GROUPS ADMITTING A FIXED-POINT-FREE GROUP OF AUTOMORPHISMS ISOMORPHIC TO S_3

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A thesis submitted to the University of Adelaide in October, 1983 for the degree of Doctor of Philosophy. The research was undertaken in the Department of Pure Mathematics at the University of Adelaide.

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SUMMARY

The aim of the research undertaken was to investigate groups of order coprime to six which admit a fixed-point-free group of automorphisms A isomorphic to S_3, and in particular to find a direct proof of the solubility of such groups without using the Feit-Thompson theorem or other 'heavy machinery'. This is a specific case of the general conjecture that a group which admits a coprime fixed-point-free group of automorphisms must be soluble.

The first chapter consists of an account of the necessary preliminary results together with some other results and examples which shed some light on the properties of groups admitting a group of automorphisms isomorphic to S_3.

In chapter two we present results (obtained mainly by Martineau and Glauberman) on the structure of maximal V-invariant $\{p,q\}$ - subgroups of a minimal counter-example to a more general conjecture than the one stated above. These results are given in the most general possible setting in order to be applicable to a wide range of hypotheses.

In chapter three we prove that a minimal counter - example to our theorem has at most three maximal A-invariant $\{p,q\}$ - subgroups. This has proved to be a useful mid-point in the deduction of solubility in

other special cases of the conjecture, but does not appear to be particularly useful in this instance.

Accordingly, a different approach was adopted, and chapter four consists of preliminary results about the maximal A-invariant subgroups of a minimal counter-example to the theorem. In the last two chapters this line of approach is developed and in a sequence of arguments the structure of these maximal A-invariant subgroups is investigated, culminating in the proof of the theorem.

STATEMENT

(a) This thesis contains no material which has been accepted for the award of any other degree or diploma in any University, and, to the best of my knowledge and belief, contains no material previously published or written by another person, except when due reference is made in the text of the thesis.

(b) I consent to this thesis being made available for photocopying and loan when it is deposited in the University Library.

B.E. Dolman

ACKNOWLEDGEMENTS

In any undertaking of this magnitude, there will be many people who have contributed in some way, either directly or indirectly, towards the ultimate realisation of the original goal. Practical limitations dictate that everyone can not be mentioned here but the contribution of those omitted is gratefully acknowledged.

At the outset, I wish to thank my parents for giving me the opportunity to pursue my education to this level and for the support and encouragement which they provided throughout.

I would also like to record my appreciation of the efforts of my supervisor, Dr. R.J. Clarke, who suggested the problem and made many useful comments on the draft versions of the thesis.

Dr. D.L. Parrott suggested the second approach to the problem, and during that stage of the work he was a constant source of inspiration and guidance, only too willing to discuss the research and provide suggestions. I shall be eternally grateful to him.

Finally, I can not give enough credit to my wife, Kaye, for the enormous contribution she has made towards this thesis. She had to make many sacrifices to enable me to pursue this goal but was still able to provide moral support and encouragement whenever it was required, and without her devotion the project would almost certainly not have been completed.

IN MEMORY OF SHANNON

INTRODUCTION

A group of automorphisms A of a group G is said to be fixed-point-free (f.p.f.) if it leaves only the identity element of G fixed. We define $C_G(A) = \{x \in G | x^a = x \ \forall a \in A\}$. Then A is f.p.f. on G iff $C_G(A) = 1$. The result that a finite group admitting a f.p.f. automorphism of order 2 is abelian was proved by Burnside late in the nineteenth century, and in 1901 Frobenius proved that a group admitting a f.p.f. automorphism of order 3 is nilpotent of class at most two (see [2]). This prompted Frobenius to pose the following conjecture:

If G is a finite group admitting a f.p.f. automorphism of order p (p a prime) then

(i) G is soluble

(ii) G is nilpotent.

The proof of (ii) assuming (i) has been attributed to Witt in about 1936, and in any case the result appears to have been known before Higman published a proof in 1957. The proof of the conjecture was then completed in 1959 by Thompson ([20]) when he proved (i). Since that time, the conjecture has been extended in various ways, and its most common form now is:

If G is a finite group admitting a f.p.f. group of automorphisms A with A cyclic or $(|G|, |A|) = 1$ then

(1) G is soluble.

(2) The nilpotent height of G is bounded above by the number of primes dividing $|A|$ (counting multiplicity).

In recent years most of the work on groups admitting such automorphism groups has been centred on (2), mainly because (1) appears to be a much deeper problem, but also because by the result of Feit and Thompson in [4], (1) follows if $|A|$ is even. There have, however, been a few successes in proving (1) for particular kinds of automorphism groups.

The first result in this direction was the proof of (1) when A is cyclic of order 4 by Gorenstein and Herstein ([10]) in 1961. In the mid-60's Bauman ([1]) proved that if A is a klein-four group and G is soluble then G' is nilpotent, and using this result Glauberman proved (1) when A is the klein-four group. An account of this work may be found in [9], p. 351-356.

In 1968 Scimemi ([18]) generalized the result of Thompson in [20] to the case where A is cyclic of composite order, though he required additional assumptions about the fixed-points. Specifically, he proved:

Let G be a finite group admitting a f.p.f. group of automorphisms $A = \langle\sigma\rangle$ of order n where n is a product of distinct primes. If the fixed-points of the non-trivial powers of σ are all in the same nilpotent Hall subgroup of G, then G is nilpotent.

In a sequence of papers from 1971-73, Martineau proved (1) (and in some cases (2) also) for the following cases (we include all results here, even though some represent improvements of others, as this gives a better indication of the hard-won progress on the proof of the conjecture):

(a) A is elementary abelian of order r^2 and $C_G(\alpha)$ is abelian for all $\alpha \in A^*$ ([12]).

(b) A is elementary abelian of order r^2 ([13]).

(c) A is elementary abelian of order r^3 ([14]).

(d) A is elementary abelian ([14]).

During this period Ralston ([17]) succeeded in proving (1) when A is cyclic of order rs, where r and s are distinct primes, and in 1973 Martineau ([15]) was able to resolve (1) when A is a soluble group whose centre contains an elementary abelian subgroup of order r^3.

In 1975, Carr ([3]) proved the result in the case when $A = \langle \phi \rangle$ is cyclic of order r^2 where r is an odd prime, under the additional assumption that either $|C_G(\phi^r)|$ is odd or G has abelian Sylow 2-subgroups. The most recent result was obtained in 1976 by Pettet ([16]). He proved (1) for the case that A is a direct product of two elementary abelian groups and $|A|$ is not divisible by a Fermat prime.

Summarizing, then, we know that if A is a f.p.f. group of automorphisms of G, with $(|G|,|A|) = 1$ if A is non-cyclic, then G is necessarily soluble under the following conditions:

I A is cyclic of order p, rs $(r \neq s)$, 4, r^2 (r odd and either $|C_G(\phi^r)|$ is odd or G has abelian Sylow 2-subgroups) or n (where n is a product of distinct primes and the fixed-points of the non-trivial powers of σ are all in the same nilpotent Hall subgroup of G).

II A is elementary abelian.

III A is a product of two elementary abelian groups and $|A|$ is not divisible by a Fermat prime.

IV A is soluble and $Z(A)$ contains an elementary abelian group of order r^3.

It should be noted that we have restricted attention above to those results which are special cases of the general conjecture, and then only to those that can be obtained in a direct manner i.e. without employing 'heavy machinery' such as the Feit-Thompson theorem. Thus we have omitted to mention the work of several authors who have worked on hypotheses not requiring A to act f.p.f. on G which also imply solubility. Also Pettet has proved several cases of the general conjecture using 'high powered' methods, Rowley has similarly removed the restriction of non-divisibility by Fermat primes in [16] and Rickman has generalized some of the results above.

Many of the proofs of the recent results listed above have a common theme. Using the facts that $C_G(A) = 1$ and either A is cyclic or $(|G|,|A|) = 1$ it can be deduced (see [9], theorems 10.1.2 and 6.2.2) that for all primes p dividing $|G|$, A leaves invariant a unique Sylow p-subgroup of G. These A-invariant Sylow subgroups are then shown to be pairwise permutable and hence to form a Sylow system. P. Hall's characterization of soluble groups ([11]) is then used to deduce that G is soluble.

The most common technique used to deduce the permutability of the A-invariant Sylow subgroups was developed by Martineau, and involves a detailed investigation of maximal A-invariant $\{p,q\}$-subgroups of a minimal counter-example to the conjecture. Martineau and Glauberman ([15],[8]) have shown that one can say a good deal about the structure of these subgroups for a minimal counter-example to the general conjecture, and using these results for specific cases of the conjecture it is shown that there are 'not many' maximal A-invariant $\{p,q\}$-subgroups of G. This result is then used to show that the A-invariant Sylow p- and q-subgroups P and Q of G must in fact permute. It is apparent, however, that even under very strong assumptions about the structure of A, it is often not possible to prove that P and Q permute by a purely 'local' argument. Indeed, having generalized the preliminary reduction used by Martineau to find a small bound for the number of maximal A-invariant $\{p,q\}$-subgroups in the special case where A is a product of two elementary abelian groups, Pettet ([16]) was forced to resort to a global argument to prove the solubility of G.

All automorphism groups considered thus far, with the exception of the paper of Martineau [15], have been abelian. Thus it is natural to ask whether any such results may be obtained when A is non-abelian, and since Shult ([19]) has proved (2) of the conjecture when $A \cong S_3$ and $(|G|,|A|) = 1$ our attention is drawn to (1) for this case. Of course, the problem has been solved by the use of high powered techniques from the theory of simple groups (see, for example, [7], corollary 7.3) even in the case when $(|G|,3) = 1$, but we are interested in finding a direct proof in the hope of shedding some light on the possible proof of the general conjecture.

The first approach to this problem was to follow the techniques developed by Martineau, and although some key results which hold in several of the special cases listed above do not hold for S_3, it was possible by this method to find a small bound for the number of maximal A-invariant $\{p,q\}$ - subgroups of a counter-example G of minimal order.

However, it then appeared to be very difficult to deduce the solubility of G from this result without making additional rather restrictive assumptions about G, and these restrictions were necessary mainly because it seemed not to be possible to otherwise guarantee the existence of any A-invariant $\{p,q\}$ - groups.

This difficulty led fairly naturally to an entirely different approach to the problem which entailed consideration of the structure of maximal A-invariant subgroups of a minimal counter-example to the theorem, and using the results of Glauberman in [5] it was possible using this method to deduce the solubility of G.

The notation for the most part is standard, taken from [9]. In addition, all groups are assumed to be finite and wherever $A \cong S_3$, we take $A = \langle \pi, \tau \mid \pi^3 = \tau^2 = 1,\ \tau^{-1}\pi\tau = \pi^2 \rangle$. For automorphisms σ_1, σ_2 of G we will denote the result of applying σ_1 followed by σ_2 to an element x of G by either $x^{\sigma_1\sigma_2}$ or $\sigma_2\sigma_1(x)$, whichever is the more convenient. Finally, for a prime p dividing $|G|$, $O^p(G)$ is defined to be the maximal p-factor group of G.

CHAPTER ONE

PRELIMINARY RESULTS

This chapter consists of a detailed account of the more basic properties of a finite group admitting a group of automorphisms $A \cong S_3$. We begin by proving two easy lemmas which will be crucial to our later work.

1.1 LEMMA Let G be a finite group admitting a group of automorphisms $A \cong S_3$. Then we have the following:

(1) $C_G(\pi)$ is an A-invariant subgroup of G.

(2) If A acts f.p.f. on G, then $C_G(\pi)$ is abelian, has odd order and is inverted by τ.

PROOF

(1) Clearly $C_G(\pi)$ is π-invariant.
If $x \in C_G(\pi)$, $\pi(\tau(x)) = \tau\pi^2(x) = \tau(x)$, so $\tau(x) \in C_G(\pi)$.
Thus $C_G(\pi)$ is τ-invariant, and hence A-invariant.

(2) If A acts f.p.f. on G, then τ must act f.p.f. on $C_G(\pi)$.
Now by [9], theorem 10.1.4, $C_G(\pi)$ is abelian and inverted by τ.
By [9], theorem 6.2.3, $|C_G(\pi)|$ is coprime to $|\langle\tau\rangle| = 2$. □

1.2 LEMMA Suppose that G is a finite group admitting a group of automorphisms $A \cong S_3$. Then

(1) If G is cyclic, π centralizes G.

(2) If $C_G(\tau) = 1$, π centralizes G.

(3) If G is a p-group, either $C_G(\pi) \neq 1$ or $C_G(\tau) \neq 1$.

(4) If $C_G(\pi) = 1$, G is nilpotent.

(5) If $C_G(\pi) = 1$ then $\langle x, x^\pi \rangle$ is an A-invariant abelian subgroup of G for all x in G satisfying either $x^\tau = x^{-1}$ or $x^\tau = x$.

PROOF

(1) If G is cyclic, $\mathrm{Aut}(G)$ is abelian. Since $\mathrm{Aut}(G)$ contains a homomorphic image of A, the result follows.

(2) By [9], theorem 6.2.3, G has odd order.
Then by [9], theorem 10.4.1 we have $G = C_G(\tau).I$ where $I = \{x \in G | x^\tau = x^{-1}\}$.
So τ inverts every element of G.
Now $\forall x \in G$, $\pi^2\tau(x) = \pi^2(x^{-1})$
and $\tau\pi(x) = \pi(x)^{-1} = \pi(x^{-1})$.
Thus $\pi^2(x^{-1}) = \pi(x^{-1})$, so that $\pi(x) = x$.
The result follows.

(3) If $C_G(\pi) = C_G(\tau) = 1$, A is a regular group of automorphisms of G, contradicting [9], theorem 5.3.14(iii).

(4) This is the result of Frobenius mentioned in the introduction.

(5) If $C_G(\pi) = 1$, then $xx^{\pi}x^{\pi^2} = x^{\pi^2}x^{\pi}x = 1$ $\forall x \in G$ by [9], theorem 10.1.1(ii).

Thus $xx^{\pi} = x^{-\pi^2} = x^{\pi}x$, so $R = \langle x,x^{\pi}\rangle$ is abelian.

As $x^{\pi^2} = x^{-1}x^{-\pi}$, clearly R is π-invariant.

Now suppose $x \in C_G(\tau)$.

Then $x^{\tau} = x$ and $x^{\pi\tau} = x^{\tau\pi^2} = x^{\pi^2} = x^{-1}x^{-\pi}$, so R is τ-invariant.

Similarly if $x^{\tau} = x^{-1}$, $x^{\pi\tau} = x^{\tau\pi^2} = (x^{-1})^{\pi^2} = xx^{\pi}$, so again R is τ-invariant. Hence in both cases R is A-invariant. □

We can now prove a structure theorem for abelian p-groups which admit S_3 f.p.f. This lemma will also be used extensively later.

<u>1.3 LEMMA</u> Suppose that G is a finite abelian p-group, p a prime different from 3, admitting a f.p.f. group of automorphisms $A \cong S_3$. Then

(1) $G = C_G(\pi) \times G_1$ where G_1 is A-invariant and $C_{G_1}(\pi) = 1$.

(2) If $p \neq 2$, $G = C_G(\pi) \times C_G(\tau) \times C_G(\tau)^{\pi}$.

<u>PROOF</u>

(1) By [9], theorem 5.2.3, we have $G = C_G(\langle\pi\rangle) \times [G,\langle\pi\rangle]$. As both G and $\langle\pi\rangle$ are A-invariant, $G_1 = [G,\langle\pi\rangle]$ is A-invariant. The result follows.

(2) In view of (1), it suffices to prove that if $C_G(\pi) = 1$ then $G = C_G(\tau) \times C_G(\tau)^{\pi}$.

Let $G^* = C_G(\tau) \times C_G(\tau)^{\pi}$. Then as in the proof of lemma 1.2(5) above, G^* is A-invariant, so A is a regular group of automorphisms of G/G^* by [9], theorem 6.2.2.

Thus $G = G^*$ by lemma 1.2(3). □

The following result on the structure of certain finite groups which admit an automorphism of order 3 will be required in our investigation of S_3-invariant $\{p,q\}$-groups.

1.4 LEMMA Suppose that G is a finite group admitting an automorphism π of order 3. Then

(1) If G is a p-group $(p \neq 3)$, $G = C_G(\pi).[G,\langle\pi\rangle]$ and $[G,\langle\pi\rangle] \lhd G$.

(2) If $G = G_1G_2$ where G_1 is centralized by π and G_2 is a π-invariant p-subgroup of G $(p \neq 3)$ with $C_{G_2}(\pi) = 1$, then $G_2 \lhd G$.

PROOF

Firstly, it is clear that $C_G(\pi)$ normalizes $[G,\langle\pi\rangle]$, since $\langle\pi\rangle$ and $C_G(\pi)$ centralize each other. Now (1) follows from [9], theorem 5.3.5.

(2) $[G,\langle\pi\rangle] = \langle g^{-1}g^{\pi}, g^{-1}g^{\pi^2} \mid g \in G\rangle$.

If $g \in G$, $g = g_1g_2$ for some $g_1 \in G_1$, $g_2 \in G_2$.

$$\begin{aligned}\therefore\quad g^{-1}g^{\pi} &= (g_1g_2)^{-1}(g_1g_2)^{\pi}\\ &= g_2^{-1}g_1^{-1}g_1^{\pi}g_2^{\pi}\\ &= g_2^{-1}g_2^{\pi} \in [G_2,\langle\pi\rangle].\end{aligned}$$

Similarly $g^{-1}g^{\pi^2} \in [G_2,\langle\pi\rangle]$, so $[G,\langle\pi\rangle] = [G_2,\langle\pi\rangle]$.

But by (1), $G_2 = [G_2,\langle\pi\rangle] = [G,\langle\pi\rangle]$.

Clearly $C_G(\pi) = G_1$, so by the remark above

$G_2 \lhd G_1G_2 = G$. □

Our next result concerns groups of odd order which admit S_3 f.p.f.

1.5 LEMMA Suppose G is a finite group of odd order admitting a f.p.f. group of automorphisms $A \cong S_3$, and let $I = \{x \in G \mid x^\tau = x^{-1}\}$.
Then $C_G(\pi) \subseteq I$ and $C_G(\pi) = I \Leftrightarrow C_G(\tau) = 1$.

PROOF

By [9], theorem 10.4.1(i), $G = C_G(\tau).I$.
By lemma 1.1(2), $C_G(\pi) \subseteq I$.
If $C_G(\tau) = 1$ then $G = I = C_G(\pi)$ by lemma 1.2(2).
Conversely, if $C_G(\pi) = I$, I is an A-invariant normal subgroup of G by [9], theorem 10.4.1(ii).
Thus $G/I \cong C_G(\tau)$ is A-invariant, so $C_G(\tau) = 1$. □

The next three results are crucial to the discussion of the structure of a minimal counter-example to the main theorem.

1.6 THEOREM Let G be a finite group with $(|G|,3) = 1$ which admits a f.p.f. group of automorphisms $A \cong S_3$. Then for all prime divisors p of $|G|$, A leaves invariant a unique Sylow p-subgroup of G.

PROOF

Let $\mathcal{P}$ be the set of all π-invariant Sylow p-subgroups of G. Then $\mathcal{P} \neq \phi$ by [9], theorem 6.2.2. Since $\pi\tau = \tau\pi^2$, τ permutes $\mathcal{P}$, and since τ has order 2 each orbit of τ on $\mathcal{P}$ has order 1 or 2. Now if $P, Q \in \mathcal{P}$, by [9], theorem 6.2.2 $\exists x \in C_G(\pi)$ such that $P^x = Q$. Thus $P = Q \Leftrightarrow P = P^x$

$$\Leftrightarrow x \in N_G(P) \cap C_G(\pi).$$

So $|\mathcal{P}| = (C_G(\pi) : C_G(\pi) \cap N_G(P)) \,\big|\, |C_G(\pi)|$.

Hence $|\mathcal{P}|$ is odd by lemma 1.1(2), so that some orbit of τ on $\mathcal{P}$ has order 1.

That is, $\exists P \in \mathcal{P}$ such that P is τ-invariant, and hence A-invariant.

Now let P, Q be any two A-invariant Sylow p-subgroups of G.

Then by [9], theorem 6.2.2 $\exists x \in C_G(\pi)$ such that $Q = P^x$.

Thus $Q = \tau(Q) = \tau(P)^{\tau(x)} = P^{x^{-1}}$, so $P^x = P^{x^{-1}}$

i.e. $x^2 \in N_G(P)$.

But $x \in C_G(\pi)$ so x has odd order by lemma 1.1(2).

Hence $x \in N_G(P)$, so that A leaves invariant a unique Sylow p-subgroup of G. □

1.7 LEMMA Let G be a finite soluble group with $(|G|, 3) = 1$ which admits a f.p.f. group of automorphisms $A \cong S_3$. Then for all factorizations $|G| = mn$ with $(m,n) = 1$, A leaves invariant a unique Hall m-subgroup of G.

PROOF

Since the analogues of (i) and (ii) of [9], theorem 6.2.2 hold for Hall π-subgroups of a soluble group (using the analogous argument), we can apply the same argument as in the previous lemma to deduce the desired result. □

1.8 LEMMA Let G be a finite group with $(|G|,3) = 1$ which admits a f.p.f. group of automorphisms $A \cong S_3$. If H is an A-invariant normal subgroup of G, then A induces a f.p.f. automorphism group of G/H.

PROOF

Suppose $C_{G/H}(A) \neq 1$. Then $C_{G/H}(A)$ has an A-invariant Sylow p-subgroup K/H for some prime p. Let P be the A-invariant Sylow p-subgroup of K, so that $K/H = PH/H$.
Then $\forall x \in P-H$, A leaves xH invariant.
Thus $\forall a \in A$, $x^a = xh$ for some $h \in H$, so that $x^{-1}x^a \in P \cap H$.
It follows that A leaves the coset $xP \cap H$ invariant.
But if $p \neq 2$, [9], theorem 5.3.15 asserts that A acts f.p.f. on $P/P \cap H$, and if $p = 2$ then π, and hence A, acts f.p.f. on $P/P \cap H$ by lemma 1.1(2).
We therefore have $x \in P \cap H$, a contradiction. □

The next lemma enables us to apply Martineau's techniques to find a small bound on the number of maximal A-invariant $\{p,q\}$ - subgroups of a minimal counter-example to our theorem.

1.9 LEMMA Let G be a finite soluble group with $(|G|,3) = 1$ and let S be a Sylow p-subgroup of G. If G admits a f.p.f. group of automorphisms $A \cong S_3$ then $G = O_{p'}(G).C_G(Z(S)).N_G(J(S))$.

PROOF

We proceed by induction on $|G|$.

If $O_{p'}(G) \neq 1$, then by lemma 1.8 and the inductive hypothesis we get

$$G/O_{p'}(G) = C_{G/O_{p'}(G)}(Z(\frac{SO_{p'}(G)}{O_{p'}(G)})).N_{G/O_{p'}(G)}(J(\frac{SO_{p'}(G)}{O_{p'}(G)}))$$

$$= C_{G/O_{p'}(G)}(\frac{Z(S)O_{p'}(G)}{O_{p'}(G)}).N_{G/O_{p'}(G)}(\frac{J(S)O_{p'}(G)}{O_{p'}(G)})$$

Thus $G = O_{p'}(G).C_G(Z(S)).N_G(J(S))$ as required (it is routine to check that if T,M are subgroups of a soluble group H such that $(|T|,|M|) = 1$ then $N_{H/M}(TM/M) = N_H(T)M/M$ and $C_{H/M}(TM/M) = C_H(T)M/M$).

Thus we may assume that $O_{p'}(G) = 1$.

Now by [6], corollary 1 we may assume that $p = 2$ (since $(|G|,3) = 1$).

But then $|\langle\pi\rangle|$ is relatively prime to $|G|$ and π has no fixed point of order 2 by lemma 1.1(2).

The result then follows by [6], corollary 2. □

The two main results obtained by Shult ([19]) on soluble groups admitting a coprime automorphism group $A \cong S_3$ which acts f.p.f. are also critical for our later work, and for the sake of completeness are repeated here.

1.10 THEOREM (Shult, [19]). If G is a soluble group of order coprime to 6 admitting a f.p.f. group of automorphisms $A \cong S_3$, then G has σ-length at most one for any collection σ of primes dividing $|G|$.

1.11 THEOREM (Shult, [19]). If G is a soluble group of order coprime to 6 admitting a f.p.f. group of automorphisms $A \cong S_3$ then G' is nilpotent.

The following lemma is included because it gives some insight into the way S_3 can act f.p.f. on certain groups of order coprime to 3, even though the result itself does not appear to be particularly useful.

1.12 LEMMA Let G be a finite group of order coprime to 3 admitting a f.p.f. group of automorphisms $A \cong S_3$. Suppose that $1 < G_1 < G$ is a normal A-invariant series of G such that G_1 and G/G_1 are elementary abelian. Suppose further that A acts irreducibly on G_1 and G/G_1. Then either G is abelian, or $|G_1| = p^2$, $|G/G_1| = q$ for primes p and q such that $q|p + 1$ if $p \equiv 5$ or 11 (12) and $q|p - 1$ if $p \equiv 1$ or 7 (12).

PROOF

Since the only irreducible representations of A over any field have degree 1 or 2, the only elementary abelian groups on which A can act irreducibly are C_p and $C_p \times C_p$.

Thus $|G_1| = p^a$ and $|G/G_1| = q^b$ where $1 \leqslant a, b \leqslant 2$.

Suppose first that $a = 1$. Then G_1 is cyclic and hence centralized by π by lemma 1.2(1).

If $b = 1$, G/G_1 is also centralized by π, and so G is centralized by π.

Hence G is abelian by lemma 1.1(2). Thus we may assume that $b = 2$.

If $p \neq q$, we can write $G = G_1Q$ where Q is the A-invariant Sylow q-subgroup of G.

But then $Q \lhd G$ by lemma 1.4(2), so that again G is abelian.

We are left with 5 remaining cases:

I $\quad a = 1, b = 2, p = q$

II $\quad a = 2, b = 1, p = q$

III $\quad a = 2, b = 1, p \neq q$

IV $\quad a = b = 2, p = q$

V $\quad a = b = 2, p \neq q$.

We deal with each case in turn.

I Clearly we may assume that G is non-abelian, so that by [9], theorem 5.5.1, G is isomorphic to one of $M_3(p)$, $M(p)$, D_3 or Q_3 where

$$M_3(p) = \langle g,h \mid g^{p^2} = h^p = 1,\ g^h = g^{p+1}\rangle$$

$$M(p) = \langle g,h,k \mid g^p = h^p = k^p = 1,\ [g,h] = [h,k] = 1 \text{ and } [g,h] = k\rangle$$

and D_3, Q_3 are respectively the dihedral and quaternion groups of order 8.

By lemma 1.1(2), π acts f.p.f. on a 2-group, so G must have odd order. Now both $M_3(p)$ and $M(p)$ are extra-special, so $G_1 = Z(G)$ in both cases.

Let $G_1 = \langle z \rangle$ and $G/G_1 = \langle xG_1, yG_1 \rangle$ where (w.l.o.g.) $z^\pi = z$, $(xG_1)^\pi = xy^{-1}G_1$, and $(xG_1)^\tau = yG_1$.

Then $x^\pi = xy^{-1}z^a$ and $y^\pi = x^{-1}z^b$ for some $a,b \in Z$.

Suppose that $[x,y] = z^i$ i.e. $xy = yxz^i$.

Applying π, we get $xy^{-1}z^a x^{-1}z^b = x^{-1}z^b xy^{-1}z^a z^i$

$$\therefore \quad xy^{-1}x^{-1}z^{a+b} = y^{-1}z^{a+b+i}$$

$$\therefore \quad xy^{-1}x^{-1} = y^{-1}z^i .$$

Now $xy^{-1} = y^{-1}xz^{-i}$, so $y^{-1}z^{-i} = y^{-1}z^i$.

O.K. but give reason

Thus $z^{2i} = 1$, and hence $z^i = 1$ since G has odd order.

Thus x and y commute and G is abelian, a contradiction.

II Again we may assume that G is non-abelian, so that as above G is isomorphic to $M_3(p)$ or $M(p)$. As both are extra-special, $Z(G)$ has order p and hence $Z(G) \leqslant G_1$. But then A does not act irreducibly on G_1, a contradiction.

III Let $G = G_1Q$ where Q is the A-invariant Sylow q-subgroup of G.

Then Q is cyclic and hence centralized by π by lemma 1.2(1).

Suppose first that p and q are both odd.

W.l.o.g. let $G_1 = \langle x,y\rangle$ where $x^\pi = x^{-1}y$, $y^\pi = x^{-1}$ and $x^\tau = y$ and $Q = \langle z\rangle$ where $z^\pi = z$.

Since $G_1 \triangleleft G$, $z^{-1}xz = x^\alpha y^\beta$ and $z^{-1}yz = x^u y^v$ for some $0 \leqslant \alpha,\beta,u,v \leqslant p-1$.

Applying π to $z^{-1}xz = x^\alpha y^\beta$ we get

$$z^{-1}x^{-1}yz = (x^{-1}y)^\alpha (x^{-1})^\beta$$

$$(z^{-1}x^{-1}z)(z^{-1}yz) = x^{-\alpha-\beta}y^\alpha$$

$$x^{-\alpha}y^{-\beta}x^u y^v = x^{-\alpha-\beta}y^\alpha$$

$$x^{u-\alpha}y^{v-\beta} = x^{-\alpha-\beta}y^\alpha .$$

Thus $u-\alpha \equiv -\alpha-\beta(p)$ and $v-\beta \equiv \alpha(p)$

i.e. $u \equiv -\beta(p)$ and $v \equiv \alpha+\beta(p)$.

So we have $z^{-1}xz = x^\alpha y^\beta$ and $z^{-1}yz = x^{-\beta}y^{\alpha+\beta}$.

Applying τ to $z^{-1}xz = x^\alpha y^\beta$, we get

$$zyz^{-1} = x^\beta y^\alpha$$

$$\begin{aligned}\therefore \quad y &= z^{-1}x^\beta y^\alpha z \\ &= (z^{-1}xz)^\beta (z^{-1}yz)^\alpha \\ &= x^{\alpha\beta}y^{\beta^2}x^{-\alpha\beta}y^{\alpha^2+\alpha\beta} \\ &= y^{\alpha^2+\alpha\beta+\beta^2}.\end{aligned}$$

Thus $\alpha^2 + \alpha\beta + \beta^2 \equiv 1(p)$.

It follows that if G is non-abelian, the action of z on $\langle x,y\rangle$ is just 'multiplication' by $\begin{pmatrix}\alpha & -\beta\\ \beta & \alpha+\beta\end{pmatrix}$ where $\alpha^2 + \alpha\beta + \beta^2 \equiv 1(p)$.

Let $W = \{X \in SL(2,p) \mid X = \begin{pmatrix}\alpha & -\beta\\ \beta & \alpha+\beta\end{pmatrix}\}$.

Then $\begin{pmatrix}\alpha & -\beta\\ \beta & \alpha+\beta\end{pmatrix}\begin{pmatrix}u & -v\\ v & u+v\end{pmatrix} = \begin{pmatrix}\alpha u-\beta v & -(\alpha v+\beta u+\beta v)\\ \alpha v+\beta u+\beta v & \alpha u+\alpha v+\beta u\end{pmatrix}$,

so W is a subgroup of $SL(2,p)$.

Define $V = \{(\alpha,\beta) \in F_p \times F_p \mid \alpha^2+\alpha\beta+\beta^2 \equiv 1(p)\}$.

Then clearly $|W| = |V|$.

Now $(\alpha,\beta) \in V \Leftrightarrow (\alpha+2\beta)^2 + 3\alpha^2 \equiv 4(p)$, and the number of solutions of $X^2 + 3Y^2 \equiv 4(p)$ is $p - (-1)^{\frac{p-1}{2}}(\frac{3}{p})$.

$$\text{Thus} \quad |W| = |V| = \begin{cases} p+1 & \text{if } p \equiv 5 \text{ or } 11(12) \\ p-1 & \text{if } p \equiv 1 \text{ or } 7(12) \end{cases}.$$

As $z \in W$, we must have $q \mid |W|$.

Thus $|G_1| = p^2$, $|G/G_1| = q$ and $q \mid p+1$ if $p \equiv 5$ or $11(12)$ and $q \mid p-1$ if $p \equiv 1$ or $7(12)$.

We have now to deal with the case where p or q are even.

Since π centralizes Q, q is odd by lemma 1.1(1).

Suppose $p = 2$, and let $G_1 = \{1,\alpha,\beta,\gamma\}$ where (w.l.o.g.) $\alpha^\pi = \gamma$, $\gamma^\pi = \beta$, $\alpha^\tau = \beta$ and $\gamma^\tau = \gamma$.

Again let $Q = \langle z\rangle$ where $z^\pi = z$.

If $z^{-1}\alpha z = \alpha$, applying π we get $z^{-1}\gamma z = \gamma$ and it follows that G is abelian.

If $z^{-1}\alpha z = \beta$, applying π we get $z^{-1}\gamma z = \alpha$

and $z^{-1}\beta z = \gamma$

$$\therefore \quad \beta = z^{-1}\alpha z = z^{-2}\gamma z^2 = z^{-3}\beta z^3 .$$

Thus z^3 centralizes β, so z must also (since $(q,3) = 1$).

But then $\gamma = z^{-1}\beta z = \beta$, a contradiction.

Similarly if $z^{-1}\alpha z = \gamma$ we get a contradiction.

IV Since G is a p-group, $Z(G) \neq 1$. As $G_1 \lhd G$, $G_1 \cap Z(G) \neq 1$.

Thus $G_1 \cap Z(G) = G_1$, since A acts irreducibly on G_1. i.e. $G_1 \leqslant Z(G)$.

Now w.l.o.g. let $G_1 = \langle x,y \rangle$ where $x^\pi = x^{-1}y$, $y^\pi = x^{-1}$ and $x^\tau = y$ and let $G/G_1 = \langle uG_1, vG_1 \rangle$ where $(uG_1)^\pi = u^{-1}vG_1$, $(vG_1)^\pi = u^{-1}G_1$, and $(uG_1)^\tau = vG_1$.

Now x,y commute and u,v commute with x and y, but u and v only commute modulo G_1.

So let $[u,v] = x^iy^j$,

$$u^\pi = u^{-1}vx^ay^b$$

and $$v^\pi = u^{-1}x^cy^d .$$

Applying π to $uv = vu\,x^iy^j$, we get

$$u^{-1}vx^ay^bu^{-1}x^cy^d = u^{-1}x^cy^du^{-1}vx^ay^bx^{-i}y^ix^{-j}$$

$$\therefore u^{-1}vu^{-1}x^{a+c}y^{b+d} = u^{-2}vx^{a+c-i-j}y^{b+d+i}$$

$$\therefore \quad u^{-1}vu^{-1} = u^{-2}vx^{-i-j}y^i .$$

Now $vu^{-1} = u^{-1}vx^iy^j$, so this gives

$$u^{-2}vx^iy^j = u^{-2}vx^{-i-j}y^i$$

$$\therefore \qquad x^i = x^{-i-j} \text{ and } y^j = y^i .$$

It follows that $2i + j \equiv 0(p)$ and $i \equiv j(p)$, so $3i \equiv 0(p)$.

Thus $i \equiv 0 \equiv j(p)$ and $[u,v] = 1$ so that G is abelian.

V Write $G = G_1Q$ where Q is the A-invariant Sylow q-subgroup of G.
Then since A acts irreducibly on both G_1 and Q, we must have $C_{G_1}(\pi) = C_Q(\pi) = 1$. It follows that $C_G(\pi) = 1$ (see lemma 1.14 below), so G is nilpotent, and hence abelian. □

The following example is included to exhibit some properties of non-soluble groups admitting a group of automorphisms isomorphic to S_3, with the purpose of gaining information which might give some insight into the best method of attempting a proof of the theorem. We examine the action of various S_3's on the simple group $PSL(3,4)$ of order 20160, and in particular calculate the Sylow p-subgroups which are left invariant by each S_3 for $p = 3,5$ and 7.

1.13 EXAMPLE

Let $G = PSL(3,4)$, $Z = Z(GL(3,4))$ and $GF(4) = \{0,1,\theta,\theta^2 = 1+\theta\}$. Then G admits the following four groups of automorphisms isomorphic to S_3:

(1) $A_1 = \langle a, f \rangle$, where a is conjugation by $\begin{pmatrix}\theta & & \\ & 1 & \\ & & 1\end{pmatrix}$ and f maps $(p_{ij})Z$ to $(p_{ij}^2)Z$. $|a| = 3$, $|f| = 2$ and $f^{-1}af = a^{-1}$.

(2) $A_2 = \langle a, * \rangle$, where $* : pZ \to (p^{-1})^t Z$. $|*| = 2$ and $*^{-1}a* = a^{-1}$.

(3) $A_3 = \langle b, f \rangle$, where b is conjugation by $\begin{pmatrix}0 & \theta & 1 \\ 1 & 1 & \theta \\ \theta & \theta^2 & 1\end{pmatrix}$. $|b| = 3$ and $f^{-1}bf = b^{-1}$.

(4) $A_4 = \langle b, h \rangle$, where $h = *g$ and g is conjugation by $\begin{pmatrix}1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0\end{pmatrix}$.

Some tedious calculation reveals the following:

$$C_G(a) = \langle \begin{pmatrix}1 & \\ & B\end{pmatrix} Z \mid B \in SL(2,4) \rangle, \quad |C_G(a)| = 2^2.3.5;$$

$$C_G(b) = \langle \begin{pmatrix}0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0\end{pmatrix}, \begin{pmatrix}1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0\end{pmatrix} \rangle Z, \quad |C_G(b)| = 3.7;$$

$$C_G(f) = \{pZ \mid p \in SL(3,2)\}, \quad |C_G(f)| = 2^3.3.7;$$

$$C_G(*) = \langle S \rangle Z \text{ where } S = \{p \in SL(3,4) \mid p^t p \in Z\},$$
$$|C_G(*)| = 2^2.3.5;$$

$$C_G(h) = \langle \begin{pmatrix}0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & x\end{pmatrix} Z \mid x \in GF(4) \rangle, \quad |C_G(h)| = 2^2.3.5.$$

Thus we can calculate:

$$C_G(A_1) = C_G(a) \cap C_G(f) = \{\begin{pmatrix}1 & \\ & B\end{pmatrix} Z \mid B \in SL(2,2)\}$$
has order 6.

$C_G(A_2) = C_G(a) \cap C_G(*) =$

$\{(^1{}_B)Z \mid B \in SL(2,4) \text{ and } B^tB \in Z\}$ has order 16.

$C_G(A_3) = C_G(b) \cap C_G(f) = C_G(b)$ has order 21.

$C_G(A_4) = \langle \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} Z \rangle$ has order 3.

I The Sylow 7-subgroups

Since the Sylow 7-subgroups of G are cyclic of order 7, it follows from lemma 1.2(1) that a Sylow 7-subgroup which admits a group of automorphisms isomorphic to S_3 must be centralized by the 3-elements of S_3, while the 2-elements either centralize or invert it. Now since $C_G(a)$ is not divisible by 7, A_1 and A_2 leave no Sylow 7-subgroup of G invariant. As $C_G(b)$ has a unique Sylow 7-subgroup, namely $\langle \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \rangle Z$, it is a simple matter to verify that f also centralizes this group while h inverts it. Thus A_3 and A_4 leave invariant a unique Sylow 7-subgroup of G.

II The Sylow 5-subgroups

As above, any subgroup of order 5 which admits a group of automorphisms isomorphic to S_3 must be centralized by the 3-elements of S_3 and either centralized or inverted by the 2-elements. Now $C_G(a)$ contains the following 6 subgroups of order 5:

$$M = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & \theta & \theta^2 \\ 0 & \theta & 0 \end{pmatrix} \right\rangle Z, \quad N = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & \theta^2 & \theta^2 \\ 0 & \theta^2 & 1 \end{pmatrix} \right\rangle Z, \quad P = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & \theta^2 & \theta \\ 0 & 1 & 1 \end{pmatrix} \right\rangle Z$$

$$Q = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & \theta^2 & 1 \\ 0 & \theta & 1 \end{pmatrix} \right\rangle Z, \quad R = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \theta \\ 0 & \theta^2 & \theta \end{pmatrix} \right\rangle Z, \quad T = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & \theta & \theta \\ 0 & \theta & 1 \end{pmatrix} \right\rangle Z$$

Since $C_G(f)$ is not divisible by 5, A_1 does not centralize any of these groups. It is easily checked that f does not invert any of them either, so that A_1 leaves no Sylow 5-subgroup of G invariant. Routine calculation reveals that $*$ inverts N and T but does not centralize or invert the other subgroups. Hence A_2 leaves two Sylow 5-subgroups of G invariant. As $C_G(b)$ is not divisible by 5, neither A_3 or A_4 can leave a Sylow 5-subgroup of G invariant.

III The Sylow 3-subgroups

The Sylow 3-subgroups of G are elementary abelian of order 9. By [9], lemma 2.6.3, any automorphism of order 3 of such a group must centralize an element or order 3. We therefore determine the Sylow 3-subgroups of G left invariant by each A_i by the following steps:

(i) Find all elements of order 3 centralized by a, and similarly for b.

(ii) Determine the centralizer of each of these elements, and hence all the Sylow 3-subgroups of G in which they are contained.

(iii) Examine these subgroups to find which are left invariant by each A_i.

We find that $C_G(a)$ has 10 subgroups of order 3, and each is contained in a unique Sylow 3-subgroup of G. Thus we have only to consider the following:

$$M = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & \theta \\ \theta & \theta^2 & \theta \\ 1 & 1 & \theta^2 \end{pmatrix} \right\rangle Z$$

$$N = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \theta^2 \\ 0 & \theta & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & \theta \\ 1 & \theta^2 & 1 \end{pmatrix} \right\rangle Z$$

$$O = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \theta \\ 0 & \theta^2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & \theta^2 \\ 1 & \theta & 1 \end{pmatrix} \right\rangle Z$$

$$P = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & \theta^2 & 0 \\ 0 & \theta^2 & \theta \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & \theta^2 & 1 \\ 1 & \theta & \theta^2 \end{pmatrix} \right\rangle Z$$

$$Q = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & \theta & 0 \\ 0 & \theta & \theta^2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & \theta^2 \\ \theta^2 & 0 & 0 \\ 1 & \theta^2 & 0 \end{pmatrix} \right\rangle Z$$

$$R = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & \theta^2 & 0 \\ 0 & 1 & \theta \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \right\rangle Z$$

$$S = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & \theta^2 & \theta \\ 0 & 0 & \theta \end{pmatrix}, \begin{pmatrix} 0 & \theta^2 & 1 \\ 1 & 0 & \theta^2 \\ \theta^2 & 0 & 0 \end{pmatrix} \right\rangle Z$$

$$T = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & \theta & \theta^2 \\ 0 & 0 & \theta^2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \theta^2 \\ \theta^2 & \theta & 1 \\ 0 & \theta^2 & \theta \end{pmatrix} \right\rangle Z$$

$$U = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & \theta^2 & 1 \\ 0 & 0 & \theta \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \right\rangle Z$$

$$V = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & \theta^2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle Z$$

Inspection now reveals that f leaves M, R, U and V invariant and $*$ leaves M and V invariant, while all 10 subgroups are left invariant by a. Hence A_1 leaves 4 Sylow 3-subgroups of G invariant and A_2 leaves 2 invariant.

Similarly, $C_G(b)$ contains 7 subgroups of order 3, and each is contained in a unique Sylow 3-subgroup of G. Thus we need only consider the following:

$$B = \left\langle \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ \theta & \theta & 1 \\ \theta & 0 & \theta^2 \end{pmatrix} \right\rangle Z$$

$$C = \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} \theta & 1 & 1 \\ 0 & \theta^2 & \theta \\ 0 & 0 & 1 \end{pmatrix} \right\rangle Z$$

$$D = \left\langle \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \theta^2 & 0 & 0 \\ \theta & 1 & 0 \\ 0 & 0 & \theta \end{pmatrix} \right\rangle Z$$

$$E = \left\langle \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \theta & 0 & 0 \\ 0 & \theta^2 & 0 \\ \theta^2 & \theta & 1 \end{pmatrix} \right\rangle Z$$

$$F = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & \theta \\ \theta & \theta^2 & \theta \\ 1 & 1 & \theta^2 \end{pmatrix} \right\rangle Z$$

$$H = \left\langle \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \theta & \theta^2 & 0 \\ 0 & 1 & 0 \\ 0 & \theta & \theta^2 \end{pmatrix} \right\rangle Z$$

$$I = \left\langle \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & \theta \\ 0 & \theta & 0 \\ 0 & 0 & \theta^2 \end{pmatrix} \right\rangle Z$$

Now b necessarily leaves all of these subgroups invariant and so does f, so A_3 leaves 7 Sylow 3-subgroups of G invariant. On the other hand, h leaves only B invariant, so A_4 leaves invariant a unique Sylow 3-subgroup of G.

The next lemma, which exhibits conditions under which there must exist an A-invariant subgroup with a non-trivial centralizer in a particular subgroup, is vital to our later work.

LEMMA 1.14 Let G be a finite group with $(|G|,6) = 1$ admitting a f.p.f. group of automorphisms $A \cong S_3$. Let X be a minimal A-invariant q-subgroup of G and Y a minimal A-invariant p-subgroup of G with $[X,Y] = 1$ for primes p and q dividing $|G|$. Let K be a soluble minimal $\langle A, X \times Y\rangle$-invariant $\{p,q\}'$ subgroup of G.

Then $\exists$ an A-invariant subgroup X_0 of $X \times Y$ with $C_K(X_0) \neq 1$ under any of the following conditions:

(a) $X \times Y \leqslant C_G(\pi)$ and $p = q$.

(b) $[X \times Y, \langle\pi\rangle] = X \times Y$ and $p = q$.

(c) $X \leqslant C_G(\pi)$ and $Y = [Y, \langle\pi\rangle]$ (in this case either $C_K(X) \neq 1$ or $C_K(Y) \neq 1$).

PROOF

Since X is minimal, we have either $X \cong Z_q$ or $X \cong Z_q \times Z_q$.

Similarly $Y \cong Z_p$ or $Z_p \times Z_p$.

(a) If $p = q$ and $X \times Y \leqslant C_G(\pi)$, the result follows from [9], theorem 6.2.4, since any subgroup of $C_G(\pi)$ is A-invariant.

(b) W.l.o.g., we may assume that $C_K(X) = C_K(Y) = 1$ and that K is an elementary abelian t-group for some prime $t \neq p,q$.

By [9], theorem 6.2.4, $\exists y \in Y$ such that $C_K(y) \neq 1$.

We show that in this case $y^\tau = y^{-1}$ (w.l.o.g.).

Suppose first that $y^\tau = y$.

Then $K = C_K(y) \times C_K(y^\pi) \times C_K(y^{\pi^2})$ as the latter group is invariant under $\langle A, X \times Y\rangle$.

Choose $y_0 \in Y$ such that $y_0^\tau = y_0^{-1}$, so that $Y = \langle y, y_0\rangle$.

Then $\langle y_0, \tau\rangle$ is a dihedral group of automorphisms of $C_K(y)$.

As y_0 acts f.p.f. an $C_K(y)$ (otherwise $C_K(Y) \neq 1$), $\exists u \in C_K(y)$ such that $u \neq 1$ and $u^\tau = u$ by [9], theorem 5.3.14(iii).

But then $uu^\pi u^{\pi^2}$ is non-trivial and is centralized by A, a contradiction.

Suppose next that y has six conjugates under A.

Then $K = C_K(y)\times C_K(y^\pi)\times C_K(y^{\pi^2})\times C_K(y^\tau)\times C_K(y^{\pi\tau})\times C_K(y^{\pi^2\tau})$.

For $x \in C_K(y)$, $(xx^\pi x^{\pi^2})^\tau \notin \langle x, x^\pi, x^{\pi^2}\rangle$.

But $xx^\pi x^{\pi^2} \in C_G(\pi)$, so $(xx^\pi x^{\pi^2})^\tau = (xx^\pi x^{\pi^2})^{-1}$, a contradiction.

Hence we may assume w.l.o.g. that $y^\tau = y^{-1}$.

By a symmetric argument, $\exists v \in X$ such that $C_K(v) \neq 1$ and $v^\tau = v^{-1}$.

Then $\langle v,\tau\rangle$ is a dihedral group of automorphisms of $C_K(y)$, and as $C_G(\tau) \cap C_K(y) = 1$ by the same argument as above, as in the proof of lemma 1.2(2) we must have $[v, C_K(y)] = 1$ i.e. $C_K(y) \leqslant C_K(v)$.

Similarly $C_K(v) \leqslant C_K(y)$, so that $C_K(v) = C_K(y)$.
Now by [9], theorem 5.2.3, $K = C_K(y) \times K_0$ where $\langle v,y\rangle$ normalizes K_0.
By [9], theorem 6.2.4, $\exists z \in \langle v,y\rangle$ such that $C_{K_0}(z) \neq 1$.
But by applying the same argument as above to z (τ inverts $\langle v,y\rangle$ and hence z) we have $C_K(z) = C_K(y)$.
This contradiction completes the proof of (b).

(c) If $C_K(\pi) \neq 1$, X centralizes $C_K(\pi)$ by lemma 1.1(2).
So we may assume that $C_K(\pi) = 1$.
But then $Y.K$ is nilpotent by lemma 1.2(4), and hence Y centralizes K. □

We now demonstrate that (b) of lemma 1.14 is valid even if K is not A-invariant, provided that $C_K(C_Y(\tau)) \neq 1$.

<u>LEMMA 1.15</u> Let Y be a p-group for some prime $p \neq 3$ which is isomorphic to $Z_p \times Z_p \times Z_p \times Z_p$ and which admits a f.p.f. group of automorphisms $A \cong S_3$ with $C_Y(\pi) = 1$. Suppose that Y acts on a p'-group K with $C_K(C_Y(\tau)) \neq 1$.
Then $\exists$ an A-invariant subgroup Y_0 of Y with $C_K(Y_0) \neq 1$.

<u>PROOF</u>

Let $C_Y(\tau) = \langle x,y\rangle$, so that $Y = \langle x,y\rangle \times \langle x^\pi, y^\pi\rangle$ by lemma 1.3(2).

Now $<x^\pi, y^\pi>$ normalizes $C_K(C_Y(\tau))$, so by [9], theorem 6.2.4, $\exists u \in <x^\pi, y^\pi>$ such that $C_K(u) \cap C_K(C_Y(\tau)) \neq 1$.
Let $u = v^\pi$ for some $v \in C_Y(\tau)$.
Then $Y_0 = <v, v^\pi>$ is A-invariant by lemma 1.2(5) and $C_K(Y_0) \neq 1$. □

The next lemma is used often in our later work.

<u>LEMMA 1.16</u> Let P be a p-group, p a prime different from 3, admitting a f.p.f. group of automorphisms $A \cong S_3$ such that $1 \neq C_P(\pi) < P$.
Then $C_P(\pi) < C_P(C_P(\pi))$.

<u>PROOF</u> $A \cong S_3$ also!

Let $A = C_P(\pi)$ and $B = N_P(A)$.
Then $[[A,B],<\pi>] \leqslant [A,<\pi>] = 1$ and $[[A,<\pi>],B] = 1$.
Thus $[[B,<\pi>],A] = 1$ by [9], theorem 2.2.3.
But $B = [B,<\pi>].C_B(\pi)$ by [9], theorem 5.3.5

$= [B,<\pi>].A$.

Thus $B \leqslant C_P(A)$, so that $A < C_P(A)$ as required. □

The next two lemmas provide information about the structure of a p-group admitting S_3 f.p.f. in particular circumstances which arise in our later discussions.

<u>LEMMA 1.17</u> Let P be a p-group, p a prime different from 3, admitting a f.p.f. group of automorphisms $A \cong S_3$. Suppose that $Z(P)$ is cyclic and that P_1 is a proper A-invariant subgroup of P with $C_P(P_1) = P_1$.

Suppose further that for every A-invariant subgroup $X \neq 1$ of P_1 with $X \cap Z(P) = 1$ we have $C_P(X) = P_1$. Then either $|N_P(P_1)/P_1| = p$ and P_1 is a characteristic subgroup of $N_P(P_1)$ or $|N_P(P_1)/P_1| = p^2$ and if $|\Omega_1(P_1)| \geq p^4$, P_1 is a characteristic subgroup of $N_P(P_1)$.

PROOF

Let $K = N_P(P_1)$.

By lemma 1.16, P_1 is not centralized by π.

Since $Z(P)$ is cyclic, we must therefore have $\Omega_1(Z(P)) < \Omega_1(P_1)$.

If $\Omega_1(Z(P)) < \Omega_1(Z(K))$, we can choose an A-invariant subgroup X of $Z(K)$ such that $X \cap Z(P) = 1$.

But then $C_P(X) = K \neq P_1$, contradicting the assumption.

Thus $\Omega_1(Z(P)) = \Omega_1(Z(K))$.

Now $\Omega_1(P_1)Z(K)/Z(K) \triangleleft K/Z(K)$ so $\Omega_1(P_1).Z(K) \cap Z_2(K) > Z(K)$.

It follows that $\Omega_1(P_1) \cap Z_2(K) > \Omega_1(Z(K)) = \Omega_1(Z(P))$.

Let X be a minimal A-invariant subgroup of $\Omega_1(P_1) \cap Z_2(K)$ with $X \cap Z(P) = 1$.

Suppose first that $X \leq C_P(\pi)$, so that $X = \langle x \rangle \cong Z_p$.

Now for $y \in K$, $(xZ(K))^y = xZ(K)$ since $x \in Z_2(K)$

i.e. $x^y \in xZ(K)$.

But $|\Omega_1(Z(K))| = p$, so x has at most p conjugates in K.

Now $|K : C_K(x)| = |K : C_K(X)| = |K : P_1|$ by assumption, so we must have $|K/P_1| = p$.

Suppose P_1 is not a characteristic subgroup of K.
Then $\exists$ a subgroup $S \neq P_1$ of K such that $S \cong P_1$.
Clearly $K = SP_1$, and since S and P_1 are self-centralizing in K, $Z(K) = S \cap P_1$.
But $Z(K)$ is cyclic and hence is centralized by π, and since $|P_1/Z(K)| = p = |K/P_1|$, it follows from [9], theorem 6.2.2, that K is centralized by π.
But then K is abelian, contradicting $C_P(P_1) = P_1$.
Hence P_1 is a characteristic subgroup of K.

Suppose next that $[X,\langle\pi\rangle] = X$, so that $|X| = p^2$.
For $x \in X$, as above $|K : C_K(x)| = p$.
Thus $|K : C_K(X)| \leqslant p^2$ i.e. $|K/P_1| = p$ or p^2.
Suppose that $|K/P_1| = p$ and take $y \in K - P_1$.
Then for all $x \in X$, $x^y = xz^i$ for some integer i where $\Omega_1(Z(P)) = \langle z\rangle$.
Now $\exists$ an integer j such that $(i,j) \equiv 1(p)$, so $(x^j)^y = x^j z$.
It follows that $\exists x_1, x_2 \in X$ such that $x_1^y = x_1 z$ and $x_2^y = x_2 z$.

Thus $x_1 x_2^{-1} \in C_X(y)$.

Now $y^\pi = yt$ for some $t \in P_1$ since $|K/P_1|$ is cyclic.
Thus $1 = [y, x_1x_2^{-1}]^\pi = [yt, (x_1x_2^{-1})^\pi] = [y, (x_1x_2^{-1})^\pi]$.
But then $\langle x_1x_2^{-1}, (x_1x_2^{-1})^\pi\rangle = X$ centralizes y, so that $y \in P_1$, a contradiction.
Hence $|K/P_1| = p^2$.
If P_1 is not a characteristic subgroup of K, $\exists$ a subgroup $S \neq P_1$ of K with $S \cong P_1$.

Since P_1 and S are self-centralizing in K,

$P_1 \cap S = Z(P_1S)$.

If P_1S is A-invariant then $\Omega_1(P_1 \cap S) = \Omega_1(Z(P))$ since otherwise we can choose an A-invariant subgroup X_0 of $Z(P_1S)$ such that $X_0 \cap Z(P) = 1$ and $C_P(X_0) = P_1S \neq P_1$.

Thus $|\Omega_1(P_1 \cap S)| = p$.

Now $|P_1 : P_1 \cap S| = |P_1S : P_1| \leqslant p^2$, so

$|\Omega_1(P_1) : \Omega_1(P_1 \cap S)| \leqslant p^2$.

Hence $|\Omega_1(P_1)| \leqslant p^3$.

If P_1S is not A-invariant, we must have $K = P_1.S.S^{\alpha}$ where $\alpha = \pi$ or τ.

But then as above $Z(K) = P_1 \cap S \cap S^{\alpha}$ and so

$|P_1 : Z(K)| \leqslant p^2$.

Since $\Omega_1(Z(K)) = \Omega_1(Z(P))$, this again yields

$|\Omega_1(P_1)| \leqslant p^3$.

It follows that if $|\Omega_1(P_1)| \geqslant p^4$, P_1 is a characteristic subgroup of K. □

<u>LEMMA 1.18</u> Let P be a p-group, p a prime different from 3, admitting a f.p.f. group of automorphisms $A \cong S_3$ such that $C_P(\pi) = 1$. If $P - Z(P)$ contains an element of order p then P contains an A-invariant subgroup $W \cong Z_p \times Z_p \times Z_p \times Z_p$.

<u>PROOF</u>

Let $\Omega_1(Z(P)) = Z_0$.

If $|Z_0| > p^2$ then $|Z_0| \geqslant p^4$ by lemma 1.3(2) since π acts f.p.f. on Z_0.

The result then follows.

Suppose $|Z_0| = p^2$ and suppose $\exists x \in P - Z_0$ of order p.

Suppose that $xx^\tau \in Z_0$ and $x^\pi x^{\pi\tau} \in Z_0$.

Then $(xZ_0)^\tau = (xZ_0)^{-1}$ and $(x^\pi Z_0)^\tau = x^{-\pi}Z_0$.

But $(x^\pi Z_0)^\tau = (xZ_0)^{\pi\tau} = (xZ_0)^{\tau\pi^2} = (xZ_0)^{-\pi^2} = (x^{-\pi}Z_0)^\pi$.

Thus π centralizes $x^{-\pi}Z_0$, a contradiction.

We may therefore assume w.l.o.g. that $xx^\tau \notin Z_0$.

If $[x,x^\tau] = 1$, $(xx^\tau)^\tau = xx^\tau$.

Thus $\exists y \in P - Z_0$ of order p such that $y^\tau = y$.

Then $\langle y,y^\pi\rangle$ is an A-invariant group of order p^2 by lemma 1.2(5) and $W = Z_0 \times \langle y,y^\pi\rangle$ is the required group.

If $[x,x^\tau] \neq 1$, we have $[x,x^\tau] = z \in Z(P)$ since P has class 2 by the result of Frobenius in [2], section 66.

Furthermore, z has order p by [9], lemma 2.2.2.

Thus $\langle x,x^\tau\rangle$ is a non-abelian group of order p^3 and exponent p.

Since τ normalizes $\langle x,x^\tau\rangle$, $\exists y \in \langle x,x^\tau\rangle$ such that $y^\tau = y$ by [9], theorem 10.1.4.

Now $z = x^{-1}x^{-\tau}xx^\tau$ so $z^\tau = x^{-\tau}x^{-1}x^\tau x = z^{-1}$.

Thus $y \notin Z_0$ and it then follows as above that $W = Z_0 \times \langle y,y^\pi\rangle$ is the required group. □

The next result is a simple application of Shult's theorem which will be used extensively later.

of order prime to 6 ?

<u>LEMMA 1.19</u> Let K be a finite soluble group admitting a f.p.f. group of automorphisms $A \cong S_3$. If π acts f.p.f. on $F(K)$ then $K = C_K(\pi).F(K)$.

<u>PROOF</u>

By theorem 1.11 K' is nilpotent, so $K' \leqslant F(K)$. Thus $K/F(K)$ is abelian. Thus $K/F(K) = C_{K/F(K)}(\pi) \times K_0/F(K)$ where $K_0/F(K)$ is an A-invariant group on which π acts f.p.f. by lemma 1.3(1).

It follows from [9], theorem 6.2.2, that $C_{K/F(K)}(\pi) = C_K(\pi).F(K)/F(K)$ and that π acts f.p.f. on K_0.

Hence K_0 is nilpotent by theorem 1.2(4) and since $K_0 \triangleleft K$ we have $K_0 \leqslant F(K)$.

It follows that $K = C_K(\pi).F(K)$. □

Finally, we conclude this section with two results which are well-known but which are not readily found in the literature.

<u>1.20 LEMMA</u> If σ is an automorphism of the groups A and B, and AB is a group with $A \cap B = 1$ then $C_{AB}(\sigma) = C_A(\sigma).C_B(\sigma)$.

<u>PROOF</u>

$\sigma(ab) = \sigma(a)\sigma(b) = ab$ iff $a^{-1}\sigma(a) = b\sigma(b^{-1})$.

As A and B are σ-invariant, $a^{-1}\sigma(a) \in A$ and $b\sigma(b^{-1}) \in B$.

Thus $\sigma(ab) = ab \Leftrightarrow \sigma(a) = a$ and $\sigma(b) = b$.

The result follows. □

<u>1.21 LEMMA</u> Let R be a p-group and M a non-cyclic abelian q-group of automorphisms of R, where p and q are distinct primes.
Then $R = \langle C_R(B) \mid M/B$ is cyclic$\rangle$.

<u>PROOF</u>

A minor modification to the proof of [9], theorem 3.3.3 will suffice to prove the above result when R is elementary abelian. The proof of the general result is then analogous to that of [9], theorem 5.3.16.

$\square$

CHAPTER TWO

MAXIMAL V-INVARIANT $\{p,q\}$-GROUPS

In this chapter we present the results obtained by Martineau and Glauberman on the maximal V-invariant $\{p,q\}$-subgroups of a minimal counter-example to the general conjecture, using the technique pioneered by Martineau in [15]. In order to maintain as much generality as possible, we assume only that G is a finite group admitting a group of automorphisms V with the following properties:

(A) If H is a V-invariant subgroup of G then V leaves invariant a unique Sylow p-subgroup of H for all prime divisors of $|H|$.

(B) If H is a soluble V-invariant subgroup of G then for all factorizations $|H| = mn$ with $(m,n) = 1$, V leaves invariant a unique Hall m-subgroup of H.

(C) If H is a soluble V-invariant subgroup of G then for all prime divisors p of $|H|$, $H = O_{p'}(H).C_H(Z(S)).N_H(J(S))$ where S is a Sylow p-subgroup of H.

It is well known that (A) and (B) hold when V is a f.p.f. group of automorphisms of G with

$(|G|,|V|) = 1$, and (C) holds in this case by [6], corollary 2. By theorem 1.6 and lemmas 1.7 and 1.9, (A), (B) and (C) hold also when $V \cong S_3$ acts f.p.f. on G and $(|G|,3) = 1$. Furthermore, Ward ([20]) has investigated hypotheses other than V acting f.p.f. on G which will enable him to deduce the results of this section, although he omitted mention of (B).

We adopt the notation that if L is a V-invariant soluble group and σ a set of primes dividing $|L|$, then L_σ denotes the V-invariant Hall σ-subgroup of L. We first prove some preliminary results from the hypotheses above which will be fundamental in our later work.

<u>2.1 LEMMA</u> Suppose G and V are as above. Then we have the following:

(i) If H is a V-invariant subgroup of G, then for all prime divisors p of $|H|$, every V-invariant p-subgroup of H is contained in the unique V-invariant Sylow p-subgroup of H.

(ii) If H is a V-invariant subgroup of G, then $H_p = H \cap P$ where P is the unique V-invariant Sylow p-subgroup of G.

(iii) If H is a V-invariant soluble subgroup of G, then for all factorizations $|H| = mn$ with $(m,n) = 1$, every V-invariant subgroup of H of order dividing m is contained in the unique V-invariant Hall m-subgroup of H.

(iv) If L and M are V-invariant subgroups of G then $(L \cap M)_p = L_p \cap M_p$.

(v) A V-invariant subgroup H of G is soluble iff the V-invariant Sylow subgroups of H are pairwise permutable.

PROOF

(i) (The proof given here is taken from [9], theorem 6.2.2 but is reproduced for the sake of completeness).
Let T be a V-invariant p-subgroup of H, and let P be a maximal V-invariant p-subgroup of H containing T.
Then $N_H(P)$ is V-invariant, and hence contains a unique Sylow p-subgroup R by (A).
But $P \leqslant R$ since $P \triangleleft N_H(P)$, so $P = R$ by our maximal choice of P.
Now certainly P is contained in a Sylow p-subgroup Q of H, and if $P \subset Q$ then $P \subset N_Q(P)$. Thus $P = Q$ is a Sylow p-subgroup of H.

(ii) Since H_p is a V-invariant p-subgroup of G, by (i) $H_p \leqslant P$. Thus $H_p \leqslant H \cap P$, so $H_p = H \cap P$.

(iii) Let T be a V-invariant subgroup of H with $|T|\,|m$, and let M be the V-invariant Hall m-subgroup of H.

If p is any prime dividing $|T|$, M contains a unique V-invariant Sylow p-subgroup by (A), and this is clearly the V-invariant Sylow p-subgroup P of H. By (ii), $T_p = T \cap P \leq P \leq M$.

It follows that $T \leq M$.

(iv) Since $(L \cap M)_p$ is a V-invariant p-subgroup of L, $(L \cap M)_p \leq L_p$ by (i).
Similarly $(L \cap M)_p \leq M_p$, so that
$(L \cap M)_p \leq L_p \cap M_p$.
But now $L_p \cap M_p$ is a V-invariant p-subgroup of $L \cap M$, so $L_p \cap M_p \leq (L \cap M)_p$ by (i).
Hence $(L \cap M)_p = L_p \cap M_p$.

(v) Suppose H is soluble, and let P and Q be respectively the V-invariant Sylow p- and q-subgroups of H of order p^α, q^β.
By (B), H contains a unique V-invariant Hall $p^\alpha q^\beta$-subgroup S, and by (iii) we have $P, Q \leq S$.
It follows that $S = PQ = QP$, so that P and Q permute.
The reverse implication follows from P. Hall's characterization of soluble groups ([11]). □

For the remainder of this section, we make the following additional assumptions:

(D) (i) G contains no non-trivial normal V-invariant subgroups.

(ii) Every proper V-invariant subgroup of G is soluble.

These are clearly the hypotheses which would be satisfied by a minimal counter-example to a conjecture of solubility of a group G admitting a group of automorphisms V such that (A), (B) and (C) are satisfied. In particular, we will most commonly use (D) to deduce that if H is a V-invariant proper subgroup of G then $N_G(H)$ is soluble.

Now let p and q be any two primes dividing $|G|$, and let P and Q be the respective V-invariant Sylow subgroups of G. If C and D are V-invariant subgroups of P which are permutable with Q, then so is $\langle C,D\rangle$. Hence we can define X to be the largest V-invariant subgroup of P which is permutable with Q, and similarly define Y to be the largest V-invariant subgroup of Q to be permutable with P.
Then PY and QX are maximal V-invariant $\{p,q\}$-subgroups of G, and $PY = QX$ iff $Q = Y$ and $P = X$.
We now define K to be the set of all maximal V-invariant $\{p,q\}$-subgroups of G, and define $\mathcal{H} = K - \{PY,QX\}$.
Our aim is to derive information about the elements of $\mathcal{H}$, and to a lesser extent K, and to utilize these results in Chapter 3 to prove that $|\mathcal{H}| \leqslant 1$ when $V \cong S_3$ and $(|G|,3) = 1$.

We begin with a result on the structure of certain elements of K, which is a consequence of our hypothesis (C).

2.2 LEMMA (Martineau, [15]). Suppose $Z(Q) \leqslant H \in K$. Then $H \cap P = O_p(H).(H \cap X)$.
Similarly the symmetric statement holds.

PROOF

Suppose the lemma is false, and let $Q_1 = Q \cap H$, the V-invariant Sylow q-subgroup of H.
By (C), $H = O_p(H)C_H(Z(Q_1))N_H(J(Q_1))$.

Hence $H \cap P = O_p(H).C_{H\cap P}(Z(Q_1)).N_{H\cap P}(J(Q_1))$.
Now $Z(Q) \leqslant H \cap Q = Q_1 \leqslant Q$, so $Z(Q) \leqslant Z(Q_1)$.

Thus $C_G(Z(Q_1)) \leqslant C_G(Z(Q))$.
By hypothesis $C_G(Z(Q))$ is soluble, and as $C_{H\cap P}(Z(Q_1))$ is a V-invariant p-subgroup of $C_G(Z(Q))$ it is contained in $C_G(Z(Q))_{p,q}$.
But $Q \leqslant C_G(Z(Q))_{p,q}$, so it follows that $C_{H\cap P}(Z(Q_1))$ is contained in a V-invariant p-subgroup of G which is permutable with Q.

Thus $C_{H\cap P}(Z(Q_1)) \leqslant X$.
Since we are assuming the lemma to be false, we therefore have $N_P(J(Q_1)) \nleqslant X$.
Now choose $Q^* \leqslant Q$ maximal subject to the following:

Q^* is V-invariant, $Z(Q) \leqslant Q^*$ and $N_P(J(Q^*)) \nleqslant X$.
$N_G(J(Q))$ is soluble, and clearly $N_G(J(Q))_{p,q} = N_P(J(Q)).Q$
Thus $N_P(J(Q)) \leqslant X$, so that $Q^* \subset Q$.

Let P^* and $\bar{Q}$ be respectively the V-invariant Sylow p- and q-subgroups of $N_G(J(Q^*))$, and let $K = P^*\bar{Q}$. By (C), $K = O_p(K).C_K(Z(\bar{Q})).N_K(J(\bar{Q}))$.

Thus $P^* = K \cap P^* = O_p(K).C_{P^*}(Z(\bar{Q})).N_{P^*}(J(\bar{Q}))$.

Now $Z(Q) \leqslant Q^* \leqslant Q$, so $Z(Q) \leqslant Z(Q^*) \leqslant J(Q^*) \leqslant O_q(K)$.

It follows that $O_p(K)$ centralizes $Z(Q)$.

Thus $O_p(K) \leqslant C_G(Z(Q))$, so that as above we get $O_p(K) \leqslant X$.

Now $Q^* \leqslant N_G(J(Q^*))$, so $Q^* \leqslant \bar{Q}$.

Therefore $Z(Q) \leqslant Q^* \leqslant \bar{Q} \leqslant Q$, so $Z(Q) \leqslant Z(\bar{Q})$.

Hence $C_{P^*}(Z(\bar{Q})) \leqslant C_{P^*}(Z(Q)) \leqslant C_G(Z(Q))$.

So $C_{P^*}(Z(\bar{Q})) \leqslant X$.

Now $Q^* < Q$, so $Q^* < N_Q(Q^*)$.

Thus $Q^* < N_Q(Q^*) \leqslant N_G(J(Q^*))$, so $Q^* < \bar{Q}$.

Then we have $Z(Q) \leqslant \bar{Q}$, $\bar{Q}$ is V-invariant and $Q^* < \bar{Q}$, so by the maximality of Q^*, $N_P(J(\bar{Q})) \leqslant X$.

Since $P^* \leqslant P$, this gives $N_{P^*}(J(\bar{Q})) \leqslant X$, and hence $P^* \leqslant X$.

This contradiction completes the proof. □

Before proceeding to the next main result, we require the following lemma, which has been attributed to Bender:

2.3 LEMMA Let t be a prime and K a t-constrained group. If T is a t-subgroup of K then $O_{t'}(N_K(T)) \leqslant O_{t'}(K)$.

PROOF

We proceed by induction on $|K|$.

If $O_{t'}(K) \neq 1$ then by induction $O_{t'}(N_{\bar{K}}(\bar{T})) \leq O_{t'}(\bar{K})$ where $\bar{K} = K/O_{t'}(K)$ and $\bar{T}$ is the image of T in $\bar{K}$.

But $O_{t'}(\bar{K}) = 1$, so $O_{t'}(N_{\bar{K}}(\bar{T})) = 1$.

Thus $O_{t'}(N_K(T)) \leq O_{t'}(K)$ as required.

Hence we may assume that $O_{t'}(K) = 1$, so that $O_t(K) \neq 1$ by the definition of t-constraint.

Now $O_{t'}(N_K(T)) \times T$ is a group of automorphisms of $O_t(K)$, and $[O_{t'}(N_K(T)), C_{O_t(K)}(T)] \leq O_{t'}(N_K(T)) \cap O_t(K) = 1$.

Thus by [9], theorem 5.3.4, $[O_t(K), O_{t'}(N_K(T))] = 1$.

But K is t-constrained, so $C_K(O_t(K)) \leq O_t(K)$.

Hence $O_{t'}(N_K(T)) = 1$ as required. □

We now prove a sufficient condition for two members of K to be equal. Surprisingly, we need only a condition on the Fitting subgroups.

2.4 LEMMA (Martineau, [13]). Let $H \in K$ and suppose that M is a V-invariant subgroup of $F(H)$ with $O_p(M) \neq 1$ and $O_q(M) \neq 1$. Then if $M \leq K \in K$, $H = K$.

PROOF

We show first that the lamma holds for $Z = Z(F(H))$ in place of M.

Suppose that Z_p and Z_q are non-trivial.

As $N_G(Z_p)$ is V-invariant and soluble and contains H,

$N_G(Z_p)_{p,q} = H$.

Hence $Z_q \leqslant O_q(N_G(Z_p)_{p,q})$.

Since $N_K(Z_p) \leqslant N_G(Z_p)_{p,q}$ we have $Z_q \leqslant O_q(N_K(Z_p))$.

Now K is soluble and hence p-constrained by [9], theorem 6.3.3.

Hence by lemma 2.3 above, $O_q(N_K(Z_p)) \leqslant O_q(K)$.

Thus $Z_q \leqslant O_q(K)$.

But then $O_p(K) \leqslant C_G(Z_q)_{p,q} \leqslant H$.

Similarly $O_q(K) \leqslant H$, so $F(K) \leqslant H$.

Now $O_q(K) \neq 1$ since $Z_q \leqslant O_q(K)$, and similarly $O_p(K) \neq 1$.

As $F(K) \leqslant H$, certainly $Z(F(K)) \leqslant H$ so by the same argument as above with H and K interchanged we obtain $F(H) \leqslant K$.

Now $O_p(H) = O_p(N_G(O_q(H))_{p,q})$, so $O_p(H) \leqslant O_p(N_K(O_q(H)))$.

Applying lemma 2.3 again, this gives $O_p(H) \leqslant O_p(K)$.

Similarly $O_q(H) \leqslant O_q(K)$, so $F(H) \leqslant F(K)$.

Now by interchanging H and K and applying the same argument we get $F(K) \leqslant F(H)$, so that $F(H) = F(K)$.

But now $H = N_G(F(H))_{p,q} = N_G(F(K))_{p,q} = K$, so the lemma holds for the particular case $M = Z(F(H))$.

Now let M be an arbitrary V-invariant subgroup of $F(H)$ with $O_p(M) \neq 1$ and $O_q(M) \neq 1$.

Since $C_G(M_q)$ is V-invariant and soluble, and $Z(F(H)) \leqslant C_G(M_q)$, we have $C_G(M_q)_{p,q} \leqslant H$.

Thus $M_p \leqslant O_p(C_G(M_q)_{p,q})$.

Now $C_K(M_q) \leqslant C_G(M_q)_{p,q}$, so $M_p \leqslant O_p(C_K(M_q)) \leqslant O_p(N_K(M_q))$.
By lemma 2.3 again, this gives $M_p \leqslant O_p(K)$.
Thus $O_q(K) \leqslant C_G(M_p)_{p,q} \leqslant H$ since $Z(F(H)) \leqslant C_G(M_p)$.
Similarly $O_p(K) \leqslant H$, so $F(K) \leqslant H$.
Now $O_p(K) \neq 1$ since $M_p \leqslant O_p(K)$, and similarly $O_q(K) \neq 1$.
Thus we can apply the first part of the proof to K to derive $K = H$ and we are done. □

We now prove a result about the Fitting subgroups of members of $\mathcal{H}$.

<u>2.5 LEMMA</u> (Martineau, [15]). Suppose $H \in \mathcal{H}$. Then $O_p(H) \neq 1$ and $O_q(H) \neq 1$.

<u>PROOF</u>

Note that since H is soluble, we must have at least one of $O_p(H)$, $O_q(H)$ non-trivial (see [9], theorem 2.4.1).
W.l.o.g., suppose $O_p(H) = 1$. Then $O_q(H) \neq 1$, and since $N_G(O_q(H))$ is V-invariant and soluble, we have $Z(Q) \leqslant N_G(O_q(H))_{p,q} = H$.
Thus by lemma 2.2 $H \cap P = H \cap X$, so that $H \leqslant QX$, a contradiction. □

Using lemma 2.2, we can now say a great deal about the structure of elements of $\mathcal{H}$.

<u>2.6 LEMMA</u> (Martineau, [15]). If $H \in \mathcal{H}$, then $Z(P) \leqslant H$, $Z(Q) \leqslant H$ and $H = F(H).(H \cap X).(H \cap Y)$.

PROOF

From lemma 2.5 we have $O_q(H) \neq 1$. Applying the same argument as in the proof of that lemma, $Z(Q) \leqslant N_G(O_q(H))_{p,q} = H$. Similarly $Z(P) \leqslant H$. Hence by lemma 2.2 we have $H \cap P = O_p(H).(H \cap X)$ and $H \cap Q = O_q(H).(H \cap Y)$.

So
$$\begin{aligned} H = (H \cap P).(H \cap Q) &= O_p(H).O_q(H).(H \cap X).(H \cap Y) \\ &= F(H).(H \cap X).(H \cap Y). \end{aligned}$$ □

2.7 LEMMA (Martineau, [15]). If $H \in \mathcal{H}$, then $X \cap F(H) = Y \cap F(H) = 1$.

PROOF

Suppose $X \cap O_p(H) \neq 1$.

Then $M = (X \cap O_p(H)).O_q(H) \leqslant XQ$ and $M \leqslant F(H) \leqslant H$. Clearly $O_p(M)$ and $O_q(M)$ are non-trivial by lemma 2.5. So by lemma 2.4 we have $XQ = H$, a contradiction. □

We can now give a refinement of lemma 2.4 for elements of $\mathcal{H}$.

2.8 LEMMA (Martineau [15]). If $H \in \mathcal{H}$ and M is a non-trivial V-invariant subgroup of $F(H)$ with $M \leqslant K \in \mathcal{H}$, then $K = H$.

PROOF

If $pq \,|\, |M|$ then $O_p(M)$, $O_q(M)$ are non-trivial since M is nilpotent, so we can apply lemma 2.4 to deduce that $H = K$.

So w.l.o.g. assume that M is a q-group.

Let $Q_1 = [Z(P), M]$.

By lemma 2.6, $Z(P) \leqslant H$ so $Z(P)$ normalizes $O_q(H)$.

But $M \leqslant O_q(H)$, so $Q_1 \leqslant O_q(H)$ and hence is a q-group.

Also $Z(P) \leqslant K$ by lemma 2.6, so $Z(P) \leqslant Z(P \cap K)$.

Now by [9], theorem 6.3.3, $Z(P \cap K) \leqslant O_{q,p}(K)$, so

$Z(P) \leqslant O_{q,p}(K)$.

But $M \leqslant K$, so $Q_1 = [Z(P), M] \leqslant O_{q,p}(K)$.

As Q_1 is a q-group, this implies that $Q_1 \leqslant O_q(K)$.

Suppose $Q_1 = 1$. Then $M \leqslant C_G(Z(P)_{p,q} \leqslant PY$, so $M \leqslant Y$.

But then $Y \cap O_q(H) \neq 1$, contradicting lemma 2.7.

Thus $Q_1 \neq 1$. Let $M^* = C_{F(H)}(Q_1)$.

Then M^* is a V-invariant subgroup of $F(H)$ with

$O_q(M^*) \neq 1$ (as $Z(Q_1) \neq 1$) and $O_p(M^*) = O_p(H) \neq 1$.

But $M^* \leqslant C_G(Q_1)_{p,q}$ so by lemma 2.4, $C_G(Q_1)_{p,q} \leqslant H$.

As $Q_1 \leqslant O_q(K)$, this implies $O_p(K) \leqslant H$.

Thus $N = O_p(K).Q_1$ is a nilpotent subgroup of $H \cap F(K)$,

so that by lemma 2.4, $H = K$. □

The results listed thus far, and in particular lemma 2.8, are sufficient to obtain our desired result in the next section viz. to show that $|\mathcal{H}| \leqslant 1$ for a minimal counter-example to the special case of the conjecture when $V \cong S_3$ acts f.p.f. on G and $(|G|, 3) = 1$. However, it is possible to gain further information about the structure of

PY and QX and their relationship with elements of $\mathcal{H}$. Most of the results listed below are due to Glauberman and Martineau ([8]). We begin by showing that the factorization of lemma 2.2 holds for PY and QX (this does not follow from lemma 2.2 as we do not have $Z(Q) \leqslant Y$ and $Z(P) \leqslant X$).

2.9 LEMMA (Glauberman and Martineau, [8]).
$P = O_p(PY).X$ and $Q = O_q(QX).Y$.

PROOF

If $O_q(PY) \neq 1$, $Z(Q) \leqslant N_G(O_q(PY))_{p,q} = PY$.
Then by lemma 2.2, $P = O_p(PY).X$.
Thus we may assume that $O_q(PY) = 1$.
Then $O_p(PY) \neq 1$ and $P = O_p(PY).C_p(Z(Y)).N_p(J(Y))$ by hypothesis (C).
Now $N_G(Z(Y))_{p,q} \leqslant H$ for some $H \in K$, and as $Y < N_Q(Y) \leqslant N_G(Z(Y))_{p,q}$, $H \neq PY$.
If $H = QX$, then $N_p(Z(Y)) \leqslant X$.
Suppose $H \neq QX$ so that $H \in \mathcal{H}$ and $Z(Q) \leqslant H$ by lemma 2.6.
Now $Q \leqslant C_G(Z(Q))_{p,q}$, so $C_G(Z(Q))_{p,q} \leqslant QX$.
In particular $C_p(Z(Q)) \leqslant X$.
Since $X \cap O_p(H) = 1$ by lemma 2.7, it follows that $C_{O_p(H)}(Z(Q)) = 1$.
Thus $O_p(H) = C_{O_p(H)}(Z(Q)).[O_p(H),Z(Q)]$ by [9], theorem 5.3.5.
i.e. $O_p(H) = [O_p(H),Z(Q)]$.

Let $Y_1 = Z(Q)O_q(H) \cap Y$.

By lemma 2.2, $Q \cap H = O_q(H).(H \cap Y) = O_q(H).Y$ as $Y \leqslant H$.

Hence $Z(Q).O_q(H) = Y_1 O_q(H)$.

Now $[Y_1, Y] \leqslant [Z(Q)O_q(H), Y] \leqslant [O_q(H), Y] \leqslant O_q(H)$.

Also $[Y_1, Y] \leqslant Y$ and since $Y \cap O_q(H) = 1$, $[Y_1, Y] = 1$ i.e. $Y_1 \leqslant Z(Y)$.

But $Z(Y) \leqslant O_{p,q}(PY)$ by [9], theorem 6.3.3, so that $Y_1 \leqslant O_{p,q}(PY)$.

$$\begin{aligned} \text{Thus} \quad O_p(H) &= [O_p(H), Z(Q)] = [O_p(H), Z(Q)O_q(H)] \\ &= [O_p(H), Y_1 O_q(H)] \\ &= [O_p(H), Y_1] \leqslant [PY, Y_1] \cap P \leqslant O_p(PY). \end{aligned}$$

Hence $N_p(Z(Y)) \leqslant H \cap P = O_p(H).(H \cap X) \leqslant O_p(PY).X$.

A similar argument yields $N_p(J(Y)) \leqslant O_p(PY).X$, so $P = O_p(PY).X$ as required.

The other result follows by symmetry. □

An easy consequence of lemma 2.9 is the following:

<u>2.10 COROLLARY</u> If P and Q do not permute, $O_p(PY) \neq 1$ and $O_q(QX) \neq 1$.

The next lemma highlights the difference between PY, QX and elements of $\mathcal{H}$ (refer to lemma 2.5).

<u>2.11 LEMMA</u> If P and Q do not permute, at least one of $O_q(PY)$, $O_p(QX)$ is trivial.

<u>PROOF</u>

Suppose that both $O_q(PY)$ and $O_p(QX)$ are non-trivial.

Then $Z(P) \leqslant N_G(O_p(QX))_{p,q} = QX$, so $Z(P) \leqslant X$.

But $O_p(PY) \neq 1$ by corollary 2.10, so $O_p(PY) \cap Z(P) \neq 1$.

Thus $T = O_p(PY) \cap X \neq 1$.

Then $TO_q(PY) \leqslant F(PY) \cap QX$, so that $PY = QX$ by lemma 2.4.

This contradiction completes the proof. □

The next lemma is a special result which will be used to find a characteristic property of PY and QX.

2.12 LEMMA (Glauberman and Martineau, [8]).

Let $H \in K$ and suppose there exists a V-invariant subgroup W of $H \cap O_p(PY)$ such that $W \nleqslant X$ and XY normalizes W. Then $XY \leqslant H$.

PROOF

Since $W \nleqslant X$, $H \neq QX$.

If $H = PY$ the result is trivial, so we may suppose that $H \in \mathcal{H}$.

Let $H^* = \langle X, Y, O_q(H)\rangle$. Then $H^* \leqslant QX$ and $WH^* = H^*W$.

So H^*W is a V-invariant $\{p,q\}$-subgroup of G.

Therefore $H^*W \leqslant K$ for some $K \in K$.

As $W \nleqslant X$ and $O_q(H) \nleqslant Y$, $K \neq PY$ or QX. Thus $K \in \mathcal{H}$.

But then $O_q(H) \leqslant K$ so $H = K$ by lemma 2.8.

Hence $XY \leqslant H$, as required. □

We can now prove that PY is the only member of K which contains $O_p(PY)$, and similarly for QX.

<u>2.13 LEMMA</u> (Glauberman and Martineau, [8]).

Suppose that P and Q do not permute. Then

(1) If $H \in K$ contains $O_p(PY)$ then $H = PY$.

(2) If $H \in K$ contains $O_q(QX)$ then $H = QX$.

<u>PROOF</u>

(1) Suppose $O_p(PY) \leqslant H$ for some $H \in K$.

By lemma 2.9, $O_p(PY) \ntrianglelefteq X$.

Since $O_p(PY) \triangleleft PY$, XY normalizes $O_p(PY)$.

So by lemma 2.12 we have $XY \leqslant H$.

Hence $XO_p(PY) = P \leqslant H$, so that $H = PY$.

(2) Follows by symmetry. □

<u>2.14 COROLLARY</u> (Glauberman and Martineau, [8]).

Let $H \in K - \{PY\}$.

Then $O_p(H) \cap Z(P) = O_p(H) \cap C_P(O_p(PY))$

$$= O_p(H) \cap Z(O_p(PY)) = 1,$$

and similarly for the symmetric statement.

<u>PROOF</u>

Let $D = O_p(H) \cap C_P(O_p(PY))$. It is sufficient to show that $D = 1$. Suppose $D \neq 1$.

Now $O_q(H) \neq 1$ if $H \in \mathcal{K}$ by lemma 2.5 or if $H = QX$ by corollary 2.10.

As $DO_q(H) \leqslant N_G(D)_{p,q}$, H is the unique maximal V-invariant $\{p,q\}$-subgroup of G containing $N_G(D)_{p,q}$ by lemma 2.4.

But $O_p(PY) \leqslant C_P(D) \leqslant N_G(D)_{p,q} \leqslant H$, so $H = PY$ by lemma 2.13.

This contradiction completes the proof. □

The next two lemmas give some information about the relationship between the centres of $O_p(PY)$ and $O_q(QX)$, and members of $\mathcal{H}$.

<u>2.15 LEMMA</u> (Glauberman and Martineau, [8]).

Suppose $H \in \mathcal{H}$ and let $Z_P = Z(O_p(PY))$ and $Z_Q = Z(O_q(QX))$. Then $Z_P \cap H \neq 1$, $Z_Q \cap H \neq 1$ and $Z_P \cap H$ centralizes $Z_Q \cap H$.

<u>PROOF</u>

As $Z_P \lhd P$, $Z_P \cap Z(P) \neq 1$.

Thus $Z_P \cap H \neq 1$ by lemma 2.6. Similarly $Z_Q \cap H \neq 1$.

Since Z_Q is normalized by Q and X, $Z_Q \cap H$ is normalized by $Q \cap H$ and $H \cap X$.

Now by lemma 2.6, $H = O_p(H).(H \cap X).(H \cap Q)$, so $O_p(H).(Z_Q \cap H) \lhd H$.

Similarly $O_q(H).(Z_P \cap H) \lhd H$.

Let $I = O_p(H).(Z_Q \cap H) \cap O_q(H).(Z_P \cap H)$.

Then I is V-invariant, so by lemma 2.1 (iv),

$I_p = O_p(H) \cap (Z_P \cap H) = 1$ by corollary 2.14.

Similarly $I_q = 1$, so $I = 1$.

Thus $[O_p(H).(Z_Q \cap H), O_q(H).(Z_P \cap H)] \leqslant I = 1$, so that in particular $[Z_Q \cap H, Z_P \cap H] = 1$ as required. $\square$

<u>2.16 LEMMA</u> (Glauberman and Martineau, [8]).

Suppose $H \in \mathcal{H}$ and let Z_P, Z_Q be defined as in the previous lemma. Then we have the following:

(1) $Z_P \cap H \leqslant X$, $Z_Q \cap H \leqslant Y$.

(2) $O_p(H) = [O_p(H), Z_Q \cap H]$ and $O_q(H) = [O_q(H), Z_P \cap H]$.

(3) $O_p(H) \leq O_p(PY)$ and $O_q(H) \leq O_q(QX)$.

PROOF

(1) As in the proof of the preceding lemma, $Z_P \cap Z(P)$ and $Z_Q \cap Z(Q)$ are non-trivial. Since $P \leq C_G(Z_P \cap Z(P))_{p,q}$, we must have $C_G(Z_P \cap Z(P))_{p,q} \leq PY$, so that $C_Q(Z_P \cap Z(P)) \leq Y$. Now $Z_P \cap Z(P) \leq Z_P \cap H$, so by lemma 2.15 we have $Z_Q \cap H \leq C_Q(Z_P \cap Z(P)) \leq Y$.
Similarly $Z_P \cap H \leq X$.

(2) Since $Z_P \cap Z(P) \leq H$, we have by [9], theorem 5.3.5 that
$O_q(H) = C_{O_q(H)}(Z_P \cap Z(P)).[O_q(H), Z_P \cap Z(P)]$.
But $C_Q(Z_P \cap Z(P) \leq Y$, and since $Y \cap O_q(H) = 1$ by lemma 2.7, we have
$O_q(H) = [O_q(H), Z_P \cap Z(P)] = [O_q(H), Z_P \cap H]$.
The other result follows by symmetry.

(3) By (1), $Z_P \cap Z(P) \leq X$, so that $Z_P \cap Z(P) \leq Z(X)$. But $Z(X) \leq O_{q,p}(QX)$ by [9], theorem 6.3.3, so that $Z_P \cap Z(P) \leq O_{q,p}(QX)$.

$$\text{Hence } O_q(H) = [O_q(H), Z_P \cap Z(P)] \leq [QX, O_{q,p}(QX)] \leq O_{q,p}(QX).$$

It follows that $O_q(H) \leq O_q(QX)$.
By symmetry we also have $O_p(H) \leq O_p(PY)$. □

We show next that the result of lemma 2.16(3) can be extended to elements of K.

2.17 LEMMA (Glauberman and Martineau, [8]).

Let $H \in K$. Then $O_p(H) \leqslant O_p(PY)$ and $O_q(H) \leqslant O_q(QX)$.

PROOF

We show $O_p(H) \leqslant O_p(PY)$; the other result follows by symmetry.

If $H = PY$ the result is trivial, and by lemma 2.16(3) it holds if $H \in \mathcal{H}$.

Hence we may assume that $H = QX$.

As $O_p(H) \lhd QX$, XY normalizes $O_p(H)$.

But by lemma 2.9 we have $P = O_p(PY).X$, so that $PY = O_p(PY)XY$.

It follows that $O_p(PY).O_p(H) \lhd PY$, so $O_p(PY).O_p(H) \leqslant O_p(PY)$.

In particular $O_p(H) \leqslant O_p(PY)$. □

Our final result for this section shows that we can deduce a certain amount of information about PY and QX when $\mathcal{H}$ is non-empty.

2.18 LEMMA (Glauberman and Martineau, [8]).

Suppose $\mathcal{H} \neq \phi$ and let $Z_p = Z(O_p(PY))$ and $Z_Q = Z(O_q(QX))$. Then

(1) $O_p(QX) = O_q(PY) = 1$.

(2) $Z(P) \leqslant Z_P \leqslant X$ and $Z(Q) \leqslant Z_Q \leqslant Y$.

PROOF

(1) Suppose $O_q(PY) \neq 1$.

By corollary 2.10, $O_p(PY) \neq 1$.

Since $O_p(H) \leqslant O_p(PY)$ by lemma 2.16(3), $O_q(PY)$ centralizes $O_p(H)$.

Thus $H = N_G(O_p(H))_{p,q}$ contains $O_p(H) \times O_q(PY)$, so by lemma 2.4 we have $H = PY$, a contradiction.

Hence $O_q(PY) = 1$, and similarly $O_p(QX) = 1$.

(2) Since $O_q(PY) = 1$, by [9], theorem 6.3.3, $Z(P) \leqslant O_p(PY)$.

Thus $Z(P) \leqslant Z(O_p(PY)) = Z_P$.

Now $O_p(H) \leqslant O_p(PY)$ by lemma 2.16(3), so Z_P centralizes $O_p(H)$.

Hence $Z_P \leqslant N_G(O_p(H))_{p,q} = H$.

But by lemma 2.16(1), $Z_P = Z_P \cap H \leqslant X$, so the result follows.

Similarly $Z(Q) \leqslant Z_Q \leqslant Y$. □

CHAPTER THREE

INFORMATION ABOUT A COUNTER-EXAMPLE

In this section we consider the following theorem (we are justified in calling this a theorem rather than a conjecture by corollary 7.3 of [7], mentioned in the introduction):

THEOREM I

Let G be a finite group with $(|G|,3) = 1$ admitting a f.p.f. group of automorphisms $A \cong S_3$. Then G is soluble.

Now let G be a minimal counter-example to the theorem. As was indicated in chapter two, theorem 1.6 and lemmas 1.7 and 1.9 imply that the hypotheses (A), (B) and (C) of that section hold, and the hypotheses of (D) are certainly satisfied for our minimal counter-example. By lemma 2.1(V), there exist primes p and q dividing $|G|$ such that the corresponding A-invariant Sylow subgroups P and Q are not permutable. Using the notation and results of chapter two, we intend in this chapter to show that $|\mathcal{H}| \leqslant 1$. We begin with some relatively easy results.

3.1 LEMMA For $H \in \mathcal{H}$, $N_{O_p(H)}(Q \cap H) = 1 = N_{O_q(H)}(P \cap H)$.

PROOF

Suppose $N_{O_p(H)}(Q \cap H) \neq 1$.

As $N_G(Q \cap H)$ is A-invariant and soluble,

$N_G(Q \cap H)_{p,q} \leqslant L$ for some $L \in K$.

Since $O_p(H) \cap L \neq 1$ and $(O_p(H) \cap L).O_q(H) \leqslant F(H) \cap L$,

we have $H = L$ by lemma 2.4.

But $N_Q(Q \cap H) > Q \cap H$, a contradiction.

Thus $N_{O_p(H)}(Q \cap H) = 1$.

Similarly $N_{O_q(H)}(P \cap H) = 1$. □

3.2 COROLLARY If $H \in \mathcal{H}$, $C_H(\pi) \neq 1$.

PROOF

If $C_H(\pi) = 1$, H is nilpotent by lemma 1.2(4), so that $O_p(H)$ normalizes $Q \cap H$, contradicting the lemma. □

3.3 COROLLARY If $H \in \mathcal{H}$ and $C_{F(H)}(\pi) = 1$, $C_{X\cap H}(\pi) \neq 1$ and $C_{Y\cap H}(\pi) \neq 1$.

PROOF

Suppose $C_{X\cap H}(\pi) = 1$.

Then $C_{X\cap H.O_p(H)}(\pi) = 1$ by lemma 1.20, so that $P \cap H.O_q(H)$ is nilpotent by lemmas 2.2, 1.20 and 1.2(4).

Thus $O_q(H)$ normalizes $P \cap H$, again contradicting the lemma.

Hence $C_{X\cap H}(\pi) \neq 1$ and similarly $C_{Y\cap H}(\pi) \neq 1$. □

To prove that $|\mathcal{H}| \leqslant 1$, it is necessary to first examine those elements of $\mathcal{H}$ for which $C_{F(H)}(\pi) = 1$, and to find a small bound for the number of these. Accordingly, we define $\mathcal{H}_1 = \{H \in \mathcal{H} | C_{F(H)}(\pi) = 1\}$. We will prove that $|\mathcal{H}_1| \leqslant 1$, but for the sake of clarity the argument will be carried out in a sequence of four lemmas. We first dispose of the case where p or q is even:

<u>3.4 LEMMA</u> If p or q is even, $\mathcal{H}_1 = \phi$.

<u>PROOF</u>

If $H \in \mathcal{H}_1$, $C_{X \cap H}(\pi) \neq 1$ and $C_{Y \cap H}(\pi) \neq 1$ by corollary 3.3.
It then follows from lemma 1.1(2) that both p and q are odd. □

Thus we may assume for the remainder of the argument that p,q are odd and w.l.o.g. $p < q$. Then we have:

<u>3.5 LEMMA</u> $|C_{O_p(H)}(\tau)| > p \quad \forall H \in \mathcal{H}_1$

<u>PROOF</u>

Let $T_1 = C_{O_p(H)}(\tau)$.
Then $T_1 \neq 1$ by lemma 1.2(2). Suppose $|T_1| = p$. As τ must centralize some element of $\Omega_1(Z(O_p(H)))$ by lemma 1.2(3), it follows that $T_1 \leqslant \Omega_1(Z(O_p(H)))$. But then A is a regular group of automorphisms of $O_p(H)/\Omega_1(Z(O_p(H)))$, so that $O_p(H) = \Omega_1(Z(O_p(H)))$ is elementary abelian (by lemma 1.2(3) again).

Now by lemma 1.3(2), $O_p(H)$ has order p^2.
But $H \cap Y$ is a q-group of automorphisms of $O_p(H)$, and as $p < q$ it follows that $H \cap Y$ centralizes $O_p(H)$.
But then $O_p(H)$ normalizes $H \cap Y.O_q(H) = H \cap Q$, contradicting lemma 3.1.
Thus $|T_1| > p$ as required. □

We show next that every element of $\mathcal{H}_1$ contains a subgroup of P which has a certain property.

<u>3.6 LEMMA</u> $\cap\, \mathcal{H}_1$ contains an A-invariant subgroup M of P with either $C_M(\pi) = M$ and M elementary abelian of order at least p^3 or $C_M(\pi) \neq M$.

<u>PROOF</u>

Let $1 \subset Z_1 \subset Z_2 \subset \ldots \subset P$ be the upper central series of P, and let $H \in \mathcal{H}_1$ be arbitrary.
We show first that if Z_j is centralized by π then $Z_j \leqslant H$.
Since Z_j char P, $Z_jO_p(H)$ is an A-invariant p-group.
Then by lemma 1.4(2), Z_j normalizes $O_p(H)$.
Thus $Z_j \leqslant N_G(O_p(H))_{p,q} = H$.
Now let i be the smallest integer such that Z_i is not centralized by π.
If $i = 1$, $Z(P)$ is not centralized by π and $Z(P) \leqslant H$ by lemma 2.6.
Thus we may assume $i > 1$.

Hence Z_{i-1} is centralized by π and $Z_{i-1} \leq H$ by the above comment.

Since Z_{i-1} is abelian by lemma 1.1(2), we may assume that $\Omega_1(Z_{i-1})$ is elementary abelian of order at most p^2. Suppose first that $|\Omega_1(Z_{i-1})| = p^2$.

Then we can write $\Omega_1(Z_{i-1}) = \langle v\rangle \times \langle w\rangle$ where $v \in Z(P)$.

By lemma 1.2(2), $C_{Z_i}(\tau) \neq 1$ and hence

$C_{Z_i}(\pi) \subset I = \{x \in Z_i | x^\tau = x^{-1}\}$ by lemma 1.5.

Choose $t \in I - C_{Z_i}(\pi)$.

Then t normalizes $\Omega_1(Z_{i-1})$, so $t^{-1}wt \in \Omega_1(Z_{i-1})$ and hence is inverted by τ by lemma 1.1(2).

$$\text{i.e.} \quad tw^{-1}t^{-1} = t^{-1}w^{-1}t .$$

Thus t^2, and hence t, centralizes w.

So t centralizes $\Omega_1(Z_{i-1})$.

But now $\Omega_1(Z_{i-1})$ is a non-cyclic abelian group of automorphisms of $O_q(H)$, so $O_q(H) = \prod_{x\in\Omega_1(Z_{i-1})} C_{O_q(H)}(x)$

by [9], theorem 5.3.16.

Thus $\exists y \in \Omega_1(Z_{i-1})$ such that $C_{O_q(H)}(y) \neq 1$.

i.e. $C_G(y)_{p,q} \cap O_q(H) \neq 1$.

Since $y \in H \cap P$ and $Z(H \cap P) \cap O_p(H) \neq 1$,

$C_G(y)_{p,q} \cap O_p(H) \neq 1$.

So by lemma 2.4, $C_G(y)_{p,q} \leq H$.

In particular, $t \in H$.

Now if M is the smallest A-invariant subgroup of P containing t then clearly $C_M(\pi) \neq M$ and $M \leq H$.

Thus we may assume that $|\Omega_1(Z_{i-1})| = p$ i.e. Z_{i-1} is cyclic.

Let $Z_{i-1} = \langle z \rangle$. Since Z_{i-1} and $O_p(H)$ normalize each other and intersect trivially (since $C_{O_p(H)}(\pi) = 1$ by definition), they must centralize each other.

As Z_i/Z_{i-1} is abelian, $Z_i/Z_{i-1} = C_{Z_i/Z_{i-1}}(\pi) \times S/Z_{i-1}$ where S/Z_{i-1} is A-invariant and $C_{S/Z_{i-1}}(\pi) = 1$ by lemma 1.3(1).

By [9], theorem 6.2.2(iv), $C_{Z_i/Z_{i-1}}(\pi) = C_{Z_i}(\pi)/Z_{i-1}$, so that S/Z_{i-1} is non-trivial and by the same theorem applied to S/Z_{i-1} we have $C_S(\pi) = Z_{i-1}$.

By [9], theorem 10.4.1(i), $S = C_S(\tau).I$ where $I = \{x \in S | x^\tau = x^{-1}\}$.

Now $C_S(\tau) \neq 1$ by lemma 1.2(2), so $Z_{i-1} = C_S(\pi) \subset I$ by lemma 1.5.

Choose $s \in I - Z_{i-1}$.

Then $sZ_{i-1} \neq Z_{i-1}$, so we can choose $t \in \langle s \rangle$ such that tZ_{i-1} has order p.

Clearly $t \in I - C_{Z_i}(\pi)$.

Since t normalizes $Z_{i-1} = \langle z \rangle$, $t^{-1}zt$ is inverted by τ.

i.e. $tz^{-1}t^{-1} = t^{-1}z^{-1}t$.

Thus t^2, and hence t, centralizes z.

Now define $W = \{y^{-1}y^t | y \in T_1{}^\pi\}$ where $T_1 = C_{O_p(H)}(\tau)$.

As $t \in Z_i$, $y^{-1}y^t = z^k$ for some k such that $1 \leqslant k \leqslant |z| = p^\alpha$ say.

i.e. $y^t = yz^k$.

Now $t^p \in Z_{i-1}$ and $y \in O_p(H)$ so $y^{t^p} = y$.

$\therefore \quad y = y^{t^p} = yz^{kp}$.

Hence $p^\alpha | kp$, so that $k = mp^{\alpha-1}$ for $1 \leqslant m \leqslant p$.

It follows that $|W| \leqslant p$.

But by lemma 3.5, $|T_1{}^\pi| > p$, so $\exists u,v \in T_1$ such that $u \neq v$ and

$$u^{-\pi}(u^\pi)^t = v^{-\pi}(v^\pi)^t$$

$$\therefore \qquad (vu^{-1})^\pi = [(vu^{-1})^\pi]^t$$

i.e. $\quad t$ centralizes $(vu^{-1})^\pi = x^\pi$ say.

Thus $t^\tau = t^{-1}$ centralizes $x^{\pi\tau} = x^{\tau\pi^2}$

$$= x^{\pi^2} \quad \text{as} \quad x = vu^{-1} \in T_1.$$

Hence t centralizes x^{π^2}.

But $x \in O_p(H)$, so $x^{\pi^2} = x^{-1}x^{-\pi}$ by [9], theorem 10.1.1.

Thus $R = \langle x^\pi, x^{\pi^2}\rangle = \langle x, x^\pi\rangle$ is an A-invariant abelian subgroup of $O_p(H)$ (by lemma 1.2(5)) which is centralized by t.

As $C_G(R)_{p,q} \geqslant RO_q(H)$, by lemma 2.4 we have $C_G(R)_{p,q} \leqslant H$.

Hence $t \in H$. Again take M to be the smallest A-invariant subgroup of P containing t, so that $M \leqslant H$ and $C_M(\pi) \neq M$.

As $H \in \mathcal{H}_1$ was arbitrary, the result follows. □

With the aid of this lemma, we can now complete the proof of our first main result:

3.7 LEMMA $|\mathcal{H}_1| \leqslant 1$.

PROOF

Suppose first that $\exists H \in \mathcal{H}_1$ with $R = M \cap O_p(H) \neq 1$.

As $N_G(R)_{p,q} \geqslant RO_q(H)$, by lemma 2.4 we have

$N_G(R)_{p,q} \leqslant H$.

Now if $K \in \mathcal{H}_1$, $M \leqslant K \cap P$ and since $Z(K \cap P) \cap O_p(K) \neq 1$,

$C_{O_p(K)}(M) \neq 1$.

Thus $O_p(K) \cap N_G(R)_{p,q} \neq 1$ so by lemma 2.8 we have $H = K$.

Hence $|\mathcal{H}_1| = 1$.

Thus we may assume that $M \cap F(H) = 1 \quad \forall H \in \mathcal{H}_1$.

Now by theorem 1.11, H' is nilpotent, so that

$H' \leqslant F(H)$.

Thus $H/F(H)$, and hence $MF(H)/F(H)$, is abelian.

But $MF(H)/F(H) \cong M/M \cap F(H) \cong M$ so M is abelian.

Suppose first that $C_M(\pi) \neq M$.

Since M is abelian, $M = C_M(\pi) \times N$ where N is A-invariant and $C_N(\pi) = 1$ by lemma 1.3(1).

Then $NO_q(H)$ is an A-invariant group on which π acts f.p.f.

Thus $NO_q(H)$ is nilpotent by lemma 1.2(4), so that

$O_q(H)$ centralizes N.

But $N \leqslant H \cap P$ and $Z(H \cap P) \cap O_p(H) \neq 1$, so

$C_{O_p(H)}(N) \neq 1$.

Hence by lemma 2.4, H is the unique maximal A-invariant $\{p,q\}$-subgroup of G containing $C_G(N)_{p,q}$.

It follows that $|\mathcal{H}_1| \leqslant 1$.

Now suppose that π centralizes M and M is elementary abelian of order at least p^3.

Let $H,K \in \mathcal{H}_1$. Then M is a non-cyclic abelian group of automorphisms of $O_q(H)$ and $O_q(K)$, so $O_q(H) = \langle C_{O_q(H)}(B) \mid [M:B] = p\rangle$ by lemma 1.21, and similarly for $O_q(K)$.

Thus there exist subgroups B,C of M of index p such that $C_{O_q(H)}(B) \neq 1$ and $C_{O_q(K)}(C) \neq 1$.

Let $u \in B \cap C$.

Then $C_G(u)_{p,q} \leqslant L$ for some $L \in \mathcal{K}$.

But $C_{O_q(H)}(u) \geqslant C_{O_q(H)}(B) \neq 1$ and $C_{O_p(H)}(u) \neq 1$ since $O_p(H) \cap Z(H \cap P) \neq 1$. So by lemma 2.4, $H = L$.

Similarly $K = L$, so that $H = K$ and $|\mathcal{H}_1| \leqslant 1$. □

We are now in a position to prove the main result of this chapter:

3.8 THEOREM

$|\mathcal{H}| \leqslant 1$.

PROOF

Let $H,K \in \mathcal{H}$ with $H \neq K$.

By lemma 3.7 we must have one of the following (w.l.o.g.)

I $\quad C_{F(H)}(\pi) \neq 1$ and $C_{F(K)}(\pi) \neq 1$.

II $\quad C_{F(H)}(\pi) = 1$ and $C_{F(K)}(\pi) \neq 1$.

Suppose first that I holds, and assume w.l.o.g. that

$P_O = C_{O_p(H)}(\pi) \neq 1.$

Then $C_G(P_O)$ is A-invariant and soluble and

$C_G(P_O)_{p,q} \geqslant P_O.O_q(H).$

Thus by lemma 2.4, H is the unique maximal A-invariant

{p,q}-subgroup of G containing $C_G(P_O)_{p,q}$.

But $C_G(\pi)$ is abelian by lemma 1.1(2), so

$C_{F(K)}(\pi) \leqslant C_G(P_O)_{p,q}$.

Thus $F(K) \cap H \neq 1$, so by lemma 2.8, $H = K$ after all.

Suppose next that II holds, and assume w.l.o.g. that

$C_{O_p(K)}(\pi) \neq 1.$

Let $N = C_{Q\cap H}(\pi)$. Then $N \neq 1$ by corollary 3.3.

As $C_G(N)$ is A-invariant and soluble, $C_G(N)_{p,q} \leqslant L$

for some $L \in \mathcal{K}$.

Since $C_G(\pi)$ is abelian, $C_{O_p(K)}(\pi) \leqslant C_G(N)_{p,q}$.

Thus $O_p(K) \cap L \neq 1$ so that $L \neq QX$ by lemma 2.7.

Also $Z(Q \cap H) \cap O_q(H) \neq 1$, so $C_{O_q(H)}(N) \neq 1$. Hence

$O_q(H) \cap L \neq 1$, so that $L \neq PY$ (again by lemma 2.7).

Thus $L \in \mathcal{H}$, and now $H = L$ by lemma 2.8 because

$O_q(H) \cap L \neq 1$, and similarly $K = L$ because

$O_p(K) \cap L \neq 1$. Thus $H = K$.

This contradiction completes the proof. □

CHAPTER FOUR

FURTHER INFORMATION ABOUT A MINIMAL COUNTER-EXAMPLE

As mentioned in the introduction, the approach pioneered by Martineau to deduce the solubility of a group G admitting a f.p.f. group of automorphisms from information about maximal A-invariant $\{p,q\}$-groups does not appear to be a fruitful way of attempting a solution in the case when $A \cong S_3$, despite the fact that in Chapter 3 we were able to show that there were very few such subgroups in a minimal counter-example.

As it happens, the approach which eventually proved to be successful in leading to a solution in this case was to examine the structure of A-invariant maximal subgroups of a minimal counter-example using Glauberman's results ([5]), and in this Chapter we present some preliminary results which will be used in the main argument presented in the following chapters. We consider the following theorem:

THEOREM II

Let G be a finite group of order coprime to 6 which admits a f.p.f. group of automorphisms $A \cong S_3$. Then G is soluble.

Throughout this chapter (and succeeding chapters) we assume that G is a minimal counter-example to this theorem, so that as in Chapter Three the hypotheses

(A), (B), (C) and (D) of Chapter Two are satisfied. We show first that the normalizer of an A-invariant Sylow subgroup of G is a maximal A-invariant subgroup of G.

<u>LEMMA 4.1</u> If M is a maximal A-invariant subgroup of G such that $N_G(ZJ(P)) \leqslant M$ for a Sylow p-subgroup P of G then $P \triangleleft M$.

<u>PROOF</u>

Let $K = N_G(ZJ(P))$ and suppose that $p \mid |M/F(M)|$. Then $p \mid |KF(M)/F(M)|$, so that $p \mid |K/K \cap F(M)|$. Now by theorem 1.11 $M/F(M)$ is abelian, so $K/K \cap F(M)$ is also abelian. Thus $O^p(K) \neq K$.
Hence by Corollary 2.2 of [5] $O^p(G) \neq G$, a contradiction.

Thus $P \leqslant F(M)$ so that $P \triangleleft M$. □

The next three lemmas provide information about maximal A-invariant subgroups of G which have non-trivial intersection.

<u>LEMMA 4.2</u> If H and M are distinct maximal A-invariant subgroups of G such that $F(M) \leqslant H$ then $H \cap M = F(M)$.

<u>PROOF</u>

Let $K = H \cap M$ and suppose that $F(M) \subset K$. Now $[K,F(M)] \neq 1$, else $K \leqslant F(M)$ by [9], theorem 6.1.3. Thus $\exists$ a prime p such that $O_p(M) \neq 1$ and $[K,O_p(M)] = P_2 \neq 1$.

As $F(M) \subset K$, $K \lhd M$ by theorem 1.11. Hence $P_2 \lhd M$.
And $P_2 \leqslant K' \leqslant H'$, so $P_2 \leqslant F(H)$ by the same theorem.
Therefore $O_{p'}(H) \leqslant C_G(P_2) \leqslant N_G(P_2) = M$.
It follows that $O_{p'}(H) \leqslant M \cap H = K$, so that
$O_{p'}(H) \leqslant O_{p'}(K)$.
If $K_0 = [K, O_{p'}(K)] \neq 1$ then $K_0 \leqslant F(H) \cap O_{p'}(K) \leqslant O_{p'}(H)$.
As $K \lhd M$, $K_0 \lhd M$ and since $O_p(H) \leqslant C_G(K_0) \leqslant M$ we have $O_p(H) \leqslant K$.
Thus $F(H) \leqslant K$, so that $K \lhd H$.
But then $H = N_G(K) = M$, a contradiction.
Therefore we must have $K_0 = 1$ i.e. $O_{p'}(K) \leqslant Z(K)$.
Let P_1 be the A-invariant Sylow p-subgroup of K.
Then $F(M) \leqslant F(K) \leqslant P_1 O_{p'}(K)$, so $P_1 O_{p'}(K) \lhd K$.
It follows that $P_1 \lhd K$, so $P_1 \lhd M$.
Thus $M = N_G(P_1)$, so that $N_H(P_1) \leqslant H \cap M = K$.
Hence P_1 must be a Sylow p-subgroup of H.
Thus $F(H) \leqslant P_1 . O_{p'}(H) \leqslant K$, so that again $K \lhd H$.

This contradiction completes the proof. $\square$

<u>LEMMA 4.3</u> Suppose that H and M are distinct maximal A-invariant subgroups of G with $F(M) \leqslant H$. Then $F(M)$ is abelian, $Z(H) = F(H) \cap F(M)$, $H = F(M).F(H)$ and $F(H) = Z(H) \times H_0$ where H_0 is an A-invariant group of order coprime to $|F(M)|$.

<u>PROOF</u>

Let q be a prime dividing $|F(M)|$ and let $Q_0 = O_q(M)$.
Then $N_H(Q_0) = F(M)$ by lemma 4.2.

Thus Q_0 is a Sylow q-subgroup of H.

Now $Q_0.F(H) \triangleleft H$ by theorem 1.11, so by [9], theorem 1.3.7, we have $H = N_H(Q_0).Q_0F(H)$

$$= F(M).F(H).$$

If $Q_0' \neq 1$, $Q_0' \leqslant H' \leqslant F(H)$.

Thus $[Q_0', O_{q'}(F(H))] = 1$, so $Q_0' \triangleleft N_H(Q_0).F(H) = H$.

But $Q_0' \triangleleft M$, so we must have $Q_0' = 1$.

Thus $F(M)$ is abelian.

Let r be a prime dividing $|F(H) \cap F(M)|$ and let R_0 be the A-invariant Sylow r-subgroup of $F(H) \cap F(M)$.

Then as above $O_r(M)$ is a Sylow r-subgroup of H, so R_0 is a Sylow r-subgroup of $F(H)$.

Since R_0 and $F(M)$ are abelian, $R_0 \leqslant Z(H)$.

Now $Z(H)$ centralizes $F(H)$ and $F(M)$, so $Z(H) \leqslant F(H) \cap F(M)$ by [9], theorem 6.1.3.

Hence $Z(H) = F(H) \cap F(M)$.

We have shown above that if q is a prime dividing $|F(M)|$, $O_q(M)$ is a Sylow q-subgroup of H.

Since $Z(H) = F(H) \cap F(M)$, it follows that $(|F(H)/Z(H)|, |Z(H)|) = 1$.

Thus we can write $F(H) = Z(H) \times H_0$ where H_0 is A-invariant and clearly $(|F(M)|, |H_0|) = 1$. $\square$

<u>LEMMA 4.4</u> Let q be a prime dividing $|G|$, Q the A-invariant Sylow q-subgroup of G and $M = N_G(Q)$. If Q is contained in another maximal A-invariant subgroup H of G then Q is abelian, H has a normal q-complement and $Q \cap F(H) \leqslant Z(H)$.

PROOF

By theorem 1.11, $Q.F(H) \lhd H$.

So by [9], theorem 1.3.7, $H = N_H(Q).F(H)$
$= N_H(Q).O_{q'}(F(H))$.

Now $Q' \lhd N_H(Q)$ and $Q' \leqslant F(H)$ by theorem 1.11 so that $[Q',O_{q'}(F(H))] = 1$. Hence $Q' \lhd H$.

But $Q' \lhd M$, so we must have $Q' = 1$. i.e. Q is abelian.

Clearly we can write $H = Q.B$ where B is a q'-group, so if $Q \cap F(H) = 1$, $[Q,B] \leqslant F(H) \leqslant B$.

Thus H has a normal q-complement B and $Q \cap F(H) \leqslant Z(H)$.

If $Q \cap F(H) \neq 1$, $F(M) \leqslant N_G(Q \cap F(H)) \leqslant H$, so $H \cap M = F(M)$ by lemma 4.2.

Thus $N_H(Q) = H \cap M = F(M)$.

But $[Q,F(M)] = 1$, so $N_H(Q) = C_H(Q)$.

Thus by [9], theorem 7.4.3, H has a normal q-complement.

Finally, $[Q \cap F(H),O_{q'}(H)] = 1$ and Q is abelian so $Q \cap F(H)$ centralizes $Q.O_{q'}(H) = H$. i.e. $Q \cap F(H) \leqslant Z(H)$. □

The next lemma will be used frequently in conjunction with lemma 1.16 to show that for any A-invariant Sylow p-subgroup P of G, $C_P(\pi) \subset C_P(C_P(\pi))$.

LEMMA 4.5 Let p be a prime dividing $|G|$ and let P be the A-invariant Sylow p-subgroup of G. Then $C_P(\pi) \subset P$.

PROOF

Suppose $C_P(\pi) = P$ and let $G^* = G.\langle\pi\rangle$, the semi-direct product of G by $\langle\pi\rangle$.

If $C_G(P) = N_G(P)$, G has a normal p-complement by [9], theorem 7.4.3.

Thus we may assume that $C_G(P) \subset N_G(P)$.

Hence $\langle\pi\rangle \leqslant C_{G^*}(P) \subset N_{G^*}(P)$ and $N_{G^*}(P) = N_G(P).\langle\pi\rangle$ is soluble.

By [9], theorem 1.3.7, $N_{G^*}(P) = L.C_{G^*}(P)$ where

$L = N_{N_{G^*}(P)}(\langle\pi\rangle) = N_{N_G(P)}(\langle\pi\rangle).\langle\pi\rangle$.

Hence $N_{G^*}(P) = L_{3'}.C_{G^*}(P)$ where $L_{3'} = N_{N_G(P)}(\langle\pi\rangle) \leqslant G$.

But $[L_{3'},\langle\pi\rangle] \leqslant G \cap \langle\pi\rangle = 1$, so $L_{3'} \leqslant C_G(\pi) \leqslant C_G(P)$.

This contradiction completes the proof. □

The next two lemmas exhibit conditions under which certain A-invariant subgroups of G are contained in specific maximal A-invariant subgroups of G.

LEMMA 4.6 Let T be an A-invariant subgroup of G and suppose that T contains a non-abelian Sylow p-subgroup P of G for some prime p. Then $T \leqslant N_G(P)$.

PROOF

Suppose $T \leqslant M^*$, a maximal A-invariant subgroup of G.

Then $P \leqslant M^*$, so by lemma 4.4 $M^* = N_G(P)$. □

LEMMA 4.7 Let p be a prime dividing $|G|$, P the A-invariant Sylow p-subgroup of G and K a subgroup of P containing $Z(P)$. Then $N_G(K) \leqslant N_G(P)$.

PROOF

Since $Z(P) \leqslant K$, $\forall g \in N_G(K)$ we have $Z(P)^g \leqslant K \leqslant P$

Thus $Z(P)^g = Z(P)$ by [5], Corollary 2.1(a).

Hence $g \in N_G(Z(P)) = N_G(P)$, so $N_G(K) \leqslant N_G(P)$. □

The next lemma, which is really a corollary of lemma 4.7, will be used frequently.

LEMMA 4.8 Let p be a prime dividing $|G|$, P the A-invariant Sylow p-subgroup of G and V a subgroup of P. Then $C_P(V)$ is a Sylow p-subgroup of $C_G(V)$.

PROOF

Let P^* be a Sylow p-subgroup of $C_G(V)$ containing $C_P(V)$.

Since $Z(P) \leqslant C_P(V)$ we have $N_G(C_P(V)) \leqslant N_G(P)$ by lemma 4.7.

In particular $N_{P*}(C_P(V)) \leqslant P$, so that $N_{P*}(C_P(V)) = C_P(V)$.

Hence $P^* = C_P(V)$ i.e. $C_P(V)$ is a Sylow p-subgroup of $C_G(V)$. □

The final two lemmas of this chapter are technical results which are necessary for our later argument.

LEMMA 4.9 Let p be a prime dividing $|G|$, P the A-invariant Sylow p-subgroup of G and $M = N_G(P)$. Suppose that $M = C_M(\pi).F(M)$ and that $\exists$ a subgroup P^* of P such that $C_M(P^*) \nleqslant F(M)$ and $C_{P*}(\tau) \neq 1$. Then for some prime $t \mid |M/F(M)|$, if T is the A-invariant Sylow t-subgroup of G we have:

(1) $\exists x \in C_{T\cap M}(\pi) - F(M)$ such that $p \mid |C_G(x)|$.

(2) If B is a maximal A-invariant subgroup of G containing $C_G(x)$ then $1 \neq C_{T\cap B}(\pi) \neq T \cap B$ and $C_{P\cap B}(\pi) \neq P \cap B$.

PROOF

As $C_M(P^*) \not\leq F(M)$, $\exists y \in M - F(M)$ such that y centralizes P^*.

W.l.o.g., we may assume that y is a t-element for some prime $t \mid |M/F(M)|$.

(1) By theorem 1.11, $T \cap M.F(M) \lhd M$ so by [9], theorem 1.3.7, $M = N_M(T \cap M).F(M)$.

Thus $(T \cap M)' \lhd M$, so if $(T \cap M)' \neq 1$ we have $T \cap M = T$.

But $t \mid |M/F(M)|$, so $M \neq N_G(T)$.

Hence T is abelian by lemma 4.4, a contradiction.

Thus $(T \cap M)' = 1$. i.e. $T \cap M$ is abelian.

Now $\exists g \in M$ such that $y^g \in T \cap M$ and by lemma 1.3 we can write $y^g = xz$ for $x \in C_{T\cap M}(\pi)$ and $z \in T \cap F(M)$.

But then y^g and z centralize $(P^*)^g$, so x must also.

Hence $p \mid |C_G(x)|$.

(2) Since $x \in C_T(\pi)$, $C_T(x) > C_T(\pi)$ by lemmas 1.16 and 4.5.

Hence $1 \neq C_{T\cap B}(\pi) \neq T \cap B$.

Suppose that $C_{P\cap B}(\pi) = P \cap B$.

Now $(P^*)^g \leq C_G(x) \leq B$, so $(P^*)^g \leq P \cap B$.

Thus $(P^*)^g \leqslant C_P(\pi)$, so $\exists v \in P^*$ such that $v^\tau = v$ and $(v^g)^\tau = (v^g)^{-1}$.

But then $g^\tau g^{-1}$ inverts v, a contradiction since $|G|$ is odd.

Hence $C_{P\cap B}(\pi) \neq P \cap B$ as required. □

<u>LEMMA 4.10</u> Let p be a prime dividing $|G|$, P the A-invariant Sylow p-subgroup of G, $M = N_G(P)$ and $P^* = ZJ(C_P(\tau))$. If $M = C_M(\pi).F(M)$ and $C_M(P^*) \leqslant F(M)$ then $C_G(\tau)$ has a normal p-complement.

<u>PROOF</u>

Let $P_0 = C_P(P^*)$.

By lemma 4.7, $N_G(ZJ(P_0)) \leqslant M$ and hence $N_G(ZJ(P_0)) \cap C_G(P^*) \leqslant F(M)$ and so has a normal p-complement.

Now by lemma 4.8 P_0 is a Sylow p-subgroup of $C_G(P^*)$, so by [5], theorem D, $C_G(P^*)$ has a normal p-complement.

As P_0 is a Sylow p-subgroup of $C_G(P^*)$ and $C_G(P^*) \lhd N_G(P^*)$, $N_G(P^*) = (N_G(P_0) \cap N_G(P^*)).C_G(P^*)$ by [9], theorem 1.3.7.

But by lemma 4.7 $N_G(P_0) \leqslant M$ so $N_G(P^*) = N_M(P^*).C_G(P^*)$.

It now follows from [9], theorem 6.2.2 that

$N_G(P^*) \cap C_G(\tau) = (N_M(P^*) \cap C_G(\tau)).(C_G(P^*) \cap C_G(\tau))$.

Clearly $N_M(P^*) \cap C_G(\tau) \leqslant F(M)$, so

$$N_M(P^*) \cap C_G(\tau) = [N_P(P^*) \times O_{p'}(F(M))] \cap C_G(\tau).$$

But $O_{p'}(F(M)) \leqslant C_G(P^*)$, so we have

$$
\begin{aligned}
N_G(P^*) \cap C_G(\tau) &= (N_P(P^*) \cap C_G(\tau)).(C_G(P^*) \cap C_G(\tau)) \\
&= (N_P(P^*) \cap C_G(\tau)).O_{p'}(C_G(P^*) \cap C_G(\tau)) \\
&= (N_P(P^*) \cap C_G(\tau)).O_{p'}(N_G(P^*) \cap C_G(\tau)).
\end{aligned}
$$

Thus $N_G(P^*) \cap C_G(\tau)$ has a normal p-complement. Therefore by [5], theorem D, $C_G(\tau)$ has a normal p-complement. □

CHAPTER FIVE

PRELIMINARY REDUCTION

In this chapter we commence the proof of the main theorem by demonstrating that the structure of maximal A-invariant subgroups of a minimal counter-example is restricted in certain ways. Throughout, G will be a minimal counter-example to Theorem II, and for the sake of clarity the argument is presented in a series of lemmas.

We begin with a lemma which provides some basic information from which the argument in this chapter is derived.

LEMMA 5.1 Let p be a prime dividing $|G|$, P the A-invariant Sylow p-subgroup of G and $M = N_G(P)$. Suppose there exists a maximal A-invariant subgroup $H \neq M$ such that $P_1 = P \cap H \neq 1$. Let $P_0 = P \cap F(H)$. Then:

(i) P_1 is abelian.

(ii) If $P_0 \neq 1$, P_1 is self-centralizing in P and $C_G(P_1) = P_1 \times O_{p'}(C_G(P_1))$.

(iii) If $P_0 \neq 1$, $N_G(P_1) \leqslant M$.

(iv) If $P_0 \neq 1$, $O_{p'}(M) = O_{p'}(N_G(P_1)) = O_{p'}(C_G(P_1))$.

(v) If $P_1 < P$, $P_0 \cap Z(P) = 1$.

(vi) If $Z(P) \leqslant P_1 \neq P$, q is a prime dividing $|H|$ and Q_1 is the A-invariant Sylow q-subgroup of H then $Z(P) \leqslant N_G(Q_1)$.

(vii) If $Z(P) \leqslant P_1 \neq P$, q is a prime dividing $|H|$, Q_1 is the A-invariant Sylow q-subgroup of H

and $Q_0 = Q_1 \cap F(H)$ then $Q_1 = C_{Q_1}(Z(P)).Q_0$.

(viii) If in addition to (vii) we have $P_0 \neq 1$ and $O_{p'}(M) = 1$ then $C_{Q_0}(Z(P)) = 1$.

PROOF

(i) By theorem 1.11 $P_1F(H) \triangleleft H$, so by [9], theorem 1.3.7, $H = N_H(P_1).F(H)$.
Since $P_1' \leq P_0$, we must have $P_1' \triangleleft H$.
If $P_1' \neq 1$, this gives $N_G(P_1') = H$ so that $P_1 = P$.
But then P is abelian by lemma 4.4, so in either case P_1 is abelian.

(ii) Since $C_P(P_1) \leq N_G(P_0) = H$, we must have $C_P(P_1) = P_1$ i.e. P_1 is self-centralizing in P.
The second assertion then follows from [9], theorem 7.4.3.

(iii) If $P_1 = P$ the result is trivial, so assume that $P_1 \neq P$.
Then $P_1 = N_P(P_0) > P_0$.
Suppose $N_G(P_1) \leq M^*$, a maximal A-invariant subgroup of G.
Let P^* be the A-invariant Sylow p-subgroup of M^*.
Then as above $M^* = N_{M^*}(P^*).F(M^*)$ and so $P^{*'} \triangleleft M^*$.
If $M^* \neq M$, it follows that $P^{*'} = 1$.
But then $N_P(P_1)$ is abelian by (i), contradicting (ii).
Thus $M^* = M$ i.e. $N_G(P_1) \leq M$.

(iv) By lemma 2.3, $O_{p'}(N_M(P_1)) \leq O_{p'}(M)$.

It follows from (iii) that $N_M(P_1) = N_G(P_1)$,

so $O_{p'}(N_G(P_1)) \leq O_{p'}(M)$.

But $O_{p'}(M) \leq N_G(P_1) \leq M$, so $O_{p'}(M) \leq O_{p'}(N_G(P_1))$.

Thus $O_{p'}(M) = O_{p'}(N_G(P_1))$.

Clearly $O_{p'}(N_G(P_1)) \leq C_G(P_1)$ so that

$O_{p'}(N_G(P_1)) = O_{p'}(C_G(P_1))$.

Thus $O_{p'}(M) = O_{p'}(N_G(P_1)) = O_{p'}(C_G(P_1))$.

(v) Clearly we may assume that $P_0 \neq 1$ and we have

$P_0 < N_P(P_0) = P_1$.

Then by (ii), $Z(P) \leq P_1$. Suppose that $P_0 \cap Z(P) \neq 1$.

Then $C_G(P_0 \cap Z(P)) \geq F(H).P_1 > F(H)$ so by lemma 4.2, $N_G(P_0 \cap Z(P)) \leq H$.

But $C_P(P_0 \cap Z(P)) = P$, a contradiction.

Thus $P_0 \cap Z(P) = 1$.

(vi) As above, $H = N_H(Q_1).F(H)$.

If $P_0 = 1$, $Z(P) \leq P_1 \leq N_H(Q_1)$.

Thus we may assume that $P_0 \neq 1$.

Now $N_G(P_1) \leq M$ by (iii), and since

$M = N_G(P) = N_G(Z(P))$ we have $N_{Q_1}(P_1) \leq N_G(Z(P))$.

Thus $[N_{Q_1}(P_1), Z(P)] \leq Z(P) \cap F(H) = 1$ by (v).

Now $H = N_H(P_1).F(H)$, so $Q_1 = N_{Q_1}(P_1).Q_1 \cap F(H)$.

Clearly $Z(P)$ normalizes $Q_1 \cap F(H)$, so $Z(P)$ normalizes Q_1.

(vii) Since $H = N_H(P_1).F(H)$, $\exists$ an A-invariant Sylow q-subgroup Q_2 of $N_H(P_1)$ such that $Q_1 = Q_2.Q_0$.

Now $[Q_2, Z(P)] \leqslant P_1 \cap F(H) = P_0$.

And $Q_2 \leqslant N_G(P_1) \leqslant M$ by (iii), so Q_2 normalizes $Z(P)$.

Thus $[Q_2, Z(P)] \leqslant P_0 \cap Z(P) = 1$ by (v).

Hence $Q_1 = C_{Q_1}(Z(P)).Q_0$.

(viii) Suppose $C_{Q_0}(Z(P)) \neq 1$. Then $C_{Q_0}(Z(P)) \leqslant M = N_G(P)$.

As $C_{Q_0}(Z(P)) \leqslant H$, $C_{Q_0}(Z(P)) \leqslant N_G(P_1)$

$\therefore$ $[C_{Q_0}(Z(P)), P_1] \leqslant Q_0 \cap P_1 = 1$.

Hence $C_{Q_0}(Z(P)) \leqslant O_{p'}(C_G(P_1))$ by (ii)

$= O_{p'}(M)$ by (iv).

The result follows. $\square$

<u>LEMMA 5.2</u> Let p be a prime dividing $|G|$, P the A-invariant Sylow p-subgroup of G and $M = N_G(P)$. Suppose $\exists$ a maximal A-invariant subgroup H of G such that $1 \neq P_0 = P \cap F(H) < P_1 = P \cap H < P$. Then

(i) $F(H) \cap M = P_0$.

(ii) If X is any A-invariant subgroup of $Z(P) \times O_{p'}(M)$, $C_G(X) \cap O_{p'}(F(H)) = 1$.

(iii) $(|O_{p'}(F(H))|, |O_{p'}(M)|) = 1$.

(iv) Either $Z(P) \times O_{p'}(M)$ is cyclic and centralized by π or $Z(P) \times O_{p'}(M) \cong Z_n \times Z_n$ for some integer n and π acts f.p.f. on $Z(P) \times O_{p'}(M)$.

<u>PROOF</u>

(i) Let Q_0 be a Sylow q-subgroup of $M \cap F(H)$ for some prime $q \neq p$.

Then $[Q_0, P_1] \leqslant P \cap O_q(H) = 1$, so

$Q_0 \leqslant O_{p'}(C_G(P_1))$ by lemma 5.1 (ii)

$= O_{p'}(M)$ by lemma 5.1 (iv).

Thus $[Q_0, P] = 1$.

Since $P_1 = C_P(P_1) < P$, P is non-abelian.

Thus by lemma 4.4 $N_G(Q_0) \leqslant M$.

It follows that $Q_0 = O_q(H)$, so that $Q_0 \lhd H$, a contradiction.

Hence $M \cap F(H) = P_0$.

(ii) Suppose X is an A-invariant subgroup of $Z(P) \times O_{p'}(M)$ with $C_G(X) \cap O_{p'}(F(H)) \neq 1$.

As $P \leqslant C_G(X)$, $N_G(X) \leqslant M$ by lemma 4.4.

But then $F(H) \cap M \supset P_0$, a contradiction.

(iii) Suppose $\exists q \mid (|O_{p'}(F(H))|, |O_{p'}(M)|)$, and let Q_1 be the A-invariant Sylow q-subgroup of H.

Then $O_q(H) \neq 1$ and $X = Z(Q_1) \cap O_q(H) \neq 1$.

But $O_{p'}(M) \leqslant C_G(P_0) \leqslant H$, so $O_q(M) \leqslant C_G(X)$, contradicting (ii).

(iv) Since $O_{p'}(M) \leqslant C_G(P\ ) \leqslant H$ and $O_{p'}(M) \cap F(H) = 1$ by (i), it follows that $O_{p'}(M)$ is abelian by theorem 1.11.

Thus $T = Z(P) \times O_{p'}(M)$ is an abelian group of automorphisms of $O_{p'}(F(H))$.

If $1 \neq C_T(\pi) < T$, we can choose minimal A-invariant subgroups X and Y of T such that $X \leqslant C_T(\pi)$ and $[Y, \langle\pi\rangle] = 1$.

But then lemma 1.14(c) implies that $\exists$ an A-invariant subgroup X_0 of T such that $C_G(X_0) \cap O_{p'}(F(H)) \neq 1$, contradicting (ii).

Thus either $C_T(\pi) = T$ or $C_T(\pi) = 1$.

If $C_T(\pi) = T$ and T is non-cyclic then some Sylow q-subgroup of T is non-cyclic. But then lemma 1.14(a) yields a contradiction as above.

Similarly if $C_T(\pi) = 1$ lemma 1.14(b) yields that each Sylow subgroup of T is isomorphic to $Z_{q^i} \times Z_{q^i}$ for some prime q, so that $T \cong Z_n \times Z_n$ for some integer n. □

<u>LEMMA 5.3</u> Let p be a prime dividing $|G|$, P the A-invariant Sylow p-subgroup of G and $M = N_G(P)$. Suppose $\exists$ a maximal A-invariant subgroup H of G such that $1 \neq P_0 = P \cap F(H) < P_1 = P \cap H < P$. Then $C_{Z(P)}(\pi) \neq 1$.

<u>PROOF</u>

Suppose that π acts f.p.f. on $Z(P)$.

We show first that $C_G(\pi) \leqslant H$. Suppose $C_G(\pi) \nleqslant H$.

Now by lemma 5.2(ii), $Z(P)$ acts f.p.f. on $O_{p'}(F(H))$ and by [9], theorem 6.1.3, $O_{p'}(F(H)) \neq 1$.

Thus $\exists x \in C_G(\pi) \cap Z(O_{p'}(F(H)))$ by lemma 1.2(4). It follows that $C_G(x) \nleqslant H$.

Suppose $C_G(x) \leqslant M^*$ where M^* is a maximal A-invariant subgroup of G different from H.

Then $F(H) \leqslant M^*$, so $C_H(x) = F(H)$ by lemma 4.2 and $M^* = F(H).F(M^*)$ and $F(H)$ is abelian by lemma 4.3.

Let $Y = C_{P_0}(\pi)$ and suppose $Y \neq 1$.

Then $C_H(Y) \geqslant F(H).P_1 > F(H)$, so by lemma 4.2 $C_G(Y) \leqslant H$.

But then $C_G(\pi) \leqslant H$, a contradiction.

Hence π must act f.p.f. on P_0.

Now $P \cap M^* = P_0$ by lemma 4.2, and as $C_G(\pi) \leqslant M^*$ we must have $C_P(\pi) = 1$.

Let L be an A-invariant Sylow q-subgroup of M, $q \neq p$, with $[L,P] \neq 1$ (L must exist else $O^p(M) \neq M$, contradicting [5], Corollary 2.2).

Then $L_0 = C_L(\pi) \neq 1$, $L_0 \leqslant M^*$ and $[L_0,P] \neq 1$.

Thus $[L_0,P_0] \leqslant F(M^*) \cap P \leqslant P_0$.

Hence $L_0 \leqslant N_G(P_0) = H$, so $L_0 \leqslant M^* \cap H = F(H)$.

Now $[P_1,L_0] \leqslant P \cap F(H) = P_0$, so $[P_1,L_0,L_0] \leqslant [P_0,L_0] = 1$.

Thus by [9], theorem 5.3.6, $[P_1,L_0] = 1$.

But then by lemma 5.1(ii), $L_0 \leqslant O_{p'}(C_G(P_1)) = O_{p'}(M)$, a contradiction.

Hence $C_G(\pi) \leqslant H$.

Choose L as above, and let Q be the A-invariant Sylow q-subgroup of G. Let $Q_1 = Q \cap H$ and suppose that $Q_1 \not\leqslant L$.

By lemma 5.1(vii), $Q_1 = C_{Q_1}(Z(P)).Q_0$ where $Q_0 = Q_1 \cap F(H)$.

If $Q_0 = 1$, $Q_1 \leqslant M$ so that $L \geqslant Q_1$, a contradiction.

Thus $Q_0 \neq 1$.

Now clearly $C_{Q_1}(Z(P)) \leqslant L \cap H$, and $[L \cap H, Z(P)] \leqslant Z(P) \cap F(H) = 1$ by lemma 5.1(v).

Thus $C_{Q_1}(Z(P)) = L \cap H$ and $Q_1 = L \cap H.Q_0$.

Let $M^* = N_G(Q)$.

Then $Z(P) \leqslant N_G(Q_1) \leqslant M^*$ by lemma 5.1(vi) and (iii).

Hence $Z(P)$ normalizes $N_Q(Q_1)/Q_1$.

Thus by [9], theorem 5.3.5,

$N_Q(Q_1)/Q_1 = C_{N_Q(Q_1)/Q_1}(Z(P)).[Z(P),\ N_Q(Q_1)/Q_1]$.

Since $C_G(\pi) \leqslant H$ and $Q \cap H = Q_1$, π acts f.p.f. on $N_Q(Q_1)/Q_1$.

Hence $[Z(P),\ N_Q(Q_1)/Q_1] = Q_1$ by lemma 1.2(4).

It follows that $N_Q(Q_1) = C_{N_Q(Q_1)}(Z(P)).Q_1$

$= N_Q(Q_1) \cap C_Q(Z(P)).Q_1$

$= N_Q(Q_1) \cap L.Q_1$

since $[N_Q(Q_1) \cap L,\ Z(P)] \leqslant Q_1 \cap Z(P) = 1$

$= N_Q(Q_1) \cap L.Q_0$

since $Q_1 = L \cap H.Q_0$.

But then $L \cap N_Q(Q_1) \leqslant N_G([Z(P),\ Q_1]) = N_G(Q_0)$ as $C_G(Z(P)) \cap Q_0 = 1$ by lemma 5.2(ii)

i.e. $L \cap N_Q(Q_1) \leqslant H$.

Thus $N_Q(Q_1) \leqslant H$, so that $Q_1 = Q$.

If $M^* = H$, $[P_1,L] \leqslant Q \cap P = 1$.

Thus $L \leqslant O_{p'}(C_G(P_1)) = O_{p'}(M)$ by lemma 5.1(ii) and (iv).

But then $[L,P] = 1$, a contradiction.

Thus $M^* \neq H$.

Now $Q = C_Q(Z(P)).Q_0$ by lemma 5.1(vii) and $Q_0 \leqslant Z(H)$ by lemma 4.4.

Thus Q centralizes $Z(P)$, so that $Q \leqslant M$ and hence $Q = L$. Contradiction.

It follows that $Q \cap H \leqslant L$.

As above $[L \cap H,\ Z(P)] = 1$.

Now L is abelian by lemma 5.1(i), so $L = C_L(\pi) \times [L,\langle\pi\rangle]$ by lemma 1.3.

Since π acts f.p.f. on $Z(P)$, $[L,Z(P)] = [C_L(\pi),Z(P)]$.
But $C_G(\pi) \leqslant H$, so $C_L(\pi) \leqslant L \cap H$.
Thus $[L, Z(P)] \leqslant [L \cap H, Z(P)] = 1$.

We show next that $Z(P) \leqslant M^* = N_G(Q)$.
Let K be a maximal A-invariant subgroup of G containing $N_G(L)$.
Then $Z(P) \leqslant K$ so we may assume $K \neq M^*$.
If $K = M$, $L = N_Q(L)$ so that $Q = L$. Hence $Z(P) \leqslant M^*$.
If $K \neq M$, $K \cap P < P$ since P is non-abelian.
But then by lemma 5.1(vi) $Z(P)$ normalizes $K \cap Q$ so we may assume that $K \cap Q < Q$.
If $Q \cap F(K) = 1$, $K \cap Q \leqslant C_Q(Z(P))$ by lemma 5.1(vii) so that $K \cap Q \leqslant M$.
Thus $K \cap Q \leqslant L$ and hence $L = Q$ as above, so again $Z(P) \leqslant M^*$.
Finally, if $Q \cap F(K) \neq 1$, $N_G(Q \cap K) \leqslant M^*$ by lemma 5.1(iii).

Thus $Z(P) \leqslant M^*$.
It follows that $Z(P)$ normalizes $N_Q(L)$, and since $C_Q(\pi) \leqslant Q \cap H \leqslant L$, π acts f.p.f. on $N_Q(L)/L$ and $Z(P)$.
Thus $[N_Q(L)/L, Z(P)] = 1$ by lemma 1.2(4).
Therefore $[N_Q(L), Z(P)] \leqslant L$, so that $[N_Q(L), Z(P), Z(P)] = 1$.
It then follows from [9], theorem 5.3.6 that $[N_Q(L), Z(P)] = 1$.
Hence $N_Q(L) \leqslant M$ so $N_Q(L) = L$ i.e. $L = Q$.
Next, let R_0 be an A-invariant Sylow r-subgroup of M^* with $[R_0,Q] \neq 1$.
As $[P \cap M^*, Q] \leqslant P \cap Q = 1$, $r \neq p$ and $M^* \neq M$.

Thus $Z(P) \leqslant N_G(R_0)$ by lemma 5.1(vi) ($P \cap M^* \neq P$ else P is abelian by lemma 4.4).

Now R_0 is abelian by lemma 5.1(i), so

$R_0 = C_{R_0}(Z(P)) \times [Z(P), R_0]$.

But $[Z(P), R_0] \leqslant R_0 \cap F(M^*)$, so $[Z(P), R_0]$ centralizes Q.

Thus $[C_{R_0}(Z(P)), Q] \neq 1$.

However, $C_{R_0}(Z(P)) \leqslant M$, so $[C_{R_0}(Z(P)), Q] \leqslant Q \cap F(M) \leqslant Z(M)$ by lemma 4.4.

Thus $[C_{R_0}(Z(P)), Q] = 1$ by [9], theorem 5.3.6.

This contradiction completes the proof. □

<u>LEMMA 5.4</u> Let p be a prime dividing $|G|$, P the A-invariant Sylow p-subgroup of G and $M = N_G(P)$.

Suppose $\exists$ a maximal A-invariant subgroup H of G such that $1 \neq P_0 = P \cap F(H) < P_1 = P \cap H < P$.

Then $O_{p'}(M) = 1$ and if E is an A-invariant complement to P in M then $[P,E] = P$ and either $E \leqslant C_G(\pi)$ or $[E, \langle\pi\rangle] = E$.

<u>PROOF</u>

By lemmas 5.2(iv) and 5.3, $Z(P) \times O_{p'}(M)$ is a cyclic subgroup of $C_G(\pi)$.

Let r be a prime dividing $|Z(P) \times O_{p'}(M)|$ and let R be the A-invariant Sylow r-subgroup of $Z(P) \times O_{p'}(M)$.

If $R \nleqslant Z(M)$, $\exists$ a minimal A-invariant t-subgroup T of M for some prime $t \neq r$ such that $[T, R] \neq 1$.

Thus by [9], theorem 5.2.4, $[T, \Omega_1(R)] \neq 1$. Since $C_G(\pi)$ is abelian, clearly T is non-cyclic.

But $\Omega_1(R)$ is cyclic of order r and hence has a cyclic automorphism group. Contradiction.

Hence $R \leqslant Z(M)$, so that $Z(P) \times O_{p'}(M) \leqslant Z(M)$.

Suppose that $O_{p'}(M) \neq 1$ and let Q_0 be the A-invariant Sylow q-subgroup of $O_{p'}(M)$ for some prime q.

Let Q be the A-invariant Sylow q-subgroup of G.

Since $Q_0 \leqslant Z(M)$, $Q_0 \neq Q_1 = Q \cap M$ else $Q_0 = Q$ and G has a normal q-complement by [9], theorem 7.4.3.

Now $Z(P) \leqslant C_G(Q_1)$, so $Z(P) \leqslant O_{q'}(M^*)$ where $M^* = N_G(Q)$ by lemma 5.1(ii) and (iv).

Thus $[Z(P), Q] = 1$, so that $Q \leqslant M$ i.e. $Q_1 = Q$.

Since $1 \neq Q_0 \triangleleft M$, $O_{q'}(M^*) \leqslant M$.

Thus $M = F(M^*).F(M)$ by lemma 4.3.

Since $[Z(P), Q] = 1$, $Z(P) \leqslant M^* \cap M = F(M^*)$.

If P^* is the A-invariant Sylow p-subgroup of M^*, $P^* \leqslant M \cap M^* = F(M^*)$ so that $P^* \triangleleft M^*$.

Thus $P^* = P$, so $M^* = M$, a contradiction.

It follows that $O_{p'}(M) = 1$.

Hence $F(M) = P$ and so E is abelian by theorem 1.11.

Now $P = C_P(E).[P,E]$ by [9], theorem 5.3.5, so $E.[P,E] \triangleleft M$.

Thus $[P,E] = P$ by [5], Corollary 2.2.

Suppose next that E contains an A-invariant Sylow q-subgroup Q of G for some prime q.

Then $[Q, Z(P)] = 1$ as $Z(P) \leqslant Z(M)$, and now a contradiction is obtained as in the last paragraph of lemma 5.3.

It follows that $\exists$ a minimal A-invariant q-subgroup C of E with $C_G(C) \not\leq M$.

Suppose that $E = C_E(\pi) \times [E, \langle\pi\rangle]$ where $C_E(\pi) \neq 1$ and $[E, \langle\pi\rangle] \neq 1$.

Then we can choose a minimal A-invariant subgroup D of E such that $1 \neq C_{C\times D}(\pi) \neq C \times D$.

As $P = F(M)$, $C_M(P) \leq P$ by [9], theorem 6.1.3.

Let $\bar{P} = P/\Phi(P)$, so that $\bar{P} = C_{\bar{P}}(D) \times [\bar{P},D]$.

If $\bar{P} = C_{\bar{P}}(D)$, D centralizes P by [9], theorem 5.1.4, a contradiction.

Thus $\bar{P}^* = [\bar{P},D] \neq 1$.

Hence $C_{\bar{P}^*}(C) \neq 1$ by lemma 1.14(c), so that $C_P(C) \neq 1$ by [9], theorem 6.2.2.

Clearly $[D, C_P(C)] \neq 1$

Let $C_G(C) \leq M^*$, a maximal A-invariant subgroup of G.

Now $[D, C_P(C)] \leq P \cap F(M^*) = P_0^*$, so $P_0^* \neq 1$.

And since $M^* \neq M$, $P_0^* < P_1^* = P \cap M^*$.

Also $P_1^* \neq P$, else P would be abelian by lemma 4.4.

Thus by lemma 5.2(ii), $C_G(Z(P)) \cap F(M^*) = P_0^*$ i.e. $M \cap F(M^*) = P_0^*$.

Hence $E \cap F(M^*) = 1$.

Let Q be the A-invariant Sylow q-subgroup of G, $L = Q \cap E$, $Q_1 = Q \cap M^*$ and $Q_0 = Q \cap F(M^*)$.

By lemma 5.1(vii), $Q_1 = C_{Q_1}(Z(P)).Q_0 = L.Q_0$ since $L = C_Q(Z(P)) \leq Q_1$.

Since $C_G(C) \not\leq M$ but $C_G(C) \leq M^*$, we must have $Q_0 \neq 1$.

And $Q_1 \neq Q$, or else $Q_0 \leq Z(M^*)$ by lemma 4.4 ($M^* \neq N_G(Q)$ because $E \cap F(M^*) = 1$) and then $[Q_0, Z(P)] = 1$.

Thus by lemma 5.1(ii) and (v), $Z(Q) \cap Q_0 = 1$.

It follows from lemma 5.2(ii) that $C_G(Z(Q)) \cap F(M^*) = Q_0$.

We show next that $[Z(Q),E] = 1$.

Let r be a prime dividing $|E|$, $r \neq q$, and let R_1 be the A-invariant Sylow r-subgroup of M^*.

By lemma 5.1(vi), $Z(Q) \leqslant N_{M^*}(R_1)$ so

$[E \cap R_1, Z(Q)] \leqslant R_1 \cap F(M^*)$.

But clearly E normalizes $Q_1 = L.Q_0$ and as $Z(Q) \leqslant Q_1$,

$[E \cap R_1, Z(Q)] \leqslant Q_1$.

Thus $[E \cap R_1, Z(Q)] = 1$ and it follows that $[E, Z(Q)] = 1$.

Let $Z_0 = \Omega_1(Z(Q))$ ($Z(Q)$ is cyclic by lemma 5.2 and 5.3) and take E_0 to be either C or D so that π acts f.p.f. on E_0.

Then $Z_0 \times E_0$ normalizes P_0^* and $C_{P_0^*}(Z_0) = 1$ since $C_G(Z(Q)) \cap F(M^*) = Q_0$.

Therefore by lemma 1.14, $[E_0,P_0^*] = 1$.

Now by [9], theorem 5.3.6, it follows that

$[E_0,P_1^*] = [E_0,P_1^*,P_1^*] \leqslant [P_0^*,P_1^*] = 1$.

Thus $E_0 \leqslant O_{p'}(C_G(P_1^*))$ by lemma 5.1(ii)

$= O_{p'}(M)$ by lemma 5.1(iv)

$= 1$ by the first part of this lemma.

This contradiction completes the proof. □

<u>LEMMA 5.5</u> Let p be a prime dividing $|G|$, P the A-invariant Sylow p-subgroup of G and $M = N_G(P)$. Suppose that $M = P.E$ where $[E,\pi] = E$ and $\exists$ a maximal A-invariant subgroup H of G such that

$1 \neq P_0 = P \cap F(H) < P_1 = P \cap H < P$. Then:

(i) $C_G(\pi) = C_P(\pi)$.

(ii) $\forall x \in C_P(\pi)$ such that $x \neq 1$, $C_G(x) \leqslant M$.

(iii) If q is a prime dividing $|E|$ and L is the A-invariant Sylow q-subgroup of E then $C_P(L)$ is cyclic.

(iv) $\exists$ a maximal A-invariant subgroup M^* of G such that $M^* = C_{M^*}(\pi).F(M^*)$, $C_{M^*}(\pi) \cap F(M^*) = 1$, $O_p(M^*) = 1$ and $E = Z(M^*)$.

(v) $H = C_H(\pi).F(H)$ where π acts f.p.f. on $F(H)$, $O_{p'}(H) \neq 1$ and is a Hall subgroup of G and $(|O_{p'}(H)|, |F(M^*)|) = 1$.

PROOF

(i) Since π centralizes $Z(P)$ by lemmas 5.2(iv) and 5.3, $C_G(\pi) \leqslant C_G(Z(P)) \leqslant M$.
But $C_E(\pi) = 1$ by assumption, so $C_G(\pi) \leqslant P$.

(ii) Suppose $\exists x \in C_P(\pi)$ such that $x \neq 1$ and $C_G(x) \nleqslant M$. W.l.o.g. we may assume that x has order p.
Now $C_G(x) \leqslant H^*$ for some maximal A-invariant subgroup $H^* \neq M$.
If $P \cap F(H^*) = 1$, $C_{F(H^*)}(\pi) = 1$ so that $H^* = C_P(\pi).F(H^*)$ by lemma 1.19.
Thus $C_P(\pi)$ is a Sylow p-subgroup of H^*.
But $C_P(\pi) < C_P(C_P(\pi)) \leqslant C_G(x)$ by lemmas 1.16 and 4.5, a contradiction.

Hence $P \cap F(H^*) \neq 1$.

Clearly $P \cap H^* \neq P$, else P would be abelian by lemma 4.4.

Thus w.l.o.g. we may assume that $C_G(x) \leqslant H$, so that $C_P(\pi) \leqslant H$.

Let X be a minimal A-invariant subgroup of P_1 such that $X \cap Z(P) = 1$ and let $Z_0 = \Omega_1(Z(P))$. Let $q \neq p$ be a prime dividing $|F(H)|$.

Then $X \times Z_0$ normalizes $B = M \cap O_q(F(H))$ and $D = N_{O_q(F(H))}(B)$.

Hence $X \times Z_0$ normalizes D/B, so by lemma 1.14 $\exists$ an A-invariant sugroup X_0 of $X \times Z_0$ such that $C_{D/B}(X_0) \neq B$.

Thus $C_D(X_0) \nleqslant B$ so $C_D(X_0) \nleqslant M$ i.e. $C_H(X_0) \nleqslant M$.

If $C_P(X) \neq P_1$ then $C_P(X_0) \neq P_1$ so that $C_G(X_0) \leqslant M^*$, a maximal A-invariant subgroup of G such that $M^* \neq H, M$.

But then $P \cap M^*$ is abelian by lemma 5.1(i), contradicting lemma 5.1(ii). Hence $C_P(X) = P_1$.

It now follows from lemma 1.17 that $|N_P(P_1)/P_1| = p^2$ since $C_P(\pi) \leqslant P_1$.

Since P_1 is abelian, $P_1 = C_{P_1}(\pi) \times [P_1, \langle\pi\rangle]$ by [9], theorem 5.2.3.

Now $x \in P_1 - Z(P)$, and $\langle x, Z_0\rangle \leqslant \Omega_1(C_{P_1}(\pi))$.

It follows that $|\Omega_1(C_{P_1}(\pi))| \geqslant p^2$.

And P_1 is self-centralizing in P, so by lemma 1.16 $C_{P_1}(\pi) < P_1$.

Thus $[P_1,\langle\pi\rangle] \neq 1$, so that $|\Omega_1([P_1,\langle\pi\rangle])| \geqslant p^2$.

Hence $|\Omega_1(P_1)| \geqslant p^4$.

Thus by lemma 1.17, P_1 is a characteristic subgroup of P.

It follows that $E \leqslant N_G(P_1)$ and $[E,P] \leqslant P_1$ since π is f.p.f. on E and P/P_1.

Since $P = C_P(E).[P,E]$ by [9], theorem 5.3.5, $C_P(E)$ is not centralized by π. Let $u \in C_P(E)$ be an element which is not centralized by π.

Let q be a prime dividing $|E|$, let Q be the A-invariant Sylow q-subgroup of G, and let $L = Q \cap E$.

Suppose $N_G(L) \leqslant M^*$ where M^* is a maximal A-invariant subgroup of G.

Let $Q_0^* = Q \cap F(M^*)$, $Q_1^* = Q \cap M^*$, $P_0^* = P \cap F(M^*)$ and $P_1^* = P \cap M^*$.

Now $u \in P \cap M^* = P_1^*$ and since $M^* \neq M$, P_1^* is abelian by lemma 5.1(i). It follows that u is contained in an A-invariant group U on which π acts f.p.f., and since π also acts f.p.f. on $O_{p'}(F(M^*))$, u centralizes $F(M^*)$.

Thus $u \in P_0^* \neq 1$ and clearly $P_0^* < P_1^* \neq P$.

By lemma 5.2(ii), $C_G(Z(P)) \cap O_{p'}(F(M^*)) = 1$.

And by lemma 5.1(vii), $Q_1^* = C_{Q_1^*}(Z(P)).Q_0^* = L.Q_0^*$.

Suppose that $Q_1^* = Q$.

If $Q_0^* \neq 1$, $Q_0^* \leqslant Z(M^*)$ by lemma 4.4, contradicting lemma 5.1(viii).

If $Q_0^* = 1$, $Q = Q_1^* = L$.

But then $N_G(Q) \leqslant M^*$ so $Q \lhd M^*$, a contradiction.

Hence $Q_1^* \neq Q$ and as $L \neq N_Q(L) \leqslant Q_1^*$, $Q_0^* \neq 1$.

But then by lemma 5.3 $C_Q(\pi) \neq 1$, a contradiction.

Hence $C_G(x) \leqslant M$.

(iii) Suppose that $C_P(L)$ is not cyclic.

Since π centralizes $Z(P)$ and acts f.p.f. on L, $Z(P)$ normalizes L by lemma 1.4(2).

Thus $[Z(P),L] \leqslant Z(P) \cap L = 1$ i.e. $Z(P) \leqslant C_P(L)$.

Choose $x \in C_P(L)$ such that $\langle x,z\rangle$ is not cyclic, where $\langle z\rangle = \Omega_1(Z(P))$.

Suppose that $N_G(L) \leqslant M^*$, a maximal A-invariant subgroup of G.

If $x \notin C_P(\pi)$, we can derive a contradiction as above (replacing u by x). Thus we may assume that $x \in C_P(\pi)$.

If $O_{p'}(F(M^*)) \leqslant M$, $[\langle x,z\rangle, O_{p'}(F(M^*))] \leqslant P \cap O_{p'}(F(M^*)) =$

Thus $O_p(F(M^*)) \neq 1$ by [9], theorem 6.1.3, and clearly $P \cap M^* \neq P$.

But then $C_G(z) \cap O_{p'}(F(M^*)) = 1$ by lemma 5.2(ii), a contradiction.

Hence $O_{p'}(F(M^*)) \nleqslant M$.

But by [9], theorem 6.2.4,

$$O_{p'}(F(M^*)) = \langle C_{O_{p'}(F(M^*))}(\alpha) \mid \alpha \in \langle x,z\rangle^\#\rangle$$
$$\leqslant M \text{ by (ii).}$$

This contradiction completes the proof that $C_P(L)$ is cyclic.

(iv) Again let q be a prime dividing $|E|$, L the A-invariant Sylow q-subgroup of E and $N_G(L) \leqslant M^*$, a maximal A-invariant subgroup of G.

If $O_p(M^*) \neq 1$ we derive a contradiction as above.

So $O_p(M^*) = 1$, and hence $C_{F(M^*)}(\pi) = 1$.

Consequently $M^* = C_{M^*}(\pi).F(M^*)$ by lemma 1.19, so that $O_q(M^*) = Q$, the A-invariant Sylow q-subgroup of G.

It follows that $\forall$ primes $r \mid |E|$, $M^* = N_G(R)$ where R is the A-invariant Sylow r-subgroup of G.

Furthermore, $[C_{M^*}(\pi), L] \leqslant P \cap F(M^*) = 1$.

Thus $[L,M^*] \leqslant Q'$ and hence $LQ'/Q' \leqslant Z(M^*/Q')$.

If $L \not\leqslant Q'$, $O^q(M^*/Q') \neq M^*/Q'$ by [9], theorem 7.4.4.

Hence $O^q(M^*) \neq M^*$, contradicting [5], corollary 2.2.

Thus $L \leqslant Q' \leqslant Z(Q)$ by [2], section 66.

It follows that $E \leqslant Z(F(M^*))$.

Now $\forall x \in C_{M^*}(\pi)$, $C_G(x) \cap F(M^*) \leqslant M \cap F(M^*) = E$.

Since $[x,E] \leqslant P \cap F(M^*) = 1$, this yields $Z(M^*) = E$.

(v) If $C_{P_0}(\pi) \neq 1$, $C_G(x) \leqslant M$ $\forall x \in C_{P_0}(\pi)$ such that $x \neq 1$ by (ii).
Thus $F(H) < P_1F(H) \leqslant M$, contradicting lemma 4.2.
Hence $C_{P_0}(\pi) = 1$, so that π acts f.p.f. on $F(H)$ and $H = C_H(\pi).F(H)$ by lemma 1.19.
Clearly $O_{p'}(H) \neq 1$ (else $F(H) = P_0 \leqslant Z(H)$ by lemma 4.3), and since $H/F(H)$ is a p-group, $O_{p'}(H)$ must be a Hall subgroup of G.
Finally, if v is a prime divisor of $|O_{p'}(H)|$ and $|F(M^*)|$ then $H = N_G(V) = M^*$ where V is the A-invariant Sylow v-subgroup of G.
But then $[P_0,E] = 1$, contradicting (iii).
Hence $(|O_{p'}(H)|, |F(M^*)|) = 1$. □

LEMMA 5.6 Let p be a prime dividing $|G|$, P the A-invariant Sylow p-subgroup of G and $M = N_G(P)$. Suppose that $\exists$ a maximal A-invariant subgroup H of G such that $1 \neq P_0 = P \cap F(H) < P_1 = P \cap H < P$. Then if E is the A-invariant complement to P in M, $E \leqslant C_G(\pi)$.

PROOF

Suppose that $E \not\leqslant C_G(\pi)$. Then by lemma 5.4 π acts f.p.f. on E and hence the results of lemma 5.5 hold. Let M^* be the maximal A-invariant subgroup of G described in (iv) of that lemma.
Let $r \neq p$ be a prime dividing $|F(H)|$, let R be the A-invariant Sylow r-subgroup of G, $R_\tau = C_R(\tau)$ ($\neq 1$ by lemma 1.2(3)) and $R^* = ZJ(R_\tau)$.

By lemma 5.5(v), $H = N_G(R) = C_H(\pi).F(H)$.

Now $\forall x \in C_H(\pi)$ such that $x \neq 1$, $C_G(x) \leqslant M$ by lemma 5.5(ii) and $r \nmid |M|$ as $E \leqslant F(M^*)$ and $(|O_{p'}(H)|, |F(M^*)|) = 1$ by (v) of the same lemma.

Thus by lemma 4.9, $C_H(R^*) \leqslant F(H)$.

It then follows from lemma 4.10 that $C_G(\tau)$ has a normal r-complement.

Next, let q be a prime dividing $|F(M^*)|$ and let $Q_\tau = C_Q(\tau)$ where Q is the A-invariant Sylow q-subgroup of G.

Then we may assume w.l.o.g. that R_τ normalizes Q_τ.

Let $Q^* = C_Q(Q_\tau)$.

Then $Z(Q) \leqslant Q^*$, so $N_Q(Q^*) \leqslant M^*$ by lemma 4.7.

But Q^* is a Sylow q-subgroup of $C_G(Q_\tau)$ by lemma 4.8, so by [9], theorem 1.3.7,

$$N_G(Q_\tau) = (N_G(Q_\tau) \cap N_G(Q^*)).C_G(Q_\tau)$$

Now $R_\tau \leqslant N_G(Q_\tau)$ and $r \nmid |M^*|$ by lemma 5.5(v) so w.l.o.g. $R_\tau \leqslant C_G(Q_\tau)$.

Take $a \in Z(R) \cap R_\tau$, so that $C_H(a) = F(H)$ by lemma 5.5.

Now R is a Sylow r-subgroup of $C_G(a)$ and $H = N_G(ZJ(R))$ so $N(ZJ(R)) \cap C_G(a) = F(H)$ has a normal r-complement.

Hence by [5], theorem D, $C_G(a)$ has a normal r-complement.

Suppose that R contains an A-invariant subgroup $W \cong E_{r^4}$ and let $\tilde{Q}$ be an R-invariant Sylow q-subgroup of $C_G(a)$ ($\tilde{Q}$ exists by theorem 6.2.2 of [9]).

Then by lemma 1.15 $\exists$ an A-invariant subgroup W_0 of W such that $C_{\tilde{Q}}(W_0) \neq 1$.

Let $N_G(W_0) \leqslant K$, a maximal A-invariant subgroup of G.

Then by lemma 1.2(4) $C_K(\pi) \neq 1$ and hence $p \mid |K|$.

If $P \cap F(K) = 1$, π acts f.p.f. on $F(K)$ and hence π centralizes $P \cap K$ by lemma 1.19.

But $P_0 \leqslant N_G(W_0) \leqslant K$ and $C_{P_0}(\pi) = 1$ by lemma 5.5(v).

Thus $P \cap F(K) \neq 1$. Since $r \nmid |M|$, $K \neq M$. Thus $P \cap K$ is abelian by lemma 5.1(i), so that $P \cap K \neq P$ by lemma 5.1(ii).

It now follows from lemma 5.5(v) that $q \nmid |K|$, contradicting $C_{\tilde{Q}}(W_0) \neq 1$.

Hence R contains no such subgroup W, so by lemma 1.18 there is no element of order r in $R-Z(R)$.

Now if Q is abelian take $x \in (Q - Q \cap E) \cap Q_\tau$ and if Q is not abelian take $x \in (Q - Z(Q)) \cap Q_\tau$ (both sets are non-empty by lemma 1.2(3) and [9], theorem 6.2.2 because π acts f.p.f. on $Q/Q \cap E$ and $Q/Z(Q)$).

In the first case it follows at once from lemma 5.5 that $C_{M^*}(x) = F(M^*)$, so $C_{M^*}(x)$ has a normal q-complement.

In the second case, suppose $C_{M^*}(x) \not\leqslant F(M^*)$.

Then $\exists y \in M^* - F(M^*)$ such that y centralizes x, and w.l.o.g. we may assume that y is a p-element.

Since $P \cap M^*(\pi)$ is a Sylow p-subgroup of M^*,
$\exists g \in M^*$ such that $y^g \in C_P(\pi)$.
But $C_G(y^g) \leqslant M$ by lemma 5.5(ii), so $x^g \leqslant Q \cap M \leqslant Z(M^*)$.
As $g \in M^*$, this implies $x \in Z(M^*)$ so that $x \in Z(Q)$,
a contradiction.
Hence $C_{M^*}(x) \leqslant F(M^*)$ and so $C_{M^*}(x)$ has a normal
q-complement in this case as well.
Now $C_Q(x)$ is a Sylow q-subgroup of $C_G(x)$ by lemma
4.8, and $N_G(ZJ(C_Q(x))) \leqslant M^*$ by lemma 4.7.
Thus $N_G(ZJ(C_Q(x))) \cap C_G(x) \leqslant C_{M^*}(x)$ and so has a
normal q-complement.
Hence by [5], theorem D, $C_G(x)$ has a normal q-complement.
Let $\tilde{R}$ be a $\langle Q,\tau\rangle$-invariant Sylow r-subgroup of
$O_{q'}(C_G(x))$.
Then $\Omega_1(Z(\tilde{R})) \leqslant \Omega_1(Z(R))$ since there is no element of
order r in $R - Z(R)$.
Clearly $|\Omega_1(Z(R))| = r^2$, so if $|\Omega_1(Z(\tilde{R}))| = r^2$ we
must have $\Omega_1(Z(\tilde{R})) = \Omega_1(Z(R))$.
But then $q \,|\, |N_G(\Omega_1(Z(R)))|$ i.e. $q \,|\, |H|$, a contradiction.
Hence $|\Omega_1(Z(\tilde{R}))| = r$, so that $\Omega_1(Z(\tilde{R})) = \langle a\rangle$.
Thus $Z(Q)$ normalizes $\langle a\rangle$. If $[Z(Q),\langle a\rangle] \neq 1$, $\exists w \in H$
such that $a^w = a^k$ for some integer $k \neq 1$ by [5],
Corollary 2.1(a).
Since $H = C_{P\cap H}(\pi).F(H)$ and $F(H)$ centralizes a, we
may assume that $w \in C_P(\pi)$, so that $a^w = (a^w)^\tau = a^{w^{-1}}$.
But then w^2, and hence w, centralizes $\langle a\rangle$.

Thus $[Z(Q),\langle a\rangle] = 1$, which yields $\langle a\rangle \leqslant M^*$.
This contradiction completes the proof. $\square$

LEMMA 5.7 Let p be a prime dividing $|G|$, P the A-invariant Sylow p-subgroup of G and $M = N_G(P)$. Suppose $\exists$ a maximal A-invariant subgroup H of G such that $1 \neq P_0 = P \cap F(H) < P_1 = P \cap H < P$, and let E be the A-invariant complement of P in M.
Then we have:

(i) $\forall x \in E$ such that $x \neq 1$, $C_P(x) = C_P(E)$.

(ii) E is cyclic.

(iii) $C_P(E)$ is cyclic and $C_P(E) = C_P(\pi)$, so that $C_G(\pi) = E.C_P(\pi)$ is cyclic.

(iv) $\forall y \in C_P(\pi)$ such that $y \neq 1$, $C_G(y) \leqslant M$.

(v) Let q be a prime dividing $|E|$, Q the A-invariant Sylow q-subgroup of G and $M^* = N_G(Q)$.
Then:

(a) $M^* = C_G(\pi).F(M^*)$.

(b) $\forall x \in C_G(\pi) - F(M^*)$, $C_Q(x) = C_Q(\pi)$.

(c) $\forall y \in C_Q(\pi)$ such that $y \neq 1$, $C_G(y) \leqslant M^*$.

PROOF

(i) By lemma 5.6, $E \leqslant C_G(\pi)$.
Let q be a prime dividing $|E|$ and take $x \in E$ of order q. Suppose that $C_P(E) < C_P(x)$.
Let $L = Q \cap E$ where Q is the A-invariant Sylow q-subgroup of G.
Then $L = C_Q(\pi)$ as $C_G(\pi) \leqslant M$.

Thus by lemmas 1.16 and 4.5, $C_Q(L) \neq L$ so that $C_G(x) \not\leqslant M$.

Let H^* be a maximal A-invariant subgroup of G containing $C_G(x)$.

Now $E \leqslant C_G(x)$ and $[E, C_P(x)] \neq 1$ by assumption.

Thus $P_0^* = P \cap F(H^*) \neq 1$.

And $P_1^* = P \cap H^* \neq P$, else $M = P.E \leqslant H^*$.

Let $Q_0^* = Q \cap F(H^*)$ and $Q_1^* = Q \cap H^*$.

Since $L \leqslant C_{Q_1^*}(Z(P)) \leqslant M \cap Q = L$, we have $C_{Q_1^*}(Z(P)) = L$.

By lemma 5.1(vii), $Q_1^* = C_{Q_1^*}(Z(P)).Q_0^* = L.Q_0^*$ and as $L < C_Q(L) \leqslant Q_1^*$ we have $Q_0^* \neq 1$.

And by lemma 5.1(viii), $Q_0^* \neq Q$.

If $Q_1^* = Q$, $Q_0^* \leqslant Z(H^*)$ by lemma 4.4.

But then Q_0^* centralizes $Z(P)$, contradicting lemma 5.1(viii).

Thus $1 \neq Q_0^* < Q_1^* < Q$, and it then follows from lemmas 5.2, 5.4 and 5.6 that $Z(Q)$ is cyclic, $M^* = N_G(Q) = Q.E^*$ where E^* is centralized by π and $O_{q'}(M^*) = 1$.

Now by lemma 5.1(ii), $C_Q(Q_1^*) = Q_1^*$.

Let Y be a minimal A-invariant subgroup of Q_1^* with $Y \cap Z(Q) = 1$ and suppose that $C_Q(Y) \neq Q_1^*$.

Then $C_G(Y) \leqslant M^*$, since otherwise $C_Q(Y)$ is abelian by lemma 5.1(i), contradicting $C_Q(Q_1^*) = Q_1^*$.

Now by lemma 1.14, $\exists$ an A-invariant subgroup Y_1 of $Y \times \Omega_1(Z(Q))$ such that $C_{P_0^*}(Y_1) \neq 1$.

If $Y_1 = \Omega_1(Z(Q))$, $C_G(Y_1) \leqslant M^*$.

But then $[C_{P_0^*}(Y_1), Z(Q)] \leqslant P_0^* \cap Q = 1$, so $C_{P_0^*}(Z(Q)) \neq 1$.

This contradicts lemma 5.1(viii), so we must have $Y_1 \neq \Omega_1(Z(Q))$.

Thus $C_Q(Y_1) = C_Q(Y)$, so that as above $C_G(Y_1) \leqslant M^*$.

But then $[Z(Q), C_{P_0^*}(Y_1)] \leqslant O_p(M^*) = 1$, so $C_{P_0^*}(Z(Q)) \neq 1$, again contradicting lemma 5.1(viii).

It follows that $C_Q(Y) = Q_1^*$.

Now $C_Q(\pi) = L < Q_1^*$, so by lemma 1.17 we have $|N_Q(Q_1^*) : Q_1^*| = q^2$.

By [9], theorem 5.2.3, $Q_1^* = C_Q(\pi) \times [Q_1^*, \langle\pi\rangle]$ and since $C_Q(\pi) \neq 1$ and $[Q_1^*, \langle\pi\rangle] \neq 1$, $|\Omega_1(Q_1^*)| \geqslant q^3$.

If $|\Omega_1(Q_1^*)| = q^3$ then $|\Omega_1(Q_0^*)| = q^2$ since $Q_0^* \cap L = 1$.

As $Z(P)$ acts f.p.f. on $\Omega_1(Q_0^*)$ by lemma 5.1(viii), we have $p | q^2 - 1$.

If $|\Omega_1(Q_1^*)| > q^3$, Q_1^* is a characteristic subgroup of Q by lemma 1.17.

Thus $|Q/Q_1^*| = q^2$ and as $C_Q(Z(P)) = L \leqslant Q_1$, $Z(P)$ acts f.p.f. on Q/Q_1. Hence again $p | q^2 - 1$.

Now by the symmetric argument applied to $Z(Q)$ and P we derive that $q | p^2 - 1$.

As p,q are odd this is a contradiction, so that $C_P(x) = C_P(E)$ after all.

Now if v is an arbitrary element of E^* and q is a prime dividing $|v|$, $C_P(v) \leq C_P(x)$ for some element x of order q.

Thus $C_P(E) \leq C_P(v) \leq C_P(x) = C_P(E)$, so that $C_P(v) = C_P(E)$ $\forall v \in E^*$.

(ii) If E is non-cyclic, $P = \langle C_P(v) \mid v \in E^* \rangle$ by [9], theorem 6.2.4.

Thus $P = C_P(E)$ by (i), contradicting lemma 5.4.

Hence E is cyclic.

(iii) Suppose that either $C_P(E)$ is not cyclic or $C_P(E) \neq C_P(\pi)$.

Let q be a prime dividing $|E|$, and again let $L = Q \cap E$ where Q is the A-invariant Sylow q-subgroup of G.

Then $C_Q(L) \neq L$ by lemmas 1.16 and 4.5, so $C_G(L) \not\leq M$.

Let $C_G(L) \leq H^*$, a maximal A-invariant subgroup of G, and suppose that $H^* \neq M^* = N_G(Q)$.

Now $1 \neq P \cap H^* \neq P$ else P would be abelian by lemma 5.1(i).

So by lemma 5.1(vii) we have

$Q \cap H^* = C_{Q\cap H^*}(Z(P)).Q \cap F(H^*)$.

Thus if $Q \cap F(H^*) = 1$, $Q \cap H^* \leq C_G(Z(P)) \leq M$, a contradiction.

So $Q \cap F(H^*) \neq 1$.

If $Q \cap H^* = Q$, $Q \cap F(H^*) \leqslant Z(H^*)$ by lemma 4.4.
But then $Z(P)$ centralizes $Q \cap H^* = Q$ so
$Q \leqslant M$, a contradiction.

Thus $1 \neq Q \cap F(H^*) < Q \cap H^* \neq Q$.

Now by lemmas 5.2 and 5.3, $Z(Q)$ is cyclic and
by (ii) $M^* = N_G(Q) = Q.E_1$ where E_1 is cyclic.
Thus $C_P(E) \leqslant C_P(Z(Q)) \leqslant M^*$, so $C_P(E) \leqslant E_1 \leqslant C_G(\pi)$,
yielding a contradiction in both cases.
Hence $C_G(L) \leqslant M^* = N_G(Q)$.
Since $M^* \neq M$, $P \cap M^* \neq P$ else P would be
abelian by lemma 5.1(i).
If $P \cap F(M^*) \neq 1$, by lemma 5.1(viii) $C_Q(Z(P)) = 1$,
a contradiction.

Hence $P \cap F(M^*) = 1$.

Suppose that $P \cap M^*$ contains an A-invariant
subgroup Y of order p^2 such that π acts
f.p.f. on Y.
Let $Z_0 = \Omega_1(Z(P))$.
Since $[L,Y] \leqslant Q \cap P = 1$, $C_Q(Y) \geqslant L = C_Q(Z_0)$.
Now by [9], theorem 5.3.5, $Q/\Phi(Q) = [Z_0, Q/\Phi(Q)] \times C_{Q/\Phi(Q)}(Z_0)$.
As Z_0 acts f.p.f. on $[Z_0, Q/\Phi(Q)]$, π must
also act f.p.f..

Hence Y centralizes $[Z_0, Q/\Phi(Q)]$ by
lemma 1.2(4).
Since also $C_Q(Z_0) \leqslant C_Q(Y)$, Y must centralize
$Q/\Phi(Q)$.

Thus by [9], lemma 5.1.4, Y centralizes Q.

If $r \neq q$ is a prime dividing $|F(M^*)|$ we may apply the same argument to $O_r(M^*)$ since $C_G(Z_0) \cap O_r(M^*) \leqslant E$ and $[E \cap O_r(M^*), Y] \leqslant O_r(M^*) \cap P = 1$, to yield $[Y, O_r(M^*)] = 1$.

Thus $[Y, F(M^*)] = 1$ so that $Y \leqslant P \cap F(M^*)$ by [9], theorem 6.1.3.

This contradiction proves that $P \cap M^* \leqslant C_P(\pi)$.

In particular $C_P(E) \leqslant C_P(\pi)$, so that $C_P(E) = C_P(\pi)$.

It remains to show that $C_P(E)$ is cyclic.

Suppose not, and take $x \in \Omega_1(C_P(E)) - Z(P)$.

As above, let $Z_0 = \Omega_1(Z(P))$. Then $C_Q(Z_0) = L$.

Since $\langle x, Z_0\rangle$ is a non-cyclic group of automorphisms of Q, by [9], theorem 5.3.16 $\exists w \in \langle x, Z_0\rangle$ such that $C_Q(w) \neq L$.

As $P \cap M^* \leqslant C_P(\pi)$, $C_P(w) \not\leqslant M^*$ by lemmas 1.16 and 4.5.

Let $C_G(w) \leqslant H^*$ for some maximal A-invariant subgroup H^* of G. Then we have shown that $H^* \neq M, M^*$.

Now $[C_Q(w), Z_0] \neq 1$ and is a q-group since $Z_0 \leqslant M^*$.

Thus $Q \cap F(H^*) \neq 1$.

It then follows as above from lemma 4.4 that $Q \cap H^* \neq Q$, so we have $1 \neq Q \cap F(H^*) < Q \cap H^* \neq Q$.

But then by (ii) $M^* = Q, E_1$ where E_1 is cyclic, so $C_P(E) \leqslant E_1$ is cyclic after all.

(iv) Since $C_P(\pi)$ is cyclic, $\Omega_1(C_P(\pi)) = \Omega_1(Z(P)) = \langle z \rangle$ say.

Now $\forall y \in C_P(\pi)$, $y^n = z$ for some integer n.

Thus $C_G(y) \leqslant C_G(z) \leqslant M$.

(v) If there exists a maximal A-invariant subgroup H^* of G with $1 \neq Q \cap F(H^*) < Q \cap H^* \neq Q$ then the results follow from lemma 5.6 and (i), (iii) and (iv) of this lemma.

Thus we may assume that no such maximal A-invariant subgroup exists, and it then follows as in (iii) that $C_G(L) \leqslant M^*$ where $L = Q \cap E$ and $P \cap M^* \leqslant C_P(\pi)$. Hence $Z(P) \leqslant P \cap M^* \neq P$.

If $r \neq p$ is a prime dividing $|M^*|$ and R is the A-invariant Sylow r-subgroup of G, by lemma 5.1(vii) we have

$$R \cap M^* = C_{R\cap M^*}(Z(P)).R \cap F(M^*) \leqslant C_R(\pi).R \cap F(M^*).$$

Since $C_G(\pi) \leqslant C_G(L) \leqslant M^*$, it follows that $M^* = C_G(\pi).F(M^*)$.

Next, let $r \neq q$ be a prime dividing $|M^*/F(M^*)|$ and let x be an element of order r in $C_G(\pi) - F(M^*)$.

If $r = p$ the result (b) follows from (iv), so we may assume $r \neq p$.

Let $C_G(x) \leqslant H^*$, a maximal A-invariant subgroup of G, and suppose that $C_Q(\pi) < C_Q(x)$.

Since $r \neq p$, $R \cap M = C_R(\pi)$ where R is the A-invariant Sylow r-subgroup of G.

Thus $C_R(x) \not\leq M$ by lemma 1.16, so that $H^* \neq M$ and hence $P \cap H^* \neq P$.

Now $Z(P)$ normalizes $Q \cap H^*$ by lemma 5.1(vi) and since $Q \cap M < Q \cap H^*$, $[Q \cap H^*, Z(P)] \neq 1$.

Hence $Q \cap F(H^*) \neq 1$.

If $H^* \neq M^*$, we must have $Q \leq H^*$ by assumption.

But then $Q \cap F(H^*) \leq Z(H^*)$ by lemma 4.4, so that by [9], theorem 5.3.6,

$$\begin{aligned} [Q, Z(P)] &= [Q, Z(P), Z(P)] \\ &\leq [Q \cap F(H^*), Z(P)] \\ &= 1, \text{ a contradiction.} \end{aligned}$$

Thus $H^* = M^*$.

But clearly $M^* \neq N_G(R)$ so by the same argument we have $1 \neq R \cap F(M^*) < R \cap M^* \neq R$.

Thus by lemma 5.6 $N_G(R) = R.E_2$ where $E_2 \leq C_G(\pi)$.

But then by (iv), $C_G(x) \leq N_G(R)$, so

$C_Q(x) \leq Q \cap N_G(R) = C_Q(\pi)$.

Thus $C_Q(x) = C_Q(\pi)$, and it follows that

$C_Q(v) = C_Q(\pi) \quad \forall v \in C_G(\pi) - F(M^*)$.

Finally, take $y \in C_Q(\pi)$.

If $y \in Z(Q)$, $C_G(y) = M^*$ as $M^* = C_G(\pi).F(M^*)$.

If $y \notin Z(Q)$, suppose $C_G(y) \leq H^*$ for some maximal A-invariant subgroup $H^* \neq M^*$ of G.

Now $C_Q(y) \neq C_Q(\pi)$ by lemmas 1.16 and 4.5, so $[Z(P), C_Q(y)] \neq 1$.

Hence $Q \cap F(H^*) \neq 1$, so that $Q \leq H^*$ by assumption.

But then Q is abelian by lemma 4.4 so $y \in Z(Q)$, a contradiction.

Thus $C_G(y) \leqslant M^*$. □

We are now in a position to show that a maximal A-invariant subgroup H of the type mentioned in lemmas 5.2 to 5.7 cannot exist.

LEMMA 5.8 Let p be a prime dividing $|G|$, P the A-invariant Sylow p-subgroup of G and $M = N_G(P)$. Then there does not exist a maximal A-invariant subgroup H of G such that $1 \neq P_0 = P \cap F(H) < P_1 = P \cap H < P$.

PROOF

Suppose that such a subgroup H exists, so that the results of lemma 5.7 hold.

Let $r \neq p$ be a prime dividing $|H|$, R the A-invariant Sylow r-subgroup of G, $R_0 = R \cap F(H)$ and $R_1 = R \cap H$.

Then by lemma 5.1(vii) and (viii), $R_1 = C_{R_1}(Z(P)).R_0$ and $C_{R_0}(Z(P)) = 1$.

But $Z(P) \leqslant C_P(\pi)$ by lemmas 5.2(iv) and 5.3.

Thus by lemma 5.7(iv), $C_{R_1}(Z(P)) \leqslant M$.

It follows that $C_{R_1}(Z(P)) \leqslant C_G(\pi)$ and $C_{R_0}(\pi) = 1$.

Suppose that $C_{P_0}(\pi) \neq 1$.

Then for $x \in C_{P_0}(\pi)$, $C_G(x) \leqslant M$ by lemma 5.7(iv).

But then $F(H) < P_1.F(H) \leqslant C_G(x) \leqslant M$, contradicting lemma 4.2.

Hence $C_{P_0}(\pi) = 1$.

It follows that $C_{F(H)}(\pi) = 1$ and therefore π centralizes $H/F(H)$ by lemma 1.19.

By lemma 4.2, $\exists$ a prime $q \neq p$ such that $Q_0 = Q \cap F(H) \neq 1$ where Q is the A-invariant Sylow q-subgroup of G. Let $Q_1 = Q \cap H$ and suppose first that $Q_0 < Q_1$.

If $Q_1 = Q$, $Q_0 \leqslant Z(H)$ by lemma 4.4, contradicting lemma 5.1(viii).

So $Q_1 < Q$ and hence the results of lemma 5.7 hold for q, $N_G(Q)$ and H.

Let X be a minimal A-invariant subgroup of P_1 such that $X \cap Z(P) = 1$ and suppose that $P_1 < C_P(X)$.

If $C_G(X) \not\leqslant M$, $C_P(X)$ is abelian by lemma 5.1(i), contradicting (ii) of that lemma.

So $C_G(X) \leqslant M$.

Now $X \times \Omega_1(Z(P))$ normalizes Q_0, so by lemma 1.14 $\exists$ an A-invariant subgroup X_1 of $X \times \Omega_1(Z(P))$ such that $C_{Q_0}(X_1) \neq 1$.

As $X_1 \neq \Omega_1(Z(P))$ by lemma 5.1(viii), we have $C_P(X_1) = C_P(X)$.

Thus $C_G(X_1) \leqslant M$ also.

But $C_{Q_0}(Z(P)) = 1$ by lemma 5.1(viii), so $[Z(P), C_{Q_0}(X_1)] \neq 1$.

Thus $O_{p'}(F(M)) \neq 1$, contradicting lemma 5.4.

Hence $P_1 = C_P(X)$.

Since $C_P(P_1) = P_1 < P$ by lemma 5.1(ii) and $Z(P)$ is cyclic by lemma 5.2(iv) and 5.3, we may apply lemma 1.17 to derive that $|N_P(P_1)/P_1| = p$ or p^2.

If $|N_P(P_1)/P_1| = p$, $N_P(P_1) = C_{N_P(P_1)}(\pi).P_0$ and so $P_0 \lhd N_P(P_1)$.

Thus $N_P(P_1) \leqslant N_G(P_0) = H$, a contradiction.

Hence $|N_P(P_1)/P_1| = p^2$.

Now clearly $|\Omega_1(P_0)| \geqslant p^2$ and $|\Omega_1(C_{P_1}(\pi))| = p$ by lemma 5.7(iii).

If $|\Omega_1(P_0)| = p^2$, since $C_{P_0}(Z(Q)) = 1$ by lemma 5.1(viii) we have $q|p^2 - 1$.

If $|\Omega_1(P_0)| > p^2$, by lemma 1.17 $N_P(P_1) = P$.

But then $Z(Q)$ normalizes P since $Z(Q) \leqslant C_Q(\pi) \leqslant M$, so $Z(Q)$ normalizes P/P_1.

Now $C_{P/P_1}(Z(Q)) = C_P(Z(Q)).P_1/P_1$ and $C_P(Z(Q)) = C_P(\pi)$ by lemma 5.7(i) and (iii).

Also $C_P(\pi) \leqslant P_1$, else $P = C_P(\pi).P_0$ and $P_0 \lhd P$.

Thus $Z(Q)$ acts non-trivially on P/P_1, so again $q|p^2 - 1$.

By the symmetric argument $p|q^2 - 1$ and as p and q are odd we have a contradiction.

Hence we may assume that $Q_0 = Q_1$, so that $Q_0 = Q$ i.e. $H = N_G(Q)$ for all primes $q \neq p$ dividing $|F(H)|$.

Now let $r \neq p$ be a prime dividing $|F(H)|$, so that $H = N_G(R)$ where R is the A-invariant Sylow r-subgroup of G.

Let $R^* = ZJ(C_R(\tau))$.

If $C_H(R^*) \not\leqslant F(H)$, by lemma 4.9 for some prime $t||H/F(H)|$, $\exists x \in C_{T\cap H}(\pi) - F(H)$ such that $r||C_G(x)|$ where T is the A-invariant Sylow t-subgroup of G.

By lemma 5.7(v), $C_G(x) \leqslant N_G(T) = C_G(\pi).F(N_G(T))$.

But π acts f.p.f. on R, so that $R \cap N_G(T) \leqslant F(N_G(T))$.

Hence $R \cap N_G(T) = R$ so that $N_G(T) = N_G(R) = H$, a contradiction.

Thus $C_H(R^*) \leqslant F(H)$ and so by lemma 4.10 $C_G(\tau)$ has a normal r-complement.

Next, let q be a prime dividing $|E|$, Q the A-invariant Sylow q-subgroup of G and $M^* = N_G(Q)$ as in lemma 5.7(v).

By lemma 4.5 $C_Q(\pi) < Q$, so $C_Q(\tau) \neq 1$.

Let $Q^* = ZJ(C_Q(\tau))$, and suppose that $C_{M^*}(Q^*) \nleqslant F(M^*)$.

Now $F(M^*) \leqslant C_{M^*}(C_{Z(Q)}(\tau))$, so if $C_{Z(Q)}(\tau) \neq 1$ we must have $F(M^*) = C_{M^*}(C_{Z(Q)}(\tau))$ by lemma 5.7(v)(b).

But $C_{M^*}(Q^*) \leqslant C_{M^*}(C_{Z(Q)}(\tau))$, so $C_{Z(Q)}(\tau) = 1$.

Hence $Z(Q)$ is centralized by π, and since $C_Q(\pi)$ is cyclic, $\Omega_1(C_Q(\pi)) = \Omega_1(Z(Q))$.

Choose $x \in Q^*$ of order q. Then $C_{M^*}(x) \nleqslant F(M^*)$.

Now by hypothesis $\exists y \notin F(M^*)$ such that y centralizes x, and w.l.o.g. we may assume that y is an s-element for some prime $s \neq q$. Let S be the A-invariant Sylow s-subgroup of G.

Since $C_S(\pi).O_s(F(M^*))$ is a Sylow s-subgroup of M^*, $\exists g \in M^*$ such that $y^g = ab$ where $a \in C_S(\pi)$ and $b \in O_s(F(M^*))$.

Now ab and b centralize x^g, so a must also.

If $a \in C_G(\pi) - F(M^*)$, by lemma 5.7(v)(b) $x^g \in C_Q(\pi)$.

But then $x^g \in \Omega_1(C_Q(\pi)) = \Omega_1(Z(Q))$, so $x \in Z(Q)$.

Contradiction.

Thus $a \in F(M^*)$, and hence $y^g \in F(M^*)$.

But it then follows that $y \in F(M^*)$, a contradiction.

Hence $C_{M^*}(Q^*) \leqslant F(M^*)$.

Thus by lemma 4.10, $C_G(\tau)$ has a normal q-complement.

Let $R_\tau = C_R(\tau)$, $Q_\tau = C_Q(\tau)$ and $W = C_G(\tau)$.

Then R_τ is a Sylow r-subgroup of W by [22], theorem 4.3.

As $R_\tau \leqslant O_{q'}(W)$, $W = N_W(R_\tau).O_{q'}(W)$ by [9], theorem 1.3.7.

Thus $Q_\tau^x \leqslant N_W(R_\tau)$ for some $x \in W$.

It follows that $Q_\tau^x \leqslant N_W(R_\tau) \cap O_{r'}(W)$, so that $[R_\tau, Q_\tau^x] = 1$.

In particular, $q \mid |C_G(R_\tau)|$ and $r \mid |C_G(Q_\tau)|$.

Now choose an element $a \in \Omega_1(C_{Z(R)}(\tau))$.

Clearly $F(H) \leqslant C_H(a)$, so if $C_H(a) \neq F(H)$ we can choose $x \in C_H(\pi) - F(H)$ such that x centralizes a.

W.l.o.g. x is a t-element for some prime t.

Now by lemma 5.7(v), $C_G(x) \leqslant N_G(T) = C_G(\pi).F(N_G(T))$, where T is the A-invariant Sylow t-subgroup of G.

Thus $a \in N_G(T)$, so that $r \mid |N_G(T)|$.

Since π acts f.p.f. on R, it follows that $R \cap N_G(T) \leqslant F(N_G(T))$.

Thus $R \cap N_G(T) = R$ and $R \lhd N_G(T) = H$, so that $x \in F(H)$, a contradiction.

Therefore $C_H(a) = F(H)$.

Now R is a Sylow r-subgroup of $C_G(a)$ and $H = N_G(ZJ(R))$ so $N(ZJ(R)) \cap C_G(a) = F(H)$ has a normal r-complement.

Hence by [5], theorem D, $C_G(a)$ has a normal r-complement.

Suppose that R contains an A-invariant subgroup $W \cong E_{r^4}$ and let $\tilde{Q}$ be an R-invariant Sylow q-subgroup of $C_G(a)$.

Then by lemma 1.15 $\exists$ an A-invariant subgroup W_0 of W such that $C_{\tilde{Q}}(W_0) \neq 1$.

Let $N_G(W_0) \leqslant K$, a maximal A-invariant subgroup of G.

Then $p \mid |K|$ because $[P_0, R] = 1$.

Suppose first that $P \cap F(K) \neq 1$.

Now $K \neq M$ since $r \nmid |M|$, so we must have $p \mid |K/F(K)|$.

And $P \cap K \neq P$, else P would be abelian by lemma 4.4, contradicting the fact that P_1 is self-centralizing in P (lemma 5.1(ii)).

Thus by the argument at the beginning of this proof applied to K we have $K = C_K(\pi).F(K)$, so that $R \cap K \leqslant F(K)$.

Hence $R \cap K = R$ and so $K = N_G(R) = H$.

Thus $C_{\tilde{Q}}(W_0) \leqslant H$ and centralizes a.

But then $C_{\tilde{Q}}(W_0) \leqslant C_H(a) = F(H)$, contradicting lemma 5.2(iii).

Thus we may assume that $P \cap F(K) = 1$.

Since $P_0 \leqslant K$ and $P_0 \cap F(K) = 1$, we must have $C_{F(K)}(\pi) \neq 1$ by lemma 1.19.

Choose a prime $s \mid |F(K)|$ such that $C_{S \cap F(K)}(\pi) \neq 1$, where S is the A-invariant Sylow s-subgroup of G.

Then $s \mid |E|$, so if $S \leqslant F(K)$ we have

$K = N_G(S) = C_G(\pi).F(K)$ by lemma 5.7(v)(a), a contradiction.

So $1 \neq S \cap F(K) < S \cap K = S_1$, and by lemma 5.1(i)

S_1 is abelian.

Thus for $x \in C_{S \cap F(K)}(\pi)$, $F(K) < S_1.F(K) \leqslant C_G(x) \leqslant N_G(S)$

by lemma 5.7(v)(c).

Hence $F(K) < K \cap N_G(S)$, contradicting lemma 4.2.

It follows that R cannot contain such a subgroup W,

so that by lemma 1.18 there is no element of order r

in $R - Z(R)$.

Finally, choose an element $b \in Q$ such that $b \neq 1$

and $b \in C_{Z(Q)}(\tau)$ if $Z(Q)$ is non-cyclic or $b \in C_Q(\tau)$

if $Z(Q)$ is cyclic.

In the first case $C_{M*}(b) = F(M*)$ by lemma 5.7(v)(b).

Suppose in the second case that $C_{M*}(b) \not\leqslant F(M*)$.

Then by lemma 4.9, for some prime $t \mid |M*/F(M*)|$,

$\exists x \in C_{T \cap M*}(\pi) - F(M*)$ such that if B is a maximal

A-invariant subgroup of G containing $C_G(x)$ then

$C_{Q \cap B}(\pi) \neq Q \cap B$.

Now by lemma 5.7(v), $C_G(x) \leqslant N_G(T) = C_G(\pi).F(N_G(T))$.

Hence $Q \cap N_G(T) \leqslant C_G(x)$, so that $C_Q(\pi) < C_Q(x)$,

contradicting lemma 5.7(v)(b).

Thus in both cases $C_{M*}(b) \leqslant F(M*)$, and $C_G(b) \cap C_Q(\pi) \neq 1$.

Let $Q_2 = C_Q(b)$. Then Q_2 is a Sylow q-subgroup of

$C_G(b)$ by lemma 4.8.

As $Z(Q) \leqslant ZJ(Q_2)$, $N_G(ZJ(Q_2)) \leqslant M*$ by lemma 4.7.

Thus $N(ZJ(Q_2)) \cap C_G(b)$ has a normal q-complement.

It then follows from [5], theorem D, that $C_G(b)$ has a normal q-complement.

Let $\tilde{R}$ be a $\langle Q_2, \tau\rangle$-invariant Sylow r-subgroup of $O_{q'}(C_G(b))$.

Then $\tilde{R}^g \leqslant R$ for some element $g \in C_G(\tau)$ by [9], theorem 6.2.2.

Thus $\Omega_1(Z(\tilde{R}^g)) \leqslant \Omega_1(Z(R))$ since there is no element of order r in $R - Z(R)$.

Clearly $|\Omega_1(Z(R))| = r^2$, so if $|\Omega_1(Z(\tilde{R}^g))| = r^2$ we must have $\Omega_1(Z(\tilde{R}^g)) = \Omega_1(Z(R))$.

Hence $q \mid |N_G(\Omega_1(Z(R)))|$ i.e. $q \mid |H|$.

Since $q \nmid |F(H)|$ by lemma 5.2(iii), this yields $Q_2^g \leqslant C_Q(\pi)$, a contradiction.

Hence $|\Omega_1(Z(\tilde{R}^g))| = r$, so that $\Omega_1(Z(\tilde{R})) = \langle a^{g^{-1}}\rangle = \langle a_1\rangle$ say.

Thus for $y \in C_G(b) \cap C_Q(\pi)$, $y^{-1}a_1y = a_1^i$ for some integer i.

Applying τ, we get $ya_1y^{-1} = a_1^i$ since $a_1 = a^{g^{-1}} \in C_G(\tau)$.

Thus y^2, and hence y, centralizes a_1.

But then $a_1 \in C_G(y) \leqslant M^*$ by lemma 5.7(v)(c).

So $r \mid |M^*|$, a contradiction which completes the proof. $\square$

CHAPTER SIX

PROOF OF THE MAIN THEOREM

In this chapter we complete the proof of theorem II which was commenced in the previous chapter. Thus we continue to examine a minimal counter-example G to theorem II, and the argument is again presented in a sequence of lemmas. We have shown in Chapter five that if p is a prime dividing $|G|$ and $H \neq N_G(P)$ is a maximal A-invariant subgroup of G such that $P \cap F(H) \neq 1$ where P is the A-invariant Sylow p-subgroup of G then $P \leq H$. We show next that in fact there cannot exist a maximal A-invariant subgroup $H \neq N_G(P)$ of G with $P \cap F(H) \neq 1$ and then use this result to complete the proof.

We first prove a result which will be used in both of these sections of this chapter.

LEMMA 6.1 Let p be a prime dividing $|G|$, P the A-invariant Sylow p-subgroup of G and $M = N_G(P)$. If $H \neq M$ is a maximal A-invariant subgroup of G containing $F(M)$ then at least one of the following does not hold:

(1) π acts f.p.f. on $M/F(M)$.

(2) $|F(M)|$ and $|M/F(M)|$ are coprime.

(3) $C_G(\pi) \not\leq M$.

(4) $C_G(\pi) \leq H$.

(5) If K is a maximal A-invariant subgroup of G such that $K \neq H$ or M then π acts f.p.f. on $F(K)$, $K = C_K(\pi).F(K)$ and $(|F(K)|, |H_0|) = 1$ where H_0 is the complement of $Z(H)$ in $F(H)$.

(6) $\forall x \in C_G(\pi)$ such that $x \neq 1$, $C_G(x) \leqslant H$.

<u>PROOF</u>

Suppose to the contrary that all six properties hold for H and M.

Then by lemma 4.3 $F(M)$ is abelian, $H = F(M).F(H)$, $Z(H) = F(H) \cap F(M)$ and $F(H) = Z(H) \times H_0$ where $(|H_0|, |F(M)|) = 1$.

Let q be a prime dividing $|H_0|$ and let Q be the A-invariant Sylow q-subgroup of G.

Let r be a prime dividing $|M/F(M)|$ and let R be the A-invariant Sylow r-subgroup of G and $N = N_G(R)$.

Since $N \neq H$ or M, $N = C_N(\pi).F(N)$ by hypothesis (5).

Let $R^* = ZJ(C_R(\tau))$ and suppose that $C_N(R^*) \nleqslant F(N)$.

Then by lemma 4.9, for some prime $t \,|\, |N/F(N)|$, $\exists x \in C_{T\cap N}(\pi) - F(N)$ such that $r \,|\, |C_G(x)|$, where T is the A-invariant Sylow t-subgroup of G.

But by hypothesis (6), $C_G(x) \leqslant H$ so that $r \,|\, |H|$, a contradiction.

Thus $C_N(R^*) \leqslant F(N)$.

It then follows from lemma 4.10 that $C_G(\tau)$ has a normal r-complement.

Thus we may assume w.l.o.g. that $R_\tau = C_R(\tau)$ normalizes

$Q_\tau = C_Q(\tau)$, since R_τ and Q_τ are Sylow subgroups of $C_G(\tau)$ by [22], theorem 4.3.

Let $\tilde{Q} = C_Q(Q_\tau)$.

Then $Z(Q) \leqslant \tilde{Q}$, so $N_G(\tilde{Q}) \leqslant H$ by lemma 4.7.

But $\tilde{Q}$ is a Sylow q-subgroup of $C_G(Q_\tau)$ by lemma 4.8, so by [9], theorem 1.3.7,

$$N_G(Q_\tau) = (N_G(Q_\tau) \cap N_G(\tilde{Q})).C_G(Q_\tau).$$

Now $R_\tau \leqslant N_G(Q_\tau)$ and $r \nmid |H|$, so $R_\tau \leqslant C_G(Q_\tau)$.

Suppose first that R contains an A-invariant subgroup $W \cong E_{r^4}$ and take $x \in C_{Z(R)}(\tau)$.

Then by hypothesis (6), $C_N(x) = F(N)$.

Now R is a Sylow r-subgroup of $C_G(x)$ and $N = N_G(ZJ(R))$.

So $N(ZJ(R)) \cap C_G(x) = F(N)$ has a normal r-complement.

Thus by [5], theorem D, $C_G(x)$ has a normal r-complement.

Let Q^* be an $\langle R,\tau\rangle$-invariant Sylow q-subgroup of $C_G(x)$ and suppose w.l.o.g. that $Q_\tau \leqslant Q^*$.

Then $W \cap R_\tau$ centralizes Q_τ, so by lemma 1.15 $\exists$ an A-invariant subgroup W_0 of W such that $C_{Q^*}(W_0) \neq 1$.

Let $N_G(W_0) \leqslant K$, a maximal A-invariant subgroup of G.

Since $r \nmid |H|$ and $q \nmid |M|$, $K \neq H$ or M.

Thus by hypothesis (5), $K = C_K(\pi).F(K)$.

Since π acts f.p.f. on R we must have $R \leqslant F(K)$ and hence $K = N$.

Thus $C_{Q^*}(W_0) \leqslant N$ and centralizes x.

But then $C_{Q*}(W_0) \leqslant C_N(x) = F(N)$, contradicting hypothesis (5).

Thus R cannot contain an A-invariant subgroup $W \cong E_{r^4}$.

It follows from lemma 1.18 that $R - Z(R)$ contains no element of order r, and since $|\Omega_1(Z(R))| = r^2$ R_τ must be cyclic.

Let $R_2 = \langle r_2 \rangle = \Omega_1(R_\tau)$. Then $\Omega_1(Z(R)) = \langle r_2, r_2^{\pi} \rangle$.

If the Sylow r-subgroup of $C_G(Q_\tau)$ is not cyclic, we must have $\Omega_1(Z(R)) \leqslant C_G(Q_\tau)$.

But then $Q_\tau \leqslant C_G(\Omega_1(Z(R))) \leqslant N$, yielding a contradiction as above.

So $C_G(Q_\tau)_r$ is cyclic, and since $R_\tau \leqslant C_G(Q_\tau)$ we have $R_\tau = C_G(Q_\tau)_r$.

By [9], theorem 1.3.7, $N_G(R_2) = (N_G(R_2) \cap N_G(R)).C_G(R_2)$

$$= N_N(R_2).C_G(R_2).$$

Clearly $F(N) \leqslant N_N(R_2)$, and if $y \in C_N(\pi)$ normalizes R_2 we must have $R_2 \leqslant C_G(y)$ since τ inverts y and centralizes R_2.

But then $R_2 \leqslant H$ by hypothesis (4), a contradiction.

So $N_N(R_2) = F(N)$ and hence $N_G(R_2) = C_G(R_2)$.

Let $t \neq r$ be a prime dividing $|C_G(Q_\tau)|$.

Then $(C_G(Q_\tau) \cap N(R_\tau))_t$ is an r'-group of automorphisms of R_τ and hence of $R_2 = \Omega_1(R_\tau)$.

So $(C_G(Q_\tau) \cap N(R_\tau))_t$ centralizes $\Omega_1(R_\tau)$, and hence centralizes R_τ by [9], theorem 5.2.4.

Since R_τ is an abelian Sylow r-subgroup of $C_G(Q_\tau)$,

this yields $R_\tau \leqslant Z(C_G(Q_\tau) \cap N(R_\tau))$ so that by [9], theorem 7.4.3, $C_G(Q_\tau)$ has a normal r-complement. Thus R_τ normalizes a Sylow q-subgroup of $C_G(Q_\tau)$, and since $\tilde{Q} = C_Q(Q_\tau)$ is a Sylow q-subgroup of $C_G(Q_\tau)$ by lemma 4.8, $r \mid |N_G(\tilde{Q})|$.
But $N_G(\tilde{Q}) \leqslant H$ by lemma 4.7, so $r \mid |H|$.
This contradiction completes the proof. $\square$

We show in the next six lemmas that the hypotheses (1) to (6) of lemma 6.1 hold for a maximal A-invariant subgroup H of G such that $1 \neq P \cap F(H) < P \cap H = P$.

LEMMA 6.2 Let p be a prime dividing $|G|$, P the A-invariant Sylow p-subgroup of G and $M = N_G(P)$. Suppose $\exists$ a maximal A-invariant subgroup H of G such that $1 \neq P_0 = P \cap F(H) < P \cap H = P$. Then π acts f.p.f. on $M/F(M)$.

PROOF

By lemma 4.4, P is abelian and $P_0 \leqslant Z(H)$. Thus $F(M) \leqslant C_G(P_0) \leqslant H$, so by lemma 4.2, $H \cap M = F(M)$. And by lemma 4.3, $F(M)$ is abelian, $H = F(H).F(M)$, $Z(H) = F(H) \cap F(M)$ and $F(H) = Z(H) \times H_0$ where $(|H_0|, |F(M)|) = 1$.
Suppose $\exists x \in M - F(M)$ such that $x \in C_G(\pi)$.
If $C_{Z(H)}(\pi) \neq 1$, $x \in C_G(C_{Z(H)}(\pi)) = H$ and so $x \in H \cap M = F(M)$, a contradiction. Thus $C_{Z(H)}(\pi) = 1$.

Let Q be an A-invariant Sylow q-subgroup of $F(H)$ for some prime q such that $q \nmid |F(M)|$ (such a subgroup must exist or else $F(H) \leqslant Z(H)$, contradicting [9], theorem 6.1.3).

Suppose first that $C_{Z(Q)}(\pi) = 1$.

If $C_{F(M)}(\pi) = 1$, $[Z(Q), F(M)] = 1$ and then $Z(Q) \leqslant Z(H)$, a contradiction.

So $C_{F(M)}(\pi) \neq 1$ and hence $C_G(\pi) \leqslant M$ by lemma 4.2.

It follows that $C_Q(\pi) = 1$ and $C_{F(H)}(\pi) = 1$.

Hence π centralizes $H/F(H)$ by lemma 1.19.

But then $P_0 = [P, \langle\pi\rangle]$ is normalized by $C_M(\pi)$, so $x \in C_M(\pi) \leqslant H$, a contradiction.

Hence we may assume that $C_{Z(Q)}(\pi) \neq 1$.

Suppose $\exists y \in C_{F(M)}(\pi)$ with $y \neq 1$.

Then $y \notin Z(H)$ since $C_{Z(H)}(\pi) = 1$, so that $F(H) < F(H).\langle y\rangle \leqslant C_G(C_{Z(Q)}(\pi))$.

It follows from lemma 4.2 that $C_G(C_{Z(Q)}(\pi)) \leqslant H$, and in particular $C_G(\pi) \leqslant H$.

But then $\langle x\rangle.F(M) \leqslant M \cap H$, contradicting the same lemma.

So $C_{F(M)}(\pi) = 1$ and therefore $M/F(M)$ is centralized by π.

Take $y \in C_{Z(Q)}(\pi)$ and suppose $C_G(y) \leqslant M^*$ where M^* is a maximal A-invariant subgroup of G.

Then clearly $M^* \neq M$ and $M^* \neq H$.

But $F(H) \leqslant M^*$, so by lemma 4.3 $M^* = F(H).F(M^*)$.

Now if $Z(H) < F(M) \cap M^*$, $F(H) < M^* \cap H$ since $Z(H) = F(M) \cap F(H)$.

Thus $F(M) \cap M^* = Z(H)$.

Let s be a prime dividing $|M/F(M)|$ and S the A-invariant Sylow s-subgroup of G. Then $C_{S \cap M}(\pi) \neq 1$.
Thus $[Z(H), C_{S \cap M}(\pi)] \leqslant F(M) \cap M^* = Z(H)$.
Hence $C_{S \cap M}(\pi) \leqslant H$ and so $F(M) < M \cap H$, a contradiction.

It follows that π acts f.p.f. on $M/F(M)$ by [9], theorem 6.2.2(iv). □

LEMMA 6.3 Let p be a prime dividing $|G|$, P the A-invariant Sylow p-subgroup of G and $M = N_G(P)$.
Suppose $\exists$ a maximal A-invariant subgroup H of G such that $1 \neq P_0 = P \cap F(H) < P \cap H = P$.
Then $|F(M)|$ and $|M/F(M)|$ are coprime.

PROOF

Suppose that $\exists$ a prime s such that $1 \neq S \cap F(M) < S \cap M = S$ where S is the A-invariant Sylow s-subgroup of G. Let $M^* = N_G(S)$.
Then by lemma 6.2 π acts f.p.f. on $M^*/F(M^*)$ and by lemmas 4.2, 4.3 and 4.4 $M = F(M^*).F(M)$.
Thus $C_{F(M^*)}(\pi) \neq 1$, and since π acts f.p.f. on $M/F(M)$ and $F(M^*) \cap F(M) = Z(M)$ by lemma 4.3 we must have $C_{Z(M)}(\pi) \neq 1$.
Thus $C_G(\pi) \leqslant F(M)$.
It follows that π must act f.p.f. on H_0 where H_0 is the A-invariant complement of $Z(H)$ in $F(H)$.
Suppose that π doesn't centralize $H/F(H)$.
Since $F(M)$ is abelian, $\exists$ an A-invariant subgroup $Y \leqslant F(M)$ such that $Y \not\leqslant Z(H)$ and $C_Y(\pi) = 1$.

But then Y centralizes H_0 by lemma 1.2(4), and so Y centralizes $H_0.Z(H) = F(H)$, contradicting [9], theorem 6.1.3.

Hence π centralizes $H/F(H)$, and it follows that $C_{Z(H)}(\pi) \neq Z(H)$ or else $F(M) = C_G(\pi)$, contradicting lemma 4.5.

Clearly $P_\tau = C_P(\tau) \leqslant Z(H)$, so if q is a prime dividing $|H_0|$ and Q is the A-invariant Sylow q-subgroup of H_0, $Q \leqslant C_G(P_\tau)$.

Thus by [9], theorem 1.3.7, $N_G(P_\tau) = N_{N_G(P_\tau)}(Q).C_G(P_\tau)$

$$= H.C_G(P_\tau) = C_G(P_\tau).$$

Since S is abelian by lemma 4.4, we can take an A-invariant subgroup $Y_1 \cong Z_s \times Z_s$ of S such that $Y_1 \not\leqslant F(M)$ and $C_{Y_1}(\pi) = 1$.

Let $Y_1 = \langle y, y^\pi \rangle$ where $y \in C_S(\tau)$.

Then $P^* = C_P(y) \times C_P(y^\pi) \times C_P(y^{\pi^2})$ is a $\langle Y_1, A\rangle$-invariant p-group.

Clearly y normalizes P_τ, so $P_\tau \leqslant C_P(y)$.

But then for $x \in P_\tau$, $1 \neq xx^\pi x^{\pi^2}$ is centralized by A.

This contradiction completes the proof. □

<u>LEMMA 6.4</u> Let p be a prime dividing $|G|$, P the A-invariant Sylow p-subgroup of G and $M = N_G(P)$. Suppose $\exists$ a maximal A-invariant subgroup H of G such that $1 \neq P_0 = P \cap F(H) < P \cap H = P$.

Then $C_G(\pi) \not\leqslant M$.

PROOF

Suppose to the contrary that $C_G(\pi) \leq M$.

Then $C_G(\pi) \leq F(M)$ by lemma 6.2.

Let r be a prime dividing $|M/F(M)|$, R the A-invariant Sylow r-subgroup of G, $M^* = N_G(R)$ and $R_1 = R \cap M$.

Then clearly π acts f.p.f. on R.

Since $F(M)$ is an abelian Hall subgroup of G, it follows that if $F(M) \cap F(M^*) \neq 1$ then $F(M) \leq M^*$.

But then $R_1.F(M) < M \cap M^*$, contradicting lemma 4.2.

So $F(M) \cap F(M^*) = 1$.

In particular, π acts f.p.f. on $F(M^*)$ and so π centralizes $M^*/F(M^*)$.

Thus $[C_{M^*}(\pi),R_1] \leq F(M) \cap R = 1$.

Let $N_G(R_1) \leq N$ where N is a maximal A-invariant subgroup of G and suppose that $N \neq M^*$.

Let $M = M_1.F(M)$ where M_1 is the A-invariant Hall subgroup of M such that $M_1 \cap F(M) = 1$.

Then M_1 is abelian by theorem 1.11, so $M_1 \leq N$.

If $r \mid |F(N)|$, we must have $1 \neq R \cap F(N) < R \cap N = R$.

But then R is abelian by lemma 4.4 and so $R_1 \leq Z(M^*)$.

It follows that $N_G(R_1) = M^*$, a contradiction.

Hence we may assume that $r \nmid |F(N)|$.

Since π acts f.p.f. on R, $\exists$ a prime $s \mid |F(N)|$ such that $C_{S\cap F(N)}(\pi) \neq 1$ where S is the A-invariant Sylow s-subgroup of G.

Hence $S \leq F(M)$, so $1 \neq S \cap F(N) < S \cap N = S$.

But then $F(M) \leqslant C_G(S \cap F(N)) \leqslant N$, so $M = M_1.F(M) \leqslant N$, a contradiction.

Thus we must have $N_G(R_1) \leqslant M^* = N_G(R)$.

Thus $[C_{M*}(\pi), M_1] \leqslant F(M) \cap F(M^*) = 1$, so $C_{M*}(\pi)$ centralizes $M_1.F(M) = M$ i.e. $C_{M*}(\pi) \leqslant Z(M)$.

Let v be a prime dividing $|C_{M*}(\pi)|$ and V the A-invariant Sylow v-subgroup of G.

Then $V \leqslant F(M)$ and so V is abelian.

Since $V \cap Z(M) \neq 1$, by [9], theorem 7.4.4(ii) $M \neq O^v(M)$.

But then by [5], Corollary 2.2, $G \neq O^v(G)$.

This contradiction completes the proof. □

LEMMA 6.5 Let p be a prime dividing $|G|$, P the A-invariant Sylow p-subgroup of G and $M = N_G(P)$.

Suppose $\exists$ a maximal A-invariant subgroup H of G such that $1 \neq P_0 = P \cap F(H) < P \cap H = P$.

Then w.l.o.g. $C_G(\pi) \leqslant H$.

PROOF

Suppose that $C_G(\pi) \nleqslant H$, so that $C_{Z(H)}(\pi) = 1$.

Choose $x \in C_{F(M)}(\pi)$. Then $C_G(x) \leqslant H^*$ for some maximal A-invariant subgroup H^* of G.

Clearly $C_G(\pi)$ and $F(M)$ are contained in H^*, so $H^* \neq H$ by hypothesis and $H^* \neq M$ by lemma 6.4.

By lemma 4.3, $H^* = F(M).F(H^*)$.

Suppose that $F(M) \cap F(H^*) = 1$.

Then $F(H^*)$ is a Hall subgroup of G.

Now choose minimal A-invariant subgroups $X \leqslant C_{F(M)}(\pi)$ and $Y \leqslant Z(H)$.

If $C_{F(H^*)}(Y) \neq 1$, $F(H^*) \cap H \neq 1$. But $H = H_0.F(M)$ where $F(H) = H_0 \times Z(H)$ and $(|H_0|,|Z(H)|) = 1$, so $F(H^*) \cap H_0 \neq 1$.

Since H_0 is a Hall subgroup of G by lemma 4.3, it follows that $H = H^*$. Thus $C_{F(H^*)}(Y) = 1$.

Now by lemma 1.14(c), X must centralize $F(H^*)$, contradicting [9], theorem 6.1.3.

Thus $F(M) \cap F(H^*) \neq 1$, so $\exists$ a prime t such that $M = N_G(T)$ where T is the A-invariant Sylow t-subgroup of G and $1 \neq T \cap F(H^*) < T \cap H^* = T$.

W.l.o.g., we may take $t = p$ and $H^* = H$. □

<u>LEMMA 6.6</u> Let p be a prime dividing $|G|$, P the A-invariant Sylow p-subgroup of G and $M = N_G(P)$. Suppose $\exists$ a maximal A-invariant subgroup H of G such that $1 \neq P_0 = P \cap F(H) < P \cap H = P$.

Let K be a maximal A-invariant subgroup of G such that $K \neq H$ or M. Then π acts f.p.f. on $F(K)$, $K = C_K(\pi).F(K)$ and $(|F(K)|,|H_0|) = 1$ where $F(H) = H_0 \times Z(H)$.

<u>PROOF</u>

If $(|F(K)|,|H|) = 1$, the result follows immediately from lemma 6.5.

So we may assume that $(|F(K)|,|H|) \neq 1$.

Suppose first that $\exists$ a prime $q \mid (|F(K)|,|H_0|)$ and let Q be the A-invariant Sylow q-subgroup of G.

Then $Q \leqslant H_0$ and so $1 \neq Q \cap F(K) < Q \cap K = Q$.

Thus by lemma 6.3, $|F(H)|$ and $|H/F(H)|$ are coprime, a contradiction.

Hence $(|F(K)|, |H_0|) = 1$ so that $F(K) \cap H \leqslant F(M)$.

Let t be a prime dividing $(|F(K)|, |F(M)|)$ and let T be the A-invariant Sylow t-subgroup of G.

Then $T \leqslant F(M)$ by lemma 6.3, so $1 \neq T \cap F(K) < T \cap K = T$.

Thus by lemmas 4.3 and 4.4, $K = F(M).F(K)$ and $Z(K) = F(M) \cap F(K)$.

If $C_{Z(K)}(\pi) \neq 1$, $C_G(\pi) \leqslant K$.

But then by lemmas 6.4 and 6.5, $H_0 \cap F(K) \neq 1$, a contradiction.

So $C_{Z(K)}(\pi) = 1$, and clearly $C_{K_0}(\pi) = 1$ where K_0 is the A-invariant complement of $Z(K)$ in $F(K)$.

Hence $C_{F(K)}(\pi) = 1$ and so π centralizes $K/F(K)$ by lemma 1.19. □

<u>LEMMA 6.7</u> Let p be a prime dividing $|G|$, P the A-invariant Sylow p-subgroup of G and $M = N_G(P)$. Suppose $\exists$ a maximal A-invariant subgroup H of G such that $1 \neq P_0 = P \cap F(H) < P \cap H = P$. Then $\forall x \in C_G(\pi)$ such that $x \neq 1$, $C_G(x) \leqslant H$.

<u>PROOF</u>

Let q be a prime dividing $|C_G(\pi)|$ and let Q be the A-invariant Sylow q-subgroup of G. Choose $1 \neq x \in C_Q(\pi)$ and let $F(H) = H_0 \times Z(H)$ where $(|H_0|, |Z(H)|) = 1$. Suppose first that $Q \leqslant H_0$ and that $C_G(x) \leqslant H^*$, a maximal A-invariant subgroup of G different from H.

If $1 \neq Q \cap F(H^*) < Q \cap H^* = Q$, by lemma 6.3 $|F(H)|$ and $|H/F(H)|$ are coprime, a contradiction.

Hence $Q \cap F(H^*) = 1$.

Now $H^* \neq H, M$ so by lemma 6.6 π acts f.p.f. on $F(H^*)$ and π centralizes $H^*/F(H^*)$.

In particular π centralizes $Q \cap H^*$.

But $C_Q(\pi) < C_Q(C_Q(\pi)) \leqslant C_Q(x)$ by lemmas 1.16 and 4.5, a contradiction.

Suppose next that $Q \leqslant F(M)$.

Then $F(M) \leqslant C_G(x)$, so if $C_G(x) \leqslant H^*$ for some maximal A-invariant subgroup H^* of G we have $H^* = F(M).F(H^*)$ by lemma 4.3 ($H^* \neq M$ since $1 \neq C_{H_0}(\pi) \leqslant C_G(x)$).

Since $1 \neq C_{H_0}(\pi) \leqslant H^*$ and $(|H_0|, |F(M)|) = 1$ by lemma 4.3, we must have $H_0 \cap F(H^*) \neq 1$.

It follows that $H^* = H$, so that $C_G(x) \leqslant H$. The result follows. □

We can now derive the main intermediate result of this chapter.

<u>LEMMA 6.8</u> Let p be a prime dividing $|G|$, P the A-invariant Sylow p-subgroup of G and $M = N_G(P)$. If H is a maximal A-invariant subgroup of G such that $H \neq M$ and $p \mid |H|$ then $P \cap F(H) = 1$.

<u>PROOF</u>

Suppose that $P_0 = P \cap F(H) \neq 1$, so that $P_0 < P \cap H = P$ by the results of chapter five.

Then the hypotheses (1) to (6) of lemma 6.1 hold by lemmas 6.2 to 6.7 respectively, and since $P_0 \leqslant Z(H)$ by lemma 4.2 we have $F(M) \leqslant H$.
But then lemma 6.1 yields a contradiction, completing the proof. $\square$

Our results thus far show that for any maximal A-invariant subgroup H of a minimal counter-example G, $F(H)$ is a Hall subgroup of G. We analyze this situation in the second half of this chapter to ultimately derive a proof of theorem II.

LEMMA 6.9 Let H be a maximal A-invariant subgroup of G and let $H = H_1.F(H)$ where H_1 is an A-invariant subgroup of H such that $(|H_1|,|F(H)|) = 1$. If K is a maximal A-invariant subgroup of G, $K \neq H$, then $[H_1 \cap K, F(H) \cap K] = 1$.

PROOF

By theorem 1.11, $[H_1 \cap K, F(H) \cap K] \leqslant F(H) \cap F(K) = 1.$ $\square$

LEMMA 6.10 Let H be a maximal A-invariant subgroup of G and let $H = H_1.F(H)$ where H_1 is an A-invariant subgroup of H such that $(|H_1|,|F(H)|) = 1$. Suppose that $1 \neq C_{H_1}(\pi) < H_1$.
Then if Y is any minimal A-invariant subgroup of H_1, $C_G(Y) \leqslant H$.

PROOF

Suppose H_1 contains a minimal A-invariant

subgroup Y with $C_G(Y) \nleq H$, and let K be a maximal A-invariant subgroup of G containing $C_G(Y)$.
Let p be a prime dividing $|F(H)|$ and let P be the A-invariant Sylow p-subgroup of G.
Then by [9], theorem 5.2.3, $P/\Phi(P) = C_{P/\Phi(P)}(Y) \times [P/\Phi(P),Y]$.
Now choose a minimal A-invariant subgroup X of H_1 such that $[X,\langle\pi\rangle] = X$ if $Y \leqslant C_G(\pi)$ and $X \leqslant C_G(\pi)$ if $[Y,\langle\pi\rangle] = Y$.
Since H_1 is abelian, $H_1 \leqslant C_G(Y) \leqslant K$.
Thus $[H_1, C_P(Y)] = 1$ by lemma 6.9.
In particular $[X, C_P(Y)] = 1$, so that X centralizes $C_{P/\Phi(P)}(Y) = C_P(Y)\Phi(P)/\Phi(P)$.
But X centralizes $[P/\Phi(P), Y]$ by lemma 1.14, so X centralizes $P/\Phi(P)$.
Hence by [9], theorem 5.1.4, X centralizes P.
It follows that X centralizes $F(H)$, contradicting [9], theorem 6.1.3. □

<u>COROLLARY 6.11</u> Let H be a maximal A-invariant subgroup of G and let $H = H_1.F(H)$ where H_1 is an A-invariant subgroup of H such that $(|H_1|, |F(H)|) = 1$.
Suppose that $1 \neq C_{H_1}(\pi) < H_1$.
Then H_1 is a Hall subgroup of G and $C_G(\pi) \leqslant H$.

<u>PROOF</u>

Let r be a prime dividing $|H_1|$, and let Y be a minimal A-invariant r-subgroup of H_1. Let R be the A-invariant Sylow r-subgroup of G.

Then by lemma 6.10, $Z(R) \leqslant C_G(Y) \leqslant H$.

But then if W is a minimal A-invariant subgroup of $Z(R)$, $R \leqslant C_G(W) \leqslant H$ by lemma 6.10.

It follows that H_1 is a Hall subgroup of G.

Since $C_{H_1}(\pi) \neq 1$, it follows at once from lemma 6.10 that $C_G(\pi) \leqslant H$. □

<u>LEMMA 6.12</u> Let H be a maximal A-invariant subgroup of G and let $H = H_1.F(H)$ where H_1 is an A-invariant subgroup of H such that $(|H_1|,|F(H)|) = 1$.

Then either π centralizes H_1 or π acts f.p.f. on H_1.

<u>PROOF</u>

Suppose that $1 \neq C_{H_1}(\pi) < H_1$.

Let r be a prime dividing $|H_1|$, R the A-invariant Sylow r-subgroup of G and $M = N_G(R)$.

We show that the hypotheses (1) to (6) of lemma 6.1 hold for M and H.

Let $M = M_1.F(M)$ where M_1 is an A-invariant subgroup of M such that $(|M_1|,|F(M)|) = 1$.

Hypothesis (2) holds trivially.

If π centralizes M_1, $M_1 \leqslant H$ by corollary 6.10.

Hence $[M_1,R] = 1$ by lemma 6.9, so that $R \leqslant Z(M)$.

But then G has a normal r-complement by [9], theorem 7.4.3, a contradiction.

If $1 \neq C_{M_1}(\pi) < M_1$, by corollary 6.11 M_1 is a Hall subgroup of G and $C_G(\pi) \leqslant M$.

Since $1 \neq C_{H_1}(\pi) < H_1$, $C_{F(H)}(\pi) \neq 1$.

Hence $\exists$ a prime $p \mid |F(H)|$ such that $C_P(\pi) \neq 1$

where P is the A-invariant Sylow p-subgroup of G.

But then $p \mid |M|$, so that $P \leqslant M$.

Since $H_1 \leqslant M$, we have $[H_1,P] = 1$ by lemma 6.9.

But then $P \leqslant Z(H)$, yielding a contradiction as above.

Hence π acts f.p.f. on M_1, so that (1) holds.

Now by lemma 6.10, $C_G(R) \leqslant H$ so $F(M) \leqslant H$.

Thus $F(M) \leqslant H_1$, and since $H_1 \leqslant M$ we must have $H_1 = F(M)$.

Clearly if K is any maximal A-invariant subgroup of G different from H and M, π must act f.p.f. on $F(K)$ and so must centralize $K/F(K)$. And $(|F(K)|,|F(H)|) = 1$ since $F(K)$ and $F(H)$ are Hall subgroups of G. Hence (5) holds. $C_G(\pi) \leqslant H$ by corollary 6.11, and since $1 \neq C_{H_1}(\pi) < H_1$ we must have $C_{F(H)}(\pi) \neq 1$ and so $C_G(\pi) \nleqslant M$. Thus (3) and (4) also hold.

Finally, let t be a prime dividing $|H|$ and let T be the A-invariant Sylow t-subgroup of G.

We show that $\forall x \in C_T(\pi)$ such that $x \neq 1$, $C_G(x) \leqslant H$.

If $T \leqslant F(M) = H_1$, the result follows from lemma 6.10.

So we may assume that $T \leqslant F(H)$.

Let $x \in C_T(\pi)$ such that $x \neq 1$ and suppose that $C_G(x) \nleqslant H$.

Let K be a maximal A-invariant subgroup of G containing $C_G(x)$.

Then $K \neq H$, so $T \nleqslant F(K)$.

Hence π centralizes $T \cap K$.

But $C_T(\pi) < C_T(C_T(\pi)) \leqslant C_G(x) \leqslant K$ by lemmas 1.16 and 4.5.

This contradiction proves that $C_G(x) \leqslant H$, and it follows that (6) holds as well.

Thus all of the hypotheses of lemma 6.1 hold for M and H, yielding a contradiction which completes the proof. □

LEMMA 6.13 Let H be a maximal A-invariant subgroup of G containing $C_G(\pi)$.

Let $H = H_1.F(H)$ where H_1 is an A-invariant subgroup of H such that $(|H_1|, |F(H)|) = 1$. Then π centralizes H_1.

PROOF

Suppose that π doesn't centralize H_1, so that by lemma 6.12 π acts f.p.f. on H_1.

Thus $C_G(\pi) \leqslant F(H)$, so that if K is a maximal A-invariant subgroup of G different from H, π must act f.p.f. on $F(K)$ and π must centralize $K/F(K)$ by lemma 1.19.

Choose a prime q dividing $|H_1|$, let Q be the A-invariant Sylow q-subgroup of G and let $K = N_G(Q)$.

Since π acts f.p.f. on Q, we must have $N_G(H_1 \cap Q) \leqslant K$ and in particular $H_1 \leqslant K$.

Let $K = K_1.F(K)$ where K_1 is an A-invariant subgroup of K such that $(|K_1|, |F(K)|) = 1$. Clearly $K_1 \leqslant F(H)$.

If $Q \leqslant C_G(K_1)$, $O^q(K) \neq K$ and hence $O^q(G) \neq G$ by [5], Corollary 2.2.

Thus $Q \nleqslant C_G(K_1)$.

Choose a prime p dividing $|K_1|$ such that $[P \cap K_1, Q] \neq 1$ where P is the A-invariant Sylow p-subgroup of G.

By [9], theorem 7.5.2, $Q \cap G' = Q \cap K'$.

Thus $Q = Q \cap [K_1, K] = [K_1Q, Q]$.

Let $\bar{Q} = Q/Z(Q)$ and $\bar{K}_1 = K_1Z(Q)/Z(Q)$.

Now it follows from [2], section 66, that Q has class $\leqslant 2$ and so $[\bar{Q},\bar{Q}] = \overline{Z(Q)}$.

Therefore $[\bar{K}_1,\bar{Q}] = [\overline{K_1Q}, \bar{Q}] = \bar{Q}$.

Hence $C_{\bar{Q}}(\bar{K}_1) = \overline{Z(Q)}$ i.e. $C_Q(K_1) \leqslant Z(Q)$.

Thus by lemma 6.9 $H_1 \cap Q \leqslant Z(Q)$.

Now take $x_1 \in H_1 \cap Q$ such that $x_1^\tau = x_1^{-1}$ and x_1 has order q.

Then $Q^* = \langle x_1, x_1^\pi \rangle$ is an A-invariant subgroup of $Z(K)$ of order q^2.

Let $P^* = C_P(x_1)$.

We show first that $K_1 \cap P < P^*$.

Let $\bar{P} = P/P'$.

Then by [9], theorem 5.2.3, $\bar{P} = C_{\bar{P}}(Q^*) \times [\bar{P},Q^*]$.

Clearly $C_P(Q^*) = K_1 \cap P$, and as in the proof of lemma 1.14 we have, w.l.o.g., $C_{[\bar{P},Q^*]}(x_1) \neq 1$.

It then follows from [9], theorem 6.2.2(iv) that $K_1 \cap P < C_P(x_1) = P^*$.

We prove next that P^* is inverted by τ.

Let $[\bar{P},Q^*] = \bar{P}_2$.

Then $C_{\bar{P}_2}(x_1) \times C_{\bar{P}_2}(x_1^{\pi}) \times C_{\bar{P}_2}(x_1^{\pi^2})$ is an A-invariant subgroup of $\bar{P}_2$ and if $\bar{y} \in C_{\bar{P}_2}(x_1)$ is centralized by τ, $\bar{y}\bar{y}^{\pi}\bar{y}^{\pi^2}$ is centralized by A, a contradiction.

Since $C_{\bar{P}_2}(x_1)$ is τ-invariant, it follows that it must be inverted by τ.

As $C_G(Q^*) \leqslant K$, $C_P(Q^*) \leqslant P \cap K_1$. Thus $C_{\bar{P}}(Q^*)$ is inverted by τ, so that $C_{\bar{P}}(x_1) = C_{\bar{P}}(Q^*) \times C_{\bar{P}_2}(x_1)$ is inverted by τ.

Now by applying the same argument to each factor of the derived series of P and then applying [9], theorem 6.2.2(iv) to each in turn in the reverse order we have that $C_P(x_1)$ is inverted by τ.

Hence P^* is abelian.

Now $C_P(P^*)$ is a Sylow p-subgroup of $C_G(P^*)$ by lemma 4.8, so by [9], theorem 1.3.7,

$N_G(P^*) = N_G(P^*) \cap N(C_P(P^*)).C_G(P^*)$.

But $Z(P) \leqslant C_P(P^*)$, so $N(C_P(P^*)) \leqslant H$ by lemma 4.7.

Now $C_G(P^*) \leqslant C_G(x)$ for $x \in K_1 \cap P$, and since $C_P(x) \geqslant C_P(C_P(\pi)) > C_P(\pi)$ by lemma 1.16, $C_G(x) \leqslant H$ by lemma 6.12.

Thus $N_G(P^*) \leqslant H$.

If P_1 is a Sylow p-subgroup of $C_G(x_1)$ containing P^*, the same argument as above yields $N_G(P_1) \leqslant H$ so that $P_1 \leqslant P$.

Thus $P_1 = P^*$ i.e. P^* is a Sylow p-subgroup of $C_G(x_1)$.

Let $N = N_G(P^*) \cap C_G(x_1)$.

Then $H_1 \leqslant N$ so $P^* \cap Z(N)$ centralizes H_1.

Hence $P^* \cap Z(N) \leqslant K_1$.

But $[P^* \cap K_1, H_1] = 1$ by lemma 6.9, and since P^* is an abelian Sylow p-subgroup of N, $P^* \cap K_1 \leqslant Z(N)$.

Thus $P^* \cap Z(N) = P^* \cap K_1$.

Thus by [9], theorem 7.4.4(ii), $\exists$ a subgroup Y_1 of $C_G(x_1)$ such that $C_G(x_1) = (P^* \cap K_1).Y_1$.

Repeating this argument for all prime divisors of $|K_1|$, we get $C_G(x_1) = K_1.Y$ where $Y \lhd C_G(x_1)$ and $Y \cap K_1 = 1$.

Now $N(ZJ(Q)) \cap Y = F(K)$ has a normal q-complement, so by [5], theorem D, Y has a normal q-complement.

Hence by [9], theorem 6.2.2(i), Y contains a Q-invariant Sylow p-subgroup P_0.

Then by [9]. theorem 1.3.7, $N_{C_G(x_1)}(P_0) = N_K(P_0).N_Y(P_0)$.

Now $N_{C_G(x_1)}(P_0)$ contains a Sylow p-subgroup of $C_G(x_1)$ and $P \cap K_1$ is a Sylow p-subgroup of K contained in $C_G(x_1)$.

Since $N_K(P_0)$ covers $N_{C_G(x_1)}(P_0)/N_Y(P_0)$, it follows that $\exists y \in K$ such that $(P \cap K_1)^y \leqslant N_{C_G(x_1)}(P_0)$.

Thus $(P \cap K_1)^y. P_0$ is a Sylow p-subgroup of $N_{C_G(x_1)}(P_0)$.

Now $C_P(x_1)$ is abelian, so $[P \cap K_1, P_0^{y^{-1}}] = 1$.

Since Q normalizes P_0 and $y \in K = N_G(Q)$, Q normalizes $P_0^{y^{-1}}$.

Let $\tilde{P} = P_0^{y^{-1}}$.

Then $[[K_1 \cap P, \tilde{P}], Q] = [1, Q] = 1$ and

$[[\tilde{P}, Q], K_1 \cap P] \leqslant [\tilde{P}, K_1 \cap P] = 1$.

Thus by [9], theorem 2.2.3, $[[K_1 \cap P, Q], \tilde{P}] = 1$.

But by assumption $Q_0 = [K_1 \cap P, Q] \neq 1$, so $\tilde{P} \leqslant C_G(Q_0)$.

Now if $N_G(Q_0) \leq H$, $Q_0 \leq H_1 \cap Q \leq Z(Q)$ and then $P \cap K_1.F(K) \leq H$, contradicting lemma 4.2.

Hence $N_G(Q_0) \not\leq H$, and so we must have $N_G(Q_0) \leq K$ since π centralizes $T/F(T)$ for every maximal A-invariant subgroup $T \neq H$.

In particular $\tilde{P} \leq K$ and hence $P_0 \leq K$ and $C_P(x_1) = P \cap K_1$.

This contradiction completes the proof. $\square$

LEMMA 6.14 $C_G(\tau)$ is nilpotent.

PROOF

Let t be a prime dividing $|G|$, T the A-invariant Sylow t-subgroup of G and $M = N_G(T) = M_1.F(M)$ where M_1 is the A-invariant complement of $F(M)$ in M.

If $C_{F(M)}(\pi) = 1$, π must centralize M_1 by lemma 1.2(4).

On the other hand if $C_Q(\pi) \neq 1$ for some A-invariant Sylow q-subgroup Q of G contained in $F(M)$, take $x \in C_Q(\pi)$.

Then by lemma 1.16 $C_Q(\pi) < C_Q(x)$, so it follows from lemma 6.12 that $C_G(x) \leq N_G(Q) = M$.

Hence $C_G(\pi) \leq M$, so by lemma 6.13 we again have that M_1 is centralized by π.

Let $T^* = ZJ(C_T(\tau))$ and suppose that $C_M(T^*) \not\leq F(M)$.

Then by lemma 4.9(1), for some prime p dividing $|M_1|$ $\exists x \in M_1 \cap P$ such that $t \mid |C_G(x)|$ where P is the A-invariant Sylow p-subgroup of G.

It follows from lemmas 1.16 and 6.12 that $C_G(x) \leq N_G(P)$ and so by lemma 4.9(2) we have $1 \neq C_{T\cap N_G(P)}(\pi) \neq T \cap N_G(P)$.

Thus by lemma 6.12 $T \leqslant F(N_G(P))$ i.e. $N_G(P) = M$.
This contradiction yields that $C_M(T^*) \leqslant F(M)$.
Now by lemma 4.10 $C_G(\tau)$ has a normal t-complement, and since t was arbitrary it follows that $C_G(\tau)$ is nilpotent. □

We are now in a position to complete the

PROOF OF THEOREM II

Let H be a maximal A-invariant subgroup of G containing $C_G(\pi)$.
Let $H = H_1.F(H)$ where H_1 is an A-invariant subgroup of H such that $(|H_1|, |F(H)|) = 1$.
Then by lemma 6.13, π centralizes H_1.
Let q be a prime dividing $|H_1|$, Q the A-invariant Sylow q-subgroup of G and $K = N_G(Q)$.
Let $K = K_1.F(K)$ where K_1 is an A-invariant subgroup of K such that $(|K_1|, |F(K)|) = 1$.
Clearly $Q \cap H_1 = C_Q(\pi)$, so by lemmas 1.16 and 6.12 we must have $N_G(Q \cap H_1) \leqslant K$.
It follows that $C_G(\pi) \leqslant K$, so that K_1 is centralized by π by lemma 6.13.
Suppose that for some prime $t \mid |H_1|$, $T \cap H_1$ is non-cyclic where T is the A-invariant Sylow t-subgroup of G.
Then by [9], theorem 5.3.16, applied to each Sylow subgroup of $F(H)$ we obtain $F(H) = \prod_{x \in T \cap H_1} C_{F(H)}(x)$.
But by lemmas 1.16 and 6.13 $C_G(x) \leqslant K$ $\forall x \in T \cap H_1$ so that $F(H) \leqslant K$, a contradiction.

It follows that H_1 is cyclic, and similarly K_1 is cyclic.

Hence $C_G(\pi)$ is cyclic.

Let p be a prime dividing $|K_1|$, P the A-invariant Sylow p-subgroup of G and assume w.l.o.g. that $H = N_G(P)$.

Choose $a \in C_P(\tau)$ and suppose that $C_H(a) \not\leqslant F(H)$.

Then by lemma 4.9(1), for some prime $s \,|\, |H_1|$ $\exists x \in S \cap H_1$ such that $p \,|\, |C_G(x)|$ where S is the A-invariant Sylow s-subgroup of G.

It follows from lemmas 1.16 and 6.12 that $C_G(x) \leqslant N_G(S)$ and from lemmas 4.9(2) and 6.12 that $N_G(S) = H$, a contradiction.

Thus $C_H(a) \leqslant F(H)$.

Let $P_2 = C_P(a)$. Then P_2 is a Sylow p-subgroup of $C_G(a)$ by lemma 4.8.

As $Z(P) \leqslant ZJ(P_2)$, $N_G(ZJ(P_2)) \leqslant H$ by lemma 4.7.

Thus $N_G(ZJ(P_2)) \cap C_G(a) \leqslant F(H)$ and so has a normal p-complement.

Hence by [5], theorem D, $C_G(a)$ has a normal p-complement.

By the symmetric argument we also have that $C_G(b)$ has a normal q-complement $\forall b \in C_Q(\tau)$.

Now suppose that P contains a characteristic non-cyclic abelian subgroup W.

Then $W = C_W(\pi) \times [W, \langle\pi\rangle]$ and since $C_G(\pi)$ is cyclic $[W, \langle\pi\rangle] \neq 1$.

If $|\Omega_1([W,<\pi>])| \geq p^4$, $[W,<\pi>]$ contains an A-invariant subgroup $E^* \cong E_{p^4}$.

Take $x \in C_W(\tau)$, so that $E^* \leq C_G(x)$ and E^* normalizes a Sylow q-subgroup $\tilde{Q} \neq 1$ of $C_G(x)$ ($C_Q(\tau) \leq C_G(x)$ by lemma 6.14).

Then by lemma 1.15, $C_{\tilde{Q}}(P_0) \neq 1$ for some A-invariant subgroup P_0 of E^*.

Since π acts f.p.f. on P_0 we must have $C_G(P_0) \leq H$.

But $C_{\tilde{Q}}(P_0) \leq C_G(x)$ and as $C_H(x) = F(H)$ we must have $C_{\tilde{Q}}(P_0) \leq F(H)$, a contradiction.

Hence $|\Omega_1([W,<\pi>])| = p^2$ and as $H_1 \cap Q$ acts f.p.f. on $\Omega_1([W,<\pi>])$ we have $q|p^2 - 1$.

Now if Q also contains a characteristic non-cyclic abelian subgroup we have $p|q^2 - 1$, a contradiction.

Hence we may assume w.l.o.g. that P does not contain a characteristic non-cyclic abelian subgroup, and that either Q does not contain such a subgroup either or that $p|q^2 - 1$.

In particular, $Z(P)$ is cyclic.

Thus by [9], theorem 5.4.9, P is the central product of an extra-special group E and a cyclic group R.

Clearly $R \leq Z(P)$ and so R is A-invariant.

Thus we may assume w.l.o.g. that E is also A-invariant.

Now $E/Z(E) = \bar{E} = C_{\bar{E}}(\pi) \times [\bar{E},<\pi>]$.

Since $H_1 \cap Q$ acts f.p.f. on $[\bar{E},<\pi>]$, if $|[\bar{E},<\pi>]| = p^2$ we have $q|p^2 - 1$.

Thus by lemma 1.3 we may assume w.l.o.g. that $|[E,<\pi>]| \geq p^4$.

But now it follows as in the proof of [9], theorem 5.5.2, that E is the central product of non-abelian groups of the type $\langle v, v^{\pi}, z\rangle$ where $v \in C_P(\tau)$ and $\langle z\rangle = Z(E)$.
Thus $C_P(v)$ is non-abelian, and so $\exists y \in C_{C_P(v)}(\tau)$ such that $y \notin Z(C_P(v))$.
Now y has p conjugates in P, and hence in $C_P(v)$.
As $\langle y, z\rangle \triangleleft C_P(v)$, the conjugates of y are contained in $\langle y, z\rangle$.
Now $C_P(v)$ is a Sylow p-subgroup of $C_G(v)$ by lemma 4.8, and since $C_G(v)$ has a normal p-complement we can select a Sylow q-subgroup $\tilde{Q}$ of $C_G(v)$ which is invariant under $\langle C_P(v), \tau\rangle$ by [9], theorem 6.2.2(i).
Let $1 = Z_0 < Z_1 < \ldots < Z_n = \tilde{Q}$ be the upper central series of $\tilde{Q}$, and let i be the least integer such that Z_{i+1} is not inverted by τ.
Then Z_i is inverted by τ, and since z normalizes Z_i and z is inverted by τ, z centralizes Z_i.
Hence $Z_i \leqslant C_G(z) \leqslant H$.
Now $C_{Z_{i+1}}(\tau) \neq 1$, so clearly $Z_{i+1} \nleqslant H$.
Since $Z_{i+1}/Z_i = C_{Z_{i+1}/Z_i}(z) \times [z, Z_{i+1}/Z_i]$ by [9], theorem 5.2.3, and $C_{Z_{i+1}}(z) \leqslant H$, it follows that $W_0/Z_i = [z, Z_{i+1}/Z_i] \neq Z_i$.
Since y normalizes W_0/Z_i, $W_0/Z_i = C_{W_0/Z_i}(y) \times [y, W_0/Z_i]$ by the same theorem.
If y centralizes W_0/Z_i, so will each conjugate of y in $C_P(v)$.

But then $\langle y,z\rangle$ centralizes W_0/Z_i, a contradiction.
Hence $[y, W_0/Z_i] \neq Z_i$.
Now $[y, W_0/Z_i]$ is invariant under $\langle \tau,z\rangle$ and since z acts f.p.f. on W_0/Z_i we have $C_{[y,\ W_0/Z_i]}(\tau) \neq 1$ by [9], theorem 5.3.14(iii).
Since $\tilde{Q}$ is τ-invariant, by [9], theorem 6.2.2, $\exists\alpha \in C_G(\tau)$ such that $C_Q(\tau)^{\alpha} \leqslant \tilde{Q}$.
But $C_Q(\tau)^{\alpha} = C_Q(\tau)$ by lemma 6.14.
Thus $C_{W_0}(\tau) \leqslant C_Q(\tau)$, so that $[y, C_{W_0/Z_i}(\tau)] = Z_i$.
Hence $C_{W_0/Z_i}(\tau) \leqslant C_{W_0/Z_i}(y)$, so that $C_{[y,\ W_0/Z_i]}(\tau) = 1$.
This contradiction completes the proof. $\square$

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ERRATA SHEET

p.2	ℓ.15	For 'klein-four group' read 'Klein four-group'
p.7	ℓ.15	For '$O^p(G)$' read '$G/O^p(G)$'
p.9	proof of (4)	Add '(see [9], theorem 10.1.5)'
p.12	lemma 1.5	Include in the hypothesis that $\lvert G\rvert$ is coprime to 6
p.14	ℓ.19	For '$xP \cap H$' read '$x(P \cap H)$'
p.22	ℓ.12	For 'lemma 1.14' read 'lemma 1.20'
p.25	ℓ.16	For 'or' read 'of'
p.28	ℓ.12	Add $x_o \neq 1$
p.31	lemma 1.16	For 'A' in the proof of the lemma read 'D'
p.36	lemma 1.19	Include in the hypothesis that $\lvert K\rvert$ is coprime to 6
p.45	ℓ.22	For 'lamma' read 'lemma'
p.126	ℓ.4 & ℓ.19	For x read x
p.127	ℓ.23	For x read x