# White Noise Analysis and Stochastic Evolution Equations

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Thesis submitted for the degree of
Doctor of Philosophy
in the Department of Pure Mathematics
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January 19, 2001

# Contents

# Abstract


Let $U$ and $H$ be real separable Hilbert spaces. This thesis is devoted to the construction of $H$-valued stochastic distributions and to stochastic evolution equations with additive noise

$$\begin{aligned} dX(t) &= AX(t)dt + BdW(t) \ , \quad t \in [0,T] \ , \\ X(0) &= \xi \in \mathcal{D}(A) \ , \end{aligned} \tag{1}$$

where $A$ is a linear operator on $H$, $B$ is a continuous linear map from $U$ to $H$ and $W(\cdot)$ is a $U$-valued Wiener process. The thesis consists of five parts. In **Chapter 1** spaces of $H$-valued stochastic test functions $S(H)_\rho$ and $H$-valued stochastic distributions $S(H)_{-\rho}$, $\rho \in [0,1]$, are constructed. Results regarding the strong topology of $S(H)_{-\rho}$, together with continuity, differentiation and integration in $S(H)_{-\rho}$ with respect to the strong topology on $S(H)_{-\rho}$ are studied.

In **Chapter 2** the Hermite transform of elements of $S(H)_{-1}$ is defined, which maps elements of $S(H)_{-1}$ to power series defined on particular subsets of $\mathbb{C}^{\mathbb{N}}$ with values in $H_{\mathbb{C}}$ ($H_{\mathbb{C}}$ being the complexification of $H$). Results regarding the strong convergence of sequences in $S(H)_{-1}$ and their Hermite transform are studied, which leads to results relating the continuity, differentiability or integrabilty of $S(H)_{-1}$ processes and their Hermite transform in the last part of the chapter.

In **Chapter 3** a sequence of independent Brownian motions $\{\beta_i(\cdot)\}_{i=1}^\infty$ is constructed so that the $U$-valued Wiener process

$$W(t) = \sum_{i=1}^{\infty} \beta_i(t) f_i \ ,$$

belongs to $S(U)_{-0}$, where $\{f_i\}_{i=1}^\infty$ is an orthonormal basis for $U$. This is used to define a generalised stochastic convolution for strongly continuous families of continuous linear maps from $U$ to $H$.

In **Chapter 4** equation (1) is shown to have a unique solution in $S(H)_{-1}$, when $A$ is the generator of a $C_0$-semigroup. The result is illustrated with the stochastic Heat and Wave equations.

In **Chapter 5** equation (1) is shown to have a unique solution in $S(H)_{-1}$, when $A$ is the generator of a non degenerate, 1-times integrated, exponentially bounded semigroup. The result is illustrated with the stochastic Wave equation.

# Statement of Originality

This work contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text.

I give consent to this copy of my thesis being made available for loan and photocopying.

Adelaide, 19 January, 2001

(Julian Sorensen)

# Acknowledgements

I would like to thank Dr. Alexei Filinkov for his guidance and supervision over the course of this research. I am especially grateful to Dr. Alexei Filinkov providing the idea for this research.

I also wish to thank my family and friends for their encouragement over the last three years.

Finally, special thanks go to Belinda for her support and understanding.

# Introduction

Many models in physics, biology, finance, etc, involve stochastic differential equations. Some stochastic evolution equations can be written in the following form

$$\begin{aligned} dX(t) &= AX(t)dt + BdW(t) \ , \quad t \in [0,T] \ , \\ X(0) &= \xi \in \mathcal{D}(A) \text{ a.e } , \end{aligned} \tag{2}$$

where $B$ is a continuous linear map from a Hilbert space $U$ to another Hilbert space $H$, $A$ is the generator of some semigroup on $H$ and $W(\cdot)$ is a $U$-valued Wiener process. One example of this is the stochastic Heat equation

$$\begin{aligned} d_tX(t,x) &= \triangle_x X(t,x)dt + dW(t,x) \ , \quad t \in [0,T] \ , \ x \in \mathcal{O} \ , \\ X(t,x) &= 0 \ , \quad t \in [0,T] \ , \ x \in \partial\mathcal{O} \ , \\ X(0,x) &= 0 \ , \quad x \in \mathcal{O} \ , \end{aligned} \tag{3}$$

where $dW(t,x)$, $t \in [0,T]$, $x \in \mathcal{O}$ is temporal and spatial white noise and

$$\mathcal{O} = \left\{ x \in \mathbb{R}^N : \ 0 < x_k < a_k \ , \ k = 1, \ldots, N \right\} \ .$$

Another is the stochastic Wave equation

$$\begin{aligned} dY_t'(t,x) &= \frac{\partial^2}{\partial x^2} Y(t,x)dt + dW(t,x) \ , \quad t > 0 \ , \ x \in \Omega = (0,1) \ , \\ Y(t,0) &= Y(t,1) = 0 \ , \quad t \in [0,T] \ , \\ Y(0,x) &= Y_0(x) \ , \ Y_t'(0,x) = Y_1(x) \ , \quad x \in \Omega \ . \end{aligned} \tag{4}$$

The semigroup approach to stochastic evolution equations has been developed by many authors. See [2], [3], [17] and references therein. It is shown in [2] that if $A$ generates a $C_0$-semigroup $\{S(t), \ t \geq 0\}$ and the operator

$$S_T x := \int_0^T S(t)S^*(t)x \ dt \ , \quad x \in H \ , \tag{5}$$

is trace class, then the process

$$X(t) = S(t)\xi + \int_0^t S(t-s)BdW(s) \ ,$$

is the unique weak $L^2(H)$ solution to (2). For the stochastic Heat equation, if $N > 1$, then $S_T$ is not trace class. If the stochastic Wave equation is considered in the space $L^2(\Omega) \times L^2(\Omega)$, then the operator $A$ generates a 1-times integrated semigroup and not a $C_0$-semigroup.

One of the alternative approaches to treating stochastic differential equations was developed in the framework of the white noise analysis. White noise analysis for $\mathbb{R}^n$ valued processes and stochastic differential equations has been developed extensively in the past three decades, see for example [8], [10], [11] and references therein. The approach of [10] has been to construct spaces of $\mathbb{R}$-valued stochastic distributions $(S)_{-\rho}$, $\rho \in [0,1]$, in which a stochastic differential equation becomes a deterministic differential or integral equation, which is then solved using standard results for deterministic differential or integral equations.

While some steps were made in [15] to consider spaces of stochastic distributions with values in particular Sobolev spaces, no one has constructed spaces of stochastic distributions with values in arbitrary Hilbert spaces. This motivates this thesis, that is developing a particular white noise framework within which one can consider stochastic evolution equations as deterministic differential or integral equations that can be solved using the semigroup theory.

We generalise the approach of [10] by constructing spaces of Hilbert space valued stochastic distributions $S(H)_{-\rho}$, $\rho \in [0,1]$, in which we wish to consider stochastic evolution equations. The first three chapters deal with the construction and properties of these spaces and particular $S(H)_{-0}$ and $S(H)_{-1}$ processes. In Chapter 4 we solve equation (2) in $S(H)_{-1}$, for the case when $A$ generates a $C_0$-semigroup, without $S_T$ being necessarily trace class. In Chapter 5 we solve equation (2) in $S(H)_{-1}$ when $A$ generates a non degenerate, $n$-times integrated, exponentially bounded semigroup, $n \geq 1$. We illustrate our results with the stochastic Heat and Wave equations.

**Chapter 1** deals with the construction and properties of the spaces of $H$-valued stochastic distributions $S(H)_{-\rho}$, $\rho \in [0,1]$. The main motivation of the chapter is to set up the spaces $S(H)_{-\rho}$ in which $H$-valued stochastic differential equations can be considered as deterministic differential or integral equations.

We start by considering the classical Wiener-Itô expansion of elements in $L^2(\mu)$ in terms of Hermite polynomials (see, for example [10]). The expansion works as follows: if $f \in L^2(\mu)$, then

$$f = \sum_{\alpha \in \mathcal{J}} c_\alpha H_\alpha \ , \quad c_\alpha \in \mathbb{R} \ ,$$

where $\mathcal{J} = (\mathbb{N}_0)^{\mathbb{N}}$, the space of finite sequences whose values belong to $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and

$$\langle H_\alpha, H_\beta \rangle_{L^2(\mu)} = \delta_{\alpha,\beta}\alpha! = \delta_{\alpha,\beta}\alpha_1!\alpha_2!\dots \ .$$

We then show that $\{H_\alpha e_i\}_{i\in\mathbb{N},\alpha\in\mathcal{J}}$ is an orthogonal basis for $L^2(H)$, that is if $f \in$

$L^2(H)$, then

$$f = \sum_{\alpha \in \mathcal{J}} \sum_{i=1}^{\infty} c_{i,\alpha} H_\alpha e_i \ , \quad c_{i,\alpha} \in \mathbb{R} \ ,$$

where

$$\langle H_\alpha e_i, H_\beta e_j \rangle_{L^2(H)} = \delta_{\alpha,\beta} \delta_{i,j} \alpha! \ .$$

We use $\{H_\alpha e_i\}_{i \in \mathbb{N}, \alpha \in \mathcal{J}}$ to construct the spaces of $H$-valued test functions $S(H)_\rho$, $\rho \in [0,1]$, being all $f = \sum_{\alpha \in \mathcal{J}} c_\alpha H_\alpha$, $c_\alpha \in H$, belonging to $L^2(H)$ such that for all $k \in \mathbb{N}$

$$\|f\|^2_{\rho,k} := \sum_{\alpha \in \mathcal{J}} (\alpha!)^{1+\rho} \|c_\alpha\|^2_H (2\mathbb{N})^{k\alpha} < \infty \ .$$

We also construct spaces of $H$-valued stochastic distributions $S(H)_{-\rho}$, $\rho \in [0,1]$, being all formal sums $f = \sum_{\alpha \in \mathcal{J}} c_\alpha H_\alpha$, $c_\alpha \in H$, such that for some $q \in \mathbb{N}$

$$\|f\|^2_{-\rho,-q} := \sum_{\alpha \in \mathcal{J}} (\alpha!)^{1-\rho} \|c_\alpha\|^2_H (2\mathbb{N})^{-q\alpha} < \infty \ .$$

These spaces have the property

$$S(H)_1 \subset S(H)_\rho \subset S(H)_0 \subset L^2(H) \subset S(H)_{-0} \subset S(H)_{-\rho} \subset S(H)_{-1} \ . \tag{6}$$

Before defining and proving results for continuity, differentiation and integration in $S(H)_{-\rho}$, results on the topology of $S(H)_\rho$ and $S(H)_{-\rho}$ are proved. To do this, the intermediary spaces $S(H)_{\rho,k}$, $k \in \mathbb{N}$ and $S(H)_{-\rho,-q}$, $q \in \mathbb{N}$ are introduced, having the properties

$$S(H)_\rho = \cap_{k=1}^{\infty} S(H)_{\rho,k} \ , \quad S(H)_{-\rho} = \cup_{q=1}^{\infty} S(H)_{-\rho,-q} \ . \tag{7}$$

By showing that $S(H)_{\rho,k}$ is a Hilbert space with the dual $S(H)_{-\rho,-k}$, we show that $S(H)_\rho$ is a countably Hilbert space with the dual $S(H)_{-\rho}$. We then prove that a sequence $\{F_n\}_{n=1}^{\infty}$ converges strongly to $F$ in $S(H)_{-\rho}$ if and only if there exists a $q \in \mathbb{N}$ such that $F_n \longrightarrow_{n \to \infty} F$ in $S(H)_{-\rho,-q}$. This result leads to the defining of continuity, differentiation and integration in $S(H)_{-\rho}$ with respect to strong convergence, so that with continuity, differentiation and integration, one need not deal directly with the (strong) topology of $S(H)_{-\rho}$, but only with one space $S(H)_{-\rho,-q}$, for some $q \in \mathbb{N}$.

**Chapter 2** deals with the Hermite transform $\mathcal{H} : S(H)_{-1} \to H_\mathbb{C}$. The main motivation of the chapter is to provide a way of transforming differential or integral equations in $S(H)_{-1}$ into differential or integral equations in $H_\mathbb{C}$. The Hermite transform also provides a convenient way of showing that an $S(H)_{-1}$ process is continuous, differentiable or integrable.

We start by considering power series with values in $H_\mathbb{C}$, defined on the sets

$$\mathbb{K}_q := \left\{ z \in \mathbb{C}^\mathbb{N} \ ; \ |z_j| < (2j)^{-q}, \ j \in \mathbb{N} \right\} \ ,$$

where $q \geq 1$. Two important results proved regarding power series needed latter on in the chapter for the Hermite transform are the following:

1. Consider $X(z) : \mathbb{K}_q \to H_\mathbb{C}$, $q > 1$, given by

$$X(z) = \sum_{\alpha \in \mathcal{J}} c_\alpha z^\alpha \ , \quad c_\alpha \in H_\mathbb{C} \ ,$$

bounded by some $M < \infty$. Then for all $z \in \mathbb{K}_{2q}$

$$\sum_{\alpha \in \mathcal{J}} \|c_\alpha\|_{H_\mathbb{C}} |z^\alpha| \leq MA(q) \ ,$$

where $A(q) := \sum_{\alpha \in \mathcal{J}} (2\mathbb{N})^{-q\alpha}$ converges if and only if $q > 1$.

2. Consider $X_n(z), X(z) : \mathbb{K}_q \to H_\mathbb{C}$, $q > 1$, given by

$$X_n(z) = \sum_{\alpha \in \mathcal{J}} c_\alpha^{(n)} z^\alpha \ , \quad X(z) = \sum_{\alpha \in \mathcal{J}} c_\alpha z^\alpha \ , \quad c_\alpha^{(n)}, c_\alpha \in H_\mathbb{C} \ ,$$

such that $X_n(\cdot) \longrightarrow_{n\to\infty} X(\cdot)$ pointwise boundedly for $z \in \mathbb{K}_q$. Then $X_n(\cdot) \longrightarrow_{n\to\infty} X(\cdot)$ uniformly on $\mathbb{K}_{2q}$.

After the results relating to power series defined on $\mathbb{K}_q$ are proved, the Hermite transform $\mathcal{H} : S(H)_{-1} \to H_\mathbb{C}$ of $F = \sum_{\alpha \in \mathcal{J}} c_\alpha H_\alpha \in S(H)_{-1}$, is then defined to be the power series

$$\mathcal{H}F(z) := \sum_{\alpha \in \mathcal{J}} c_\alpha z^\alpha \ , \tag{8}$$

for $z \in \mathbb{C}^\mathbb{N}$ such that the series converges in $H_\mathbb{C}$. Using the two results above, we prove the following results:

1. If $F \in S(H)_{-1}$, then there exists a $q \in \mathbb{N} \setminus \{1\}$ such that $\mathcal{H}F(z)$ exists for all $z \in \mathbb{K}_q$ and

$$\|\mathcal{H}F(z)\|_{H_\mathbb{C}} \leq \|F\|_{-1,-q} A(q)^{1/2} \ .$$

2. Consider $X(z) : \mathbb{K}_q \to H_\mathbb{C}$, $q \in \mathbb{N} \setminus \{1\}$, given by

$$X(z) = \sum_{\alpha \in \mathcal{J}} c_\alpha z^\alpha \ , \quad c_\alpha \in H \ ,$$

bounded by some $M < \infty$. Then

$$F = \sum_{\alpha \in \mathcal{J}} c_\alpha H_\alpha \ ,$$

belongs to $S(H)_{-1,-4q}$ and

$$\|F\|_{-1,-4q} \leq MA(q) \ .$$

3. A sequence $\{F_n\}_{n=1}^\infty$ converges strongly to $F$ in $S(H)_{-1}$ if and only if there exists a $q \in \mathbb{N} \setminus \{1\}$ such that $\mathcal{H}F_n(z), \mathcal{H}F(z)$ exist for all $z \in \mathbb{K}_q$ and $\mathcal{H}F_n(\cdot) \longrightarrow_{n\to\infty} \mathcal{H}F(\cdot)$ pointwise boundedly in $\mathbb{K}_q$.

The last part of the chapter uses these three results to prove results regarding continuity, differentiation and integration of a $S(H)_{-1}$ process and it's Hermite transform. For instance, for a $S(H)_{-1}$ process $F(t) : [a,b] \to S(H)_{-1}$, the following statements are equivalent:

1. $F(\cdot)$ is a continuous $S(H)_{-1}$ process on $[a,b]$ .
2. There exists a $q \in \mathbb{N} \setminus \{1\}$ such that
   (a) $\mathcal{H}F(t,z)$ exists for all $(t,z) \in [a,b] \times \mathbb{K}_q$.
   (b) $\mathcal{H}F(\cdot,z)$ is continuous with respect to $t$ on $[a,b]$ for all $z \in \mathbb{K}_q$ and bounded by some $M < \infty$, for $(t,z) \in [a,b] \times \mathbb{K}_q$.

**Chapter 3** defines generalised stochastic convolution in $S(H)_{-0}$ for strongly continuous families of continuous linear maps from $U$ to $H$.

We start by defining a sequence of independent Brownian motions $\{\beta_i(\cdot)\}_{i=1}^{\infty}$, so that the $U$-valued Wiener process

$$W(t) = \sum_{i=1}^{\infty} \beta_i(t) f_i \ ,$$

belongs to $S(U)_{-0}$ for all $t \geq 0$ ($\{f_i\}_{i=1}^{\infty}$ is an orthonormal basis for $U$). We also define $\mathbb{W}(\cdot)$, which we call a $U$-valued singular white noise process. We show that $\mathbb{W}(\cdot)$ is the continuous derivative of $W(\cdot)$ in $S(U)_{-0}$. We also show that $\mathbb{W}(\cdot)$ is differentiable up to any order in $S(U)_{-0}$.

Before developing generalised stochastic convolution, the Pettis integral for $S(H)_{-0}$ processes is defined. A $S(H)_{-0}$ process $F(t) : \mathbb{R} \to S(H)_{-\rho}$ is said to be Pettis integrable if $\langle F(\cdot), f\rangle$ belongs to $L^1(\mathbb{R})$ for all $f \in S(H)_0$. The Pettis integral of $F(\cdot)$ is defined to be the unique element, denoted $\int_{\mathbb{R}} F(t)dt$ belonging to $S(H)_{-0}$ such that

$$\langle \int_{\mathbb{R}} F(t)dt, f\rangle = \int_{\mathbb{R}} \langle F(t), f\rangle dt \ ,$$

for all $f \in S(H)_0$.

We define the extension of a linear map from $U$ to $H$ to a linear map from $S(U)_{-\rho}$ to $S(H)_{-\rho}$, for $\rho \in [0,1]$. This allows us to define generalised stochastic convolution for a strongly continuous family of continuous linear maps from $U$ to $H$, $\{S(t),\ t \geq 0\}$, as the Pettis integral

$$\int_0^t S(s)\delta W(s) := \int_0^t S(s)\mathbb{W}(s)ds \ , \quad t \in [0,T] \ , \tag{9}$$

assuming that $S(\cdot)\mathbb{W}(\cdot)$ is Pettis integrable on $[0,T]$.

For the case $U = H$ (the case for when $U \neq H$ is the same proof), we prove that if the operator

$$S_T := \int_0^T S(s)S^*(s)x \, ds \ , \quad x \in H \ ,$$

is trace class, then for $t \in [0,T]$, we have that $\int_0^t S(s)\delta W(s)$ belongs to $L^2(H)$ and agrees with the notion of stochastic convolution found in [2], that is

$$\int_0^t S(s)\delta W(s) = \int_0^t S(s)dW(s) = \sum_{k=1}^{\infty} \int_0^t S(s)e_k d\beta_k(s) \ ,$$

where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for $H$, $\int_0^t S(s)dW(s)$ is the Wiener integral of $\{S(t),\ t \geq 0\}$ with respect to the $H$-valued Wiener process $W(\cdot)$ and $\int_0^t S(s)e_k d\beta_k(s)$ is the Itô integral of $\{S(t)e_k,\ t \geq 0\}$ with respect to $\beta_k(\cdot)$.

The final part of the chapter defines generalised $n^{th}$ stochastic convolution for a strongly continuous family of continuous linear maps from $U$ to $H$, $\{V(t),\ t \geq 0\}$, as the Pettis integral

$$\int_0^t V(s)\delta W^{(n)}(s) := \int_0^t V(s)\mathbb{W}^{(n)}(s)dt \ , \quad t \in [0,T] \ ,$$

assuming that $V(\cdot)\mathbb{W}^{(n)}(\cdot)$ is Pettis integrable on $[0,T]$.

In **Chapter 4** we consider the differential equation in $S(H)_{-1}$

$$\begin{aligned} \frac{dX(t)}{dt} &= AX(t) + B\mathbb{W}(t) \ , \quad t \in [0,T] \ , \\ X(0) &= \xi \in \mathcal{D}(A)_{-1} \ , \end{aligned} \tag{10}$$

where $A$ generates a $C_0$-semigroup $\{S(t),\ t \geq 0\}$ on $H$ and $B$ is a continuous linear map from $U$ to $H$. We give the definition of a solution to (10) in $S(H)_{-1}$ and for the case $U = H$, show that it has the unique solution in $S(H)_{-1}$

$$X(t) := S(t)\xi + \int_0^t S(t-s)B\delta W(s) \ .$$

We also show this is the unique solution to the problem

$$\begin{aligned} X(t) &= X(0) + \int_0^t AX(s)ds + BW(t) \ , \quad t \in [0,T] \ , \\ X(0) &= \xi \in \mathcal{D}(A)_{-1} \ . \end{aligned}$$

In the last part of the chapter we illustrate our results with the stochastic Heat and Wave equations.

In **Chapter 5** we consider two problems. The first is

$$\begin{aligned} X(t) &= X(0) + A\int_0^t X(s)ds + BW(t) \ , \quad t \in [0,T] \ , \\ X(0) &= \xi \in \mathcal{D}(A)_{-1} \ , \end{aligned} \tag{11}$$

where $A$ is a closed, densely defined operator on $H$, generating a non degenerate, 1-times integrated, exponentially bounded semigroup $\{V(t),\ t \geq 0\}$ and $B$ is a continuous linear map from $U$ to $H$. We give the definition of a solution to (11) in $S(H)_{-1}$ and for the case $U = H$, show that it has the unique solution in $S(H)_{-1}$

$$X(t) := \frac{d}{dt}V(t)\xi + \int_0^t V(t-s)B\delta W^{(1)}(s) + V(t)B\mathbb{W}(0) \ . \tag{12}$$

We illustrate our results with the Wave equation.

The second problem we consider is

$$\begin{aligned} X(t) &= \frac{t^n}{n!}X(0) + A\int_0^t X(s)ds + \int_0^t \frac{(t-s)^n}{n!}B\delta W(s) \ , \quad t \in [0,T] \ , \\ X(0) &= \xi \in S(H)_{-1} \ , \end{aligned} \tag{13}$$

where $A$ is a closed, densely defined operator on $H$, generating a non degenerate, $n$-times integrated, exponentially bounded semigroup $\{V(t),\ t \geq 0\}$ and $B$ is a continuous linear map from $U$ to $H$. We give the definition of a solution to (13) in $S(H)_{-1}$ and for the case $U = H$, show that it has the unique solution in $S(H)_{-1}$

$$X(t) := V(t)\xi + \int_0^t V(t-s)B\delta W(s) \ , \quad t \in [0,T] \ . \tag{14}$$

# Chapter 1

# Spaces of $H$-valued Stochastic Test Functions and Distributions

## 1.1 Preliminaries

The probability space used in this thesis is $(S'(\mathbb{R}^d), \mathcal{B}(S'(\mathbb{R}^d)), \mu)$, where $S'(\mathbb{R}^d)$ is the space of tempered distributions and $\mathcal{B}(S'(\mathbb{R}^d))$ is the $\sigma$-algebra generated by the opens sets generated by the weak star topology on $S'(\mathbb{R}^d)$. The measure $\mu$ is the unique probability measure on $(S'(\mathbb{R}^d), \mathcal{B}(S'(\mathbb{R}^d)))$ satisfying

$$\int_{S'(\mathbb{R}^d)} e^{i\langle\omega,\phi\rangle} d\mu(\omega) = e^{-1/2\|\phi\|_{L^2(\mathbb{R}^d)}} , \tag{1.1}$$

where $\phi \in S(\mathbb{R}^d)$ and $\langle\omega, \phi\rangle$ denotes the action of $\omega \in S'(\mathbb{R}^d)$ on $\phi \in S(\mathbb{R}^d)$. The proof of the existence and uniqueness of $\mu$ can be found in [18]. By $L^2(\mu)$ we denote the space $L^2(S'(\mathbb{R}^d), \mathcal{B}(S'(\mathbb{R}^d)), \mu; \mathbb{R})$.

We let $H$ be a real separable Hilbert space with the orthonormal basis $\{e_i\}_{i=1}^{\infty}$. We denote the space $L^2(S'(\mathbb{R}^d), \mathcal{B}(S'(\mathbb{R}^d)), \mu; H)$ by $L^2(H)$.

## 1.2 Orthogonal Basis for $L^2(\mu)$

We use the classical Wiener-Itô chaos expansion of elements of $L^2(\mu)$ in terms of Hermite polynomials and functions (see, for example, [10]). The Hermite polynomials are defined as

$$h_n(x) = (-1)^n e^{1/2x^2} \frac{d^n}{dx^n}(e^{-1/2x^2}) , \quad n = 0, 1, 2, \dots . \tag{1.2}$$

The Hermite functions are defined as

$$\xi_n(x) = \pi^{-1/4}((n-1)!)^{-1/2} e^{-1/2x^2} h_{n-1}(x) , \quad n = 1, 2, \dots . \tag{1.3}$$

The Hermite functions form an orthonormal basis for $L^2(\mathbb{R})$. This gives rise to the following orthonormal basis for $L^2(\mathbb{R}^d)$, the family of tensor products

$$\eta_i := \xi_{\delta_1^{(i)}} \otimes \ldots \otimes \xi_{\delta_d^{(i)}} , \quad i = 1, 2, \ldots , \tag{1.4}$$

where $\delta_1^{(i)}, \ldots, \delta_d^{(i)} \in \mathbb{N}$ are chosen such that if $i < j$

$$\delta_1^{(i)} + \delta_2^{(i)} + \ldots + \delta_d^{(i)} \leq \delta_1^{(j)} + \delta_2^{(j)} + \ldots + \delta_d^{(j)} .$$

The classical Wiener-Itô chaos expansion of elements of $L^2(\mu)$ is the expansion of elements in $L^2(\mu)$ with respect to the following orthogonal basis $\{H_\alpha\}_{\alpha\in\mathcal{J}}$ for $L^2(\mu)$, defined by

$$H_\alpha(\omega) := \prod_{i=1}^{\infty} h_{\alpha_i}(\langle\omega, \eta_i\rangle) , \quad \omega \in S'(\mathbb{R}^d) , \tag{1.5}$$

where $\mathcal{J} = (\mathbb{N}_0^{\mathbb{N}})_c$ is the space of sequences $\alpha = (\alpha_1, \alpha_2, \ldots)$, $\alpha_1, \alpha_2, \ldots \in \mathbb{N}_0$, such that there are only finitely many $\alpha_i \neq 0$. The norm in $L^2(\mu)$ of these $\{H_\alpha\}_{\alpha\in\mathcal{J}}$ is

$$||H_\alpha||^2_{L^2(\mu)} = \alpha! := \alpha_1!\alpha_2!\ldots . \tag{1.6}$$

### 1.2.1 Some Notation

We introduce the notation

$$\text{index } \alpha := \sup\{k \mid \alpha_k \neq 0\} ,$$
$$\Gamma_n := \{\alpha \in \mathcal{J} \mid \alpha_i \leq n, \ \forall i \leq n \text{ and } \alpha_i = 0, \ \forall i > n\} .$$

Take $f \in L^2(\mu)$. As $\{H_\alpha\}_{\alpha\in\mathcal{J}}$ is an orthogonal basis for $L^2(\mu)$, $f$ has the following expansions in $L^2(\mu)$

$$f = \lim_{k\to\infty} \sum_{\text{index } \alpha \leq k} c_\alpha H_\alpha = \lim_{n\to\infty} \sum_{\alpha\in\Gamma_n} c_\alpha H_\alpha ,$$

where

$$c_\alpha = (\alpha!)^{-1}\langle f, H_\alpha\rangle_{L^2(\mu)} .$$

We denote $\lim_{k\to\infty}\sum_{\text{index } \alpha\leq k}$ and $\lim_{n\to\infty}\sum_{\alpha\in\Gamma_n}$ by $\sum_{\alpha\in\mathcal{J}}$ unless the application needs a specific choice.

## 1.3 Orthogonal Basis for $L^2(H)$

In this section we use the functions $\{H_\alpha\}_{\alpha\in\mathcal{J}}$ to construct an orthogonal basis for $L^2(H)$. With this basis we construct the spaces of $H$-valued stochastic test functions and distributions.

The following lemma is needed.

**Lemma 1.1** *For a random variable $f(\omega) : S'(\mathbb{R}^d) \to H$, belonging to $L^2(H)$, there exists random variables $a_i(\omega) : S'(\mathbb{R}^d) \to \mathbb{R}$, $i \in \mathbb{N}$ belonging to $L^2(\mu)$, such that the series*

$$f = \sum_{i=1}^{\infty} a_i e_i \; , \tag{1.7}$$

*converges in $L^2(H)$.*

**Proof:** Define $a_i(\omega) : S'(\mathbb{R}^d) \to \mathbb{R}$ almost everywhere to be

$$a_i(\omega) := \langle f(\omega), e_i \rangle_H \; .$$

We firstly show that $a_i$ is measurable. Now $f(\omega) : S'(\mathbb{R}^d) \to H$ is measurable and $\langle \cdot, e_i \rangle_H : H \to \mathbb{R}$ is continuous. Hence $a_i$ is measurable. Note that by continuity of scalar multiplication, $a_i e_i$ is then measurable.

We secondly show that $a_i$ belongs to $L^2(\mu)$. Now

$$\int_{S'(\mathbb{R}^d)} a_i(\omega)^2 d\mu(\omega) \;\; \leq \;\; \int_{S'(\mathbb{R}^d)} \sum_{j=1}^{\infty} a_j(\omega)^2 d\mu(\omega) = \int_{S'(\mathbb{R}^d)} \|f(\omega)\|_H^2 d\mu(\omega) < \infty \; .$$

Note that $a_i e_i$ then belongs to $L^2(H)$.

We lastly show that the series in equation (1.7) converges in $L^2(H)$. Using the Dominated Convergence Theorem

$$\begin{aligned} \int_{S'(\mathbb{R}^d)} \|f(\omega)\|_H^2 d\mu(\omega) &= \int_{S'(\mathbb{R}^d)} \sum_{j=1}^{\infty} a_j(\omega)^2 d\mu(\omega) = \sum_{j=1}^{\infty} \int_{S'(\mathbb{R}^d)} a_j(\omega)^2 d\mu(\omega) \\ &= \sum_{i=1}^{\infty} \|a_i\|_{L^2(\mu)}^2 \end{aligned}$$

Hence for any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$

$$\begin{aligned} \epsilon^2 &> \left| \|f\|_{L^2(H)}^2 - \sum_{i=1}^{n} \|a_i\|_{L^2(\mu)}^2 \right| = \left| \sum_{i=1}^{\infty} \|a_i\|_{L^2(\mu)}^2 - \sum_{i=1}^{n} \|a_i\|_{L^2(\mu)}^2 \right| \\ &= \sum_{i=n+1}^{\infty} \|a_i\|_{L^2(\mu)}^2 = \sum_{i=n+1}^{\infty} \int_{S'(\mathbb{R}^d)} a_i(\omega)^2 d\mu(\omega) = \int_{S'(\mathbb{R}^d)} \sum_{i=n+1}^{\infty} a_i(\omega)^2 d\mu(\omega) \\ &= \int_{S'(\mathbb{R}^d)} \left\| \sum_{i=n+1}^{\infty} a_i(\omega) e_i \right\|_H^2 d\mu(\omega) = \int_{S'(\mathbb{R}^d)} \left\| \sum_{i=1}^{\infty} a_i(\omega) e_i - \sum_{i=1}^{n} a_i(\omega) e_i \right\|_H^2 d\mu(\omega) \\ &= \int_{S'(\mathbb{R}^d)} \left\| f(\omega) - \sum_{i=1}^{n} a_i(\omega) e_i \right\|_H^2 d\mu(\omega) = \left\| f - \sum_{i=1}^{n} a_i e_i \right\|_{L^2(H)}^2 \; , \end{aligned}$$

as required. ■

This expansion helps give rise to an orthogonal basis for $L^2(H)$.

**Propositon 1.1** *The family of functions* $\{H_\alpha e_i\}_{i\in\mathbb{N},\alpha\in\mathcal{J}}$ *is an orthogonal basis for* $L^2(H)$*, with norm in* $L^2(H)$

$$\|H_\alpha e_i\|^2_{L^2(H)} = \alpha! \ . \tag{1.8}$$

**Proof:** The family of functions $\{H_\alpha e_i\}_{i\in\mathbb{N},\alpha\in\mathcal{J}}$ is orthogonal as

$$\begin{aligned} & \langle H_\alpha e_i, H_\beta e_j\rangle_{L^2(H)} \\ = & \int_{S'(\mathbb{R}^d)} \langle H_\alpha(\omega)e_i, H_\beta(\omega)e_j\rangle_H d\mu = \int_{S'(\mathbb{R}^d)} \delta_{i,j} H_\alpha(\omega)H_\beta(\omega)d\mu = \delta_{i,j}\ \delta_{\alpha,\beta}\ \alpha! \ . \end{aligned}$$

We secondly show that the family of functions $\{H_\alpha e_i\}_{i\in\mathbb{N},\alpha\in\mathcal{J}}$ spans $L^2(H)$. Take any $f \in L^2(\mu)$. From the previous lemma

$$f = \sum_{i=1}^{\infty} a_i e_i \ ,$$

where each $a_i$ belongs to $L^2(\mu)$ and the series converges in $L^2(H)$. Each $a_i$ has the following expansion in $L^2(\mu)$

$$a_i = \sum_{\alpha\in\mathcal{J}} c_{i,\alpha} H_\alpha \ , \quad c_{i,\alpha} \in \mathbb{R} \ .$$

Hence it remains to show that

$$a_i e_i = \sum_{\alpha\in\mathcal{J}} c_{i,\alpha} H_\alpha e_i \ ,$$

where the series is in $L^2(H)$. Now for any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$

$$\begin{aligned} \epsilon^2 &> \left\| a_i - \sum_{\alpha\in\Gamma_n} c_{i,\alpha} H_\alpha \right\|^2_{L^2(\mu)} = \int_{S'(\mathbb{R}^d)} \left( a_i(\omega) - \sum_{\alpha\in\Gamma_n} c_{i,\alpha} H_\alpha(\omega) \right)^2 d\mu \\ &= \int_{S'(\mathbb{R}^d)} \left\| \left( a_i(\omega) - \sum_{\alpha\in\Gamma_n} c_{i,\alpha} H_\alpha(\omega) \right) e_i \right\|^2_H d\mu \\ &= \left\| a_i e_i - \sum_{\alpha\in\Gamma_n} c_{i,\alpha} H_\alpha e_i \right\|^2_{L^2(H)} \ , \end{aligned}$$

as required. ■

Note that the order of summation can be changed as $\{H_\alpha e_i\}_{i\in\mathbb{N},\alpha\in\mathcal{J}}$ is an orthogonal basis for $L^2(H)$. That is

$$\begin{aligned} f &= \sum_{i=1}^{\infty} \sum_{\alpha\in\mathcal{J}} (\alpha!)^{-1} \langle f, H_\alpha e_i\rangle_{L^2(H)} H_\alpha e_i \\ &= \sum_{\alpha\in\mathcal{J}} \sum_{i=1}^{\infty} (\alpha!)^{-1} \langle f, H_\alpha e_i\rangle_{L^2(H)} H_\alpha e_i \ . \end{aligned}$$

**Propositon 1.2** *Consider $f \in L^2(H)$ with form*

$$f = \sum_{\alpha \in \mathcal{J}} \sum_{i=1}^{\infty} c_{i,\alpha} H_\alpha e_i \ .$$

*Then*

$$\|f\|^2_{L^2(H)} = \sum_{\alpha \in \mathcal{J}} \alpha! \|c_\alpha\|^2_H \ , \tag{1.9}$$

*where*

$$c_\alpha := \sum_{i=1}^{\infty} c_{i,\alpha} e_i \ ,$$

*converges in $H$, for all $\alpha \in \mathcal{J}$.*

**Proof:** To do this, we need only show that

$$\sum_{i=1}^{\infty} c^2_{i,\alpha} < \infty \ ,$$

for all $\alpha \in \mathcal{J}$. This is true as

$$\|f\|^2_{L^2(H)} = \sum_{\alpha \in \mathcal{J}} \sum_{i=1}^{\infty} \alpha! \ c^2_{i,\alpha} = \sum_{\alpha \in \mathcal{J}} \alpha! \left( \sum_{i=1}^{\infty} c^2_{i,\alpha} \right) < \infty \ ,$$

as required. ■

Note also that if $f \in L^2(H)$ has the form

$$f = \sum_{\alpha \in \mathcal{J}} \sum_{i=1}^{\infty} c_{i,\alpha} H_\alpha e_i \ , \quad c_{i,\alpha} \in \mathbb{R} \ ,$$

then we can write

$$f = \sum_{\alpha \in \mathcal{J}} c_\alpha H_\alpha \ , \quad c_\alpha \in H \ ,$$

where the series is in $L^2(H)$ and $c_\alpha = \sum_{i=1}^{\infty} c_{i,\alpha} e_i$.

## 1.4 $H$-valued Stochastic Test Functions and Distributions

We start by introducing the following Kondratiev spaces of $\mathbb{R}$-valued stochastic test functions and distributions which are standard in the white noise calculus (see [10] and references therein).

**Definition 1.1** *1. For $\rho \in [0,1]$, define $(S)_\rho$ to consist of all*

$$f = \sum_{\alpha \in \mathcal{J}} c_\alpha H_\alpha \ , \quad c_\alpha \in \mathbb{R} \ ,$$

*in $L^2(\mu)$ such that*

$$|f|^2_{\rho,k} := \sum_{\alpha \in \mathcal{J}} c_\alpha^2 (\alpha!)^{1+\rho} (2\mathbb{N})^{k\alpha} < \infty \ , \tag{1.10}$$

*for all $k \in \mathbb{N}$.*

*2. For $\rho \in [0,1]$, define $(S)_{-\rho}$ to consist of all formal sums*

$$F = \sum_{\alpha \in \mathcal{J}} c_\alpha H_\alpha \ , \quad c_\alpha \in \mathbb{R} \ ,$$

*such that*

$$|F|^2_{-\rho,-q} := \sum_{\alpha \in \mathcal{J}} c_\alpha^2 (\alpha!)^{1-\rho} (2\mathbb{N})^{-q\alpha} < \infty \ , \tag{1.11}$$

*for some $q \in \mathbb{N}$.*

In a similar way, we define the spaces $S(H)_\rho$ of $H$-valued stochastic test functions and the spaces $S(H)_{-\rho}$ of $H$-valued stochastic distributions.

**Definition 1.2** *1. For $\rho \in [0,1]$, define $S(H)_\rho$ to consist of all*

$$f = \sum_{\alpha \in \mathcal{J}} \sum_{i=1}^{\infty} c_{i,\alpha} H_\alpha e_i \ , \quad c_{i,\alpha} \in \mathbb{R} \ ,$$

*in $L^2(H)$ such that*

$$\|f\|^2_{\rho,k} := \sum_{\alpha \in \mathcal{J}} \sum_{i=1}^{\infty} c_{i,\alpha}^2 (\alpha!)^{1+\rho} (2\mathbb{N})^{k\alpha} < \infty \ , \tag{1.12}$$

*for all $k \in \mathbb{N}$.*

*2. For $\rho \in [0,1]$, define $S(H)_{-\rho}$ to consist of all formal sums*

$$F = \sum_{\alpha \in \mathcal{J}} \sum_{i=1}^{\infty} c_{i,\alpha} H_\alpha e_i \ , \quad c_{i,\alpha} \in \mathbb{R} \ ,$$

*such that*

$$\|F\|^2_{-\rho,-q} := \sum_{\alpha \in \mathcal{J}} \sum_{i=1}^{\infty} c_{i,\alpha}^2 (\alpha!)^{1-\rho} (2\mathbb{N})^{-q\alpha} < \infty \ , \tag{1.13}$$

*for some $q \in \mathbb{N}$.*

For $\rho \in [0,1]$, and $f$ and $F$ belonging to $S(H)_\rho$ and $S(H)_{-\rho}$ respectively, we can write

$$f = \sum_{\alpha \in \mathcal{J}} \sum_{i=1}^{\infty} c_{i,\alpha} H_\alpha e_i = \sum_{\alpha \in \mathcal{J}} c_\alpha H_\alpha = \sum_{i=1}^{\infty} f_i e_i \ ,$$
$$F = \sum_{\alpha \in \mathcal{J}} \sum_{i=1}^{\infty} d_{i,\alpha} H_\alpha e_i = \sum_{\alpha \in \mathcal{J}} d_\alpha H_\alpha = \sum_{i=1}^{\infty} F_i e_i \ ,$$

where for all $\alpha \in \mathcal{J}$ and $i \in \mathbb{N}$

$$c_\alpha = \sum_{i=1}^{\infty} c_{i,\alpha} e_i \in H \ , \quad d_\alpha = \sum_{i=1}^{\infty} d_{i,\alpha} e_i \in H$$
$$f_i = \sum_{\alpha \in \mathcal{J}} c_{i,\alpha} H_\alpha \in (S)_\rho \ , \quad F_i = \sum_{\alpha \in \mathcal{J}} d_{i,\alpha} H_\alpha \in (S)_{-\rho} \ .$$

For $k, q \in \mathbb{N}$

$$||f||^2_{\rho,k} = \sum_{\alpha \in \mathcal{J}} (\alpha!)(2\mathbb{N})^{k\alpha} ||c_\alpha||^2_H = \sum_{i=1}^{\infty} |f_i|^2_{\rho,k} \ ,$$
$$||F||^2_{-\rho,-q} = \sum_{\alpha \in \mathcal{J}} (\alpha!)(2\mathbb{N})^{-q\alpha} ||d_\alpha||^2_H = \sum_{i=1}^{\infty} |F_i|^2_{-\rho,-q} \ .$$

We can see that for $\rho \in [0,1]$

$$S(H)_1 \subset S(H)_\rho \subset S(H)_0 \subset L^2(H) \subset S(H)_{-0} \subset S(H)_{-\rho} \subset S(H)_{-1} \ ,$$

and

$$||f||_{\rho,1} \leq ||f||_{\rho,2} \leq \cdots \leq ||f||_{\rho,k} \leq \cdots \ ,$$

for all $f \in S(H)_\rho$.

## 1.5 Topologies of $S(H)_\rho$ and $S(H)_{-\rho}$

We wish to consider stochastic differential equations as deterministic differential or integral equations in $S(H)_{-\rho}$. In order to do this, we firstly need to investigate the topologies of $S(H)_\rho$ and $S(H)_{-\rho}$. The main results we prove in this section are:

1. For all $\rho \in [0,1]$, $S(H)_\rho$ is a countably Hilbert space.
2. For all $\rho \in [0,1]$, $S(H)_{-\rho}$ is the dual of $S(H)_\rho$.
3. For all $\rho \in [0,1]$, a sequence $\{F_n\}_{n=1}^{\infty}$ converges strongly to $F$ in $S(H)_{-\rho}$ if and only if there exists a $q \in \mathbb{N}$ such that $F_n$ converges to $F$ in $S(H)_{-\rho,-q}$.

Appendix A contains the definitions and results regarding linear topological spaces which we need for this section.

The following result found in [22] is needed.

**Lemma 1.2** *The series*

$$A(q) := \sum_{\alpha\in\mathcal{J}} (2\mathbb{N})^{-q\alpha} ,$$

*converges if and only if* $q > 1$.

The following spaces are needed.

**Definition 1.3** *1. For* $\rho \in [0,1]$ *and* $k \in \mathbb{N}$*, define* $S(H)_{\rho,k}$ *to to be*

$$\{f \in L^2(H);\ \|f\|_{\rho,k} < \infty\} . \tag{1.14}$$

*Define the inner product of two elements*

$$f = \sum_{\alpha\in\mathcal{J}} c_\alpha H_\alpha , \quad g = \sum_{\alpha\in\mathcal{J}} d_\alpha H_\alpha ,$$

*in* $S(H)_{\rho,k}$ *to be*

$$\langle f, g\rangle_{\rho,k} := \sum_{\alpha\in\mathcal{J}} \langle c_\alpha, d_\alpha\rangle_H (\alpha!)^{1+\rho} (2\mathbb{N})^{k\alpha} . \tag{1.15}$$

*2. For* $\rho \in [0,1]$ *and* $q \in \mathbb{N}$*, define* $S(H)_{-\rho,-q}$ *to be*

$$\{f \in S(H)_{-\rho};\ \|f\|_{-\rho,-q} < \infty\} . \tag{1.16}$$

For $\rho \in [0,1]$ and $k \in \mathbb{N}$, $\langle f, g\rangle_{\rho,k}$ is well defined as

$$\begin{aligned} &|\langle f, g\rangle_{\rho,k}| \\ \leq\ & \sum_{\alpha\in\mathcal{J}} |\langle c_\alpha, d_\alpha\rangle_H|\ (\alpha!)^{1+\rho}(2\mathbb{N})^{k\alpha} \leq \sum_{\alpha\in\mathcal{J}} \|c_\alpha\|_H\ \|d_\alpha\|_H \left((\alpha!)^{\frac{1+\rho}{2}}(2\mathbb{N})^{\frac{k\alpha}{2}}\right)^2 \\ \leq\ & \left(\sum_{\alpha\in\mathcal{J}} \|c_\alpha\|_H^2 (\alpha!)^{1+\rho}(2\mathbb{N})^{k\alpha}\right)^{1/2} \left(\sum_{\alpha\in\mathcal{J}} \|d_\alpha\|_H^2 (\alpha!)^{1+\rho}(2\mathbb{N})^{k\alpha}\right)^{1/2} < \infty . \end{aligned}$$

We can see that for $\rho \in [0,1]$

$$S(H)_\rho = \cap_{k=1}^{\infty} S(H)_{\rho,k} , \quad S(H)_{-\rho} = \cup_{q=1}^{\infty} S(H)_{-\rho,-q} .$$

The proof of the next result follows the proof of Lemma 2.7.2 found in [10].

**Lemma 1.3** *For* $\rho \in [0,1]$ *and* $k \in \mathbb{N}$*,* $S(H)_{\rho,k}$ *equipped with* $\langle\cdot,\cdot\rangle_{\rho,k}$ *is a separable Hilbert space.*

**Proof:** We start with the completeness. Take a Cauchy sequence $\{f_n\}_{n=1}^{\infty}$ in $S(H)_{\rho,k}$, where

$$f_n = \sum_{\alpha\in\mathcal{J}} c_\alpha^{(n)} H_\alpha \ .$$

Now for any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n_1, n_2 \geq N$

$$\|f_{n_1} - f_{n_2}\|_{\rho,k}^2 \ = \ \sum_{\alpha\in\mathcal{J}} (\alpha!)^{1+\rho}(2\mathbb{N})^{k\alpha} \left\|c_\alpha^{(n_1)} - c_\alpha^{(n_2)}\right\|_H^2 < \epsilon^2 \ .$$

For this $N$ and $n_1, n_2 \geq N$

$$\left\|c_\alpha^{(n_1)} - c_\alpha^{(n_2)}\right\|_H^2 < \frac{\epsilon^2}{(\alpha!)^{1+\rho}(2\mathbb{N})^{k\alpha}} \leq \epsilon^2 \ , \quad \forall \alpha \in \mathcal{J} \ .$$

Therefore $\{c_\alpha^{(n)}\}_{n=1}^{\infty}$ is a Cauchy sequence in $H$ and hence converges to some $c_\alpha$ in $H$, for all $\alpha \in \mathcal{J}$. Using these $c_\alpha$, define

$$f = \sum_{\alpha\in\mathcal{J}} c_\alpha H_\alpha \ .$$

We need to show that $f \in S(H)_{\rho,k}$ and $\|f_n - f\|_{\rho,k} \longrightarrow_{n\to\infty} 0$.

We start by showing that $f \in S(H)_{\rho,k}$. Now $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $S(H)_{\rho,k}$, so $\|f_n\|_{\rho,k}$ is bounded by some $M < \infty$. So for all $n, n_1 \in \mathbb{N}$

$$\sum_{\alpha\in\Gamma_n} (\alpha!)^{1+\rho}(2\mathbb{N})^{k\alpha} \left\|c_\alpha^{(n_1)}\right\|_H^2 \ \leq \ \sum_{\alpha\in\mathcal{J}} (\alpha!)^{1+\rho}(2\mathbb{N})^{k\alpha} \left\|c_\alpha^{(n_1)}\right\|_H^2 \leq M^2 < \infty \ .$$

This implies that

$$\begin{aligned}
\lim_{n_1\to\infty} \left( \sum_{\alpha\in\Gamma_n} (\alpha!)^{1+\rho}(2\mathbb{N})^{k\alpha} \left\|c_\alpha^{(n_1)}\right\|_H^2 \right) \ &= \ \sum_{\alpha\in\Gamma_n} (\alpha!)^{1+\rho}(2\mathbb{N})^{k\alpha} \left\| \lim_{n_1\to\infty} c_\alpha^{(n_1)} \right\|_H^2 \\
&= \ \sum_{\alpha\in\Gamma_n} (\alpha!)^{1+\rho}(2\mathbb{N})^{k\alpha} \|c_\alpha\|_H^2 \leq M^2 < \infty \ .
\end{aligned}$$

Therefore

$$\lim_{n\to\infty} \left( \sum_{\alpha\in\Gamma_n} (\alpha!)^{1+\rho}(2\mathbb{N})^{k\alpha} \|c_\alpha\|_H^2 \right) \ = \ \sum_{\alpha\in\mathcal{J}} (\alpha!)^{1+\rho}(2\mathbb{N})^{k\alpha} \|c_\alpha\|_H^2 \leq M^2 < \infty \ ,$$

as required.

We secondly show that $\|f_n - f\|_{\rho,k} \longrightarrow_{n\to\infty} 0$. Now for all $n \in \mathbb{N}$ and $n_1, n_2 \geq N$

$$\sum_{\alpha\in\Gamma_n} (\alpha!)^{1+\rho}(2\mathbb{N})^{k\alpha} \left\|c_\alpha^{(n_1)} - c_\alpha^{(n_2)}\right\|_H^2 \ \leq \ \sum_{\alpha\in\mathcal{J}} (\alpha!)^{1+\rho}(2\mathbb{N})^{k\alpha} \left\|c_\alpha^{(n_1)} - c_\alpha^{(n_2)}\right\|_H^2 < \epsilon^2 \ .$$

This implies that for all $n_2 \geq N$

$$\lim_{n_1 \to \infty} \left( \sum_{\alpha \in \Gamma_n} (\alpha!)^{1+\rho} (2\mathbb{N})^{k\alpha} \left\| c_\alpha^{(n_1)} - c_\alpha^{(n_2)} \right\|_H^2 \right)$$

$$= \sum_{\alpha \in \Gamma_n} (\alpha!)^{1+\rho} (2\mathbb{N})^{k\alpha} \left\| \lim_{n_1 \to \infty} c_\alpha^{(n_1)} - c_\alpha^{(n_2)} \right\|_H^2$$

$$= \sum_{\alpha \in \Gamma_n} (\alpha!)^{1+\rho} (2\mathbb{N})^{k\alpha} \left\| c_\alpha - c_\alpha^{(n_2)} \right\|_H^2 < \epsilon^2 .$$

Therefore for all $n_2 \geq N$

$$\lim_{n \to \infty} \left( \sum_{\alpha \in \Gamma_n} (\alpha!)^{1+\rho} (2\mathbb{N})^{k\alpha} \left\| c_\alpha - c_\alpha^{(n_2)} \right\|_H^2 \right) = \sum_{\alpha \in \mathcal{J}} (\alpha!)^{1+\rho} (2\mathbb{N})^{k\alpha} \left\| c_\alpha - c_\alpha^{(n_2)} \right\|_H^2 < \epsilon^2 ,$$

as required.

The separability of $S(H)_{\rho,k}$ follows from the observation that the set $\{H_\alpha e_i\}_{i \in \mathbb{N}, \alpha \in \mathcal{J}}$ is a countable, linearly dense subset of $S(H)_{\rho,k}$. ■

**Lemma 1.4** *For $\rho \in [0,1]$, the system of inner products $\langle \cdot, \cdot \rangle_{\rho,k}$ on $S(H)_\rho$ are compatible.*

**Proof:** Take a sequence $\{f_n\}_{n=1}^\infty$ in $S(H)_\rho$, where

$$f_n = \sum_{\alpha \in \mathcal{J}} c_\alpha^{(n)} H_\alpha ,$$

that converges to 0 in some norm $\|\cdot\|_{\rho,k_1}$ and is a Cauchy sequence in another norm $\|\cdot\|_{\rho,k_2}$. We need to show that $\{f_n\}_{n=1}^\infty$ also converges to 0 in the norm $\|\cdot\|_{\rho,k_2}$.

We note that for all $\alpha \in \mathcal{J}$, $c_\alpha^{(n)} \longrightarrow_{n \to \infty} 0$ in $H$ as $\|f_n\|_{\rho,k_1} \longrightarrow_{n \to \infty} 0$.

As $\{f_n\}_{n=1}^\infty$ is a Cauchy sequence in the norm $\|\cdot\|_{\rho,k_2}$, it is also a Cauchy sequence in $S(H)_{\rho,k_2}$. By the completeness of $S(H)_{\rho,k_2}$, there exists an element $f \in S(H)_{\rho,k_2}$ having form

$$f = \sum_{\alpha \in \mathcal{J}} c_\alpha H_\alpha ,$$

such that $f_n \longrightarrow_{n \to \infty} f$ in $S(H)_{\rho,k_2}$. Suppose that $f \neq 0$. Then there exists an $\alpha \in \mathcal{J}$ such that $c_\alpha \neq 0$. However this is not possible as $c_\alpha^{(n)} \longrightarrow_{n \to \infty} c_\alpha$ in $H$. Hence $f_n \longrightarrow_{n \to \infty} 0$ in $\|\cdot\|_{\rho,k_2}$. ■

**Theorem 1.1** *For $\rho \in [0,1]$, $S(H)_\rho$ equipped with the countable collection of inner products $\{\langle \cdot, \cdot \rangle_{\rho,k}\}_{k=1}^\infty$ is a countably Hilbert space.*

**Proof:** $S(H)_{\rho,k}$ is a countably Hilbert space if and only if

$$S(H)_\rho = \cap_{k=1}^\infty (S(H)_\rho)_k \ ,$$

where $(S(H)_\rho)_k$ is the completion of $S(H)_\rho$ with respect to norm $\|\cdot\|_k$. We have that

$$S(H)_\rho = \cap_{k=1}^\infty S(H)_{\rho,k} \ .$$

So it remains to show that $S(H)_{\rho,k}$ is the completion of $S(H)_\rho$ with respect to the norm $\|\cdot\|_k$. This is the case because $\{H_\alpha e_i\}_{i\in\mathbb{N},\alpha\in\mathcal{J}} \subset S(H)_\rho$ is linearly dense in $S(H)_{\rho,k}$ (with respect to $\langle\cdot,\cdot\rangle_{\rho,k}$), which is complete. ■

**Corollary 1.1** *For $\rho \in [0,1]$, $S(H)_\rho$ is a Frechet space.*

**Proof:** A countably Hilbert space is a Frechet space. ■

**Lemma 1.5** *For all $\rho \in [0,1]$ and $r \in \mathbb{N}$, $S(H)_{-\rho,-r}$ is the dual of $S(H)_{\rho,r}$.*

**Proof:** To prove, we set up a one-to-one correspondence between $S(H)_{-\rho,-r}$ and $S(H)_{\rho,r}$, then use the Riez Representation theorem to set up a one-to-one correspondence between $S(H)_{-\rho,-r}$ and $S(H)'_{\rho,r}$ which preserves norms.

Take an element $F \in S(H)_{-\rho,-r}$ having form

$$F = \sum_{\alpha\in\mathcal{J}} c_\alpha H_\alpha \ .$$

By letting $b_\alpha = (\alpha!)^{-\rho}(2\mathbb{N})^{-r\alpha}c_\alpha$, we can identify this with $f \in S(H)_{\rho,r}$ having form

$$f = \sum_{\alpha\in\mathcal{J}} b_\alpha H_\alpha \ .$$

We have that $f$ belongs to $S(H)_{\rho,r}$ as

$$\begin{aligned}
\sum_{\alpha\in\mathcal{J}} \|b_\alpha\|_H^2 (\alpha!)^{1+\rho}(2\mathbb{N})^{r\alpha} &= \sum_{\alpha\in\mathcal{J}} \|(\alpha!)^{-\rho}(2\mathbb{N})^{-r\alpha}c_\alpha\|_H^2(\alpha!)^{1+\rho}(2\mathbb{N})^{r\alpha} \\
&= \sum_{\alpha\in\mathcal{J}} \|c_\alpha\|_H^2(\alpha!)^{-2\rho}(2\mathbb{N})^{-2r\alpha}(\alpha!)^{1+\rho}(2\mathbb{N})^{r\alpha} \\
&= \sum_{\alpha\in\mathcal{J}} \|c_\alpha\|_H^2(\alpha!)^{1-\rho}(2\mathbb{N})^{-r\alpha} < \infty \ .
\end{aligned}$$

Similarly any element $f \in S(H)_{\rho,r}$ with form

$$f = \sum_{\alpha\in\mathcal{J}} b_\alpha H_\alpha \ ,$$

can be identified with an element $F \in S(H)_{-\rho,-r}$ having form

$$F = \sum_{\alpha\in\mathcal{J}} c_\alpha H_\alpha \ ,$$

where $c_\alpha = (\alpha!)^\rho (2\mathbb{N})^{r\alpha} b_\alpha$.

Now a linear functional $\Psi$ on $S(H)_{\rho,k}$ is continuous if and only if there exists an unique $f \in S(H)_{\rho,k}$ such that

$$\Psi[g] = \langle g, f\rangle_{\rho,r} = \sum_{\alpha\in\mathcal{J}} \langle a_\alpha, b_\alpha\rangle_H (\alpha!)^{1+\rho}(2\mathbb{N})^{r\alpha} ,$$

for all $g \in S(H)_{\rho,k}$, where

$$f = \sum_{\alpha\in\mathcal{J}} b_\alpha H_\alpha , \quad g = \sum_{\alpha\in\mathcal{J}} a_\alpha H_\alpha .$$

So a continuous linear functional $\Psi$ can be identified with an element $F \in S(H)_{-\rho,-r}$ via $f$. Similarly an element $F \in S(H)_{-\rho,-r}$ can be identified with a continuous linear functional $\Psi$. The norms of $\Psi$ and $F$ coincide as

$$\begin{aligned} \|\Psi\|_{S(H)'_{\rho,r}} &= \|f\|_{\rho,r} = \sum_{\alpha\in\mathcal{J}} \|b_\alpha\|_H^2 (\alpha!)^{1+\rho}(2\mathbb{N})^{r\alpha} \\ &= \sum_{\alpha\in\mathcal{J}} \|(\alpha!)^{-\rho}(2\mathbb{N})^{-r\alpha} c_\alpha\|_H^2 (\alpha!)^{1+\rho}(2\mathbb{N})^{r\alpha} \\ &= \sum_{\alpha\in\mathcal{J}} \|c_\alpha\|_H^2 (\alpha!)^{-2\rho}(2\mathbb{N})^{-2r\alpha}(\alpha!)^{1+\rho}(2\mathbb{N})^{r\alpha} \\ &= \sum_{\alpha\in\mathcal{J}} \|c_\alpha\|_H^2 (\alpha!)^{1-\rho}(2\mathbb{N})^{-r\alpha} = \|F\|_{-\rho,-r} , \end{aligned}$$

as required. ■

**Propositon 1.3** *For $\rho \in [0,1]$, $S(H)_{-\rho}$ is the dual of $S(H)_\rho$.*

**Proof:** For a countably Hilbert space, $\Phi = \cap_{k=1}^\infty \Phi_k$, the dual is

$$\Phi' = \cup_{k=1}^\infty \Phi'_k .$$

Now for $\rho \in [0,1]$ and $k \in \mathbb{N}$, $S(H)_{-\rho,-k}$ is the dual of $S(H)_{\rho,k}$. Therefore

$$S(H)_{-\rho} = \cup_{k=1}^\infty S(H)_{-\rho,-k} ,$$

is the dual of $S(H)_\rho$. ■

**Lemma 1.6** *Take $\rho \in [0,1]$. Consider $F \in S(H)_{-\rho}$ and $f \in S(H)_\rho$, having forms*

$$F = \sum_{\alpha\in\mathcal{J}} c_\alpha H_\alpha , \quad f = \sum_{\alpha\in\mathcal{J}} a_\alpha H_\alpha .$$

*The action of $F$ on $f$ is*

$$\langle F, f\rangle = \sum_{\alpha\in\mathcal{J}} \alpha! \langle a_\alpha, c_\alpha\rangle_H .$$

**Proof:** There exists a $q \in \mathbb{N}$ such that $F \in S(H)_{-\rho,-q}$. Since $S(H)_{-\rho}$ is the dual of $S(H)_\rho$, the action of $F$ on $f$ is just the action of $F$ on $f$ when $F$ and $f$ are considered as elements of $S(H)_{-\rho,-q}$ and $S(H)_{\rho,q}$ respectively. Hence

$$\langle F, f \rangle = \sum_{\alpha \in \mathcal{J}} \langle a_\alpha, c_\alpha (\alpha!)^{-\rho} (2\mathbb{N})^{-q\alpha} \rangle_H (\alpha!)^{1+\rho} (2\mathbb{N})^{q\alpha} = \sum_{\alpha \in \mathcal{J}} \alpha! \langle a_\alpha, c_\alpha \rangle_H ,$$

as required. ■

**Lemma 1.7** *Take $\rho \in [0,1]$. Consider a sequence $\{F_n\}_{n=1}^\infty$ and $F$ belonging to $S(H)_{-\rho}$, having forms*

$$F_n = \sum_{\alpha \in \mathcal{J}} c_\alpha^{(n)} H_\alpha , \quad F = \sum_{\alpha \in \mathcal{J}} c_\alpha H_\alpha .$$

*If $F_n \longrightarrow_{n\to\infty} F$ strongly in $S(H)_{-\rho}$, then*

$$\lim_{n \to \infty} c_\alpha^{(n)} = c_\alpha , \tag{1.17}$$

*in $H$, for all $\alpha \in \mathcal{J}$.*

**Proof:** As $F_n \longrightarrow_{n\to\infty} F$ strongly in $S(H)_{-\rho}$, there exists a $q \in \mathbb{N}$ such that $\{F_n\}_{n=1}^\infty$ and $F$ belong to $S(H)_{-\rho,-q}$ and are bounded by some $M < \infty$ in the norm $\|\cdot\|_{-\rho,-q}$. It follow that

$$(\alpha!)^{1-\rho} \|c_\alpha^{(n)}\|_H^2 (2\mathbb{N})^{-q\alpha} , \ (\alpha!)^{1-\rho} \|c_\alpha\|_H^2 (2\mathbb{N})^{-q\alpha} \leq M^2 < \infty .$$

Therefore

$$\|c_\alpha^{(n)}\|_H^2 , \ \|c_\alpha\|_H^2 \leq (\alpha!)^{\rho-1} (2\mathbb{N})^{q\alpha} M^2 = C_\alpha^2 < \infty ,$$

for all $\alpha \in \mathcal{J}$. Now for fixed $\alpha \in \mathcal{J}$, the set $\{hH_\alpha\}_{\|h\|_H \leq 2C_\alpha}$ is a bounded set in $S(H)_\rho$ as

$$\|hH_\alpha\|_{\rho,k}^2 = (\alpha!)^{1+\rho} (2\mathbb{N})^{k\alpha} \|h\|_H^2 = 4(\alpha!)^{1+\rho} (2\mathbb{N})^{k\alpha} C_\alpha^2 < \infty ,$$

for all $k \in \mathbb{N}$. Hence

$$\langle F_n - F, hH_\alpha \rangle = \alpha! \langle c_\alpha^{(n)} - c_\alpha, h \rangle_H \underset{n\to\infty}{\longrightarrow} 0 ,$$

uniformly on the set $\{hH_\alpha\}_{\|h\|_H \leq 2C_\alpha}$, that is, for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$

$$|\langle F_n - F, hH_\alpha \rangle| = \left| \alpha! \langle c_\alpha^{(n)} - c_\alpha, h \rangle_H \right| < \frac{\epsilon^2}{\alpha!} ,$$

on the set $\{hH_\alpha\}_{\|h\|_H \leq 2C_\alpha}$. For this $N$, we have that for all $n \geq N$

$$\begin{aligned} \|c_\alpha^{(n)} - c_\alpha\|_H^2 &= \left| \langle c_\alpha^{(n)} - c_\alpha, c_\alpha^{(n)} - c_\alpha \rangle_H \right| \\ &\leq \sup_{\|h\|_H \leq 2C_\alpha} \left| \langle c_\alpha^{(n)} - c_\alpha, h \rangle_H \right| < \epsilon^2 . \end{aligned}$$

as required. ■

**Propositon 1.4** *Take $\rho \in [0,1]$. Consider a sequence $\{F_n\}_{n=1}^{\infty}$ and $F$ belonging to $S(H)_{-\rho}$, such that $F_n \longrightarrow_{n\to\infty} F$ strongly in $S(H)_{-\rho}$. Then there exists a $q \in \mathbb{N}$ such that $F_n, F$ belong to $S(H)_{-\rho,-q}$ and*

$$\|F_n - F\|_{-\rho,-q} \underset{n\to\infty}{\longrightarrow} 0 \ . \tag{1.18}$$

**Proof:** Let $F_n, F$ have forms

$$F_n = \sum_{\alpha\in\mathcal{J}} c_\alpha^{(n)} H_\alpha \ , \quad F = \sum_{\alpha\in\mathcal{J}} c_\alpha H_\alpha \ .$$

As $F_n \longrightarrow_{n\to\infty} F$ strongly in $S(H)_{-\rho}$, there exists a $q \in \mathbb{N}$ such that $\{F_n\}_{n=1}^{\infty}$ and $F$ belong to $S(H)_{-\rho,-q}$ and are bounded by some $M < \infty$ in the norm $\|\cdot\|_{-\rho,-q}$. Since $\{F_n\}_{n=1}^{\infty}$, $F$ belong to $S(H)_{-\rho,-q}$, they also belong to $S(H)_{-\rho,-q-2}$. Now

$$\begin{aligned}
&\|F_n - F\|_{-\rho,-q-2}^2 \\
= &\sum_{\alpha\in\mathcal{J}} \left\|c_\alpha^{(n)} - c_\alpha\right\|_H^2 (\alpha!)^{1-\rho}(2\mathbb{N})^{-(q+2)\alpha} \\
= &\sum_{\alpha\in\Gamma_k} \left\|c_\alpha^{(n)} - c_\alpha\right\|_H^2 (\alpha!)^{1-\rho}(2\mathbb{N})^{-(q+2)\alpha} \\
&+ \sum_{\alpha\notin\Gamma_k} \left\|c_\alpha^{(n)} - c_\alpha\right\|_H^2 (\alpha!)^{1-\rho}(2\mathbb{N})^{-q\alpha}(2\mathbb{N})^{-2\alpha} \\
\le &\left(\max_{\alpha\in\Gamma_k} (\alpha!)^{1-\rho} \left\|c_\alpha^{(n)} - c_\alpha\right\|_H^2\right) \sum_{\alpha\in\Gamma_k} (2\mathbb{N})^{-(q+2)\alpha} + 4M^2 \sum_{\alpha\notin\Gamma_k} (2\mathbb{N})^{-2\alpha} \ .
\end{aligned}$$

Take $\epsilon > 0$. Choose $k$ such that

$$\sum_{\alpha\notin\Gamma_k} (2\mathbb{N})^{-2\alpha} < \frac{\epsilon^2}{8M^2} \ .$$

Now there exists an $N \in \mathbb{N}$ such that for all $n \ge N$

$$\max_{\alpha\in\Gamma_k} (\alpha!)^{1-\rho} \left\|c_\alpha^{(n)} - c_\alpha\right\|_H^2 < \frac{\epsilon^2}{2A(q+2)} \ .$$

For $k$ and $N$ chosen this way

$$\|F_n - F\|_{-1,-q-2}^2 \ < \ \frac{\epsilon^2}{2A(q+2)} A(q+2) + 4M^2 \frac{\epsilon^2}{8M^2} = \epsilon^2 \ ,$$

for all $n \ge N$, as required. ■

## 1.6 Analysis in $S(H)_{-\rho}$

In order to consider stochastic differential equations as deterministic differential or integral equations in $S(H)_{-\rho}$, definitions and results for continuity, differentiability and integration in $S(H)_{-\rho}$ need to be set up.

In this section $[a,b]$ will be a closed, bounded interval in $\mathbb{R}$.

### 1.6.1 Continuity

**Definition 1.4** *A function $F(t) : [a,b] \to S(H)_{-\rho}$ is said to be continuous at $t_0 \in [a,b]$ if for each bounded set $E \subset S(H)_\rho$ and $\epsilon > 0$, there exists a $\delta > 0$ such that*

$$s \in [a,b],\ |t_0 - s| < \delta \Rightarrow |\langle F(t_0) - F(s), f\rangle| < \epsilon\ , \tag{1.19}$$

*for all $f \in E$.*

*We say $F(\cdot)$ is a continuous $S(H)_{-\rho}$ process on $[a,b]$ if it is continuous at each point $t \in [a,b]$.*

**Propositon 1.5** *A function $F(t) : [a,b] \to S(H)_{-\rho}$ is continuous at $t_0 \in [a,b]$ if and only if for all $f \in S(H)_\rho$ and sequences $\{t_n\}_{n=1}^{\infty} \subset [a,b]$ such that $t_n \longrightarrow_{n\to\infty} t_0$, we have that*

$$\langle F(t_0) - F(t_n), f\rangle \underset{n\to\infty}{\longrightarrow} 0\ , \tag{1.20}$$

*uniformly on all bounded subsets of $S(H)_\rho$.*

**Proof:** ($\Rightarrow$) By continuity at $t_0$, for any bounded set $E \subset S(H)_\rho$ and $\epsilon > 0$, there exists a $\delta > 0$ such that

$$s \in [a,b],\ |t_0 - s| < \delta \Rightarrow |\langle F(t_0) - F(s), f\rangle| < \epsilon\ ,$$

for all $f \in E$. Consider a sequence $\{t_n\}_{n=1}^{\infty} \subset [a,b]$ such that $t_n \longrightarrow_{n\to\infty} t_0$. Now there exists an $N \in \mathbb{N}$ such that for all $n \geq N$

$$|t_0 - t_n| < \delta\ .$$

This gives

$$|\langle F(t_0) - F(t_n), f\rangle| < \epsilon\ ,$$

for all $f \in E$ and $n \geq N$, as required.

($\Leftarrow$) Suppose that $F(\cdot)$ is not continuous at $t_0$. Then there exists a bounded set $E \in S(H)_\rho$ and $\epsilon > 0$ such that for all $n \in \mathbb{N}$, there exists a $t_n \in (t_0 - 1/n, t_0 + 1/n)$ and $f_n \in E$ such that

$$|\langle F(t_n) - F(t_0), f_n\rangle| \geq \epsilon\ .$$

However, $\{f_n\}_{n=1}^{\infty} \subset E$ is a bounded set and $t_n \longrightarrow_{n\to\infty} t_0$, so there exists an $N \in \mathbb{N}$ such that for all $n \geq N$

$$|\langle F(t_n) - F(t_0), f\rangle| < \epsilon\ ,$$

for all $f \in \{f_n\}_{n=1}^{\infty}$, which is a contradiction, as required. ∎

**Corollary 1.2** *Consider $F(t) : [a,b] \to S(H)_{-\rho}$ with form*

$$F(t) = \sum_{\alpha\in\mathcal{J}} c_\alpha(t) H_\alpha\ .$$

*If $F(\cdot)$ is a continuous $S(H)_{-\rho}$ process on $[a,b]$, then the functions $c_\alpha(t) : [a,b] \to H$ are uniformly continuous on $[a,b]$, for all $\alpha \in \mathcal{J}$.*

**Proof:** Take any $t_0 \in [a,b]$ and any sequence $\{t_n\}_{n=1}^{\infty} \subset [a,b]$ such that $t_n \longrightarrow_{n\to\infty} t_0$. Then for all $f \in S(H)_\rho$ we have that

$$\langle F(t_0) - F(t_n), f \rangle \underset{n\to\infty}{\longrightarrow} 0 ,$$

uniformly on all bounded subsets of $S(H)_1$. By Lemma 1.7

$$\|c_\alpha(t_0) - c_\alpha(t_n)\|_H \underset{n\to\infty}{\longrightarrow} 0 ,$$

for all $\alpha \in \mathcal{J}$. This gives continuity on $[a,b]$. Uniform continuity follows from $[a,b]$ being a closed, bounded interval. ■

**Lemma 1.8** *If a function $F(t) : [a,b] \to S(H)_{-\rho}$ is a continuous $S(H)_{-\rho}$ process on $[a,b]$, then the set $\{F(t)\}_{t\in[a,b]}$ is a bounded set in $S(H)_{-\rho}$.*

**Proof:** Take a bounded set $E \in S(H)_\rho$, any point $t \in [a,b]$ and $\epsilon > 0$. Then there exists a $\delta_t > 0$ such that for all $f \in E$

$$s \in [a,b],\ |t-s| < \delta_t \Rightarrow |\langle F(t) - F(s), f\rangle| < \epsilon .$$

Therefore, for all $f \in E$

$$s \in [a,b],\ |t-s| < \delta_t \Rightarrow |\langle F(s), f\rangle| < \epsilon + |\langle F(t), f\rangle| \leq M_t < \infty ,$$

as $F(t)$ is a single element in $S(H)_{-\rho}$, single elements being bounded sets. Now consider the collection of balls $\{B(t,\delta_t)\}_{t\in[a,b]}$ that cover $[a,b]$. Since $[a,b]$ is closed and bounded, there exists a finite subcover

$$\{B(t_1,\delta_1), B(t_2,\delta_2), \ldots, B(t_n,\delta_n)\} ,$$

covering $[a,b]$. For each $B(t_i,\delta_i)$, there exists a $M_i < \infty$ such that $|\langle F(t), f\rangle| \leq M_i$, for all $t \in B(t_i,\delta_i) \cap [a,b]$ and $f \in E$. Let $M = \max_{i=1,\ldots n} M_i$. Then $|\langle F(t), f\rangle| \leq M$ for all $t \in [a,b]$ and $f \in E$. Hence the set $\{F(t)\}_{t\in[a,b]}$ is bounded on any bounded set of $S(H)_\rho$, as required. ■

**Propositon 1.6** *Consider a function $F(t) : [a,b] \to S(H)_{-\rho}$. The following statements are equivalent:*

1. *$F(\cdot)$ is a continuous $S(H)_{-\rho}$ process on $[a,b]$.*
2. *There exists a $q \in \mathbb{N}$ such that:*
   (a) *$F(t) \in S(H)_{-\rho,-q}$ for all $t \in [a,b]$.*
   (b) *For all $\epsilon > 0$, there exists a $\delta > 0$ such that*

$$s,t \in [a,b],\ |t-s| < \delta \Rightarrow \|F(t) - F(s)\|_{-\rho,-q} < \epsilon . \quad (1.21)$$

**Proof:** $(1 \Rightarrow 2)$ $(a)$ From the previous lemma we know that $\{F(t)\}_{t\in[a,b]}$ is a bounded set in $S(H)_{-\rho}$. Hence there exists a $q \in \mathbb{N}$ such that $F(t) \in S(H)_{-\rho,-q}$ and is bounded by some $M < \infty$ in the norm $\|\cdot\|_{-\rho,-q}$, for all $t \in [a,b]$.

$(b)$ Let $F(\cdot)$ have form

$$F(t) = \sum_{\alpha\in\mathcal{J}} c_\alpha(t) H_\alpha \ .$$

Note that as $F(t) \in S(H)_{-\rho,-q}$ for all $t \in [a,b]$, then $F(t) \in S(H)_{-\rho,-q-2}$ for all $t \in [a,b]$. Take $s,t \in [a,b]$. Then

$$\begin{aligned}
&\|F(t) - F(s)\|^2_{-\rho,-q-2} \\
= \ & \sum_{\alpha\in\mathcal{J}} (\alpha!)^{1-\rho} \|c_\alpha(t) - c_\alpha(s)\|^2_H (2\mathbb{N})^{-(q+2)\alpha} \\
= \ & \sum_{\alpha\in\Gamma_k} (\alpha!)^{1-\rho} \|c_\alpha(t) - c_\alpha(s)\|^2_H (2\mathbb{N})^{-(q+2)\alpha} \\
& + \sum_{\alpha\notin\Gamma_k} (\alpha!)^{1-\rho} \|c_\alpha(t) - c_\alpha(s)\|^2_H (2\mathbb{N})^{-q\alpha} (2\mathbb{N})^{-2\alpha} \\
\leq \ & \left( \max_{\alpha\in\Gamma_k} (\alpha!)^{1-\rho} \|c_\alpha(t) - c_\alpha(s)\|^2_H \right) \sum_{\alpha\in\Gamma_k} (2\mathbb{N})^{-(q+2)\alpha} + 4M^2 \sum_{\alpha\notin\Gamma_k} (2\mathbb{N})^{-2\alpha} \ .
\end{aligned}$$

Take $\epsilon > 0$. Choose $k \in \mathbb{N}$ such that

$$\sum_{\alpha\notin\Gamma_k} (2\mathbb{N})^{-2\alpha} < \frac{\epsilon^2}{8M^2} \ .$$

From the uniform continuity of $c_\alpha(\cdot)$ on $[a,b]$ for each $\alpha \in \mathcal{J}$, there exists a $\delta > 0$ such that for all $s,t \in [a,b]$ with $|t-s| < \delta$

$$\max_{\alpha\in\Gamma_k} (\alpha!)^{1-\rho} \|c_\alpha(t) - c_\alpha(s)\|^2_H < \frac{\epsilon^2}{2A(q+2)} \ .$$

For $k$ and $\delta$ chosen this way

$$\|F(t) - F(s)\|^2_{-\rho,-q-2} \ < \ \frac{\epsilon^2}{2A(q+2)} A(q+2) + 4M^2 \frac{\epsilon^2}{8M^2} = \epsilon^2 \ ,$$

for all $s,t \in [a,b]$ with $|t-s| < \delta$, as required.

$(2 \Rightarrow 1)$ Take any $t_0 \in [a,b]$ and any sequence $\{t_n\}_{n=1}^\infty \subset [a,b]$ such that $t_n \longrightarrow_{n\to\infty} t_0$. Then for all $f \in S(H)_\rho$

$$|\langle F(t_0) - F(t_n), f \rangle| \leq \|F(t_0) - F(t_n)\|_{-\rho,-q} \, \|f\|_{\rho,q} \ .$$

Hence $F(t_0) - F(t_n) \longrightarrow_{n\to\infty} 0$ strongly in $S(H)_{-\rho}$. So by Proposition 1.5, we have continuity at $t_0$ and hence continuity on $[a,b]$, as required. ■

### 1.6.2 Differentiability

**Definition 1.5** *A function $X(t) : [a,b] \to S(H)_{-\rho}$ is said to be differentiable at $t_0 \in [a,b]$ if for all $f \in S(H)_\rho$ and sequences $\{t_n\}_{n=1}^{\infty} \subset [a,b]$, $t_n \neq t_0$ such that $t_n \longrightarrow_{n\to\infty} t_0$, we have that*

$$\langle \frac{X(t_n) - X(t_0)}{t_n - t_0}, f \rangle \ , \tag{1.22}$$

*converges uniformly as $n \to \infty$ on all bounded subsets of $S(H)_\rho$.*

*We say $X(\cdot)$ is a differentiable $S(H)_{-\rho}$ process on $[a,b]$ if it is differentiable at each point $t \in [a,b]$.*

**Lemma 1.9** *If a function $X(t) : [a,b] \to S(H)_{-\rho}$ is differentiable at $t_0 \in [a,b]$, then there exists a unique $F(t_0)$ such that for all $f \in S(H)_\rho$ and sequences $\{t_n\}_{n=1}^{\infty} \subset [a,b]$, $t_n \neq t_0$ such that $t_n \longrightarrow_{n\to\infty} t_0$, we have that*

$$\langle \frac{X(t_n) - X(t_0)}{t_n - t_0} - F(t_0), f \rangle \underset{n\to\infty}{\longrightarrow} 0 \ , \tag{1.23}$$

*uniformly on all bounded subsets of $S(H)_\rho$.*

*We call $F(t_0)$ the derivative of $X(\cdot)$ at $t_0$.*

**Proof:** Firstly existence. From the completeness of $S(H)_{-\rho}$ with respect to strong convergence, for a particular sequence $\{t_n\}_{n=1}^{\infty} \subset [a,b]$, $t_n \neq t_0$ such that $t_n \longrightarrow_{n\to\infty} t_0$, there exists a unique $F(t_0) \in S(H)_{-\rho}$ such that for all $f \in S(H)_\rho$

$$\langle \frac{X(t_n) - X(t_0)}{t_n - t_0} - F(t_0), f \rangle \underset{n\to\infty}{\longrightarrow} 0 \ ,$$

uniformly on all bounded subsets of $S(H)_\rho$.

Secondly uniqueness. Take two sequences $\{t_n^{(1)}, \{t_n^{(2)}\}_{n=1}^{\infty} \subset [a,b]$, where $t_n^{(1)}, t_n^{(2)} \neq t_0$ and both sequences converge to $t_0$. From these two sequences define a new sequence $\{t_n\}_{n=1}^{\infty}$ by

$$t_n = \begin{cases} t^{(1)}_{\frac{n+1}{2}} & n \text{ odd} \\ \\ t^{(2)}_{\frac{n}{2}} & n \text{ even} \end{cases} .$$

Now $t_n \longrightarrow_{n\to\infty} t_0$, so there exists a unique $F(t_0) \in S(H)_{-\rho}$ such that for all $f \in S(H)_\rho$

$$\langle \frac{X(t_n) - X(t_0)}{t_n - t_0} - F(t_0), f \rangle \underset{n\to\infty}{\longrightarrow} 0 \ ,$$

uniformly on all bounded subsets of $S(H)_\rho$. Therefore for all $f \in S(H)_\rho$

$$\langle \frac{X(t_n^{(1)}) - X(t_0)}{t_n^{(1)} - t_0} - F(t_0), f \rangle \underset{n^{(1)}\to\infty}{\longrightarrow} 0$$

$$\langle \frac{X(t_n^{(2)}) - X(t_0)}{t_n^{(2)} - t_0} - F(t_0), f \rangle \underset{n^{(2)}\to\infty}{\longrightarrow} 0 \ ;$$

uniformly on all bounded subsets of $S(H)_\rho$, as required. ∎

**Corollary 1.3** *If $X(t) : [a,b] \to S(H)_{-\rho}$ is a differentiable $S(H)_{-\rho}$ process on $[a,b]$, then there exists a unique function $F(t) : [a,b] \to S(H)_{-\rho}$ such that $F(\cdot)$ is the derivative of $X(\cdot)$ on $[a,b]$.*

**Lemma 1.10** *Consider functions $X(t), F(t) : [a,b] \to S(H)_{-\rho}$ having forms*

$$X(t) = \sum_{\alpha \in \mathcal{J}} c_\alpha(t) H_\alpha \ , \quad F(t) = \sum_{\alpha \in \mathcal{J}} d_\alpha(t) H_\alpha \ .$$

*If $X(\cdot)$ is a differentiable $S(H)_{-\rho}$ process on $[a,b]$, with derivative $F(\cdot)$, then the functions $c_\alpha(t) : [a,b] \to H$ are differentiable on $[a,b]$ with derivative $d_\alpha(\cdot)$, for all $\alpha \in \mathcal{J}$ and therefore*

$$\frac{dX(t)}{dt} = \sum_{\alpha \in \mathcal{J}} c'_\alpha(t) H_\alpha \ . \tag{1.24}$$

**Proof:** Take any $t_0 \in [a,b]$ and any sequence $\{t_n\}_{n=1}^\infty \subset [a,b]$, $t_n \neq t_0$ such that $t_n \longrightarrow_{n\to\infty} t_0$. Then for all $f \in S(H)_\rho$

$$\langle \frac{X(t_n) - X(t_0)}{t_n - t_0} - F(t_0), f \rangle \underset{n\to\infty}{\longrightarrow} 0 \ ,$$

uniformly on all bounded subsets of $S(H)_\rho$. By Lemma 1.7

$$\left\| \frac{c_\alpha(t_n) - c_\alpha(t_0)}{t_n - t_0} - d_\alpha(t_0) \right\|_H \underset{n\to\infty}{\longrightarrow} 0 \ ,$$

for all $\alpha \in \mathcal{J}$, as required. ∎

**Lemma 1.11** *If a function $X(t) : [a,b] \to S(H)_{-\rho}$ is differentiable at $t_0 \in [a,b]$, then it is continuous at $t_0$.*

**Proof:** We wish to show that for any sequence $\{t_n\}_{n=1}^\infty \subset [a,b]$ such that $t_n \longrightarrow_{n\to\infty} t_0$, we have that for all $f \in S(H)_\rho$

$$\langle X(t_n) - X(t_0), f \rangle \underset{n\to\infty}{\longrightarrow} 0 \ ,$$

uniformly on all bounded subsets of $S(H)_\rho$.

Consider a sequence $\{t_n\}_{n=1}^\infty \subset [a,b]$, $t_n \neq t_0$ such that $t_n \longrightarrow_{n\to\infty} t_0$. Take any $\epsilon > 0$ and any bounded set $E \subset S(H)_\rho$. Since $X(\cdot)$ is differentiable at $t_0$, there exists an $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$

$$\left| \langle \frac{X(t_n) - X(t_0)}{t_n - t_0} - F(t_0), f \rangle \right| < \epsilon \ ,$$

for all $f \in E$. Therefore, for all $n \geq N$

$$|\langle X(t_n) - X(t_0), f \rangle| < |t_n - t_0| \left(\epsilon + |\langle F(t_0), f \rangle|\right) \ ,$$

for all $f \in E$. Now $F(t_0)$ is a single element of $S(H)_{-\rho}$, so there exists a $M < \infty$ such that

$$|\langle F(t_0), f\rangle| \le M \ ,$$

for all $f \in E$. Also $t_n \longrightarrow_{n\to\infty} t_0$, so there exists an $N_2 \in \mathbb{N}$ such that for all $n \ge N_2$

$$|t_n - t_0| < \frac{\epsilon}{\epsilon + M} \ .$$

So for all $n \ge \max\{N_1, N_2\}$

$$|\langle X(t_n) - X(t_0), f\rangle| < \frac{\epsilon}{\epsilon + M}(\epsilon + M) = \epsilon \ ,$$

for all $f \in E$.

Now consider any sequence $\{t_n\}_{n=1}^{\infty} \subset [a,b]$ such that $t_n \longrightarrow_{n\to\infty} t_0$. From the previous argument

$$\langle X(t_n) - X(t_0), f\rangle \underset{n\to\infty}{\longrightarrow} 0 \ ,$$

uniformly on all bounded subsets of $S(H)_{\rho}$, as required. ■

**Corollary 1.4** *If a function $X(t) : [a,b] \to S(H)_{-\rho}$ is a differentiable $S(H)_{-\rho}$ process on $[a,b]$, then it is also a continuous $S(H)_{-\rho}$ process on $[a,b]$.*

**Propositon 1.7** *Consider functions $X(t), F(t) : [a,b] \to S(H)_{-\rho}$. The following statements are equivalent:*

1. *$X(\cdot)$ is a differentiable $S(H)_{-\rho}$ process on $[a,b]$ with a continuous derivative $F(\cdot)$ on $[a,b]$ in $S(H)_{-\rho}$.*
2. *There exists a $q \in \mathbb{N}$ such that:*
   *(a) $X(t)$, $F(t) \in S(H)_{-\rho,-q}$, for all $t \in [a,b]$.*
   *(b) For all $\epsilon > 0$, there exists a $\delta > 0$ such that*
   $$s,t \in [a,b],\ |t-s| < \delta \Rightarrow \|F(t) - F(s)\|_{-\rho,-q} < \epsilon \ . \tag{1.25}$$
   *(c) For any $t_0 \in [a,b]$ and $\epsilon > 0$, there exists a $\delta > 0$ such that*
   $$s \in [a,b],\ 0 < |t_0 - s| < \delta \Rightarrow \left\| \frac{X(t_0) - X(s)}{t_0 - s} - F(t_0) \right\|_{-\rho,-q} < \epsilon \ . \tag{1.26}$$

**Proof:** Let $X(\cdot)$ and $F(\cdot)$ have forms

$$X(t) = \sum_{\alpha\in\mathcal{J}} c_\alpha(t) H_\alpha \ , \quad F(t) = \sum_{\alpha\in\mathcal{J}} d_\alpha(t) H_\alpha \ .$$

$(1 \Rightarrow 2)$ $(a)$ As $X(\cdot)$ and $F(\cdot)$ are both continuous on $[a,b]$, there exists a $q \in \mathbb{N}$ such that $X(t)$ and $F(t)$ belong to $S(H)_{-\rho,-q}$ and are bounded by some $M < \infty$ in the norm $\|\cdot\|_{-\rho,-q}$, for all $t \in [a,b]$.

(*b*) This follows from Proposition 1.6.

(*c*) Consider $s, t \in [a,b]$ with $s \neq t$. Now

$$\begin{aligned} &\left\| \frac{X(t)-X(s)}{t-s} - F(t) \right\|^2_{-\rho,-q-2} \\ = &\sum_{\alpha\in\mathcal{J}} (\alpha!)^{1-\rho} \left\| \frac{c_\alpha(t)-c_\alpha(s)}{t-s} - d_\alpha(t) \right\|^2_H (2\mathbb{N})^{-(q+2)\alpha} . \end{aligned}$$

By Lemma 1.10, $c_\alpha(\cdot)$ is differentiable on $[a,b]$, with derivative $d_\alpha(\cdot)$, for all $\alpha \in \mathcal{J}$. So by Langranges inequality, for all $t, s \in [a,b]$ with $s \neq t$

$$\left\| \frac{c_\alpha(t)-c_\alpha(s)}{t-s} \right\|_H \leq \sup_{t\in[a,b]} \|d_\alpha(t)\|_H ,$$

implying

$$\begin{aligned} &(\alpha!)^{1-\rho} \left\| \frac{c_\alpha(t)-c_\alpha(s)}{t-s} \right\|^2_H (2\mathbb{N})^{-q\alpha} \\ \leq &\sup_{t\in[a,b]} \left( (\alpha!)^{1-\rho} \|d_\alpha(t)\|_H (2\mathbb{N})^{-q\alpha} \right) \\ \leq &\sup_{t\in[a,b]} \left( \sum_{\alpha\in\mathcal{J}} (\alpha!)^{1-\rho} \|d_\alpha(t)\|^2_H (2\mathbb{N})^{-q\alpha} \right) \leq M^2 . \end{aligned}$$

Therefore

$$\begin{aligned} &\left\| \frac{X(t)-X(s)}{t-s} - F(t) \right\|^2_{-\rho,-q-2} \\ = &\sum_{\alpha\in\Gamma_k} (\alpha!)^{1-\rho} \left\| \frac{c_\alpha(t)-c_\alpha(s)}{t-s} - d_\alpha(t) \right\|^2_H (2\mathbb{N})^{-(q+2)\alpha} \\ &+ \sum_{\alpha\notin\Gamma_k} (\alpha!)^{1-\rho} \left\| \frac{c_\alpha(t)-c_\alpha(s)}{t-s} - d_\alpha(t) \right\|^2_H (2\mathbb{N})^{-q\alpha} (2\mathbb{N})^{-2\alpha} \\ \leq &\left( \max_{\alpha\in\Gamma_k} (\alpha!)^{1-\rho} \left\| \frac{c_\alpha(t)-c_\alpha(s)}{t-s} - d_\alpha(t) \right\|^2_H \right) \sum_{\alpha\in\mathcal{J}} (2\mathbb{N})^{-(q+2)\alpha} + 4M^2 \sum_{\alpha\notin\Gamma_k} (2\mathbb{N})^{-2\alpha} . \end{aligned}$$

Take $\epsilon > 0$ and $t_0 \in [a,b]$. Choose $k \in \mathbb{N}$ such that

$$\sum_{\alpha\notin\Gamma_k} (2\mathbb{N})^{-2\alpha} < \frac{\epsilon^2}{8M^2} ,$$

and $\delta > 0$ such that

$$\begin{aligned} &s \in [a,b],\ 0 < |t_0 - s| < \delta \\ \Rightarrow &\max_{\alpha\in\Gamma_k} (\alpha!)^{1-\rho} \left\| \frac{c_\alpha(t_0)-c_\alpha(s)}{t_0-s} - d_\alpha(t_0) \right\|^2_H < \frac{\epsilon^2}{2A(q+2)} . \end{aligned}$$

For $k$ and $\delta$ chosen this way

$$\left\|\frac{X(t_0)-X(s)}{t_0-s}-F(t_0)\right\|^2_{-\rho,-q-2} < \frac{\epsilon^2}{2A(q+2)}A(q+2)+4M^2\frac{\epsilon^2}{8M^2}=\epsilon^2 ,$$

for $s \in [a,b]$ with $0<|t_0-s|<\delta$, as required.

($2 \Rightarrow 1$) Continuity follows from Proposition 1.6.

Differentiability. Take any $t_0 \in [a,b]$ and any sequence $\{t_n\}_{n=1}^{\infty} \subset [a,b]$, $t_n \neq t_0$ such that $t_n \longrightarrow_{n\to\infty} t_0$. We have that for all $f \in S(H)_\rho$

$$\begin{aligned} & \left|\langle\frac{X(t_n)-X(t_0)}{t_n-t_0}-F(t_0),f\rangle\right| \\ \leq\ & \left\|\frac{X(t_n)-X(t_0)}{t_n-t_0}-F(t_0)\right\|_{-\rho,-q}\|f\|_{\rho,q} . \end{aligned}$$

This implies

$$\frac{X(t_n)-X(t_0)}{t_n-t_0}-F(t_0)\underset{n\to\infty}{\longrightarrow}0 ,$$

strongly in $S(H)_{-\rho}$, as required. ■

### 1.6.3 Integration

**Definition 1.6** *A function $F(t):[a,b]\to S(H)_{-\rho}$ is said to be integrable on $[a,t']$, $t'\in[a,b]$ if there exists a $X(t')\in S(H)_{-\rho}$ such that for all $f\in S(H)_\rho$*

$$\lim_{n\to\infty} < \sum_{k=0}^{n-1} F(t_k^*)\,(t_{k+1}-t_k)\,,f\rangle = \langle X(t'),f\rangle \ , \tag{1.27}$$

*uniformly on all bounded subsets of $S(H)_\rho$, for any set of partitions*

$$\{a=t_0<t_1<\ldots<t_n=t'\}_{n=1}^{\infty} ,$$

*of $[a,t']$, where:*

1. $\lim_{n\to\infty}\left(\max_{k\in\{0,\ldots,n-1\}}(t_{k+1}-t_k)\right)=0$.
2. $t_k^*\in[t_k,t_{k+1})$.

*If $F(\cdot)$ is integrable on $[0,t']$, then we define*

$$\int_a^{t'} F(s)ds := X(t') . \tag{1.28}$$

**Propositon 1.8** *Consider $F(t) : [a,b] \to S(H)_{-\rho}$ with form*

$$F(t) = \sum_{\alpha \in \mathcal{J}} c_\alpha(t) H_\alpha \ .$$

*If $F(\cdot)$ is a continuous $S(H)_{-\rho}$ process on $[a,b]$, then $F(\cdot)$ is also an integrable $S(H)_{-\rho}$ process on $[a,b]$ and*

$$\int_a^t F(s)ds = \sum_{\alpha \in \mathcal{J}} \left( \int_a^t c_\alpha(s) ds \right) H_\alpha \ , \tag{1.29}$$

*for all $t \in [a,b]$.*

**Proof:** As $F(\cdot)$ is continuous on $[a,b]$, there exists a $q \in \mathbb{N}$ such that $F(t) \in S(H)_{-\rho,-q}$ and is bounded by some $M < \infty$ in the norm $\|\cdot\|_{-\rho,-q}$, for all $t \in [a,b]$. Note that $c_\alpha(\cdot)$ is continuous on $[a,b]$ and hence integrable on $[a,b]$, for all $\alpha \in \mathcal{J}$. Now for any $t \in [a,b]$

$$\begin{aligned}
& \left\| \sum_{\alpha \in \mathcal{J}} \left( \int_a^t c_\alpha(s) ds \right) H_\alpha \right\|_{-\rho,-q-2}^2 \\
= \ & \sum_{\alpha \in \mathcal{J}} (\alpha!)^{1-\rho} \left\| \int_a^t c_\alpha(s) ds \right\|_H^2 (2\mathbb{N})^{-(q+2)\alpha} \\
\leq \ & \sum_{\alpha \in \mathcal{J}} (\alpha!)^{1-\rho} \left( (b-a) \left( \sup_{t \in [a,b]} \|c_\alpha(t)\|_H \right) \right)^2 (2\mathbb{N})^{-(q+2)\alpha} \\
\leq \ & (b-a)^2 \sum_{\alpha \in \mathcal{J}} \left( \sup_{t \in [a,b]} (\alpha!)^{1-\rho} \|c_\alpha(t)\|_H^2 (2\mathbb{N})^{-q\alpha} \right) (2\mathbb{N})^{-2\alpha} \\
\leq \ & (b-a)^2 M^2 \sum_{\alpha \in \mathcal{J}} (2\mathbb{N})^{-2\alpha} = (b-a)^2 M^2 A(2) = M_1^2 < \infty \ .
\end{aligned}$$

Take any $t' \in [a,b]$ and a set of partitions $\{a = t_0 < t_1 < \ldots < t_N = t'\}_{N=1}^{\infty}$. Now

$$\begin{aligned}
& \left\| \sum_{k=0}^{N-1} F(t_k^*)(t_{k+1} - t_k) - \sum_{\alpha \in \mathcal{J}} \left( \int_a^{t'} c_\alpha(s) ds \right) H_\alpha \right\|_{-\rho,-q-4}^2 \\
= \ & \sum_{\alpha \in \mathcal{J}} (\alpha!)^{1-\rho} \left\| \sum_{k=0}^{N-1} c_\alpha(t_k^*)(t_{k+1} - t_k) - \int_a^{t'} c_\alpha(s) ds \right\|_H^2 (2\mathbb{N})^{-(q+4)\alpha} \\
= \ & \sum_{\alpha \in \Gamma_n} (\alpha!)^{1-\rho} \left\| \sum_{k=0}^{N-1} c_\alpha(t_k^*)(t_{k+1} - t_k) - \int_a^{t'} c_\alpha(s) ds \right\|_H^2 (2\mathbb{N})^{-(q+4)\alpha} \\
& + \sum_{\alpha \notin \Gamma_n} (\alpha!)^{1-\rho} \left\| \sum_{k=0}^{N-1} c_\alpha(t_k^*)(t_{k+1} - t_k) - \int_a^{t'} c_\alpha(s) ds \right\|_H^2 (2\mathbb{N})^{-(q+4)\alpha}
\end{aligned}$$

$$\leq \left( \max_{\alpha\in\Gamma_n} (\alpha!)^{1-\rho} \left\| \sum_{k=0}^{N-1} c_\alpha(t_k^*)(t_{k+1}-t_k) - \int_a^{t'} c_\alpha(s)ds \right\|_H^2 \right) \left( \sum_{\alpha\in\Gamma_n} (2\mathbb{N})^{-(q+4)\alpha} \right)$$

$$+ \sum_{\alpha\notin\Gamma_n} \left( (\alpha!)^{\frac{1-\rho}{2}} \left( \left\| \sum_{k=0}^{N-1} c_\alpha(t_k^*)(t_{k+1}-t_k) \right\|_H \right.\right.$$

$$\left.\left. + \left\| \int_a^{t'} c_\alpha(s)ds \right\|_H \right) (2\mathbb{N})^{-\frac{q+2}{2}\alpha} \right)^2 (2\mathbb{N})^{-2\alpha}$$

$$\leq \left( \max_{\alpha\in\Gamma_n} (\alpha!)^{1-\rho} \left\| \sum_{k=0}^{N-1} c_\alpha(t_k^*)(t_{k+1}-t_k) - \int_a^{t'} c_\alpha(s)ds \right\|_H^2 \right) A(q+4)$$

$$+ \sum_{\alpha\notin\Gamma_n} \left( \left( \sum_{k=0}^{N-1} M(t_{k+1}-t_k) \right) + M_1 \right)^2 (2\mathbb{N})^{-2\alpha}$$

$$\leq \left( \max_{\alpha\in\Gamma_n} (\alpha!)^{1-\rho} \left\| \sum_{k=0}^{N-1} c_\alpha(t_k^*)(t_{k+1}-t_k) - \int_a^{t'} c_\alpha(s)ds \right\|_H^2 \right) A(q+4)$$

$$+ M_2 \sum_{\alpha\notin\Gamma_n} (2\mathbb{N})^{-2\alpha} .$$

Take any $\epsilon > 0$. Choose $n \in \mathbb{N}$ such that

$$\sum_{\alpha\notin\Gamma_n} (2\mathbb{N})^{-2\alpha} < \frac{\epsilon^2}{2M_2} ,$$

and $N_n \in \mathbb{N}$ such that for all $N \geq N_n$

$$\left( \max_{\alpha\in\Gamma_n} (\alpha!)^{1-\rho} \left\| \sum_{k=0}^{N-1} c_\alpha(t_k^*)(t_{k+1}-t_k) - \int_a^{t'} c_\alpha(s)ds \right\|_H^2 \right) < \frac{\epsilon^2}{2A(q+4)} .$$

For $n$ and $N_n$ chosen this way

$$\left\| \sum_{k=0}^{N-1} F(t_k^*)(t_{k+1}-t_k) - \sum_{\alpha\in\mathcal{J}} \left( \int_a^{t'} c_\alpha(s)ds \right) H_\alpha \right\|_{-\rho,-q-4}^2$$

$$< \frac{\epsilon^2}{2A(q+4)} A(q+4) + \frac{\epsilon^2}{2M_2} M_2 = \epsilon^2 ,$$

for all $N \geq N_n$, as required. ■

**Corollary 1.5** *If $F(t) : [a,b] \to S(H)_{-\rho}$ is a continuous $S(H)_{-\rho}$ process on $[a,b]$, then there exists a $q \in \mathbb{N}$ such that $F(t)$, $\int_a^t F(s)ds \in S(H)_{-\rho,-q}$ and are bounded by some $M < \infty$ in the norm $\|\cdot\|_{-\rho,-q}$, for $t \in [a,b]$.*

**Proof:** From the proof of Proposition 1.8, there exists a $q \in \mathbb{N}$ such that $F(t) \in S(H)_{-\rho,-q}$ and $\int_a^t F(s)ds \in S(H)_{-\rho,-q-2}$, both bounded in the $\|\cdot\|_{-\rho,-q-2}$, for all $t \in [a,b]$. ■

**Propositon 1.9** *Consider a continuous $S(H)_{-\rho}$ process $F(t) : [a,b] \to S(H)_{-\rho}$ and a differentiable $S(H)_{-\rho}$ process $X(t) : [a,b] \to S(H)_{-\rho}$, which has a continuous derivative on $[a,b]$. Then*

$$X(t) = \int_a^t F(s)ds + X(a) , \tag{1.30}$$

*if and only if*

$$\frac{dX(t)}{dt} = F(t) , \tag{1.31}$$

*for all $t \in [a,b]$.*

**Proof:** Let $F(\cdot)$ and $X(\cdot)$ have forms

$$F(t) = \sum_{\alpha \in \mathcal{J}} c_\alpha(t) H_\alpha , \quad X(t) = \sum_{\alpha \in \mathcal{J}} d_\alpha(t) H_\alpha .$$

($\Rightarrow$) As $F(\cdot)$ is continuous on $[a,b]$

$$d_\alpha(t) = \int_a^t c_\alpha(s)ds + d_\alpha(a) ,$$

for all $t \in [a,b]$ and $\alpha \in \mathcal{J}$. Therefore $d'_\alpha(t) = c_\alpha(t)$ for all $t \in [a,b]$ and

$$\frac{dX(t)}{dt} = \sum_{\alpha \in \mathcal{J}} d'_\alpha(t) H_\alpha = \sum_{\alpha \in \mathcal{J}} c_\alpha(t) H_\alpha .$$

($\Leftarrow$) As $X'(t) = F(t)$ for all $t \in [a,b]$, we have that $d'_\alpha(t) = c_\alpha(t)$ for all $\alpha \in \mathcal{J}$. Hence

$$d_\alpha(t) = \int_a^t c_\alpha(s)ds + d_\alpha(a) .$$

Hence

$$\begin{aligned} X(t) &= \sum_{\alpha \in \mathcal{J}} d_\alpha(t) H_\alpha = \sum_{\alpha \in \mathcal{J}} \left( \int_a^t c_\alpha(s)ds + d_\alpha(a) \right) H_\alpha \\ &= \int_a^t F(s)ds + X(a) , \end{aligned}$$

as required. ■

**Propositon 1.10** *Consider a continuous $S(H)_{-\rho}$ process $F(t) : [a,b] \to S(H)_{-\rho}$. Then the function $X(t) : [a,b] \to S(H)_{-\rho}$, defined by*

$$X(t) = \int_a^t F(s)ds , \tag{1.32}$$

*is a differentiable $S(H)_{-\rho}$ process on $[a,b]$ with derivative $F(\cdot)$.*

**Proof:** Take any $t_0 \in [a,b]$ and any sequence $\{t_n\}_{n=1}^{\infty} \subset [a,b]$, $t_n \neq t_0$ such that $t_n \longrightarrow_{n\to\infty} t_0$. We need to show that for all $f \in S(H)_\rho$

$$\langle \frac{X(t_0)-X(t_n)}{t_0-t_n}, f\rangle \underset{n\to\infty}{\longrightarrow} \langle F(t_0), f\rangle ,$$

uniformly on all bounded subsets of $S(H)_\rho$. To do this we shall show that there exists a $q \in \mathbb{N}$ such that

$$\left\| \frac{X(t_0)-X(t_n)}{t_0-t_n} - F(t_0) \right\|_{-\rho,-q} \underset{n\to\infty}{\longrightarrow} 0 .$$

Now we know that there exists a $q \in \mathbb{N}$ such that $F(t), X(t) \in S(H)_{-\rho,-q}$ and are bounded by some $M < \infty$ in the norm $\|\cdot\|_{-\rho,-q}$, for all $t \in [a,b]$. Now

$$\begin{aligned}
&\left\| \frac{X(t_0)-X(t_n)}{t_0-t_n} - F(t_0) \right\|^2_{-\rho,-q-2} \\
= &\left\| \frac{\int_a^{t_0} F(s)ds - \int_a^{t_n} F(s)ds}{t_0-t_n} - F(t_0) \right\|^2_{-\rho,-q-2} \\
= &\sum_{\alpha\in\mathcal{J}} (\alpha!)^{1-\rho} \left\| \frac{\int_a^{t_0} c_\alpha(s)ds - \int_a^{t_n} c_\alpha(s)ds}{t_0-t_n} - c_\alpha(t_0) \right\|^2_H (2\mathbb{N})^{-(q+2)\alpha} \\
= &\sum_{\alpha\in\Gamma_k} (\alpha!)^{1-\rho} \left\| \frac{\int_{t_n}^{t_0} c_\alpha(s)ds}{t_0-t_n} - c_\alpha(t_0) \right\|^2_H (2\mathbb{N})^{-(q+2)\alpha} \\
&+ \sum_{\alpha\notin\Gamma_k} (\alpha!)^{1-\rho} \left\| \frac{\int_{t_n}^{t_0} c_\alpha(s)ds}{t_0-t_n} - c_\alpha(t_0) \right\|^2_H (2\mathbb{N})^{-(q+2)\alpha} \\
\leq &\left( \max_{\alpha\in\Gamma_k} (\alpha!)^{1-\rho} \left\| \frac{\int_{t_n}^{t_0} c_\alpha(s)ds}{t_0-t_n} - c_\alpha(t_0) \right\|^2_H \right) \left( \sum_{\alpha\in\Gamma_k} (2\mathbb{N})^{-(q+2)\alpha} \right) \\
&+ \sum_{\alpha\notin\Gamma_k} \left( (\alpha!)^{\frac{1-\rho}{2}} \left( \left\| \frac{\int_{t_n}^{t_0} c_\alpha(s)ds}{t_0-t_n} \right\|_H + \|c_\alpha(t_0)\|_H \right) (2\mathbb{N})^{-\frac{q\alpha}{2}} \right)^2 (2\mathbb{N})^{-2\alpha} \\
\leq &\left( \max_{\alpha\in\Gamma_k} (\alpha!)^{1-\rho} \left\| \frac{\int_{t_n}^{t_0} c_\alpha(s)ds}{t_0-t_n} - c_\alpha(t_0) \right\|^2_H \right) \left( \sum_{\alpha\in\Gamma_k} (2\mathbb{N})^{-(q+2)\alpha} \right) \\
&+ \sum_{\alpha\notin\Gamma_k} \left( (\alpha!)^{\frac{1-\rho}{2}} \left( \left| \frac{(t_0-t_n)\left(\sup_{s\in[t_n,t_0]} \|c_\alpha(s)\|_H\right)}{t_0-t_n} \right| \right.\right. \\
&\left.\left. + \|c_\alpha(t_0)\|_H \right) (2\mathbb{N})^{-\frac{q\alpha}{2}} \right)^2 (2\mathbb{N})^{-2\alpha} \\
\leq &\left( \max_{\alpha\in\Gamma_k} (\alpha!)^{1-\rho} \left\| \frac{\int_{t_n}^{t_0} c_\alpha(s)ds}{t_0-t_n} - c_\alpha(t_0) \right\|^2_H \right) A(q+2) + \sum_{\alpha\notin\Gamma_k} (M+M)^2 (2\mathbb{N})^{-2\alpha} ,
\end{aligned}$$

where we denote $[t_0, t_n]$ by $[t_n, t_0]$ if $t_n > t_0$.

Take $\epsilon > 0$. Choose $k \in \mathbb{N}$ such that

$$\sum_{\alpha \notin \Gamma_k} (2\mathbb{N})^{-2\alpha} < \frac{\epsilon^2}{8M^2} ,$$

and $N_k \in \mathbb{N}$ such that for all $n \geq N_k$

$$\max_{\alpha \in \Gamma_k} (\alpha!)^{1-\rho} \left\| \frac{\int_{t_n}^{t_0} c_\alpha(s) ds}{t_0 - t_n} - c_\alpha(t_0) \right\|_H^2 < \frac{\epsilon^2}{2A(q+2)} .$$

For $k$ and $N_k$ chosen this way

$$\left\| \frac{X(t_0) - X(t_n)}{t_0 - t_n} - F(t_0) \right\|_{-\rho,-q-2}^2 < \frac{\epsilon^2}{2A(q+2)} A(q+2) + \frac{\epsilon^2}{8M^2} 4M^2 = \epsilon^2 ,$$

for all $n \geq N_k$, as required. ■

# Chapter 2

# Hermite Transform

In this chapter we develop the Hermite transform for elements of $S(H)_{-1}$ and results involving it. However before defining the Hermite transform, some results on power series are needed.

## 2.1 Results for Power Series Defined on $\mathbb{C}^{\mathbb{N}}$

$H_{\mathbb{C}}$ denotes the complexification of $H$.

The following neighbourhoods of zero are used for the Hermite transform.

**Definition 2.1** *For $q > 0$ define the following sets:*

1. $\mathbb{K}_q^n := \{z \in (\mathbb{C}^{\mathbb{N}})_c \, ; \, |z_j| < (2j)^{-q}, \ j \le n \text{ and } z_j = 0, \ j > n\}$,
   $\mathbb{K}_q := \{z \in \mathbb{C}^{\mathbb{N}} \, ; \, |z_j| < (2j)^{-q}, \ j \in \mathbb{N}\}$.
2. $\overline{\mathbb{K}_q^n} := \{z \in (\mathbb{C}^{\mathbb{N}})_c \, ; \, |z_j| \le (2j)^{-q}, \ j \le n \text{ and } z_j = 0, \ j > n\}$,
   $\overline{\mathbb{K}}_q := \{z \in \mathbb{C}^{\mathbb{N}} \, ; \, |z_j| \le (2j)^{-q}, \ j \in \mathbb{N}\}$.

We can regard $\mathbb{K}_q^n$ and $\overline{\mathbb{K}_q^n}$ as subsets of $\mathbb{C}^n$.

**Lemma 2.1** *Consider $f(z) : \overline{\mathbb{K}_q^n} \to \mathbb{C}^k$, $q > 0$, having form*

$$f(z) = (f_1(z), \ldots, f_k(z)) \ ,$$

*where*

$$f_l(z) = \sum_{\text{index } \alpha \le n} c_{l,\alpha} z^\alpha \ , \quad c_{l,\alpha} \in \mathbb{C} \ , \ l = 1, \ldots, k \ ,$$

*bounded by some $M < \infty$. Then for all $\alpha \in \mathcal{J}$ such that index $\alpha \le n$ and $z \in \overline{\mathbb{K}_q^n}$*

$$\left( \sum_{l=1}^{k} |c_{l,\alpha}|^2 \right)^{1/2} |z^\alpha| \le M \ . \tag{2.1}$$

**Proof:** For all $\alpha \in \mathcal{J}$ such that index $\alpha \leq n$

$$c_{l,\alpha} = \left(\frac{1}{2\pi i}\right)^n \int_\Gamma \frac{f_l(z)}{z^{\alpha+1}} dz \ , \quad l = 1, \ldots, k \ ,$$

where $\Gamma = (|z_1| = 2^{-q}, |z_2| = (2.2)^{-q}, \ldots, |z_n| = (2.n)^{-q})$ and $dz = dz_1 \ldots dz_n$.

Set $c_\alpha = (c_{1,\alpha}, \ldots, c_{k,\alpha})$, noting that

$$c_\alpha \ = \ \left(\frac{1}{2\pi i}\right)^n \left(\int_\Gamma \frac{f_1(z)}{z^{\alpha+1}} dz, \ldots, \int_\Gamma \frac{f_k(z)}{z^{\alpha+1}} dz\right) \ .$$

Now reparametrize $z = z(\theta)$ by

$$z_j = (2j)^{-q} e^{i\theta_j} \ , \qquad 0 \leq \theta \leq 2\pi \ , \ 1 \leq j \leq n \ .$$

This gives

$$\begin{aligned}
\int_\Gamma \frac{f_l(z)}{z^{\alpha+1}} dz \ &= \ \int_{[0,2\pi]} \cdots \int_{[0,2\pi]} \frac{f_l(z(\theta))}{z(\theta)^{\alpha+1}} \left(\prod_{j=1}^n \frac{\partial z_j}{\partial \theta_j}\right) d\theta_1 \ldots d\theta_n \\
&= \ \int_{[0,2\pi]^n} \frac{f_l(z(\theta))}{\left(\prod_{j=1}^n (2j)^{-q(\alpha_j+1)} e^{i\theta_j(\alpha_j+1)}\right)} \left(\prod_{j=1}^n i(2j)^{-q} e^{i\theta_j}\right) d\theta \\
&= \ \int_{[0,2\pi]^n} \frac{f_l(z(\theta)) \ \left(\prod_{j=1}^n ie^{-i\theta_j\alpha_j}\right)}{\left(\prod_{j=1}^n (2j)^{-q\alpha_j}\right)} d\theta \\
&= \ \frac{i^n}{(2\mathbb{N})^{-q\alpha}} \int_{[0,2\pi]^n} f_l(z(\theta)) \ e^{-i\sum_{j=1}^n \theta_j\alpha_j} d\theta \ ,
\end{aligned}$$

where $d\theta = d\theta_1 \ldots d\theta_n$. So

$$\begin{aligned}
&c_\alpha \\
&= \ \frac{(2\mathbb{N})^{q\alpha}}{(2\pi)^n} \left(\int_{[0,2\pi]^n} f_1(z(\theta)) \ e^{-i\sum_{j=1}^n \theta_j\alpha_j} d\theta, \ldots, \int_{[0,2\pi]^n} f_k(z(\theta)) \ e^{-i\sum_{j=1}^n \theta_j\alpha_j} d\theta\right) \ .
\end{aligned}$$

Hence

$$\begin{aligned}
|c_\alpha| \ &= \ \left[\sum_{l=1}^k |c_{l,\alpha}|^2\right]^{1/2} \ = \ \frac{(2\mathbb{N})^{q\alpha}}{(2\pi)^n} \left[\sum_{l=1}^k \left|\int_{[0,2\pi]^n} f_l(z(\theta)) \ e^{-i\sum_{j=1}^n \theta_j\alpha_j} d\theta\right|^2\right]^{1/2} \\
&\leq \ \frac{(2\mathbb{N})^{q\alpha}}{(2\pi)^n} \left[\sum_{l=1}^k \left(\int_{[0,2\pi]^n} \left|f_l(z(\theta)) \ e^{-i\sum_{j=1}^n \theta_j\alpha_j}\right| d\theta\right)^2\right]^{1/2} \\
&= \ \frac{(2\mathbb{N})^{q\alpha}}{(2\pi)^n} \left[\sum_{l=1}^k \left(\int_{[0,2\pi]^n} |f_l(z(\theta))| \ d\theta\right)^2\right]^{1/2} \\
&\leq \ \frac{(2\mathbb{N})^{q\alpha}}{(2\pi)^n} \left[\sum_{l=1}^k \left(\left\{\int_{[0,2\pi]^n} |f_l(z(\theta))|^2 \ d\theta\right\}^{1/2} \left\{\int_{[0,2\pi]^n} 1^2 d\theta\right\}^{1/2}\right)^2\right]^{1/2}
\end{aligned}$$

$$= \frac{(2\mathbb{N})^{q\alpha}}{(2\pi)^n}\left[\sum_{l=1}^{k}\left(\int_{[0,2\pi]^n}|f_l(z(\theta))|^2\,d\theta\right)\;(2\pi)^n\right]^{1/2}$$

$$= \frac{(2\mathbb{N})^{q\alpha}}{(2\pi)^{n/2}}\left[\int_{[0,2\pi]^n}\sum_{l=1}^{k}|f_l(z(\theta))|^2\,d\theta\right]^{1/2} \leq \frac{(2\mathbb{N})^{q\alpha}}{(2\pi)^{n/2}}\left[\int_{[0,2\pi]^n}M^2 d\theta\right]^{1/2}$$

$$= \frac{(2\mathbb{N})^{q\alpha}}{(2\pi)^{n/2}}\left[M^2\int_{[0,2\pi]^n}1\;d\theta\right]^{1/2} = \frac{(2\mathbb{N})^{q\alpha}}{(2\pi)^{n/2}}M(2\pi)^{n/2} = M(2\mathbb{N})^{q\alpha}\;.$$

Therefore for all $\forall z \in \overline{\mathbb{K}}_q^n$, we have that

$$|c_\alpha|\;|z^\alpha| \leq M\;,$$

as for $z \in \overline{\mathbb{K}}_q^n$, $|z| \leq (2\mathbb{N})^{-q\alpha}$. ■

**Corollary 2.1** *Consider $X(z) : \mathbb{K}_q \to \mathbb{C}^k$, $q > 0$, having form*

$$X(z) = \sum_{\alpha\in\mathcal{J}} c_\alpha z^\alpha = \sum_{\alpha\in\mathcal{J}}(c_{1,\alpha},\ldots,c_{k,\alpha})z^\alpha\;,\quad c_\alpha \in \mathbb{C}^k\;,$$

*bounded by some $M < \infty$. Then for all $\alpha \in \mathcal{J}$ and $z \in \overline{\mathbb{K}}_q$*

$$|c_\alpha|\;|z^\alpha| \leq M\;. \tag{2.2}$$

**Proof:** Take an $\alpha \in \mathcal{J}$. Let $n =$ index $\alpha$. For $z \in \overline{\mathbb{K}}_q$, define $w \in \overline{\mathbb{K}}_q^n$ as

$$w_j = \begin{cases} z_j\;, & j \leq n \\ 0\;, & j > n \end{cases}\;.$$

Now there exists a sequence $\{w^{(m)}\}_{m=1}^\infty$ such that $w^{(m)} \in \overline{\mathbb{K}}_{q+1/m}^n$, and $\{|(w^{(m)})^\alpha|\}_{m=1}^\infty$ is an increasing sequence such that $|(w^{(m)})^\alpha| \longrightarrow_{m\to\infty} |w^\alpha|$. From Lemma 2.1

$$|c_\alpha|\;|(w^{(m)})^\alpha| \leq M\;.$$

Therefore

$$|c_\alpha|\;|w^\alpha| \leq M\;.$$

Now it follows from the construction of $w$ that $|w^\alpha| = |z^\alpha|$. Therefore

$$|c_\alpha|\;|z^\alpha| \leq M\;,$$

as required. ■

**Lemma 2.2** *Consider a $z \in \mathbb{C}^\mathbb{N}$ such that*

$$\sum_{\alpha\in\mathcal{J}}\sum_{i=1}^{\infty} c_{i,\alpha} z^\alpha e_i\;,\quad c_{i,\alpha} \in \mathbb{C}\;,$$

*converges in $H_{\mathbb{C}}$. For this $z$*

$$\sum_{\alpha\in\mathcal{J}}\sum_{i=1}^{\infty} c_{i,\alpha} z^{\alpha} e_i = \sum_{i=1}^{\infty}\sum_{\alpha\in\mathcal{J}} c_{i,\alpha} z^{\alpha} e_i \ , \tag{2.3}$$

*and therefore*

$$\left\|\sum_{\alpha\in\mathcal{J}}\sum_{i=1}^{\infty} c_{i,\alpha} z^{\alpha} e_i\right\|_{H_{\mathbb{C}}}^2 = \sum_{i=1}^{\infty}\left|\sum_{\alpha\in\mathcal{J}} c_{i,\alpha} z^{\alpha}\right|^2 \ . \tag{2.4}$$

**Proof:** Using the fact that the functional $\langle \cdot, e_j\rangle_{H_{\mathbb{C}}}$ is continuous, we have that

$$\begin{aligned}
&\sum_{\alpha\in\mathcal{J}}\sum_{i=1}^{\infty} c_{i,\alpha} z^{\alpha} e_i \\
=\ & \sum_{j=1}^{\infty}\langle\sum_{\alpha\in\mathcal{J}}\sum_{i=1}^{\infty} c_{i,\alpha} z^{\alpha} e_i, e_j\rangle_{H_{\mathbb{C}}} e_j = \sum_{j=1}^{\infty}\sum_{\alpha\in\mathcal{J}}\sum_{i=1}^{\infty}\langle c_{i,\alpha} z^{\alpha} e_i, e_j\rangle_{H_{\mathbb{C}}} e_j \\
=\ & \sum_{j=1}^{\infty}\sum_{\alpha\in\mathcal{J}}\sum_{i=1}^{\infty}\delta_{i,j}\ c_{i,\alpha} z^{\alpha} e_j = \sum_{j=1}^{\infty}\sum_{\alpha\in\mathcal{J}} c_{j,\alpha} z^{\alpha} e_j \ ,
\end{aligned}$$

as required. ■

**Corollary 2.2** *Consider $X(z) : \mathbb{K}_q \to H_{\mathbb{C}}$, $q > 0$, having form*

$$X(z) = \sum_{\alpha\in\mathcal{J}} c_{\alpha} z^{\alpha} = \sum_{\alpha\in\mathcal{J}}\sum_{i=1}^{\infty} c_{i,\alpha} z^{\alpha} e_i \ , \quad c_{\alpha} \in H_{\mathbb{C}} \ ,$$

*bounded by some $M < \infty$. Then for all $\alpha \in \mathcal{J}$ and $z \in \overline{\mathbb{K}}_q$*

$$\|c_{\alpha}\|_{H_{\mathbb{C}}} |z^{\alpha}| = \left(\sum_{i=1}^{\infty}|c_{i,\alpha}|^2\right)^{1/2} |z^{\alpha}| \le M \ . \tag{2.5}$$

**Proof:** For $z \in \mathbb{K}_q$ and $k \in \mathbb{N}$, define

$$G_k(z) = \sum_{\alpha\in\mathcal{J}}(c_{1,\alpha}, \ldots, c_{k,\alpha}) z^{\alpha} = \sum_{\alpha\in\mathcal{J}} c_{\alpha}^{(k)} z^{\alpha} \ ,$$

where $c_{\alpha}^{(k)} = (c_{1,\alpha}, \ldots, c_{k,\alpha})$. This is well defined as

$$|G_k(z)|^2 \ = \ \sum_{i=1}^{k}\left|\sum_{\alpha\in\mathcal{J}} c_{i,\alpha} z^{\alpha}\right|^2 \le \sum_{i=1}^{\infty}\left|\sum_{\alpha\in\mathcal{J}} c_{i,\alpha} z^{\alpha}\right|^2 = \|X(z)\|_{H_{\mathbb{C}}}^2 \le M^2 \ ,$$

using Lemma 2.2. From Corollary 2.1, for $z \in \overline{\mathbb{K}}_q$

$$\left|c_{\alpha}^{(k)}\right| \ |z^{\alpha}| \le M \ .$$

So the increasing sequence

$$\left\{ \left| c_\alpha^{(k)} \right| \ |z^\alpha| \right\}_{k=1}^{\infty} ,$$

has a limit and

$$\lim_{k\to\infty} \left| c_\alpha^{(k)} \right| \ |z^\alpha| \ = \ \left\| \sum_{i=1}^{\infty} c_{i,\alpha} e_i \right\|_{H_\mathbb{C}} |z^\alpha| \le M ,$$

as required. ■

**Propositon 2.1** *Consider $X(z) : \mathbb{K}_q \to H_\mathbb{C}$, $q > 1$, having form*

$$X(z) = \sum_{\alpha \in \mathcal{J}} c_\alpha z^\alpha , \quad c_\alpha \in H_\mathbb{C} ,$$

*bounded by some $M < \infty$. Then for all $z \in \overline{\mathbb{K}}_{2q}$*

$$\sum_{\alpha \in \mathcal{J}} \|c_\alpha\|_{H_\mathbb{C}} |z^\alpha| \le M A(q) . \tag{2.6}$$

**Proof:** For $w \in \overline{\mathbb{K}}_q$ and $\alpha \in \mathcal{J}$

$$\|c_\alpha\|_{H_\mathbb{C}} \ |w^\alpha| \le M ,$$

from Corollary 2.2. Take $z \in \overline{\mathbb{K}}_{2q}$. Define $w$ by

$$w_j = (2j)^q z_j .$$

We have that $w \in \overline{\mathbb{K}}_q$ as

$$|w_j| = (2j)^q |z_j| \le (2j)^q (2j)^{-2q} = (2j)^{-q} .$$

So

$$\begin{aligned}
\sum_{\alpha \in \mathcal{J}} \|c_\alpha\|_{H_\mathbb{C}} |z^\alpha| &= \sum_{\alpha \in \mathcal{J}} \|c_\alpha\|_{H_\mathbb{C}} \ |z^\alpha| (2\mathbb{N})^{\frac{q\alpha}{2}} (2\mathbb{N})^{-\frac{q\alpha}{2}} \\
&\le \left( \sum_{\alpha \in \mathcal{J}} \|c_\alpha\|_{H_\mathbb{C}}^2 \ |z^\alpha|^2 (2\mathbb{N})^{q\alpha} \right)^{1/2} \left( \sum_{\alpha \in \mathcal{J}} (2\mathbb{N})^{-q\alpha} \right)^{1/2} \\
&= \left( \sum_{\alpha \in \mathcal{J}} \|c_\alpha\|_{H_\mathbb{C}}^2 \ |w^\alpha|^2 (2\mathbb{N})^{-2q\alpha} (2\mathbb{N})^{q\alpha} \right)^{1/2} A(q)^{1/2} \\
&\le \left( M^2 \sum_{\alpha \in \mathcal{J}} (2\mathbb{N})^{-q\alpha} \right)^{1/2} A(q)^{1/2} = M A(q) ,
\end{aligned}$$

as required. ■

**Propositon 2.2** *Consider* $X_1(z), X_2(z) : \mathbb{K}_q \to H_{\mathbb{C}}$, $q > 0$, *having form*

$$X_1(z) = \sum_{\alpha \in \mathcal{J}} c_\alpha z^\alpha \ , \ X_2(z) = \sum_{\alpha \in \mathcal{J}} d_\alpha z^\alpha \ , \quad c_\alpha, d_\alpha \in H_{\mathbb{C}} \ .$$

*If* $X_1(z) = X_2(z)$ *for all* $z \in \mathbb{K}_q^n$ *and* $n \in \mathbb{N}$, *then for all* $\alpha \in \mathcal{J}$, $c_\alpha = d_\alpha$. *Moreover,* $X_1(z) = X_2(z)$ *for all* $z \in \mathbb{K}_q$.

**Proof:** Take any $\alpha \in \mathcal{J}$. Let $n = \text{index } \alpha$. If $X_1(z) = X_2(z)$ for all $z \in \mathbb{K}_q^n$, then

$$\langle X_1(z), e_i \rangle_{H_{\mathbb{C}}} = \langle X_2(z), e_i \rangle_{H_{\mathbb{C}}} \ ,$$

for all $z \in \mathbb{K}_q^n$. Now

$$\langle X_1(z), e_i \rangle_{H_{\mathbb{C}}} \ = \ \langle \sum_{\text{index } \beta \le n} c_\beta z^\beta, e_i \rangle_{H_{\mathbb{C}}} = \sum_{\text{index } \beta \le n} \langle c_\beta, e_i \rangle_{H_{\mathbb{C}}} z^\beta \ ,$$

and similarly

$$\langle X_2(z), e_i \rangle_{H_{\mathbb{C}}} \ = \ \sum_{\text{index } \beta \le n} \langle d_\beta, e_i \rangle_{H_{\mathbb{C}}} z^\beta \ .$$

By standard results for power series in $\mathbb{C}^n$ with values in $\mathbb{C}$, we have that

$$\langle c_\alpha, e_i \rangle_{H_{\mathbb{C}}} = \langle d_\alpha, e_i \rangle_{H_{\mathbb{C}}} \ .$$

Therefore

$$c_\alpha \ = \ \sum_{i=1}^{\infty} \langle c_\alpha, e_i \rangle_{H_{\mathbb{C}}} e_i = \sum_{i=1}^{\infty} \langle d_\alpha, e_i \rangle_{H_{\mathbb{C}}} e_i = d_\alpha \ ,$$

as required. ■

## 2.2 Convergence Theorems for Power Series

**Definition 2.2** *Consider* $X_n(z), X(z) : \mathbb{K}_q \to H_{\mathbb{C}}$, $q > 0$, $n \in \mathbb{N}$, *having form*

$$X_n(z) = \sum_{\alpha \in \mathcal{J}} c_\alpha^{(n)} z^\alpha \ , \ X(z) = \sum_{\alpha \in \mathcal{J}} c_\alpha z^\alpha \ , \quad c_\alpha^{(n)}, c_\alpha \in H_{\mathbb{C}} \ .$$

*The sequence* $\{X_n(\cdot)\}_{n=1}^{\infty}$ *is said to converge pointwise boundedly to* $X(\cdot)$ *for* $z \in \mathbb{K}_q$ *if:*

1. *For all* $z \in \mathbb{K}_q$
$$\|X_n(z) - X(z)\|_{H_{\mathbb{C}}} \underset{n \to \infty}{\longrightarrow} 0 \ .$$

*2. There exists a $M < \infty$ such that*

$$\|X_n(z)\|_{H_{\mathbb{C}}} \ , \ \|X(z)\|_{H_{\mathbb{C}}} \leq M \ ,$$

*for all $z \in \mathbb{K}_q$ and $n \in \mathbb{N}$.*

**Lemma 2.3** *Consider $X_n(z), X(z) : \mathbb{K}_q \to H_{\mathbb{C}}$, $q > 0$, $n \in \mathbb{N}$, having form*

$$X_n(z) = \sum_{\alpha \in \mathcal{J}} c_\alpha^{(n)} z^\alpha \ , \ X(z) = \sum_{\alpha \in \mathcal{J}} c_\alpha z^\alpha \ , \quad c_\alpha^{(n)}, c_\alpha \in H_{\mathbb{C}} \ .$$

*If the sequence $\{X_n(\cdot)\}_{n=1}^\infty$ converges pointwise to $X(\cdot)$ for $z \in \mathbb{K}_q$, then for all $\alpha \in \mathcal{J}$*

$$c_\alpha^{(n)} \underset{n \to \infty}{\longrightarrow} c_\alpha \ , \tag{2.7}$$

*in $H_{\mathbb{C}}$.*

**Proof:** Take $\beta \in \mathcal{J}$. Let $k = \text{index } \beta$. By standard results for power series in $\mathbb{C}^k$ with values in $\mathbb{C}$ and the continuity of the operator $< \cdot, e_j >_{H_{\mathbb{C}}}$, we have that

$$\begin{aligned}
c_\beta &= \sum_{i=1}^\infty \langle c_\beta, e_i \rangle_{H_{\mathbb{C}}} e_i = \sum_{i=1}^\infty \left( \frac{\partial^\beta}{\partial z^\beta} \sum_{\text{index } \alpha \leq k} \langle c_\alpha, e_i \rangle_{H_{\mathbb{C}}} z^\alpha \Big|_{z=0} \right) e_i \\
&= \sum_{i=1}^\infty \left( \frac{\partial^\beta}{\partial z^\beta} \sum_{\text{index } \alpha \leq k} \langle c_\alpha z^\alpha, e_i \rangle_{H_{\mathbb{C}}} \Big|_{z=0} \right) e_i \\
&= \sum_{i=1}^\infty \left( \frac{\partial^\beta}{\partial z^\beta} \langle \sum_{\text{index } \alpha \leq k} c_\alpha z^\alpha, e_i \rangle_{H_{\mathbb{C}}} \Big|_{z=0} \right) e_i \\
&= \sum_{i=1}^\infty \left( \frac{\partial^\beta}{\partial z^\beta} \langle \lim_{n \to \infty} \sum_{\text{index } \alpha \leq k} c_\alpha^{(n)} z^\alpha, e_i \rangle_{H_{\mathbb{C}}} \Big|_{z=0} \right) e_i \\
&= \sum_{i=1}^\infty \left( \frac{\partial^\beta}{\partial z^\beta} \left( \lim_{n \to \infty} \sum_{\text{index } \alpha \leq k} \langle c_\alpha^{(n)}, e_i \rangle_{H_{\mathbb{C}}} z^\alpha \right) \Big|_{z=0} \right) e_i \\
&= \sum_{i=1}^\infty \left( \lim_{n \to \infty} \left( \frac{\partial^\beta}{\partial z^\beta} \sum_{\text{index } \alpha \leq k} \langle c_\alpha^{(n)}, e_i \rangle_{H_{\mathbb{C}}} z^\alpha \Big|_{z=0} \right) \right) e_i \\
&= \sum_{i=1}^\infty \lim_{n \to \infty} \left( \frac{\partial^\beta}{\partial z^\beta} \sum_{\text{index } \alpha \leq k} \langle c_\alpha^{(n)} z^\alpha, e_i \rangle_{H_{\mathbb{C}}} e_i \Big|_{z=0} \right) \\
&= \sum_{i=1}^\infty \lim_{n \to \infty} \left( \sum_{j=1}^\infty \langle \frac{\partial^\beta}{\partial z^\beta} \sum_{\text{index } \alpha \leq k} \langle c_\alpha^{(n)} z^\alpha, e_i \rangle_{H_{\mathbb{C}}} e_i \Big|_{z=0}, e_j \rangle_{H_{\mathbb{C}}} e_j \right)
\end{aligned}$$

$$= \sum_{i=1}^{\infty} \lim_{n\to\infty} \left( \sum_{j=1}^{\infty} \left( \frac{\partial^\beta}{\partial z^\beta} \sum_{\text{index } \alpha \le k} \langle c_\alpha^{(n)} z^\alpha, e_i \rangle_{H_\mathbb{C}} \Bigg|_{z=0} \right) \langle e_i, e_j \rangle_{H_\mathbb{C}} e_j \right)$$

$$= \lim_{n\to\infty} \sum_{j=1}^{\infty} \left( \frac{\partial^\beta}{\partial z^\beta} \sum_{\text{index } \alpha \le k} \langle c_\alpha^{(n)} z^\alpha, e_j \rangle_{H_\mathbb{C}} \Bigg|_{z=0} \right) e_j$$

$$= \lim_{n\to\infty} \sum_{j=1}^{\infty} \left( \frac{\partial^\beta}{\partial z^\beta} \sum_{\text{index } \alpha \le k} \langle c_\alpha^{(n)}, e_j \rangle_{H_\mathbb{C}} z^\alpha \Bigg|_{z=0} \right) e_j$$

$$= \lim_{n\to\infty} \sum_{j=1}^{\infty} \langle c_\beta^{(n)}, e_j \rangle_{H_\mathbb{C}} e_j = \lim_{n\to\infty} c_\beta^{(n)} ,$$

as required. ■

**Propositon 2.3** *Consider* $X_n(z), X(z) : \mathbb{K}_q \to H_\mathbb{C}$, $q > 1$, $n \in \mathbb{N}$, *having form*

$$X_n(z) = \sum_{\alpha \in \mathcal{J}} c_\alpha^{(n)} z^\alpha \ , \ X(z) = \sum_{\alpha \in \mathcal{J}} c_\alpha z^\alpha \ , \quad c_\alpha^{(n)}, c_\alpha \in H_\mathbb{C} \ .$$

*If the sequence* $\{X_n(\cdot)\}_{n=1}^{\infty}$ *converges pointwise boundedly to* $X(\cdot)$ *for* $z \in \mathbb{K}_q$, *then* $X_n(\cdot)$ *converges uniformly to* $X(\cdot)$ *for* $z \in \overline{\mathbb{K}}_{2q}$.

**Proof:** Take $z \in \overline{\mathbb{K}}_{2q}$. Using $z$, define $w \in \overline{\mathbb{K}}_q$ by

$$w_j = z_j (2j)^q \ .$$

Now by the definition of pointwise convergence, there exists a $M < \infty$ such that $||X_n(z)||_{H_\mathbb{C}}, ||X(z)||_{H_\mathbb{C}} \le M$ for all $z \in \mathbb{K}_q$ and $n \in \mathbb{N}$. Therefore

$$||X_n(z) - X(z)||_{H_\mathbb{C}}$$

$$= \left\| \sum_{\alpha \in \mathcal{J}} (c_\alpha^{(n)} - c_\alpha) z^\alpha \right\|_{H_\mathbb{C}} \le \sum_{\alpha \in \Gamma_k} ||c_\alpha^{(n)} - c_\alpha||_{H_\mathbb{C}} \, |z^\alpha| + \sum_{\alpha \notin \Gamma_k} ||c_\alpha^{(n)} - c_\alpha||_{H_\mathbb{C}} \, |z^\alpha|$$

$$\le \sum_{\alpha \in \Gamma_k} ||c_\alpha^{(n)} - c_\alpha||_{H_\mathbb{C}} \, (2\mathbb{N})^{-q\alpha} + \sum_{\alpha \notin \Gamma_k} ||c_\alpha^{(n)} - c_\alpha||_{H_\mathbb{C}} \, |w^\alpha| (2\mathbb{N})^{-q\alpha}$$

$$\le \left( \max_{\alpha \in \Gamma_k} ||c_\alpha^{(n)} - c_\alpha||_{H_\mathbb{C}} \right) \sum_{\alpha \in \Gamma_k} (2\mathbb{N})^{-q\alpha} + 2M \sum_{\alpha \notin \Gamma_k} (2\mathbb{N})^{-q\alpha} ,$$

using Corollary 2.2.

Take $\epsilon > 0$. Choose $k \in \mathbb{N}$ such that

$$\sum_{\alpha \notin \Gamma_k} (2\mathbb{N})^{-q\alpha} < \frac{\epsilon}{4M} ,$$

and $N_k \in \mathbb{N}$ such that for all $n \ge N_k$

$$\max_{\alpha \in \Gamma_k} ||c_\alpha^{(n)} - c_\alpha||_{H_\mathbb{C}} < \frac{\epsilon}{2A(q)} ,$$

for all $\alpha \in \Gamma_k$. With this $k$ and $N_k$

$$\|X_n(z) - X(z)\|_{H_\mathbb{C}} < \frac{\epsilon}{2A(q)}A(q) + 2M\frac{\epsilon}{4M} = \epsilon ,$$

for all $n \geq N_k$, as required. ■

**Propositon 2.4** *Consider $X(t,z) : [a,b] \times \mathbb{K}_q \to H_\mathbb{C}$, $q > 1$, having form*

$$X(t,z) = \sum_{\alpha \in \mathcal{J}} c_\alpha(t) z^\alpha , \quad c_\alpha(t) \in H_\mathbb{C} ,$$

*bounded by some $M < \infty$, where the functions $\{c_\alpha(\cdot)\}_{\alpha \in \mathcal{J}}$ are continuous on $[a,b]$. Then $X(\cdot, z)$ is continuous with respect to $t$ on $[a,b]$ for all $z \in \overline{\mathbb{K}}_{2q}$.*

**Proof:** Take $z \in \overline{\mathbb{K}}_{2q}$. Using $z$, define $w \in \overline{\mathbb{K}}_q$ by

$$w_j = z_j (2j)^q .$$

Now

$$\begin{aligned}
&\|X(t,z) - X(s,z)\|_{H_\mathbb{C}} \\
= &\left\| \sum_{\alpha \in \mathcal{J}} (c_\alpha(t) - c_\alpha(s)) z^\alpha \right\|_{H_\mathbb{C}} \\
\leq &\sum_{\alpha \in \Gamma_k} \|c_\alpha(t) - c_\alpha(s)\|_{H_\mathbb{C}} \, |z^\alpha| + \sum_{\alpha \notin \Gamma_k} \|c_\alpha(t) - c_\alpha(s)\|_{H_\mathbb{C}} \, |z^\alpha| \\
\leq &\sum_{\alpha \in \Gamma_k} \|c_\alpha(t) - c_\alpha(s)\|_{H_\mathbb{C}} \, (2\mathbb{N})^{-q\alpha} + \sum_{\alpha \notin \Gamma_k} \|c_\alpha(t) - c_\alpha(s)\|_{H_\mathbb{C}} \, |w^\alpha| (2\mathbb{N})^{-q\alpha} \\
\leq &\left( \max_{\alpha \in \Gamma_k} \|c_\alpha(t) - c_\alpha(s)\|_{H_\mathbb{C}} \right) \sum_{\alpha \in \Gamma_k} (2\mathbb{N})^{-q\alpha} + 2M \sum_{\alpha \notin \Gamma_k} (2\mathbb{N})^{-q\alpha} ,
\end{aligned}$$

using Corollary 2.2. Take $\epsilon > 0$. Choose $k \in \mathbb{N}$ such that

$$\sum_{\alpha \notin \Gamma_k} (2\mathbb{N})^{-q\alpha} < \frac{\epsilon}{4M} ,$$

and $\delta_k > 0$ such that

$$s, t \in [a,b], \ |t - s| < \delta_k \Rightarrow \max_{\alpha \in \Gamma_k} \|c_\alpha(t) - c_\alpha(s)\|_{H_\mathbb{C}} < \frac{\epsilon}{2A(q)} .$$

With this $k$ and $\delta_k$

$$\|X(t,z) - X(s,z)\|_{H_\mathbb{C}} < \frac{\epsilon}{2A(q)}A(q) + 2M\frac{\epsilon}{4M} = \epsilon ,$$

for all $s, t \in [a,b]$ with $|t - s| < \delta_k$, as required. ■

and $\delta_k > 0$ such that

$$s \in [a,b],\ 0 < |t-s| < \delta_k \Rightarrow \left\| \frac{c_\alpha(t) - c_\alpha(s)}{t-s} - d_\alpha(t) \right\|_{H_\mathbb{C}} < \frac{\epsilon}{2A(q)} ,$$

for all $\alpha \in \Gamma_k$. With this $k$ and $\delta_k$

$$\left\| \frac{X(t,z) - X(s,z)}{t-s} - F(t,z) \right\|_{H_\mathbb{C}} < \frac{\epsilon}{2A(q)} A(q) + 2M \frac{\epsilon}{4M} = \epsilon ,$$

for $s \in [a,b]$ with $0 < |t-s| < \delta_k$, as required. ■

## 2.3 The Hermite Transform

In this section we define the Hermite transform for elements of $S(H)_{-1}$ and establish results regarding $S(H)_{-1}$ processes and their Hermite transform.

### 2.3.1 Characterisation Theorems

We let $F, G \in S(H)_{-1}$ in this section have the form

$$F = \sum_{\alpha \in \mathcal{J}} c_\alpha H_\alpha , \quad G = \sum_{\alpha \in \mathcal{J}} d_\alpha H_\alpha .$$

**Definition 2.3** *Define the Hermite transform of $F \in S(H)_{-1}$ as*

$$\mathcal{H}F(z) = \widetilde{F}(z) := \sum_{\alpha \in \mathcal{J}} c_\alpha z^\alpha , \tag{2.9}$$

*for $z = \mathbb{C}^\mathbb{N}$ such that the above series exists in $H_\mathbb{C}$.*

**Propositon 2.6** *If $F \in S(H)_{-1,-q}$, where $q \in \mathbb{N} \setminus \{1\}$, then for all $z \in \overline{\mathbb{K}}_q$, $\mathcal{H}F(z)$ converges absolutely and*

$$\sum_{\alpha \in \mathcal{J}} \|c_\alpha\|_{H_\mathbb{C}} |z^\alpha| \le \|F\|_{-1,-q}\, A(q)^{1/2} . \tag{2.10}$$

**Proof:** Note that as $F \in S(H)_{-1,-q}$

$$\|F\|_{-1,-q} < \infty .$$

Now for all $z \in \overline{\mathbb{K}}_q$

$$\sum_{\alpha \in \mathcal{J}} \|c_\alpha z^\alpha\|_{H_\mathbb{C}}$$

$$
\begin{aligned}
&= \sum_{\alpha\in\mathcal{J}} \|c_\alpha\|_{H_\mathbb{C}} |z^\alpha| (2\mathbb{N})^{-\frac{q\alpha}{2}} (2\mathbb{N})^{\frac{q\alpha}{2}} \\
&\le \left(\sum_{\alpha\in\mathcal{J}} \|c_\alpha\|^2_{H_\mathbb{C}} (2\mathbb{N})^{-q\alpha}\right)^{1/2} \left(\sum_{\alpha\in\mathcal{J}} |z^\alpha|^2 (2\mathbb{N})^{q\alpha}\right)^{1/2} \\
&\le \|F\|_{-1,-q} \left(\sum_{\alpha\in\mathcal{J}} (2\mathbb{N})^{-2q\alpha} (2\mathbb{N})^{q\alpha}\right)^{1/2} = \|F\|_{-1,-q} \left(\sum_{\alpha\in\mathcal{J}} (2\mathbb{N})^{-q\alpha}\right)^{1/2} \\
&= \|F\|_{-1,-q} A(q)^{1/2} < \infty ,
\end{aligned}
$$

as required. ■

**Propositon 2.7** *Consider* $X(z) : \mathbb{K}_q \to H_\mathbb{C}$, $q \in \mathbb{N} \setminus \{1\}$, *with form*

$$
X(z) = \sum_{\alpha\in\mathcal{J}} c_\alpha z^\alpha , \quad c_\alpha \in H ,
$$

*bounded by some* $M < \infty$. *Then the formal sum*

$$
F := \sum_{\alpha\in\mathcal{J}} c_\alpha H_\alpha ,
$$

*belongs to* $S(H)_{-1,-4q}$ *and*

$$
\|F\|_{-1,-4q} \le MA(q) . \tag{2.11}
$$

*Also* $\mathcal{H}F(z) = X(z)$ *for all* $z \in \mathbb{K}_q$.

**Proof:** As $X(\cdot)$ is bounded by $M < \infty$ for $z \in \mathbb{K}_q$, it follows from Proposition 2.1 that

$$
\sum_{\alpha\in\mathcal{J}} \|c_\alpha\|_H (2\mathbb{N})^{-2q\alpha} \le MA(q) ,
$$

as $((2.1)^{-2q}, (2.2)^{-2q}, (2.3)^{-3q}, \ldots) \in \overline{\mathbb{K}}_{2q}$. So

$$
\begin{aligned}
\|F\|^2_{-1,-4q} &= \sum_{\alpha\in\mathcal{J}} \|c_\alpha\|^2_H (2\mathbb{N})^{-4q\alpha} \le \sum_{\alpha\in\mathcal{J}} MA(q) \|c_\alpha\|_H (2\mathbb{N})^{-2q\alpha} \\
&= MA(q) \sum_{\alpha\in\mathcal{J}} \|c_\alpha\|_H (2\mathbb{N})^{-2q\alpha} \le (MA(q))^2 < \infty ,
\end{aligned}
$$

as required.

We just take the Hermite transform of $F$ to show that $\mathcal{H}F(z) = X(z)$ for all $z \in \mathbb{K}_q$. ■

**Propositon 2.8** *Consider* $F, G \in S(H)_{-1}$. *If there exists a* $q \in \mathbb{N}$ *such that* $\mathcal{H}F(z) = \mathcal{H}G(z)$ *for all* $z \in \mathbb{K}_q$, *then* $F = G$.

**Proof:** We need to show that $c_\alpha = d_\alpha$. This is the case by Proposition 2.2. ■

This proposition allows us to use the notation $\mathcal{H}^{-1}$ in the following contexts:

1. If $F \in S(H)_{-1}$, then there exists a $q \in \mathbb{N} \setminus \{1\}$ such that for all $z \in \mathbb{K}_q$

$$F = \mathcal{H}^{-1}\left(\mathcal{H}F(z)\right) .$$

2. If $X(z) : \mathbb{K}_q \to H_{\mathbb{C}}$, $q \in \mathbb{N} \setminus \{1\}$ given by

$$X(z) = \sum_{\alpha \in \mathcal{J}} c_\alpha z^\alpha , \quad c_\alpha \in H ,$$

is bounded by some $M < \infty$, then

$$X(z) = \mathcal{H}\left(\mathcal{H}^{-1}X\right)(z) ,$$

for all $z \in \mathbb{K}_q$.

### 2.3.2 Convergence Theorems

**Propositon 2.9** *Consider a sequence $\{F_n\}_{n=1}^{\infty}$ and $F$ belonging to $S(H)_{-1}$. The following statements are equivalent:*

1. *$F_n \longrightarrow_{n\to\infty} F$ strongly in $S(H)_{-1}$.*
2. *There exists a $q \in \mathbb{N} \setminus \{1\}$ such that $\widetilde{F}_n(\cdot) \longrightarrow_{n\to\infty} \widetilde{F}(\cdot)$ pointwise boundedly in $\mathbb{K}_q$.*

**Proof:** $(1 \Rightarrow 2)$ We start by finding a $q \in \mathbb{N} \setminus \{1\}$ such that $\{\widetilde{F}_n(z)\}_{n=1}^{\infty}$ and $\widetilde{F}(z)$ exist and are bounded by some $M < \infty$, for $z \in \mathbb{K}_q$.

Now $F_n \longrightarrow_{n\to\infty} F$ strongly in $S(H)_{-1}$. So by Proposition 1.4 there exists a $q \in \mathbb{N} \setminus \{1\}$ such that $F_n \longrightarrow_{n\to\infty} F$ in $S(H)_{-1,-q}$. This implies that $\{||F_n||_{-1,-q}\}_{n=1}^{\infty}$ and $||F||_{-1,-q}$ are bounded by some $M < \infty$. By Proposition 2.6, for $z \in \overline{\mathbb{K}}_q$

$$||\widetilde{F}_n(z)||_{H_{\mathbb{C}}} \;=\; ||F_n||_{-1,-q}A(q)^{1/2} \leq MA(q)^{1/2} < \infty ,$$

and similarly for $\widetilde{F}(\cdot)$.

We now show that $\widetilde{F}_n(z) \longrightarrow_{n\to\infty} \widetilde{F}(z)$ for all $z \in \mathbb{K}_q$. By Proposition 2.6, for all $z \in \overline{\mathbb{K}}_q$

$$\left\|\widetilde{F}_n(z) - \widetilde{F}(z)\right\|_{H_{\mathbb{C}}} \;\leq\; ||F_n - F||_{-1,-q}\; A(q)^{1/2} \underset{n\to\infty}{\longrightarrow} 0 ,$$

as required.

$(2 \Rightarrow 1)$ To do this we show that $F_n \longrightarrow_{n\to\infty} F$ in $S(H)_{-1,-8q}$.

By Proposition 2.7, $\{F_n\}_{n=1}^{\infty}$ and $F$ belong to $S(H)_{-1,-4q}$ and hence $S(H)_{-1,-8q}$.

By Proposition 2.3, $\widetilde{F}_n(\cdot) \longrightarrow_{n\to\infty} \widetilde{F}(\cdot)$ uniformly on $\overline{\mathbb{K}}_{2q}$. So for any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$

$$||\widetilde{F}_n(z) - \widetilde{F}(z)||_{H_\mathbb{C}} < \frac{\epsilon}{A(2q)+1} ,$$

for $z \in \overline{\mathbb{K}}_{2q}$. With this $N$ and Proposition 2.7

$$||F_n - F||_{-1,-8q} = ||F_n - F||_{-1,-4.2q} \leq \frac{\epsilon}{A(2q)+1} A(2q) < \epsilon ,$$

for all $n \geq N$, as required. ■

**Theorem 2.1** *Consider a $S(H)_{-1}$ process $F(t) : [a,b] \to S(H)_{-1}$. The following statements are equivalent:*

1. *$F(\cdot)$ is a continuous $S(H)_{-1}$ process on $[a,b]$.*
2. *There exists a $q \in \mathbb{N} \setminus \{1\}$ such that:*
   (a) *$\widetilde{F}(t,z)$ exists for all $(t,z) \in [a,b] \times \mathbb{K}_q$.*
   (b) *$\widetilde{F}(\cdot,z)$ is continuous with respect to $t$ on $[a,b]$ for all $z \in \mathbb{K}_q$ and bounded by some $M < \infty$ for $(t,z) \in [a,b] \times \mathbb{K}_q$.*

**Proof:** $(1 \Rightarrow 2)$ $(a)$ By Proposition 1.6, there exists a $q \in \mathbb{N} \setminus \{1\}$ such that $F(t) \in S(H)_{-1,-q}$ for all $t \in [a,b]$. From Proposition 2.6, $\widetilde{F}(t,z)$ exists for $(t,z) \in [a,b] \times \overline{\mathbb{K}}_q$.

$(b)$ Firstly continuity. Take any $t \in [a,b]$. By Proposition 1.6, for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$s \in [a,b],\ |t-s| < \delta \Rightarrow ||F(t) - F(s)||_{-1,-q} < \frac{\epsilon}{A(q)^{1/2}+1} .$$

Hence it follows from Proposition 2.6 that

$$\left\|\widetilde{F}(t,z) - \widetilde{F}(s,z)\right\|_{H_\mathbb{C}} \leq ||F(t) - F(s)||_{-1,-q}\ A(q)^{1/2} < \epsilon ,$$

for $s \in [a,b]$ with $|t-s| < \delta$ and $z \in \mathbb{K}_q$, as required.

Secondly boundedness. As $F(\cdot)$ is continuous with respect to the norm $||\cdot||_{-1,-q}$, $F(\cdot)$ must also be bounded by some $M < \infty$ in that norm, as $[a,b]$ is a closed, bounded interval. Hence $\widetilde{F}(\cdot,\cdot)$ is bounded on $(t,z) \in [a,b] \times \mathbb{K}_q$ by the inequality

$$\left\|\widetilde{F}(t,z)\right\|_{H_\mathbb{C}} \leq ||F(t)||_{-1,-q}\ A(q)^{1/2} \leq MA(q)^{1/2} < \infty ,$$

by Proposition 2.6.

$(2 \Rightarrow 1)$ Take any $t \in [a,b]$ and any sequence $\{t_n\}_{n=1}^{\infty} \subset [a,b]$ such that $t_n \longrightarrow_{n\to\infty} t$. Then by $(b)$

$$\left\|\widetilde{F}(t,z) - \widetilde{F}(t_n,z)\right\|_{H_\mathbb{C}} \underset{n\to\infty}{\longrightarrow} 0 ,$$

pointwise boundedly on $\mathbb{K}_q$. By Proposition 2.9, $F(t_n) \longrightarrow_{n\to\infty} F(t)$ strongly in $S(H)_{-1}$. By Proposition 1.5, this gives continuity at $t$, as required. ■

**Theorem 2.2** *Consider two $S(H)_{-1}$ processes $X(t), F(t) : [a,b] \to S(H)_{-1}$. The following statements are equivalent:*

1. *$X(\cdot)$ is a differentiable $S(H)_{-1}$ process on $[a,b]$, and $F(\cdot)$ is a continuous $S(H)_{-1}$ process on $[a,b]$, such that*

$$\frac{dX(t)}{dt} = F(t) \ , \tag{2.12}$$

*for all $t \in [a,b]$.*

2. *There exists a $q \in \mathbb{N} \setminus \{1\}$ such that:*

   *(a) $\widetilde{X}(t,z)$ and $\widetilde{F}(t,z)$ exist for all $(t,z) \in [a,b] \times \mathbb{K}_q$.*

   *(b) $\widetilde{F}(\cdot,z)$ is continuous with respect to $t$ on $[a,b]$ for all $z \in \mathbb{K}_q$ and bounded by some $M < \infty$ for $(t,z) \in [a,b] \times \mathbb{K}_q$.*

   *(c) $\widetilde{X}(\cdot,z)$ is differentiable with respect to $t$ on $[a,b]$ for all $z \in \mathbb{K}_q$ and*

$$\frac{d\widetilde{X}(t,z)}{dt} = \widetilde{F}(t,z) \ , \tag{2.13}$$

   *for all $(t,z) \in [a,b] \times \mathbb{K}_q$.*

**Proof:** $(1 \Rightarrow 2)$. $(a)$ By Proposition 1.7, there exists a $q \in \mathbb{N} \setminus \{1\}$ such that $X(t)$ and $F(t)$ belong to $S(H)_{-1,-q}$, for all $t \in [a,b]$. From Proposition 2.6, $\widetilde{X}(t,z)$ and $\widetilde{F}(t,z)$ exist for $z \in \overline{\mathbb{K}}_q$.

$(b)$ Follows from Theorem 2.1.

$(c)$ Take any $t \in [a,b]$ and $\epsilon > 0$. By Proposition 1.7, there exists a $\delta > 0$ such that

$$s \in [a,b],\ 0 < |t-s| < \delta \Rightarrow \left\| \frac{X(t)-X(s)}{t-s} - F(t) \right\|_{-1,-q} < \frac{\epsilon}{A(q)^{1/2}+1} \ .$$

Hence it follows from Proposition 2.6 that

$$\left\| \frac{\widetilde{X}(t,z)-\widetilde{X}(s,z)}{t-s} - \widetilde{F}(t,z) \right\|_{H_{\mathbb{C}}} \leq \left\| \frac{X(t)-X(s)}{t-s} - F(t) \right\|_{-1,-q} A(q)^{1/2} < \epsilon \ ,$$

for $s \in [a,b]$ with $0 < |t-s| < \delta$ and $z \in \mathbb{K}_q$, as required.

$(2 \Rightarrow 1)$ Continuity of $F(\cdot)$ on $[a,b]$ follows from Theorem 2.1.

We now show that $X(\cdot)$ is differentiable with derivative $F(\cdot)$ on $[a,b]$. Take any $t \in [a,b]$ and any sequence $\{t_n\}_{n=1}^{\infty} \subset [a,b]$, $t_n \neq t$ such that $t_n \longrightarrow_{n\to\infty} t$. Now

$$\left\| \frac{\widetilde{X}(t_n,z)-\widetilde{X}(t,z)}{t_n-t} - \widetilde{F}(t,z) \right\|_{H_{\mathbb{C}}} \underset{n\to\infty}{\longrightarrow} 0 \ .$$

This convergence is pointwise boundedly for $z \in \mathbb{K}_q$ as

$$\left\| \frac{\widetilde{X}(t_n, z) - \widetilde{X}(t, z)}{t_n - t} \right\|_{H_\mathbb{C}} \leq \sup_{(t,z)\in[a,b]\times\mathbb{K}_q} \|\widetilde{F}(t, z)\|_{H_\mathbb{C}} \leq M ,$$

by Lagranges theorem. Hence, by Proposition 2.9

$$\frac{X(t_n) - X(t)}{t_n - t} - F(t) \underset{n\to\infty}{\longrightarrow} 0 ,$$

strongly in $S(H)_{-1}$, as required. ■

**Theorem 2.3** *Consider a function $X(t) : [a, b] \to S(H)_{-1}$, with a $q \in \mathbb{N}\setminus\{1\}$, such that $\widetilde{X}(t, z)$ exists and is bounded by some $M < \infty$, for $(t, z) \in [a, b] \times \mathbb{K}_q$. If*

$$\int_a^t \widetilde{X}(s, z)ds ,$$

*exists for all $(t, z) \in [a, b] \times \mathbb{K}_q$, then $X(\cdot)$ is an integrable $S(H)_{-1}$ process on $[a, b]$ and*

$$\mathcal{H}\left(\int_a^t X(s)ds\right)(z) = \int_a^t \widetilde{X}(s, z)ds , \tag{2.14}$$

*for all $(t, z) \in [a, b] \times \mathbb{K}_q$.*

**Proof:** Take any $t \in [a, b]$ and a set of partitions

$$\{a = t_0 < t_1 < \ldots t_n = t\}_{n=1}^{\infty} ,$$

of $[a, t]$, such that

$$\lim_{n\to\infty} \left( \max_{k\in\{0,\ldots,n-1\}} (t_{k+1} - t_k) \right) = 0 .$$

For $n \in \mathbb{N}$ define

$$S_n := \sum_{k=0}^{n-1} X(t_k^*)(t_{k+1} - t_k) ,$$

where $t_k^* \in [t_t, t_{k+1})$. Now

$$\begin{aligned} \left\|\widetilde{S}_n(z)\right\|_{H_\mathbb{C}} &= \left\|\sum_{k=0}^{n-1} \widetilde{X}(t_k^*, z)(t_{k+1} - t_k)\right\|_{H_\mathbb{C}} \leq \sum_{k=0}^{n-1} \left\|\widetilde{X}(t_k^*, z)(t_{k+1} - t_k)\right\|_{H_\mathbb{C}} \\ &\leq \sum_{k=0}^{n-1} M(t_{k+1} - t_k) = M(t - a) \leq M(b - a) < \infty . \end{aligned}$$

Therefore $\widetilde{S}_n(\cdot) \longrightarrow_{n\to\infty} \int_a^t \widetilde{X}(s, \cdot)ds$ pointwise boundedly, for $z \in \mathbb{K}_q$. So by Proposition 2.9

$$S_n \underset{n\to\infty}{\longrightarrow} \mathcal{H}^{-1}\left(\int_a^t \widetilde{X}(s, z)ds\right) ,$$

strongly in $S(H)_{-1}$, as required.

Equation (2.14) follows from

$$\int_a^t \widetilde{X}(s,z)ds = \mathcal{H}\left(\mathcal{H}^{-1}\int_a^t \widetilde{X}(s,z)ds\right) = \mathcal{H}\left(\int_a^t X(s)ds\right)(z) ,$$

as required. ■

**Theorem 2.4** *For a continuous process $F(t) : [a,b] \to S(H)_{-1}$, there exists a $q \in \mathbb{N}\setminus\{1\}$ such that $\widetilde{F}(t,z), \mathcal{H}\left(\int_a^t F(s)ds\right)(z)$ exist for $(t,z) \in [a,b]\times\mathbb{K}_q$ and*

$$\mathcal{H}\left(\int_a^t F(s)ds\right)(z) = \int_a^t \widetilde{F}(s,z)ds , \tag{2.15}$$

*for all $(t,z) \in [a,b]\times\mathbb{K}_q$.*

**Proof:** As $F(\cdot)$ is continuous on $[a,b]$, there exists a $q \in \mathbb{N}$ (and hence $\mathbb{N}\setminus\{1\}$) such that $X(t)$ and $F(t)$ belong to $S(H)_{-1,-q}$, for all $t \in [a,b]$. From Proposition 2.6, $\widetilde{F}(t,z), \mathcal{H}\left(\int_a^t F(s)ds\right)(z)$ exist for all $(t,z) \in [a,b]\times\mathbb{K}_q$.

Take $t \in [a,b]$ and a set of partitions

$$\{a = t_0 < t_1 < \ldots t_n = t\}_{n=1}^{\infty} ,$$

of $[a,t]$ such that

$$\lim_{n\to\infty}\left(\max_{k\in\{0,\ldots,n-1\}}(t_{k+1}-t_k)\right) = 0 .$$

For $n \in \mathbb{N}$ define

$$S_n := \sum_{k=0}^{n-1} F(t_k^*)(t_{k+1}-t_k) .$$

where $t_k^* \in [t_k, t_{k+1})$. So by Proposition 2.6

$$\left\|\widetilde{S}_n(z) - \mathcal{H}\left(\int_a^t F(s)ds\right)(z)\right\|_{H_{\mathbb{C}}} \le \left\|S_n - \int_a^t F(s)ds\right\|_{-1,-q} A(q)^{1/2} \underset{n\to\infty}{\longrightarrow} 0 ,$$

for all $z \in \mathbb{K}_q$, as required. ■

# Chapter 3

# Stochastic Convolution

## 3.1 Introduction

In chapters 4 and 5 we consider the stochastic evolution equation with additive noise

$$\begin{aligned} dX(t) &= AX(t)dt + BdW(t)\ , \quad t \in [0,T]\ , \\ X(0) &= \xi \in \mathcal{D}(A)\ , \end{aligned}$$

where $A$ generates a $C_0$-semigroup $\{S(t),\ t \geq 0\}$ on $H$ and $B$ is a continuous linear map from $U$ (a separable Hilbert space) to $H$. Here $W(\cdot)$ is a $U$-valued Wiener process, being the formal sum

$$W(t) = \sum_{i=1}^{\infty} \beta_i(t) f_i\ ,$$

where $\{\beta_i(\cdot)\}_{i=1}^{\infty}$ is a sequence of independent Brownian motions and $\{f_i\}_{i=1}^{\infty}$ is an orthonormal basis for $U$. In [2], they define particular notions of weak and strong solutions for this problem. In both cases the solutions involve the stochastic convolution

$$\int_0^t S(t-s)dW(s) := \sum_{i=1}^{\infty} \int_0^t S(t-s) f_i d\beta_i(s)\ .$$

However to guarantee $\int_0^t S(t-s)dW(s)$ converges in $L^2(H)$, the following operator must be trace class

$$S_T x := \int_0^T S(s)S^*(s)x\ ds\ .$$

We define generalised stochastic convolution in $S(H)_{-0}$ that does not require this condition, but agrees with the above stochastic convolution when $S_T$ is trace class. To do this we start by generating a sequence of independent Brownian motions.

## 3.2 Generating a Sequence of Independent Brownian Motions

Let $\epsilon_n = (0, \ldots, 0, 1, 0, \ldots)$, a sequence with a 1 in the $n$-th place and 0 elsewhere. Define the function $n(i,j)$ by the following table

| $i$ \ $j$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | $\cdots$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 3 | 6 | 10 | 15 | 21 | 28 | $\cdots$ |
| 2 | 2 | 5 | 9 | 14 | 20 | 27 | | |
| 3 | 4 | 8 | 13 | 19 | 26 | | | |
| 4 | 7 | 12 | 18 | 25 | | | | |
| 5 | 11 | 17 | 24 | | | | | $n(i,j)$ |
| 6 | 16 | 23 | | | | | | |
| 7 | 22 | | | | | | | |
| | $\cdots$ | | | | | | | |

(3.1)

Take $i \in \mathbb{N}$. Define $\beta_i(t) : [0,\infty) \to L^2(\mu)$ by

$$\beta_i(t) = \sum_{j=1}^{\infty} \left( \int_0^t \xi_j(s)ds \right) H_{\epsilon_{n(i,j)}} \ , \tag{3.2}$$

where $\{\xi_j(\cdot)\}_{j=1}^{\infty}$ are the Hermite functions found in section 1.2. This series converges in $L^2(\mu)$ for each $t \geq 0$ as

$$\begin{aligned} \sum_{j=1}^{\infty} (\epsilon_{n(i,j)}!) \left( \int_0^t \xi_j(s)ds \right)^2 &= \sum_{j=1}^{\infty} (1!) \left( \int_{\mathbb{R}} I_{[0,t]}(s)\xi_j(s)ds \right)^2 \\ &= \sum_{j=1}^{\infty} \left( \langle I_{[0,t]}(s), \xi_j(s) \rangle_{L^2(\mathbb{R})} \right)^2 \\ &= \| I_{[0,t]}(s) \|_{L^2(\mathbb{R})}^2 = t^2 < \infty \ . \end{aligned}$$

**Lemma 3.1** *For all $i \in \mathbb{N}$ and $t \in [0,\infty)$*

$$\beta_i(t) = \langle \omega, \sum_{j=1}^{\infty} \left( \int_0^t \xi_j(s)ds \right) \eta_{n(i,j)} \rangle \ , \tag{3.3}$$

*for almost all $\omega \in S'(\mathbb{R}^d)$, where $\{\eta_n\}_{n=1}^{\infty}$ are the functions defined in equation (1.4).*

**Proof:** Take $t \in [0,\infty)$. By the definition of $\{\beta_i(\cdot)\}_{i=1}^{\infty}$, for almost all $\omega \in S'(\mathbb{R}^d)$

$$\begin{aligned} \beta_i(t) &= \sum_{j=1}^{\infty} \left( \int_0^t \xi_j(s)ds \right) H_{\epsilon_{n(i,j)}} = \sum_{j=1}^{\infty} \left( \int_0^t \xi_j(s)ds \right) \langle \omega, \eta_{n(i,j)} \rangle \\ &= \langle \omega, \sum_{j=1}^{\infty} \left( \int_0^t \xi_j(s)ds \right) \eta_{n(i,j)} \rangle \ , \end{aligned}$$

as required. ■

With this lemma, we now show that these $\{\beta_i(\cdot)\}_{i=1}^{\infty}$ have the properties required to be Brownian motion.

Before stating the next proposition, we introduce the following notation. If $X$ is a random variable defined on a probability space $(\Omega, \mathcal{F}, \mu)$, with values in $\mathbb{R}$, then $P[X = a]$ is defined to be

$$P[X = a] = \mu\{\omega;\ X(\omega) = a\} .$$

**Propositon 3.1** *The random variables $\{\beta_i(\cdot)\}_{i=1}^{\infty}$ have the following properties for all $i \in \mathbb{N}$:*

1. *$P[\beta_i(0) = 0] = 1$.*
2. *$\beta_i(\cdot)$ has independent increments.*
3. *For $0 \leq s \leq t$, the random variable $\beta_i(t) - \beta_i(s)$ is $N(0, t-s)$ distributed.*

**Proof:** (1) For all $i \in \mathbb{N}$

$$P[\beta_i(0) = 0] = P\left[\sum_{j=1}^{\infty}\left(\int_0^0 \xi_j(s)ds\right)\langle\omega, \eta_{n(i,j)}\rangle = 0\right] = P[0 = 0] = 1 .$$

(2) Take any set of increments $0 = t_0 < t_1 < \ldots < t_n$ and $i \in \mathbb{N}$. We show that the random variables from the collection $\{\beta_i(t_{k+1}) - \beta_i(t_k)\}_{k=0}^{n-1}$ are mutually independent. Now, the collection of functions

$$\left\{\left(\frac{1}{t_{k+1} - t_k}\right)^{1/2}\sum_{j=1}^{\infty}\left(\int_{t_k}^{t_{k+1}} \xi_j(u)du\right)\eta_{n(i,j)}\right\}_{k=0}^{n-1},$$

are orthonormal in $L^2(\mathbb{R}^d)$ as

$$\begin{aligned}
&\langle\sum_{j=1}^{\infty}\left(\int_{t_l}^{t_{l+1}} \xi_j(u)du\right)\eta_{n(i,j)}\ ,\ \sum_{j=1}^{\infty}\left(\int_{t_k}^{t_{k+1}} \xi_j(u)du\right)\eta_{n(i,j)}\rangle_{L^2(\mathbb{R}^d)} \\
&= \sum_{j=1}^{\infty}\left(\int_{t_l}^{t_{l+1}} \xi_j(u)du\right)\left(\int_{t_k}^{t_{k+1}} \xi_j(u)du\right) \\
&= \langle\sum_{j=1}^{\infty}\left(\int_{t_l}^{t_{l+1}} \xi_j(u)du\right)\xi_j\ ,\ \sum_{j=1}^{\infty}\left(\int_{t_k}^{t_{k+1}} \xi_j(u)du\right)\xi_j\rangle_{L^2(\mathbb{R})} \\
&= \langle\sum_{j=1}^{\infty}\left(\int_{\mathbb{R}} I_{[t_l,t_{l+1}]}(u)\xi_j(u)du\right)\xi_j\ ,\ \sum_{j=1}^{\infty}\left(\int_{\mathbb{R}} I_{[t_k,t_{k+1}]}(u)\xi_j(u)du\right)\xi_j\rangle_{L^2(\mathbb{R})} \\
&= \langle I_{[t_l,t_{l+1}]}(u), I_{[t_k,t_{k+1}]}(u)\rangle_{L^2(\mathbb{R})} = \int_{-\infty}^{\infty} I_{[t_l,t_{l+1}]}(u)\ I_{[t_k,t_{k+1}]}(u)du \\
&= \delta_{l,k}(t_{k+1} - t_k) .
\end{aligned}$$

So by Lemma 2.1.2 in [10], the random variable

$$\omega \rightarrow \left(\langle\omega, \left(\frac{1}{t_1 - t_0}\right)^{1/2} \sum_{j=1}^{\infty} \left(\int_{t_0}^{t_1} \xi_j(u)du\right) \eta_{n(i,j)}\rangle \, , \ldots ,\right.$$

$$\left.\langle\omega, \left(\frac{1}{t_n - t_{n-1}}\right)^{1/2} \sum_{j=1}^{\infty} \left(\int_{t_{n-1}}^{t_n} \xi_j(u)du\right) \eta_{n(i,j)}\right) \, ,$$

has normalised Gaussian measure

$$d\lambda_n(x) = (2\pi)^{-n/2} e^{-1/2|x|^2} dx_1 \ldots dx_n \, , \qquad x \in \mathbb{R}^n \, .$$

Therefore

$$\left\{\langle\omega, \left(\frac{1}{t_{k+1} - t_k}\right)^{1/2} \sum_{j=1}^{\infty} \left(\int_{t_k}^{t_{k+1}} \xi_j(u)du\right) \eta_{n(i,j)}\rangle\right\}_{k=0}^{n-1} \, ,$$

are independently distributed $N(0,1)$ random variables. Now

$$\begin{aligned} & \langle\omega, \sum_{j=1}^{\infty} \left(\int_{t_k}^{t_{k+1}} \xi_j(u)du\right) \eta_{n(i,j)}\rangle \\ = & \langle\omega, \sum_{j=1}^{\infty} \left(\int_{0}^{t_{k+1}} \xi_j(u)du\right) \eta_{n(i,j)} - \sum_{j=1}^{\infty} \left(\int_{0}^{t_k} \xi_j(u)du\right) \eta_{n(i,j)}\rangle \\ = & \beta_i(t_{k+1}) - \beta_i(t_k) \, . \end{aligned}$$

Hence the random variables from the collection $\{\beta_i(t_{k+1}) - \beta_i(t_k)\}_{k=0}^{n-1}$ are mutually independent, $N(0, t_{k+1} - t_k)$ distributed.

(3) From above we see that the random variable $\beta_i(t) - \beta_i(s)$ has a $N(0, t-s)$ distribution, for all $i \in \mathbb{N}$ ■

For all $i \in \mathbb{N}$ we choose a continuous version of $\beta_i(\cdot)$ which we know to exist by Kolorgorov's extension theorem (see for instance [14]) because

$$\mathrm{E}\left[\left(\frac{1}{t-s}\right)^2 (\beta_i(t) - \beta_i(s))^4\right] = 1 \, ,$$

as

$$\left(\frac{1}{t-s}\right)^{1/2} (\beta_i(t) - \beta_i(s)) \sim N(0,1) \, .$$

**Propositon 3.2** *The sequence of* $\mathbb{R}$*-valued processes* $\{\beta_i(\cdot)\}_{i=1}^{\infty}$ *is a sequence of independent Brownian motions.*

**Proof:** For all $t \geq 0$ and $i \in \mathbb{N}$, we have that $\beta_i(t)$ is $N(0,t)$ distribution. Hence it is sufficient to show that

$$\mathrm{E}[\beta_i(s)\beta_k(t)] = 0 \, ,$$

for all $s, t \geq 0$ and $i \neq k$. Now

$$\begin{aligned} & \mathrm{E}[\beta_i(s)\beta_k(t)] \\ = & \langle \sum_{j=1}^{\infty} \left( \int_0^s \xi_j(u)du \right) H_{\epsilon_{n(i,j)}} , \sum_{j=1}^{\infty} \left( \int_0^t \xi_j(u)du \right) H_{\epsilon_{n(k,j)}} \rangle_{L^2(\mu)} = 0 , \end{aligned}$$

as $n(i,j)$ is a one-to-one function. ■

## 3.3 $H$-valued Wiener Processes

To define generalised stochastic convolution, we need the $S(H)_{-0}$ processes defined in this section.

If $\{\beta_i(\cdot)\}_{i=1}^{\infty}$ is a sequence of independent Brownian motions and $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis for $H$, the formal sum

$$W(t) = \sum_{i=1}^{\infty} \beta_i(t) e_i , \quad t \geq 0 , \tag{3.4}$$

is called an *$H$-valued Wiener process*. We use the Brownian motions defined in the previous section. In this case

$$W(t) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left( \int_0^t \xi_j(s)ds \right) H_{\epsilon_{n(i,j)}} e_i . \tag{3.5}$$

This is equivalent to the formal sum

$$W(t) \;=\; \sum_{k=1}^{\infty} \delta_{n(i,j),k} \left( \int_0^t \xi_j(s)ds \; e_i \right) H_{\epsilon_k} = \sum_{k=1}^{\infty} \theta_k(t) H_{\epsilon_k} , \tag{3.6}$$

where

$$\theta_{i,k}(t) = \theta_k(t) = \int_0^t \xi_j(s)ds \; e_i , \quad k = n(i,j) . \tag{3.7}$$

**Lemma 3.2** *For all $t \geq 0$, $W(t)$ belongs to $S(H)_{-0,-2}$ and hence $S(H)_{-0}$.*

**Proof:** Now

$$\begin{aligned} & \|W(t)\|_{-0,-2}^2 \\ = & \sum_{k=1}^{\infty} (\epsilon_k!) \|\theta_k(t)\|_H^2 (2\mathbb{N})^{-2\epsilon_k} = \sum_{k=1}^{\infty} \delta_{n(i,j),k} (1!) \left\| \left( \int_0^t \xi_j(s)ds \right) e_i \right\|_H^2 (2k)^{-2} \\ = & \sum_{k=1}^{\infty} \delta_{n(i,j),k} \left( \int_{\mathbb{R}} I_{[0,t]}(s)\xi_j(s)ds \right)^2 (2k)^{-2} \\ = & \sum_{k=1}^{\infty} \delta_{n(i,j),k} \langle I_{[0,t]}(s), \xi_j(s) \rangle_{L^2(\mathbb{R})}^2 (2k)^{-2} \\ \leq & \sum_{k=1}^{\infty} \delta_{n(i,j),k} t^2 (2k)^{-2} = t^2 \sum_{k=1}^{\infty} (2k)^{-2} < \infty , \end{aligned}$$

as required. ∎

### 3.3.1 $H$-valued Singular White Noise Process

Define $\mathbb{W}(\cdot)$ to be the formal sum

$$\mathbb{W}(t) := \sum_{k=1}^{\infty} \kappa_k(t) H_{\epsilon_k} \ , \quad t \in \mathbb{R} \ , \tag{3.8}$$

where

$$\kappa_{i,k}(t) = \kappa_k(t) = \xi_j(t) e_i \ , \quad k = n(i,j) \ . \tag{3.9}$$

We call $\mathbb{W}(\cdot)$ an *$H$-valued singular White noise process.*

**Lemma 3.3** *For all $t \in \mathbb{R}$, $\mathbb{W}(t)$ belongs to $S(H)_{-0,-2}$ and hence $S(H)_{-0}$.*

**Proof:** Now

$$\begin{aligned}
& ||\mathbb{W}(t)||^2_{-0,-2} \\
= & \sum_{k=1}^{\infty} (\epsilon_k!) ||\kappa_k(t)||^2_H (2\mathbb{N})^{-2\epsilon_k} = \sum_{k=1}^{\infty} \delta_{n(i,j),k}(1!) \, ||\xi_j(t) e_i||^2_H \, (2k)^{-2} \\
\leq & \left( \sup_{j \in \mathbb{N}, t \in \mathbb{R}} |\xi_j(t)| \right) \left( \sum_{k=1}^{\infty} \delta_{n(i,j),k} (2k)^{-2} \right) \leq C \sum_{k=1}^{\infty} (2k)^{-2} < \infty \ ,
\end{aligned}$$

where $C = \sup_{j \in \mathbb{N}, t \in \mathbb{R}} |\xi_j(t)|$, as required. ∎

We now show that $W(\cdot)$ is a differentiable $S(H)_{-1}$ process with continuous derivative $\mathbb{W}(\cdot)$.

**Propositon 3.3** *For any closed bounded interval $[a,b]$, $a \geq 0$, $\mathbb{W}(\cdot)$ is the continuous derivative of $W(\cdot)$ on $[a,b]$ in $S(H)_{-1}$.*

**Proof:** We need to show that there exists a $q \in \mathbb{N} \setminus \{1\}$, so that conditions $(a)$, $(b)$ and $(c)$ of Theorem 2.2 are fulfilled.

$(a)$ For all $t \in [a,b]$, $W(t)$ and $\mathbb{W}(t)$ belong to $S(H)_{-0,-2}$, and hence $S(H)_{-1,-4}$. Hence $\widetilde{W}(t,z)$ and $\widetilde{\mathbb{W}}(t,z)$ exist for all $(t,z) \in [a,b] \times \mathbb{K}_4$.

$(b)$ We need to show the continuity with respect to $t$ on $[a,b]$ and the boundedness of $\widetilde{\mathbb{W}}(\cdot,\cdot)$. Now the functions $\{\kappa_k(\cdot) = \delta_{n(i,j),k} \xi_j(\cdot) e_i\}_{k=1}^{\infty}$ are continuous on $[a,b]$ and from the proof of the previous lemma

$$\begin{aligned}
\left\| \widetilde{\mathbb{W}}(t,z) \right\|_{H_{\mathbb{C}}} & \leq \ ||\mathbb{W}(t)||_{-1,-2} \ A(2)^{1/2} \leq ||\mathbb{W}(t)||_{-0,-2} \ A(2)^{1/2} \\
& \leq \left( C \sum_{k=1}^{\infty} (2k)^{-2} \right)^{1/2} A(2)^{1/2} \leq (CA(2))^{1/2} \ A(2)^{1/2} = CA(2) \ ,
\end{aligned}$$

where $C = \sup_{j\in\mathbb{N},t\in[a,b]} |\xi_j(t)|$. Hence by Proposition 2.4, $\widetilde{\mathbb{W}}(\cdot, z)$ is continuous with respect to $t$ on $[a, b]$ for all $z \in \mathbb{K}_4$.

(c) Now $\widetilde{\mathbb{W}}(\cdot, \cdot)$ is bounded on $[a, b] \times \mathbb{K}_4$ and

$$\frac{d\theta_k(t)}{dt} = \delta_{n(i,j),k} \frac{d\left(\int_0^t \xi_j(s)ds\right)}{dt} e_i = \delta_{n(i,j),k}\xi_j(t)e_i = \kappa_k(t) .$$

By Proposition 2.5, $\widetilde{W}(\cdot, z)$ is differentiable with respect to $t$ on $[a, b]$ for all $z \in \mathbb{K}_4$ and

$$\frac{d\widetilde{W}(t,z)}{dt} = \widetilde{\mathbb{W}}(t, z)$$

for all $(t, z) \in [a, b] \times \mathbb{K}_4$. ■

**Corollary 3.1** *For any closed bounded interval $[a, b]$, $a \geq 0$, $\mathbb{W}(\cdot)$ is the continuous derivative of $W(\cdot)$ on $[a, b]$ in $S(H)_{-0}$.*

**Proof:** For $q \in \mathbb{N}$

$$\begin{aligned} ||W(t)||^2_{-0,-q} &= \sum_{k=1}^{\infty} (\epsilon_k!) ||\theta_k(t)||^2_H (2\mathbb{N})^{-q\epsilon_k} \\ &= \sum_{k=1}^{\infty} ||\theta_k(t)||^2_H (2\mathbb{N})^{-q\epsilon_k} = ||W(t)||^2_{-1,-q} , \end{aligned}$$

and similarly $||\mathbb{W}(t)||_{-0,-q} = ||\mathbb{W}(t)||_{-1,-q}$. Result then follows from Proposition 3.3 and Proposition 1.7. ■

### 3.3.2 $n^{th}$ Derivative of $\mathbb{W}(t)$

For $n \in \mathbb{N}_0$, consider the formal sum

$$\mathbb{W}^{(n)}(t) := \sum_{k=1}^{\infty} \kappa_k^{(n)}(t) H_{\epsilon_k} , \quad t \in \mathbb{R} . \tag{3.10}$$

**Lemma 3.4** *For all $n \in \mathbb{N}_0$ and $t \in \mathbb{R}$, $\mathbb{W}^{(n)}(t)$ belongs to $S(H)_{-0,-2[\![\frac{n+1}{2}]\!]-2}$. Moreover, for any interval $[a, b]$*

$$\left|\left|\mathbb{W}^{(n)}(t)\right|\right|_{-0,-2[\![\frac{n+1}{2}]\!]-2} \leq K_{a,b,n} < \infty , \tag{3.11}$$

*where the constant $K_{a,b,n}$ depends only on $a, b$ and $n$.*

**Proof**: Consider an interval $[a, b]$. Then for all $t \in [a, b]$

$$\left|\left|\mathbb{W}^{(n)}(t)\right|\right|^2_{-0,-2[\![\frac{n+1}{2}]\!]-2}$$

$$
\begin{aligned}
&= \sum_{k=1}^{\infty}(\epsilon_k!)\left\|\kappa_k^{(n)}(t)\right\|_H^2 (2\mathbb{N})^{-(2[\![\frac{n+1}{2}]\!]+2)\epsilon_k} \\
&= \sum_{k=1}^{\infty}\delta_{n(i,j),k}\left\|\xi_j^{(n)}(t)e_i\right\|_H^2 (2k)^{-2[\![\frac{n+1}{2}]\!]-2} \\
&\leq \sum_{k=1}^{\infty}\delta_{n(i,j),k}\left(\sup_{t\in[a,b]}|\xi_j^{(n)}(t)|^2\right)(2k)^{-2[\![\frac{n+1}{2}]\!]-2} \\
&= \sum_{k=1}^{\infty}\delta_{n(i,j),k}\left(C_{a,b,n}(2j)^{[\![\frac{n+1}{2}]\!]}\right)^2(2k)^{-2[\![\frac{n+1}{2}]\!]-2} \\
&\leq \sum_{k=1}^{\infty}\delta_{n(i,j),k}\left(C_{a,b,n}(2k)^{[\![\frac{n+1}{2}]\!]}\right)^2(2k)^{-2[\![\frac{n+1}{2}]\!]-2} = C_{a,b,n}^2\sum_{k=1}^{\infty}(2k)^{-2} \\
&\leq C_{a,b,n}^2 A(2) < \infty ,
\end{aligned}
$$

using Proposition B.2, as required. ■

**Propositon 3.4** *For all $n \in \mathbb{N}_0$ and any closed bounded interval $[a,b]$, $\mathbb{W}^{(n)}(\cdot)$ is a differentiable $S(H)_{-1}$ process on $[a,b]$, with continuous derivative $\mathbb{W}^{(n+1)}(\cdot)$.*

**Proof:** We need to show that there exists a $q \in \mathbb{N}\setminus\{1\}$, so that conditions $(a)$, $(b)$ and $(c)$ of Theorem 2.2 are fulfilled.

$(a)$ For $t \in \mathbb{R}$, $\mathbb{W}^{(n)}(t)$ and $\mathbb{W}^{(n+1)}(t)$ belong to $S(H)_{-0,-q}$, and hence $S(H)_{-0,-q} \subset S(H)_{-1,-q}$, where $q = 2[\![\frac{n+2}{2}]\!] + 2$. Hence $\mathcal{H}\mathbb{W}^{(n)}(t,z)$ and $\mathcal{H}\mathbb{W}^{(n+1)}(t,z)$ exist for all $(t,z) \in [a,b] \times \mathbb{K}_q$.

$(b)$ We need to show continuity with respect to $t$ on $[a,b]$ and boundedness of $\mathcal{H}\mathbb{W}^{(n+1)}(\cdot,\cdot)$. Now the functions

$$
\left\{\kappa_k^{(n+1)}(\cdot) = \delta_{n(i,j),k}\xi_j^{(n+1)}(\cdot)e_i\right\}_{k=1}^{\infty} ,
$$

are continuous on $[a,b]$ and from Lemma 3.4

$$
\begin{aligned}
\left\|\mathcal{H}\mathbb{W}^{(n+1)}(t,z)\right\|_{H_{\mathbb{C}}} &\leq \|\mathbb{W}^{(n+1)}(t)\|_{-1,-q}\, A(q)^{1/2} \leq \|\mathbb{W}^{(n+1)}(t)\|_{-0,-q}\, A(q)^{1/2} \\
&\leq K_{a,b,n+1}A(q)^{1/2} < \infty ,
\end{aligned}
$$

for all $(t,z) \in [a,b]\times\mathbb{K}_q$. Hence by Proposition 2.4, $\mathcal{H}\mathbb{W}^{(n+1)}(\cdot,z)$ is continuous with respect to $t$ on $[a,b]$ for all $z \in \mathbb{K}_{2q}$.

$(c)$ Now $\mathcal{H}\mathbb{W}^{(n+1)}(\cdot,\cdot)$ is bounded on $[a,b]\times\mathbb{K}_q$ and

$$
\frac{d\kappa_k^{(n)}(t)}{dt} = \kappa_k^{(n+1)}(t) .
$$

It follows from Proposition 2.5 that $\mathcal{H}\mathbb{W}^{(n)}(\cdot,z)$ is differentiable with respect to $t$ on $[a,b]$ for all, $z \in \mathbb{K}_{2q}$ and

$$
\frac{\mathcal{H}\mathbb{W}^{(n)}(t,z)}{dt} = \mathcal{H}\mathbb{W}^{(n+1)}(t,z) ,
$$

for all $(t,z) \in [a,b] \times \mathbb{K}_{2q}$, as required. ■

**Corollary 3.2** *For any closed bounded interval $[a,b]$, $\mathbb{W}^{(n)}(\cdot)$ is a differentiable $S(H)_{-0}$ process on $[a,b]$, with continuous derivative $\mathbb{W}^{(n+1)}(\cdot)$.*

**Proof:** For $q \in \mathbb{N}$

$$\begin{aligned} \|\mathbb{W}^{(n)}(t)\|_{-0,-q}^2 &= \sum_{k=1}^{\infty} (\epsilon_k!) \|\kappa_k^{(n)}(t)\|_H^2 (2\mathbb{N})^{-q\epsilon_k} \\ &= \sum_{k=1}^{\infty} \|\kappa_k^{(n)}(t)\|_H^2 (2\mathbb{N})^{-q\epsilon_k} = \|\mathbb{W}^{(n)}(t)\|_{-1,-q}^2 , \end{aligned}$$

and similarly $\|\mathbb{W}^{(n+1)}(t)\|_{-0,-q} = \|\mathbb{W}^{(n+1)}(t)\|_{-1,-q}$. Result then follows from Proposition 3.4 Proposition 1.7. ■

### 3.3.3 $Q$-Wiener Process

While we won't use $Q$-Wiener processes ourselves, it's of interest to see their expansion in $L^2(H)$ using the Brownian motions constructed earlier.

Let $Q$ be a positive, trace class operator on $H$ with positive eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ and eigenvectors $\{e_i\}_{i=1}^{\infty}$. If $\{\beta_i(\cdot)\}_{i=1}^{\infty}$ is a sequence of independent Brownian motions, the following sum

$$W_Q(t) := \sum_{i=1}^{\infty} \sqrt{\lambda_i}\beta_i(t)e_i , \quad t \geq 0 , \tag{3.12}$$

is called a *$Q$-Wiener process.* If we use the Brownian motions constructed earlier in this chapter, $W_Q(\cdot)$ is equivalent to the formal sum

$$\begin{aligned} W_Q(t) &= \sum_{k=1}^{\infty} \delta_{n(i,j),k} \left( \sqrt{\lambda_i} \int_0^t \xi_j(s)ds \; e_i \right) H_{\epsilon_k}(\omega) \\ &= \sum_{k=1}^{\infty} \vartheta_k(t) H_{\epsilon_k}(\omega) , \end{aligned} \tag{3.13}$$

where

$$\vartheta_k(t) = \sqrt{\lambda_i} \int_0^t \xi_j(s)ds \; e_i , \quad k = n(i,j) . \tag{3.14}$$

For all $t \geq 0$, $W_Q(t)$ belongs to $L^2(H)$ as

$$\begin{aligned} \|W_Q(t)\|_{L^2(H)}^2 &= \sum_{k=1}^{\infty} (\epsilon_k!) \|\vartheta_k\|_H^2 = \sum_{k=1}^{\infty} \delta_{n(i,j),k} \left( \sqrt{\lambda_i} \int_0^t \xi_j(s)ds \right)^2 \\ &\leq \sum_{i=1}^{\infty} \lambda_i \sum_{j=1}^{\infty} \left( \int_{\mathbb{R}} I_{[0,t]}(s)\xi_j(s)ds \right)^2 = \sum_{i=1}^{\infty} \lambda_i \|I_{[0,t]}\|_{L^2(\mathbb{R})}^2 < \infty . \end{aligned}$$

## 3.4 Wick Product

**Definition 3.1** *1. Take $F, G \in S(H)_{-1}$ given by*

$$F = \sum_{\alpha \in \mathcal{J}} \sum_{i=1}^{\infty} b_{i,\alpha} H_\alpha e_i \ , \quad b_{i,\alpha} \in \mathbb{R} \ ,$$

$$G = \sum_{\alpha \in \mathcal{J}} \sum_{i=1}^{\infty} c_{i,\alpha} H_\alpha e_i \ , \quad c_{i,\alpha} \in \mathbb{R} \ .$$

*Define the Wick product of $F$ and $G$ as*

$$\begin{aligned} F \diamond G &:= \sum_{\gamma \in \mathcal{J}} \left( \sum_{i=1}^{\infty} \sum_{\alpha+\beta=\gamma} b_{i,\alpha} c_{i,\beta} \ e_i \right) H_\gamma \\ &= \sum_{\gamma \in \mathcal{J}} g_\gamma H_\gamma(\omega) \ , \end{aligned} \tag{3.15}$$

*where*

$$g_\gamma = \sum_{i=1}^{\infty} \sum_{\alpha+\beta=\gamma} b_{i,\alpha} c_{i,\beta} \ e_i \ .$$

*2. Take $F \in (S)_{-1}$ and $G \in S(H)_{-1}$ given by*

$$F = \sum_{\alpha \in \mathcal{J}} b_\alpha H_\alpha \ , \quad b_\alpha \in \mathbb{R}$$

$$G = \sum_{\alpha \in \mathcal{J}} c_\alpha H_\alpha \ , \quad c_\alpha \in H \ .$$

*Define the Wick product of $F$ and $G$ as*

$$F \diamond G := \sum_{\gamma \in \mathcal{J}} \left( \sum_{\alpha+\beta=\gamma} b_\alpha c_\beta \right) H_\gamma = \sum_{\gamma \in \mathcal{J}} g_\gamma H_\gamma \ , \tag{3.16}$$

*where*

$$g_\gamma = \sum_{\alpha+\beta=\gamma} b_\alpha c_\beta \ .$$

**Propositon 3.5** *The following properties hold:*

1. *If $F, G \in S(H)_{-1}$, then $F \diamond G \in S(H)_{-1}$.*
2. *If $F \in (S)_{-1}$ and $G \in S(H)_{-1}$, then $F \diamond G \in S(H)_{-1}$.*
3. *If $F \in S(H)_{-0}$, then $F \diamond \mathbb{W}(t) \in S(H)_{-0}$.*

**Proof:** (1) Let

$$F = \sum_{\alpha \in \mathcal{J}} \sum_{i=1}^{\infty} b_{i,\alpha} H_\alpha e_i \ , \quad G = \sum_{\alpha \in \mathcal{J}} \sum_{i=1}^{\infty} c_{i,\alpha} H_\alpha e_i \ .$$

There exists $q_1, q_2 \in \mathbb{N}$ so that $F \in S(H)_{-1,-q_1}$ and $G \in S(H)_{-1,-q_2}$. Now for $k > 1$

$$
\begin{aligned}
& \|F \diamond G\|^2_{-1,-(q_1+q_2+k)} \\
= & \sum_{\gamma\in\mathcal{J}} (2\mathbb{N})^{-(q_1+q_2+k)\gamma} \left( \sum_{i=1}^{\infty} \left( \sum_{\alpha+\beta=\gamma} b_{i,\alpha} c_{i,\beta} \right)^2 \right) \\
\leq & \sum_{\gamma\in\mathcal{J}} (2\mathbb{N})^{-(q_1+q_2+k)\gamma} \left( \sum_{i=1}^{\infty} \left( \sum_{\alpha+\beta=\gamma} b_{i,\alpha}^2 \right) \left( \sum_{\alpha+\beta=\gamma} c_{i,\beta}^2 \right) \right) \\
\leq & \sum_{\gamma\in\mathcal{J}} (2\mathbb{N})^{-(q_1+q_2+k)\gamma} \left( \sum_{i=1}^{\infty} \sum_{\alpha+\beta=\gamma} b_{i,\alpha}^2 \right) \left( \sum_{i=1}^{\infty} \sum_{\alpha+\beta=\gamma} c_{i,\beta}^2 \right) \\
= & \sum_{\gamma\in\mathcal{J}} (2\mathbb{N})^{-k\gamma} \left( \sum_{\alpha+\beta=\gamma} (2\mathbb{N})^{-q_1\gamma} \left( \sum_{i=1}^{\infty} b_{i,\alpha}^2 \right) \right) \left( \sum_{\alpha+\beta=\gamma} (2\mathbb{N})^{-q_2\gamma} \left( \sum_{i=1}^{\infty} c_{i,\beta}^2 \right) \right) \\
\leq & \sum_{\gamma\in\mathcal{J}} (2\mathbb{N})^{-k\gamma} \left( \sum_{\alpha+\beta=\gamma} (2\mathbb{N})^{-q_1\alpha} \left( \sum_{i=1}^{\infty} b_{i,\alpha}^2 \right) \right) \left( \sum_{\alpha+\beta=\gamma} (2\mathbb{N})^{-q_2\beta} \left( \sum_{i=1}^{\infty} c_{i,\beta}^2 \right) \right) \\
\leq & \left( \sum_{\gamma\in\mathcal{J}} (2\mathbb{N})^{-k\gamma} \right) \left( \sum_{\alpha\in\mathcal{J}} (2\mathbb{N})^{-q_1\alpha} \left( \sum_{i=1}^{\infty} b_{i,\alpha}^2 \right) \right) \left( \sum_{\beta\in\mathcal{J}} (2\mathbb{N})^{-q_1\beta} \left( \sum_{i=1}^{\infty} c_{i,\beta}^2 \right) \right) \\
< & \infty ,
\end{aligned}
$$

as required.

(2) Let

$$
F = \sum_{\alpha\in\mathcal{J}} b_\alpha H_\alpha , \quad G = \sum_{\alpha\in\mathcal{J}} c_\alpha H_\alpha .
$$

There exists $q_1, q_2 \in \mathbb{N}$ so that $F \in (S)_{-1,-q_1}$ and $G \in S(H)_{-1,-q_2}$. Now for $k > 1$

$$
\begin{aligned}
& \|F \diamond G\|^2_{-1,-(q_1+q_2+k)} \\
= & \sum_{\gamma\in\mathcal{J}} (2\mathbb{N})^{-(q_1+q_2+k)\gamma} \left\| \sum_{\alpha+\beta=\gamma} b_\alpha c_\beta \right\|_H^2 \\
\leq & \sum_{\gamma\in\mathcal{J}} (2\mathbb{N})^{-(q_1+q_2+k)\gamma} \left( \sum_{\alpha+\beta=\gamma} \|b_\alpha c_\beta\|_H \right)^2 \\
\leq & \sum_{\gamma\in\mathcal{J}} (2\mathbb{N})^{-k\gamma} \left( \sum_{\alpha+\beta=\gamma} (2\mathbb{N})^{-q_1\gamma} b_\alpha^2 \right) \left( \sum_{\alpha+\beta=\gamma} (2\mathbb{N})^{-q_2\gamma} \|c_\beta\|_H^2 \right) \\
\leq & \sum_{\gamma\in\mathcal{J}} (2\mathbb{N})^{-k\gamma} \left( \sum_{\alpha+\beta=\gamma} (2\mathbb{N})^{-q_1\alpha} b_\alpha^2 \right) \left( \sum_{\alpha+\beta=\gamma} (2\mathbb{N})^{-q_2\beta} \|c_\beta\|_H^2 \right) \\
\leq & \left( \sum_{\gamma\in\mathcal{J}} (2\mathbb{N})^{-k\gamma} \right) \left( \sum_{\alpha\in\mathcal{J}} (2\mathbb{N})^{-q_1\alpha} b_\alpha^2 \right) \left( \sum_{\beta\in\mathcal{J}} (2\mathbb{N})^{-q_2\beta} \|c_\beta\|_H^2 \right) \\
< & \infty ,
\end{aligned}
$$

as required.

(3) Let

$$F = \sum_{\alpha \in \mathcal{J}} \sum_{i=1}^{\infty} c_{i,\alpha} H_\alpha e_i .$$

There exists a $q \in \mathbb{N}$ so that $F \in S(H)_{-0,-q}$. For $k > 1$

$$\begin{aligned}
&\|F \diamond \mathbb{W}(t)\|^2_{-0,-(q+2+k+2)} \\
= &\sum_{\gamma \in \mathcal{J}} \gamma!(2\mathbb{N})^{-(q+2+k+2)\gamma} \left( \sum_{i=1}^{\infty} \left( \sum_{\alpha+\epsilon_k=\gamma} c_{i,\alpha}\kappa_{i,k} \right)^2 \right) \\
= &\sum_{\gamma \in \mathcal{J}} (2\mathbb{N})^{-(q+2+k)\gamma} \left( \sum_{i=1}^{\infty} \left( \sum_{\alpha+\epsilon_k=\gamma} \left(\gamma!(2\mathbb{N})^{-2\gamma)}\right)^{1/2} c_{i,\alpha}\kappa_{i,k} \right)^2 \right) \\
\le &\sum_{\gamma \in \mathcal{J}} (2\mathbb{N})^{-(q+2+k)\gamma} \left( \sum_{i=1}^{\infty} \left( \sum_{\alpha+\epsilon_k=\gamma} \left(\alpha!(\alpha_k+1)(2\mathbb{N})^{-2(\alpha+\epsilon_k)}\right) c^2_{i,\alpha} \right) \left( \sum_{\alpha+\epsilon_k=\gamma} \kappa^2_{i,k} \right) \right) \\
\le &\sum_{\gamma \in \mathcal{J}} (2\mathbb{N})^{-(q+2+k)\gamma} \left( \sum_{i=1}^{\infty} \left( \sum_{\alpha+\epsilon_k=\gamma} \alpha! c^2_{i,\alpha} \right) \left( \sum_{\alpha+\epsilon_k=\gamma} \kappa^2_{i,k} \right) \right) \\
\le &\sum_{\gamma \in \mathcal{J}} (2\mathbb{N})^{-k\gamma} \left( \sum_{i=1}^{\infty} \left( \sum_{\alpha+\epsilon_k=\gamma} \alpha!(2\mathbb{N})^{-q\gamma} c^2_{i,\alpha} \right) \right) \left( \sum_{i=1}^{\infty} \left( \sum_{\alpha+\epsilon_k=\gamma} (2\mathbb{N})^{-2\gamma} \kappa^2_{i,k} \right) \right) \\
\le &\sum_{\gamma \in \mathcal{J}} (2\mathbb{N})^{-k\gamma} \left( \sum_{\alpha+\epsilon_k=\gamma} \alpha!(2\mathbb{N})^{-q\alpha} \left( \sum_{i=1}^{\infty} c^2_{i,\alpha} \right) \right) \left( \sum_{\alpha+\epsilon_k=\gamma} (2\mathbb{N})^{-2\epsilon_k} \left( \sum_{i=1}^{\infty} \kappa^2_{i,k} \right) \right) \\
\le &\left( \sum_{\gamma \in \mathcal{J}} (2\mathbb{N})^{-k\gamma} \right) \left( \sum_{\alpha \in \mathcal{J}} \alpha!(2\mathbb{N})^{-q\alpha} \left( \sum_{i=1}^{\infty} c^2_{i,\alpha} \right) \right) \left( \sum_{k=0}^{\infty} (2k)^{-2} \kappa^2_{i,k} \right) \\
< &\ \infty ,
\end{aligned}$$

as required. ■

**Propositon 3.6** *1. Consider $F,\ G \in S(H)_{-1}$ given by*

$$F = \sum_{i=1}^{\infty} F_i e_i , \quad G = \sum_{i=1}^{\infty} G_i e_i .$$

*Then*

$$\mathcal{H}(F \diamond G)(z) = \sum_{i=1}^{\infty} \mathcal{H}F_i(z)\ \mathcal{H}G_i(z)\ e_i ,$$

*for all $z \in \mathbb{C}^{\mathbb{N}}$ such that both $\mathcal{H}F_i(z)$ and $\mathcal{H}G_i(z)$ exist, for all $i \in \mathbb{N}$.*

*2. Consider $F \in (S)_{-1}, G \in S(H)_{-1}$. Then*

$$\mathcal{H}(F \diamond G)(z) = \mathcal{H}F(z)\ \mathcal{H}G(z) ,$$

*for all $z \in \mathbb{C}^{\mathbb{N}}$ such that both $\mathcal{H}F(z)$ and $\mathcal{H}G(z)$ exist.*

**Proof:** (1) Let

$$F = \sum_{\alpha\in\mathcal{J}} \sum_{i=1}^{\infty} c_{i,\alpha} H_\alpha e_i \ , \quad G = \sum_{\alpha\in\mathcal{J}} \sum_{i=1}^{\infty} d_{i,\alpha} H_\alpha e_i \ .$$

Using Lemma 2.2

$$\begin{aligned}
&\mathcal{H}(F \diamond G)(z) \\
&= \sum_{\gamma\in\mathcal{J}} \sum_{i=1}^{\infty} \left( \sum_{\alpha+\beta=\gamma} c_{i,\alpha} d_{i,\beta} z^{\alpha+\beta} \right) e_i = \sum_{i=1}^{\infty} \sum_{\gamma\in\mathcal{J}} \left( \sum_{\alpha+\beta=\gamma} c_{i,\alpha} d_{i,\beta} z^{\alpha+\beta} \right) e_i \\
&= \sum_{i=1}^{\infty} \left( \sum_{\alpha\in\mathcal{J}} c_{i,\alpha} z^\alpha \right) \left( \sum_{\beta\in\mathcal{J}} d_{i,\beta} z^\beta \right) e_i = \sum_{i=1}^{\infty} \mathcal{H}F_i(z) \ \mathcal{H}G_i(z) \ e_i \ ,
\end{aligned}$$

as required.

(2) Let

$$F = \sum_{\alpha\in\mathcal{J}} c_\alpha H_\alpha \ , \quad G = \sum_{\alpha\in\mathcal{J}} d_\alpha H_\alpha \ .$$

Now

$$\begin{aligned}
\mathcal{H}(F \diamond G)(z) &= \sum_{\gamma\in\mathcal{J}} \left( \sum_{\alpha+\beta=\gamma} c_\alpha d_\beta z^{\alpha+\beta} \right) \\
&= \left( \sum_{\alpha\in\mathcal{J}} c_\alpha z^\alpha \right) \left( \sum_{\beta\in\mathcal{J}} d_\beta z^\beta \right) = \mathcal{H}F(z) \ \mathcal{H}G(z) \ ,
\end{aligned}$$

as required. ■

Following the ideas of [10], we define the generalised expectation of elements in $S(H)_{-1}$.

**Definition 3.2** *Consider $F \in S(H)_{-1}$ having form*

$$F = \sum_{\alpha\in\mathcal{J}} c_\alpha H_\alpha \ .$$

*Define the generalised expectation of $F$ as*

$$E[F] := c_{(0,\dots)} = \widetilde{F}(0) \in H \ .$$

Clearly we have for all $F, G \in S(H)_{-1}$

$$E[F \diamond G] = \sum_{i=1}^{\infty} E[F_i] \ E[G_i] \ e_i \ ,$$

and for all $F \in (S)_{-1}$ and $G \in S(H)_{-1}$

$$E[F \diamond G] = E[F] \ E[G] \ .$$

## 3.5 Pettis Integral

**Definition 3.3** *We say $F(t) : \mathbb{R} \to S(H)_{-0}$ is Pettis integrable if*

$$\langle F(\cdot), f \rangle \in L^1(\mathbb{R}, dt) \ , \tag{3.17}$$

*for all $f \in S(H)_0$. In addition, we say $F(\cdot)$ is Pettis integrable on $E \subset \mathbb{R}$ if $I_E(\cdot)F(\cdot)$ is Pettis integrable.*

**Propositon 3.7** *If $F(t) : \mathbb{R} \to S(H)_{-0}$ is Pettis integrable, then there exists a unique element in $S(H)_{-0}$, denoted $\int_{\mathbb{R}} F(t)dt$, such that*

$$\langle \int_{\mathbb{R}} F(t)dt, f \rangle = \int_{\mathbb{R}} \langle F(t), f \rangle dt \ , \tag{3.18}$$

*for all $f \in S(H)_0$.*

*If $F(\cdot)$ is Pettis integrable, then we call $\int_{\mathbb{R}} F(t)dt$ the Pettis integral of $F(\cdot)$.*

**Proof:** This proof follows the proof of Proposition 8.1 in [8].

To do this, we only need to show that

$$\int_{\mathbb{R}} \langle F(t), \cdot \rangle dt \ ,$$

is a continuous, linear functional on $S(H)_0$, that is, it is an element of $S(H)_{-0}$.

Linearity follows from the linear property of the integral and the linearity of $F(t)$, for each $t \in \mathbb{R}$.

To show continuity, we need the linear functional $X : S(H)_0 \to L^1(\mathbb{R})$ defined by

$$X[f] := \langle F(\cdot), f \rangle \ .$$

We start by showing that $X$ is closed. Take a sequence $\{f_n\}_{n=1}^{\infty} \subset S(H)_0$ such that $f_n \longrightarrow_{n\to\infty} f$ in $S(H)_0$ and

$$||X[f_n] - \phi(t)||_{L^1(\mathbb{R})} \underset{n\to\infty}{\longrightarrow} 0 \ ,$$

for some $\phi(t) \in L^1(\mathbb{R})$. Note that for almost all $t \in \mathbb{R}$

$$\lim_{n\to\infty} \langle F(t), f_n \rangle = \phi(t) \ ,$$

where the convergence pointwise in $\mathbb{R}$. Also note that as for each $t \in \mathbb{R}$, $F(t) \in S(H)_{-0}$, then

$$\lim_{n\to\infty} \langle F(t), f_n - f \rangle \ = \ \langle F(t), \lim_{n\to\infty} (f_n - f) \rangle = 0 \ .$$

Hence

$$
\begin{aligned}
& \|X[f]-\phi(t)\|_{L^1(\mathbb{R})} \\
= & \int_{\mathbb{R}} |X[f]-\phi(t)|\,dt = \int_{\mathbb{R}} |\langle F(t), f\rangle - \phi(t)|\,dt \\
= & \int_{\mathbb{R}} \left|\langle F(t), f\rangle - \phi(t) + \lim_{n\to\infty}\langle F(t), f_n - f_n\rangle\right| dt \\
= & \int_{\mathbb{R}} \left|\lim_{n\to\infty}\langle F(t), f - f_n\rangle + \lim_{n\to\infty}\langle F(t), f_n\rangle - \phi(t)\right| dt \\
= & \ 0 \,.
\end{aligned}
$$

Therefore $X[f] = \phi(t)$ in $L^1(\mathbb{R})$. Hence $X$ is closed.

Now $S(H)_0$ is Frechet space, where closedness is equivalent to continuity. This gives continuity for $X$. Hence

$$
\begin{aligned}
\lim_{n\to\infty}\left|\int_{\mathbb{R}}\langle F(t), f_n - f\rangle dt\right| & \le \lim_{n\to\infty}\int_{\mathbb{R}}|\langle F(t), f_n - f\rangle|\,dt \\
& = \lim_{n\to\infty}\|X[f_n - f]\|_{L^1(\mathbb{R})} = 0 \,,
\end{aligned}
$$

as required. ■

Note that if $F(\cdot)$ is Pettis integrable on $E \subset \mathbb{R}$, then for measurable $G \subset E$ we define

$$
\int_G F(t)dt := \int_E I_G(t)F(t)dt \,. \tag{3.19}
$$

**Propositon 3.8** *Take $F(t) : \mathbb{R} \to S(H)_{-0}$ with expansion*

$$
F(t) = \sum_{\alpha\in\mathcal{J}} c_\alpha(t) H_\alpha \,,
$$

*such that for some $q \in \mathbb{N}$*

$$
\sum_{\alpha\in\mathcal{J}} \alpha!(2\mathbb{N})^{-q\alpha}\left(\int_{\mathbb{R}}\|c_\alpha(t)\|_H dt\right)^2 < \infty \,. \tag{3.20}
$$

*In this case, $F(\cdot)$ is Pettis integrable and*

$$
\int_{\mathbb{R}} F(t)dt = \sum_{\alpha\in\mathcal{J}}\left(\int_{\mathbb{R}} c_\alpha(t)dt\right) H_\alpha \,. \tag{3.21}
$$

**Proof:** We firstly show that $F(\cdot)$ is Pettis integrable. Take $f \in S(H)_0$ with form

$$
f = \sum_{\alpha\in\mathcal{J}} a_\alpha H_\alpha \,.
$$

Using Lemma 1.6

$$
\begin{aligned}
&\int_{\mathbb{R}} |\langle F(t), f\rangle| dt \\
= & \int_{\mathbb{R}} \left| \sum_{\alpha\in\mathcal{J}} \alpha! \langle a_\alpha, c_\alpha(t)\rangle_H \right| dt \le \int_{\mathbb{R}} \sum_{\alpha\in\mathcal{J}} \alpha! \left|\langle a_\alpha, c_\alpha(t)\rangle_H\right| dt \\
\le & \int_{\mathbb{R}} \sum_{\alpha\in\mathcal{J}} \alpha! \|a_\alpha\|_H \|c_\alpha(t)\|_H dt = \sum_{\alpha\in\mathcal{J}} \alpha! (2\mathbb{N})^{\frac{q\alpha}{2}} (2\mathbb{N})^{\frac{-q\alpha}{2}} \|a_\alpha\|_H \int_{\mathbb{R}} \|c_\alpha(t)\|_H dt \\
\le & \left( \sum_{\alpha\in\mathcal{J}} \alpha! (2\mathbb{N})^{q\alpha} \|a_\alpha\|_H^2 \right)^{1/2} \left( \sum_{\alpha\in\mathcal{J}} \alpha! (2\mathbb{N})^{-q\alpha} \left( \int_{\mathbb{R}} \|c_\alpha(t)\|_H dt \right)^2 \right)^{1/2} \\
< & \ \infty ,
\end{aligned}
$$

as required.

Secondly show equation (3.21). Start by noting that the formal sum

$$
\sum_{\alpha\in\mathcal{J}} \left( \int_{\mathbb{R}} c_\alpha(t) dt \right) H_\alpha ,
$$

belongs to $S(H)_{-0,-q}$ by equation (3.20). Using Proposition 1.6 again

$$
\begin{aligned}
& \langle \int_{\mathbb{R}} F(t) dt, f\rangle \\
= & \langle \int_{\mathbb{R}} F(t) dt, \sum_{\alpha\in\mathcal{J}} a_\alpha H_\alpha \rangle = \sum_{\alpha\in\mathcal{J}} \langle \int_{\mathbb{R}} F(t) dt, a_\alpha H_\alpha \rangle \\
= & \sum_{\alpha\in\mathcal{J}} \int_{\mathbb{R}} \langle F(t), a_\alpha H_\alpha\rangle dt = \sum_{\alpha\in\mathcal{J}} \int_{\mathbb{R}} \alpha! \langle c_\alpha(t), a_\alpha\rangle_H dt \\
= & \sum_{\alpha\in\mathcal{J}} \alpha! \langle \int_{\mathbb{R}} c_\alpha(t) dt, a_\alpha\rangle_H = \langle \sum_{\alpha\in\mathcal{J}} \left( \int_{\mathbb{R}} c_\alpha(t) dt \right) H_\alpha, f\rangle ,
\end{aligned}
$$

as required. ∎

**Definition 3.4** *Consider a $S(H)_{-0}$ (or $(S)_{-0}$) process, $F(t) : \mathbb{R} \to S(H)_{-0}$ (or $(S)_{-0}$). If $F(\cdot) \diamond \mathbb{W}(\cdot)$ is Pettis integrable, define the abstract Hitsuda-Skorohod integral of $F(\cdot)$ as the Pettis integral*

$$
\int_{\mathbb{R}} F(t) \delta W(t) := \int_{\mathbb{R}} F(t) \diamond \mathbb{W}(t) dt . \tag{3.22}
$$

**Propositon 3.9** *Consider $F(t) : \mathbb{R} \to S(H)_{-0}$ with form*

$$
F(t) = \sum_{\alpha\in\mathcal{J}} \sum_{i=1}^{\infty} c_{i,\alpha}(t) H_\alpha e_i = \sum_{\alpha\in\mathcal{J}} c_\alpha(t) H_\alpha ;
$$

*such that for some $q \in \mathbb{N}$*

$$K := \sup_{\alpha \in \mathcal{J}} \left\{ \alpha!(2\mathbb{N})^{-q\alpha} \int_{\mathbb{R}} \|c_\alpha(t)\|_H^2 dt \right\} < \infty . \tag{3.23}$$

*Then $F(\cdot) \diamond \mathbb{W}(\cdot)$ satisfies the condition of Proposition 3.8, and hence $F(\cdot)$ is Hitsuda-Skorohod integrable.*

**Proof:** $F(\cdot) \diamond \mathbb{W}(\cdot)$ has expansion

$$F(t) \diamond \mathbb{W}(t) = \sum_{\gamma \in \mathcal{J}} \left( \sum_{i=1}^{\infty} \sum_{\alpha+\epsilon_k=\gamma} c_{i,\alpha}(t) \kappa_{i,k}(t)\ e_i \right) H_\gamma =: \sum_{\gamma \in \mathcal{J}} p_\gamma(t) H_\gamma .$$

We need show that there exists a $p \in \mathbb{N}$ such that

$$\sum_{\gamma \in \mathcal{J}} (\gamma!) \left( \int_{\mathbb{R}} \|p_\gamma(t)\|_H dt \right)^2 (2\mathbb{N})^{-p\gamma} < \infty .$$

Now

$$\begin{aligned}
&\sum_{\gamma \in \mathcal{J}} (\gamma!)(2\mathbb{N})^{-(q+8)\gamma} \left( \int_{\mathbb{R}} \|p_\gamma(t)\|_H dt \right)^2 \\
= &\sum_{\gamma \in \mathcal{J}} (\gamma!)(2\mathbb{N})^{-(q+8)\gamma} \left( \int_{\mathbb{R}} \left( \sum_{i=1}^{\infty} \left( \sum_{\alpha+\epsilon_k=\gamma} c_{i,\alpha}(t)\kappa_{i,k}(t) \right)^2 \right)^{1/2} dt \right)^2 \\
\leq &\sum_{\gamma \in \mathcal{J}} (\gamma!)(2\mathbb{N})^{-(q+8)\gamma} \left( \int_{\mathbb{R}} \left( \sum_{i=1}^{\infty} \left( \sum_{\alpha+\epsilon_k=\gamma} c_{i,\alpha}(t)^2 \right) \left( \sum_{\alpha+\epsilon_k=\gamma} \kappa_{i,k}(t)^2 \right) \right)^{1/2} dt \right)^2 \\
\leq &\sum_{\gamma \in \mathcal{J}} (\gamma!)(2\mathbb{N})^{-(q+8)\gamma} \left( \int_{\mathbb{R}} \left( \sum_{i=1}^{\infty} \sum_{\alpha+\epsilon_k=\gamma} c_{i,\alpha}(t)^2 \right)^{1/2} \left( \sum_{i=1}^{\infty} \sum_{\alpha+\epsilon_k=\gamma} \kappa_{i,k}(t)^2 \right)^{1/2} dt \right)^2 \\
\leq &\sum_{\gamma \in \mathcal{J}} (\gamma!)(2\mathbb{N})^{-(q+8)\gamma} \left( \int_{\mathbb{R}} \sum_{i=1}^{\infty} \sum_{\alpha+\epsilon_k=\gamma} c_{i,\alpha}(t)^2 dt \right) \left( \int_{\mathbb{R}} \sum_{i=1}^{\infty} \sum_{\alpha+\epsilon_k=\gamma} \kappa_{i,k}(t)^2 dt \right) \\
\leq &\sum_{\gamma \in \mathcal{J}} (2\mathbb{N})^{-(q+6)\gamma} \left( \int_{\mathbb{R}} \sum_{\alpha+\epsilon_k=\gamma} (\alpha!)(\alpha_k+1)(2\mathbb{N})^{-2\gamma} \|c_\alpha(t)\|_H^2 dt \right) \\
&\left( \int_{\mathbb{R}} \sum_{\alpha+\epsilon_k=\gamma} \|\kappa_k(t)\|_H^2 dt \right) \\
\leq &\sum_{\gamma \in \mathcal{J}} (2\mathbb{N})^{-(q+6)\gamma} \left( \int_{\mathbb{R}} \sum_{\alpha+\epsilon_k=\gamma} (\alpha!) \|c_\alpha(t)\|_H^2 dt \right) \\
&\left( \int_{\mathbb{R}} \sum_{\alpha+\epsilon_k=\gamma} \|\kappa_k(t)\|_H^2 dt \right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\gamma\in\mathcal{J}}(2\mathbb{N})^{-2\gamma}\left(\int_{\mathbb{R}}\sum_{\alpha+\epsilon_k=\gamma}(\alpha!)(2\mathbb{N})^{-(q+2)\gamma}\|c_\alpha(t)\|_H^2dt\right) \\
&\quad \left(\int_{\mathbb{R}}\sum_{\alpha+\epsilon_k=\gamma}(2\mathbb{N})^{-2\gamma}\|\kappa_k(t)\|_H^2dt\right) \\
&\leq \sum_{\gamma\in\mathcal{J}}(2\mathbb{N})^{-2\gamma}\left(\int_{\mathbb{R}}\sum_{\alpha\in\mathcal{J}}(\alpha!)(2\mathbb{N})^{-(q+2)\alpha}\|c_\alpha(t)\|_H^2dt\right)\left(\int_{\mathbb{R}}\sum_{k=1}^{\infty}(2k)^{-2}\|\kappa_k(t)\|_H^2dt\right) \\
&= A(2)\left(\sum_{\alpha\in\mathcal{J}}(\alpha!)(2\mathbb{N})^{-(q+2)\alpha}\int_{\mathbb{R}}\|c_\alpha(t)\|_H^2dt\right)\left(\sum_{k=1}^{\infty}\delta_{n(i,j),k}(2k)^{-2}\int_{\mathbb{R}}\xi_j(t)^2dt\right) \\
&\leq A(2)\left(K\sum_{\alpha\in\mathcal{J}}(2\mathbb{N})^{-q\alpha}\right)\left(\sum_{k=1}^{\infty}(2k)^{-2}\right) \leq KA(2)^3 < \infty\,,
\end{aligned}
$$

as required. ■

## 3.6 Operators on $S(H)_{-\rho}$

For this section we take $\rho \in [0,1]$, letting $F, G \in S(H)_{-\rho}$ have expansions

$$F = \sum_{\alpha\in\mathcal{J}} c_\alpha H_\alpha\,, \quad G = \sum_{\alpha\in\mathcal{J}} d_\alpha H_\alpha\,.$$

We let $\mathcal{L}(H)$ denote the space of bounded linear operators on $H$.

**Definition 3.5** *For a linear operator $A$ on $H$ define the domain of $A$ in $S(H)_{-\rho}$ as*

$$
\begin{aligned}
&\mathcal{D}(A)_{-\rho} \\
&:= \left\{F(\omega) \in S(H)_{-\rho}\ : \exists q \in \mathbb{N} \text{ such that } \sum_{\alpha\in\mathcal{J}}(\alpha!)^{1-\rho}\|Ac_\alpha\|_H^2(2\mathbb{N})^{-q\alpha} < \infty\right\}\,.
\end{aligned}
$$

*For $F \in \mathcal{D}(A)_{-\rho}$, define the action of $A$ on $F$ as*

$$AF := \sum_{\alpha\in\mathcal{J}}(Ac_\alpha)H_\alpha\,.$$

Let $H_1$ be another separable Hilbert space with inner product $\langle\cdot,\cdot\rangle_{H_1}$ and norm $\|\cdot\|_{H_1}$. Let $\mathcal{L}(H, H_1)$ denote the space of continuous linear maps from $H$ to $H_1$.

**Definition 3.6** *For a linear map $A : H \to H_1$, define the domain of $A$ in $S(H)_{-\rho}$ as*

$$
\begin{aligned}
&\mathcal{D}(A)_{-\rho} \\
&:= \left\{F(\omega) \in S(H)_{-\rho}\ : \exists q \in \mathbb{N} \text{ such that } \sum_{\alpha\in\mathcal{J}}(\alpha!)^{1-\rho}\|Ac_\alpha\|_{H_1}^2(2\mathbb{N})^{-q\alpha} < \infty\right\}\,.
\end{aligned}
$$

*For $F \in \mathcal{D}(A)_{-\rho}$, define the action of $A$ on $F$ as*

$$AF := \sum_{\alpha \in \mathcal{J}} (Ac_\alpha) H_\alpha \ .$$

**Propositon 3.10** *Let $A$ be either a linear operator on $H$, or a linear map from $H$ to $H_1$. If $F, G \in \mathcal{D}(A)_{-\rho}$, then $F + G \in \mathcal{D}(A)_{-\rho}$ and*

$$A[F+G] = AF + AG \ . \tag{3.24}$$

**Proof:** The proof of either case will be the same, so we just prove case for when $A$ is a linear operator on $H$.

Take $F, G \in \mathcal{D}(A)_{-\rho}$. There exists $q_1, q_2 \in \mathbb{N}$ such that

$$\sum_{\alpha \in \mathcal{J}} (\alpha!)^{1-\rho} \|Ac_\alpha\|_H^2 (2\mathbb{N})^{-q_1 \alpha} < \infty \ , \quad \sum_{\alpha \in \mathcal{J}} (\alpha!)^{1-\rho} \|Ad_\alpha\|_H^2 (2\mathbb{N})^{-q_2 \alpha} < \infty \ .$$

Let $q = \max\{q_1, q_2\}$. Then

$$\begin{aligned}
& \sum_{\alpha \in \mathcal{J}} (\alpha!)^{1-\rho} \|A(c_\alpha + d_\alpha)\|_H^2 (2\mathbb{N})^{-q\alpha} \\
\leq \ & \sum_{\alpha \in \mathcal{J}} (\alpha!)^{1-\rho} \left(\|Ac_\alpha\|_H + \|Ad_\alpha\|_H\right)^2 (2\mathbb{N})^{-q\alpha} \\
= \ & \sum_{\alpha \in \mathcal{J}} (\alpha!)^{1-\rho} \left(\|Ac_\alpha\|_H^2 + 2\|Ac_\alpha\|_H \|Ad_\alpha\|_H + \|Ad_\alpha\|_H^2\right) (2\mathbb{N})^{-q\alpha} \\
\leq \ & \sum_{\alpha \in \mathcal{J}} (\alpha!)^{1-\rho} \|Ac_\alpha\|_H^2 (2\mathbb{N})^{-q\alpha} + \sum_{\alpha \in \mathcal{J}} (\alpha!)^{1-\rho} \|Ad_\alpha\|_H^2 (2\mathbb{N})^{-q\alpha} \\
& + 2 \left( \sum_{\alpha \in \mathcal{J}} (\alpha!)^{1-\rho} \|Ac_\alpha\|_H^2 (2\mathbb{N})^{-q\alpha} \right)^{1/2} \left( \sum_{\alpha \in \mathcal{J}} (\alpha!)^{1-\rho} \|Ad_\alpha\|_H^2 (2\mathbb{N})^{-q\alpha} \right)^{1/2} \\
< \ & \infty \ .
\end{aligned}$$

Also

$$\begin{aligned}
A[F+G] &= \sum_{\alpha \in \mathcal{J}} A\left(c_\alpha + d_\alpha\right) H_\alpha = \sum_{\alpha \in \mathcal{J}} \left(Ac_\alpha + Ad_\alpha\right) H_\alpha \\
&= \sum_{\alpha \in \mathcal{J}} Ac_\alpha H_\alpha + \sum_{\alpha \in \mathcal{J}} Ad_\alpha H_\alpha \ ,
\end{aligned}$$

as required. ■

**Propositon 3.11** *Let $B$ belong to either $\mathcal{L}(H)$ or $\mathcal{L}(H, H_1)$. In both cases we have that $\mathcal{D}(B)_{-\rho} = S(H)_{-\rho}$.*

**Proof:** The proof of either case will be the same, so we just prove case for when $B$ belongs to $\mathcal{L}(H)$.

Take $F \in S(H)_{-\rho}$. There exists a $q \in \mathbb{N}$ such that $F \in S(H)_{-\rho,-q}$. For this $q$

$$\begin{aligned}\sum_{\alpha\in\mathcal{J}}(\alpha!)^{1-\rho}\|Bc_\alpha\|_H^2(2\mathbb{N})^{-q\alpha} &\leq \sum_{\alpha\in\mathcal{J}}(\alpha!)^{1-\rho}\|B\|^2\|c_\alpha\|_H^2(2\mathbb{N})^{-q\alpha} \\ &= \|B\|^2\ \|F\|_{-1,-q}^2 < \infty\ ,\end{aligned}$$

as required. ■

**Propositon 3.12** *Let $A$ be either a closed operator on $H$ or a closed linear map from $H$ to $H_1$. Let $F \in \mathcal{D}(A)_{-1}$. Then there exists a $q \in \mathbb{N}\setminus\{1\}$ such that*

$$\mathcal{H}(AF)(z) = A\widetilde{F}(z)\ ,$$

*for all $z \in \mathbb{K}_q$.*

**Proof:** The proof of either case will be the same, so we just prove case for when $A$ is a closed operator on $H$.

Since $F \in \mathcal{D}(A)_{-1}$, there exists a $q \in \mathbb{N}\setminus\{1\}$ such that for all $z \in \mathbb{K}_q$

$$\sum_{\alpha\in\mathcal{J}} c_\alpha z^\alpha\ ,\quad \sum_{\alpha\in\mathcal{J}} Ac_\alpha z^\alpha\ ,$$

both converge absolutely in $H_{\mathbb{C}}$, by Proposition 2.6. For each $n \in \mathbb{N}$, define

$$\widetilde{F}_n(z) = \sum_{\alpha\in\Gamma_n} c_\alpha z^\alpha\ .$$

Now for each $z \in \mathbb{K}_q$, $\widetilde{F}_n(z) \longrightarrow_{n\to\infty} \widetilde{F}(z)$ and $A\widetilde{F}_n(z) \longrightarrow_{n\to\infty} \mathcal{H}(AF)(z)$ in $H_{\mathbb{C}}$. Hence by closedness of $A$, for each $z \in \mathbb{K}_q$, $\widetilde{F}(z) \in \mathcal{D}(A) \subseteq H_{\mathbb{C}}$ and $\mathcal{H}(AF)(z) = A\widetilde{F}(z)$, as required. ■

## 3.7 Generalised Stochastic Convolution

For a strongly continuous family of continuous linear maps from $U$ to $H$, $\{S(t),\ t \geq 0\}$, such that $S(\cdot)\mathbb{W}(\cdot)$ is Pettis integrable on $[0,T]$, define

$$\int_0^t S(s)\delta W(s) := \int_0^t S(s)\mathbb{W}(s)ds\ ,\quad t \in [0,T]\ . \tag{3.25}$$

We call $\int_0^t S(s)\delta W(s)$ generalised stochastic convolution. Note that in the above equation, $\mathbb{W}(\cdot)$ belongs to $S(U)_{-0}$, being defined in the same way as in section 3.3.1 with respect to $\{f_i\}_{i=1}^\infty$.

In the rest of this section we consider the case the when $U = H$, so that $\{S(t),\ t \geq 0\}$ is a strongly continuous family of bounded operators on $H$. However, all of the results proved in this section will apply equally to the more general case of when $U \neq H$.

**Propositon 3.13** *Suppose that for some $T \in (0, \infty)$*

$$\int_0^T \|S(t)\|^2 dt < \infty . \tag{3.26}$$

*Then we have the following:*

1. *$S(\cdot)\mathbb{W}(\cdot)$ is Pettis integrable on $[0, T]$ and for all $t \in [0, T]$*

$$\int_0^t S(s)\delta W(s) = \sum_{k=1}^{\infty} \left( \int_0^t S(s)\kappa_k(s)ds \right) H_{\epsilon_k} . \tag{3.27}$$

2. *$\int_0^t S(s)\delta W(s)$ is a continuous $S(H)_{-1}$ process on $[0, T]$.*

**Proof:** (1) We show that $I_{[0,T]}(\cdot)S(\cdot)\mathbb{W}(\cdot)$ satisfies equation (3.20) of Proposition 3.8. Now

$$\begin{aligned}
&\sum_{k=1}^{\infty} (\epsilon_k!) \left( \int_{\mathbb{R}} I_{[0,T]}(t) \|S(t)\kappa_k(t)\|_H dt \right)^2 (2\mathbb{N})^{-2\epsilon_k} \\
\leq &\sum_{k=1}^{\infty} (1!) \left( \int_0^T \|S(t)\| \; \|\kappa_k(t)\|_H dt \right)^2 (2k)^{-2} \\
\leq &\sum_{k=1}^{\infty} \delta_{n(i,j),k} \left( \int_0^T \|S(t)\|^2 dt \right) \left( \int_{\mathbb{R}} \|\xi_j(t)e_i\|_H^2 dt \right) (2k)^{-2} \\
\leq &\left( \int_0^T \|S(t)\|^2 dt \right) \sum_{k=1}^{\infty} (2k)^{-2} \leq \left( \int_0^T \|S(t)\|^2 dt \right) A(2) < \infty ,
\end{aligned}$$

as required.

(2) Note that from the proof of part (1), we have that $\int_0^t S(s)\delta W(s)$ belongs to $S(H)_{-0,-2} \subset S(H)_{-1,-2}$, for $t \in [0, T]$. So by Proposition 2.7, we have that for $z \in \mathbb{K}_2$

$$\begin{aligned}
&\left\| \mathcal{H} \left( \int_0^t S(s)\delta W(s) \right) (z) \right\|_{H_{\mathbb{C}}} \\
\leq &\left\| \int_0^t S(s)\delta W(s) ds \right\|_{-1,-2} A(2)^{1/2} \\
= &\left( \sum_{k=1}^{\infty} \left\| \int_0^t S(s)\kappa_k(s)ds \right\|_H^2 (2\mathbb{N})^{-2\epsilon_k} \right)^{1/2} A(2)^{1/2} \\
\leq &\left( \sum_{k=1}^{\infty} \left( \int_0^t \|S(s)\| \; \|\kappa_k(s)\|_H ds \right)^2 (2k)^{-2} \right)^{1/2} A(2)^{1/2} \\
\leq &\left( \sum_{k=1}^{\infty} \left( \int_0^t \|S(s)\|^2 ds \right) \left( \int_0^t \|\kappa_k(s)\|_H^2 ds \right) (2k)^{-2} \right)^{1/2} A(2)^{1/2}
\end{aligned}$$

$$\leq \left(\int_0^T \|S(s)\|^2 ds\right)^{1/2} \left(\sum_{k=1}^{\infty} \delta_{n(i,j),k} \left(\int_{\mathbb{R}} \|\xi_j(s)e_i\|_H^2 ds\right) (2k)^{-2}\right)^{1/2} A(2)^{1/2}$$

$$= \left(\int_0^T \|S(s)\|^2 ds\right)^{1/2} \left(\sum_{k=1}^{\infty} (2k)^{-2}\right)^{1/2} A(2)^{1/2}$$

$$\leq \left(\int_0^T \|S(t)\|^2 dt\right)^{1/2} A(2)^{1/2} A(2)^{1/2} \leq K < \infty ,$$

where $K$ is a constant depending only on $T$. Hence $\mathcal{H}\left(\int_0^t S(s)\delta W(s)\right)(z)$ is continuous with respect to $t$ on $[0,T]$ for all $z \in \mathbb{K}_4$, by Proposition 2.4, as the functions

$$\left\{\int_0^t S(s)\kappa_k(s)ds\right\}_{k=1}^{\infty} ,$$

are continuous on $[0,T]$. Hence $\int_0^t S(s)\delta W(s)$ is a continuous $S(H)_{-1}$ process on $[0,T]$, by Theorem 2.1. ■

Note that from the previous proof, $\int_0^t S(t)\delta W(s) \in S(H)_{-0,-2} \subset S(H)_{-1,-2}$.

**Corollary 3.3** *Suppose again that for some $T \in (0,\infty)$*

$$\int_0^T \|S(t)\|^2 dt < \infty . \tag{3.28}$$

*Then $\int_0^t S(s)\delta W(s)$ is a continuous $S(H)_{-0}$ process on $[0,T]$.*

**Proof:** For $q \in \mathbb{N}$

$$\left\|\int_0^t S(s)\delta W(s)\right\|_{-0,-q}^2$$

$$= \sum_{k=1}^{\infty} (\epsilon_k!) \left\|\int_0^t S(s)\kappa_k(s)ds\right\|_H^2 (2\mathbb{N})^{-q\epsilon_k}$$

$$= \sum_{k=1}^{\infty} \left\|\int_0^t S(s)\kappa_k(s)ds\right\|_H^2 (2\mathbb{N})^{-q\epsilon_k} = \left\|\int_0^t S(s)\delta W(s)\right\|_{-1,-q}^2 .$$

Result then follows from Proposition 3.13 and Proposition 1.7. ■

**Propositon 3.14** *Suppose that for some $T \in (0,\infty)$, the linear operator $S_T$ defined by*

$$S_T x := \int_0^T S(s)S^*(s)x\, ds , \quad x \in H , \tag{3.29}$$

*is trace class. Then $S(\cdot)\mathbb{W}(\cdot)$ is Pettis integrable on $[0,T]$ and for all $t \in [0,T]$*

$$\int_0^t S(s)\delta W(s) = \sum_{k=1}^{\infty} \left(\int_0^t S(s)\kappa_k(s)ds\right) H_{\epsilon_k} , \tag{3.30}$$

*belongs to $L^2(H)$.*

**Proof:** To show that $S(\cdot)\mathbb{W}(\cdot)$ is Pettis integrable on $[0,T]$ and equation (3.30), we show that $I_{[0,T]}(t)S(\cdot)\mathbb{W}(\cdot)$ satisfies equation (3.20) of Proposition 3.8. Now

$$\begin{aligned}
&\sum_{k=1}^{\infty}(\epsilon_k!)\left(\int_{\mathbb{R}} I_{[0,T]}(t)\|S(t)\kappa_k(t)\|_H dt\right)^2 (2\mathbb{N})^{-2\epsilon_k} \\
&= \sum_{k=1}^{\infty}\delta_{n(i,j),k}(1!)\left(\int_0^T \|S(t)\xi_j(t)e_i\|_H dt\right)^2 (2k)^{-2} \\
&= \sum_{k=1}^{\infty}\delta_{n(i,j),k}\left(\int_0^T \|S(t)e_i\|_H|\xi_j(t)|dt\right)^2 (2k)^{-2} \\
&\leq \sum_{k=1}^{\infty}\delta_{n(i,j),k}\left(\int_0^T \|S(t)e_i\|_H^2 dt\right)\left(\int_0^T |\xi_j(t)|^2 dt\right)(2k)^{-2} \\
&= \sum_{k=1}^{\infty}\delta_{n(i,j),k}\left(\int_0^T \langle S(t)e_i, S(t)e_i\rangle_H dt\right)\left(\int_{\mathbb{R}} \xi_j(t)^2 dt\right)(2k)^{-2} \\
&= \sum_{k=1}^{\infty}\delta_{n(i,j),k}\left(\int_0^T \langle S^*(t)e_i, S^*(t)e_i\rangle_H dt\right)(2k)^{-2} \\
&= \sum_{k=1}^{\infty}\delta_{n(i,j),k}\left(\int_0^T \langle S(t)S^*(t)e_i, e_i\rangle_H dt\right)(2k)^{-2} \\
&= \sum_{k=1}^{\infty}\delta_{n(i,j),k}\left(\langle\int_0^T S(t)S^*(t)e_i\, dt, e_i\rangle_H\right)(2k)^{-2} \\
&\leq \sum_{k=1}^{\infty}\delta_{n(i,j),k}\mathrm{Tr}(S_T)(2k)^{-2} = \mathrm{Tr}(S_T)A(2) < \infty\ ,
\end{aligned}$$

as required.

To show that $\int_0^t S(s)\delta W(s)$ belongs to $L^2(H)$ for all $t \in [0,T]$, consider

$$\begin{aligned}
&\left\|\int_0^t S(s)\delta W(s)\right\|_{L^2(H)}^2 \\
&= \left\|\sum_{k=1}^{\infty}\left(\int_0^t S(s)\kappa_k(s)ds\right)H_{\epsilon_k}\right\|_{L^2(H)}^2 \\
&= \sum_{\alpha\in\mathcal{J}}\sum_{l=1}^{\infty}(\alpha!)^{-1}\langle\sum_{k=1}^{\infty}\left(\int_0^t S(s)\kappa_k(s)ds\right)H_{\epsilon_k}, H_\alpha e_l\rangle_{L^2(H)}^2 \\
&= \sum_{\alpha\in\mathcal{J}}\sum_{l=1}^{\infty}(\alpha!)^{-1}\left(\sum_{k=1}^{\infty}\langle\left(\int_0^t S(s)\kappa_k(s)ds\right)H_{\epsilon_k}, H_\alpha e_l\rangle_{L^2(H)}\right)^2 \\
&= \sum_{\alpha\in\mathcal{J}}\sum_{l=1}^{\infty}(\alpha!)^{-1}\left(\sum_{k=1}^{\infty}\mathrm{E}\left[\langle\left(\int_0^t S(s)\kappa_k(s)ds\right)H_{\epsilon_k}, H_\alpha e_l\rangle_H\right]\right)^2
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\alpha\in\mathcal{J}}\sum_{l=1}^{\infty}(\alpha!)^{-1}\left(\sum_{k=1}^{\infty}\langle\left(\int_0^t S(s)\kappa_k(s)ds\right), e_l\rangle_H \mathrm{E}\left[H_{\epsilon_k}H_\alpha\right]\right)^2 \\
&= \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\langle\left(\int_0^t S(s)\kappa_k(s)ds\right), e_l\rangle_H^2 = \sum_{k=1}^{\infty}\left\|\int_0^t S(s)\kappa_k(s)ds\right\|_H^2 \\
&= \sum_{k=1}^{\infty}\delta_{n(i,j),k}\left\|\int_0^t S(s)e_i\ \xi_j(s)ds\right\|_H^2 \le \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left\|\int_0^t S(s)e_i\ \xi_j(s)ds\right\|_H^2 \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\langle\int_0^t S(s)e_i\ \xi_j(s)ds, e_l\rangle_H^2 = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\left(\int_0^t\langle S(s)e_i\ \xi_j(s), e_l\rangle_H ds\right)^2 \\
&= \sum_{i=1}^{\infty}\sum_{l=1}^{\infty}\sum_{j=1}^{\infty}\left(\int_{\mathbb{R}}\xi_j(s)\langle I_{[0,t]}(s)S(s)e_i, e_l\rangle_H ds\right)^2 \\
&= \sum_{i=1}^{\infty}\sum_{l=1}^{\infty}\left\|\langle I_{[0,t]}(s)S(s)e_i, e_l\rangle_H\right\|_{L^2(\mathbb{R})}^2 = \sum_{i=1}^{\infty}\sum_{l=1}^{\infty}\left(\int_0^t\langle S(s)e_i, e_l\rangle_H ds\right)^2 \\
&= \sum_{i=1}^{\infty}\sum_{l=1}^{\infty}\langle\int_0^t S(s)e_i\ ds, e_l\rangle_H^2 = \sum_{i=1}^{\infty}\left\|\int_0^t S(s)e_i\ ds\right\|_H^2 \le \sum_{i=1}^{\infty}\left(\int_0^t\|S(s)e_i\|_H\ ds\right)^2 \\
&\le \sum_{i=1}^{\infty}\left(\int_0^T\|S(s)e_i\|_H\ ds\right)^2 \le \sum_{i=1}^{\infty}\left(\int_0^T\|S(s)e_i\|_H^2\ ds\right)\left(\int_0^T 1ds\right) \\
&= T\sum_{i=1}^{\infty}\left(\int_0^T\|S(s)e_i\|_H^2\ ds\right) = T\sum_{i=1}^{\infty}\int_0^T\langle S(s)e_i, S(s)e_i\rangle_H ds \\
&= T\sum_{i=1}^{\infty}\int_0^T\langle S^*(s)e_i, S^*(s)e_i\rangle_H ds = T\sum_{i=1}^{\infty}\int_0^T\langle S(s)S^*(s)e_i, e_i\rangle_H ds \\
&= T\sum_{i=1}^{\infty}\langle\int_0^T S(s)S^*(s)e_i\ ds, e_i\rangle_H = T\ \mathrm{Tr}(S_T) < \infty\ ,
\end{aligned}
$$

as required. ■

**Theorem 3.1** *Suppose again that for some $T \in (0,\infty)$, the linear operator $S_T$ defined by*

$$S_T x := \int_0^T S(t)S^*(t)x\ dt\ , \quad x \in H\ , \tag{3.31}$$

*is trace class. Then for all $t \in [0,T]$, we have that in $L^2(H)$*

$$\int_0^t S(s)\delta W(s) = \sum_{i=1}^{\infty}\int_0^t S(s)e_i\ d\beta_i(s)\ , \tag{3.32}$$

*where summation is in $L^2(H)$ and $\int_0^t S(s)e_i\ d\beta_i(s)$ is the normal Ito integral with respect to $\beta_i(\cdot)$.*

**Proof:** Now $\int_0^t S(s)\delta W(s)$ has the following expansion in $L^2(H)$

$$\sum_{k=1}^{\infty}\left(\int_0^t S(s)\kappa_k(s)ds\right) H_{\epsilon_k} \;=\; \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\int_0^t S(s)e_i\;\xi_j(s)ds\right) H_{\epsilon_{n(i,j)}} \,.$$

So it remains to show that in $L^2(H)$

$$\int_0^t S(s)e_i\;d\beta_i(s) \;=\; \sum_{j=1}^{\infty}\left(\int_0^t S(s)e_i\;\xi_j(s)ds\right) H_{n(i,j)}\;,$$

where series is in $L^2(H)$.

Take any $t \in [0,T]$, and a set of partitions

$$\{0 = t_0 < t_1 < \ldots < t_N = t\}_{N=1}^{\infty}\;,$$

of $[0,t]$ such that

$$\lim_{N\to\infty}\left(\max_{k\in\{0,\ldots,N-1\}}(t_{k+1}-t_k)\right) = 0\;.$$

Now

$$\lim_{N\to\infty} \mathrm{E}\left[\int_0^t \left\|S(s)e_i - \sum_{k=0}^{N-1} I_{[t_k,t_{k+1})}(s)S(t_k^*)e_i\right\|_H^2 ds\right]$$
$$= \lim_{N\to\infty}\int_0^t \left\|S(s)e_i - \sum_{k=0}^{N-1} I_{[t_k,t_{k+1})}(s)S(t_k^*)e_i\right\|_H^2 ds\;,$$

where $t_k^* \in [t_k,t_{k+1})$. Since $\{S(t),\; t \geq 0\}$ is strongly continuous, the functions $\{S(\cdot)e_i\}_{i=1}^{\infty}$ are continuous and hence bounded by some $M < \infty$ on $[0,T]$. So by the Dominated Convergence theorem

$$\lim_{N\to\infty}\int_0^t \left\|S(s)e_i - \sum_{k=0}^{N-1} I_{[t_k,t_{k+1})}(s)S(t_k^*)e_i\right\|_H^2 ds$$
$$= \int_0^t \lim_{N\to\infty}\left\|S(s)e_i - \sum_{k=0}^{N-1} I_{[t_k,t_{k+1})}(s)S(t_k^*)e_i\right\|_H^2 ds = \int_0^t 0\;ds = 0\;,$$

for all $i \in \mathbb{N}$ as the functions $\{S(\cdot)e_i\}_{i=1}^{\infty}$ are continuous. Therefore

$$\int_0^t S(s)e_i\;d\beta_i(s) \;=\; \lim_{N\to\infty}\sum_{k=0}^{N-1} S(t_k^*)e_i\,(\beta_i(t_{k+1}) - \beta_i(t_k))\;,$$

where limit is in $L^2(H)$. Now in $L^2(H)$

$$\int_0^t S(s)e_i\;d\beta_i(s)$$

$$= \sum_{\alpha\in\mathcal{J}}\sum_{l=1}^{\infty}(\alpha!)^{-1}\langle\int_0^t S(s)e_i\ d\beta_i(s), H_\alpha e_l\rangle_{L^2(H)}H_\alpha e_l$$

$$= \sum_{\alpha\in\mathcal{J}}\sum_{l=1}^{\infty}(\alpha!)^{-1}\langle\lim_{N\to\infty}\sum_{k=0}^{N-1} S(t_k^*)e_i\left(\beta_i(t_{k+1})-\beta_i(t_k)\right), H_\alpha e_l\rangle_{L^2(H)}H_\alpha e_l$$

$$= \sum_{\alpha\in\mathcal{J}}\sum_{l=1}^{\infty}(\alpha!)^{-1}\left(\lim_{N\to\infty}\sum_{k=0}^{N-1}\langle S(t_k^*)e_i\left(\beta_i(t_{k+1})-\beta_i(t_k)\right), H_\alpha e_l\rangle_{L^2(H)}\right)H_\alpha e_l\ . \tag{3.33}$$

Now

$$\begin{aligned}
&\langle S(t_k^*)e_i\left(\beta_i(t_{k+1})-\beta_i(t_k)\right), H_\alpha e_l\rangle_{L^2(H)}\\
&= \int_{S'(\mathbb{R}^d)}\langle S(t_k^*)e_i\left(\beta_i(t_{k+1})-\beta_i(t_k)\right), H_\alpha(\omega)e_l\rangle_H d\mu(\omega)\\
&= \int_{S'(\mathbb{R}^d)} H_\alpha(\omega)\left(\beta_i(t_{k+1})-\beta_i(t_k)\right)\langle S(t_k^*)e_i, e_l\rangle_H d\mu(\omega)\\
&= \langle S(t_k^*)e_i, e_l\rangle_H\ \langle H_\alpha, \left(\sum_{j=1}^{\infty}\left(\int_{t_k}^{t_{k+1}}\xi_j(s)ds\right)H_{\epsilon_{n(i,j)}}\right)\rangle_{L^2(\mu)}\\
&= \langle S(t_k^*)e_i, e_l\rangle_H\ \left(\sum_{j=1}^{\infty}\left(\int_{t_k}^{t_{k+1}}\xi_j(s)ds\right)\langle H_\alpha, H_{\epsilon_{n(i,j)}}\rangle_{L^2(\mu)}\right)\\
&= \langle S(t_k^*)e_i, e_l\rangle_H\ \left(\sum_{j=1}^{\infty}\left(\int_{t_k}^{t_{k+1}}\xi_j(s)ds\right)\delta_{\epsilon_{n(i,j)},\alpha}\right)\\
&= \sum_{j=1}^{\infty}\delta_{\epsilon_{n(i,j)},\alpha}\langle\left(\int_{t_k}^{t_{k+1}}\xi_j(s)ds\right)S(t_k^*)e_i, e_l\rangle_H\\
&= \sum_{j=1}^{\infty}\delta_{\epsilon_{n(i,j)},\alpha}\langle\int_{t_k}^{t_{k+1}}S(t_k^*)e_i\ \xi_j(s)ds, e_l\rangle_H\ .
\end{aligned}$$

Putting this into equation (3.33) gives

$$\begin{aligned}
&\int_0^t S(s)e_i\ d\beta_i(s)\\
&= \sum_{\alpha\in\mathcal{J}}\sum_{l=1}^{\infty}(\alpha!)^{-1}\left(\lim_{N\to\infty}\sum_{k=0}^{N-1}\langle S(t_k^*)e_i\left(\beta_i(t_{k+1})-\beta_i(t_k)\right), H_\alpha e_l\rangle_{L^2(H)}\right)H_\alpha e_l\\
&= \sum_{\alpha\in\mathcal{J}}\sum_{l=1}^{\infty}(\alpha!)^{-1}\left(\lim_{N\to\infty}\sum_{k=0}^{N-1}\sum_{j=1}^{\infty}\delta_{\epsilon_{n(i,j)},\alpha}\langle\int_{t_k}^{t_{k+1}}S(t_k^*)e_i\ \xi_j(s)ds, e_l\rangle_H\right)H_\alpha e_l\\
&= \sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\left(\lim_{N\to\infty}\sum_{k=0}^{N-1}\langle\int_{t_k}^{t_{k+1}}S(t_k^*)e_i\ \xi_j(s)ds, e_l\rangle_H\right)H_{\epsilon_{n(i,j)}}e_l\\
&= \sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\langle\lim_{N\to\infty}\sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}}I_{[t_k,t_{k+1})}(s)S(t_k^*)e_i\ \xi_j(s)ds, e_l\rangle_H H_{\epsilon_{n(i,j)}}e_l
\end{aligned}$$

$$= \sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\langle \lim_{N\to\infty} \int_0^t \sum_{k=0}^{N-1} I_{[t_k,t_{k+1})}(s) S(t_k^*) e_i \ \xi_j(s) ds, e_l\rangle_H H_{\epsilon_{n(i,j)}} e_l$$

$$= \sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\langle \int_0^t S(s) e_i \ \xi_j(s) ds, e_l\rangle_H H_{\epsilon_{n(i,j)}} e_l \ .$$

Now

$$\left\| \sum_{l=1}^{\infty}\langle \int_0^t S(s)e_i \ \xi_j(s)ds, e_l\rangle_H H_{\epsilon_{n(i,j)}} e_l \right.$$
$$\left. - \left(\int_0^t S(s) e_i \ \xi_j(s) ds\right) H_{\epsilon_{n(i,j)}} \right\|_{L^2(H)}^2$$
$$= \int_{S'(\mathbb{R}^d)} \left\| \sum_{l=1}^{\infty}\langle \int_0^t S(s)e_i \ \xi_j(s)ds, e_l\rangle_H H_{\epsilon_{n(i,j)}}(\omega) e_l \right.$$
$$\left. - \left(\int_0^t S(s) e_i \ \xi_j(s) ds\right) H_{\epsilon_{n(i,j)}}(\omega) \right\|_H^2 d\mu(\omega)$$
$$= \int_{S'(\mathbb{R}^d)} \left\| \sum_{l=1}^{\infty}\langle \int_0^t S(s)e_i \ \xi_j(s)ds, e_l\rangle_H H_{\epsilon_{n(i,j)}}(\omega) e_l \right.$$
$$\left. - \sum_{l=1}^{\infty}\langle \int_0^t S(s)e_i \ \xi_j(s)ds, e_l\rangle_H H_{\epsilon_{n(i,j)}}(\omega) e_l \right\|_H^2 d\mu(\omega)$$
$$= \int_{S'(\mathbb{R}^d)} \|0\|_H^2 d\mu(\omega) = 0 \ .$$

Hence

$$\int_0^t S(s) e_i \ d\beta_i(s) = \sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\langle \int_0^t S(s) e_i \xi_j(s) ds, e_l\rangle_H H_{\epsilon_{n(i,j)}} e_l$$
$$= \sum_{j=1}^{\infty}\left(\int_0^t S(s) e_i \xi_j(s) ds\right) H_{\epsilon_{n(i,j)}} \ ,$$

as required. ■

## 3.8 Generalised $n^{th}$ Stochastic Convolution

For a strongly continuous family of continuous linear maps from $U$ to $H$, $\{V(t), \ t \geq 0\}$, such that $V(\cdot)\mathbb{W}^{(n)}(\cdot)$ is Pettis integrable on $[0,T]$, define

$$\int_0^t V(s)\delta W^{(n)}(s) := \int_0^t V(s)\mathbb{W}^{(n)}(s) ds \ , \quad t \in [0,T] \ . \tag{3.34}$$

We call $\int_0^t V(s)\delta W^{(n)}(s)$ generalised $n^{th}$ stochastic convolution. Note that in the above equation, $\mathbb{W}(\cdot)$ belongs to $S(U)_{-0}$, being defined in the same way as in section 3.3.1 with respect to $\{f_i\}_{i=1}^{\infty}$.

In the rest of this section we consider the case the when $U = H$, so that $\{V(t),\ t \geq 0\}$ is a strongly continuous family of bounded operators on $H$. However, all of the results proved in this section will apply equally to the more general case of when $U \neq H$.

**Propositon 3.15** *Suppose that for some* $T \in (0, \infty)$

$$\int_0^T \|V(t)\|^2 dt < \infty \,. \tag{3.35}$$

*Then we have the following:*

1. $V(\cdot)\mathbb{W}^{(n)}(\cdot)$ *is Pettis integrable on* $[0,T]$ *and for all* $t \in [0,T]$

$$\int_0^t V(s)\delta W^{(n)}(s) = \sum_{k=1}^{\infty} \left( \int_0^t V(s)\kappa_k^{(n)}(s) ds \right) H_{\epsilon_k} \,. \tag{3.36}$$

2. $\int_0^t V(s)\delta W^{(n)}(s)$ *is a continuous* $S(H)_{-1}$ *process on* $[0,T]$.

**Proof:** (1) We just need $I_{[0,T]}(\cdot)V(\cdot)\mathbb{W}^{(n)}(\cdot)$ to satisfy equation (3.20) of Proposition 3.8. Now

$$\begin{aligned}
& \sum_{k=1}^{\infty} (\epsilon_k!) \left( \int_{\mathbb{R}} I_{[0,T]}(t) \left\| V(t)\kappa_k^{(n)}(t) \right\|_H dt \right)^2 (2\mathbb{N})^{-(2[\![\frac{n+1}{2}]\!]+2)\epsilon_k} \\
\leq\ & \sum_{k=1}^{\infty} (1!) \left( \int_0^T \|V(t)\| \left\| \kappa_k^{(n)}(t) \right\|_H dt \right)^2 (2k)^{-2[\![\frac{n+1}{2}]\!]-2} \\
\leq\ & \sum_{k=1}^{\infty} \left( \int_0^T \|V(t)\|^2 dt \right) \left( \int_0^T \left\| \kappa_k^{(n)}(t) \right\|_H^2 dt \right) (2k)^{-2[\![\frac{n+1}{2}]\!]-2} \\
\leq\ & K_1 \sum_{k=1}^{\infty} \delta_{n(i,j),k} \left( \int_0^T \left\| \xi_j^{(n)}(t) e_i \right\|_H^2 dt \right) (2k)^{-2[\![\frac{n+1}{2}]\!]-2} \\
\leq\ & K_1 \sum_{k=1}^{\infty} \delta_{n(i,j),k} T \left( \sup_{j \in \mathbb{N}, t \in [0,T]} \left( \xi_j^{(n)}(t) \right)^2 \right) (2k)^{-2[\![\frac{n+1}{2}]\!]-2} \\
\leq\ & K_1 \sum_{k=1}^{\infty} \delta_{n(i,j),k} T \left( C_{0,T,n} (2j)^{[\![\frac{n+1}{2}]\!]} \right)^2 (2k)^{-2[\![\frac{n+1}{2}]\!]-2} \\
\leq\ & K_1 \sum_{k=1}^{\infty} \delta_{n(i,j),k} T \left( C_{0,T,n} (2k)^{[\![\frac{n+1}{2}]\!]} \right)^2 (2k)^{-2[\![\frac{n+1}{2}]\!]-2} \\
=\ & K_2 \sum_{k=1}^{\infty} (2k)^{-2} = K_2 A(2) < \infty \,,
\end{aligned}$$

using Proposition B.2. Note that $K_1$ and $K_2$ are constants depending only on $T$.

(2) Note that from the proof of part (1), we have that for all $t \in [0,T]$

$$\int_0^t V(s)\delta W^{(n)}(s) \in S(H)_{-0,-2[\![\frac{n+1}{2}]\!]-2} \subset S(H)_{-1,-2[\![\frac{n+1}{2}]\!]-2} \cdot$$

Let $q = 2[\![\frac{n+1}{2}]\!] + 2$.

By Proposition 2.6, we have that for $z \in \mathbb{K}_q$

$$\begin{aligned}\left\|\mathcal{H}\left(\int_0^t V(s)\delta W^{(n)}(s)\right)(z)\right\|_{H_C} &\leq \left\|\int_0^t V(s)\delta W^{(n)}(s)ds\right\|_{-1,-q} A(q)^{1/2} \\ &\leq K_2 A(2) A(q)^{1/2} .\end{aligned}$$

Hence $\mathcal{H}\left(\int_0^t V(s)\delta W^{(n)}(s)\right)(z)$ is continuous with respect to $t$ on $[0,T]$ for all $z \in \mathbb{K}_{2q}$, by Proposition 2.4, as the functions

$$\left\{\int_0^t V(s)\kappa_k^{(n)}(s)ds\right\}_{k=1}^{\infty} ,$$

are continuous on $[0,T]$. Hence $\int_0^t V(s)\delta W^{(n)}(s)$ is a continuous $S(H)_{-1}$ process on $[0,T]$ by Theorem 2.1. ■

**Corollary 3.4** *Suppose again that for some $T \in (0,\infty)$*

$$\int_0^T \|V(t)\|^2 dt < \infty . \tag{3.37}$$

*Then $\int_0^t V(s)\delta W^{(n)}(s)$ is a continuous $S(H)_{-0}$ process on $[0,T]$.*

**Proof:** For $q \in \mathbb{N}$

$$\begin{aligned}\left\|\int_0^t V(s)\delta W^{(n)}(s)\right\|_{-0,-q}^2 &= \sum_{k=1}^{\infty}(\epsilon_k!)\left\|\int_0^t V(s)\kappa_k^{(n)}(s)ds\right\|_H^2 (2\mathbb{N})^{-q\epsilon_k} \\ &= \sum_{k=1}^{\infty}\left\|\int_0^t V(s)\kappa_k^{(n)}(s)ds\right\|_H^2 (2\mathbb{N})^{-q\epsilon_k} \\ &= \left\|\int_0^t V(s)\delta W^{(n)}(s)\right\|_{-1,-q}^2 .\end{aligned}$$

Result then follows from Proposition 3.15 and Proposition 1.7. ■

# Chapter 4

# Stochastic Evolution Equation when $A$ Generates $C_0$-Semigroup

## 4.1 Stochastic Evolution Equation as a Differential Equation in $S(H)_{-1}$

Consider the following differential equation in $S(H)_{-1}$

$$\begin{aligned} &\frac{dX(t)}{dt} = AX(t) + B\mathbb{W}(t) \ , \quad t \in [0,T] \ , \\ &X(0) = \xi \in \mathcal{D}(A)_{-1} \ , \end{aligned} \tag{4.1}$$

where we have that:

1. $A$ is the generator of a $C_0$-semigroup $\{S(t), \ t \geq 0\}$ on $H$.
2. $B$ is a continuous linear map from $U$ to $H$.

Note that as $\{S(t), \ t \geq 0\}$ is a $C_0$-semigroup, there exists a $K \geq 0$ and $a \in \mathbb{R}$ such that $||S(t)|| \leq Ke^{at}$.

Note also that in the above equation, $\mathbb{W}(\cdot)$ belongs to $S(U)_{-0}$, being defined in the same way as in section 3.3.1 with respect to $\{f_i\}_{i=1}^{\infty}$, an orthonormal basis for $U$.

In the rest of this section we consider the case the when $U = H$. However, all of the results proved in this section and section 4.2 will apply equally to the more general case of when $U \neq H$.

**Definition 4.1** *A function $X(t) : [0,T] \to S(H)_{-1}$ is said to be a solution of (4.1) if:*

*1. $X(t) \in \mathcal{D}(A)_{-1}$ for all $t \in [0,T]$.*

2. *$X(\cdot)$ is a differentiable $S(H)_{-1}$ process on $[0,T]$ with continuous derivative on $[0,T]$.*

3. *$X(\cdot)$ satisfies (4.1).*

**Propositon 4.1** *We have that:*

1. *For all $t \in (0,\infty)$, $S(t-s)B\mathbb{W}(s)$ is Pettis integrable with respect to $s$ on $[0,t]$ and*

$$W_A(t) := \int_0^t S(t-s)B\delta W(s) = \sum_{k=1}^{\infty} \left( \int_0^t S(t-s)B\kappa_k(s)ds \right) e_k \ . \tag{4.2}$$

2. *For all $t \in [0,\infty)$, $W_A(t) \in \mathcal{D}(A)_{-1}$.*

3. *For all $T \in (0,\infty)$, $AW_A(\cdot) + B\mathbb{W}(\cdot)$ is a continuous $S(H)_{-1}$ process on $[0,T]$.*

4. *For all $T \in (0,\infty)$, $W_A(\cdot)$ is a differentiable $S(H)_{-1}$ process on $[0,T]$ and for all $t \in [0,T]$*

$$\frac{dW_A(t)}{dt} = AW_A(t) + B\mathbb{W}(t) \ . \tag{4.3}$$

**Proof:** (1) For any $t \geq 0$

$$\int_0^t \|S(t-s)B\|^2 ds \leq \int_0^t \left( Ke^{a(t-s)}\|B\|\right)^2 ds < \infty \ .$$

So by Proposition 3.13, $S(t-s)B\mathbb{W}(s)$ is Pettis integrable with respect to $s$ on $[0,t]$ and

$$\int_0^t S(t-s)B\delta W(s) = \sum_{k=1}^{\infty} \left( \int_0^t S(t-s)B\kappa_k(s)ds \right) H_{\epsilon_k} \ .$$

(2) Using integration by parts and property (5) of Proposition C.1

$$\begin{aligned}
& A\left( \int_0^t S(t-s)B\kappa_k(s)ds \right) \\
= \ & A\left( \int_0^t S(s)B\kappa_k(t-s)ds \right) \\
= \ & A\left( \left[ \left( \int_0^s S(r)Bdr \right) \kappa_k(t-s) \right]_0^t + \int_0^t \left( \int_0^s S(r)Bdr \right) \kappa_k'(t-s)ds \right) \\
= \ & A\left( \int_0^t S(r)B\kappa_k(0)dr \right) + \int_0^t A\left( \int_0^s S(r)B\kappa_k'(t-s)dr \right) ds \\
= \ & \left(S(t)-I\right)B\kappa_k(0) + \int_0^t \left(S(s)-I\right)B\kappa_k'(t-s)\ ds \ .
\end{aligned}$$

So

$$\begin{aligned}
&\left\|\sum_{k=1}^{\infty} A\left(\int_0^t S(t-s)B\kappa_k(s)ds\right) H_{\epsilon_k}\right\|_{-1,-4}^2 \\
= \;& \sum_{k=1}^{\infty}\left\|A\left(\int_0^t S(t-s)B\kappa_k(s)ds\right)\right\|_H^2 (2\mathbb{N})^{-4\epsilon_k} \\
\leq \;& \sum_{k=1}^{\infty}\left(\|S(t)-I\|\,\|B\|\|\kappa_k(0)\|_H + \int_0^t \|S(s)-I\|\|B\|\|\kappa_k'(t-s)\|_H ds\right)^2 (2k)^{-4} \\
\leq \;& \sum_{k=1}^{\infty}\delta_{n(i,j),k}\|B\|^2\Bigg((Ke^{at}+1)C \\
& + \left(\sup_{s\in[0,t]}\|\xi_j'(t-s)e_i\|_H\right)\int_0^t (Ke^{as}+1)ds\Bigg)^2 (2k)^{-4} \\
\leq \;& \sum_{k=1}^{\infty}\delta_{n(i,j),k}\|B\|^2\left((Ke^{at}+1)C + C_{0,t,1}j\int_0^t (Ke^{as}+1)ds\right)^2 (2k)^{-4} \\
\leq \;& \sum_{k=1}^{\infty}\delta_{n(i,j),k}\|B\|^2\left((Ke^{at}+1)C + C_{0,t,1}k\int_0^t (Ke^{as}+1)ds\right)^2 (2k)^{-4} \\
\leq \;& \sum_{k=1}^{\infty}k^2\|B\|^2\left((Ke^{at}+1)C + C_{0,t,1}\int_0^t (Ke^{as}+1)ds\right)^2 (2k)^{-4} \\
\leq \;& D\sum_{k=1}^{\infty}k^2(2k)^{-4} \leq DA(2) < \infty\,,
\end{aligned}$$

as required. Note that for $t \in [0,T]$, the constant $D$ need only depend on $T$.

(3) Take any $T \in (0,\infty)$. We firstly show that $AW_A(\cdot)$ is a continuous $S(H)_{-1}$ process on $[0,T]$. Now $AW_A(t) \in S(H)_{-0,-4} \subset S(H)_{-1,-4}$. So by Proposition 2.6, we have that $\mathcal{H}(AW_A(t))(z)$ exists, for all $(t,z) \in [0,T]\times\mathbb{K}_4$. Now

$$\begin{aligned}
\mathcal{H}(AW_A(t))(z) &= \sum_{k=1}^{\infty} A\left(\int_0^t S(t-s)B\kappa_k(s)ds\right) z^{\epsilon_k} \\
&= \sum_{k=1}^{\infty}\left((S(t)-I)\,B\kappa_k(0) + \int_0^t (S(s)-I)\,B\kappa_k'(t-s)\;ds\right) z^{\epsilon_k}\,.
\end{aligned}$$

The functions

$$\left\{(S(t)-I)\,B\kappa_k(0) + \int_0^t (S(s)-I)\,B\kappa_k'(t-s)\;ds\right\}_{k=1}^{\infty},$$

are continuous on $[0,T]$, for all $T \geq 0$. From the proof of (2) above, we see that there exists a $M_T < \infty$ such that

$$\|AW_A(t)\|_{-1,-4} \leq M_T\,,$$

for all $t \in [0,T]$. So by Proposition 2.6, we have that

$$\|\mathcal{H}(AW_A(t))(z)\|_{H_{\mathbb{C}}} \leq M_T A(4)^{1/2} ,$$

for all $(t,z) \in [0,T] \times \mathbb{K}_4$. So by Proposition 2.4, $\mathcal{H}(AW_A(\cdot))(z)$ is continuous with respect to $t$ on $[0,T]$ for all $z \in \mathbb{K}_8$. By Theorem 2.1, $AW_A(\cdot)$ is a continuous $S(H)_{-1}$ process on $[0,T]$.

We secondly show that $B\mathbb{W}(\cdot)$ is continuous on $[0,T]$. Now for $(t,z) \in [0,T] \times \mathbb{K}_2$

$$\begin{aligned}\left\| \sum_{k=1}^{\infty} B\kappa_k(t) z^{\epsilon_k} \right\|_{H_{\mathbb{C}}} &\leq \sum_{k=1}^{\infty} \|B\kappa_k(t) z^{\epsilon_k}\|_{H_{\mathbb{C}}} \leq \sum_{k=1}^{\infty} \|B\| \, \|\kappa_k(t)\|_H (2k)^{-2} \\ &\leq \|B\| \sum_{k=1}^{\infty} C(2k)^{-2} \leq BCA(2) < \infty ,\end{aligned}$$

where $C$ depends only on $T$. Now the functions $\{B\kappa_k(\cdot)\}_{k=1}^{\infty}$ are continuous on $[0,T]$, so by Proposition 2.4, $\mathcal{H}(B\mathbb{W}(\cdot))(z)$ is continuous with respect to $t$ on $[0,T]$ for all $z \in \mathbb{K}_4$. By Theorem 2.1, $B\mathbb{W}(\cdot)$ is a continuous $S(H)_{-1}$ process on $[0,T]$.

(4) Take any $T \in (0,\infty)$. Now from the proof of Proposition 3.13 and (1) of this proposition, we have that $W_A(t)$ belongs to $S(H)_{-0,-2} \subset S(H)_{-1,-2}$, for all $t > 0$. So by Proposition 2.6 we have that $\mathcal{H}(W_A(t))(z)$ exists, for all $(t,z) \in [0,T] \times \mathbb{K}_2$.

From the proof of part (3) of this proposition, we see that $\mathcal{H}\left(AW_A(t) + B\mathbb{W}(t)\right)(z)$ exists and is bounded for $(t,z) \in \mathbb{K}_4 \times [0,T]$, for all $T > 0$. Also

$$\frac{d}{dt}\int_0^t S(t-s)B\kappa_k(s)ds = A\int_0^t S(t-s)B\kappa_k(s)ds + B\kappa_k(t) .$$

So by Proposition 2.5, $\mathcal{H}(W_A(\cdot))(z)$ is differentiable with respect to $t$ on $[0,T]$ for all $z \in \mathbb{K}_8$ and for all $(t,z) \in [0,T] \times \mathbb{K}_8$

$$\begin{aligned}&\frac{d}{dt}\left(\sum_{k=1}^{\infty}\left(\int_0^t S(t-s)B\kappa_k(s)ds\right) z^{\epsilon_k}\right) \\ &= \sum_{k=1}^{\infty}\frac{d}{dt}\left(\int_0^t S(t-s)B\kappa_k(s)ds\right) z^{\epsilon_k} \\ &= \sum_{k=1}^{\infty}\left(A\left(\int_0^t S(t-s)B\kappa_k(s)ds\right) + B\kappa_k(t)\right) z^{\epsilon_k} \\ &= \mathcal{H}\left(\sum_{k=1}^{\infty} A\left(\int_0^t S(t-s)B\kappa_k(s)ds\right) H_{\epsilon_k} + B\kappa_k(t)H_{\epsilon_k}\right)(z) \\ &= \mathcal{H}\left(AW_A(t) + B\mathbb{W}(t)\right)(z) .\end{aligned}$$

So by Theorem 2.2, $W_A(\cdot)$ is a differentiable $S(H)_{-1}$ process on $[0,T]$, with continuous derivative $AW_A(\cdot) + B\mathbb{W}(\cdot)$. ■

**Propositon 4.2** *We have that:*

1. *For all* $t \in [0, \infty)$, $S(t)\xi$ *belongs to* $\mathcal{D}(A)_{-1}$.
2. *For all* $T \in (0, \infty)$, $AS(\cdot)\xi$ *is a continuous* $S(H)_{-1}$ *process on* $[0, T]$.
3. *For all* $T \in (0, \infty)$, $S(\cdot)\xi$ *is a differentiable* $S(H)_{-1}$ *process on* $[0, T]$ *and for all* $t \in [0, T]$

$$\frac{dS(t)\xi}{dt} = AS(t)\xi \; . \tag{4.4}$$

**Proof:** (1) Let $\xi$ have expansion

$$\xi = \sum_{\alpha \in \mathcal{J}} c_\alpha H_\alpha \; .$$

As $\xi \in \mathcal{D}(A)_{-1}$, there exists a $q \in \mathbb{N} \setminus \{1\}$ such that

$$\sum_{\alpha \in \mathcal{J}} \|Ac_\alpha\|_H^2 (2\mathbb{N})^{-q\alpha} < \infty \; .$$

For this $q$

$$\begin{aligned} \sum_{\alpha \in \mathcal{J}} \|AS(t)c_\alpha\|_H^2 (2\mathbb{N})^{-q\alpha} &= \sum_{\alpha \in \mathcal{J}} \|S(t)Ac_\alpha\|_H^2 (2\mathbb{N})^{-q\alpha} \\ &\leq \sum_{\alpha \in \mathcal{J}} \|S(t)\|^2 \; \|Ac_\alpha\|_H^2 (2\mathbb{N})^{-q\alpha} < \infty \; , \end{aligned}$$

since $c_\alpha \in \mathcal{D}(A)$ for all $\alpha \in \mathcal{J}$.

(2) Take any $T \in (0, \infty)$. We firstly show that $\mathcal{H}(AS(\cdot)\xi)(z)$ is continuous with respect to $t$ on $[0, T]$ for all $z \in \mathbb{K}_{2q}$. By Proposition 2.6, we have that $\mathcal{H}(AS(t)\xi)(z)$ exists for all $(t, z) \in [0, T] \times \mathbb{K}_q$ (and hence $\mathbb{K}_{2q}$). Now for all $(t, z) \in [0, T] \times \mathbb{K}_q$

$$\begin{aligned} \left\| \sum_{\alpha \in \mathcal{J}} AS(t)c_\alpha z^\alpha \right\|_{H_\mathbb{C}} &= \left\| \sum_{\alpha \in \mathcal{J}} S(t)Ac_\alpha z^\alpha \right\|_{H_\mathbb{C}} \leq \sum_{\alpha \in \mathcal{J}} \|S(t)\| \; \|Ac_\alpha\|_H \; |z^\alpha| \\ &\leq Ke^{at} \sum_{\alpha \in \mathcal{J}} \|Ac_\alpha\|_H \; |z^\alpha| \leq Ke^{at} \|A\xi\|_{-1,-q} A(q)^{1/2} \\ &\leq M < \infty \; , \end{aligned}$$

where $M$ is a constant depending only on $T$. Also, the functions

$$\{AS(\cdot)c_\alpha = S(\cdot)Ac_\alpha\}_{\alpha \in \mathcal{J}} \; ,$$

are continuous on $[0, T]$. So by Proposition 2.4, $\mathcal{H}(AS(\cdot)\xi)(z)$ is continuous with respect to $t$ on $[0, T]$ for all $z \in \mathbb{K}_{2q}$. By Theorem 2.1, $AS(\cdot)\xi$ is a continuous $S(H)_{-1}$ process on $[0, T]$.

(3) Take any $T \in (0, \infty)$. Now there exists a $q_1 \in \mathbb{N} \setminus \{1\}$ such that $\xi \in S(H)_{-1,-q_1}$ and hence that $S(t)\xi \in S(H)_{-1,-q_1}$. By Proposition 2.6, we have that $\mathcal{H}(S(t)\xi)(z)$ exists for all $z \in \mathbb{K}_{q_1}$.

Now for $(t, z) \in [0, T] \times \mathbb{K}_{\max\{q,q_1\}}$ we have that from the proof of part (2) of this Proposition, $\mathcal{H}(AS(t)\xi)(z)$ exists and is bounded. Also

$$\frac{d}{dt} S(t) c_\alpha = AS(t) c_\alpha ,$$

for all $\alpha \in \mathcal{J}$. So by Proposition 2.5, $\mathcal{H}(S(\cdot)\xi)(z)$ is differentiable with respect to $t$ on $[0, T]$ for all $z \in \mathbb{K}_{2q_1}$ and for $(t, z) \in [0, T] \times \mathbb{K}_{\max\{2q,2q_1\}}$

$$\begin{aligned} \frac{d}{dt}\left(\sum_{\alpha\in\mathcal{J}} S(t)c_\alpha z^\alpha\right) &= \sum_{\alpha\in\mathcal{J}} \frac{d}{dt}\left(S(t)c_\alpha z^\alpha\right) = \sum_{\alpha\in\mathcal{J}} AS(t)c_\alpha z^\alpha \\ &= \mathcal{H}\left(\sum_{\alpha\in\mathcal{J}} AS(t)c_\alpha z^\alpha\right)(z) = \mathcal{H}\left(AS(t)\xi\right)(z) . \end{aligned}$$

So by Theorem 2.2, $S(\cdot)\xi$ is a differentiable $S(H)_{-1}$ process on $[0, T]$ with continuous derivative $AS(\cdot)\xi$. ■

**Theorem 4.1** *The $S(H)_{-1}$ process*

$$X(t) = S(t)\xi + W_A(t) , \tag{4.5}$$

*is a solution to (4.1).*

**Proof:** We need to show that $X(\cdot)$ satisfies the requirements for a solution to (4.1).

(1) From the two previous propositions, we have that for all $t \in [0, T]$, $S(t)\xi$, $W_A(t) \in \mathcal{D}(A)_{-1}$. So by Proposition 3.10, for all $t \in [0, T]$

$$X(t) = S(t)\xi + W_A(t) ,$$

belongs to $\mathcal{D}(A)_{-1}$.

(2) From the two previous propositions, we have that $S(\cdot)\xi$ and $W_A(\cdot)$ are differentiable $S(H)_{-1}$ processes on $[0, T]$. So $X(\cdot)$ is a differentiable $S(H)_{-1}$ process on $[0, T]$.

For all $t \in [0, T]$

$$\begin{aligned} AX(t) + B\mathbb{W}(t) &= A\left(S(t)\xi + W_A(t)\right) + B\mathbb{W}(t) \\ &= AS(t)\xi + AW_A(t) + B\mathbb{W}(t) . \end{aligned}$$

We know from the two previous propositions that $AS(\cdot)\xi$, $AW_A(\cdot)$ and $B\mathbb{W}(\cdot)$ are continuous $S(H)_{-1}$ processes on $[0, T]$. Hence $AX(\cdot)+B\mathbb{W}(\cdot)$ is a continuous $S(H)_{-1}$ process on $[0, T]$.

(3) For all $t \in [0,T]$

$$\begin{aligned}\frac{dX(t)}{dt} &= \frac{d}{dt}\left(S(t)\xi + W_A(t)\right) = \frac{d}{dt}\left(S(t)\xi\right) + \frac{d}{dt}\left(W_A(t)\right) \\ &= AS(t)\xi + AW_A(t) + B\mathbb{W}(t) = AX(t) + B\mathbb{W}(t) ,\end{aligned}$$

as required. ■

**Propositon 4.3** *If $X(\cdot)$ is a solution to (4.1), then*

$$X(t) = S(t)\xi + W_A(t) . \tag{4.6}$$

**Proof:** Suppose

$$X(t) = \sum_{\alpha\in\mathcal{J}} d_\alpha(t)H_\alpha ,$$

is a solution to (4.1). By the conditions placed on a solution and Theorem 2.2, there exists a $q \in \mathbb{N}\setminus\{1\}$ such that for all $(t,z) \in [0,T]\times\mathbb{K}_q$

$$\begin{aligned}&\frac{d\widetilde{X}(t,z)}{dt} \\ &= \mathcal{H}\left(AX(t) + B\mathbb{W}(t)\right)(z) = \mathcal{H}\left(\sum_{\alpha\in\mathcal{J}} Ad_\alpha(t)H_\alpha + \sum_{k=1}^{\infty} B\kappa_k(t)H_{\epsilon_k}\right)(z) \\ &= \sum_{\alpha\in\mathcal{J}} Ad_\alpha(t)z^\alpha + \sum_{k=1}^{\infty} B\kappa_k(t)z^{\epsilon_k} = A\left(\sum_{\alpha\in\mathcal{J}} d_\alpha(t)z^\alpha\right) + B\left(\sum_{k=1}^{\infty}\kappa_k(t)z^{\epsilon_k}\right) \\ &= A\widetilde{X}(t,z) + B\widetilde{\mathbb{W}}(t,z) ,\end{aligned}$$

as $A$ is closed and $B$ is bounded. So by Theorem C.1, for $z \in \mathbb{K}_q$

$$\widetilde{X}(t,z) = S(t)\widetilde{\xi}(z) + \int_0^t S(t-s)B\widetilde{\mathbb{W}}(t,z) .$$

Hence for all $n \in \mathbb{N}$, for $z \in \mathbb{K}_q^n$

$$\begin{aligned}\widetilde{X}(t,z) &= S(t)\widetilde{\xi}(z) + \int_0^t S(t-s)B\widetilde{\mathbb{W}}(t,z) \\ &= S(t)\sum_{\alpha\in\mathcal{J}} c_\alpha z^\alpha + \int_0^t S(t-s)B\left(\sum_{k=1}^{\infty}\kappa_k(s)z^{\epsilon_k}\right)ds \\ &= S(t)\sum_{\alpha\in\mathcal{J}} c_\alpha z^\alpha + \int_0^t S(t-s)B\left(\sum_{k=1}^{n}\kappa_k(s)z^{\epsilon_k}\right)ds \\ &= S(t)\sum_{\alpha\in\mathcal{J}} c_\alpha z^\alpha + \sum_{k=1}^{n}\left(\int_0^t S(t-s)B\kappa_k(s)ds\right)z^{\epsilon_k} \\ &= \sum_{\alpha\in\mathcal{J}} S(t)c_\alpha z^\alpha + \sum_{k=1}^{\infty}\left(\int_0^t S(t-s)B\kappa_k(s)ds\right)z^{\epsilon_k} .\end{aligned}$$

So by Proposition 2.2

$$X(t) = \sum_{\alpha\in\mathcal{J}} S(t)c_\alpha H_\alpha + \sum_{k=1}^{\infty}\left(\int_0^t S(t-s)B\kappa_k(s)ds\right)H_{\epsilon_k}$$
$$= S(t)\xi + W_A(t) ,$$

as required. ■

## 4.2 Stochastic Evolution Equation as an Integral Equation in $S(H)_{-1}$

Consider the following integral equation in $S(H)_{-1}$

$$X(t) = X(0) + \int_0^t AX(s)ds + BW(t) , \quad t \in [0,T] ,$$
$$X(0) = \xi \in \mathcal{D}(A)_{-1} , \tag{4.7}$$

where $A$, $B$, and $\{S(t),\ t \geq 0\}$ are the same as in the previous section and $U = H$.

**Definition 4.2** *A function $X(t) : [0,T] \to S(H)_{-1}$ is said to be a solution of (4.7) if:*

1. *$X(t) \in \mathcal{D}(A)_{-1}$ for all $t \in [0,T]$.*
2. *$X(\cdot)$ is a differentiable $S(H)_{-1}$ process on $[0,T]$ with continuous derivative on $[0,T]$.*
3. *$X(\cdot)$ satisfies (4.7).*

**Propositon 4.4** *A $S(H)_{-1}$ process $X(t) : [0,T] \to S(H)_{-1}$ is solution to (4.1) if and only if it is a solution to (4.7)*

**Proof:** Conditions (1) and (2) for a solution of (4.1) correspond to conditions (1) and (2) of (4.7)

Condition (3) of (4.1) is equivalent to condition (3) of (4.7) by Proposition 1.9, that is

$$\frac{dX(t)}{dt} = AX(t) + B\mathbb{W}(t) ,$$

if and only if

$$\int_0^t \frac{dX(s)}{ds} = X(t) - X(0) = \int_0^t AX(s)ds + \int_0^t B\mathbb{W}(s)ds$$
$$= \int_0^t AX(s)ds + BW(t) ,$$

as required. ■

## 4.3 Examples

### 4.3.1 Stochastic Heat Equation

Consider the stochastic Heat equation

$$\begin{aligned} &d_tX(t,x) = \Delta_x X(t,x)dt + dW(t,x) \ , \quad t \in [0,T] \ , \ x \in \mathcal{O} \ , \\ &X(t,x) = 0 \ , \quad t \in [0,T] \ , \ x \in \partial\mathcal{O} \ , \\ &X(0,x) = 0 \ , \quad x \in \mathcal{O} \ , \end{aligned} \tag{4.8}$$

where $dW(t,x)$ is temporal and spatial white noise. $\mathcal{O}$ is the open, bounded set in $\mathbb{R}^N$

$$\mathcal{O} = \left\{ x \in \mathbb{R}^N \ ; \ 0 < x_i < a_i \ , \ i = 1,2\ldots,N \right\} \ .$$

Equation (4.8) can be considered in the abstract form in $L^2(\mathcal{O})$

$$\begin{aligned} &dX(t) = AX(t)dt + dW(t) \ , \quad t \in [0,T] \ , \\ &X(0) = 0 \ , \end{aligned} \tag{4.9}$$

where

$$\begin{aligned} &\mathcal{D}(A) := H^2(\mathcal{O}) \cap H_0^1(\mathcal{O}) \ , \\ &A = \Delta_x \ , \end{aligned}$$

where the Laplace operator $\Delta_x$ is understood in the sense of distributions.

$A$ is a self adjoint, strictly non-negative operator on $L^2(\mathcal{O})$ with eigenbasis $\{e_k\}_{k=1}^{\infty}$ and real negative eigenvalues $-\lambda_1 \geq -\lambda_2 \geq \ldots$. $A$ is also the generator of a $C_0$-semigroup $\{S(t), \ t \geq 0\}$ on $L^2(\mathcal{O})$ given by

$$S(t)u := \sum_{k=1}^{\infty} u_k e^{-\lambda_k t} e_k \ ,$$

where $u$ has the expansion in $L^2(\mathcal{O})$

$$u = \sum_{i=1}^{\infty} u_k e_k \ .$$

See section D.1 of appendix D for more details regarding $A$ and $\{S(t), \ t \geq 0\}$.

Let $H = L^2(\mathcal{O})$. We consider equation (4.9) as the following integral equation in $S(H)_{-1}$

$$\begin{aligned} &X(t) = \int_0^t AX(s)ds + W(t) \ , \quad t \in [0,T] \ , \\ &X(0) = 0 \in \mathcal{D}(A)_{-1} \ , \end{aligned} \tag{4.10}$$

where we have

$$W(t) = \sum_{i=1}^{\infty} \beta_i(t) e_i \ ,$$

with

$$\beta_i(t) = \sum_{j=1}^{\infty} \left( \int_0^t \xi_j(s) ds \right) H_{\epsilon_{n(i,j)}} \ .$$

By Theorem 4.1, Proposition 4.3 and Proposition 4.4, equation (4.10) has the unique solution in $S(H)_{-1}$

$$X(t) = \int_0^t S(t-s) \delta W(s) = W_A(t) \ .$$

Note that for all $t \in [0,T]$, $X(t)$ belongs to $S(H)_{-0}$ by definition of generalised stochastic convolution.

Now for $N = 1$, the operator $S_T$ defined by

$$S_T u := \int_0^T S(s) S^*(s) u \ ds \ , \quad u \in L^2(\mathcal{O}) \ ,$$

is trace class as

$$\begin{aligned} \mathrm{Tr}(S_T) &= \sum_{k=1}^{\infty} \langle S_T e_k, e_k \rangle_{L^2(\mathcal{O})} = \sum_{k=1}^{\infty} \langle \int_0^T S(t) S^*(t) e_k \ dt, e_k \rangle_{L^2(\mathcal{O})} \\ &= \sum_{k=1}^{\infty} \int_0^T \langle S(t) S^*(t) e_k, e_k \rangle_{L^2(\mathcal{O})} dt = \sum_{k=1}^{\infty} \int_0^T \langle S^*(t) e_k, S^*(t) e_k \rangle_{L^2(\mathcal{O})} dt \\ &= \sum_{k=1}^{\infty} \int_0^T \| S(t) e_k \|^2_{L^2(\mathcal{O})} ds = \sum_{k=1}^{\infty} \int_0^T e^{-2\lambda_k t} dt = \sum_{k=1}^{\infty} \left[ \frac{e^{-2\lambda_k t}}{-2\lambda_k} \right]_0^T \\ &= \sum_{k=1}^{\infty} \frac{1}{2\lambda_k} \left( 1 - e^{-2\lambda_k T} \right) \leq \sum_{k=1}^{\infty} \frac{1}{2\lambda_k} < \infty \ . \end{aligned}$$

So by Proposition 3.14 and Theorem 3.1, for $N = 1$, $X(\cdot) = W_A(\cdot)$ belongs to $L^2(H)$ and agrees with the normal notion of stochastic convolution, that is

$$X(t) = W_A(t) \ = \ \int_0^t S(t-s) dW(t) := \sum_{i=1}^{\infty} \int_0^t S(t-s) e_i \ d\beta_i(s) \ ,$$

where summation is in $L^2(H)$. Hence if $N = 1$, $W_A(\cdot)$ agrees with the weak solution of equation (4.9) found in [2].

### 4.3.2 Stochastic Wave Equation

Consider the stochastic Wave equation

$$\begin{aligned} & dY'_t(t,x) = \frac{\partial^2}{\partial x^2} Y(t,x) dt + dW(t,x) \ , \quad t \in [0,T] \ , \ x \in \Omega = (0,1) \ , \\ & Y(t,0) = Y(t,1) = 0 \ , \quad t \in [0,T] \ , \\ & Y(0,x) = Y_0(x) \ , \ Y'_t(0,x) = Y_1(x) \ , \quad x \in \Omega \ , \end{aligned} \tag{4.11}$$

where $dW(t,x)$ is temporal and spatial white noise. Let

$$X(t) = \begin{pmatrix} Y(t,x) \\ Y_t'(t,x) \end{pmatrix} , \quad X(0) = \begin{pmatrix} Y_0(x) \\ Y_1(x) \end{pmatrix} ,$$

with $X(t) \in H_0^1(\Omega) \times L^2(\Omega)$ for all $t \in [0,T]$. Also define the operator $A$ on $L^2(\Omega)$ as

$$\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega) \subset L^2(\Omega) ,$$
$$A = \frac{d^2}{dx^2} ,$$

where $\frac{d^2}{dx^2}$ is understood in the sense of distributions. If $Y_0 \in \mathcal{D}(A)$ and $Y_1 \in H_0^1(\Omega)$, then we can write equation (4.11) in the abstract form in $H_0^1(\Omega) \times L^2(\Omega)$

$$\begin{aligned} dX(t) &= \mathcal{A}_1 X(t)dt + BdW(t) , \quad t \in [0,T] , \\ X(0) &= \xi \in \mathcal{D}(\mathcal{A}_1) , \end{aligned} \tag{4.12}$$

where

$$\mathcal{D}(\mathcal{A}_1) = \mathcal{D}(A) \times H_0^1(\Omega) ,$$
$$\mathcal{A}_1 = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix} ,$$

and $B : L^2(\Omega) \to H_0^1(\Omega) \times L^2(\Omega)$ is defined by

$$\mathcal{D}(B) = L^2(\Omega) ,$$
$$Bu = \begin{pmatrix} 0 \\ u \end{pmatrix} .$$

Note that $W(\cdot)$ is a $L^2(\Omega)$-valued Wiener process.

$\mathcal{A}_1$ generates a $C_0$-semigroup $U(t)$ on $H_0^1(\Omega) \times L^2(\Omega)$. See section D.2 of appendix D for more details regarding $\mathcal{A}_1$ and $U(t)$.

Let $H = H_0^1(\Omega) \times L^2(\Omega)$. We consider equation (4.12) as the following integral equation in $S(H)_{-1}$

$$\begin{aligned} X(t) &= X(0) + \int_0^t \mathcal{A}_1 X(s)ds + BW(t) , \quad t \in [0,T] , \\ X(0) &= \xi \in \mathcal{D}(\mathcal{A}_1)_{-1} , \end{aligned} \tag{4.13}$$

where we have that

$$W(t) = \sum_{i=1}^{\infty} \beta_i(t) f_i ,$$

where $\{f_i\}_{i=1}^{\infty}$ is an orthonormal basis for $L^2(\Omega)$ and

$$\beta_i(t) = \sum_{j=1}^{\infty} \left( \int_0^t \xi_j(s)ds \right) H_{\epsilon_{n(i,j)}} .$$

Hence by Theorem 4.1, Proposition 4.3 and Proposition 4.4, equation (4.13) has the unique solution in $S(H)_{-1}$

$$X(t) = U(t)\xi + \int_0^t U(t-s)B\delta W(s) .$$

# Chapter 5

# Stochastic Evolution Equation when $A$ Generates an $n$-times Integrated Semigroup

## 5.1 Stochastic Evolution Equation when $A$ Generates a 1-times Integrated Semigroup

Consider the integral equation in $S(H)_{-1}$

$$\begin{aligned} X(t) &= X(0) + A\int_0^t X(s)ds + BW(t)\ , \quad t \in [0,T]\ , \\ X(0) &= \xi \in \mathcal{D}(A)_{-1}\ , \end{aligned} \tag{5.1}$$

where we have that:

1. $A$ is a closed, densely defined operator on $H$.
2. $A$ generates a non degenerate, 1-times integrated, exponentially bounded semigroup $\{V(t),\ t \geq 0\}$.
3. $B$ is a continuous linear map from $U$ to $H$.

Note that as $\{V(t),\ t \geq 0\}$ is exponentially bounded, there exists a $M \geq 0$ and $a \in \mathbb{R}$ such that for all $t \in [0,\infty)$, we have that $||V(t)|| \leq Me^{at}$.

Note also that in the above equation, $W(\cdot)$ belongs to $S(U)_{-0}$, being defined in the same way as in section 3.3 with respect to $\{f_i\}_{i=1}^{\infty}$, an orthonormal basis for $U$.

In the rest of this section we consider the case the when $U = H$. However, all of the results proved in this section will apply equally to the more general case of when $U \neq H$.

**Definition 5.1** *A function $X(t) : [0,T] \to S(H)_{-1}$ is said to be a solution of (5.1) if:*

1. *$X(\cdot)$ is a continuous $S(H)_{-1}$ process on $[0,T]$.*
2. *For all $t \in [0,T]$, we have that*

$$\int_0^t X(s)ds \in \mathcal{D}(A)_{-1} . \tag{5.2}$$

3. *$X(\cdot)$ satisfies equation (5.1).*

**Propositon 5.1** *For all $T \in (0,\infty)$:*

1. *$V(\cdot)B\mathbb{W}(0)$ is a continuous $S(H)_{-1}$ process on $[0,T]$.*
2. *For all $t \in [0,T]$, $\int_0^t V(s)B\mathbb{W}(0)ds \in \mathcal{D}(A)_{-1}$.*

**Proof:** (1) We know that $\mathbb{W}(0) \in S(H)_{-0,-4} \subset S(H)_{-1,-4}$, so for $(t,z) \in [0,T] \times \mathbb{K}_4$

$$\begin{aligned}
&\|\mathcal{H}(V(t)B\mathbb{W}(0))(z)\|_{H_{\mathbb{C}}} \\
= &\left\|\sum_{k=1}^{\infty} V(t)B\kappa_k(0)z^{\epsilon_k}\right\|_{H_{\mathbb{C}}} \leq \|V(t)\| \; \|B\| \left\|\sum_{k=1}^{\infty} \kappa_k(0)z^{\epsilon_k}\right\|_{H_{\mathbb{C}}} \\
\leq &\; Me^{at}\|B\| \; \|\mathbb{W}(0)\|_{-1,-4}A(4)^{1/2} \leq K < \infty ,
\end{aligned}$$

where the constant $K$ depends only on $T$. So by Proposition 2.4, $\mathcal{H}(V(\cdot)B\mathbb{W}(0))(z)$ is continuous with respect to $t$ on $[0,T]$ for all $z \in \mathbb{K}_8$, as the functions $\{V(\cdot)B\kappa_k(0)\}_{k=1}^{\infty}$ are continuous on $[0,T]$. Hence $V(\cdot)B\mathbb{W}(0)$ is a continuous $S(H)_{-1}$ process on $[0,T]$, by Theorem 2.1.

(2) Since $V(\cdot)B\mathbb{W}(0)$ is a continuous $S(H)_{-1}$ process on $[0,T]$, it is also an integrable $S(H)_{-1}$ process on $[0,T]$ such that

$$\int_0^t V(s)B\mathbb{W}(0)ds = \sum_{k=1}^{\infty}\left(\int_0^t V(s)B\kappa_k(0)ds\right) H_{\epsilon_k} .$$

Now for all $k \in \mathbb{N}$, each $B\kappa_k(0) \in \overline{\mathcal{D}(A)} = H$. So by property (2) of Proposition C.2, each $\int_0^t V(s)B\kappa_k(0)ds \in \mathcal{D}(A)$ and

$$\begin{aligned}
&\sum_{k=1}^{\infty}\left\|A\int_0^t V(s)B\kappa_k(0)ds\right\|_H^2 (2\mathbb{N})^{-4\epsilon_k} \\
= &\sum_{k=1}^{\infty}\|V(t)B\kappa_k(0) - tB\kappa_k(0)\|_H^2 (2\mathbb{N})^{-4\epsilon_k} = \|V(t)B\mathbb{W}(0) - tB\mathbb{W}(0)\|_{-1,-4}^2 \\
\leq &\; (\|V(t)B\mathbb{W}(0)\|_{-1,-4} + \|tB\mathbb{W}(0)\|_{-1,-4})^2 < \infty ,
\end{aligned}$$

as required. ■

**Propositon 5.2** *We have that:*

1. *For all* $t \in [0, \infty)$, $V(t-s)B\mathbb{W}^{(1)}(s)$ *is Pettis integrable with respect to* $s$ *on* $[0, t]$ *and*

$$W_A^{(1)}(t) := \int_0^t V(t-s)B\delta W^{(1)}(s) = \sum_{k=1}^{\infty}\left(\int_0^t V(t-s)B\kappa_k'(s)ds\right)e_k \ . \quad (5.3)$$

2. *For all* $T \in (0, \infty)$, $W_A^{(1)}(\cdot)$ *is a continuous* $S(H)_{-1}$ *process on* $[0, T]$.

3. *For all* $t \in [0, \infty)$, $\int_0^t W_A^{(1)}(\tau)d\tau \in \mathcal{D}(A)_{-1}$ *and*

$$W_A^{(1)}(t) + V(t)B\mathbb{W}(0) = A\int_0^t \left(W_A^{(1)}(s) + V(s)B\mathbb{W}(0)\right) ds + BW(t) \ . \quad (5.4)$$

**Proof:** (1) and (2). For any $t \geq 0$

$$\int_0^t \|V(t-s)B\|^2 ds \leq \int_0^t (\|V(t-s)\| \ \|B\|)^2 ds \leq \int_0^t \left(Me^{a(t-s)}\|B\|\right)^2 ds < \infty \ .$$

So by Proposition 3.15, $S(t-s)B\mathbb{W}^{(1)}(s)$ is Pettis integrable with respect to $s$ on $[0, t]$ and

$$\int_0^t V(t-s)B\delta W^{(1)}(s) = \sum_{k=1}^{\infty}\left(\int_0^t V(t-s)B\kappa_k'(s)ds\right)H_{\epsilon_k} \ ,$$

is a continuous $S(H)_{-1}$ process on $[0, T]$, for all $T \in (0, \infty)$.

(3) From Proposition C.5 we have that

$$\int_0^t \int_0^\tau V(\tau - s)B\kappa_k'(s)ds \ d\tau \in \mathcal{D}(A) \ .$$

and

$$\begin{aligned}
&\int_0^t V(t-s)B\kappa_k'(s)ds + V(t)B\kappa_k(0) \\
&= A\int_0^t \left(\int_0^\tau V(\tau - s)B\kappa_k'(s)ds + V(\tau)B\kappa_k(0)\right) d\tau + \int_0^t B\kappa_k(s)ds \\
&= A\int_0^t \left(\int_0^\tau V(\tau - s)B\kappa_k'(s)ds + V(\tau)B\kappa_k(0)\right) d\tau + B\theta_k(t) \ .
\end{aligned}$$

So

$$\begin{aligned}
&\sum_{k=1}^{\infty}\left\|A\int_0^t\int_0^\tau V(\tau - s)B\kappa_k'(s)ds \ d\tau\right\|_H^2 (2\mathbb{N})^{-4\epsilon_k} \\
&= \sum_{k=1}^{\infty}\left\|\int_0^t V(t-s)B\kappa_k'(s)ds + V(t)B\kappa_k(0)\right.
\end{aligned}$$

$$-A\int_0^t V(s)B\kappa_k(0)ds - B\theta_k(t)\Big\|_H^2 (2\mathbb{N})^{-4\epsilon_k}$$

$$= \left\|W_A^{(1)}(t) + V(t)B\mathbb{W}(0) - A\int_0^t V(s)B\mathbb{W}(0)ds - BW(t)\right\|_{-1,-4}^2$$

$$\leq \Big(\|W_A^{(1)}(t)\|_{-1,-4} + \|V(t)B\mathbb{W}(0)\|_{-1,-4}$$

$$+ \left\|A\int_0^t V(s)B\mathbb{W}(0)ds\right\|_{-1,-4} + \|BW(t)\|_{-1,-4}\Big)^2$$

$$< \infty .$$

Hence $\int_0^t W_A^{(1)}(\tau)d\tau \in \mathcal{D}(A)_{-1}$, using

$$\int_0^t W_A^{(1)}(\tau)d\tau = \sum_{k=1}^{\infty}\left(\int_0^t\int_0^\tau V(\tau-s)B\kappa_k'(s)ds\, d\tau\right)H_{\epsilon_k} ,$$

as $W_A^{(1)}(\cdot)$ is a continuous $S(H)_{-1}$ process on $[0,T]$. Now

$$W_A^{(1)}(t) + V(t)\mathbb{W}(0)$$

$$= \sum_{k=1}^{\infty}\left(\int_0^t V(t-s)B\kappa_k'(s)ds + V(t)B\kappa_k(0)\right)H_{\epsilon_k}$$

$$= \sum_{k=1}^{\infty}\left(A\int_0^t\left(\int_0^\tau V(\tau-s)B\kappa_k'(s)ds + V(\tau)B\kappa_k(0)\right)d\tau + B\theta_k(t)\right)H_{\epsilon_k}$$

$$= A\int_0^t\left(W_A^{(1)}(\tau) + V(\tau)B\mathbb{W}(0)\right)d\tau + BW(t) ,$$

as required. ■

**Propositon 5.3** *For all $T \in (0,\infty)$:*

1. *For all $t \in [0,T]$, $V(t)\xi$ belongs to $\mathcal{D}(A)_{-1}$ and*

$$AV(t)\xi = V(t)A\xi . \tag{5.5}$$

2. *$AV(\cdot)\xi$ is a continuous $S(H)_{-1}$ process on $[0,T]$.*
3. *$V(\cdot)\xi$ is a differentiable $S(H)_{-1}$ process on $[0,T]$ with continuous derivative on $[0,T]$ given by*

$$\frac{d}{dt}V(t)\xi = AV(t)\xi + \xi , \tag{5.6}$$

*for all $t \in [0,T]$.*

4. *For all $t \in [0,T]$*

$$V(t)\xi = \int_0^t \frac{d}{d\tau}V(\tau)\xi d\tau . \tag{5.7}$$

**Proof:** (1) Let $\xi$ have form

$$\xi = \sum_{\alpha \in \mathcal{J}} b_\alpha H_\alpha \ .$$

As $\xi \in \mathcal{D}(A)_{-1}$, there exists a $q \in \mathbb{N} \setminus \{1\}$ such that

$$\sum_{\alpha \in \mathcal{J}} \|Ab_\alpha\|_H^2 (2\mathbb{N})^{-q\alpha} < \infty \ .$$

Note that for all $\alpha \in \mathcal{J}$, $b_\alpha \in \mathcal{D}(A)$. So by property (1) of Proposition C.2, $V(t)Ab_\alpha = AV(t)b_\alpha$, which gives

$$\begin{aligned}\sum_{\alpha \in \mathcal{J}} \|AV(t)b_\alpha\|_H^2 (2\mathbb{N})^{-q\alpha} &= \sum_{\alpha \in \mathcal{J}} \|V(t)Ab_\alpha\|_H^2 (2\mathbb{N})^{-q\alpha} \\ &\le \sum_{\alpha \in \mathcal{J}} \|V(t)\|^2 \|Ab_\alpha\|_H^2 (2\mathbb{N})^{-q\alpha} < \infty \ .\end{aligned}$$

Therefore $V(t)\xi \in \mathcal{D}(A)_{-1}$ and

$$AV(t)\xi = \sum_{\alpha \in \mathcal{J}} AV(t)b_\alpha H_\alpha = \sum_{\alpha \in \mathcal{J}} V(t)Ab_\alpha H_\alpha = V(t)A\xi \ ,$$

as required.

(2) We show that $V(\cdot)A\xi$ is a continuous $S(H)_{-1}$ process on $[0,T]$. Now the functions

$$\{V(\cdot)Ab_\alpha\}_{\alpha \in \mathcal{J}} \ ,$$

are continuous on $[0,T]$ and for $(t,z) \in [0,T] \times \mathbb{K}_q$

$$\begin{aligned}\left\| \sum_{\alpha \in \mathcal{J}} V(t)Ab_\alpha z^\alpha \right\|_{H_{\mathbb{C}}} &\le \|V(t)\| \left\| \sum_{\alpha \in \mathcal{J}} Ab_\alpha z^\alpha \right\|_{H_{\mathbb{C}}} \le \|V(t)\| \ \|A\xi\|_{-1,-q} A(q)^{1/2} \\ &\le Me^{at} \|A\xi\|_{-1,-q} A(q)^{1/2} \le K < \infty \ ,\end{aligned}$$

where $K$ is a constant depending only on $T$. So by Proposition 2.4, $\mathcal{H}\left( (V(\cdot)A\xi) \right)(z)$ is continuous with respect to $t$ on $[0,T]$ for all $z \in \mathbb{K}_{2q}$. Hence $V(\cdot)A\xi$ is a continuous $S(H)_{-1}$ process on $[0,T]$, by Theorem 2.1.

(3) For all $\alpha \in \mathcal{J}$, $b_\alpha \in \mathcal{D}(A)$. So by property (3) of Proposition C.2

$$\frac{d}{dt} V(t)b_\alpha = V(t)Ab_\alpha + b_\alpha = AV(t)b_\alpha + b_\alpha \ .$$

By part (1) and (2) of this proof, we can see that $\mathcal{H}\left( AV(\cdot)\xi + \xi \right)(\cdot)$ is bounded for $(t,z) \in [0,T] \times \mathbb{K}_q$. Now the functions

$$\{V(\cdot)b_\alpha\}_{\alpha \in \mathcal{J}} \ ,$$

are differentiable on $[0, T]$. Therefore by Proposition 2.5, $\mathcal{H}(V(\cdot)\xi)(z)$ is differentiable with respect to $t$ on $[0, T]$ for all $z \in \mathbb{K}_{2q}$ and

$$\begin{aligned}\frac{d}{dt}\mathcal{H}(V(t)\xi)(z) &= \sum_{\alpha \in \mathcal{J}} (AV(t)b_\alpha + b_\alpha) z^\alpha \\ &= \mathcal{H}(AV(t)\xi + \xi)(z) ,\end{aligned}$$

for $(t, z) \in [0, T] \times \mathbb{K}_{2q}$. Hence by Theorem 2.2, $V(t)\xi$ is a differentiable $S(H)_{-1}$ process on $[0, T]$ with continuous derivative $AV(t)\xi + \xi$, as required.

(4) Take $t \in [0, T]$. By part (3) of this proposition

$$\begin{aligned}\frac{d}{dt}V(t)\xi &= AV(t)\xi + \xi = \sum_{\alpha \in \mathcal{J}} AV(t)b_\alpha H_\alpha + \sum_{\alpha \in \mathcal{J}} b_\alpha H_\alpha \\ &= \sum_{\alpha \in \mathcal{J}} (AV(t)b_\alpha + b_\alpha) H_\alpha = \sum_{\alpha \in \mathcal{J}} \left(\frac{d}{dt}V(t)b_\alpha\right) H_\alpha .\end{aligned}$$

Now $V'(\cdot)\xi$ is a continuous $S(H)_{-1}$ process on $[0, T]$, so

$$\begin{aligned}\int_0^t \frac{d}{d\tau}V(\tau)\xi d\tau &= \sum_{\alpha \in \mathcal{J}} \left(\int_0^t \frac{d}{d\tau}V(\tau)b_\alpha d\tau\right) H_\alpha = \sum_{\alpha \in \mathcal{J}} V(t)b_\alpha H_\alpha \\ &= V(t)\xi ,\end{aligned}$$

as required. ■

**Theorem 5.1** *For any $T \in (0, \infty)$, the $S(H)_{-1}$ process*

$$X(t) := \frac{d}{dt}V(t)\xi + W_A^{(1)}(t) + V(t)B\mathbb{W}(0) , \tag{5.8}$$

*is a solution to (5.1).*

**Proof:** We need to show that $X(\cdot)$ satisfies the requirements for a solution to (5.1).

(1) By Propositions 5.1, 5.2 and 5.3 we have that $V'(\cdot)\xi$, $W_A^{(1)}(\cdot)$ and $V(\cdot)B\mathbb{W}(0)$ are continuous $S(H)_{-1}$ processes on $[0, T]$. Therefore

$$X(t) = \frac{d}{dt}V(t)\xi + W_A^{(1)}(t) + V(t)B\mathbb{W}(0) ,$$

is a continuous $S(H)_{-1}$ process on $[0, T]$.

(2) By Propositions 5.1, 5.2 and 5.3 we have that for all $t \in [0, T]$

$$\int_0^t \frac{d}{ds}V(s)\xi ds , \int_0^t W_A^{(1)}(s)ds , \int_0^t V(s)B\mathbb{W}(0)ds \in \mathcal{D}(A)_{-1} .$$

Therefore

$$\int_0^t X(s)ds = \int_0^t \left(\frac{d}{ds}V(s)\xi + W_A^{(1)}(s) + V(s)B\mathbb{W}(0)\right) ds ,$$

belongs to $\mathcal{D}(A)_{-1}$ for all $t \in [0,T]$.

(3) Using Propositions 5.2 and 5.3, we have that for $t \in [0,T]$

$$\begin{aligned}
& X(t) \\
= \;& \frac{d}{dt}V(t)\xi + W_A^{(1)}(t) + V(t)B\mathbb{W}(0) \\
= \;& (AV(t)\xi + \xi) + \left(A\int_0^t \left(W_A^{(1)}(s) + V(s)B\mathbb{W}(0)\right) ds + BW(t)\right) \\
= \;& \left(A\int_0^t \frac{d}{ds}V(s)\xi ds + \xi\right) \\
& + \left(A\int_0^t \left(W_A^{(1)}(s) + V(s)B\mathbb{W}(0)\right) ds + BW(t)\right) \\
= \;& \xi + A\int_0^t \left(\frac{d}{ds}V(s)\xi + W_A^{(1)}(s) + V(s)B\mathbb{W}(0)\right) ds + BW(t) \\
= \;& X(0) + A\int_0^t X(s)ds + BW(t) \; ,
\end{aligned}$$

as required. ■

**Propositon 5.4** *Take $T > 0$. If $X(\cdot)$ is a solution to (5.1), then*

$$X(t) = \frac{d}{dt}V(t)\xi + W_A^{(1)}(t) + V(t)B\mathbb{W}(0) \; , \tag{5.9}$$

*for all $t \in [0,T]$.*

**Proof:** Let $X(\cdot)$ and $\xi$ have forms

$$X(t) = \sum_{\alpha\in\mathcal{J}} c_\alpha(t)H_\alpha \; , \quad X(0) = \xi = \sum_{\alpha\in\mathcal{J}} b_\alpha H_\alpha \; .$$

As $X(\cdot)$, $X(0)$, $BW(\cdot)$ are continuous $S(H)_{-1}$ processes on $[0,T]$, we have that

$$A\int_0^t X(s)ds = X(t) - X(0) - BW(t) \; ,$$

is a continuous $S(H)_{-1}$ process on $[0,T]$. So there exists a $q \in \mathbb{N}\setminus\{1\}$ such that $X(t)$, $A\int_0^t X(s)ds$, $BW(t)$, $X(0) \in S(H)_{-1,-q}$ for all $t \in [0,T]$ and $X(\cdot)$, $A\int_0^t X(s)ds$, $BW(\cdot)$ are continuous with respect to $t$ in the norm $\|\cdot\|_{-1,-q}$ on $[0,T]$. Note that as $X(\cdot)$ is a continuous $S(H)_{-1}$ process on $[0,T]$, then for all $t \in [0,T]$

$$A\int_0^t X(s)ds = A\left(\sum_{\alpha\in\mathcal{J}}\left(\int_0^t c_\alpha(s)ds\right)H_\alpha\right) = \sum_{\alpha\in\mathcal{J}} A\left(\int_0^t c_\alpha(s)ds\right)H_\alpha \; .$$

Now for $(t,z) \in [0,T]\times\mathbb{K}_q$

$$\widetilde{X}(t,z) \;=\; \mathcal{H}\left(X(0) + A\int_0^t X(s)ds + BW(t)\right)(z) \; .$$

That is

$$\sum_{\alpha\in\mathcal{J}} c_\alpha(t) z^\alpha = \sum_{\alpha\in\mathcal{J}} \left( b_\alpha + A\int_0^t c_\alpha(s)ds \right) z^\alpha + \sum_{k=1}^{\infty} B\left( \int_0^t \kappa_k(s)ds \right) z^{\epsilon_k} .$$

So by Proposition 2.2, for all $\alpha \in \mathcal{J}$ and $k \in \mathbb{N}$

$$c_\alpha(t) = b_\alpha + A\int_0^t c_\alpha(s)ds + \delta_{\epsilon_k,\alpha}\left( B\int_0^t \kappa_k(s)ds \right) .$$

So by Corollary C.1 and Proposition C.5

$$\begin{aligned} & c_\alpha(t) \\ = \ & \frac{d}{dt}V(t)b_\alpha + \int_0^t V(t-s)\kappa_k'(s)ds + \delta_{\epsilon_k,\alpha}\left(V(t)B\kappa_k(0)\right) , \quad \alpha\in\mathcal{J} , \ k\in\mathbb{N} . \end{aligned}$$

Hence

$$\begin{aligned} & X(t) \\ = \ & \sum_{\alpha\in\mathcal{J}} c_\alpha(t) H_\alpha \\ = \ & \sum_{\alpha\in\mathcal{J}} \frac{d}{dt}V(t)b_\alpha H_\alpha + \sum_{k=1}^{\infty}\left( \int_0^t V(t-s)B\kappa_k'(s)ds + V(t)B\kappa_k(0) \right) H_{\epsilon_k} \\ = \ & \frac{d}{dt}V(t)\xi + W_A^{(1)}(t) + V(t)B\mathbb{W}(0) , \end{aligned}$$

as required. ■

### 5.1.1 Example

Consider again the stochastic Wave equation

$$\begin{aligned} & dY_t'(t,x) = \frac{\partial^2}{\partial x^2}Y(t,x)dt + dW(t,x) , \quad t\in[0,T] , \ x\in\Omega=(0,1) , \\ & Y(t,0) = Y(t,1) = 0 , \quad t\in[0,T] , \\ & Y(0,x) = Y_0(x) , \ Y_t'(0,x) = Y_1(x) , \quad x\in\Omega . \end{aligned} \tag{5.10}$$

Again let

$$X(t) = \begin{pmatrix} Y(t,x) \\ Y_t'(t,x) \end{pmatrix} , \quad X(0) = \begin{pmatrix} Y_0(x) \\ Y_1(x) \end{pmatrix} ,$$

but in this case $X(t) \in L^2(\Omega)\times L^2(\Omega)$. If $Y_0 \in \mathcal{D}(A)$ and $Y_1 \in L^2(\Omega)$, then we can write equation (5.10) in the abstract form in $L^2(\Omega)\times L^2(\Omega)$

$$\begin{aligned} & dX(t) = \mathcal{A}_2 X(t)dt + BdW(t) , \quad t\in[0,T] , \\ & X(0) \in \mathcal{D}(\mathcal{A}_2) , \end{aligned} \tag{5.11}$$

where

$$\mathcal{D}(\mathcal{A}_2) = \mathcal{D}(A) \times L^2(\Omega) ,$$
$$\mathcal{A}_2 = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix} ,$$

and $B : L^2(\Omega) \to L^2(\Omega) \times L^2(\Omega)$ is defined by

$$\mathcal{D}(B) = L^2(\Omega) ,$$
$$Bu = \begin{pmatrix} 0 \\ u \end{pmatrix} .$$

$\mathcal{A}_2$ generates a non degenerate, 1-times integrated, exponentially bounded semigroup $\{V(t),\ t \geq 0\}$ on $L^2(\Omega) \times L^2(\Omega)$. See section D.2 of appendix D for more details regarding $\mathcal{A}_2$ and $\{V(t),\ t \geq 0\}$.

Let $H = L^2(\Omega) \times L^2(\Omega)$. We consider equation (5.11) as the following integral equation in $S(H)_{-1}$

$$\begin{aligned} X(t) &= X(0) + \mathcal{A}_2 \int_0^t X(s)ds + BW(t) , \quad t \in [0,T] , \\ X(0) &= \xi \in \mathcal{D}(\mathcal{A}_2)_{-1} , \end{aligned} \tag{5.12}$$

where we have that

$$W(t) = \sum_{i=1}^{\infty} \beta_i(t) f_i ,$$

where $\{f_i\}_{i=1}^{\infty}$ is an orthonormal basis for $L^2(\Omega)$ and

$$\beta_i(t) = \sum_{j=1}^{\infty} \left( \int_0^t \xi_j(s)ds \right) H_{\epsilon_{n(i,j)}} .$$

By Theorem 5.1 and Proposition 5.4, equation (5.12) has the unique solution in $S(H)_{-1}$

$$X(t) = \frac{d}{dt}V(t)\xi + \int_0^t V(t-s)B\delta W^{(1)}(s) + V(t)B\mathbb{W}(0) .$$

## 5.2 $n$-Integrated Solution

Consider the integral equation in $S(H)_{-1}$

$$\begin{aligned} X(t) &= \frac{t^n}{n!}X(0) + A\int_0^t X(s)ds + \int_0^t \frac{(t-s)^n}{n!} B\delta W(s) , \quad t \in [0,T] , \\ X(0) &= \xi \in S(H)_{-1} , \end{aligned} \tag{5.13}$$

where we have that:

1. $A$ is a closed, densely defined operator on $H$.
2. $A$ generates a non degenerate, $n$-times integrated, exponentially bounded semigroup $\{V(t),\ t \geq 0\}$.
3. $B$ is a continuous linear map from $U$ to $H$.

Note that in the above equation, $W(\cdot)$ belongs to $S(U)_{-0}$, being defined in the same way as in section 3.3.1 with respect to $\{f_i\}_{i=1}^{\infty}$, an orthonormal basis for $U$.

In the rest of this section we consider the case the when $U = H$. However, all of the results proved in this section will apply equally to the more general case of when $U \neq H$.

Note that by Proposition 3.13 it follows that as $B$ is a bounded linear operator on $H$, the $S(H)_{-0}$ process $\frac{(t-s)^n}{n!}B\mathbb{W}(s)$ is Pettis integrable with respect to $s$ on $[0,t]$, for all $t \geq 0$ and

$$\int_0^t \frac{(t-s)^n}{n!} B\delta W(s) = \sum_{k=1}^{\infty} \left( \int_0^t \frac{(t-s)^n}{n!} B\kappa_k(s)ds \right) H_{\epsilon_k} . \tag{5.14}$$

As $\{V(t),\ t \geq 0\}$ is exponentially bounded, there exists a $M \geq 0$ and $a \in \mathbb{R}$ such that for all $t \in [0,\infty)$, we have that $\|V(t)\| \leq Me^{at}$.

**Definition 5.2** *A function $X(t) : [0,T] \to S(H)_{-1}$ is said to be a solution of (5.13) if:*

1. *$X(\cdot)$ is a continuous $S(H)_{-1}$ process on $[0,T]$.*
2. *For all $t \in [0,T]$*
$$\int_0^t X(s)ds \in \mathcal{D}(A)_{-1} . \tag{5.15}$$
3. *$X(\cdot)$ satisfies equation (5.13).*

**Propositon 5.5** *For all $T \in (0,\infty)$ we have that:*

1. *$V(\cdot)\xi$ is a continuous $S(H)_{-1}$ process on $[0,T]$.*
2. *For all $t \in [0,T]$*
$$\int_0^t V(s)\xi \in \mathcal{D}(A)_{-1} . \tag{5.16}$$
3. *For all $t \in [0,T]$*
$$V(t)\xi = \frac{t^n}{n!}\xi + A\int_0^t V(s)\xi ds . \tag{5.17}$$

**Proof:** (1) Let $X(0) = \xi$ have expansion

$$\xi = \sum_{\alpha \in \mathcal{J}} b_\alpha H_\alpha .$$

There exists a $q \in \mathbb{N} \setminus \{1\}$ so that $\xi \in S(H)_{-1,-q}$. Now for all $t \in [0,T]$, $V(t)\xi \in S(H)_{-1,-q}$ and for all $(t,z) \in \mathbb{K}_q$

$$\begin{aligned} \|\mathcal{H}(V(t)\xi)(z)\|_{H_\mathbb{C}} &= \left\|\sum_{\alpha\in\mathcal{J}} V(t)b_\alpha z^\alpha\right\|_{H_\mathbb{C}} \le \|V(t)\| \left\|\sum_{\alpha\in\mathcal{J}} b_\alpha z^\alpha\right\|_{H_\mathbb{C}} \\ &\le Me^{at}\|\xi\|_{-1,-q}A(q)^{1/2} \le K < \infty , \end{aligned}$$

where $K$ is a constant depending only on $T$. It follows from Proposition 2.4 that $\mathcal{H}(V(\cdot)\xi)(z)$ is continuous with respect to $t$ on $[0,T]$ for all $z \in \mathbb{K}_{2q}$, as the functions $\{V(\cdot)b_\alpha\}_{\alpha\in\mathcal{J}}$ are continuous on $[0,T]$. Hence by Theorem 2.1, $V(\cdot)\xi$ is a continuous $S(H)_{-1}$ process on $[0,T]$.

(2) As $V(\cdot)\xi$ is a continuous $S(H)_{-1}$ process on $[0,T]$

$$\int_0^t V(s)\xi ds = \sum_{\alpha\in\mathcal{J}} \left(\int_0^t V(s)b_\alpha ds\right) H_\alpha .$$

Now for all $\alpha \in \mathcal{J}$, $b_\alpha \in \overline{\mathcal{D}(A)} = H$. So by property (2) of Proposition C.2,

$$\int_0^t V(s)b_\alpha ds \in \mathcal{D}(A) ,$$

and for all $t \in [0,T]$

$$\begin{aligned} &\sum_{\alpha\in\mathcal{J}} \left\|A\int_0^t V(s)b_\alpha ds\right\|_H^2 (2\mathbb{N})^{-q\alpha} \\ &= \sum_{\alpha\in\mathcal{J}} \left\|V(t)b_\alpha - \frac{t^n}{n!}b_\alpha\right\|_H^2 (2\mathbb{N})^{-q\alpha} = \left\|V(t)\xi - \frac{t^n}{n!}\xi\right\|_{-1,-q}^2 \\ &\le \left(\|V(t)\xi\|_{-1,-q} + \frac{t^n}{n!}\|\xi\|_{-1,-q}\right)^2 < \infty , \end{aligned}$$

as required.

(3) For all $t \in [0,T]$

$$\begin{aligned} V(t)\xi &= \sum_{\alpha\in\mathcal{J}} V(t)b_\alpha H_\alpha = \sum_{\alpha\in\mathcal{J}} \left(\frac{t^n}{n!}b_\alpha + A\int_0^t V(s)b_\alpha ds\right) H_\alpha \\ &= \frac{t^n}{n!}\xi + A\int_0^t V(s)\xi ds , \end{aligned}$$

as required. ■

**Propositon 5.6** *We have that:*

1. *For all $t \in (0,\infty)$, $V(t-s)B\mathbb{W}(s)$ is Pettis integrable with respect to $s$ on $[0,t]$ and*

$$
\begin{aligned}
W_A(t) &:= \int_0^t V(t-s)B\delta W(s) \\
&= \sum_{k=1}^{\infty}\left(\int_0^t V(t-s)B\kappa_k(s)ds\right)H_{\epsilon_k} . \qquad (5.18)
\end{aligned}
$$

2. *For all $T \in (0,\infty)$, $W_A(\cdot)$ is a continuous $S(H)_{-1}$ process on $[0,T]$.*

3. *For all $t \in [0,\infty)$, $\int_0^t W_A(s)ds \in \mathcal{D}(A)_{-1}$.*

4. *For all $t \in [0,\infty)$*

$$
W_A(t) = A\int_0^t W_A(s)ds + \int_0^t \frac{(t-s)^n}{n!}B\delta W(s)ds . \qquad (5.19)
$$

**Proof:** (1) and (2). For any $t \geq 0$

$$
\int_0^t \|V(t-s)B\|^2 ds \leq \int_0^t \left(Me^{a(t-s)}\|B\|\right)^2 ds < \infty .
$$

So by Proposition 3.13, $V(t-s)B\mathbb{W}(s)$ is Pettis integrable with respect to $s$ on $[0,t]$ and

$$
\int_0^t V(t-s)B\delta W(s) = \sum_{k=1}^{\infty}\left(\int_0^t V(t-s)B\kappa_k(s)ds\right)H_{\epsilon_k} ,
$$

is a continuous $S(H)_{-1}$ process on $[0,T]$.

(3) Now

$$
\begin{aligned}
&\int_0^t V(t-s)B\kappa_k(s)ds \\
&= A\int_0^t\int_0^\tau V(\tau-s)B\kappa_k(s)dsd\tau + \int_0^t \frac{(t-s)^n}{n!}B\kappa_k(s)ds .
\end{aligned}
$$

Now $W_A(\cdot)$ is a continuous $S(H)_{-1}$ process, so

$$
\int_0^t W_A(s)ds = \sum_{k=1}^{\infty}\left(\int_0^t\int_0^\tau V(\tau-s)B\kappa_k(s)dsd\tau\right)H_{\epsilon_k} .
$$

We now show that $\int_0^t W_A(\tau)d\tau \in \mathcal{D}(A)_{-1}$. We have that

$$
\begin{aligned}
&\sum_{k=1}^{\infty}\left\|A\int_0^t\int_0^\tau V(\tau-s)B\kappa_k(s)dsd\tau\right\|_H^2 (2\mathbb{N})^{-4\epsilon_k} \\
&= \sum_{k=1}^{\infty}\left\|\int_0^t V(t-s)B\kappa_k(s)ds - \int_0^t \frac{(t-s)^n}{n!}B\kappa_k(s)ds\right\|_H^2 (2k)^{-4}
\end{aligned}
$$

$$= \left\| W_A(t) - \int_0^t \frac{(t-s)^n}{n!} B\delta W(s)ds \right\|_{-1,-4}^2$$

$$\leq \left( \|W_A(t)\|_{-1,-4} + \left\| \int_0^t \frac{(t-s)^n}{n!} B\delta W(s)ds \right\|_{-1,-4} \right)^2 < \infty ,$$

as required.

(4) For all $t \in [0,T]$

$$\begin{aligned} W_A(t) &= \sum_{k=1}^{\infty} \left( \int_0^t V(t-s)\kappa_k(s)ds \right) H_{\epsilon_k} \\ &= \sum_{k=1}^{\infty} \left( A \int_0^t \int_0^\tau V(\tau - s)\kappa_k(s)dsd\tau + \int_0^t \frac{(t-s)^n}{n!} B\kappa_k(s)ds \right) H_{\epsilon_k} \\ &= A \int_0^t W_A(s)ds + \int_0^t \frac{(t-s)^n}{n!} B\delta W(s)ds , \end{aligned}$$

as required. ■

**Theorem 5.2** *For any $T \in (0,\infty)$, the $S(H)_{-1}$ process*

$$X(t) := V(t)\xi + W_A(t) , \quad t \in [0,T] , \tag{5.20}$$

*is a solution of (5.13).*

**Proof:** We need to show that $X(\cdot)$ satisfies the requirements for a solution to (5.13).

(1) By Propositions 5.5 and 5.6, we have that $V(\cdot)\xi$ and $W_A(\cdot)$ are both continuous $S(H)_{-1}$ processes on $[0,T]$. Therefore $X(\cdot) = V(\cdot)\xi + W_A(\cdot)$ is a continuous $S(H)_{-1}$ process on $[0,T]$.

(2) By Propositions 5.5 and 5.6, we have that for all $t \in [0,T]$

$$\int_0^t V(s)\xi ds , \int_0^t W_A(s)ds \in \mathcal{D}(A)_{-1} .$$

Therefore

$$\int_0^t X(s)ds = \int_0^t (V(s)\xi + W_A(s))\, ds \in \mathcal{D}(A)_{-1} ,$$

for all $t \in [0,T]$.

(3) By Propositions 5.5 and 5.6, we have that for all $t \in [0,T]$

$$\begin{aligned} & X(t) \\ = \ & V(t)\xi + W_A(t) \\ = \ & \left( \frac{t^n}{n!}\xi + A \int_0^t V(s)\xi ds \right) + \left( A \int_0^t W_A(s)ds + \int_0^t \frac{(t-s)^n}{n!} B\delta W(s) \right) \\ = \ & \frac{t^n}{n!}\xi + A \int_0^t (V(s)\xi + W_A(s))\, ds + \int_0^t \frac{(t-s)^n}{n!} B\delta W(s) , \end{aligned}$$

as required. ■

**Propositon 5.7** *If $X(\cdot)$ is a solution to (5.13), then*

$$X(t) = V(t)\xi + W_A(t) \ , \tag{5.21}$$

*for all $t \in [0,T]$.*

**Proof:** Let $X(\cdot)$ and $\xi$ have forms

$$X(t) = \sum_{\alpha\in\mathcal{J}} c_\alpha(t)H_\alpha \ , \quad X(0) = \xi = \sum_{\alpha\in\mathcal{J}} b_\alpha H_\alpha \ .$$

As $X(\cdot)$, $\frac{t^n}{n!}X(0)$ and $\int_0^t \frac{(t-s)^n}{n!}B\delta W(s)$ are continuous $S(H)_{-1}$ processes on $[0,T]$, then

$$A\int_0^t X(s)ds = X(t) - \frac{t^n}{n!}X(0) - \int_0^t \frac{(t-s)^n}{n!}B\delta W(s) \ ,$$

is a continuous $S(H)_{-1}$ process on $[0,T]$. So there exists a $q \in \mathbb{N}\setminus\{1\}$ such that $X(t)$, $A\int_0^t X(s)ds$, $\frac{t^n}{n!}X(0)$ and $\int_0^t \frac{(t-s)^n}{n!}B\delta W(s)$ belong to $S(H)_{-1,-q}$ for all $t\in[0,T]$ and are continuous with respect to $t$ on $[0,T]$ in the norm $\|\cdot\|_{-1,-q}$. Note that as $X(\cdot)$ is a continuous $S(H)_{-1}$ process on $[0,T]$, then for all $t \in [0,T]$

$$A\int_0^t X(s)ds = A\left(\sum_{\alpha\in\mathcal{J}}\left(\int_0^t c_\alpha(s)ds\right)H_\alpha\right) = \sum_{\alpha\in\mathcal{J}} A\left(\int_0^t c_\alpha(s)ds\right)H_\alpha \ .$$

Now for all $(t,z) \in [0,T]\times\mathbb{K}_q$

$$\widetilde{X}(t,z) \ = \ \mathcal{H}\left(\frac{t^n}{n!}\xi + A\int_0^t X(s)ds + \int_0^t \frac{(t-s)^n}{n!}B\delta W(s)\right)(z) \ .$$

That is

$$\begin{aligned}&\sum_{\alpha\in\mathcal{J}} c_\alpha(t)z^\alpha \\ &= \sum_{\alpha\in\mathcal{J}}\left(\frac{t^n}{n!}b_\alpha + A\int_0^t c_\alpha(s)ds\right)z^\alpha + \sum_{k=1}^{\infty}\left(\int_0^t \frac{(t-s)^n}{n!}B\kappa_k(s)ds\right)z^{\epsilon_k} \ .\end{aligned}$$

So by Proposition 2.2, for all $\alpha \in \mathcal{J}$ and $k \in \mathbb{N}$

$$c_\alpha(t) \ = \ \frac{t^n}{n!}b_\alpha + A\int_0^t c_\alpha(s)ds + \delta_{\epsilon_k,\alpha}\int_0^t \frac{(t-s)^n}{n!}B\kappa_k(s)ds \ .$$

So

$$c_\alpha(t) \ = \ V(t)b_\alpha + \delta_{\epsilon_k,\alpha}\int_0^t V(t-s)B\kappa_k(s)ds \ , \quad \alpha\in\mathcal{J} \ , \ k\in\mathbb{N} \ .$$

Hence

$$\begin{aligned}&X(t) \\ &= \sum_{\alpha\in\mathcal{J}} c_\alpha(t)H_\alpha = \sum_{\alpha\in\mathcal{J}} V(t)b_\alpha H_\alpha + \sum_{k=1}^{\infty}\left(\int_0^t V(t-s)B\kappa_k(s)ds\right)H_{\epsilon_k} \\ &= V(t)\xi + W_A(t) \ ,\end{aligned}$$

as required. ■

# Appendix A

# Linear Topological Spaces

The theory found in this chapter on linear topological spaces follows Chapter 1 of [7]. We need the results on countably normed spaces to show that $S(H)_\rho$, $\rho \in [0,1]$ is a countably Hilbert space and $S(H)_{-\rho}$ is it's dual.

Throughout this chapter, $\Phi$ is at least a vector space.

## A.1 Linear Topological Spaces

**Definition A.1** *A vector space $\Phi$ equipped with a topology such that addition and multiplication are continuous in this topology is called a linear topological space.*

By a topology we mean that there is a system of (open) neighbourhoods $\{U\}$ such that:

1. For every point $\phi \in \Phi$, there exists a neighbourhood $U = U(\phi)$ such that $\phi \in U(\phi)$.
2. If $\phi$ belongs to two neighbourhoods $U$ and $V$, then there exists a neighbourhood $W$, such that $\phi \in W \subset U \cap V$.
3. For any pair of points $\phi \neq \psi$, there exists a neighbourhood $U$ such that $\phi \in U$ but $\psi \notin U$.

The topology is the set of all finite and infinite unions of this system of neighbourhoods $\{U\}$.

By continuity of addition we mean that if

$$\phi \pm \psi = \chi \, ,$$

and $U$ is any neighbourhood of $\chi$, there exists a neighbourhood $V$ of $\phi$ and $W$ of $\psi$ such that $V + W \subset U$.

By continuity of multiplication we mean that if $\lambda_0\phi = \psi$ and $U$ is any neighbourhood of $\psi$, then there exists a number $\epsilon > 0$ and neighbourhood $V$ of $\phi$ such that $|\lambda - \lambda_0| < \epsilon$ implies $\lambda V \subset U$.

Note that the topology in $\Phi$ induced by the system of neighbourhoods $\{U\}$ can be reconstructed by all possible translations of the neighbourhoods of zero.

**Definition A.2** *A sequence $\{\phi_n\}_{n=1}^{\infty}$ is said to converge to an element $\phi$ if each neighbourhood of $\phi$ contains all but a finite number of $\{\phi_n\}_{n=1}^{\infty}$.*

**Definition A.3**

1. *A system of neighbourhoods $\{V\}$ containing $\phi$ is said to be a basis of neighbourhoods of $\phi$ if each neighbourhood of $\phi$ contains at least one neighbourhood from $\{V\}$.*
2. *We say a point $\phi \in \Phi$ satisfies the first axiom of countablity if it has a countable neighbourhood basis.*

**Propositon A.1** *If there exists one point in $\Phi$ having a countable neighbourhood basis, then every other point in $\Phi$ also has a countable neighbourhood basis.*

**Theorem A.1** *Suppose that on a vector space $\Phi$ there exists a system $C$ of sets all containing the zero element such that:*

1. *For any $U, V \in C$, there exists a $W \in C$ such that $W \subset U \cap V$.*
2. *For any $\phi \neq 0$, there exists a $U \in C$ such that $\phi \notin U$.*
3. *For any set $U \in C$, there exists a $W \in C$ such that $W \pm W \subset U$.*
4. *If $\phi \in U \in C$, then there exists a $V \in C$ such that $\phi + V \subset U$.*
5. *For any $U \in C$ and any number $\alpha$, there exists a $V \in C$ such that $\alpha V \subset U$.*
6. *For any $U \in C$ and any point $\phi$, there exists an $\epsilon > 0$ such that $\delta\phi \in U$ for $|\delta| < \epsilon$.*
7. *For any $U \in C$, there exists an $\epsilon > 0$ such that $\delta U \subset U$ for $|\delta| < \epsilon$.*

*Then there exists a system of (open) neighbourhoods $\{U\}$ in $\Phi$ such that $\Phi$ is a linear topological space and $C$ is a neighbourhood basis of zero.*

Two distinct systems of sets $C_1$ and $C_2$ satisfying the above theorem lead to the same topology if for any $U \in C_1$ there exists a $V \in C_2$ such that $V \subset U$ and for any $V \in C_2$ there exists a $U \in C_1$ such that $U \subset V$. We say $C_1$ and $C_2$ are equivalent systems if this is the case.

## A.2 Countably Normed Spaces

### A.2.1 Comparability and Compatibility of Norms

**Definition A.4** *1. Take two norms $||\cdot||_1$ and $||\cdot||_2$ on a vector space $\Phi$. The norm $||\cdot||_1$ is weaker than $||\cdot||_2$ (and $||\cdot||_2$ stronger than $||\cdot||_1$) if there exists a constant $C < \infty$ such that for all $\phi \in \Phi$*

$$||\phi||_1 \leq C||\phi||_2 .$$

*2. Take two norms $||\cdot||_1$ and $||\cdot||_2$ on a vector space $\Phi$. The norms $||\cdot||_1$ and $||\cdot||_2$ are said to be compatible if for any sequence that converges to zero in one norm and is a Cauchy sequence in the other will also converge to zero in that norm.*

If $||\cdot||_k$ is a norm on $\Phi$, we denote the completion of $\Phi$ with respect to this norm by $\Phi_k$.

**Lemma A.1** *If $||\cdot||_1$ and $||\cdot||_2$ are compatible norms on $\Phi$ and $||\cdot||_1$ is weaker than $||\cdot||_2$, then*

$$\Phi_1 \supset \Phi_2 \supset \Phi .$$

### A.2.2 Countably Normed Spaces

**Propositon A.2** *If $\Phi$ has a system of norms $\{||\cdot||_k\}_{k=1}^{\infty}$, then the collection of sets*

$$\{\phi \in \Phi;\ ||\phi||_1 < \epsilon, ||\phi||_2 < \epsilon, \ldots, ||\phi||_p < \epsilon\}_{p\in\mathrm{N},\ \epsilon>0} ,$$

*satisfies the conditions of Theorem A.1.*

**Lemma A.2** *The collection of sets*

$$\left\{\phi \in \Phi;\ ||\phi||_1 < \frac{1}{m}, ||\phi||_2 < \frac{1}{m}, \ldots, ||\phi||_p < \frac{1}{m}\right\}_{m,p\in\mathrm{N}} ,$$

*is an equivalent system of sets to*

$$\{\phi \in \Phi;\ ||\phi||_1 < \epsilon, ||\phi||_2 < \epsilon, \ldots, ||\phi||_p < \epsilon\}_{p\in\mathrm{N},\ \epsilon>0} .$$

**Definition A.5** *If the system of norms $\{||\cdot||_k\}_{k=1}^{\infty}$ are compatible, we call $\Phi$ equipped with the topology introduced in Proposition A.2 a countably normed space.*

Note that by Lemma A.2, a countably normed space satisfies the first axiom of countablity.

We also assume that the system of norms are non-decreasing, that is for all $\phi \in \Phi$

$$||\phi||_1 \leq ||\phi||_2 \leq ||\phi||_3 \leq \ldots .$$

This means that

$$\Phi_1 \supset \Phi_2 \supset \Phi_3 \supset \ldots \supset \Phi .$$

**Definition A.6** *1. A sequence in a countably normed space is said to be a Cauchy sequence if it is a Cauchy sequence in each norm.*

*2. We say a countably normed space is complete if every Cauchy sequence converges.*

**Propositon A.3** *A countably normed space is complete if and only if*

$$\Phi = \cap_{k=1}^{\infty} \Phi_k .$$

From now on, we assume that any countably normed space is complete.

**Definition A.7** *A countably normed space is called a countably Hilbert space if the system norms $\{|| \cdot ||_p\}_{p=1}^{\infty}$ correspond to a system of inner products $\{\langle \cdot, \cdot \rangle_p\}_{p=1}^{\infty}$, that is for all $p \in \mathbb{N}$*

$$||\phi||_p^2 = \langle \phi, \phi \rangle_p ,$$

*for all $\phi \in \Phi$.*

**Bounded sets in Topological Linear Spaces**

**Definition A.8** *A set $E$ contained in a topological linear space $\Phi$, is said to be bounded if for each neighbourhood $U$ of zero, there exists a $\lambda > 0$ such that*

$$\lambda E \subset U .$$

**Propositon A.4** *A set $E$ contained in a countably normed space $\Phi$, is bounded if for all $p \in \mathbb{N}$ and $\phi \in E$*

$$||\phi||_p \leq C_p < \infty . \tag{A.1}$$

**Propositon A.5** *Any convergent sequence in a topological linear space is a bounded set.*

## A.3 The Dual of a Linear Topological Space

### A.3.1 Continuous Linear Functionals

**Definition A.9** *A linear functional $(f, \cdot)$ on $\Phi$, a topological linear space, is said to be continuous if for any $\epsilon > 0$, there exists a neighbourhood $U$ of zero such that*

$$|(f, \phi)| < \epsilon , \tag{A.2}$$

*for all* $\phi \in U$.

We denote the space of continuous linear functionals on a topological linear space $\Phi$ by $\Phi'$.

**Propositon A.6** *If* $\Phi$ *satisfies the first axiom of countablity, then* $f$ *is continuous if and only if for each* $\phi \in \Phi$

$$\lim_{n\to\infty}(f, \phi_n) = (f, \phi) ,$$

*for all convergent sequences* $\{\phi_n\}_{n=1}^{\infty} \subset \Phi$ *such that* $\phi_n \longrightarrow_{n\to\infty} \phi$.

For a countably normed space, denote $(\Phi_k)'$ by $\Phi_{-k}$ and the dual norm on $(\Phi_k)'$ by $||\cdot||_{-k}$.

**Propositon A.7** *For a countably normed space*

$$\Phi_{-1} \subset \Phi_{-2} \subset \Phi_{-3} \subset \ldots \subset \Phi' , \tag{A.3}$$

*and*

$$\Phi' = \cup_{k=1}^{\infty}\Phi_{-k} . \tag{A.4}$$

*If* $f \in \Phi'$, *then there exists a* $p \in \mathbb{N}$ *such that* $f \in \Phi_{-p}$ *and*

$$\infty > ||f||_{-p} \geq ||f||_{-(p+1)} \geq \cdots . \tag{A.5}$$

### A.3.2 Strong Topology on $\Phi'$

**Definition A.10** *For* $\Phi'$, *the dual of a linear topological space* $\Phi$, *define the strong neighbourhoods of zero as*

$$\left\{ f \in \Phi';\ \sup_{\phi\in E}|(f,\phi)| < \epsilon \right\} ,$$

*where* $E \subset \Phi$ *is bounded and* $\epsilon > 0$.

**Propositon A.8** *The strong neighbourhoods of zero satisfy the conditions of Theorem A.1.*

We call the topology induced on $\Phi'$ by the strong neighbourhoods of zero, the strong topology on $\Phi'$.

**Definition A.11** *Let* $\{f_k\}_{k=1}^{\infty}$ *be any sequence contained in* $\Phi'$.

1. *We say* $\{f_k\}_{k=1}^{\infty}$ *converges strongly to* $f$ *in* $\Phi'$ *if* $(f_k, \phi) \longrightarrow_{k\to\infty}(f, \phi)$ *uniformly on all bounded sets in* $\Phi$.

2. $\Phi'$ *is said to be complete if* $(f_k, \phi)$ *converging uniformly on all bounded sets in* $\Phi$ *implies there exists a* $f \in \Phi'$ *such that* $f_k \longrightarrow_{k\to\infty} f$ *strongly.*

**Propositon A.9** *If the first axion of countablity is satisfied for the space* $\Phi$*, then* $\Phi'$ *is complete with respect to strong convergence.*

**Corollary A.1** *If* $\Phi$ *is a countably normed space, then* $\Phi'$ *is complete with respect to strong convergence.*

### Strongly Bounded Sets

**Definition A.12** *A set* $F \subset \Phi'$ *is said to be strongly bounded if for every strong neighbourhood* $U$ *of the zero functional, there exists a* $\lambda > 0$ *such that* $\lambda F \subset U$.

**Lemma A.3** *A set* $F \subset \Phi'$ *is strongly bounded if and only if* $F$ *is bounded on every bounded set* $E \subset \Phi$.

**Corollary A.2** *If* $\Phi$ *satisfies the first axiom of countablity, then every strongly bounded set* $F \subset \Phi'$ *is bounded on some neighbourhood of zero* $U \subset \Phi$.

**Corollary A.3** *For a countably normed space, a set* $F \subset \Phi'$ *is strongly bounded if and only there exists a* $p \in \mathbb{N}$ *such that* $F \subset \Phi_{-p}$ *and bounded in the norm* $||\cdot||_{-p}$.

**Lemma A.4** *A strongly convergent sequence* $\{f_k\}_{k=1}^{\infty}$ *in* $\Phi'$ *is strongly bounded.*

**Corollary A.4** *For every strongly convergent sequence* $\{f_k\}_{k=1}^{\infty}$ *in* $\Phi'$*, there exists a* $p \in \mathbb{N}$ *such that* $\{f_k\}_{k=1}^{\infty}$ *is contained in* $\Phi_{-p}$ *and bounded in the norm* $||\cdot||_{-p}$.

## A.4 Frechet Spaces

The material found in this section on Frechet spaces can be found in [19].

For a countably normed space $\Phi$, define the following metric on $\rho(\cdot,\cdot)$

$$\rho(\phi,\psi) := \sum_{p=1}^{\infty} \frac{1}{2^p} \frac{||\phi-\psi||_p}{1+||\phi-\psi||_p}, \quad \phi,\psi \in \Phi .$$

**Propositon A.10** $\rho(\cdot,\cdot)$ *is a complete invariant metric on* $\Phi$*, inducing the same topology on* $\Phi$ *as* $\{||\cdot||_p\}_{p=1}^{\infty}$.

**Definition A.13** *A set* $U \subset \Phi$ *is called convex if for all* $\phi,\psi \in U$ *and* $\alpha \in [0,1]$

$$\alpha\phi + (1-\alpha)\psi \in U .$$

**Definition A.14** *A linear topological space* $\Phi$ *is called a Frechet space if:*

1. *There exists a neighbourhood basis of zero such that it's members are convex.*
2. *The topology on $\Phi$ is induced by a complete invariant metric.*

**Propositon A.11** *A countably normed space is a Frechet space.*

**Propositon A.12** *In Frechet space, a closed functional is also a continuous functional.*

# Appendix B

# The Hermite Polynomials and Functions

## B.1 Properties of Hermite Polynomials and Functions

The Hermite polynomials $h_n(\cdot)$ are defined by

$$h_n(x) = (-1)^n e^{1/2x^2} \frac{d^n}{dx^n} \left(e^{-1/2x^2}\right) , \quad n = 0, 1, 2, \dots . \tag{B.1}$$

The Hermite functions $\xi_n(\cdot)$ are defined by

$$\xi_n(x) = \pi^{-1/4}((n-1)!)^{-1/2} e^{-1/2x^2} h_{n-1}\left(\sqrt{2}x\right) , \quad n = 1, 2, \dots . \tag{B.2}$$

The following properties of the Hermite polynomials and functions can be found in [9] and [10]:

1. For $n = 0, 1, 2, \dots$
$$\frac{dh_n(x)}{dx} = nh_{n-1}(x) ,$$
where we take $h_{-1}(0) = 0$.
2. $\{\xi_n(\cdot)\}_{n=1}^{\infty}$ is an orthonormal basis for $L^2(\mathbb{R})$.
3. For $n = 1, 2, \dots$
$$\sup_{x\in\mathbb{R}} |\xi_n(x)| = \mathcal{O}(n^{-1/12}) .$$
4. For $n = 1, 2, \dots$
$$-\frac{d^2\xi_n(x)}{dx^2} + x^2\xi_n(x) = 2n\xi_n(x) .$$

## B.2 Growth Estimates for the Derivatives of Hermite Functions

The following results we demonstrate in this section provide growth estimates for the derivatives of $\xi_n(\cdot)$ up to any order. We use these results in Chapter 4 to show that $W(t)$ is differentiable in $S(H)_{-1}$ up to any order, on any interval $[a, b]$.

**Lemma B.1** *For all $n \in \mathbb{N}$*

$$\frac{d\xi_n(x)}{dx} = -x\xi_n(x) + (2n-2)^{1/2}\xi_{n-1}(x) \ , \tag{B.3}$$

*where we take $\xi_0(x) = 0$.*

**Proof:** Using the definition of the Hermite functions and property (1) above, we have

$$\begin{aligned}
&\frac{d\xi_n(x)}{dx} \\
= \ & \pi^{-1/4}((n-1)!)^{-1/2}\left(-\frac{1}{2}2xe^{-1/2x^2}h_{n-1}\left(\sqrt{2}x\right) + e^{-1/2x^2}\sqrt{2}\left.\frac{dh_{n-1}(x)}{dx}\right|_{\sqrt{2}x}\right) \\
= \ & \pi^{-1/4}((n-1)!)^{-1/2}\left(-xe^{-1/2x^2}h_{n-1}\left(\sqrt{2}x\right) + e^{-1/2x^2}\sqrt{2}(n-1)h_{n-2}\left(\sqrt{2}x\right)\right) \\
= \ & -x\xi_n(x) + (n-1)^{1/2}\sqrt{2}\left(\pi^{-1/4}((n-2)!)^{-1/2}e^{-1/2x^2}h_{n-2}\left(\sqrt{2}x\right)\right) \\
= \ & -x\xi_n(x) + (2n-2)^{1/2}\xi_{n-1}(x) \ ,
\end{aligned}$$

as required. ■

**Propositon B.1** *For all $n \in \mathbb{N}$ and $a, b \in \mathbb{R}$*

$$\sup_{x\in[a,b]}\left|\frac{d\xi_n(x)}{dx}\right| \leq C_{a,b}n \ , \tag{B.4}$$

*where $C_{a,b}$ is a constant depending only on $a$ and $b$.*

**Proof:** Using Lemma B.1 and property (3) above, we have

$$\begin{aligned}
\sup_{x\in[a,b]}\left|\frac{d\xi_n(x)}{dx}\right| &\leq \sup_{x\in[a,b]}|x\xi_n(x)| + \sup_{x\in[a,b]}\left|(2n-2)^{1/2}\xi_{n-1}(x)\right| \\
&= M_{a,b,1} + M_2(2n-2)^{1/2} < 2n(M_{a,b,1} + M_2) = C_{a,b}n \ ,
\end{aligned}$$

where the constants $M_{a,b,1}$ and $M_2$ and hence $C_{a,b}$ do not depend on $n$, as required. ■

**Lemma B.2** *For all $k \geq 2$*

$$\frac{d^k\xi_n(x)}{dx^k} = \frac{d^{k-2}\xi_n(x)}{dx^{k-2}}(x^2 - 2n) + \sum_{j=0}^{k-3}\frac{d^j\xi_n(x)}{dx^j}(a_j + b_jn + c_jx + d_jx^2) \ ,$$

*where the constants $a_j, b_j, c_j, d_j \in \mathbb{R}$ do not depend on $n$.*

**Proof:** By property (4) above, we have for $k = 2$

$$\frac{d^2\xi_n(x)}{dx^2} = \xi_n(x)\left(x^2 - 2n\right) = \xi_n(x)\left(x^2 - 2n\right) + 0 .$$

Therefore true for $k = 2$.

Assume true for some $k \geq 2$. Then

$$\frac{d^k\xi_n(x)}{dx^k} = \frac{d^{k-2}\xi_n(x)}{dx^{k-2}}\left(x^2 - 2n\right) + \sum_{j=0}^{k-3} \frac{d^j\xi_n(x)}{dx^j}\left(a_j + b_j n + c_j x + d_j x^2\right) .$$

So

$$\begin{aligned}
&\frac{d^{k+1}\xi_n(x)}{dx^{k+1}} \\
= \; & \frac{d^{k-1}\xi_n(x)}{dx^{k-1}}\left(x^2 - 2n\right) + \frac{d^{k-2}\xi_n(x)}{dx^{k-2}}(2x) \\
&+ \sum_{j=0}^{k-3} \frac{d^{j+1}\xi_n(x)}{dx^{j+1}}\left(a_j + b_j n + c_j x + d_j x^2\right) + \sum_{j=0}^{k-3} \frac{d^j\xi_n(x)}{dx^j}\left(c_j + 2d_j x\right) \\
= \; & \frac{d^{k-1}\xi_n(x)}{dx^{k-1}}\left(x^2 - 2n\right) + \frac{d^{k-2}\xi_n(x)}{dx^{k-2}}(2x) \\
&+ \sum_{j=1}^{k-2} \frac{d^j\xi_n(x)}{dx^j}\left(a_{j-1} + b_{j-1} n + c_{j-1} x + d_{j-1} x^2\right) + \sum_{j=0}^{k-3} \frac{d^j\xi_n(x)}{dx^j}\left(c_j + 2d_j x\right) \\
= \; & \frac{d^{k-1}\xi_n(x)}{dx^{k-1}}\left(x^2 - 2n\right) + \frac{d^{k-2}\xi_n(x)}{dx^{k-2}}\left(a_{k-3} + b_{k-3} n + (2 + c_{k-3})x + d_{k-3} x^2\right) \\
&+ \sum_{j=1}^{k-3} \frac{d^j\xi_n(x)}{dx^j}\left((a_{j-1} + c_j) + b_{j-1} n + (c_{j-1} + 2d_j)x + d_{j-1} x^2\right) \\
&+ \xi_n(x)(c_0 + 2d_0 x) .
\end{aligned}$$

So if $p(k)$ is true, then $p(k+1)$ is true, as required. ■

**Propositon B.2** *For any interval $[a, b]$, and $n, k \in \mathbb{N}$*

$$\sup_{x\in[a,b]} \left|\frac{d^k\xi_n(x)}{dx^k}\right| \leq C_{a,b,k}(2n)^{[\![\frac{k+1}{2}]\!]} < \infty , \tag{B.5}$$

*where the constant $C_{a,b,k}$ depends only on $a, b$ and $k$.*

**Proof:** True for $k = 0$ as

$$\sup_{x\in[a,b]} |\xi_n(x)| \leq \sup_{x\in\mathbb{R}} |\xi_n(x)| \leq C < \infty .$$

True for $k = 1$ as

$$\sup_{x\in[a,b]} \left|\frac{d\xi_n(x)}{dx}\right| \leq C_{a,b} n = C_{a,b,1}(2n)^{[\![\frac{1+1}{2}]\!]} .$$

Assume true for $0, 1, \ldots, k$, where $k \geq 1$. From Lemma B.2

$$\frac{d^{k+1}\xi_n(x)}{dx^{k+1}} = \frac{d^{k-1}\xi_n(x)}{dx^{k-1}}(x^2 - 2n) + \sum_{j=0}^{k-2} \frac{d^j \xi_n(x)}{dx^j}(a_j + b_j n + c_j x + d_j x^2) .$$

So

$$\begin{aligned}
&\sup_{x\in[a,b]}\left|\frac{d^{k+1}\xi_n(x)}{dx^{k+1}}\right| \\
\leq\ & \left(\sup_{x\in[a,b]}\left|\frac{d^{k-1}\xi_n(x)}{dx^{k-1}}\right|\right)\left(\sup_{x\in[a,b]}\left|x^2-2n\right|\right) \\
&+\left(\sup_{j\in\{0,1,\ldots,k-2\}}\left(\sup_{x\in[a,b]}\left|\frac{d^j\xi_n(x)}{dx^j}\right|\right)\right)\sum_{j=0}^{k-2}\left(\sup_{x\in[a,b]}\left|a_j+b_jn+c_jx+d_jx^2\right|\right) \\
\leq\ & \left(C_{a,b,k-1}(2n)^{[\![\frac{(k-1)+1}{2}]\!]}\right)\left(K_{a,b,k-1}(2n)\right) \\
&+\left(\sup_{j\in\{0,1,\ldots,k-2\}} C_{a,b,j}(2n)^{[\![\frac{j+1}{2}]\!]}\right)\left(\sum_{j=0}^{k-2}K_{a,b,j}(2n)\right) \\
\leq\ & C_{a,b,k-1}K_{a,b,k-1}(2n)^{[\![\frac{k}{2}]\!]+1}+(2n)^{[\![\frac{(k-2)+1}{2}]\!]}\left(\sup_{j\in\{0,1,\ldots,k-2\}}C_{a,b,j}\right)(2n)\left(\sum_{j=0}^{k-2}K_{a,b,j}\right) \\
=\ & M_{a,b,1}(2n)^{[\![\frac{k}{2}]\!]+1}+M_{a,b,2}(2n)^{[\![\frac{k-1}{2}]\!]+1}=M_{a,b,1}(2n)^{[\![\frac{k+2}{2}]\!]}+M_{a,b,2}(2n)^{[\![\frac{k-1+2}{2}]\!]} \\
\leq\ & M_{a,b,1}(2n)^{[\![\frac{k+2}{2}]\!]}+M_{a,b,2}(2n)^{[\![\frac{k+2}{2}]\!]}=C_{a,b,k+1}(2n)^{[\![\frac{(k+1)+1}{2}]\!]} ,
\end{aligned}$$

for some $C_{a,b,k+1} < \infty$ that only depends on $a$, $b$ and $k$. So if $p(0), \ldots, p(k)$ are true, then $p(k+1)$ is true, as required. ■

# Appendix C

# Semigroups

## C.1 $C_0$-Semigroups

The material regarding $C_0$-semigroups in this section can be found in [5].

$X$ is a Banach space throughout this chapter.

**Definition C.1** *A one-parameter family of bounded linear operators $\{S(t),\ t \geq 0\}$ on $X$ is called a $C_0$-semigroup if*

**(S1)** *For all $s, t \geq 0$*

$$S(t+s) = S(t)S(s)\ . \tag{C.1}$$

**(S2)** $S(0) = I$.

**(S3)** $\{S(t),\ t \geq 0\}$ *is strongly continuous with respect to $t \geq 0$.*

By strongly continuous we mean that for all $x \in X$, $S(\cdot)x$ is continuous with respect to $t \geq 0$.

**Definition C.2** *For a $C_0$-semigroup $\{S(t),\ t \geq 0\}$, define the operator*

$$\mathcal{D}(A) := \left\{ x \in X;\ \exists \lim_{h \to 0} \frac{S(h) - I}{h} x \right\}\ ,$$
$$A(x) := \lim_{h \to 0} \frac{S(h) - I}{h} x\ , \quad x \in \mathcal{D}(A)\ .$$

*We call $A$ the generator of $\{S(t),\ t \geq 0\}$.*

**Propositon C.1** *For a $C_0$-semigroup $\{S(t),\ t \geq 0\}$ with generator $A$, we have that:*

1. *For all $s, t \geq 0$*

$$S(s)S(t) = S(t)S(s)\ . \tag{C.2}$$

2. *There exists a $K > 0$ and $\omega \in \mathbb{R}$ such that for all $t \geq 0$*

$$||S(t)|| \leq Ke^{\omega t} \ . \tag{C.3}$$

3. *$A$ is a densely defined, closed operator on $X$.*

4. *For all $x \in \mathcal{D}(A)$ and $t \geq 0$*

$$U'(t)x = U(t)Ax = AU(t)x \ . \tag{C.4}$$

5. *For all $x \in X$ and $t \geq 0$, $\int_0^t U(s)ds \in \mathcal{D}(A)$ and*

$$A\int_0^t U(s)ds = U(t)x - x \ . \tag{C.5}$$

6. *For all $\lambda \in \mathbb{C}$ with $Re\lambda > \omega$, $(\lambda - A)$ is invertible and for all $x \in X$*

$$R_A(\lambda)x := (\lambda - A)^{-1}x = \int_0^\infty e^{-\lambda t}S(t)x \ dt \ . \tag{C.6}$$

## C.2 $n$-times Integrated Semigroups

The material regarding integrated semigroups in this section can be found in [12], [1] and [13].

**Definition C.3** *Let $n \in \mathbb{N}$. A one-parameter family of bounded linear operators $\{V(t),\ t \geq 0\}$ is called an $n$-times integrated, exponentially bounded semigroup if*

**(V1)** *For all $s, t \geq 0$*

$$\frac{1}{(n-1)!}\int_0^s [\,(s-r)^{n-1}V(t+r) - (t+s-r)^{n-1}V(r)\,]\,dr = V(t)V(s) \ . \tag{C.7}$$

**(V2)** *$\{V(t),\ t \geq 0\}$ is strongly continuous with respect to $t \geq 0$.*

**(V3)** *There exists a $K > 0$ and $a \in \mathbb{R}$ such that for all $t \geq 0$*

$$||V(t)|| \ \leq Ke^{at} \ . \tag{C.8}$$

*In addition, $\{V(t),\ t \geq 0\}$ is said to be non degenerate if*

$$\forall t \geq 0 \ , \ V(t)x = 0 \Rightarrow x = 0 \ .$$

If $\{V(t),\ t \geq 0\}$, an $n$-times integrated, exponentially bounded semi-group is non degenerate, then $V(0) = 0$. Also, the operator

$$R(\lambda) := \int_0^\infty \lambda^n e^{-\lambda t} V(t)\, dt\ , \quad \mathrm{Re}\lambda > a\ , \tag{C.9}$$

is invertible. There exists a unique operator $A$ such that for all $x \in$

$$(\lambda - A)^{-1} x = \int_0^\infty \lambda^n e^{-\lambda t} V(t) x\, dt\ , \tag{C.10}$$

with domain equal to the range of $(\lambda - A)^{-1}$. $A$ is called the *generator* of $\{V(t),\ t \geq 0\}$. Note that $A$ does not depend on the choice of $\lambda$.

**Propositon C.2** *Take $n \in \mathbb{N}$. For a non degenerate, n-times integrated, exponentially bounded semigroup $\{V(t),\ t \geq 0\}$ with generator generator $A$, we have that:*

1. *For all $x \in \mathcal{D}(A)$ and $t \geq 0$*

$$\begin{aligned} V(t)x \in \mathcal{D}(A)\ , \quad & AV(t)x = V(t)Ax\ , \\ V(t)x = \left(\frac{t^n}{n!}x\right) + & \int_0^t V(s)Ax\, ds\ . \end{aligned} \tag{C.11}$$

2. *For all $x \in \overline{\mathcal{D}(A)}$ and $t \geq 0$*

$$\begin{aligned} & \int_0^t V(s)x\, ds \in \mathcal{D}(A)\ , \\ & A\int_0^t V(s)x\, ds = V(t)x - \left(\frac{t^n}{n!}x\right)\ . \end{aligned} \tag{C.12}$$

3. *For all $x \in \mathcal{D}(A^n)$ and $t \geq 0$*

$$V^{(n)}(t)x = V(t)A^n x + \sum_{k=0}^{n-1} \left(\frac{t^k}{k!}\right) A^k x\ . \tag{C.13}$$

4. *For all $x \in \mathcal{D}(A^{n+1})$ and $t \geq 0$*

$$\frac{d}{dt}V^{(n)}(t)x = AV^{(n)}(t)x = V^{(n)}(t)Ax\ . \tag{C.14}$$

## C.3 The Abstract Cauchy Problem

The material regarding the abstract Cauchy problem in this section can be found in [1], [5], [12] [13] and [20].

Consider the Cauchy problem

$$\begin{aligned} u'(t) &= Au(t)\ , \quad t \geq 0\ , \\ u(0) &= x\ , \end{aligned} \tag{C.15}$$

where $A$ is a linear operator on $X$ with $\mathcal{D}(A) \subseteq X$.

**Definition C.4** *A function $u(t) : [0,\infty) \to X$ is called a solution of the Cauchy problem if:*

1. *$u(\cdot) \in C^1\{[0,\infty), X\} \cap C\{[0,\infty), \mathcal{D}(A)\}$.*
2. *$u(\cdot)$ satisfies equation (C.15).*

**Definition C.5** *The Cauchy problem is said to be uniformly well-posed on $E \subset X$ (where $\overline{E} = X$) if:*

1. *A unique solutions exists for any $x \in E$.*
2. *For any $T > 0$, the solution is uniformly stable for $t \in [0,T]$ with respect to the initial data.*

**Theorem C.1** *Suppose that $A$ is a closed, densely defined operator on $X$. Then the following statements are equivalent:*

1. *The Cauchy problem is uniformly well-posed on $\mathcal{D}(A)$.*
2. *The operator $A$ is the generator of a $C_0$-semigroup $\{S(t),\ t \geq 0\}$.*
3. *The Miyadera-Feller-Phillips-Hille-Yosida (MFPHY) conditions are fulfilled, that is, there exists a $K > 0$ and $\omega \in \mathbb{R}$ such that*

$$\left\| (Re\lambda - \omega)^{n+1} R_A^{(n)}(\lambda)/n! \right\| \leq K\ , \quad Re\lambda > \omega\ ,\ n = 0, 1, 2, \ldots\ .$$

*In this case, the solution of the Cauchy problem is given by*

$$u(t) = S(t)x\ .$$

If $A$ is a linear, closed, densely defined operator on $X$, then define $[\mathcal{D}(A^n)]$ as the Banach space

$$\{\mathcal{D}(A^n)\ ,\ \|x\|_n := \|x\| + \|Ax\| + \ldots \|A^n x\|\}\ .$$

**Definition C.6** *The Cauchy problem is said to be $(n,\omega)$-well-posed if for any $x \in \mathcal{D}(A^{n+1})$:*

1. *There exists a unique solution $u(\cdot)$.*
2. *There exists a $K > 0$ and $\omega \in \mathbb{R}$ such that*

$$\|u(t)\| \leq Ke^{\omega t}\ \|x\|_n\ .$$

**Theorem C.2** *Let $A$ be a densely defined linear operator on $X$ with nonempty resolvent set. Then the following statements are equivalent:*

1. *$A$ is the generator of a non degenerate, $n$-times integrated, exponentially bounded semigroup $\{V(t),\ t \geq 0\}$.*
2. *The Cauchy problem is $(n, \omega)$-well-posed.*

*In this case, the solution of the Cauchy problem is given by*

$$u(t) = V^{(n)}(t)x \ .$$

**Definition C.7** *A function $v(\cdot) \in C\{[0, \infty), X\}$ is called an $n$-integrated solution of (C.15) if for all $t \geq 0$, we have that $\int_0^t v(s)ds \in \mathcal{D}(A)$ and*

$$v(t) = \frac{t^n}{n!}x + A \int_0^t v(s)ds \ . \tag{C.16}$$

**Theorem C.3** *If $A$ generates an $n$-times integrated semigroup $\{V(t),\ t \geq 0\}$ (not necessarily non degenerate), then for any $x \in X$, (C.15) has unique $n$-times integrated solution, given by*

$$v(t) = V(t)x \ . \tag{C.17}$$

Consider the problem

$$\begin{aligned} u(t) &= A \int_0^t u(s)ds + x \ , \quad t \in [0, T] \ , \\ u(0) &= x \in \mathcal{D}(A) \ , \end{aligned} \tag{C.18}$$

where $A$ is a closed, densely defined operator, generating a non degenerate, 1-times integrated, exponentially bounded semigroup $\{V(t),\ t \geq 0\}$.

**Definition C.8** *A function $u(t) : [0, \infty) \to X$ is called a solution of (C.18) if:*

1. *$u(\cdot) \in C\{[0, \infty), X\}$.*
2. *$\int_0^t u(s)ds \in \mathcal{D}(A)$ for all $t \geq 0$.*
3. *$u(\cdot)$ satisfies equation (C.18).*

**Propositon C.3** *The function*

$$u(t) := \frac{d}{dt}V(t)x \ , \quad t \geq 0 \ , \tag{C.19}$$

*is a solution to (C.18).*

**Proof:** We show that $u(\cdot)$ satisfies the requirements to be a solution of equation (C.18).

(1) By property (3) of Proposition C.2, we have that

$$\frac{d}{dt}V(t)x = V(t)Ax + x \ .$$

Now $\{V(t),\ t \geq 0\}$ is strongly continuous, so $V(\cdot)Ax$ is continuous and therefore so to is $V'(\cdot)x$.

(2) As $V'(\cdot)x$ is continuous, we have that

$$\int_0^t \frac{d}{ds}V(s)xds = [V(s)x]_0^t = V(t)x \ .$$

Now by property (1) of Proposition C.2, we have that $V(t)x \in \mathcal{D}(A)$ as $x \in \mathcal{D}(A)$, as required.

(3) By property (1) and (3) of Proposition C.2, we have that

$$\frac{d}{dt}V(t)x \;=\; AV(t)x + x = A\int_0^t \frac{d}{ds}V(s)xds + x \ ,$$

as required. ■

**Propositon C.4** . *If $u(\cdot)$ is a solution to (C.18), then*

$$v(t) := \int_0^t u(s)ds \ , \quad t \geq 0 \ , \tag{C.20}$$

*is an 1-times integrated solution to (C.15).*

**Proof:** We need to show that $v(\cdot)$ satisfies the requirements for a solution to (C.15). As $u(\cdot)$ belongs to $C\{[0,\infty), X\}$, then $v(\cdot)$ belongs to $C\{[0,\infty), X\}$

We have that $\int_0^t u(s)ds \in \mathcal{D}(A)$ and

$$A\int_0^t u(s)ds = u(t) - x \ .$$

Therefore $\int_0^t \left(A\int_0^s u(r)ds\right)dr$ exists and because $A$ is closed

$$\int_0^t v(s)ds = \int_0^t\int_0^s u(r)drds \in \mathcal{D}(A) \ .$$

Finally, $v(\cdot)$ satisfies equation (C.16) as

$$\begin{aligned} v(t) &= \int_0^t u(s)ds = \int_0^t \left(A\int_0^s u(r)dr + x\right)ds \\ &= A\int_0^t \left(\int_0^s u(r)dr\right)ds + tx = A\int_0^t v(s)ds + \frac{t}{1!}x \ , \end{aligned}$$

since $A$ is closed. ■

**Corollary C.1** *The function*

$$u(t) = \frac{d}{dt}V(t)x \ , \quad t \geq 0 \ ,$$

*is the unique solution to (C.18).*

**Proof:** Follows from Proposition C.3, Proposition C.4 and Theorem C.3. ■

## C.4 Some Additional Results for a 1-times Integrated Semigroups

The following results we demonstrate in this section are needed in Chapter 5.

In this section $A$ is closed, densely defined operator, generating $\{V(t),\ t \geq 0\}$, a non degenerate, 1-times exponentially bounded semigroup.

**Lemma C.1** *If $f(t) : [0,T] \to X$ is continuously differentiable on $[0,T]$, then*

$$u(t) := \int_0^t V(t-s)f'(s)ds \ , \quad t \in [0,T] \ , \tag{C.21}$$

*satisfies the equation*

$$u(t) = \int_0^t Au(s)ds + \int_0^t f(s)ds - tf(0) \ , \quad t \in [0,T] \ . \tag{C.22}$$

**Proof:** For $t \in [0,T]$, define

$$v_t(s) = \int_0^{t-s} V(r)f'(s)dr \ .$$

Now

$$v_t'(s) \ = \ -V(t-s)f'(s) + \int_0^{t-s} V(r)f''(s)dr \ .$$

This implies

$$\begin{aligned} v_t(t) - v_t(0) &= \int_0^t v_t'(s)ds = -\int_0^t V(t-s)f'(s)ds + \int_0^t \int_0^{t-s} V(r)f''(s)dr\ ds \\ &= -u(t) + \int_0^t \int_0^{t-s} V(r)f''(s)dr\ ds \ . \end{aligned}$$

Therefore

$$u(t) \ = \ \int_0^t V(r)f'(0)dr + \int_0^t \int_0^{t-s} V(r)f''(s)dr\ ds \ .$$

Now $\int_0^t V(r)f'(0)dr \in \mathcal{D}(A)$ and

$$A\int_0^t V(r)f'(0)dr = V(t)f'(0) - tf'(0) .$$

Therefore

$$\int_0^t A\left(\int_0^\tau V(r)f'(0)dr\right)d\tau = \int_0^t V(\tau)f'(0)d\tau - \int_0^t \tau f'(0)d\tau .$$

Now $\int_0^{t-s} V(r)f''(s)dr \in \mathcal{D}(A)$ and

$$A\int_0^{t-s} V(r)f''(s)dr = V(t-s)f''(s) - (t-s)f''(s) .$$

Hence, by using closedness of $A$, $\int_0^t \int_0^{t-s} V(r)f''(s)dr\, ds \in \mathcal{D}(A)$ and

$$\begin{aligned}
&\int_0^t A\left(\int_0^\tau \int_0^{\tau-s} V(r)f''(s)dr\, ds\right)d\tau \\
&= \int_0^t \int_0^\tau A\left(\int_0^{\tau-s} V(r)f''(s)dr\right)ds\, d\tau \\
&= \int_0^t \int_0^\tau V(\tau-s)f''(s)ds\, d\tau - \int_0^t \int_0^\tau (\tau-s)f''(s)ds\, d\tau .
\end{aligned}$$

Now

$$\begin{aligned}
&\int_0^t \int_0^\tau V(\tau-s)f''(s)ds\, d\tau \\
&= \int_0^t \int_0^t I_{[0,s]}(\tau)V(\tau-s)f''(s)ds\, d\tau = \int_0^t \int_0^t I_{[0,s]}(\tau)V(\tau-s)f''(s)d\tau\, ds \\
&= \int_0^t \int_s^t V(\tau-s)f''(s)d\tau\, ds = \int_0^t \int_0^{t-s} V(\tau)f''(s)d\tau\, ds \\
&= u(t) - \int_0^t V(r)f'(0)dr ,
\end{aligned}$$

and

$$\begin{aligned}
\int_0^t \int_0^\tau (\tau-s)f''(s)ds\, d\tau &= \int_0^t \left([(\tau-s)f'(s)]_0^\tau + \int_0^\tau f'(s)ds\right)d\tau \\
&= \int_0^t (-\tau f'(0) + f(\tau) - f(0))\, d\tau \\
&= -\int_0^t \tau f'(0)d\tau + \int_0^t f(\tau)d\tau - tf(0) .
\end{aligned}$$

So $u(t) \in \mathcal{D}(A)$ for all $t \in [0,T]$ and

$$\begin{aligned}
\int_0^t Au(\tau)d\tau &= \left[\int_0^t V(\tau)f'(0)d\tau - \int_0^t \tau f'(0)d\tau\right] + \left[\left(u(t) - \int_0^t V(r)f'(0)dr\right)\right. \\
&\quad \left. - \left(-\int_0^t \tau f'(0)d\tau + \int_0^t f(\tau)d\tau - tf(0)\right)\right] \\
&= u(t) - \int_0^t f(\tau)d\tau + tf(0) ,
\end{aligned}$$

as required. ■

**Propositon C.5** *If $f(t) : [0,T] \to X$ is continuously differentiable on $[0,T]$, then*

$$u(t) := \int_0^t V(t-s)f'(s)ds + V(t)f(0) \ , \quad t \in [0,T] \ , \tag{C.23}$$

*satisfies the equation*

$$u(t) = A\int_0^t u(s)ds + \int_0^t f(s)ds \ , \quad t \in [0,T] \ . \tag{C.24}$$

**Proof:** We know that

$$\int_0^t V(t-s)f'(s)ds = \int_0^t \left(A\int_0^\tau V(\tau-s)f'(s)ds\right) d\tau + \int_0^t f(s)ds - tf(0) \ .$$

Therefore by closedness of $A$

$$\int_0^t V(t-s)f'(s)ds = A\int_0^t \int_0^\tau V(\tau-s)f'(s)ds\, d\tau + \int_0^t f(s)ds - tf(0) \ ,$$

Therefore

$$\begin{aligned}
& u(t) \\
= & \int_0^t V(t-s)f'(t)ds + V(t)f(0) \\
= & \left(A\int_0^t \int_0^\tau V(\tau-s)f'(s)ds\, d\tau + \int_0^t f(s)ds - tf(0)\right) \\
& + (V(t)f(0) - tf(0) + tf(0)) \\
= & \left(A\int_0^t \int_0^\tau V(\tau-s)f'(s)ds\, d\tau + \int_0^t f(s)ds - tf(0)\right) \\
& + \left(A\int_0^t V(\tau)f(0)d\tau - tf(0)\right) \\
= & A\int_0^t \left(\int_0^\tau V(\tau-s)f'(s)ds + V(\tau)f(0)\right) d\tau + \int_0^t f(s)ds \\
= & A\int_0^t u(\tau)d\tau + \int_0^t f(s)ds \ ,
\end{aligned}$$

as required. ■

# Appendix D

# Examples of Semigroups

The examples in this chapter are based on material from [4], [6], [16], [21] and [20].

## D.1 Heat Equation

We start this section by defining operators $A$ and $\{S(t),\ t \geq 0\}$.

Let $\mathcal{O}$ be the following open, bounded set in $\mathbb{R}^N$

$$\mathcal{O} = \left\{x \in \mathbb{R}^N \ ; \ 0 < x_i < a_i \ , \ i = 1, 2 \ldots, N\right\} \ .$$

Define the operator $A$ on $L^2(\mathcal{O})$ by

$$\begin{aligned} \mathcal{D}(A) &= H^2(\mathcal{O}) \cap H_0^1(\mathcal{O}) \subset L^2(\mathcal{O}) \ , \\ A &= \triangle_x \ , \end{aligned}$$

where the Laplace operator $\triangle_x$ is understood in the sense of distributions. Note that $H^2(\mathcal{O})$ and $H_0^1(\mathcal{O})$ are the classical Sobolev spaces

$$\begin{aligned} H_0^1(\mathcal{O}) &= \left\{u \in L^2(\mathcal{O}) \ ; \ \frac{\partial u}{\partial x_i} \in L^2(\mathcal{O}) \ , \ i = 1, 2, \ldots, N \text{ and } u = 0 \text{ on } \partial\mathcal{O}\right\} \ , \\ H^2(\mathcal{O}) &= \left\{u \in L^2(\mathcal{O}) \ ; \ \triangle_x u \in L^2(\mathcal{O})\right\} \ . \end{aligned}$$

$A$ is a self adjoint, closed, densely define operator on $L^2(\mathcal{O})$, with eigenvalues

$$-\sum_{i=1}^{N} \frac{k_i^2 \pi^2}{a_i^2} \ , \quad k_i \in \mathbb{N} \ , \ i = 1, 2, \ldots, N \ ,$$

and eigenvectors

$$w_{k_1,\ldots,k_N} = \prod_{i=1}^{N} \sqrt{\frac{2}{a_i}} \sin\left(\frac{k_i \pi x_i}{a_i}\right) \ , \quad k_i \in \mathbb{N} \ , \ i = 1, 2, \ldots, N \ .$$

Let $\{-\lambda_k\}_{k=1}^{\infty}$ and $\{e_k\}_{k=1}^{\infty}$ be an ordering of the eigenvalues and eigenvectors of $A$ respectively such that

$$-\lambda_1 \geq -\lambda_2 \geq \dots .$$

Note that $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for $L^2(\mathcal{O})$. We have that $A$ generates a $C_0$-semigroup $\{S(t),\ t \geq 0\}$ on $L^2(\mathcal{O})$, given by

$$S(t)v = \sum_{k=1}^{\infty} v_k e^{-\lambda_k t} e_k \ ,$$

where $v$ has the expansion in $L^2(\mathcal{O})$

$$v = \sum_{k=1}^{\infty} v_k e_k \ .$$

Consider the problem

$$\begin{aligned} &\frac{\partial u(t,x)}{\partial t} = \triangle_x u(t,x) \ , \quad t \in [0,T] \ , \ x \in \mathcal{O} \ , \\ &u(t,x) = 0 \ , \quad x \in \partial\mathcal{O} \ , \\ &u(0,x) = u^0(x) \ , \quad x \in \mathcal{O} \ . \end{aligned} \tag{D.1}$$

If $u^0 \in \mathcal{D}(A)$, then equation (D.1) can be can considered as the operator equation

$$\begin{aligned} &u'(t) = Au(t) \ , \quad t \in [0,T] \ , \\ &u(0) \in \mathcal{D}(A) \ , \end{aligned} \tag{D.2}$$

in the space $L^2(\mathcal{O})$, with unique solution given by

$$u(t) = S(t)u(0) \ , \quad t \in [0,T] \ .$$

## D.2 Wave Equation

We start this section by defining operators $A$, $\{\mathcal{C}(t),\ t \geq 0\}$, $\{\mathcal{S}(t),\ t \geq 0\}$, $\mathcal{A}_1$, $\{U(t),\ t \geq 0\}$, $\mathcal{A}_2$ and $\{V(t),\ t \geq 0\}$.

Let $\Omega = (0,1)$.

Define the operator $A$ on $L^2(\Omega)$ by

$$\begin{aligned} &\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega) \subset L^2(\Omega) \ , \\ &A = \frac{d^2}{dx^2} \ , \end{aligned}$$

where $\frac{d^2}{dx^2}$ is understood in the sense of distributions. Note that $H^2(\Omega)$ and $H_0^1(\Omega)$ are the classical Sobolev spaces

$$H_0^1(\Omega) = \left\{ u \in L^2(\Omega) \ ; \ \frac{du}{dx} \in L^2(\Omega) \ , u(0) = u(1) = 0 \right\} \ ,$$
$$H^2(\Omega) = \left\{ u \in L^2(\Omega) \ ; \ \frac{d^2u}{dx^2} \in L^2(\Omega) \right\} \ .$$

$A$ is a self adjoint, closed, densely defined operator on $L^2(\Omega)$, with eigenvalues $-\mu_k$ and eigenvectors $e_k$ where

$$\mu_k = k^2\pi^2 \ , \ e_k = \sqrt{2}\sin(k\pi x) \ , \quad k \in \mathbb{N} \ .$$

Define the operators on $\{\mathcal{C}(t), \ t \geq 0\}$ and $\{\mathcal{S}(t), \ t \geq 0\}$ on $L^2(\Omega)$ by

$$\mathcal{C}(t)v := \sum_{k=1}^{\infty} \cos(\sqrt{\mu_k}t)v_k e_k \ , \quad \mathcal{S}(t)v := \sum_{k=1}^{\infty} \frac{\sin(\sqrt{\mu_k}t)}{\sqrt{\mu_k}} v_k e_k \ ,$$

where $v$ has the expansion in $L^2(\Omega)$

$$v = \sum_{k=1}^{\infty} v_k e_k \ .$$

Define the operator $\mathcal{A}_1$ on $H_0^1(\Omega) \times L^2(\Omega)$ by

$$\mathcal{D}(\mathcal{A}_1) = \mathcal{D}(A) \times H_0^1(\Omega) \ ,$$
$$\mathcal{A}_1 = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix} \ .$$

$\mathcal{A}_1$ generates a $C_0$-semigroup $\{U(t), \ t \geq 0\}$ on $H_0^1(\Omega) \times L^2(\Omega)$, where

$$U(t) = \begin{pmatrix} \mathcal{C}(t) & \mathcal{S}(t) \\ \mathcal{C}'(t) & \mathcal{C}(t) \end{pmatrix} \ , \quad t \geq 0 \ .$$

Define the operator $\mathcal{A}_2$ on $L^2(\Omega) \times L^2(\Omega)$ by

$$\mathcal{D}(\mathcal{A}_2) = \mathcal{D}(A) \times L^2(\Omega) \ ,$$
$$\mathcal{A}_2 = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix} \ .$$

$\mathcal{A}_2$ generates a non degenerate, 1-times integrated, exponentially bounded semigroup $\{V(t), \ t \geq 0\}$ on $L^2(\Omega) \times L^2(\Omega)$, where

$$V(t) = \begin{pmatrix} \mathcal{S}(t) & \int_0^t \mathcal{S}(s)ds \\ \mathcal{C}(t) - I & \mathcal{S}(t) \end{pmatrix} \ , \quad t \geq 0 \ .$$

Consider the problem

$$\begin{aligned}
&\frac{\partial^2 u(t,x)}{\partial t^2} = \frac{\partial^2 u(t,x)}{\partial x^2} \ , \quad t \in [0,T] \ , \ x \in \Omega \ , \\
&u(t,0) = u(t,1) = 0 \ , \quad t \in [0,T] \ , \\
&u(0,x) = u^0(x) \ , \ \frac{\partial u}{\partial t}(0,x) = u^1(x) \ , \quad x \in \Omega \ .
\end{aligned} \tag{D.3}$$

Let

$$w(t) = \begin{pmatrix} u(t) \\ u'(t) \end{pmatrix} \ , \quad w(0) = \begin{pmatrix} u^0 \\ u^1 \end{pmatrix} \ .$$

If $u^0 \in \mathcal{D}(A)$ and $u^1 \in H_0^1(\Omega)$, then equation (D.3) can be can considered as the operator equation

$$\begin{aligned}
&w'(t) = \mathcal{A}_1 w(t) \ , \quad t \in [0,T] \ , \\
&w(0) \in \mathcal{D}(\mathcal{A}_1) \ ,
\end{aligned} \tag{D.4}$$

in the space $H_0^1(\Omega) \times L^2(\Omega)$, with unique solution given by

$$w(t) = U(t)w(0) \ , \quad t \in [0,T] \ .$$

If $u^0 \in \mathcal{D}(A)$ and $u^1 \in L^2(\Omega)$, then equation (D.3) can be can considered as the operator equation

$$\begin{aligned}
&w(t) = tw(0) + \int_0^t \mathcal{A}_2 w(s) ds \ , \quad t \in [0,T] \ , \\
&w(0) \in \mathcal{D}(\mathcal{A}_2) \ ,
\end{aligned} \tag{D.5}$$

in the space $L^2(\Omega) \times L^2(\Omega)$, with unique solution given by

$$w(t) = V(t)w(0) \ , \quad t \in [0,T] \ .$$

If $u^0 \in \mathcal{D}(A^2)$ and $u^1 \in L^2(\Omega)$, then equation (D.3) can be can considered as the operator equation

$$\begin{aligned}
&w'(t) = \mathcal{A}_2 w(t) \ , \quad t \in [0,T] \ , \\
&w(0) \in \mathcal{D}(\mathcal{A}_2^2) \ ,
\end{aligned} \tag{D.6}$$

in the space $L^2(\Omega) \times L^2(\Omega)$, with unique solution given by

$$w(t) = V^{(1)}(t)w(0) \ , \quad t \in [0,T] \ .$$

# List of Symbols

| | |
|---|---|
| $S(\mathbb{R}^d)$ | space of tempered test functions |
| $S'(\mathbb{R}^d)$ | space of tempered distributions |
| $\mathcal{B}(S'(\mathbb{R}^d))$ | weak star topology on $S'(\mathbb{R}^d)$ |
| $\mu$ | probability measure on $(S'(\mathbb{R}^d), \mathcal{B}(S'(\mathbb{R}^d))$, see Section 1.1 |
| $H$ | a separable Hilbert Space |
| $\{e_i\}_{i=1}^{\infty}$ | orthonormal basis for $H$ |
| $\|\cdot\|_H$ | norm on $H$ |
| $\langle \cdot\,, \cdot \rangle_H$ | inner product on $H$ |
| $H_{\mathbb{C}}$ | complexification of $H$ |
| $\|\cdot\|_{H_{\mathbb{C}}}$ | norm on $H_{\mathbb{C}}$ |
| $\langle \cdot\,, \cdot \rangle_{H_{\mathbb{C}}}$ | inner product on $H_{\mathbb{C}}$ |
| $L^2(\mu)$ | space of square integrable functions with values in $\mathbb{R}$, see Section 1.1 |
| $\|\cdot\|_{L^2(\mu)}$ | usual norm on $L^2(\mu)$ |
| $\langle \cdot, \cdot \rangle_{L^2(\mu)}$ | usual inner product on $L^2(\mu)$ |
| $L^2(H)$ | space of square integrable functions with values in $H$, see Section 1.1 |
| $\|\cdot\|_{L^2(H)}$ | usual norm on $L^2(H)$ |
| $\langle \cdot, \cdot \rangle_{L^2(H)}$ | usual inner product on $L^2(H)$ |
| $\|\cdot\|_{L^2(\mathbb{R})}$ | usual norm on $L^2(\mathbb{R})$ |
| $\langle \cdot, \cdot \rangle_{L^2(\mathbb{R})}$ | usual inner product on $L^2(\mathbb{R})$ |
| $\|\cdot\|_{L^2(\mathbb{R}^d)}$ | usual norm on $L^2(\mathbb{R}^d)$ |
| $\langle \cdot, \cdot \rangle_{L^2(\mathbb{R}^d)}$ | usual inner product on $L^2(\mathbb{R}^d)$ |
| $h_n(\cdot)$ | Hermite Polynomials, see appendix B |
| $\xi_n(\cdot)$ | Hermite Functions, see appendix B |
| $C_{a,b,k}$ | see Proposition B.2 |
| $\eta_i$ | see Section 1.2 |
| $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ | |
| $\mathcal{J} = (\mathbb{N}_0^{\mathbb{N}})_c$ | space of finite sequences $\alpha = (\alpha_1, \alpha_2, \ldots)$, $\alpha_i \in \mathbb{N}_0$, see Section 1.2 |
| $\alpha! = \alpha_1!\alpha_2!\ldots$ | where $\alpha \in \mathcal{J}$ |
| $\{H_\alpha\}_{\alpha\in\mathcal{J}}$ | orthogonal basis for $L^2(\mu)$, see Section 1.2 |
| index $\alpha$ | see Section 1.2.1 |
| $\Gamma_n$ | see Section 1.2.1 |

| | |
|---|---|
| $(S)_\rho$ | spaces of $\mathbb{R}$-valued stochastic test functions, see Definition 1.1 |
| $\lvert\cdot\rvert_{\rho,k}$ | see Definition 1.1 |
| $(S)_{-\rho}$ | spaces of $\mathbb{R}$-valued stochastic distributions, see Definition 1.1 |
| $\lvert\cdot\rvert_{-\rho,-q}$ | see Definition 1.1 |
| $S(H)_\rho$ | spaces of $H$-valued stochastic test functions, see Definition 1.2 |
| $S(H)_{\rho,k}$ | see Definition 1.3 |
| $\lVert\cdot\rVert_{\rho,k}$ | see Definition 1.3 |
| $\langle\cdot,\cdot\rangle_{\rho,k}$ | see Definition 1.3 |
| $S(H)_{-\rho}$ | spaces of $H$-valued stochastic distributions, see Definition 1.2 |
| $S(H)_{-\rho,-q}$ | see Definition 1.3 |
| $\lVert\cdot\rVert_{-\rho,-q}$ | see Definition 1.3 |
| $A(q)$ | see Lemma 1.2 |
| $\mathbb{K}_q^n$ | neighbourhood of zero, see Definition 2.1 |
| $\mathbb{K}_q$ | neighbourhood of zero, see Definition 2.1 |
| $\overline{\mathbb{K}}_q^n$ | neighbourhood of zero, see Definition 2.1 |
| $\overline{\mathbb{K}}_q$ | neighbourhood of zero, see Definition 2.1 |
| $\mathcal{H}F(z)$ | Hermite transform of $F \in S(H)_{-1}$, see Definition 2.3 |
| $\widetilde{F}(z)$ | Hermite transform of $F \in S(H)_{-1}$, see Definition 2.3 |
| $\{\beta_i(\cdot)\}_{i=1}^\infty$ | a sequence of independent Brownian motions |
| $W(t)$ | $H$ or $U$-valued Wiener process, see Sections 3.1 and 3.3 |
| $n(i,j)$ | function defined in equation (3.1) |
| $\epsilon_n$ | see Section 3.2 |
| $N(0,\sigma^2)$ | the normal distribution with mean 0 and variance $\sigma^2$ |
| $I_{[a,b]}(u)$ | indicator function, equals 1 on $[a,b]$ and 0 elsewhere |
| $\theta_{i,k}(t)$ | see equations (3.6) and (3.7) |
| $\theta_k(t)$ | see equations (3.6) and (3.7) |
| $\mathbb{W}(t)$ | $H$ or $U$-valued singular white noise, see Section 3.3.1 |
| $\kappa_{i,k}(t)$ | see equations (3.8) and (3.9) |
| $\kappa_k(t)$ | see equations (3.8) and (3.9) |
| $\kappa_k^{(n)}(t)$ | $n$th derivative of $\kappa_k(t)$ |
| $\mathbb{W}^{(n)}(t)$ | $n$th derivative of $\mathbb{W}(t)$ in $S(U)_{-1}$, or $S(U)_{-1}$, see Section 3.3.2 |
| $[\![n]\!]$ | greatest integer less than or equal to $n$ |
| $K_{a,b,n}$ | see Lemma 3.4 |
| $W_Q(t)$ | $Q$-Wiener process, see Section 3.3.3 |
| $\vartheta_k(t)$ | see equations (3.13) and (3.14) |
| $F \diamond G$ | wick product, see Definition 3.1 |
| $\mathrm{E}[F]$ | generalised expectation, see Definition 3.2 |
| $\int_{\mathbb{R}} F(t)\delta W(t)$ | Hitsuda-Skorohod integral, see Definition 3.4 |
| $\mathcal{D}(A)_{-\rho}$ | see Definition 3.5 |
| $\{S(t),\ t \geq 0\}$ | generally a $C_0$-semigroup, see Definition C.1 |
| $\{U(t),\ t \geq 0\}$ | usually a $C_0$-semigroup |
| $\{V(t),\ t \geq 0\}$ | usually an $n$-times integrated semigroup, see Definition C.3 |

| | |
|---|---|
| $\mathcal{L}(H)$ | space of bounded linear operators on $H$ |
| $H_1$ | a separable Hilbert space |
| $\mathcal{L}(H, H_1)$ | space of continuous linear maps from $H$ to $H_1$ |
| $U$ | a separable Hilbert space |
| $\{f_i\}_{i=1}^{\infty}$ | orthonormal basis for $U$ |
| $\int_0^t S(s)\delta W(s)$ | generalised stochastic convolution, see equation (3.25) |
| $\int_0^t V(s)\delta W^{(n)}(s)$ | generalised $n^{th}$ stochastic convolution, see equation (3.34) |
| $W_A(t)$ | see Propositions 4.1 and 5.6 |
| $W_A^{(1)}(t)$ | see Proposition 5.2 |
| $\Phi$ | usually a linear topological space |
| $\Phi'$ | dual space of $\Phi$, see Section A.3.1 |
| $\mathcal{A}_1$ | see Section D.2 |
| $\mathcal{A}_2$ | see Section D.2 |
| $H_0^1(\mathcal{O})$ | see Section D.1 |
| $H^2(\mathcal{O})$ | see Section D.1 |
| $H_0^1(\Omega)$ | see Section D.2 |
| $H^2(\Omega)$ | see Section D.2 |

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