
2.499	24.515	0.07	7.00E-05	350217	377157	326869
2.987	29.302	0.082	8.20E-05	357347	380552	336810
3.489	34.227	0.097	9.70E-05	352857	372034	335560
3.99	39.142	0.11	1.10E-04	355835	372780	340364
4.492	44.067	0.124	1.24E-04	355375	370307	341601
4.994	48.991	0.14	1.40E-04	349937	362897	337870
5.487	53.827	0.154	1.54E-04	349529	361258	338538
5.986	58.723	0.168	1.68E-04	349540	360262	339437
6.482	63.588	0.18	1.80E-04	353269	363362	343721
6.984	68.513	0.195	1.95E-04	351349	360595	342565
7.482	73.398	0.21	2.10E-04	349516	358041	341388
7.979	78.274	0.224	2.24E-04	349437	357415	341808
8.471	83.101	0.239	2.39E-04	347701	355130	340576
8.972	88.015	0.252	2.52E-04	349267	356337	342472
9.472	92.920	0.263	2.63E-04	353309	360156	346718
9.973	97.835	0.278	2.78E-04	351925	358370	345707
10.472	102.730	0.293	2.93E-04	350615	356703	344733
10.973	107.645	0.31	3.10E-04	347242	352935	341731
11.476	112.580	0.323	3.23E-04	348544	354024	343230
11.976	117.485	0.339	3.39E-04	346562	351750	341525
12.474	122.370	0.35	3.50E-04	349628	354695	344704
12.972	127.255	0.363	3.63E-04	350566	355462	345803
13.472	132.160	0.38	3.80E-04	347790	352428	343274
13.973	137.075	0.391	3.91E-04	350576	355117	346149
14.471	141.961	0.407	4.07E-04	348797	353136	344564
14.965	146.807	0.42	4.20E-04	349540	353751	345427

Spring 3					Error	
Mass (Kg)	Force	Dist. (mm)	Dist. (m)	Stiffness	Lower Bound	Upper Bound
0.494	4.846	0.01	1.00E-05	484614	969228	323076
0.993	9.741	0.026	2.60E-05	374657	463873	314236
1.489	14.607	0.04	4.00E-05	365177	417345	324602
1.984	19.463	0.054	5.40E-05	360427	397205	329882
2.481	24.339	0.069	6.90E-05	352733	380291	328900
2.983	29.263	0.08	8.00E-05	365790	390176	344273
3.485	34.188	0.094	9.40E-05	363701	384133	345332
3.988	39.122	0.109	1.09E-04	358920	376176	343178
4.489	44.037	0.123	1.23E-04	358025	373196	344040
4.99	48.952	0.137	1.37E-04	357313	370848	344732
5.481	53.769	0.151	1.51E-04	356094	368278	344671
5.98	58.664	0.165	1.65E-04	355538	366649	345081
6.477	63.539	0.178	1.78E-04	356963	367280	347210
6.972	68.395	0.192	1.92E-04	356226	365750	347184
7.469	73.271	0.206	2.06E-04	355684	364532	347255
7.971	78.196	0.221	2.21E-04	353826	362016	345998
8.473	83.120	0.235	2.35E-04	353703	361392	346334
8.974	88.035	0.249	2.49E-04	353554	360799	346594
9.477	92.969	0.261	2.61E-04	356204	363162	349509
9.976	97.865	0.278	2.78E-04	352031	358478	345811
10.478	102.789	0.295	2.95E-04	348438	354445	342631
10.975	107.665	0.308	3.08E-04	349561	355329	343977
11.476	112.580	0.322	3.22E-04	349626	355141	344280
11.975	117.475	0.338	3.38E-04	347558	352777	342492
12.472	122.350	0.351	3.51E-04	348576	353614	343681
12.97	127.236	0.368	3.68E-04	345749	350512	341114
13.472	132.160	0.383	3.83E-04	345066	349630	340619
13.968	137.026	0.4	4.00E-04	342565	346901	338336
14.464	141.892	0.411	4.11E-04	345236	349487	341086
14.965	146.807	0.422	4.22E-04	347883	352054	343809

APPENDIX B:

Raw Data from Chi-squared Tests

Configuration 1

Node	Mass (kg)	Expt Freq (Hz)	k= 378000 Freq (Hz)	ΔF	Chi squ'd	K=350000 Freq (Hz)	ΔF	Chi squ'd	K=340000 Freq (Hz)	ΔF	Chi squ'd
1	6.195	6.800	8.090	1.290	0.245	7.788	0.988	0.143	7.788	0.988	0.143
2	4.867	21.800	24.450	2.650	0.322	23.531	1.731	0.137	23.192	1.392	0.089
3	3.772	35.300	36.570	1.270	0.046	35.185	0.115	0.000	34.679	0.621	0.011
4	4.641	50.600	51.080	0.480	0.005	49.152	1.449	0.041	43.444	7.156	1.012
5	6.503	62.000	62.170	0.170	0.000	59.825	2.175	0.076	58.964	3.036	0.149
6	5.361	73.700	72.790	0.910	0.011	70.044	3.656	0.181	69.036	4.664	0.295
7	3.628	80.500	80.390	0.110	0.000	77.359	3.141	0.123	76.246	4.254	0.225
8	3.291	91.200	89.910	1.290	0.018	86.517	4.683	0.240	85.272	5.928	0.385
9	2.078	103.500	100.630	2.870	0.080	96.834	6.666	0.429	95.441	8.060	0.628
				Sum =	0.727		Sum =	1.373		Sum =	2.937

Configuration 2

Node	Mass (kg)	Expt Freq (Hz)	k= 378000 Freq (Hz)	ΔF	Chi squ'd	K=350000 Freq (Hz)	ΔF	Chi squ'd	K=340000 Freq (Hz)	ΔF	Chi squ'd
1	5.928	7.000	7.958	0.958	0.131	7.672	0.672	0.065	7.562	0.562	0.045
2	4.600	21.458	24.085	2.626	0.321	23.166	1.708	0.136	22.833	1.375	0.088
3	3.505	35.757	36.083	0.326	0.003	34.726	1.031	0.030	34.227	1.530	0.066
4	4.374	50.350	50.380	0.030	0.000	48.478	1.872	0.070	47.781	2.569	0.131
5	6.236	62.167	61.352	0.815	0.011	59.035	3.132	0.158	58.185	3.982	0.255
6	5.094	73.450	71.814	1.636	0.036	69.107	4.343	0.257	68.113	5.337	0.388
7	3.361	81.000	79.211	1.789	0.040	76.214	4.786	0.283	75.117	5.883	0.427
8	3.024	92.563	87.882	4.681	0.237	85.184	7.378	0.588	83.958	8.604	0.800
9	1.944	99.500	98.792	0.708	0.005	95.066	4.435	0.198	93.698	5.802	0.338
				Sum =	0.784		Sum =	1.783		Sum =	2.538

Configuration 3

Node	Mass (kg)	Expt Freq (Hz)	k= 378000 Freq (Hz)	ΔF	Chi squ'd	K=350000 Freq (Hz)	ΔF	Chi squ'd	K=340000 Freq (Hz)	ΔF	Chi squ'd
1	4.735	8.194	11.254	3.060	1.142	10.828	2.633	0.846	10.672	2.477	0.749
2	1.196	30.708	35.553	4.844	0.764	34.225	3.517	0.403	33.733	3.024	0.298
3	0.220	46.286	50.354	4.069	0.358	48.462	2.177	0.102	47.765	1.479	0.047
4	0.805	65.750	71.158	5.408	0.445	68.473	2.723	0.113	67.487	1.737	0.046
5	0.464	92.583	100.671	8.088	0.707	96.866	4.282	0.198	95.472	2.888	0.090
6	0.874	119.922	123.281	3.359	0.094	118.269	1.653	0.023	116.922	3.000	0.075
7	2.265	146.861	142.353	4.509	0.138	136.960	9.901	0.668	134.989	11.872	0.960
8	0.759	180.438	174.331	6.107	0.207	167.750	12.687	0.892	165.336	15.101	1.264
9	3.852	210.792	201.348	9.443	0.423	193.728	17.063	1.381	190.941	19.851	1.869
				Sum =	4.278		Sum =	4.626		Sum =	5.398

Configuration 4

Node	Mass (kg)	Expt Freq (Hz)	k= 378000 Freq (Hz)	ΔF	Chi squ'd	K=350000 Freq (Hz)	ΔF	Chi squ'd	K=340000 Freq (Hz)	ΔF	Chi squ'd
1	3.776	9.569	11.141	1.571	0.258	10.738	1.168	0.143	10.583	1.014	0.107
2	2.659	28.944	32.187	3.243	0.363	30.964	2.020	0.141	30.524	1.579	0.086
3	3.490	45.972	47.560	1.588	0.055	45.745	0.227	0.001	45.087	0.886	0.017
4	3.856	63.056	63.920	0.865	0.012	61.475	1.581	0.040	60.591	2.465	0.096
5	4.332	80.597	81.185	0.587	0.004	77.972	2.626	0.086	76.850	3.748	0.174
6	0.802	87.056	92.049	4.993	0.286	88.561	1.505	0.026	87.286	0.231	0.001
7	1.430	101.406	98.868	2.538	0.064	95.101	6.305	0.392	93.733	7.673	0.581
8	0.351	145.078	141.809	3.269	0.074	136.445	8.634	0.514	134.481	10.597	0.774
9	0.703	186.292	184.983	1.309	0.009	178.026	8.265	0.367	175.465	10.827	0.629
				Sum =	1.125		Sum =	1.708		Sum =	2.466

Configuration 5

Node	Mass (kg)	Expt Freq (Hz)	k= 378000 Freq (Hz)	ΔF	Chi squ'd	K=350000 Freq (Hz)	ΔF	Chi squ'd	K=340000 Freq (Hz)	ΔF	Chi squ'd
1	6.797	9.000	10.473	1.473	0.241	10.079	1.079	0.129	9.934	0.934	0.097
2	3.557	26.000	27.731	1.731	0.115	26.683	0.683	0.018	26.299	0.299	0.003
3	3.684	42.083	43.163	1.080	0.028	41.534	0.549	0.007	40.937	1.147	0.031
4	3.235	56.167	57.025	0.859	0.013	54.872	1.295	0.030	54.082	2.085	0.077
5	2.347	72.771	72.018	0.753	0.008	69.299	3.472	0.166	68.301	4.469	0.275
6	2.584	88.667	86.793	1.874	0.040	83.517	5.150	0.299	82.315	6.352	0.455
7	1.757	98.188	96.339	1.848	0.035	92.701	5.487	0.307	91.367	6.821	0.474
8	1.086	113.479	109.465	4.015	0.142	105.331	8.149	0.585	103.815	9.664	0.823
9	1.204	142.313	139.292	3.021	0.064	134.033	8.280	0.482	132.104	10.208	0.732
				Sum =	0.685		Sum =	2.022		Sum =	2.968

Configuration 6

Node	Mass (kg)	Expt Freq (Hz)	k= 378000 Freq (Hz)	ΔF	Chi squ'd	K=350000 Freq (Hz)	ΔF	Chi squ'd	K=340000 Freq (Hz)	ΔF	Chi squ'd
1	3.498	9.500	10.747	1.247	0.164	10.336	0.836	0.074	10.188	0.688	0.050
2	2.017	31.281	34.541	3.259	0.340	32.382	1.101	0.039	31.916	0.635	0.013
3	1.558	48.031	49.625	1.593	0.053	47.753	0.278	0.002	47.066	0.966	0.019
4	2.019	65.766	67.085	1.320	0.026	64.553	1.213	0.022	63.624	2.142	0.070
5	3.064	82.031	81.788	0.244	0.001	78.700	3.332	0.135	77.567	4.464	0.243
6	3.070	98.000	96.400	1.600	0.026	92.761	5.239	0.280	91.426	6.574	0.441
7	2.285	106.422	103.717	2.705	0.069	99.804	6.618	0.412	98.368	8.054	0.610
8	0.960	129.906	124.885	5.021	0.194	120.171	9.735	0.730	118.442	11.465	1.012
9	0.877	146.156	145.142	1.014	0.007	139.672	6.485	0.288	137.662	8.494	0.494
				Sum =	0.879		Sum =	1.981		Sum =	2.951

Configuration 7

Node	Mass (kg)	Expt Freq (Hz)	k= 378000 Freq (Hz)	ΔF	Chi squ'd	K=350000 Freq (Hz)	ΔF	Chi squ'd	K=340000 Freq (Hz)	ΔF	Chi squ'd
1	4.011	9.500	10.981	1.481	0.231	10.565	1.065	0.119	10.413	0.913	0.088
2	1.332	31.472	33.807	2.335	0.173	32.528	1.056	0.035	32.060	0.588	0.011
3	1.829	48.444	50.168	1.723	0.061	48.272	0.172	0.001	47.578	0.867	0.016
4	1.205	65.969	67.788	1.819	0.050	65.228	0.741	0.008	64.289	1.679	0.043
5	3.019	78.016	77.945	0.070	0.000	75.001	3.015	0.117	73.921	4.094	0.215
6	1.769	109.819	106.789	3.030	0.084	102.752	7.068	0.455	101.273	8.546	0.665
7	2.929	120.431	116.734	3.696	0.113	112.319	8.112	0.546	110.702	9.729	0.786
8	0.974	139.450	134.260	5.190	0.193	129.185	10.265	0.756	127.327	12.124	1.054
9	1.175	144.611	139.376	5.236	0.190	134.111	10.500	0.762	132.181	12.430	1.068
				Sum =	1.095		Sum =	2.799		Sum =	3.945

Configuration 8

Node	Mass (kg)	Expt Freq (Hz)	k= 378000 Freq (Hz)	ΔF	Chi squ'd	K=350000 Freq (Hz)	ΔF	Chi squ'd	K=340000 Freq (Hz)	ΔF	Chi squ'd
1	6.529	8.781	10.078	1.297	0.192	9.692	0.911	0.094	9.553	0.772	0.068
2	1.496	29.250	31.691	2.441	0.204	30.492	1.242	0.053	30.053	0.803	0.022
3	1.371	42.000	43.818	1.818	0.079	42.163	0.163	0.001	41.555	0.445	0.005
4	1.673	61.714	61.441	0.273	0.001	59.118	2.597	0.109	58.267	3.447	0.193
5	3.094	80.688	79.361	1.327	0.022	76.360	4.327	0.232	75.262	5.426	0.365
6	3.827	96.906	94.290	2.616	0.071	90.726	6.180	0.394	89.420	7.486	0.578
7	2.426	108.188	106.756	1.431	0.019	102.724	5.463	0.276	101.243	6.944	0.446
8	2.005	124.781	123.373	1.408	0.016	118.627	6.155	0.304	116.920	7.862	0.495
9	0.886	136.475	133.562	2.913	0.062	128.508	7.967	0.465	126.660	9.815	0.706
				Sum =	0.665		Sum =	1.928		Sum =	2.877

APPENDIX C:

Raw Data from Experimental Tests of Algorithm 4.2

RESULTS FROM TESTING OF ALGORITHM

NOTES:
These results have used the spring constant K = 378000 N/m
The lumped parameter model has been used i.e. one third of the spring mass has been lumped at the nodes
All entries marked with a * either had poor coherence (below 85%) or were poorly defined in the Frequency Responce Function
Entries labled 'not done' were not done because the results were either not necesary or because the results had poor coherence for several entries
All lambda values below have been multiplied by the constant E(-5)

Experiment 1

Node	Desired Λ (rad/s)	Desired f(Hz)	Given by algorithm Λ (rad/s)	Given by algorithm f (Hz)	Given by algorithm m (kg)	Experimental Results (Hz) mass at node (kg)	Striking Location 1	2	3	4	5	6	7	8	9	Value Used 1	2	3	4	5	6	7	8	9	Av. Freq (Hz)	Spread (Hz)
1	0.025	7.96	0.025	7.96	6.330	5.928	7	not	7	7	7	not	7	not	7	7	not	7	7	7	not	7	not	7	7.0	0
2	0.250	25.16	0.229	24.08	5.002	4.600	21.5	done	21.5	21.25	21.5	done	21.5	done	21.5	21.5	done	21.5	21.25	21.5	done	21.5	done	21.5	21.5	0.25
3	0.500	35.59	0.514	36.08	3.907	3.505	35.75		35.75	35.75	35.785		-		35.75	35.75		35.75	35.75	35.785		-		35.75	35.8	0.035
4	1.000	50.33	1.002	50.38	4.776	4.374	50.25-50.5		50.5-50.25	50.5	50.25*		50.25		50.25	50.375		50.375	50.5	-		50.25		50.25	50.4	0.25
5	1.500	61.64	1.486	61.35	6.638	6.236	62.25		62	62.25	62.25		62.25		62	62.25		62	62.25	62.25		62.25		62	62.2	0.25
6	2.000	71.18	2.036	71.81	5.496	5.094	73.5		-	73.5	73.5		73.5		73.25	73.5		-	73.5	73.5		73.5		73.25	73.5	0.25
7	2.500	79.58	2.477	79.21	3.763	3.361	81		81	81	81		81		81	81		81	81	81		81		81	81.0	0
8	3	87.17	3.049	87.88	3.426	3.024	92.5		92.25	92.5	93		-		-	92.5		92.25	92.5	93		-		-	92.6	0.75
9	3.750	97.46	3.853	98.79	2.145	1.844	99.5		99.5	99.5	99.5		99.5		99.5	99.5		99.5	99.5	99.5		99.5		99.5	99.5	0

Experiment 2

Node	Desired Λ (rad/s)	Desired f(Hz)	Given by algorithm Λ (rad/s)	Given by algorithm f (Hz)	Given by algorithm m (kg)	Experimental Results (Hz) mass at node (kg)	Striking Location 1	2	3	4	5	6	7	8	9	Value Used 1	2	3	4	5	6	7	8	9	Av. Freq (Hz)	Spread (Hz)
1	0.050	11.25	0.050	11.25	5.137	4.735	8.25	8.25	8.25	8.25	8.25	8-8.25	8.25	8-8.25	8	8.25	8.25	8.25	8.25	8.25	8.125	8.25	8.125	8	8.2	0.25
2	0.500	35.59	0.499	35.55	1.598	1.196	30.75	30.75	30.75	30.75	30.75	30.5-30.75	30.5	30.75	30.75	30.75	30.75	30.75	30.75	30.75	30.625	30.5	30.75	30.75	30.7	0.25
3	1.000	50.33	1.001	50.35	0.622	0.220	46.5	46.25	-	46.25	46.25	46.25	46.25	-	46.25	46.5	46.25	-	46.25	46.25	46.25	46.25	-	46.25	46.3	0.25
4	2.000	71.18	1.999	71.16	1.207	0.805	65.75	65.75	65.75	65.75	65.75	65.75	65.75	65.5	66	65.75	65.75	65.75	65.75	65.75	65.75	65.75	65.5	66	65.8	0.5
5	4.000	100.66	4.001	100.67	0.866	0.464	92.5	92-92.25	92.75	92.25-92.5	92.75	92	93	92.25	93.5	92.5	92.125	92.75	92.375	92.75	92	93	92.25	93.5	92.6	1.5
6	6.000	123.28	6.000	123.28	1.276	0.874	119.75	119.75-120	120	119.75	120.25	119.75	120	119.5	120.5	119.75	119.875	-	119.75	120.25	119.75	120	119.5	120.5	119.9	1
7	8.000	142.35	8.000	142.35	2.667	2.265	146.75	147	147	147	147	146.75	146.75	146.5	147	146.75	147	147	147	147	146.75	146.75	146.5	147	146.9	0.5
8	12.000	174.35	11.998	174.33	1.162	0.759	-	174.25*	173.75-174*	178.5-179.25	179.25-179.75*	179-179.75*	180.25-180.75*	182.25	-	-	-	-	178.625	-	-	-	182.25	-	180.4	3.625
9	16.000	201.32	16.005	201.35	4.053	3.652	-	210.75	210.5-210.75	211	-	-	-	200*	200*	-	210.75	210.625	211	-	-	-	-	-	210.8	0.375

Experiment 3

Node	Desired Λ (rad/s)	Desired f(Hz)	Given by algorithm Λ (rad/s)	Given by algorithm f (Hz)	Given by algorithm m (kg)	Experimental Results (Hz) mass at node (kg)	Striking Location 1	2	3	4	5	6	7	8	9	Value Used 1	2	3	4	5	6	7	8	9	Av. Freq (Hz)	Spread (Hz)
1	0.050	11.25	0.049	11.14	4.178	3.776	9.75	9.5	9.5-9.75	9.5	9.5	9.5	9.5	9.5-9.75	9.5-9.75	9.75	9.5	9.625	9.5	9.5	9.5	9.5	9.625	9.625	9.6	0.25
2	0.500	35.59	0.409	32.19	3.061	2.659	29	29	29	29	28.5	29	29	29	29	29	29	29	29	28.5	29	29	29	29	28.9	0.5
3	1.000	50.33	0.893	47.56	3.892	3.490	46	46.25	45.75	46	46	45.75	46	46	46	46	46.25	45.75	46	46	45.75	46	46	46	46.0	0.5
4	1.500	61.64	1.613	63.92	4.258	3.856	63	63	63	63	63.25	63	62.5	63.5	63.25	63	63	63	63	63.25	63	62.5	63.5	63.25	63.1	1
5	2.500	79.58	2.602	81.18	4.734	4.332	80.5	80.5	80.5	81	80-80.25	80.75	80.5	81	80.5	80.5	80.5	80.5	81	80.125	80.75	80.5	81	80.5	80.6	0.875
6	3.000	87.17	3.345	92.05	1.204	0.802	87	87-87.25	86.75	87	87.25	87.25	87-87.25	86.75	87.25	87	87.125	86.75	87	87.25	87.25	87.125	86.75	87.25	87.1	0.5
7	3.500	94.16	3.859	98.87	1.832	1.430	101.5	101-101.25	101.25	101.75	-	-	-	-	-	101.5	101.125	101.25	101.75	-	-	-	-	-	101.4	0.625
8	8.000	142.35	7.939	141.81	0.753	0.351	145.25	-	145.25	144.5-144.75	144.75	145-145.5	145.5	144.75	145.25	145.25	-	145.25	144.625	144.75	145.25	145.5	144.75	145.25	145.1	0.875
9	13.000	181.46	13.509	184.98	0.904	0.703	186.5-186.75	-	-	186.75	-	185.75-186	185.75	186.25	186.5	186.625	-	-	186.75	-	185.875	185.75	186.25	186.5	186.3	1

Experiment 4

Node	Desired Λ (rad/s)	Desired f(Hz)	Given by algorithm Λ (rad/s)	Given by algorithm f (Hz)	Given by algorithm m (kg)	Experimental Results (Hz) mass at node (kg)	Striking Location 1	2	3	4	5	6	7	8	9	Value Used 1	2	3	4	5	6	7	8	9	Av. Freq (Hz)	Spread (Hz)
1	0.030	8.72	0.030	8.72	3.797	3.395	7.75	7.75	7.75	7.75	7.75	7.75	7.75	7.75	7.75	7.75	7.75	7.75	7.75	7.75	7.75	7.75	7.75	7.75	7.8	0
2	0.300	27.57	0.300	27.57	3.728	3.326	26.5	26.5	26.5	26.5	26.5	26.5	26.5	26.5	26.5	26.5	26.5	26.5	26.5	26.5	26.5	26.5	26.5	26.5	26.5	0
3	0.750	43.59	0.750	43.59	2.537	2.135	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45.0	0
4	1.500	61.64	1.500	61.64	1.866	1.464	61.75	61.75	61.75	61.75	61.75	62	61.75-62	61.75	62	61.75	61.75	61.75	61.75	61.75	62	61.875	61.75	62	61.8	0.25
5	2.250	75.49	2.250	75.49	0.911	0.509	78.75	79	78.75	79	78.75	78.75	79	78.75	79	78.75	79	78.75	79	78.75	78.75	79	78.75	79	78.9	0.25
6	3.000	87.17	3.000	87.17	2.620	2.218	88.25	88.25	-	88.25	87.5	88.5	88-88.25	88.25	88.5	88.25	88.25	-	88.25	87.5	88.5	88.125	88.25	88.5	88.2	1
7	3.750	97.48	3.750	97.48	3.436	3.034	101.5	101.5	101.5	101.25	101.5	101.5	101.5	101.5	101.75	101.5	101.5	101.5	101.25	101.5	101.5	101.5	101.5	101.75	101.5	0.5
8	5.000	112.54	5.000	112.54	3.857	3.455	114.75	114.75	114.75	114.75	114.75	115	115	115.25	-	114.75	114.75	114.75	114.75	114.75	115	115	115.25	-	114.9	0.5
9	10.500	163.09	10.500	163.09	3.523	3.322	164.5	-	164.75	164.75	164.5	164.75	-	164.5	164.25	164.5	-	164.75	164.75	164.5	164.75	-	164.5	164.25	164.6	0.5

Experiment 5

Node	Desired Λ (rad/s)	Desired f(Hz)	Given by algorithm Λ (rad/s)	Given by algorithm f (Hz)	Given by algorithm m (kg)	Experimental Results (Hz) mass at node (kg)	Striking Location 1	2	3	4	5	6	7	8	9	Value Used 1	2	3	4	5	6	7	8	9	Av. Freq (Hz)	Spread (Hz)
1	0.050	11.25	0.043	10.47	7.199	6.797	not	9	9	not	9	not	9	9	9	not	9	9	not	9	not	9	9	9	9.0	0
2	0.300	27.57	0.304	27.73	3.959	3.557	done	26	26	done	26	done	26	26	26	done	26	26	done	26	done	26	26	26	26.0	0
3	0.750	43.59	0.736	43.16	4.086	3.684		42	42.25		42		42.25	42	42		42	42.25		42		42.25	42	42	42.1	0.25
4	1.250	56.27	1.284	57.03	3.637	3.235		56.25	56.25		56.25		56	56.25	56		56.25	56.25		56.25		56	56.25	56	56.2	0.25
5	2.000	71.18	2.048	72.02	2.749	2.347		72.75	72.75		72.75		72.75	73	2.5-72.75		72.75	72.75		72.75		72.75	73	72.625	72.8	0.375
6	3.000	87.17	2.974	88.79	2.986	2.584		88.75	88.75		88.75		88.75	88.5	88.5		88.75	88.75		88.75		88.75	88.5	88.5	88.7	0.25
7	3.500	94.16	3.664	98.34	2.160	1.757		98.25	98.25		98.25		98.25	98	98-98.25		98.25	98.25		98.25		98.25	98	98.125	98.2	0.25
8	4.500	106.76	4.731	109.46	1.488	1.086		113.5	113.5		113.5		113.5	113.25	.25-113.5		113.5	113.5		113.5		113.5	113.25	113.625	113.5	0.375
9	7.500	137.83	7.660	139.29	1.405	1.204		-	-		142.5		142.25	142.25	142.25		-	-		142.5		142.25	142.25	142.25	142.3	0.25

RESULTS FROM TESTING OF ALGORITHM

NOTES:
These results have used the spring constant K = 378000 N/m
The lumped parameter model has been used. i.e. one third of the spring mass has been lumped at the nodes
All entries marked with a * either had poor coherence (below 85%) or were poorly defined in the Frequency Responce Function
Entries labled 'not done' were not done because the results were either not necesary or because the results had poor coherence for several entries
All lambda values below have been multiplied by the constant E(-5)

Experiment 6

Node	Desired Λ (rad/s)	Desired f(Hz)	Given by algorithm Λ (rad/s)	Given by algorithm f (Hz)	Given by algorithm m (kg)	mass at node (kg)	Experimental Results (Hz) Striking Location 1	2	3	4	5	6	7	8	9	Value Used 1	2	3	4	5	6	7	8	9	Av. Freq (Hz)	Spread (Hz)
1	0.050	11.25	0.046	10.75	3.900	3.498	not done	9.5	9.5	9.5	9.5	9.5	9.5	9.5	9.5	not done	9.5	9.5	9.5	9.5	9.5	9.5	9.5	9.5	9.5	0
2	0.500	35.59	0.471	34.54	2.419	2.017		31.25	31.25	31.25	31.25	31.5	31.25	31.25	31.25		31.25	31.25	31.25	31.25	31.5	31.25	31.25	31.25	31.3	0.25
3	1.000	50.33	0.972	49.62	1.960	1.558		48	48	48	48	48	48.25	48	48		48	48	48	48	48	48.25	48	48	48.0	0.25
4	1.750	66.58	1.777	67.09	2.421	2.019		65.75	65.75	65.75	65.75	65.75	65.75	65.75-66	65.75		65.75	65.75	65.75	65.75	65.75	65.75	65.875	65.75	65.8	0.125
5	2.500	79.58	2.641	81.79	3.466	3.064		82	81.75	81.5-81.75	82	82	81.75-82	82.25	81.75		82	81.75	81.625	82	82	82.875	82.25	81.75	82.0	1.25
6	3.500	94.16	3.669	96.40	3.472	3.070		98	98	98	98.25	98	98	97.75	98		98	98	98	98.25	98	98	97.75	98	98.0	0.5
7	4.000	100.66	4.247	103.72	2.687	2.285		106.25	106.5	106.25	106.25-106.5	106.5	106.75	106.25	106.5		106.25	106.5	106.25	106.375	106.5	106.75	106.25	106.5	106.4	0.5
8	6.000	123.28	6.157	124.89	1.362	0.960		129.75	129.75	129.75-130	130.25	130.25*	-	-	-		129.75	129.75	129.875	130.25	-	-	-	-	129.9	0.5
9	8.000	142.35	8.317	145.14	1.076	0.877		146.25	146.25	146.25	146.25	146.25	146-146.25	145.75-146	146		146.25	146.25	146.25	146.25	146.25	146.125	145.875	146	146.2	0.375

Experiment 7

Node	Desired Λ (rad/s)	Desired f(Hz)	Given by algorithm Λ (rad/s)	Given by algorithm f (Hz)	Given by algorithm m (kg)	mass at node (kg)	Experimental Results (Hz) Striking Location 1	2	3	4	5	6	7	8	9	Value Used 1	2	3	4	5	6	7	8	9	Av. Freq (Hz)	Spread (Hz)
1	0.050	11.25	0.048	10.98	4.413	4.011	9.5	9.5	9.5	9.5	9.5	9.5	9.5	9.5	9.5	9.5	9.5	9.5	9.5	9.5	9.5	9.5	9.5	9.5	9.5	0
2	0.500	35.59	0.451	33.81	1.734	1.332	31.5	31.5	31.5	31.5	31.5	31.25	31.5	31.5	31.5	31.5	31.5	31.5	31.5	31.5	31.25	31.5	31.5	31.5	31.5	0.25
3	1.000	50.33	0.994	50.17	2.231	1.829	48.5	48.5	48.5	48.5	48.5	48.5	48	48.5	48.5	48.5	48.5	48.5	48.5	48.5	48.5	48	48.5	48.5	48.4	0.5
4	1.750	66.58	1.814	67.79	1.607	1.205	66	-	66	66	66	66	66	66	65.75	66	-	66	66	66	66	66	66	65.75	66.0	0.25
5	2.250	75.49	2.399	77.95	3.421	3.019	78	78	78	78	78.25	77.75	78-78.25	-	78	78	78	78	78	78.25	77.75	78.125	-	78	78.0	0.5
6	4.500	106.76	4.502	106.79	2.171	1.769	110	109.75	109.75	109.75	109.75-110	109.75	110	109.5	110	110	109.75	109.75	109.75	109.875	109.75	110	109.5	110	109.8	0.5
7	5.250	115.32	5.380	116.73	3.331	2.929	120.25-120.5	120.25	120.5	120-120.25	120.5	120.5-120.75	121	120	120.5	120.375	120.25	120.5	120.125	120.5	120.625	121	120	120.5	120.4	1
8	7.000	133.16	7.116	134.26	1.376	0.974	139.5	139.25	139.5	139.25	139.75	-	-	-	-	139.5	139.25	139.5	139.25	139.75	-	-	-	-	139.5	0.5
9	7.500	137.83	7.669	139.38	1.376	1.175	145	144.75	145	144.5	144.75	144.5	144.5	144.25	144.25	145	144.75	145	144.5	144.75	144.5	144.5	144.25	144.25	144.6	0.75

Experiment 8

Node	Desired Λ (rad/s)	Desired f(Hz)	Given by algorithm Λ (rad/s)	Given by algorithm f (Hz)	Given by algorithm m (kg)	mass at node (kg)	Experimental Results (Hz) Striking Location 1	2	3	4	5	6	7	8	9	Value Used 1	2	3	4	5	6	7	8	9	Av. Freq (Hz)	Spread (Hz)
1	0.040	10.07	0.040	10.08	6.931	6.529	not done	8.75	8.75	9	8.75	8.75	8.75	8.75	8.75	not done	8.75	8.75	9	8.75	8.75	8.75	8.75	8.75	8.8	0.25
2	0.400	31.83	0.397	31.69	1.898	1.496		29.25	29.25	29.25	29.25	29*	29.25	29.25	29.25		29.25	29.25	29.25	29.25	-	29.25	29.25	29.25	29.3	0
3	0.750	43.59	0.758	43.82	1.774	1.371		42	41.75*	42	42	42	42.25*	42	42		42	-	42	42	42	-	42	42	42.0	0
4	1.500	61.64	1.490	61.44	2.075	1.673		61.75	61.75	61.75	61.25*	61.75	61.75	61.75	61.5		61.75	61.75	61.75	-	61.75	61.75	61.75	61.5	61.7	0.25
5	2.500	79.58	2.486	79.38	3.496	3.094		80.75	80.5	80.75	80.75	80.75	80.75	80.75	80.5		80.75	80.5	80.75	80.75	80.75	80.75	80.75	80.5	80.7	0.25
6	3.500	94.16	3.510	94.29	4.230	3.827		97	97	97	96.75	97	97	96.75	96.75		97	97	97	96.75	97	97	96.75	96.75	96.9	0.25
7	4.500	106.76	4.499	106.76	2.828	2.426		108.25	108.25-108.5	107.75-108	108	108.25	108.5	107.75	108.5		108.25	108.375	107.875	108	108.25	108.5	107.75	108.5	108.2	0.75
8	6.000	123.28	6.009	123.37	2.407	2.005		124.25	125	125	125	124.75	124.75	124.75	124.75		124.25	125	125	125	124.75	124.75	124.75	124.75	124.8	0.75
9	7.000	133.16	7.043	133.56	1.087	0.888		136.5	136.5	136.25	136.25	-	136.5*	136.75-137*	136.75*		136.5	136.5	136.25	136.25	-	-	136.875	-	136.5	0.625

Experiment 9

Node	Desired Λ (rad/s)	Desired f(Hz)	Given by algorithm Λ (rad/s)	Given by algorithm f (Hz)	Given by algorithm m (kg)	mass at node (kg)	Experimental Results (Hz) Striking Location 1	2	3	4	5	6	7	8	9	Value Used 1	2	3	4	5	6	7	8	9	Av. Freq (Hz)	Spread (Hz)
1	0.030	8.72	0.031	8.82	2.202	1.800		7.5	7.5	7.5	7.5	7.5	7.5	7.5	7.5	-	7.5	7.5	7.5	7.5	7.5	7.5	7.5	7.5	7.5	0
2	0.300	27.57	0.299	27.51	2.780	2.378		24.75	24.75	24.75	24.75	24.75	25	25	24.75	-	24.75	24.75	24.75	24.75	24.75	25	25	24.75	24.8	0.25
3	0.750	43.59	0.742	43.35	3.469	3.067		42.5-42.75	42.75	42.5	42.75	42.75	42.5-42.75	42.75	42.5	-	42.625	42.75	42.5	42.75	42.75	42.625	42.75	42.5	42.7	0.25
4	1.500	61.64	1.493	61.49	2.530	2.128		61.25	61.5	61.25	61.25	61.5	61.25	61.25	61.25	-	61.25	61.5	61.25	61.25	61.5	61.25	61.25	61.25	61.3	0.25
5	2.250	75.49	2.235	75.24	3.137	2.735		75.25	75.25	75.25	75.25	75.25	75.25-75.5	75.25	75.25-75.5	-	75.25	75.25	75.25	75.25	75.25	75.375	75.25	75.375	75.3	0.125
6	3.000	87.17	3.007	87.27	4.050	3.648		89	88.75-89	89	89	89	89-89.25	88.75	89	-	89	88.875	89	89	89	89.125	88.75	89	89.0	0.375
7	4.000	100.66	3.978	100.39	3.870	3.468		104.25	104.25	104.5	104.5	104.25	104.25	104.25	104.5	-	104.25	104.25	104.5	104.5	104.25	104.25	104.25	104.5	104.3	0.25
8	4.500	106.76	4.478	106.50	2.833	2.431		108.5	-	108.5	108.75	108.75	109	109	109	-	108.5	-	108.5	108.75	108.75	109	109	109	108.8	0.5
9	5.000	112.54	5.610	119.21	3.423	3.222	115.5	115.75	115.75	-	-	-	-	-	-	115.5	115.75	115.75	-	-	-	-	-	-	115.7	0.25

Experiment 10

Node	Desired Λ (rad/s)	Desired f(Hz)	Given by algorithm Λ (rad/s)	Given by algorithm f (Hz)	Given by algorithm m (kg)	mass at node (kg)	Experimental Results (Hz) Striking Location 1	2	3	4	5	6	7	8	9	Value Used 1	2	3	4	5	6	7	8	9	Av. Freq (Hz)	Spread (Hz)
1	0.040	10.07	0.038	9.79	1.000	0.598	8	8	8	8	8	8	8	7.75-8	7.75	8	8	8	8	8	8	8	7.875	7.75	8.0	0.25
2	0.300	27.57	0.298	27.39	5.788	5.386	24.75	24.75	24.75	24.75	24.75	24.75	24.75	24.75	24.75	24.75	24.75	24.75	24.75	24.75	24.75	24.75	24.75	24.75	24.8	0
3	0.750	43.59	0.760	43.87	1.007	0.605	42.5	42.5	42.5	42.5	42.5	42.5	42.25	42.5	42.25-42.5	42.5	42.5	42.5	42.5	42.5	42.5	42.25	42.5	42.375	42.5	0.25
4	1.500	61.64	1.533	62.32	4.040	3.638	61.25	61.25	61.25	61.25	61	61.25	61.25	61	61.25	61.25	61.25	61.25	61.25	61	61.25	61.25	61	61.25	61.2	0.25
5	2.500	79.58	2.533	80.10	3.532	3.130	81.25	81	81.25	81	81	81	81.25	81	81.25	81.25	81	81.25	81	81	81	81.25	81	81.25	81.1	0.25
6	4.000	100.66	4.083	101.70	1.014	0.612	106.5	106.25	106.5	106.25	106.25	106.25	106.25	106	106	106.5	106.25	106.5	106.25	106.25	106.25	106.25	106	106	106.3	0.5
7	7.750	140.11	7.746	140.07	3.390	2.988	135.25	135.75	135.75	135.75	-	-	-	-	136-136.25	135.25	135.75	135.75	135.75	-	-	-	-	136.125	135.7	0.875
8	8.500	146.73	8.517	146.88	2.826	2.424	150*	149.75	149.5	149.75	-	149.5	149.5	-	-	-	149.75	149.5	149.75	-	149.5	149.5	-	-	149.6	0.25
9	8.750	148.88	8.760	148.96	2.424	2.223	152.25	152.5	152.5-152.75	152.75	153	152.75	153	-	-	152.25	152.5	152.625	152.75	153	152.75	153	-	-	152.7	0.75