VARIOUS PROBLEMS

IN

INHOMOGENEOUS DIOPHANTINE APPROXIMATION

by

Peter E. BLANKSBY B.Sc. Hons (Adel.)

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Department of Mathematics,

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CONTENTS

SUMMARY

This thesis deals with several problems from the theory of two dimensional inhomogeneous diophantine approximation. The opening chapter introduces the problems considered by tracing their history, and the development of some closely related questions.

The problems are attacked by the "divided cell method", which was devised by Barnes and Swinnerton-Dyer [1], [2]. If $f(x,y)$ is an indefinite binary quadratic form, then there corresponds to each form f, and real point $P = P(x_0,y_0)$ an inhomogeneous lattice, or grid. By considering a subset of these grid points, it is possible to evaluate certain inhomogeneous minima of f. The geometry of the grid gives rise to a semi-regular continued fraction expansion of the roots of the form f. This approach is expounded in Chapters II and III of the thesis.

One of the main problems considered is a hybrid of the two classical results of Hurwitz and Minkowski on the minima of indefinite binary quadratic forms. Suppose $f(x,y)$ is the form

$$f(x,y) = (\alpha x + \beta y)(\gamma x + \delta y),$$

with determinant $\Delta = |\alpha\delta - \beta\gamma|$. Then, for any real, non-zero constant, η, we define

$$M(f;\eta) = \inf_{\substack{x,y \text{ integral} \\ \neq 0,0}} |(\alpha x + \beta y)(\gamma x + \delta y + \eta)|.$$

Chapter IV provides a systematic method for the evaluation of the function $M(f;\eta)$, for forms that do not represent zero. The method is a modification of the divided cell procedure described in the

previous chapter.

This method enables us to obtain the best possible constant for the mixed form problem. Let

$$k = \frac{(3/49)(366458018\,\phi - 7320551)}{(8238730\,\theta + 392361)\phi - (164581\,\theta + 7838)},$$

$$= 0\cdot 23425\ldots$$

where $\phi = \dfrac{147 + \sqrt{21651}}{6}$ and $\theta = \dfrac{104250 + 2\sqrt{10}}{9005}$.

Then, for all forms that do not represent zero, and all non-zero η,

$$M(f;\eta) \le \Delta k,$$

where equality holds only for an equivalence class of forms. Chapter V is devoted to the proof of the result.

One immediately wonders whether this constant k is an isolated value. A complete answer to this question is provided by the following theorem, which constitutes Chapter VI. Suppose that k' is such that $0 \le k' < k$, then there exist uncountably many forms, each for which there is a corresponding η, such that

$$M(f;\eta) = \Delta k'.$$

In Chapter VII we define a new function, $M^*(f)$, which is an inhomogeneous minimum of f, under certain restrictive conditions. $M^*(f)$ is connected with the function

$$k^+(\phi,\alpha) = \liminf_{x \to +\infty} x|\phi x + y + \alpha|,$$

for ϕ irrational and α real, which was examined in detail by Cassels [3], and Descombes [4]. We show, together with several other results, that $M^*(f) \le \dfrac{27\Delta}{28\sqrt{7}}$, which is a best possible inequality.

Let $$k^+(\phi) = \sup_\alpha k^+(\phi,\alpha),$$

where α is such that $\phi x + y + \alpha$ does not represent zero in integers x,y. Then in Chapter VIII we evaluate $k^+(\phi)$, for all ϕ equivalent to $\frac{\sqrt{5}+1}{2}$. We show that

$$k^+\left(\frac{\sqrt{5}+1}{2}\right) = \frac{3\sqrt{5}-5}{10}$$

$$= 0\cdot1708\ldots,$$

which improves the upper bound of 0•2114... given by Godwin [5], for Khintchine's absolute constant.

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This thesis contains no material which has been accepted for the award of any other degree or diploma in any University. To the best of my knowledge and belief, the thesis contains no material previously published or written by any other person, except where due reference is made in the text of the thesis.

P. E. Blanksby.

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CHAPTER I

INTRODUCTION AND HISTORICAL REVIEW OF SOME TWO DIMENSIONAL PROBLEMS IN INHOMOGENEOUS DIOPHANTINE APPROXIMATION

1. The inhomogeneous approximation problems

If ϕ is an arbitrary irrational number, the homogeneous approximation problem asks the question: how closely can ϕ be approximated by rational numbers, in terms of the square of their denominators? It seeks values of h for which the following inequality is true for infinitely many integer pairs x,y.

$$|\phi - \frac{y}{x}| < \frac{h}{x^2} .$$

This is equivalent to asking the same question of

$$\liminf_{|x| \to \infty} [\inf_{y} |x(\phi x - y)|] \leq h. \qquad (1.1)$$

We may simplify this expression by using the permanent notation $\|x\|$ to denote the distance from x to the nearest integer. We may then replace the lefthand side of (1.1) by the function

$$h(\phi) = \liminf_{|x| \to \infty} |x|.\|\phi x\|. \qquad (1.2)$$

Dirichlet [32] showed that $h(\phi)$ is bounded by 1 for all irrational ϕ, and Hurwitz [32] later showed that the supremum of values taken by $h(\phi)$ over all irrational ϕ is $\frac{1}{\sqrt{5}}$, where equality occurs whenever ϕ is equivalent (in a sense discussed later) to the number $\frac{\sqrt{5}+1}{2}$. If we exclude this equivalence class of irrationals, then for all other ϕ,

$$h(\phi) \le \frac{1}{2\sqrt{2}} .$$

It was shown by Markov [46], [45], [10], that $h(\phi)$ can take only a sequence of discrete values greater than 1/3, and that there are uncountably many ϕ for which $h(\phi) = 1/3$. For each value in this sequence $\frac{1}{\sqrt{5}}, \frac{1}{2\sqrt{2}}, \frac{5}{\sqrt{221}}, \frac{13}{\sqrt{1517}}, \ldots$ there is an equivalence class of ϕ for which $h(\phi)$ takes this value. This problem is closely connected to a similar problem for homogeneous forms (discussed in §2).

Part of this thesis will be concerned with an inhomogeneous analogue of this problem. The *inhomogeneous approximation problem* may be approached from the following geometric ideas.

Suppose C is the circle with unit circumference with some point 0 on it, taken as origin. Consider all those points on the circumference whose arc-lengths from 0 are $|\phi x|$, where x takes all integral values. Denote this set of points by S. Let B be the point whose arc-distance from 0 is β, where the positive sense is taken to be anticlockwise. Then the inhomogeneous approximation problem is concerned with the manner in which the points of S accumulate about the point B. If B is the point 0, then we have again the homogeneous problem.

We will formulate this problem algebraically by defining the function

$$k(\phi,\alpha) = \liminf_{|x| \to \infty} |x| . \| \phi x + \alpha \| , \tag{1.3}$$

where ϕ is irrational, α real, and x integral. It may be assumed that $\|\phi x + \alpha\|$ is never zero, else the problem reverts to the homogeneous type. Various mathematicians (Kronecker, Hermite for example) gave bounds on the function $k(\phi,\alpha)$, and at the turn of the last century Minkowski [47] proved that for all ϕ, α of the above type,

$$k(\phi,\alpha) \le \frac{1}{4} . \tag{1.4}$$

Grace [31] then showed that this result was best possible in the sense that, for each $\varepsilon > 0$, he constructed an irrational ϕ such that

$$k(\phi,\tfrac{1}{2}) > \frac{1}{4 + \varepsilon} .$$

In fact it has been shown by Morimoto [50], Barnes [7], that for each d with $0 \le d \le \frac{1}{4}$, there exist uncountably many pairs ϕ,α such that

$$k(\phi,\alpha) = d.$$

Hartman [33], Descombes [23], have investigated this problem in the case when α is restricted to be rational, say $\alpha = -t/s$. The latter author has shown that this is in fact, equivalent to the problem (1.1) with the inclusion of the additional conditions,

$$x \equiv t(\text{mod } s), \quad y \equiv 0(\text{mod } s).$$

This suggests the seemingly more general problem of evaluating (1.1), under the conditions

$$x \equiv a(\text{mod } s), \quad y \equiv b(\text{mod } s),$$

where $(a,b) = 1$. But it is shown (see Théorème 1, [23]) that, in a sense, this reduces to the case $a = t$, $b = 0$ above.

Referring to the geometrical interpretation of the general problem, one would not expect the same results to follow through if the set S were restricted to those points corresponding to positive integers x. This is the case, and it leads to an examination of the function

$$k^+(\phi,\alpha) = \liminf_{x \to +\infty} x.\|\phi x + \alpha\|. \qquad (1.5)$$

We will call this the *positive inhomogeneous approximation problem*.

It may again be supposed that $\|\phi x + \alpha\| \neq 0$ for any integer x. (1.6)

We now seek values of c, such that for all such ϕ,α,

$$k^+(\phi,\alpha) \leq c. \qquad (1.7)$$

In 1926 Khintchine [44] proved that (1.7) was valid for $c = \frac{1}{\sqrt{5}}$, equality holding for $\phi = \frac{\sqrt{5}+1}{2}$, $\alpha = 0$; but this is only the first step in Markov's homogeneous chain, already discussed, and so we will exclude this case. Cole [15] proved that under the condition (1.6), the inequality (1.7) is valid with $c = \frac{3}{1 + 2\sqrt{10}} = 0{\cdot}409\ldots$ The best possible constant $c = \frac{27}{28\sqrt{7}}$, was found by Cassels [11], who constructed an ingenious algorithm, involving the ordinary continued fraction expansion of ϕ, for the evaluation of $k^+(\phi,\alpha)$. Once this constant had been determined, it became a question of whether it was isolated or not. Descombes [24], using this algorithm, showed that there existed only a sequence of discrete values of $k^+(\phi,\alpha)$, greater than $1/\gamma = 0{\cdot}352\ldots$ This is an analogous result to that of Markov. Vera Sôs (Turan) [61] showed that Cassel's arithmetic algorithm can be obtained from the geometric procedure already considered.

At the other end of the range, Barnes [7] has shown that for every d with $0 \leq d \leq \frac{1}{4}$, there are uncountably many pairs ϕ, α for which

$$k^+(\phi,\alpha) = d.$$

Very little is known of the values taken by this function in the range $(\frac{1}{4}, 1/\gamma)$.

Let us define the following functions of ϕ;

$$k(\phi) = \sup_{\alpha} k(\phi,\alpha), \quad k^+(\phi) = \sup_{\alpha} k^+(\phi,\alpha), \tag{1.8}$$

where the suprema are taken over all non-zero α satisfying the requirement (1.6). We have then, for all irrational ϕ,

$$k(\phi) \le \frac{1}{4}, \quad k^+(\phi) \le \frac{27}{28\sqrt{7}}.$$

The following surprising theorem was proved by Khintchine [43].

<u>THEOREM 1.1.</u> *There exists a positive absolute constant δ, such that, for any real number ϕ, there exists at least one number α, for which*

$$x|\phi x + y + \alpha| > \delta, \tag{1.9}$$

for all integers x, y with $x > 0$.

Morimoto [50], Davenport [17], Prasad [55], and Godwin [30], using methods differing from those of Khintchine, were able to determine successively better estimates of possible values taken by δ.

Define, for x integral,

$$c(\phi,\alpha) = \inf_{x} |x|.\|\phi x + \alpha\|, \tag{1.10}$$

$$c^+(\phi,\alpha) = \inf_{x>0} x.\|\phi x + \alpha\|, \tag{1.11}$$

and then

$$c(\phi) = \sup_{\alpha} c(\phi,\alpha), \tag{1.12}$$

$$c^+(\phi) = \sup_{\alpha} c^+(\phi,\alpha), \tag{1.13}$$

where the suprema are taken over non-zero α, under the condition (1.6).

Now clearly from (1.8),

$$c(\phi) \le k(\phi), \quad c^+(\phi) \le k^+(\phi). \tag{1.14}$$

Define the following constants, where the infima are taken over all irrational ϕ;

$$\left.\begin{aligned} c &= \inf_\phi c(\phi), & k &= \inf_\phi k(\phi), \\ c^+ &= \inf_\phi c^+(\phi), & k^+ &= \inf_\phi k^+(\phi). \end{aligned}\right\} \tag{1.15}$$

Then, after Godwin [30] and (1.14),

$$\left.\begin{aligned} 0{\cdot}1407 < \frac{68}{483} < c^+ \le k^+, \\ \text{and} \qquad c^+ < 0{\cdot}2114\ldots \end{aligned}\right\} \tag{1.16}$$

As Davenport (p. 79, [19]) noted, a study of Khintchine's original proof of Theorem 1.1 reveals that the result also holds for negative x, implying that $c > 0$. Cassels [10] has shown that

$$\frac{1}{45{\cdot}2} \le c \le \frac{1}{12}.$$

The earlier results, proving the existence of some of these constants, while of interest, were often of little use in evaluating them, or even for obtaining bounds. The basis for proof of the more recent results is usually some form of continued fraction development. More often ordinary continued fractions were employed, but for some problems the Hurwitz [36] algorithm had certain advantages [17], [55]. In parts of this thesis we will use a more general type of semi-regular continued fraction to attack two of these problems.

2. The inhomogeneous form problems

Suppose $f(x,y)$ is a binary quadratic form given by

$$f(x,y) = ax^2 + bxy + cy^2. \tag{1.17}$$

If $D = b^2 - 4ac > 0$, then f is called indefinite, and we will assume this to be the case throughout this thesis. We will also suppose that f does not represent zero; that is, there do not exist integers $(x',y') \neq (0,0)$, for which $f(x',y') = 0$.

The *homogeneous form problem* is concerned with the infimum of values taken by $f(x,y)$, when (x,y) is an integral point, not the origin. Let

$$m(f) = \inf_{\substack{(x,y)\neq(0,0)\\ \text{integral}}} |f(x,y)|. \tag{1.18}$$

$m(f)$ is called the *homogeneous minimum* of the form f.

Markov showed that $m(f)/\Delta$, where $\Delta = \sqrt{D}$, takes only a countable number of discrete values exceeding 1/3, and that there are uncountably many f, for which $\frac{m(f)}{\Delta} = \frac{1}{3}$. This sequence is identical with that of the homogeneous approximation problem in §1. The corresponding forms are called Markov forms.

We will now define $M(f)$, the *inhomogeneous minimum* of the form f. Suppose $P = P(x_0,y_0)$ is a real two-dimensional point, not the origin. Then

$$M(f;P) = \inf_{(x,y)\ \text{int.}} |f(x + x_0, y + y_0)|, \tag{1.19}$$

and

$$M(f) = \sup_{P} M(f;P), \tag{1.20}$$

where the supremum need only be extended over a complete set of incongruent points (mod 1).

If the supremum in (1.20) is attained, then suppose that C is the set of points P such that

$$M(f) = M(f;P).$$

Let

$$M_2(f) = \sup_{P \text{ not in } C} M(f;P);$$

then if $M_2(f) < M(f)$, it is called the *second minimum* of f; and so on.

It is clear that for any constant K,

$$M(Kf) = |K|.M(f), \tag{1.21}$$

and a very simple argument [52] shows that

$$M(f) \geq \tfrac{1}{4}m(f). \tag{1.22}$$

The basic result on the inhomogeneous minima of forms was proved by Minkowski [47].

THEOREM 1.2. *If f is any indefinite binary quadratic form, then*

$$M(f) \leq \frac{\Delta}{4},$$

and inequality holds for all forms which do not represent zero. Equality holds only for forms "equivalent" to $f = xy$, *in which case*

$$M(f;P) = M(f) = \frac{\Delta}{4},$$

where P is the point $(\tfrac{1}{2},\tfrac{1}{2})$.

The fact that ¼ is best possible, even for forms that do not represent zero, is seen by the sequence of forms

$$h_k(x,y) = x^2 - 2kxy + y^2,$$

with k integral, which can be shown to have inhomogeneous minimum arbitrarily close to $\Delta/4$, for large enough k [16].

It is reasonable to expect that for many forms, very much stronger results than Theorem 1.2 are true, results in which Δ is replaced by other functions of the coefficients a, b and c. Many theorems of this kind have been given by Barnes [4], Heinhold [35], Davenport [16], Inkeri [39], Rogers [57], Bambah [2], Chalk [14], Mordell [48], and others. One particular result of this kind was proved by Barnes:

<u>THEOREM 1.3.</u> *If $f(x,y)$ is given by (1.17), and*

$$\mu(f) = \max \{ |a|, |c|, \min |a \pm b + c| \},$$

then

$$M(f) \le \frac{\mu(f)}{4},$$

where equality can hold only when $M(f) = M(f;P)$, *and* $2P \equiv 0 (\mathrm{mod}\ 1)$.

For a deeper discussion of results of this type consult [52].

Much of the motivation for a study of the function M(f) arose from the desire to determine those integers m for which the quadratic field $k(\sqrt{m})$ (see Chapter 14,[32]) possesses a Euclidean algorithm. The existence of a Euclidean algorithm is equivalent to the existence of an integer ρ of $k(\sqrt{m})$ corresponding to each element ω of the field, with the property that

$$|\mathrm{norm}(\omega + \rho)| < 1. \qquad (1.22)$$

The norm of $k(\sqrt{m})$ is an indefinite binary quadratic form, with rational coefficients, say $f_m(x,y)$. It follows that $k(\sqrt{m})$ is Euclidean if and only if

$$M(f_m;P) < 1 \qquad (1.24)$$

for all rational points P. Consequently if

$$M(f_m) < 1,$$

then $k(\sqrt{m})$ is Euclidean.

The fact that there are only a finite number of such Euclidean fields, is an immediate consequence of the following generalization, due to Davenport [18], of a result considered in §1.

THEOREM 1.4. *Let $f(x,y)$ be a binary quadratic form which does not represent zero, then there exist real x_0 , y_0 such that*

$$|f(x + x_0, y + y_0)| > \frac{\Delta}{128} \tag{1.25}$$

holds for all integers x, y.

In fact the following result is also shown to be true.

THEOREM 1.5. *If $f(x,y)$ has integral coefficients, then there exist rational x_0 , y_0 such that (1.25) holds.*

As a consequence, since $f_m(x,y)$ has integral coefficients, (1.24) cannot hold whenever $\Delta > 128$. Since the determinant of f_m is either $2\sqrt{m}$ or $\sqrt{m}$, then Euclid's algorithm cannot exist for $m > (128)^2$.

We define the absolute quantity M, now known as Davenport's constant, as follows:

$$M = \inf_f{}' M(f)/\Delta, \tag{1.26}$$

where the infimum is taken over all forms which do not represent zero. The best known bounds on M, given by Ennola [28] and Pitman [54], are

$$\frac{1}{30\cdot 69} < M < \frac{1}{12\cdot 92} . \tag{1.27}$$

The methods used to obtain such bounds usually rests on some semi-regular continued fraction expansion of the roots of the forms. Davenport and Ennola used the Hurwitz expansion already mentioned, while Pitman used the more general semi-regular expansions of Barnes and Swinnerton-Dyer [9], a detailed discussion of which will appear in Chapter III. In each case, rules are given for the construction of a chain of integers associated with the appropriate continued fraction development, and these together describe a point (x_0,y_0) with the required property.

During the past twenty years, various methods have been given for the evaluation of the functions M(f;P) and M(f), for a given form f and point P. Davenport [16] obtained M(f) for some of the early Markov forms, and gave the infinite sequence of isolated minima $M_1(f)$, $M_2(f)$, $M_3(f)$,.... for the form $x^2 + xy - y^2$. Varnavides [63], [64], [65] used Davenport's method to evaluate $M(f_m)$ for m = 2, 7, 11, where $f_m(x,y) = x^2 - my^2$. Bambah [1] gave new geometric proofs in the cases m = 7, 11, and using this method obtained $M_2(f_7)$. Barnes and Swinnerton-Dyer [8] considered a more general method applicable to forms with rational coefficients, and which was also used for the determination of the successive minima of certain norm forms.

However, by far the most general and powerful method devised for the evaluation of M(f), where f is an arbitrary indefinite binary quadratic form, is the divided cell method, developed by Barnes and Swinnerton-Dyer [9], [5]. To every form f and point P, there corresponds a two dimensional grid. From the geometry of this grid it is possible to construct a sequence of forms $f_n(x,y)$ $(-\infty<n<\infty)$, all related to f by an integral, unimodular substitution. It is proved that only four values of each of these forms need be evaluated to determine $M(f;P)$. This method will provide the basis for all problems examined in this thesis, and a description of it appears in Chapter III.

The method was used by Barnes and Swinnerton-Dyer [9] for evaluating the critical determinant of certain asymmetric hyperbolic regions, and by Barnes [5] for calculating $M(f)$ for two difficult norm forms. Barnes also used modifications of the method in [6], [7], and Pitman [53], [54] used the method with a great deal of success to calculate the inhomogeneous minima of a subsequence of the symmetric Markov forms, thus extending the work by Davenport.

3. The mixed form problem

In the previous section we discussed certain minima of indefinite forms, where the linear products were either both homogeneous, or both inhomogeneous. Chalk [13] showed that if

$$X = \alpha x + \beta y$$

$$Y = \gamma x + \delta y$$

were linear forms in x, y, then for non-zero real c, there exist *co-prime* integers x, y, such that

$$|(X + c)Y| < \frac{\Delta}{4}, \tag{1.28}$$

where $\Delta = |\alpha\delta - \beta\gamma|$ is the determinant of the forms.

Davenport [21] showed that this result is best possible, in the sense that for every $\varepsilon > 0$, there exist linear forms X and Y, and a non-zero constant c, such that

$$|(X + c)Y| > \frac{\Delta}{4 + \varepsilon},$$

for all co-prime integers x, y. The examples that are given in [21] are perhaps natural ones, in that Y is chosen to be badly approximable homogeneously, and X is badly approximable inhomogeneously.

The question which now arises is whether, on omitting the condition of co-primality on x and y, the constant ¼ still remains best possible. Davenport provides a negative answer to this question by showing that the inequality

$$|(X + c)Y| < \frac{\Delta}{4\cdot 1} \tag{1.29}$$

is always soluble in integers $(x,y) \neq (0,0)$. However if infinitely many solutions are required, then Theorem 1 of [21] indicates that ¼ is then the appropriate best possible constant.

The problem has further been investigated by Kanagasabapathy [41],[42], who successively improved the constant on the right-hand side of (1.29) from $\frac{1}{4\cdot 1}$ to $\frac{1}{4\cdot 25777}$. He also gave $\frac{1}{4\cdot 2847}$ as a lower bound on the best possible constant. Mention of their approach will be made in Chapter IV. A section of this thesis will be devoted to a general arithmetic formulation of this problem, and this will lead to an actual evaluation of the best possible constant, and critical forms.

A further question which, to my knowledge, has not been investigated, is the evaluation of

$$\inf_{Y>0} |(X + c)|Y,$$

where the infimum is extended over integral x, y, such that Y>0. The methods developed in this thesis, with certain modifications, would be suitable to handle this question too.

4. Summary of subsequent chapters

Chapter II of this thesis contains a brief comparison of some of the properties of ordinary and semi-regular continued fractions. Several results that are needed for following chapters will be proved. A discussion of the semi-regular continued fraction to the integer above is undertaken, and a method for transforming such expansions into ordinary continued fractions is introduced.

The basic tool for the results obtained in this thesis will be the divided cell method of Barnes and Swinnerton-Dyer. The method is expounded in Chapter III, and an arithmetic formulation of M(f;P) in terms of semi-regular continued fractions is given. The chapter concludes with several results that provide upper bounds on the value of M(f;P), and a theorem which relates the semi-regular expansions of equivalent (in the usual sense) quadratic irrationals.

A description of a modification of this method which will put the mixed form problem on an arithmetic basis appears in Chapter IV. In Chapter V we will show that the best possible constant for the problem is given by:

$$k = \frac{(3/49)(366458018\ \phi - 7320551)}{(8238730\ \theta + 392361)\phi - (164581\ \theta + 7838)}, \tag{1.30}$$

where $$\phi = \frac{147 + \sqrt{21651}}{6} \quad \text{and} \quad \theta = \frac{104250 + 2\sqrt{10}}{9005}. \tag{1.31}$$

Chapter VI investigates the distribution of the infimum of values taken by mixed forms. In fact we show that for all k' with $0 \leq k' < k$, there exist homogeneous forms X and Y, and non-zero constants c, for which

$$\inf_{(x,y)\neq(0,0)} |(X + c)Y| = k'\Delta.$$

In Chapter VII, further modifications of the divided cell method enable the investigation of a problem for forms, analogous to the Cassels-Descombes' positive approximation problem. For binary quadratic forms f, we define a restricted inhomogeneous minimum, which we denote by $M^*(f)$, and show that

$$M^*(f) \le \frac{27\Delta}{28\sqrt{7}},$$

a best possible result. Extensions of this result are indicated.

The final chapter is devoted to the evaluation of $k^+(\phi)$, where ϕ is equivalent to $\frac{\sqrt{5}+1}{2}$. We prove that

$$k^+\left(\frac{\sqrt{5}+1}{2}\right) = \frac{3\sqrt{5}-5}{10}$$

$$= 0\cdot1708\ldots,$$

which sharpens Godwin's bound of $0\cdot2114\ldots$ on c^+.

CHAPTER II

PROPERTIES OF SEMI-REGULAR CONTINUED FRACTIONS

1. Ordinary continued fractions

Throughout this thesis, extensive use will be made of the so-called semi-regular continued fraction. But first, for reasons of comparison, we will recall a few of the important features of the ordinary continued fraction (O.C.F.). Suppose $\alpha > 0$ is real, then we will denote the O.C.F. expansion of α by

$$\begin{aligned}\alpha &= (a_1, a_2, a_3, \ldots\ldots.) \\ &= a_1 + \frac{1}{a_2+}\ \frac{1}{a_3+}\cdots\cdots,\end{aligned} \tag{2.1}$$

where $a_1 \geq 0$, and $a_i > 0$ for $i > 1$.

The algorithm which produces the *partial quotients* a_i, is as follows: In this chapter, let $[x]$ denote, as is usual, the integral part of x. Then

$$\left.\begin{aligned}\alpha = \alpha_1 &= a_1 + \frac{1}{\alpha_2}, &\text{where } a_1 &= [\alpha_1] \\ \alpha_n &= a_n + \frac{1}{\alpha_{n+1}}, &\text{where } a_n &= [\alpha_n].\end{aligned}\right\} \tag{2.2}$$

Consequently $\alpha = (a_1, a_2, a_3, \ldots\ldots, a_{n-1}, \alpha_n)$. The α_i are called the *complete quotients*. Clearly a given α produces a unique sequence of integers $\{a_n\}$, provided $\alpha_n > 1$, for all $n > 1$.

<u>LEMMA 2.1.</u> *α is rational if and only if its O.C.F. expansion terminates.*

This result is equivalent to the termination of the Euclidean algorithm, and the proof may be found in say [51].

A continued fraction is said to be periodic if two partial quotients are identical. If $\alpha_r = \alpha_{r+n}$, then we write

$$\alpha = (a_1,a_2,\ldots,a_{r-1},\overline{a_r,\ldots,a_{r+n-1}}). \tag{2.3}$$

<u>LEMMA 2.2.</u> *α is a quadratic irrational if and only if its O.C.F. expansion has a non-trivial period.*

The proof may be found in [51].

Suppose α, β are two numbers connected by the relation

$$\alpha = \frac{p\beta + q}{r\beta + s}, \tag{2.4}$$

where p,q,r,s, are integers with $ps - rq = \pm 1$, then α and β are said to be *equivalent.* If $ps - rq = 1$, the equivalence is called *proper* (if not, then *improper*).

<u>LEMMA 2.3.</u> *Two equivalent numbers have identical O.C.F. expansions from some point onwards.*

The proof is given in [51].

A quadratic irrational α is said to be *reduced* (in the sense of Gauss), if

$$\alpha > 1, \quad -1 < \overline{\alpha} < 0, \tag{2.5}$$

where $\overline{\alpha}$ is the algebraic conjugate of α.

<u>LEMMA 2.4.</u> *Let α be a quadratic irrational, then α is reduced if and only if the O.C.F. expansion of α is purely periodic; that is, $\alpha = \alpha_n$, for some $n > 1$.*

A discussion of this kind of reduction may be found in [34], [51].

Now, after Lemma 2.1, we may denote the finite continued fraction

$$(a_1,a_2,\ldots,a_n) = \frac{P_n}{Q_n}, \tag{2.6}$$

where the fraction P_n/Q_n is in its lowest terms. These fractions are called the *convergents* of the O.C.F. development of α, and are obtained by the following recurrence relations [51]:

$$\left.\begin{aligned} &P_0 = 1, \quad Q_0 = 0, \quad P_1 = a_1, \quad Q_1 = 1, \\ &P_{n+1} = a_{n+1}P_n + P_{n-1}, \\ &Q_{n+1} = a_{n+1}Q_n + Q_{n-1}. \end{aligned}\right\} \quad . \tag{2.7}$$

Clearly the P_n and Q_n are positive integers which become arbitrarily large, as $n \to \infty$.

<u>LEMMA 2.5.</u> *If α is irrational, and $\alpha = (a_1,a_2,a_3,\ldots\ldots)$, then*

$$\lim_{n\to\infty} P_n/Q_n = \alpha.$$

The proof (e.g. [51]) is a consequence of the following three results:

$$P_nQ_{n-1} - Q_nP_{n-1} = (-1)^n, \tag{2.8}$$

$$\alpha = (a_1,a_2,\ldots,a_n,\alpha_{n+1}) = \frac{\alpha_{n+1}P_n + P_{n-1}}{\alpha_{n+1}Q_n + Q_{n-1}}, \tag{2.9}$$

which together imply

$$|\alpha - P_n/Q_n| = \frac{1}{Q_n(\alpha_{n+1}Q_n + Q_{n-1})}, \tag{2.10}$$

from which Dirichlet's theorem follows immediately.

2. <u>Semi-regular continued fractions</u>

In this section, a brief outline of semi-regular continued fractions (S.R.C.F.) will be given. The classical notation of Perron [51] will not be used, but a notation which arises naturally from the geometry of the problem will be considered.

The O.C.F. development of a real number α is based on the extraction of the integral part of each complete quotient as it arises, and an inversion of the fractional part to produce the next complete quotient. This yields a unique sequence of integers $\{a_n\}$.

The rules for the development of the kind of S.R.C.F. in which we will be interested are as follows: At each step of the algorithm, instead of always extracting the integer below, we usually allow ourselves the choice of either the integer below, or the integer above. Given a real number α, $|\alpha| > 1$, we define the sequence $\{a_n\}$ by;

$$\left.\begin{aligned}&\text{(i)}\quad \alpha = \alpha_1 = a_1 - \frac{1}{\alpha_2}\,,\ \text{where } a_1 = [\alpha_1] \text{ or } [\alpha_1]+1,\\ &\qquad \text{provided that } |a_1| \ge 2,\\ &\text{(ii)}\quad \alpha_n = a_n - \frac{1}{\alpha_{n+1}}\,,\ \text{where } a_n = [\alpha_n] \text{ or } [\alpha_n]+1,\\ &\qquad \text{provided that } |a_n| \ge 2.\end{aligned}\right\} \tag{2.11}$$

We will denote the S.R.C.F. in square brackets (to distinguish it from the O.C.F., but not to be confused with the integral part notation).

$$\left.\begin{aligned}\alpha &= [a_1, a_2, a_3, \ldots\ldots, a_n, \alpha_{n+1}]\\ &= a_1 - \frac{1}{a_2 -}\,\frac{1}{a_3 -}\cdots\cdots\frac{1}{a_n -}\,\frac{1}{\alpha_{n+1}}\,.\end{aligned}\right\} \tag{2.12}$$

The a_i are again called the *partial quotients*, and the α_i, *complete quotients*. Note that a_n may take either sign. We define the sequence of *convergents*, $\{p_n/q_n\}$, as for the O.C.F., by

$$[a_1, a_2, \ldots, a_n] = p_n/q_n \tag{2.13}$$

where

$$\left.\begin{aligned}&p_0 = 1,\quad q_0 = 0,\quad p_1 = a_1,\quad q_1 = 1,\\ &p_{n+1} = a_{n+1}p_n - p_{n-1},\\ &q_{n+1} = a_{n+1}q_n - q_{n-1}.\end{aligned}\right\} \tag{2.14}$$

Formulae analogous to (2.8), (2.9) and (2.10) can be shown to hold, by induction [9].

$$p_{n-1}q_n - p_nq_{n-1} = 1, \tag{2.15}$$

$$\alpha = [a_1, a_2, \ldots, a_n, \alpha_{n+1}] = \frac{\alpha_{n+1}p_n - p_{n-1}}{\alpha_{n+1}q_n - q_{n-1}}, \tag{2.16}$$

which together imply

$$\frac{p_n}{q_n} - \alpha = \frac{1}{q_n(\alpha_{n+1}q_n - q_{n-1})}. \tag{2.17}$$

Note that if α is irrational, and $1 < \alpha_n < 2$ for all n, (or $1 < -\alpha_n < 2$ for all n), then this would imply $\alpha = [2,2,\ldots.] = 1$, which is a contradiction. It therefore follows that we may expand each irrational α in uncountably many ways as a S.R.C.F. of this type.

Various properties of these S.R.C.F. are discussed and proved in [9]. We select the following results which will be required in later chapters.

<u>LEMMA 2.6.</u> *Suppose that we are given a sequence of integers* $\{a_n\}$, *with* $|a_n| \geq 2$, *and* a_n *not constantly equal to 2 (or to -2) for large n; then*

(i) *the sequences* $\{p_n\}$, $\{q_n\}$ *as defined by (2.14) are such that*

$$|p_n| \geq n + 1, \quad |q_n| \geq n, \quad \left|\frac{p_n}{q_n}\right| \geq 1 + \frac{1}{n},$$

and $\{|p_n|\}$ $\{|q_n|\}$ *are strictly increasing sequences of integers.*

(ii) *the infinite S.R.C.F.* $[a_1, a_2, a_3, \ldots\ldots.]$, *whose value is defined as* $\lim_{n \to \infty} p_n/q_n$, *converges to a real number* α, *with* $|\alpha| > 1$.

(iii) *if* $a_n > 0$, *for all* $n \geq 1$, *then* $\{p_n/q_n\}$ *is a strictly decreasing sequence of positive fractions.*

(iv) the S.R.C.F. $\alpha = [a_1, a_2, \ldots\ldots]$ *of positive partial quotients is increased if, for some n,* $a_1, a_2, \ldots, a_{n-1}$ *remain constant and* a_n *is increased, while* a_r *for* $r > n$ *take arbitrary integral values.*

These results form part of §4 of [9], where the proofs are indicated. The strict monotonicity of $\{|p_n|\}$, and $\{|q_n|\}$ is a simple inductive consequence of (2.14). (iv) results from (2.14), (2.15) and the following lemma.

LEMMA 2.7. *Suppose that* $f(x)$ *is the linear fractional form*

$$f(x) = \frac{ax + b}{cx + d},$$

where a, b, c, d are real numbers, then $f(x)$ *is a monotone function in any interval which does not include the point* $x = -d/c$, *and increases if and only if* $ad - bc > 0$.

The proof follows immediately from the inequality

$$(ad - bc).\frac{df}{dx} > 0, \qquad \text{for } f \neq b/d.$$

Any irrational number α, $|\alpha| > 1$, may be developed in S.R.C.F. of the above type in infinitely many ways, each yielding, by (2.11), a sequence of integers $\{a_n\}$, with $|a_n| \geq 2$, and a_n not constantly equal to 2 (or -2) for large n. Conversely, Lemma 2.6 shows that any such sequence of partial quotients converges, in the above sense, to a real number α, with $|\alpha| > 1$.

The following relation, which links the complete quotients with the convergents, will be required in the next chapter.

LEMMA 2.8. *Let* $\alpha = [a_1, a_2, \ldots\ldots]$ *and as usual* $\alpha_i = [a_i, a_{i+1}, \ldots\ldots]$; *then* $q_n\alpha_{n+1} - q_{n-1} = \alpha_2\alpha_3\cdots\cdots\alpha_{n+1}$.

PROOF. When n = 1, the result follows from (2.14). Suppose that for some m, with m ≥ 2, we have

$$q_{m-1}\alpha_m - q_{m-2} = \alpha_2\alpha_3\cdot\ldots\cdot\alpha_m,$$

then by (2.11), (2.14),

$$\alpha_2\alpha_3\cdot\ldots\cdot\alpha_{m+1} = \alpha_{m+1}\left(q_{m-1}(a_m - \frac{1}{\alpha_{m+1}}) - q_{m-2}\right)$$
$$= \alpha_{m+1}q_m - q_{m-1},$$

and the lemma follows by induction.

It will be of interest in Chapter VII to have a result, analogous to Lemma 2.3, for the S.R.C.F. expansion of equivalent numbers. Such a theorem is proved in §6 of Chapter III. There are many more interesting results about S.R.C.F. expansions which we could include here, but as they will not contribute to the main theme of this thesis, they will be omitted (see [51], [59]).

We conclude this section on general properties of S.R.C.F. with the following result (compare with a similar theorem in [34]).

THEOREM 2.1. *Any expansion of a quadratic irrational α, in S.R.C.F., contains some complete quotient α_n, such that $|\bar{\alpha}_n| < 1$.*

PROOF. Having written α as a linear fractional form in α_n, by (2.16), then we may solve for α_n, and take the algebraic conjugate, obtaining

$$\bar{\alpha}_n = \frac{q_{n-2}\bar{\alpha} - p_{n-2}}{q_{n-1}\bar{\alpha} - p_{n-1}}.$$

From (2.15),

$$|\bar{\alpha}_n| = \left|\frac{q_{n-2}}{q_{n-1}}\right|\cdot\left|\left(\bar{\alpha} - \frac{p_{n-2}}{q_{n-2}}\right)/\left(\bar{\alpha} - \frac{p_{n-1}}{q_{n-1}}\right)\right|$$
$$\le \left|\frac{q_{n-2}}{q_{n-1}}\right|\cdot\left(1 + |q_{n-1}q_{n-2}(\bar{\alpha} - p_{n-1}/q_{n-1})|^{-1}\right).$$

Now since $\lim_{n\to\infty} p_n/q_n = \alpha \neq \bar{\alpha}$, and $\lim_{n\to\infty} |q_n| = \infty$, then for large enough n, we have

$$|\bar{\alpha}_n| \leq \left|\frac{q_{n-2}}{q_{n-1}}\right|.\left(1 + \frac{1}{|q_{n-1}|}\right).$$

But Lemma 2.6 implies that $|q_{n-2}| \leq |q_{n-1}| - 1$, and so

$$|\bar{\alpha}_n| \leq \left(1 - \frac{1}{|q_{n-1}|}\right).\left(1 + \frac{1}{|q_{n-1}|}\right) < 1,$$

provided n is large enough.

3. Expansions to the integer above

In this section, we will examine one particular expansion from the infinitely many S.R.C.F. expansions of an irrational $\alpha > 1$. At each step of the algorithm, we will always choose the integer above; consequently, in (2.11), we have $a_n = [\alpha_n] + 1$, for all n, and $a_n \geq 2$, (and not constantly equal to 2 for large n). We will call this the A-expansion (A.C.F.) of α. The A-expansion of an irrational is unique.

Various transformations have been given [51], [59], for converting one type of continued fraction to another. In Chapter VII, we will require the A-expansions of certain reduced quadratic irrationals, knowing their O.C.F. expansions. We will now prove a result which will enable us to obtain these. We will use the following permanent notation (for all types of continued fractions). If any segment of chain is repeated s times, this segment may be enclosed in brackets and subscripted with an s, as follows:

$$[a_1,a_2,\ldots,a_n,(b_1,b_2,\ldots,b_r)_s,a_{n+rs+1},\ldots].$$

In the above notation, we will use the convention that $s = 0$ implies this segment of chain is deleted. Thus

$$\left.\begin{aligned}&[a_1,\ldots,a_n,(b_1,\ldots,b_r)_0,a_{n+1},\ldots\ldots] \\ &\qquad = [a_1,\ldots\ldots,a_n,a_{n+1},\ldots\ldots].\end{aligned}\right\} \qquad (2.18)$$

<u>LEMMA 2.9.</u> *If* $\alpha = [(2)_r,\beta]$, $r \geq 0$, *then*

$$\alpha = \frac{(r+1)\beta - r}{r\beta - (r-1)}.$$

<u>PROOF.</u> Application of (2.14) implies $p_r = r + 1$, $q_r = r$; the result follows from (2.11).

<u>THEOREM 2.2.</u> *If in O.C.F.,*

$$\alpha = (a, r+1, x), \quad \textit{where} \quad r \geq 0, \quad x > 1, \quad a \geq 0,$$

then in A.C.F.,

$$\alpha + 1 = [a+2, (2)_r, x+1].$$

<u>PROOF.</u>

$$\alpha = a + \cfrac{1}{(r+1) + \cfrac{1}{x}}$$

$$= a + 1 - \cfrac{1}{\cfrac{(r+1)x+1}{rx+1}}.$$

Clearly $\dfrac{(r+1)x+1}{rx+1} = \dfrac{(r+1)(x+1) - r}{r(x+1) - (r-1)} > 1$, and the result then follows from Lemma 2.9.

This theorem enables us to convert an O.C.F. expansion into a S.R.C.F. expansion, and vice versa. Using the convention (2.18) and inserting an appropriate $(2)_0$ into the S.R.C.F. expansion, if necessary, we have: if $\alpha = (a_1,a_2,a_3,a_4,\ldots\ldots)$, then

$$\left.\alpha + 1 = [a_1+2,(2)_{a_2-1},a_3+2,(2)_{a_4-1},\ldots\ldots].\right\} \qquad (2.19)$$

This relationship enables many of the properties of the

A-expansion to be deduced from the corresponding results for O.C.F. It is also possible to obtain expressions for the period of the A-expansion of a quadratic irrational, in terms of the period of the O.C.F. expansion and its partial quotients; but we will not include this theory.

If α is a quadratic irrational, then we will call α *A-reduced* if

$$\alpha > 1, \quad 0 < \overline{\alpha} < 1. \tag{2.20}$$

By (2.5), if β is reduced and $\alpha = \beta + 1$, then α is A-reduced.

LEMMA 2.10. *The A-expansion of α is periodic if and only if α is a quadratic irrational.*

PROOF. This follows from the conversion (2.19), and Lemma 2.2.

LEMMA 2.11. *If α is A-reduced, then all complete quotients in its A.C.F. expansion are A-reduced.*

PROOF. Now, α satisfies the inequalities (2.20), and

$$\alpha = \alpha_1 = a_1 - \frac{1}{\alpha_2},$$

where $a_1 \geq 2$, and $\alpha_2 > 1$. But $\overline{\alpha} = a_1 - \frac{1}{\overline{\alpha}_2}$, which implies $0 < \overline{\alpha}_2 < 1$, and hence α_2 is A-reduced. The result follows by induction.

LEMMA 2.12. *If the A-expansion of α is pure periodic, then α is A-reduced.*

PROOF. The lemma may be proved in a similar manner to the O.C.F case, by showing that if

$$\alpha = [\overline{a_1, a_2, \ldots, a_n}], \quad \beta = [\overline{a_n, a_{n-1}, \ldots, a_1}],$$

then $\beta = \frac{1}{\overline{\alpha}}$. However, we will deduce the result from Lemma 2.4.

Now, if α is periodic, then each complete quotient has an A-expansion that is pure periodic. Since all partial quotients cannot equal 2, then there exists an n for which $\alpha_n > 2$. But by (2.19),

$\alpha_n - 1$ will have a pure periodic O.C.F. expansion, and by Lemma 2.4 will be reduced, implying that $0 < \bar{\alpha}_n < 1$. Thus α_n is A-reduced; but α is a complete quotient of α_n, and hence by Lemma 2.11, α is A-reduced.

The converse result is also true, as may be shown using a similar technique to [20] (p. 100).

LEMMA 2.13. *If α is A-reduced, then its A-expansion is purely periodic.*

PROOF. Lemmas 2.10, 2.11 imply that the A-expansion of α is, in fact, periodic, and that each complete quotient is A-reduced. Hence for some $n < m$, $\alpha_n = \alpha_m$. (2.21)

For arbitrary $i > 0$, $\alpha_i = a_i - \dfrac{1}{\alpha_{i+1}}$, and so

$$1/(\bar{\alpha}_{i+1}) = a_i - \frac{1}{1/\bar{\alpha}_i}\ ;$$

since $\dfrac{1}{\bar{\alpha}_i} > 1$, then

$$a_i = [1/\bar{\alpha}_{i+1}] + 1,$$

implying $a_{n-1} = a_{m-1}$, and so $\alpha_{n-1} = \alpha_{m-1}$. By an inductive argument,

$$\alpha = \alpha_1 = \alpha_{m-n+1},$$

proving the pure periodicity of α.

CHAPTER III

THE DIVIDED CELL METHOD

1. Introduction

This chapter will be devoted to a description and explanation of the divided cell method, constructed by Barnes and Swinnerton-Dyer. This expository chapter is included because the problems examined in this thesis will be treated by modifications of the method, for which details of the analysis will be required. Many of the results from [5], [9], [52], [53], will be referred to without proof. The theory of the divided cell was first described in [5], [9], and was later treated in depth, by Pitman [52].

As far as possible, the notations used in the literature referred to will be employed, and any changes explicitly noted.

Suppose for real α, β, γ, δ, ξ_0, η_0, we consider the set of points, L, in the ξ, η-plane, given by

$$L: \quad \left.\begin{aligned} \xi &= \xi_0 + \alpha x + \beta y \\ \eta &= \eta_0 + \gamma x + \delta y \end{aligned}\right\} , \qquad (3.1)$$

where x, y take all integral values, and the *determinant* $\Delta = |\alpha\delta - \beta\gamma|$ is non-zero.

If the set L contains the point $(\xi,\eta) = (0,0)$, then it is called an *homogeneous lattice*. If not, then we call L an *inhomogeneous lattice*, or a *grid*. We will use the latter terminology. We will be interested in those grids which have no points on either axis, and so we will assume this to be the case throughout the chapter. Later we will see that this condition is of great importance to the general method.

If A, B are any two points of L (called grid or lattice points), then we will call the line AB a *lattice line*, and any line segment AB which contains no other grid point, a *lattice step*.

Suppose that we have a parallelogram, whose vertices are the grid points A, B, C, D, then if its area is Δ as previously defined, then ABCD is called a *cell* of the grid. The following remarks are either clear, or follow by arguments similar to those in [32] (pp. 26-29)

(i) Any edge of a cell is a lattice step.

(ii) A cell contains no points of L in its interior. (3.2)

(iii) Any two adjacent edges of a cell, together with some point of L, generate the whole grid [9].

Let a, b, p, q, r, s, be integers with ps - rq = ±1; then the integral, unimodular transformation of variables from (x,y) to (X,Y), defined by

$$\left.\begin{aligned} x &= a + pX + qY \\ y &= b + rX + sY \end{aligned}\right\}, \qquad (3.3)$$

when applied to the grid L of (3.1), produces a grid L';

$$L': \quad \begin{aligned} \xi &= \xi_0' + \alpha' X + \beta' Y \\ \eta &= \eta_0' + \gamma' X + \delta' Y , \end{aligned}$$

for integral X, Y, where the constants with the prime are simply related to the constants without the prime. In fact, it is easily seen that the grids L and L' are identical.

If, however, a grid L' is defined by

$$L': \quad \left.\begin{aligned} \xi' &= K\xi \\ \eta' &= \frac{1}{K}\eta \end{aligned}\right\}, \qquad (3.4)$$

where (ξ,η) is a point of L, and K is a non-zero constant, then the grids L and L' are said to be *similar*.

There exists a definite connection between grids and indefinite binary quadratic forms. Suppose that f is such a form, then we may denote it, for real a, b, c, α, β, γ, δ, by

$$\left.\begin{aligned} f(x,y) &= ax^2 + bxy + cy^2 \\ &= (\alpha x + \beta y)(\gamma x + \delta y) \end{aligned}\right\}, \qquad (3.5)$$

where $\Delta = |\alpha\delta - \beta\gamma| = (b^2 - 4ac)^{\frac{1}{2}} = \sqrt{D}$. Δ is called the *determinant* of the form, and D its *discriminant*.

If x_0, and y_0 are any real numbers,

$$f(x + x_0, y + y_0) = (\xi_0 + \alpha x + \beta y)(\eta_0 + \gamma x + \delta y), \qquad (3.6)$$

with $\xi_0 = \alpha x_0 + \beta y_0$, and $\eta_0 = \gamma x_0 + \delta y_0$.

Comparing this with (3.1), the set of values taken by $f(x + x_0, y + y_0)$, for integral x, y, is the same as the set of values which are the products of coordinates of the lattice points of the grid L, or any grid similar to it. Hence the value of $M(f;x_0,y_0) = M(f;P)$, defined by (1.19), is identical with the supremum of real numbers m, with the property that there is no point of L in the hyperbolic region R;

$$R: \qquad |\xi\eta| < m. \qquad (3.7)$$

It also follows that

$$M(f;P) = \inf \{|\xi\eta|; \ (\xi,\eta) \text{ a point of } L\}. \qquad (3.8)$$

Thus for any pairs of forms and points, say f, P and f_1, P_1, to which there corresponds identical or similar grids, we have

$$M(f;P) = M(f_1;P_1)$$

It also follows that the inhomogeneous minima, defined in (1.20), $M(f)$ and $M(f_1)$ are equal, whenever f is related to f_1 by a substitution of the form (3.3), with $a = b = 0$.

Any direct evaluation of $M(f;P)$ by (3.8) would involve a calculation of all the values $|\xi\eta|$, where (ξ,η) is a point of L. Clearly there will be many points of L at which this product is very large. In fact, those points for which $|\xi\eta|$ is small, will be near the origin or one of the axes. Hence the basis of many methods to evaluate $M(f;P)$ is to provide an algorithm which picks out a suitable sequence of grid points of L, which have the "smallest" values of $|\xi\eta|$.

2. Outline of the method

A cell of a grid is said to be *divided* if it has a vertex in each of the four quadrants (and not on an axis). The basic result for grids and divided cells was proven by Delauney [22], (see also [3], [49], [56]).

THEOREM 3.1. *Every two dimensional grid, with no point on either axis, has at least one divided cell.*

Delauney's proof was sketched by Barnes and Swinnerton-Dyer [9], and relies on the fact that the grids under consideration have no point on either axis. The proof will not be given here, but in Chapter IV a similar result will be proved for a special type of grid.

Assuming the existence of a divided cell, it is possible to construct from it a unique chain $\{S_n\}$, $-\infty < n < \infty$, of divided cells, which flatten out against the ξ-axis as $n \to \infty$, and against the η-axis as $n \to -\infty$. Barnes and Swinnerton-Dyer developed the construction of this chain of cells by means of a simple algorithm, which generates a corresponding chain pair of integers $\{a_{n+1}, \varepsilon_n\}$, satisfying a set of conditions.

The following theorem justifies calling $\{S_n\}$ the chain of divided cells of the grid.

THEOREM 3.2. *The chain of cells $\{S_n\}$ arising from some divided cell of a grid L, contains all the divided cells of L.*

Although this result is of considerable interest, it is not explicitly used in the applications. The proof follows immediately from Theorems 2.1, 2.2 of [52].

Conversely, suppose that we take any chain pair $\{a_{n+1}, \varepsilon_n\}$, satisfying the required conditions, then it can be shown that it determines the chain of divided cells of a grid, which is itself unique except for a constant multiple of each coordinate.

The essential step in the argument is that the infimum (3.8) need only be taken over those grid points of L which are vertices of some divided cell of the chain $\{S_n\}$. This leads to an arithmetic formulation of $M(f;P)$, in terms of the chain pair $\{a_{n+1}, \varepsilon_n\}$ arising from a corresponding grid. Thus the problem of evaluating $M(f)$ may be attacked independently of the grids, by considering all possible chain pairs satisfying the required conditions, and then applying the arithmetic formulation. §§ 3, 4, 5 of this chapter will indicate in detail (but usually without proof) the steps in the above argument, with reference to their appearance in the literature.

3. The algorithm

Suppose we have a grid L given by (3.1), and S_0 is a divided cell $A_0B_0C_0D_0$, whose existence is guaranteed by Theorem 3.1. Let us suppose that the vertices of S_0 are named in a clockwise direction, and that A_0 is in either the first, or the third, quadrant. We now define two more divided cells of L, S_1 the *successor* of S_0, and S_{-1} the *predecessor* of S_0. Define the two integers h_0, k_0, as follows:

(a) If the line segment A_0D_0 is parallel to the ξ-axis, put by convention $h_0 = k_0 = -\infty$.

(b) If not, then A_0D_0 (and B_0C_0), produced in some direction, will intersect the ξ-axis. Let $|h_0|$ be the number of lattice steps of length $|A_0D_0|$ that must be taken from S_0, on the infinite line A_0D_0, in order to intersect the ξ-axis, and let $|k_0|$ be the corresponding number of lattice steps of the same length on the line B_0C_0. Let these intersecting lattice steps be A_1B_1 and C_1D_1 respectively, and give h_0 and k_0 the sign of the slope of A_0D_0.

This process clearly defines a unique new divided cell with vertices A_1, B_1, C_1, D_1. In symbols, if we put $\underset{\sim}{V}_0 = A_0 - D_0 = B_0 - C_0$, then

$$\left.\begin{aligned} A_1 &= A_0 - (h_0 + 1)\underset{\sim}{V}_0 \\ B_1 &= A_0 - h_0\underset{\sim}{V}_0 \\ C_1 &= C_0 + (k_0 + 1)\underset{\sim}{V}_0 \\ D_1 &= C_0 + k_0\underset{\sim}{V}_0 \end{aligned}\right\} \tag{3.9}$$

Now h_0 and k_0 are non-zero integers of the same sign. Note that A_1 is either in the first or third quadrant. If A_0 is in the first quadrant, so too is A_1, whenever $h_0 < 0$. (3.10)

By considering the intersections of A_0B_0 and C_0D_0 with the η-axis, then a unique divided cell S_{-1}, with vertices $A_{-1}, B_{-1}, C_{-1}, D_{-1}$, may be obtained, and non-zero integers h_{-1}, k_{-1} defined by considering S_0 as the successor of S_{-1}.

Thus, provided that there is no lattice line parallel to an axis, a doubly infinite chain of divided cells, $\{S_n\}$, $-\infty < n < \infty$, with vertices A_n, B_n, C_n, D_n, can be constructed, such that A_n is always in either the first or the third quadrant.

Associated with this chain is the sequence of non-zero integer pairs $\{h_n, k_n\}$, obtained from the following formulae:

$$\begin{aligned} A_{n+1} &= A_n - (h_n + 1)\underset{\sim}{V}_n \\ B_{n+1} &= A_n - h_n\underset{\sim}{V}_n \\ C_{n+1} &= C_n + (k_n + 1)\underset{\sim}{V}_n \\ D_{n+1} &= C_n + k_n\underset{\sim}{V}_n \end{aligned} \qquad (3.11)$$

where $\underset{\sim}{V}_n = A_n - D_n = B_n - C_n = B_{n+1} - A_{n+1} = C_{n+1} - D_{n+1}$. (3.12)

Denote the η-coordinate of the point P by $\eta(P)$, and so on. The following lemmas will provide information on the integers h_n, k_n, and the vertices of the divided cells.

<u>LEMMA 3.1.</u> *h_n and k_n are non-zero integers, which have the same sign.*

This has already become evident in the above discussion.

<u>LEMMA 3.2.</u> *It is impossible that*

(i) for all $n \geq n_0$ (or $n \leq n_0$), either

(a) $h_n = -1$, or (b) $k_n = -1$.

(ii) for all $r \geq 0$ (or $r \leq 0$), and some n_0,

$$h_{n_0+2r} = k_{n_0+2r+1} = 1.$$

The proof appears in [9] (Lemma 1), and relies on the fact that there are no lattice points on the axes.

Let us assume from now on that there is no lattice line parallel to an axis, which is equivalent to supposing the ratios α/β, γ/δ in (3.1) to be irrational; then h_n, k_n, are finite for all n.

LEMMA 3.3. *$\eta(\underset{\sim}{V}_n)$, $\eta(A_n)$, $\eta(B_n)$, $\eta(C_n)$, $\eta(D_n)$,*

and $\xi(\underset{\sim}{V}_{-n})$, $\xi(A_{-n})$, $\xi(B_{-n})$, $\xi(C_{-n})$, $\xi(D_{-n})$,

approach zero as $n \to \infty$, and take arbitrarily large values as $n \to -\infty$.

The proofs are found in [9] (Lemma 2), and [52] (Lemma 2.3). This lemma proves the intuitive concept of the chain of cells flattening out against an axis, for large values of $|n|$.

It so happens that the integer pair h_n, k_n, is not convenient to work with, and therefore we define a further integer pair by:

$$\left.\begin{aligned} a_{n+1} &= h_n + k_n \\ \\ \varepsilon_n &= h_n - k_n \end{aligned}\right\} \qquad (3.13)$$

LEMMA 3.4. *a_{n+1} and ε_n have the same parity. $|a_n| \geq 2$ for all n, but a_n not constantly equal to 2 (or to -2) for large n (positive or negative).*

Further, $|\varepsilon_n| \leq |a_{n+1}| - 2$, but $a_{n+1} + \varepsilon_n$ or $a_{n+1} - \varepsilon_n$ is not constantly equal to -2 for all large n (positive or negative).

The lemma follows from (3.13) and Lemma 3.2. We will see later that for the special kind of grids considered in Chapter IV, Lemmas 3.2, 3.4, do not always hold.

Now if f is a binary quadratic form, and P a point, associated with the grid L, then:

THEOREM 3.3.

$$M(f;P) = \inf \{ |\xi\eta|;\ (\xi,\eta) \text{ is a vertex of } S_n, \text{ for some } n \}$$

The theorem results from [9] (Theorem 5), and is given in full in [52] (Theorem 2.1). The proof rests on the fact that if P_n and P_{n+1} are different vertices of successive divided cells in a given quadrant, then there is no point of L within the interior of the triangle formed by the axes and the line P_nP_{n+1}. Since, by construction,

$$|\xi(P_n)| \leq |\xi(P_{n+1})|, \qquad |\eta(P_{n+1})| \leq |\eta(P_n)|,$$

we can readily see that there is no point within the region formed by the axes, and the straight line segments P_nP_{n+1}, for all n. It follows by convexity arguments that, in a fixed quadrant, the region R of (3.7), contains a lattice point of L if and only if it contains a vertex of a divided cell of $\{S_n\}$. The theorem follows.

In order that this theorem be of some practical use in the evaluation of M(f;P), we will find arithmetic expressions for the products of the coordinates of the vertices of the divided cells in the chain $\{S_n\}$, in terms of the associated chain pair sequence $\{a_{n+1},\varepsilon_n\}$. In so doing, a certain type of semi-regular continued fraction (Chapter II) arises naturally from the geometry of the grid, and its chain of divided cells.

4. Arithmetic formulation of the vertices of the divided cells

Denote the vertices of the divided cell S_n by:

$$\left.\begin{aligned} C_n &= (\beta_n\xi_n, \gamma_n\eta_n) \\ B_n &= (\beta_n(\xi_n + \theta_n), \gamma_n(\eta_n + 1)) \\ D_n &= (\beta_n(\xi_n + 1), \gamma_n(\eta_n + \phi_n)) \\ A_n &= (\beta_n(\xi_n + \theta_n + 1), \gamma_n(\eta_n + 1 + \phi_n)) \end{aligned}\right\} . \tag{3.14}$$

This completely defines the real numbers β_n, ξ_n, γ_n, η_n, ϕ_n, θ_n, for each n. Note that the ξ_n and η_n are not quite the same as those of [9] (p. 204), and $\theta_n = \alpha_n/\beta_n$, $\phi_n = \delta_n/\gamma_n$, but these changes are consistent with the change of notation made in [5] (p. 243).

By (3.2) (iii), the grid L is given, for all n, by

$$L: \quad \left.\begin{aligned} \xi &= \beta_n(\xi_n + \theta_n x + y) \\ \eta &= \gamma_n(\eta_n + x + \phi_n y) \end{aligned}\right\} , \tag{3.15}$$

where x, y take all integral values. Clearly

$$\Delta = |\beta_n\gamma_n(\theta_n\phi_n - 1)|. \tag{3.16}$$

By (3.12),

$$\underset{\sim}{V}_n = (\beta_n\theta_n, \gamma_n) = (-\beta_{n+1}, -\gamma_{n+1}\phi_{n+1}). \tag{3.17}$$

Now, as in [9], it follows that (3.11), (3.12) and (3.13) imply

$$\underset{\sim}{V}_{n+1} = -a_{n+1}\underset{\sim}{V}_n - \underset{\sim}{V}_{n-1}. \tag{3.18}$$

By equating coordinates, we obtain from (3.17),

$$\left.\begin{aligned} \beta_{n+1}\theta_{n+1} &= -a_{n+1}\beta_n\theta_n - \beta_{n-1}\theta_{n-1} \\ \gamma_{n+1} &= -a_{n+1}\gamma_n - \gamma_{n-1} \end{aligned}\right\} \tag{3.19}$$

A doubly infinite sequence of integer pairs $\{p_n, q_n\}$, may be defined by putting

$$\underset{\sim}{V}_n = (\beta_n\theta_n, \gamma_n) = (-1)^n(\beta_0(\theta_0 p_n - q_n), \gamma_0(p_n - \phi_0 q_n)). \tag{3.20}$$

Now (3.19) implies that this sequence of integer pairs satisfies the recurrence relations, for all n:

$$\left.\begin{array}{l} p_{-1} = 0, \quad q_{-1} = -1; \quad p_0 = 1, \quad q_0 = 0; \quad p_1 = a_1, \quad q_1 = 1; \\ p_{n+1} = a_{n+1}p_n - p_{n-1} \\ q_{n+1} = a_{n+1}q_n - q_{n-1} \end{array}\right\}. \qquad (3.21)$$

Referring to (2.12), (2.14) and [9], it is clear that the ratios p_n/q_n are convergents of a semi-regular continued fraction, and that for $n \geq 1$,

$$\frac{p_n}{q_n} = [a_1,a_2,\ldots\ldots,a_n],$$

and for $n \geq 2$,

$$\frac{p_{-n}}{q_{-n}} = [a_0,a_{-1},\ldots\ldots,a_{-n+2}].$$

Various properties of these continued fractions were discussed in Chapter II. We now state the following result which links these S.R.C.F. developments having partial quotients from the sequence $\{a_n\}$, with the vertices of the divided cells.

<u>*LEMMA 3.4.*</u>

$$\phi_0 = [a_1,a_2,a_3,\ldots\ldots],$$

$$\theta_0 = [a_0,a_{-1},a_{-2},\ldots\ldots].$$

<u>*PROOF.*</u> Since Lemma 3.3 implies $|\eta(\underset{\sim}{V}_n)| \to 0$, as $n \to \infty$, then (3.20) indicates $\lim_{n\to\infty} |\eta(\underset{\sim}{V}_n)| = \lim_{n\to\infty} |q_n\gamma_0(\phi_0 - p_n/q_n)| = 0.$

The result follows from Lemma 2.6. The result for θ_0 follows similarly.

Since we may take any step of the sequence $\{a_n\}$ as a reference point, it follows as a corollary that, for all n:

$$\left.\begin{array}{l} \phi_n = [a_{n+1},a_{n+2},\ldots\ldots] \\ \theta_n = [a_n,a_{n-1},\ldots\ldots]. \end{array}\right\}. \qquad (3.22)$$

Following [7], we will make the definitions,

$$\left.\begin{array}{l} \lambda_n = 2\xi_n + \theta_n + 1 \\ \mu_n = 2\eta_n + 1 + \phi_n \end{array}\right\}. \qquad (3.23)$$

Note that these notations differ from the σ_n and τ_n used in [52], [53].

THEOREM 3.4. *For all n,*

$$\left.\begin{aligned} \lambda_n &= \varepsilon_{n-1} + \sum_{r=1}^{\infty} \frac{(-1)^r \varepsilon_{n-r-1}}{\theta_{n-1}\theta_{n-2}\cdots\theta_{n-r}} \\ \mu_n &= \varepsilon_n + \sum_{r=1}^{\infty} \frac{(-1)^r \varepsilon_{n+r}}{\phi_{n+1}\phi_{n+2}\cdots\phi_{n+r}} \end{aligned}\right\}. \tag{3.24}$$

It is sufficient to consider the results for the case n = 0, as the general result follows identically, except for a constant shift of subscript. The proof for this case appears in detail in [9] (Theorem 2). The method is to express A_0 and C_0 as a linear combination of $\underset{\sim}{V}_0, \underset{\sim}{V}_1, \ldots, \underset{\sim}{V}_{n-1}$, for arbitrary n > 0. Then after addition, extraction of the η-coordinate, and allowing $n \to \infty$, we obtain the result for μ_0. The proof for λ_0 follows analogously.

THEOREM 3.5. *Suppose that the grid L corresponds to some form f, and point (x',y'), then f(x + x',y + y') is equivalent (i.e. it is related by a substitution of the type (3.31) to each of the forms* $f_n(x + x_n, y + y_n) = \frac{\pm\Delta}{\theta_n\phi_n - 1}(\theta_n x + y + \xi_n)(x + \phi_n y + \eta_n)$, *for a certain sequence of points* $\{x_n, y_n\}$.

This is a consequence of (3.14), (3.15), and (3.16); the proof is given in [52] (Theorem 2.3). Recurrence relations satisfied by the $\{x_n, y_n\}$ are given, but we will not need these.

We will now quote the theorem which provides the required arithmetic formulation of the vertices of the divided cells.

THEOREM 3.6. *If* $\{a_{n+1}, \varepsilon_n\}$ *is the sequence of integers connected with the grid L (corresponding to the form f, and point P), and*

$$\left.\begin{aligned}
M_n^{(1)} &= \frac{\Delta}{4|\theta_n\phi_n - 1|}\,|(\theta_n + 1 - \lambda_n)(\phi_n + 1 - \mu_n)| \\
M_n^{(2)} &= \frac{\Delta}{4|\theta_n\phi_n - 1|}\,|(\theta_n - 1 - \lambda_n)(\phi_n - 1 + \mu_n)| \\
M_n^{(3)} &= \frac{\Delta}{4|\theta_n\phi_n - 1|}\,|(\theta_n - 1 + \lambda_n)(\phi_n - 1 - \mu_n)| \\
M_n^{(4)} &= \frac{\Delta}{4|\theta_n\phi_n - 1|}\,|(\theta_n + 1 + \lambda_n)(\phi_n + 1 + \mu_n)|
\end{aligned}\right\}, \qquad (3.25)$$

then

$$M(f;P) = \inf_n M_n ,$$

where

$$M_n = M_n(f;P) = \inf\,\{M_n^{(i)};\ i = 1,2,3,4\}.$$

This result ([7], [52]) follows from Theorem 3.3, (3.14), and (3.15), after the variables have been changed according to (3.23). The $M_n^{(i)}$ are derived from C_n, D_n, B_n, A_n, in that order.

<u>Summary of the method so far</u>

Suppose that we have some grid, associated with a form f, and point P, then we construct a sequence of divided cells, which yields a sequence of integer pairs $\{a_{n+1},\varepsilon_n\}$, with the following properties [52].

$$\left.\begin{array}{l}
\text{(i)}\quad |a_n| \ge 2,\ a_n \text{ not constantly equal 2 (or } -2) \text{ for large n of either sign.} \\
\text{(ii)}\quad |\varepsilon_n| \le |a_{n+1}| - 2, \text{ and } \varepsilon_n \text{ has the same parity as } a_{n+1}. \\
\text{(iii)}\quad \text{neither } a_{n+1} + \varepsilon_n \text{ nor } a_{n+1} - \varepsilon_n \text{ is constantly equal to } -2 \text{ for large n of either sign.} \\
\text{(iv)}\quad \text{for any n, and all } r \ge 0 \text{ (or } r \le 0), \text{ we cannot have} \\
\qquad a_{n+2r+1} + \varepsilon_{n+2r} = a_{n+2r+2} - \varepsilon_{n+2r+1} = 2.
\end{array}\right\} \qquad (3.26)$$

By means of this sequence pair, Theorem 3.6 enables a calculation of M_n, and hence M(f;P).

The question which is of importance now is: Given a chain pair $\{a_{n+1}, \varepsilon_n\}$ satisfying (3.26), does this uniquely determine the sequence of divided cells for some grid, and can we evaluate M(f;P) by (3.25)?

In order to answer these questions we state the following two theorems.

THEOREM 3.7. *The two series*

$$\sum_{r=1}^{\infty} \frac{|a_{n+r+1}| - 2}{|\phi_{n+1}\phi_{n+2}\cdots\phi_{n+r}|}, \quad \sum_{r=1}^{\infty} \frac{|a_{n-r}| - 2}{|\theta_{n-1}\theta_{n-2}\cdots\theta_{n-r}|},$$

are convergent for all n, and

$$|a_{n+1}| - 2 + \sum_{r=1}^{\infty} \frac{|a_{n+r+1}| - 2}{|\phi_{n+1}\phi_{n+2}\cdots\phi_{n+r}|} \le |\phi_n| - 1,$$

$$|a_n| - 2 + \sum_{r=1}^{\infty} \frac{|a_{n-r}| - 2}{|\theta_{n-1}\theta_{n-2}\cdots\theta_{n-r}|} \le |\theta_n| - 1.$$

Equality holds in the last assertions if and only if all the a_r involved have constant sign.

Again it is only necessary to consider the case when n = 0. The proof for the first in each pair of results, may be found in [52]; the other results follow by symmetry.

THEOREM 3.8. *For all n,*

$$\left.\begin{aligned} |\lambda_n| &< |\theta_n| - 1 \\ |\mu_n| &< |\phi_n| - 1 \end{aligned}\right\}. \qquad (3.27)$$

The proof is a consequence of (3.26) and Theorems 3.4, 3.7, and may be found in [52], [5].

If the sequence pair $\{a_{n+1},\varepsilon_n\}$ satisfies (3.26), then for all n, the values of θ_n, ϕ_n, λ_n, μ_n may be computed by the formulae (3.22) and (3.24). Consequently, we can find ξ_n, η_n from (3.23). If, as usual, the function sgn x is defined by the equation $x.\text{sgn } x = |x|$, then we have the following formulae ([5], p. 241):

$$\left.\begin{array}{l} \text{sgn } \xi_n = \text{sgn }(\xi_n + 1) = -\text{sgn } \theta_n \\ \text{sgn }(\xi_n + \theta_n) = \text{sgn }(\xi_n + \theta_n + 1) = \text{sgn } \theta_n \end{array}\right\}, \quad (3.28)$$

$$\left.\begin{array}{l} \text{sgn } \eta_n = \text{sgn }(\eta_n + 1) = -\text{sgn } \phi_n \\ \text{sgn }(\eta_n + \phi_n) = \text{sgn }(\eta_n + \phi_n + 1) = \text{sgn } \phi_n \end{array}\right\}. \quad (3.29)$$

Consider, then, the four points, A_n, B_n, C_n, D_n, defined by (3.14). The formulae (3.28), (3.29) imply that these points are vertices of a divided cell of the grid L of (3.15). If we then calculate the coordinates of the four points A_{n+1}, B_{n+1}, C_{n+1}, D_{n+1}, using (3.11), we obtain for $\beta = -\theta_n$, $\gamma = -1/\phi_{n+1}$, the points

$$\left.\begin{array}{l} \big(\beta\beta_n(\xi_{n+1} + \theta_{n+1} + 1),\gamma\gamma_n(\eta_{n+1} + 1 + \phi_{n+1})\big) \\ \big(\beta\beta_n(\xi_{n+1} + \theta_{n+1}),\gamma\gamma_n(\eta_{n+1} + 1)\big) \\ \big(\beta\beta_n\xi_{n+1},\gamma\gamma_n\eta_{n+1}\big) \\ \big(\beta\beta_n(\xi_{n+1} + 1),\gamma\gamma_n(\eta_{n+1} + \phi_{n+1})\big) \end{array}\right\}. \quad (3.30)$$

Now

$$\frac{\Delta/(\theta_{n+1}\phi_{n+1} - 1)}{\Delta/(\theta_n\phi_n - 1)} = \frac{\theta_n(\phi_n - 1/\theta_n)}{\theta_{n+1}\phi_{n+1} - 1}$$

$$= \frac{\theta_n(\theta_{n+1} - 1/\phi_{n+1})}{\theta_{n+1}\phi_{n+1} - 1}$$

$$= \beta\gamma.$$

Hence by (3.16), the four points (3.30) generate a grid which is similar to L. In fact, by the same argument as before, the points form the vertices of a divided cell of this similar grid. Thus any sequence pair $\{a_{n+1}, \varepsilon_n\}$, satisfying (3.26) determines a grid whose chain of divided cells satisfy the relations (3.11), and this grid is unique, apart from similarity [5] (p. 242).

Any indefinite binary quadratic form may be written

$$f(x,y) = \frac{\pm\Delta}{\theta\phi - 1}(\theta x + y)(x + \phi y). \tag{3.31}$$

Such an f, which does not represent zero, is called *I-reduced* (inhomogeneously reduced) if

$$|\theta| > 1, \quad |\phi| > 1. \tag{3.32}$$

If, in addition, we have $\theta\phi < 0$, then f is said to be *Gauss-reduced.* This is the classical form of reduction.

<u>LEMMA 3.6.</u> *If $f(x,y)$ is an indefinite binary quadratic form which does not represent zero, then there exists an I-reduced form equivalent to it.*

This follows from the analogous result for Gauss-reduced forms [5], [26]. The following theorem [5] also provides a parallel with the classical results.

<u>LEMMA 3.7.</u> *If $f(x,y)$ is proportional to a form with integral coefficients and does not represent zero, then there are only a finite number of I-reduced forms equivalent to $f(x,y)$.*

Suppose that we wish to evaluate the inhomogeneous minimum of a form which does not represent zero. By Lemma 3.6, there is an I-reduced form equivalent to it, say f_0, and furthermore $M(f) = M(f_0)$. Now, for the form f_0, and any point P, there corresponds a grid, with its chain of divided cells and associated sequence $\{a_{n+1}, \varepsilon_n\}$;

Theorem 3.6 then enables us to calculate M(f;P). If

$$f_0(x,y) = \frac{\pm\Delta}{\theta_0\phi_0 - 1}(\theta_0 x + y)(x + \phi_0 y),$$

and we take any semi-regular expansion $\theta_0 = [a_0, a_{-1}, \ldots.]$ and $\phi_0 = [a_1, a_2, \ldots.]$ (of which there are uncountably many), then using the rules (3.26) we may choose a companion sequence $\{\varepsilon_n\}$. This chain pair will correspond to the chain of divided cells of a grid associated with f_0, and some point P_0. Hence, as in [5],

$$\begin{aligned} M(f) &= \sup_P M(f_0;P) \\ &= \sup M(\{a_{n+1},\varepsilon_n\}), \end{aligned}$$

where the latter supremum is extended over all possible chain pairs associated with the form f_0, in the above way.

As we have already seen, there are infinitely many form chains $\{f_n\}$ of I-reduced forms, all equivalent to a given form, f. In contrast to the case of Gauss-reduction, we can no longer guarantee that every I-reduced form equivalent to f_0, will belong to one of these chains.

In the above context, $\{a_n\}$ is called an a-chain *of the form f, from the form f_0.* The problem of determining all the form chains of the form f, and all the I-reduced forms equivalent to it, will be discussed later in this chapter.

5. Bounds on the value of M(f;P)

We prove the following extension of [5] (Lemma 3.2).

THEOREM 3.9. *For any chain $\{a_{n+1},\varepsilon_n\}$, satisfying (3.26),*

$$M(f;P) \le M_n \le \frac{\Delta}{4|\theta_n\phi_n - 1|}\min\{|(\theta_n - 1)(\phi_n - 1)|,\ |(\theta_n + 1)(\phi_n + 1)|,$$
$$|\theta_n(\phi_n \pm \mu_n)|,\ |(\theta_n \pm \lambda_n)\phi_n|\}. \qquad (3.34)$$

PROOF. As a consequence of (3.27), all the four terms $(\theta_n \pm 1 \pm \lambda_n)$ have the same sign, as do the four terms $(\phi_n \pm 1 \pm \mu_n)$. (3.35)

From Theorem 3.6 we have

$$M_n = \min \{M_n^{(i)};\ i = 1,2,3,4\}. \tag{3.36}$$

Now,

$$\begin{aligned} M_n &\le \min \{M_n^{(1)}, M_n^{(2)}\} \\ &\le \left(M_n^{(1)} M_n^{(2)}\right)^{\frac{1}{2}}. \end{aligned} \tag{3.37}$$

Substitute from (3.25), and use the inequality between the geometric and arithmetic means, to give

$$\begin{aligned} M_n &\le \frac{\Delta\left|(\theta_n + 1 - \lambda_n)(\theta_n - 1 - \lambda_n)(\phi_n + 1 - \mu_n)(\phi_n - 1 + \mu_n)\right|^{\frac{1}{2}}}{4\left|\theta_n\phi_n - 1\right|} \\ &\le \frac{\Delta}{4\left|\theta_n\phi_n - 1\right|}\left|(\theta_n - \lambda_n)\phi_n\right|. \end{aligned}$$

The other five components of (3.34) are obtained similarly, from different pairings of the $M_n^{(i)}$.

The following statement of Minkowski's theorem is a corollary.

THEOREM 3.10. *If f is an indefinite, binary quadratic form which does not represent zero, then*

$$M(f) < \tfrac{1}{4}\Delta.$$

PROOF. For any I-reduced form f_0 equivalent to f, and any associated chain pair $\{a_{n+1}, \varepsilon_n\}$, one of $\theta_n\phi_n > 0$, or $\theta_n\phi_n < 0$ is true for each n. In the first case, by the first two alternatives of (3.34), using Lemma 2.7,

$$M_n \le \frac{\Delta(|\theta_n| - 1)(|\phi_n| - 1)}{4(|\theta_n\phi_n| - 1)|} < \tfrac{1}{4}\Delta.$$

In the other case, by one of the third alternatives in (3.34),

$$M_n \le \frac{\Delta|\theta_n|(|\phi_n| - |\mu_n|)}{4(|\theta_n\phi_n| + 1)} < \tfrac{1}{4}\Delta.$$

In Chapter VII, we will further extend Theorem 3.9 to deal with a more complicated situation. The following result will be used several times throughout this thesis.

THEOREM 3.11. *Suppose that we are considering chain pairs with the property that, for all n,*

$$|\theta_n| > A > 1, \quad |\phi_n| > B > 1.$$

Suppose two such chains have a common segment, that agrees for at least $2r+2$ consecutive values of the chain pair (r large). Then, if F is any (fixed) one of the four alternatives (3.25) at the central step of the common segment, we have

$$F = F' + O\left(\frac{1}{r}\right),$$

where the prime is used to distinguish the variables in the two chains, and the constant implied by the order notation is a function of A and B only.

PROOF. We may suppose, without loss of generality, that the common chain segment is

$$(a_{-r+i}, \varepsilon_{-r+i-1}), \quad i = 0,1,\ldots\ldots,2r+1. \tag{3.38}$$

Then $$\phi_0 = [a_1,a_2,\ldots\ldots,a_r,\phi_r] = \frac{p_r\phi_r - p_{r-1}}{q_r\phi_r - q_{r-1}},$$

and $$\phi_0' = [a_1,a_2,\ldots\ldots,a_r,\phi_r'] = \frac{p_r\phi_r' - p_{r-1}}{q_r\phi_r' - q_{r-1}},$$

since, by (2.14), the definitions of p_r, q_r, depend only on the values a_1, a_2,......, a_r. Hence (2.15) and Lemma 2.6 (i) imply

$$|\phi_0 - \phi_0'| = \frac{|\phi_r - \phi_r'|}{|(q_r\phi_r - q_{r-1})(q_r\phi_r' - q_{r-1})|} = O(\frac{1}{r^2}). \tag{3.39}$$

Similarly

$$|\theta_0 - \theta_0'| = O(\frac{1}{r^2}). \tag{3.40}$$

Let n = [r/2], (integer part notation); then by Lemma 2.8, and since $|q_n| \geq n$,

$$|\mu_0 - \mu_0'| \leq |\varepsilon_1| \left|\frac{1}{\phi_1} - \frac{1}{\phi_1'}\right| + \cdots\cdots + |\varepsilon_n| \left|\frac{1}{\phi_1 \cdots \phi_n} - \frac{1}{\phi_1' \cdots \phi_n'}\right| + O(\frac{1}{n}).$$

Now, the first term on the right hand side of this inequality is clearly $O(\frac{1}{n^2})$, as a result of (3.26) and (3.39). By a simple inductive argument, it follows that the remaining terms are also $O(\frac{1}{n^2})$. Consequently,

$$|\mu_0 - \mu_0'| = O(\frac{1}{n}) = O(\frac{1}{r}). \tag{3.41}$$

Similarly,

$$|\lambda_0 - \lambda_0'| = O(\frac{1}{r}). \tag{3.42}$$

Now, F is of the form $F = \frac{x_0 y_0}{|\theta_0\phi_0 - 1|}$, where, by (3.39) to (3.42),

$$x_0 = x_0' + O(\frac{1}{r}),$$
$$y_0 = y_0' + O(\frac{1}{r}).$$

Using (3.26), (3.27), (3.39) and the above results,

$$|F - F'| = \frac{|\theta_0\phi_0 - \theta_0'\phi_0'|}{|\theta_0\phi_0 - 1|} + O(\frac{1}{r})$$
$$= O(\frac{1}{r}).$$

Note that at each step of the argument, the constant implied by the O notation depends only on A and B.

The following theorem will often enable much chain exclusion to be done, without excessive splitting of cases.

THEOREM 3.12. *The value of $M(f;P)$ remains unchanged after the following elementary chain operations on the chain pair $\{a_{r+1}, \varepsilon_r\}$;*

(i) reversing the chain pair about some point; i.e. for some m, interchange for all r, $(a_{m+r}, \varepsilon_{m+r-1})$ and $(a_{m-r+1}, \varepsilon_{m-r})$.

(ii) replacing the chain pair by $\{a_{r+1}, -\varepsilon_r\}$.

(iii) replacing the chain pair by $\{-a_{r+1}, (-1)^{m+r}\varepsilon_r\}$, for a fixed integer m.

The proof is given in [5]. In Chapter VII, we will further examine this result, in the light of a slightly different problem.

6. Semi-regular expansions for equivalent quadratic irrationals

We will follow the notation of [52], [53], and write the integral, unimodular linear transformation as the matrix,

$$T = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \tag{3.43}$$

where $ps - rq = \pm 1$; if the substitution

$$x = pX + qY$$

$$y = rX + sY$$

gives $$f(x,y) = F(X,Y),$$

then we write

$$F = fT = f\begin{bmatrix} p & q \\ r & s \end{bmatrix}. \tag{3.44}$$

Pitman [53] (p. 92) has shown, by examples, that it is not always possible to obtain all the I-reduced forms equivalent to f by taking all the forms that occur in the chains from f. Nor is it always possible to obtain all chains of equivalent forms of f, by taking all chains from some one form, even in the case when it is a Gauss-reduced form, with integral coefficients. The following result gives answers to some of the questions that arise from this discussion.

THEOREM 3.13. *Suppose f is any I-reduced form (given, say, by (3.31)), and g any Gauss-reduced form, properly equivalent to f; i.e.*

$$g(x,y) = \frac{\pm\Delta}{\omega\omega' - 1}(\omega x + y)(x + \omega' y), \tag{3.45}$$

with $\omega < -1$, *and* $\omega' > 1$. *Then every form chain from f contains at least one of the three forms*

$$g, \quad g\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad g\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}. \tag{3.46}$$

The proof is given in both [52], [53].

THEOREM 3.14. *If the f and g in the above theorem have integral coefficients, then every form chain from f contains infinitely many occurrences of a form of the triad (3.46), and the distance between these occurrences (not necessarily the same form) is bounded.*

PROOF. (sketch) There exists a transformation T, of the type (3.43), with $ps - qr = 1$, such that $f = gT$. If f and g are given by (3.31) and (3.45) respectively, then as in [26] (p. 99), the "roots" of f and g are related by

$$\theta = \frac{p\omega + r}{q\omega + s}, \quad \phi = \frac{s\omega' + q}{r\omega' + p}. \tag{3.47}$$

In the proof of Theorem 3.13, two types of T are distinguished, those which imply that any chain leads forward, and those which imply any chain leads backwards, to one of the triad (3.46). There are only a finite number of forms equivalent to f (Lemma 3.7), and the existence of the fundamental automorph of f, [26], implies the existence of a T of both types mentioned. The result follows from this. It is Theorem 2 of [54].

Suppose ψ is an arbitrary quadratic irrational with $|\psi| > 1$, then we are interested in the complete quotients encountered in an arbitrary semi-regular expansion of it. The following result gives a semi-regular counterpart to Lemma 2.2.

THEOREM 3.15. *Suppose ψ, $|\psi| > 1$, is a quadratic irrational, and α is a reduced number equivalent to it (see (2.5)), then any semi-regular expansion of ψ has as a complete quotient one of the numbers*

$$\alpha, \quad \alpha + 1, \quad \frac{\alpha}{1 - \alpha}, \tag{3.48}$$

or their negatives.

PROOF. The existence of such an α follows from the well-known result (intimated in the proof of Lemma 3.6)) that to every indefinite binary quadratic form, there exists a Gauss-reduced form equivalent to it. We may assume that $\alpha > 1$, by making the transformation $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, if necessary.

Now, by Theorem 2.1, any semi-regular expansion of ψ leads forward to a complete quotient, say ϕ, such that $|\overline{\phi}| < 1$. If $\theta = 1/\overline{\phi}$, $\omega = 1/\overline{\alpha}$, and $\omega' = \alpha$, then the forms f and g given by (3.31) and (3.45) respectively, are equivalent, and both are multiples of integral forms.

Suppose that f is properly equivalent to g, then since g is Gauss-reduced, we may apply Theorem 3.13, which implies that every form chain from f, contains at least one of the triad (3.46). But by Theorem 3.14, any chain in fact leads forward to one of these forms.

Hence, by (3.47), any semi-regular expansion of ϕ,(and thus ψ), leads to one of ω', $\omega' + 1$, or $\frac{\omega'}{1 - \omega'}$, which is the required result.

If f is improperly equivalent to g, then $f\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$ is properly equivalent to g, and the same argument implies that every expansion of $(-\phi)$ leads forward to one of the triad (3.48) as a complete quotient. The result again follows since, if $\phi = [a_1,\ldots.,a_n,\phi_n]$, then $-\phi = [-a_1,\ldots.,-a_n,-\phi_n]$.

CHAPTER IV

FORMULATION OF THE MIXED FORM PROBLEM

1. Outline of the method

In this chapter we will develop the apparatus necessary to solve the mixed form problem enunciated in Chapter I. Suppose that $(\theta x + y)$ and $(x + \phi y)$ are two linear forms that do not represent zero (θ and ϕ are therefore irrational), then our aim will be to evaluate the constant k, defined by

$$\sup_{\theta,\phi,\alpha} \left(\inf_{(x,y)\neq(0,0)} \left| \frac{(\theta x + y)(x + \phi y + \alpha)}{\theta\phi - 1} \right| \right) = k \qquad (4.1)$$

where α is real and non-zero , and x,y are integral.

We may suppose without loss of generality that there do not exist integers x', y', such that $x' + \phi y' + \alpha = 0$, else the infimum in (4.1) is clearly zero, a trivial case.

The method used by Davenport [21], and Kanagasabapathy [41], [42], involved an examination of three particular values taken by the mixed form. Assuming that these values were greater than k' (> k), for all θ, ϕ, α, they obtained contradictions. While this method has yielded a close approximation to k ($k' = \frac{1}{4\cdot25777} = 0\cdot2348...$), it is not suitable for evaluating the best possible constant.

Our approach will follow similar lines to that of Chapter III. For each mixed form we will define a grid, and construct a doubly infinite chain of cells, the vertices of which will supply us with a suitable sequence of points at which we may evaluate the products of the coordinates.

2. P-grids and p-divided cells

Suppose that we have an I-reduced form of determinant 1, and a non-zero real number α. Define, for such a form f,

$$M(f;\alpha) = \inf_{(x,y)\neq(0,0)} \left|\frac{(\theta x + y)(x + \phi y + \alpha)}{\theta\phi - 1}\right|, \tag{4.2}$$

where the infimum is extended over integral pairs (x,y), not the origin.

For each such f and α, we may define a grid L as follows. Suppose $\beta > 0$, $\gamma > 0$. and $\beta\gamma = \frac{1}{|\theta\phi - 1|}$, then put

$$L: \quad \left.\begin{array}{l} \xi = \beta(\theta x + y) \\ \eta = \gamma(x + \phi y + \alpha) \end{array}\right\}, \tag{4.3}$$

where x, y take all integral values. L has unit determinant. If we assume that $x + \phi y + \alpha$ does not represent zero, then it follows that L is a grid with:

(i) only the point $(\xi,\eta) = (0,\gamma\alpha)$ on the axes.

(ii) no lattice line parallel to an axis (since θ, ϕ are irrational).

We will call such a set of points a *p-grid*. Note that these p-grids were excluded from the considerations of Chapter III. If ABCD is a cell of a p-grid, and if it has one vertex on an axis, and the other three vertices in different open quadrants, then we will call ABCD a pseudo-divided cell, or a *p-divided cell*.

We will now sketch the proof of an analogous result to Theorem 3.1. Although we will not explicitly use this result, it is of interest.

THEOREM 4.1. *Every two dimensional p-grid has at least one p-divided cell.*

PROOF. By symmetry, we may suppose that the point on the axis is

$P(0,\eta_0)$, where $\eta_0 < 0$. Now, in any bounded region of the plane, there is only a finite number of grid points; hence there exists a positive number K, such that there are no grid points except P, in the region R, defined by the inequalities

$$R: \qquad |\xi| < K, \quad |\eta - \eta_0| < |\eta_0|.$$

By Minkowski's fundamental theorem in the geometry of numbers, if K is increased continuously, then there is some value, say $K = \xi_1$, for which there is a grid point Q on the boundary of R, but not within it. Clearly Q cannot lie on the line $\eta = 2\eta_0$, else its reflection in P (i.e. the point 2P - Q) lies on the ξ-axis, which is impossible. Hence $\xi(Q) = \xi_1$, $\eta(Q) = \eta_1$, where $2\eta_0 < \eta_1 < 0$.

Suppose first that $2\eta_0 < \eta_1 < \eta_0$, then QP is a lattice step, and there exist lattice lines parallel to it, equal distances apart. If ℓ is the next lattice line parallel to QP on the origin side, then the origin must lie between QP and ℓ; for if not, ℓ would contain a line segment of length greater than QP, with no grid points on it. Now ℓ contains two grid points with $|\xi| < |\xi_1|$, and since there are no points of L in R, then there is a lattice step TU on ℓ, with T and U in the first and second quadrants. Then P, Q, T, U, are the vertices of a p-divided cell.

If, however, $\eta_0 < \eta_1 < 0$, then replace Q by Q', its reflection in P, and the proof follows as before.

3. The algorithm

We refer to a result by Davenport [21] (Lemma 2),which will enable us to assume the existence of a p-divided cell of a particular shape in a p-grid.

LEMMA 4.1. *If X and Y are homogeneous linear forms of unit determinant, which do not represent zero for integers, and c is any non-zero constant, then there exists an integral unimodular transformation into the new variables x, y, which transforms $(X + c)Y$ into*

$$\frac{\pm(x + \theta y - \alpha)(x - \phi y)}{\theta + \phi},$$

where $\theta > 1, \quad 0 < \phi < 1, \quad 1 \le \alpha < \theta.$

COROLLARY. *Changing the notation to be consistent with (4.1), we may consider, without loss of generality, the forms*

$$\left.\begin{array}{l} \dfrac{\pm(\theta x + y)(x + \phi y + \alpha)}{|\theta\phi| + 1}, \\ \text{where} \quad \phi > 1, \quad \theta < -1, \quad -\phi < \alpha \le -1. \end{array}\right\}, \tag{4.4}$$

This is an immediate consequence of Lemma 4.1, after replacing ϕ, θ, and α by $-1/\theta$, ϕ, and $-\alpha$ respectively. We may also suppose that $\alpha < -1$; for if $\alpha = -1$, then $(x,y) = (1,0)$ reduces the inhomogeneous factor to zero, which contradicts one of our assumptions.

Putting $\beta\gamma = \dfrac{1}{|\theta\phi| + 1}$, for $\beta > 0$, $\gamma > 0$, let us consider the p-grid L, defined by (4.3), under the conditions (4.4). If

$$\left.\begin{array}{ll} C_0 = \big(\beta\theta, \gamma(1 + \alpha)\big), & B_0 = \big(0, \gamma\alpha\big) \\ D_0 = \big(\beta(1 + \theta), \gamma(1 + \alpha + \phi)\big), & A_0 = \big(\beta, \gamma(\phi + \alpha)\big) \end{array}\right\}, \tag{4.5}$$

then $A_0B_0C_0D_0$ is a p-divided cell, in which A_0D_0 has negative slope. For a corresponding f and α, it is clear that

$$M(f;\alpha) = \inf\{|\xi\eta|;\ (\xi,\eta) \text{ is a point of } L, \text{ not } B_0\}. \tag{4.6}$$

Using a similar method to that of Chapter III, we will now construct a doubly infinite sequence of divided and p-divided cells, and prove that the infimum in (4.6) need only be taken over those grid points which are vertices of cells in this chain.

We define the *successor* of $A_0B_0C_0D_0$ in exactly the same way as (3.9). It follows from the geometry of the p-grid that $A_1B_1C_1D_1$ will be a genuine divided cell, and that the construction yields an integer pair (h_0,k_0), which has both components negative, implying

$$a_1 = h_0 + k_0 < 0. \tag{4.7}$$

From the genuine divided cell $A_1B_1C_1D_1$, denoted by S_1, we can construct its successor, S_2, also a genuine divided cell; and so on, giving rise to a one-sided chain of divided cells $\{S_n\}$, $n \geq 1$, and a sequence of integer pairs $\{h_n,k_n\}$, $n \geq 0$, satisfying the equations (3.11). No h_n or k_n will be infinite, since we are supposing θ and ϕ to be irrational.

Now we can construct a cell S_{-1}, which we will call the *predecessor* of S_0, by using the same formal process described in Chapter III, considering B_0 to be that vertex of S_0 which is in the fourth quadrant. Denote the lattice step of length $|C_0D_0|$ on the line C_0D_0, intersecting the positive η-axis, by $B_{-1}C_{-1}$; take $A_{-1}D_{-1}$ to be the lattice step of the same length on the line A_0B_0, such that A_{-1} is in the open third quadrant, and D_{-1} coincides with B_0. Then $A_{-1}B_{-1}C_{-1}D_{-1}$ is a p-divided cell, and the formulae (3.11) define the integer pair h_{-1}, k_{-1}. Clearly $h_{-1} = 1$.

Similarly, by considering D_{-1} to be the representative of the fourth quadrant, we can define the p-divided cell S_{-2} to be that cell

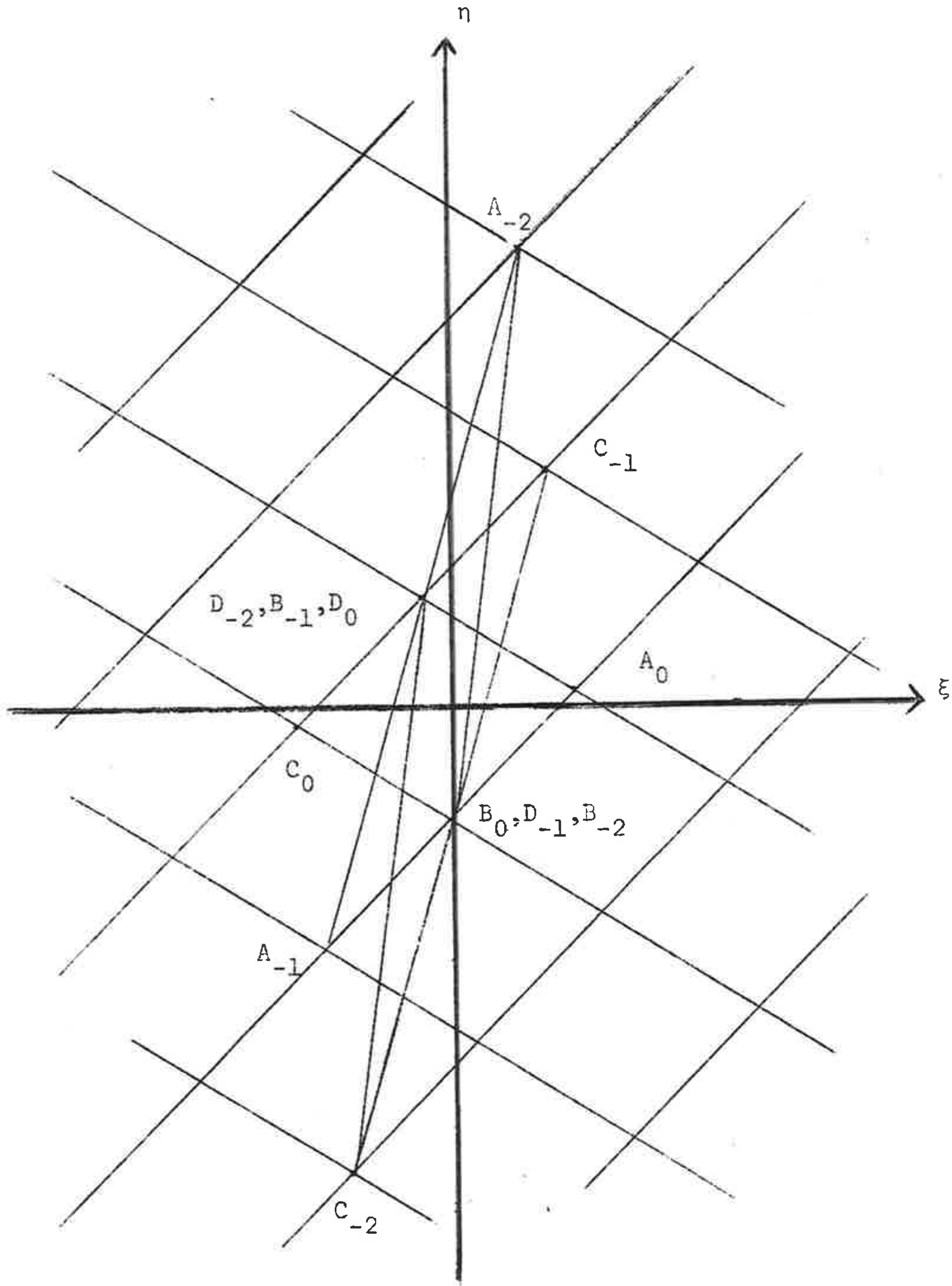

Figure 1.

for which S_{-1} is the successor. In this case $B_{-2} = D_{-1} = B_0$, and $k_{-2} = 1$. We can repeat this process indefinitely, defining the sequence of p-divided cells $\{S_n\}$, $n \leq 0$. Note that S_0 is the only member of this sequence for which A_nD_n has negative slope. We have $B_{2n} = D_{2n-1} = B_{2n-2}$ for all $n \leq 0$. In fact, the labelling of vertices is such that a given point (say A_n) alternates in opposite quadrants as n decreases; thus A_{2n} is in the first quadrant and A_{2n-1} is in the third quadrant for all negative n. This is a consequence of the fact that the point B_0 is a vertex of each p-divided cell in $\{S_n\}$, and

$$h_{2n-1} = k_{2n-2} = 1, \quad (n \leq 0), \tag{4.8}$$

implying

$$a_n = h_{n-1} + k_{n-1} > 0, \quad (n \leq 0). \tag{4.9}$$

As in Chapter III, we will be interested in what restrictions are imposed on the doubly infinite sequence $\{h_n, k_n\}$. We consider two cases separately.

Case 1: $n < 0$ The equations (4.8), (4.9), with the notation (3.13), imply that, for $n < 0$,

$$\varepsilon_n = (-1)^n(a_{n+1} - 2). \tag{4.10}$$

This result is contrary to (3.26) (iv), and arises because of the existence of a grid point on an axis.

Suppose that the equations

$$h_{2n} = k_{2n-1} = 1 \tag{4.11}$$

hold, in addition to (4.8), for all $n \leq n_0 \leq 0$. Then, as in the proof of Lemma 3.2 ([9], Lemma 1), and from the geometry of the algorithm (see figure 1), we have for all $n \leq n_0$,

$$D_{2n} = B_{2n-1} = D_{2n-2} = D, \text{ say.}$$

D is in the second quadrant; suppose P_n is the vertex of S_n in the first quadrant. Then for all $n \le n_0$, the triangle DB_0P_n has constant area (equal to $\frac{1}{2}\Delta = \frac{1}{2}$), and consequently the P_n lie in a bounded region of the first quadrant. Thus there are only a finite number of different points P_n, none of which lie on the η-axis. By the construction of the cells, we have for all $n \le n_0$,

$$P_n - D = P_{n+1} - B_0;$$

thus in the usual notation, since $\xi(B_0) = 0$ and $\xi(P_n).\xi(D) < 0$,

$$\xi(P_n) < \xi(P_{n+1}),$$

which contradicts the fact that there are only a finite number of different P_n, $n \le n_0$. Hence (4.11) cannot hold.

Thus for this case 1, we have $a_n \ge 2$, $n \le 0$, with strict inequality infinitely often. Also, (4.8) implies that the condition (3.26) (ii) is still valid for negative n.

Case 2: $n \ge 0$ $\{S_n\}$, $n > 0$, is a sequence of genuine divided cells, and the proof Lemma 1, [9], is valid, since there are no grid points on the ξ-axis.

Combining these two cases we state the following lemma.

LEMMA 4.2. *If f is an indefinite binary quadratic form, and α any real non-zero number, both satisfying the conditions (4.4), then there corresponds to them a p-grid, associated with which there is a doubly infinite sequence of p-divided and divided cells, generating the integer pair sequence $\{a_{n+1}, \varepsilon_n\}$. Furthermore, the following conditions are satisfied:*

$$\left.\begin{array}{l}
\text{(i)}\quad |a_n| \ge 2, \text{ and } a_n \text{ is not constantly equal to 2 (or } -2) \\
\qquad \text{for large } n \text{ of either sign. In fact, } a_n \ge 2 \text{ for all} \\
\qquad n \le 0, \text{ and } a_1 < 0. \\
\text{(ii)}\quad |\varepsilon_n| \le |a_{n+1}| - 2, \text{ and } \varepsilon_n \text{ has the same parity as } a_{n+1}. \\
\qquad \text{For } n < 0,\ \ \varepsilon_n = (-1)^n(a_{n+1} - 2). \\
\text{(iii) neither } a_{n+1} + \varepsilon_n \text{ nor } a_{n+1} - \varepsilon_n \text{ is constantly equal} \\
\qquad \text{to } -2 \text{ for large } n \text{ of either sign.} \\
\text{(iv)}\quad \text{for any } m, \text{ the relation} \\
\qquad\quad a_{m+2r+1} + \varepsilon_{m+2r} = a_{m+2r+2} - \varepsilon_{m+2r+1} = 2, \\
\qquad \text{does not hold for all } r \ge 0; \text{ nor does it hold for} \\
\qquad \text{all } r \le 0, \text{ if } n \text{ is even.}
\end{array}\right\} \tag{4.12}$$

These results correspond to those of (3.26) in the general divided cell method. Lemma 3.3 has its counterpart in the following:

__LEMMA 4.3.__ *Using the notation (3.12),*

$$\eta(\underset{\sim}{V}_n),\ \eta(A_n),\ \eta(B_n),\ \eta(C_n),\ \eta(D_n), \tag{4.13}$$

$$\xi(\underset{\sim}{V}_{-n}),\ \xi(A_{-n}),\ \xi(B_{-n}),\ \xi(C_{-n}),\ \xi(D_{-n}), \tag{4.14}$$

each approach zero, as $n \to \infty$. As $n \to -\infty$, they all take arbitrarily large values (positive or negative), with the exception of that vertex of the p-divided cells which is fixed on the η-axis.

__PROOF.__ Since $\{S_n\}$, $n > 0$, are genuine divided cells, the results for positive subscripts follow identically with Lemma 2 of [9]. For $n \le 0$, let R_n and Q_n be the vertices of S_n in the second and third quadrants respectively. Then

$$|\xi(\underset{\sim}{V}_n)| = |\xi(Q_n)|; \tag{4.15}$$

hence from (3.12), as in Lemma 2 of [9], if we can prove that $\xi(Q_n) \to 0$

as $n \to -\infty$, the complete result holds. Now

$$|\xi(Q_n)| = |\xi(Q_{n+1} - R_{n+1})|$$
$$< |\xi(Q_{n+1} - B_0)| = |\xi(Q_{n+1})|.$$

It is easily seen that equation (3.18) still holds for this problem (since the algorithm is described by the formulae (3.11) for all n), and so

$$|a_{n+1}\xi(Q_n)| \le |\xi(Q_{n+1})| + |\xi(Q_{n-1})|$$
$$< |\xi(Q_{n+1})| + |\xi(Q_n)|,$$

implying, for all $n < 0$, $\quad |\xi(Q_n)| < \dfrac{1}{|a_{n+1}| - 1} |\xi(Q_{n+1})|.$

Now since by (4.12) (i), $a_n \ge 3$ for infinitely many negative n,

$$|\xi(Q_n)| < \tfrac{1}{2}|\xi(Q_{n+1})|,$$

for infinitely many negative n; this implies the result.

The final assertion of the lemma for $n \to -\infty$, follows for (4.14) in an identical manner to Lemma 2.3 of [52]. Let $\{P_n\}$ be the sequence of vertices in one of the first, second or third quadrants, then if $|\eta(P_n)|$ does not become arbitrarily large as $n \to -\infty$, there exists a constant K, such that for all $n < 0$, the points P_n lie within the square $|\xi| < K$, $|\eta| < K$. Now, this implies that there are only a finite number of different P_n, none of which lie on the axes, and this contradicts the fact that $\xi(P_n) \to 0$ as $n \to -\infty$. This completes the lemma.

We will now show that the vertices of the sequence of cells $\{S_n\}$, provide a suitable set of grid points over which the infimum in (4.6) may be taken.

THEOREM 4.2. *Suppose that there is a point of L in the region R*

$$R: \qquad |\xi\eta| < m, \quad \xi \neq 0, \tag{4.16}$$

then there is a vertex of a p-divided or divided cell of the chain $\{S_n\}$, in R.

PROOF. As we have previously noted, the proof of the corresponding result (Theorem 3.3) rested on the fact that the triangle formed by the axes and the line joining different consecutive vertices of divided cells contained no point of L in its interior. Each quadrant was considered separately. Since all the required conditions are satisfied in the first and second quadrants, the result of the theorem follows for the upper half-plane. That is, if $P = P(\xi_0,\eta_0)$ is a point of L in the upper half-plane, and $|\xi_0\eta_0| < m$, then there is a vertex of a cell of $\{S_n\}$, say $Q = Q(\xi_1,\eta_1)$ with $|\xi_1\eta_1| < m$.

Suppose, however, that $P(\xi_0,\eta_0)$ is in R, and in the lower half-plane. If $\eta_0 < \eta(B_0)$, then the reflection of P in B_0 (say, $P'(-\xi_0,\eta_2)$) is also in R, since

$$|\xi_0\eta_2| < |\xi_0\eta_0| < m.$$

Thus we need consider only those P for which $\eta(B_0) < \eta_0 < 0$.

Now there is no point of L in the rectangle $\eta(B_0) < \eta < 0$, $\xi(C_0) < \xi < 0$, and so we need only consider that part of the third quadrant for which $\xi < \xi(C_0)$, $\eta(B_0) < \eta < 0$. If $\{P_n\}$, $n \geq 0$, is the sequence of vertices of divided cells of $\{S_n\}$ in the third quadrant, then it follows, as before, that there is no point of L in the region bounded by the ξ-axis, the line $\eta = \eta(C_0)$, and the infinite polygonal curve which is the join of P_0, P_1, P_2,.... Thus by the strict convexity

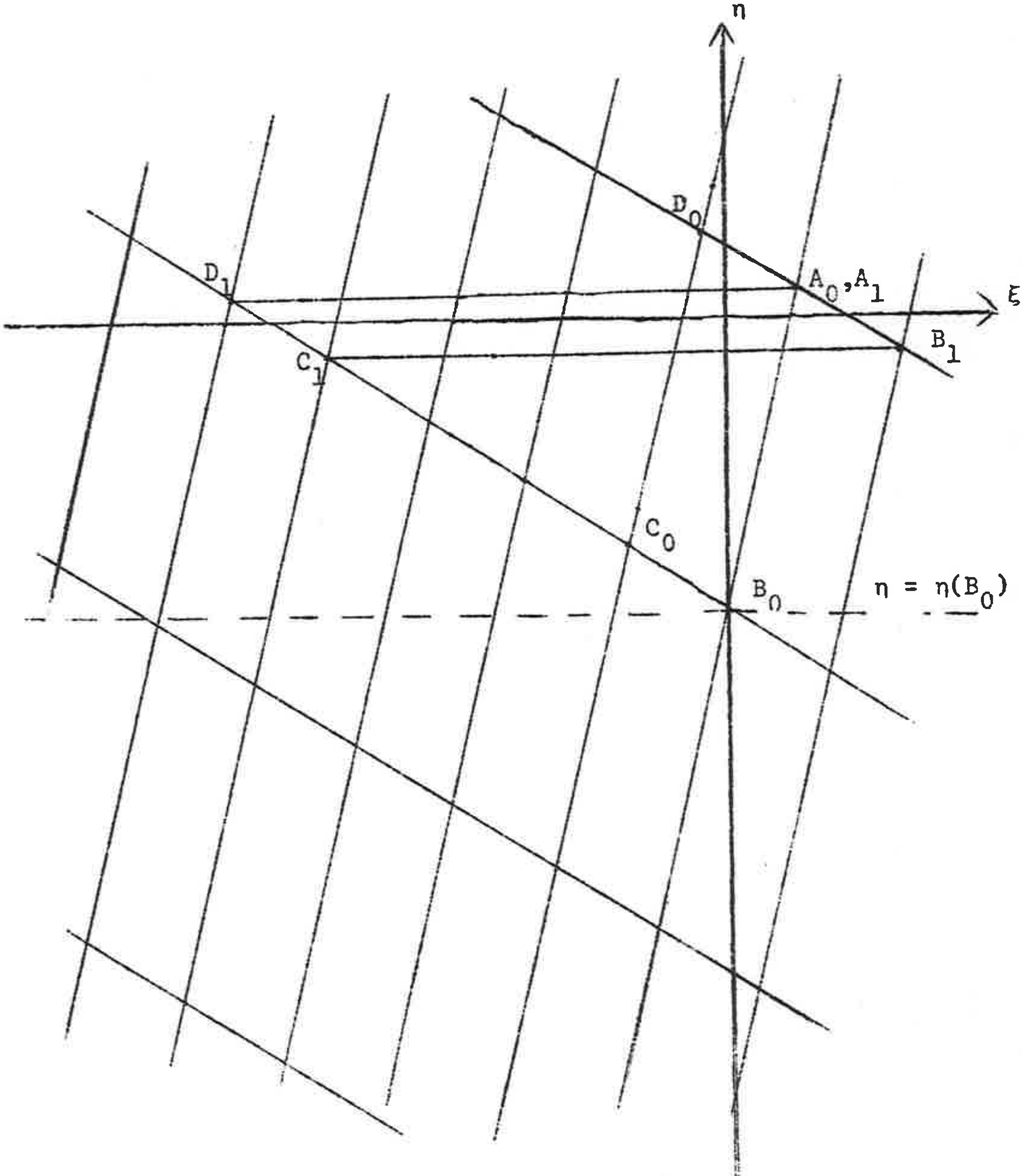

Figure 2.

of the region $|\xi\eta| \geq m$ in the third quadrant, any region R of the type (4.16), which contains a point of L, also contains a point P_n, for some n.

Similarly the theorem holds in the fourth quadrant, since there are no points of L in the rectangle $0 < \xi < \xi(B_1)$, $\eta(B_0) < \eta < 0$.

COROLLARY. *We may replace (4.6) by*

$$M(f;\alpha) = \inf\ \{|\xi\eta|; (\xi,\eta) \text{ a vertex of } S_n, \text{ for some } n, \text{ and } \xi \neq 0\}.$$

4. Vertices of p-divided cells

We now obtain the arithmetic formulation for the vertices of the chain of cells $\{S_n\}$. Denote these vertices, as in (3.14), by

$$\left.\begin{aligned} C_n &= (\beta_n\xi_n, \gamma_n\eta_n) \\ B_n &= (\beta_n(\xi_n + \theta_n), \gamma_n(\eta_n + 1)) \\ D_n &= (\beta_n(\xi_n + 1), \gamma_n(\eta_n + \phi_n)) \\ A_n &= (\beta_n(\xi_n + \theta_n + 1), \gamma_n(\eta_n + 1 + \phi_n)) \end{aligned}\right\}, \tag{4.17}$$

and this uniquely defines, for all n, the β_n, γ_n, ξ_n, η_n, ϕ_n, θ_n.

Note. For $n = 0$, these formulae coincide with (4.5), when we put

$\xi_0 = -\theta_0$, $\theta_0 = -\theta$, $\phi_0 = -\phi$, $\eta_0 = -(1 + \alpha)$, $\beta_0 = \beta$, $\gamma_0 = -\gamma$.

The p-grid is then described, for each n, by

$$L: \left.\begin{aligned} \xi &= \beta_n(\xi_n + \theta_n x + y) \\ \eta &= \gamma_n(\eta_n + x + \phi_n y) \end{aligned}\right\} \tag{4.18}$$

for all integral values of x, y; and since $\Delta = 1$,

$$|\beta_n\gamma_n| = \frac{1}{|\theta_n\phi_n - 1|}. \tag{4.19}$$

We have already noted that the equation (3.18) is still valid, for $\underset{\sim}{V}_n$ defined by (3.12). As in the previous chapter, the sequences

$\{p_n\}$, $\{q_n\}$ may be defined by (3.20), and the recurrence relations (3.21) apply, indicating that the p_n/q_n are convergents of semi-regular continued fractions. Since Lemma 4.3 is valid, we have, following Lemma 3.5 and its corollary,

$$\left.\begin{aligned} \phi_n &= [a_{n+1}, a_{n+2}, \ldots.] \\ \theta_n &= [a_n, a_{n-1}, \ldots\ldots] \end{aligned}\right\}. \tag{4.20}$$

Recall the following definitions (3.23).

$$\left.\begin{aligned} \lambda_n &= 2\xi_n + \theta_n + 1 \\ \mu_n &= 2\eta_n + 1 + \phi_n \end{aligned}\right\}. \tag{4.21}$$

Now, since the recurrence relations (3.11) satisfied by the vertices of the cells are the same for the p-grids of this chapter, reference to the proof of Theorem 3.4 readily confirms that the result still holds for the modified algorithm. Hence

<u>THEOREM 4.3.</u> *For all n,*

$$\lambda_n = \varepsilon_{n-1} + \sum_{r=1}^{\infty} \frac{(-1)^r \varepsilon_{n-r-1}}{\theta_{n-1}\theta_{n-2}\cdot\ldots\cdot\theta_{n-r}},$$

$$\mu_n = \varepsilon_n + \sum_{r=1}^{\infty} \frac{(-1)^r \varepsilon_{n+r}}{\phi_{n+1}\phi_{n+2}\cdot\ldots\cdot\phi_{n+r}}.$$

<u>LEMMA 4.4.</u> *If $\{a_{n+1}, \varepsilon_n\}$ satisfy the conditions (4.12), then*

for $n \le 0$, $\quad \lambda_n = (-1)^{n-1}(\theta_n - 1),$ (4.22)

for $n > 0$, $\quad |\lambda_n| < |\theta_n| - 1,$ (4.23)

for all n, $\quad |\mu_n| < |\phi_n| - 1.$ (4.24)

<u>PROOF.</u> Notice that (4.22) contradicts its counterpart (3.27), and this crystallizes the basic difference between the general formulation of the problem, and the modification considered in this chapter.

The conditions (4.12) and (3.26) are identical for the case $n \to \infty$; consequently (4.24) follows as in Theorems 3.7, 3.8. For $n < 0$, however, (4.12) (ii) implies

$$\varepsilon_{n-1} = (-1)^{n-1}|\varepsilon_{n-1}|.$$

But the a_n are all positive for non-positive n, and hence we may apply Theorem 3.7;

$$\lambda_n = \varepsilon_{n-1} + \sum_{r=1}^{\infty} \frac{(-1)^r \varepsilon_{n-r-1}}{\theta_{n-1}\theta_{n-2}\cdots\theta_{n-r}}$$

$$= (-1)^{n-1}(a_n - 2) + \sum_{r=1}^{\infty} \frac{(-1)^{n-1}(a_{n-r} - 2)}{\theta_{n-1}\theta_{n-2}\cdots\theta_{n-r}}$$

$$= (-1)^{n-1}(\theta_n - 1).$$

There remains only the inequality (4.23). This follows Theorems 3.7, 3.8, since for $n > 0$, the sequence $a_n, a_{n-1}, a_{n-2}, \ldots\ldots$ cannot have constant sign ($a_1 < 0$), and since for all r,

$$|\varepsilon_r| \leq |a_{r+1}| - 2.$$

<u>*THEOREM 4.4.*</u> *Suppose that $\{a_{n+1}, \varepsilon_n\}$ satisfies (4.12) and corresponds to the form f and the real non-zero number α, satisfying (4.4); Let,*

(i) for $n \geq 1$, $M_n = M_n(f;\alpha) = \inf\{M_n^{(i)};\ i = 1,2,3,4\}$, where $M_n^{(i)}$ is defined by (3.25).

(ii) for $n = 0$, $M_0 = M_0(f;\alpha) = \inf\{M_0^{(i)};\ i = 1,2,3\}$, where

$$\left.\begin{aligned} M_0^{(1)} &= \frac{1}{2(|\theta_0\phi_0| + 1)}\,\theta_0(|\phi_0| - 1 + \mu_0) \\ M_0^{(2)} &= \frac{1}{2(|\theta_0\phi_0| + 1)}\,(\theta_0 - 1)(|\phi_0| + 1 - \mu_0) \\ M_0^{(3)} &= \frac{1}{2(|\theta_0\phi_0| + 1)}\,(|\phi_0| - 1 - \mu_0) \end{aligned}\right\}. \qquad (4.25)$$

(iii) for $n < 0$, $M_n = M_n(f;\alpha) = \inf\{M_n^{(i)};\ i = 1,2,3\}$, where

$$\left.\begin{aligned} M_n^{(1)} &= \frac{1}{2(\theta_n\phi_n - 1)}\,\theta_n(\phi_n + 1 + (-1)^{n+1}\mu_n) \\ M_n^{(2)} &= \frac{1}{2(\theta_n\phi_n - 1)}\,(\theta_n - 1)(\phi_n - 1 + (-1)^n\mu_n) \\ M_n^{(3)} &= \frac{1}{2(\theta_n\phi_n - 1)}\,(\phi_n + 1 + (-1)^n\mu_n) \end{aligned}\right\}. \qquad (4.25)'$$

Then we have

$$M = M(f;\alpha) = \inf_n M_n.$$

PROOF. (i) The result for this case holds analogously to Theorem 3.6, taking $\Delta = 1$.

(ii) For $n = 0$, equation (4.22) implies $\lambda_0 = 1 - \theta_0$, from which follows $\xi_0 + \theta_0 = 0$, and so by (4.17), $\xi(B_0) = 0$. The result follows by evaluating the products of the coordinates at the remaining three vertices of (4.17), using the formulae (4.21) and (4.22).

(iii) If we substitute the formula for λ_n, (4.22), in the transformed equations (3.25) (which are of course still valid for p-grids), then we obtain the required result by considering the two cases, n even or odd.

The assertion of the theorem follows from the corollary to Theorem 4.2, since the $M_n^{(i)}$ are values of $|\xi\eta|$, where (ξ,η) is a vertex of the cell S_n.

Summary of the method so far

Given any form f which does not represent zero, and an α satisfying (4.4), then by constructing the chain of cells $\{S_n\}$ of an associated p-grid L, a sequence pair of integers $\{a_{n+1},\varepsilon_n\}$, satisfying

(4.12), is obtained. Theorem 4.4 then enables the evaluation of $M(f;\alpha)$. We will now show that the converse result is true.

Suppose that we have a doubly infinite sequence pair $\{a_{n+1},\varepsilon_n\}$, satisfying the conditions (4.12). We may define ϕ_n, θ_n by (4.20), and λ_n, μ_n by the corresponding expressions in Theorem 4.3, and consequently uniquely define ξ_n, η_n by the relations (4.21).

Consider the four points A_n B_n, C_n, D_n, defined by (4.17). In the same way to that indicated in the previous chapter, the values of h_n and k_n enable us to compute the coordinates of the four points A_{n+1}, B_{n+1}, C_{n+1}, D_{n+1}, from the formulae (3.11). These are given by (3.30).

Whenever $n > 0$, (4.23) and (4.24) imply the validity of the relations (3.28) and (3.29). Consequently A_n, B_n, C_n, D_n, and A_{n+1}, B_{n+1}, C_{n+1}, D_{n+1}, are vertices of divided cells of similar grids.

For $n < 0$, (4.24) implies the validity of equations (3.29), and (4.22) implies

$$\left.\begin{aligned} \xi_n + \theta_n &= 0, \quad \text{for } n \text{ even}, \\ \xi_n + 1 &= 0, \quad \text{for } n \text{ odd}. \end{aligned}\right\} \tag{4.26}$$

Since $\theta_n > 0$, $\phi_n > 0$, it follows that both A_n, B_n, C_n, D_n, and A_{n+1}, B_{n+1}, C_{n+1}, D_{n+1}, are the four vertices of p-divided cells in similar p-grids.

If $n = 0$, then $\xi_0 + \theta_0 = 0$ implies that B_0 is on the η-axis and A_0, B_0, C_0, D_0, are the vertices of a p-divided cell of some p-grid. Since $|\lambda_1| < |\theta_1| - 1$, $|\mu_1| < |\phi_1| - 1$, then both (3.28) and (3.29) hold, and hence A_1, B_1, C_1, D_1, are vertices of a genuine divided cell from a similar p-grid.

Thus, the chain pair $\{a_{n+1}, \varepsilon_n\}$ corresponds to a chain $\{S_n\}$ of p-divided and divided cells from some p-grid L, which is unique apart from similarity. Hence to every f, and α satisfying (4.4), there corresponds an integer chain pair $\{a_{n+1}, \varepsilon_n\}$, satisfying (4.12), and conversely. For convenience, the chain pair will be displayed as follows:

$$\begin{array}{r|l} \ldots\ldots a_{-2},\ a_{-1},\ a_0 & a_1,\ a_2,\ a_3\ldots\ldots \\ \ldots\ldots \varepsilon_{-3},\ \varepsilon_{-2},\ \varepsilon_{-1} & \varepsilon_0,\ \varepsilon_1,\ \varepsilon_2\ldots\ldots \end{array} \tag{4.27}$$

The vertical line (called the *centre* of the chain) separates the homogeneous and the inhomogeneous character of the chain, in a sense that will be amplified later.

5. A useful lemma

We first note that the upper bounds for M_n which are given in Theorem 3.9 are still valid when $n \geq 1$. Theorem 3.10 shows that the k defined in (4.1) satisfies $k \leq \frac{1}{4}$. Clearly Theorem 3.11 also holds for chains of the type (4.12).

For large negative n, Theorem 3.11 and (4.22) suggest that $|\mu_n|$ is close to $\phi_n - 1$. The following lemma gives a useful explicit formulation for $|\mu_n|$.

LEMMA 4.5. *For $n < -1$,*

$$\phi_n - 1 = |\mu_n| + \frac{\phi_{-1} - 1 + \mu_{-1}}{\phi_{n+1}\phi_{n+2}\cdots\phi_{-1}} .$$

PROOF. Since $\phi_n > 0$, $\varepsilon_n = (-1)^n(a_{n+1} - 2)$, for all $n > 0$, then

$$|\mu_n| = (a_{n+1} - 2) + \frac{a_{n+2} - 2}{\phi_{n+1}} + \cdots + \frac{a_{-1} - 2}{\phi_{n+1}\phi_{n+2}\cdots\phi_{-2}} - \frac{\mu_{-1}}{\phi_{n+1}\phi_{n+2}\cdots\phi_{-1}}.$$

Now $\phi_n = a_{n+1} - 1/\phi_{n+1}$, implies that, for all n,

$$\phi_n - 1 = a_{n+1} - 2 + \frac{\phi_{n+1} - 1}{\phi_{n+1}};$$

hence

$$\phi_n - 1 = (a_{n+1} - 2) + \frac{a_{n+2} - 2}{\phi_{n+1}} + \cdots + \frac{a_{-1} - 2}{\phi_{n+1}\phi_{n+2}\cdots\phi_{-2}} + \frac{\phi_{-1} - 1}{\phi_{n+1}\phi_{n+2}\cdots\phi_{-1}}.$$

The result follows by combining these two expressions.

<u>Remark</u> If we consider a chain $\{a_n\}$, with $a_n \geq 2$ and $\varepsilon_n = (-1)^n(a_{n+1} - 2)$ for all n, then both $|\mu_n| = \phi_n - 1$ and $|\lambda_n| = \theta_n - 1$. It is easily shown that the point B_0 is then the **origin**, and the "grid" L is a homogeneous lattice. We could then derive an alternative formulation for the homogeneous form problem, in terms of the continued fraction to the integer above. The three products at each step are

$$\frac{\theta_n}{\theta_n\phi_n - 1}, \quad \frac{\phi_n}{\theta_n\phi_n - 1}, \quad \frac{(\theta_n - 1)(\phi_n - 1)}{\theta_n\phi_n - 1}. \tag{4.28}$$

This geometric interpretation of the A-expansion of an irrational in terms of cells of homogeneous lattices, is closely connected to a geometric setting for ordinary continued fractions which was briefly considered by Cassels in his book An Introduction to the Geometry of Numbers (p. 301).

Thus in chains of the type (4.12), we distinguish two trends.

(i) As $n \to -\infty$, the appropriate three products of Theorem 4.4 are asymptotic to the three alternatives (4.28).

(ii) For positive n, the four alternatives of Theorem 4.4 are identical with those of the general inhomogeneous case discussed in Chapter III.

As a consequence we would expect to be able to construct chains with the property:

(i) $M_n \sim \frac{1}{\sqrt{5}}$ as $n \to -\infty$, and

(ii) $M_n \sim \frac{1}{4}$ as $n \to \infty$.

For this type of chain, the values of M_n approaching $M(f;\alpha)$ occur for small values of n (i.e. near the centre of the chain). We will, in fact, prove that the critical chain has the property $M_1 = k$.

6. Application of the method

The method that we have described will be applied in the next chapter to evaluate the best possible constant k, defined by (4.1). Clearly

$$k = \sup_{f,\alpha} M(f;\alpha),$$

where the supremum is taken over all forms that do not represent zero, and all non-zero α.

By Lemma 4.1 and its corollary, we need only consider those f and α which satisfy (4.4); such forms are characterized by a chain pair $\{a_{n+1},\varepsilon_n\}$. Thus

$$M = M(f;\alpha) = M(\{a_{n+1},\varepsilon_n\}),$$

which may be calculated by (4.25).

$$k = \sup M(\{a_{n+1},\varepsilon_n\}),$$

where the supremum extends over all chain pairs satisfying (4.12).

If there exists a chain pair for which $M(\{a_{n+1},\varepsilon_n\}) = k$, then it will be called a *critical chain*, and the corresponding f and α, a *critical form.* For example, if θ_0, ϕ_0, η_0, are values taken from a critical chain, then by (4.17), since $\theta_0 + \xi_0 = 0$, a critical form is given by

$$\pm \frac{(\theta x + y)(x + \phi y + \alpha)}{\theta\phi - 1}, \tag{4.29}$$

where $\theta = -\theta_0$, $\phi = -\phi_0$, and $\alpha = -(1 + \eta_0)$. If we change the variables in (4.29) by the integral unimodular transformation (3.3), then the equivalent form obtained has the same value of the infimum. (4.30)

In the next chapter we will show that k has the value given by the formulae (1.30), (1.31).

CHAPTER V

EVALUATION OF THE CONSTANT k

1. Introductory lemmas

The purpose of this chapter is to determine the best possible constant, k, by the method formulated in Chapter IV. We have seen that it is sufficient to consider those chain pairs with the specialized properties (4.12). Moving in a step-wise process from the centre of the chain, the values of each member of the chain pair will be isolated by the inequality $M_n \geq k$, for all n, where $k = 0{\cdot}234254343\ldots$ is the constant defined in (1.30), (1.31). This will lead to a unique chain, whose minimum will be evaluated in §3.

Hence from the outset we will suppose that $M_n \geq k$, for all n, thus enabling us to exclude from consideration any chain which implies for some n and i, $M_n^{(i)} < k$. For convenience, $M(f;\alpha)$ will be abbreviated to M, provided that there is no ambiguity.

The following lemmas, giving bounds on some of the variables, will be used consistently throughout the proof. We define the two new variables

$$\tau_n = \mu_n/\phi_n, \quad \sigma_n = \lambda_n/\theta_n . \tag{5.1}$$

LEMMA 5.1. *For $n \geq 1$,*

(i) if $\theta_n\phi_n > 0$, then $|\theta_n| > \frac{1}{1 - 4k}$, and $|\phi_n| > \frac{1}{1 - 4k} > 15{\cdot}87$.

(ii) if $\theta_n\phi_n < 0$, then $|\phi_n| > 2$; furthermore if $|\theta_n| > 30$, then $|\phi_n| > 10$.

LEMMA 5.2. *For $n \geq 1$,*

(i) if $\theta_n\phi_n > 0$, then $|\tau_n| < 0{\cdot}0668$, $|\sigma_n| < 0{\cdot}0668$.

(ii) if $\theta_n\phi_n < 0$, then $|\tau_n| < 1 - 4k$, $|\sigma_n| < 1 - 4k < 0{\cdot}063$.

PROOF OF LEMMA 5.1.

(i) If $\theta_n\phi_n > 0$, then by Theorem 3.9 and Lemma 2.7, whenever $|\theta_n| \leq 1/(1 - 4k)$,

$$M_n \leq \frac{(|\theta_n| - 1)(|\phi_n| - 1)}{4(|\theta_n\phi_n| - 1)}$$

$$< \frac{|\theta_n| - 1}{4|\theta_n|} \leq k.$$

Thus $|\theta_n| > 1/(1 - 4k)$. By symmetry, the result also holds for $|\phi_n|$.

(ii) If $\theta_n\phi_n < 0$, suppose without loss of generality that $a_{n+1} > 0$; then $\phi_n \geq 2 - 1/\phi_{n+1} > 2$ when $\phi_{n+1} < 0$. If $\phi_{n+1} > 0$, then by (i), $\theta_{n+1} > 15$, and the result again holds. If, however, $|\theta_n| > 30$, then by Theorem 3.9 and Lemma 2.7, whenever $|\phi_n| < 10$,

$$M_n \leq \frac{(|\theta_n| + 1)(|\phi_n| - 1)}{4(|\theta_n\phi_n| + 1)} < \frac{(31)(9)}{(4)(301)} < k.$$

Thus we have $|\phi_n| > 10$.

PROOF OF LEMMA 5.2.

(i) If $\theta_n\phi_n > 0$, then the previous lemma implies that $\theta_n\phi_n > (15{\cdot}87)^2 > 250$. Now if $|\tau_n| \geq 0{\cdot}0668$, Theorem 3.9 implies

$$M_n \leq \frac{\theta_n\phi_n(1 - |\tau_n|)}{4(\theta_n\phi_n - 1)} < \frac{(250)(0{\cdot}9332)}{(4)(249)} < k.$$

Thus $|\tau_n| < 0{\cdot}0668$. By the symmetry of Theorem 3.9, the result also holds for $|\sigma_n|$.

(ii) If $\theta_n\phi_n < 0$, then whenever $|\tau_n| \geq 1 - 4k$,

$$M_n \leq \frac{|\theta_n\phi_n|(1 - |\tau_n|)}{4(|\theta_n\phi_n| + 1)} < \frac{1 - |\tau_n|}{4} \leq k.$$

Similarly, the result holds for $|\sigma_n|$.

Throughout the remainder of this chapter, we will often use the result in Lemma 2.7 without specific reference to it

2. Chains with $\varepsilon_0 \geq 0$

This section will be devoted to the proof that $\varepsilon_0 < 0$ in any chain for which $M \geq k$.

Now for $\varepsilon_0 \geq 0$, we have $\mu_0 > -1$; thus whenever $\theta_0 > 2{\cdot}14$, we have, by (4.25),

$$M_0^{(3)} < \frac{|\phi_0|}{2(\theta_0|\phi_0| + 1)} < \frac{1}{2\theta_0} < k.$$

Hence $\theta_0 < 2{\cdot}14$ and consequently $a_0 = 2$ or 3.

When $a_0 = 3$, $\theta_{-1} = \frac{1}{3 - \theta_0}$. By the general form of the chain $\varepsilon_{-1} = -(a_0 - 2) = -1$, and so $-2 < \mu_{-1} < 0$; now since $a_1 < 0$, then $\phi_{-1} > 3$. Hence by (4.25), since $\theta_{-1} < 1{\cdot}17$,

$$M_{-1}^{(2)} < \frac{(\theta_{-1} - 1)(\phi_{-1} + 1)}{2(\theta_{-1}\phi_{-1} - 1)} < \frac{(0{\cdot}17)(4)}{5{\cdot}02} < k.$$

When $a_0 = 2$, then $\varepsilon_{-1} = 0$. Using the same argument as in Lemma 5.1 (i), it follows that $|\phi_0| > 2$. Now if $\mu_0 > 0$, then $\mu_{-1} = -\tau_0 > 0$; if, however $\mu_0 < 0$, then since $\varepsilon_0 \geq 0$, we have $\varepsilon_0 = 0$. Hence $|\mu_{-1}| = |\tau_0| = \left|\frac{\tau_1}{\phi_0}\right| < \frac{{\cdot}07}{2} < 0{\cdot}04$, by Lemma 5.2. Thus, in both cases, $\mu_{-1} > -0{\cdot}04$.

Now if $\theta_{-1} < 2{\cdot}2$, then since $\phi_{-1} < 3$,

$$M_{-1}^{(2)} < \frac{(\theta_{-1} - 1)(\phi_{-1} - 0{\cdot}96)}{2(\theta_{-1}\phi_{-1} - 1)} < \frac{(1{\cdot}2)(2{\cdot}04)}{11{\cdot}2} < k.$$

If $\theta_{-1} > 4$, then since $\phi_{-1} > 2$, and $\mu_{-1} > -0{\cdot}04$, we have

$$M_{-1}^{(3)} < \frac{\phi_{-1} + 1{\cdot}04}{2(\theta_{-1}\phi_{-1} - 1)} < \frac{3{\cdot}04}{14} < k.$$

Thus $2{\cdot}2 < \theta_{-1} < 4$, which implies that $1{\cdot}54 < \theta_0 < 1{\cdot}75$.

Now, by Lemma 5.2, $\mu_0 > -0{\cdot}07$, implying

(i) if $|\phi_0| < 5$, $M_0^{(3)} < \dfrac{5 - 0{\cdot}93}{2\{(1{\cdot}54)(5) + 1\}} < k$,

(ii) if $|\phi_0| > 5$, $M_0^{(2)} < \dfrac{(1{\cdot}75 - 1)(5 + 1{\cdot}07)}{2\{(1{\cdot}75)(5) + 1\}} < k$.

We have now excluded all possible cases when $\varepsilon_0 \geq 0$. Hence we have that $\mu_0 < 0$. We have by the formula (4.12), $\varepsilon_n = (-1)^n|\varepsilon_n|$, for $n < 0$; if $\varepsilon_n = 0$, then $\mu_n = -\tau_{n+1}$, indicating that μ_n and μ_{n+1} are of opposite sign. Thus, for all $n < 0$, $\mu_n = (-1)^n|\mu_n|$; this enables us to rewrite part of Theorem 4.4.

<u>*THEOREM 5.1.*</u>

$$M_0^{(1)} = \frac{1}{2(\theta_0|\phi_0| + 1)}\,\theta_0(|\phi_0| - 1 - |\mu_0|),$$

$$M_0^{(2)} = \frac{1}{2(\theta_0|\phi_0| + 1)}\,(\theta_0 - 1)(|\phi_0| + 1 + |\mu_0|),$$

$$M_0^{(3)} = \frac{1}{2(\theta_0|\phi_0| + 1)}\,(|\phi_0| - 1 + |\mu_0|),$$

and for $n < 0$, (5.2)

$$M_n^{(1)} = \frac{1}{2(\theta_n\phi_n - 1)}\,\theta_n(\phi_n + 1 - |\mu_n|),$$

$$M_n^{(2)} = \frac{1}{2(\theta_n\phi_n - 1)}\,(\theta_n - 1)(\phi_n - 1 + |\mu_n|),$$

$$M_n^{(3)} = \frac{1}{2(\theta_n\phi_n - 1)}\,(\phi_n + 1 + |\mu_n|);$$

The expressions for $M_n^{(i)}$, $n > 0$, *are given by* (3.25).

3. Evaluation of the minimum of a certain chain (C)

In this section we will prove that the infimum of the following chain (designated by (C)) is k. We use the notation of the previous chapter, and dividing the chain at the centre, we write each half on separate lines:

$$\infty\begin{bmatrix} 5, & 5, & 2, & 5, & 5, & 2 \\ 3, & -3, & 0, & -3, & 3, & 0 \end{bmatrix} \begin{matrix} 3, & 4, & 4, & 2 \\ 1, & -2, & 2, & 0 \end{matrix} \Bigg|$$

$$\Bigg| \begin{matrix} -11, & 21, & 461, & -17, & 50 \\ -1, & -1, & -29, & -1, & -2. \end{matrix} \begin{bmatrix} 49, & -42, & 49, & -42 \\ -3, & 0, & 3, & 0 \end{bmatrix}\infty$$

Suppose that $x = [\overline{2,5,5}]$, then $8x^2 - 16x + 3 = 0$, implying that $x = \theta_{-4} = \dfrac{4 + \sqrt{10}}{4} = 1{\cdot}790569\ldots$ Consequently

$$\theta_0 = [2,4,4,3,x] = \frac{71x - 26}{41x - 15} = \frac{5195 - 2\sqrt{10}}{2997}. \tag{5.3}$$

Suppose that $y = [\overline{49,-42}]$, then $42y^2 - 2058y - 49 = 0$, implying that

$$y = \phi_5 = \frac{147 + \sqrt{21651}}{6} = 49{\cdot}0237979\ldots \tag{5.4}$$

We will now prove a general lemma, the full force of which will not be used in this section, but will be of considerable significance in the latter part of this chapter.

LEMMA 5.3. *If we have a half-chain of the following form:*

$$\Bigg| \begin{matrix} a, & -a_2, & a, & -a_4, & a\ldots\ldots \\ \varepsilon, & 0, & -\varepsilon, & 0, & \varepsilon\ldots\ldots \end{matrix}$$

where $a_{2n+1} = a$, $\varepsilon_{2n} = (-1)^n\varepsilon$, $\varepsilon_{2n+1} = 0$, *for all* $n \geq 0$, *and* a, ε, a_{2n}, *are all positive and of the correct parity and size; then*

$$|\mu_0/\phi_0| = |\tau_0| = a/\varepsilon.$$

PROOF. We are supposing that the chain commences at $a_1 = a$, $\varepsilon_0 = \varepsilon$. It is clear from the sign pattern of the chain that

$$\mu_0 = \varepsilon\Big(1 + \sum_{r=1}^{\infty} \frac{1}{|\phi_1\phi_2\cdots\cdot\phi_{2r}|}\Big),$$

since $|\mu_{2n}| = \varepsilon + \left|\frac{\tau_{2n+2}}{\phi_{2n+1}}\right|$. Thus

$$|\tau_{2n}| = \frac{\varepsilon + |\tau_{2n+2}/\phi_{2n+1}|}{a + 1/|\phi_{2n+1}|},$$

and so

$$|\tau_{2n}| - \varepsilon/a = \frac{|\tau_{2n+2}| - \varepsilon/a}{a|\phi_{2n+1}| + 1}.$$

Now for all n we have $a|\phi_{2n+1}| + 1 > 3$, and hence for every integer r,

$$|\tau_0 - \varepsilon/a| < (\tfrac{1}{3})^r.$$

The result follows from this.

<u>COROLLARY.</u> *For the chain (C) under consideration,*

$$|\tau_5| = 3/49.$$

We will now use (5.3), (5.4), the fact from (4.22) that $\lambda_0 = 1 - \theta_0$, and the basic recurrence relations between the variables, in order to compute the values set out on the following table. The values are truncated.

<u>*TABLE 5.1.*</u>

n	$\lvert\theta_n\rvert$	$\lvert\sigma_n\rvert$	$\lvert\phi_n\rvert$	$\lvert\tau_n\rvert$
0	1·7312	0·4223	11·0476	0·0864
1	11·5776	0·0498	20·9978	0·0446
2	21·0863	0·0497	461·0587	0·0630
3	460·9525	0·0628	17·0200	0·0564
4	17·0021	0·0551	49·9796	0·0387
5	50·0588	0·0410	49·0237	0·0612
6	48·9800	0·0604	42·0203	0·0014

We will now proceed to examine, in turn, the four (or three) alternatives from Theorem 5.1, at each step of the chain, with the intent of showing that $M_n > k$ for all $n \neq 1$, and that $M_1 = k$. It will be necessary to treat those steps in the chain which have small values of n separately, since there is no pattern in $\{a_{n+1}, \varepsilon_n\}$ which readily leads to general methods. We note that there is usually at least one of the products which obviously exceeds k. We commence with the right hand chain ($n > 1$).

(i) Proof that $M_2 > k$

Clearly $M_2^{(1)} > k$. Now, by (3.25), $M_2^{(2)} > M_2^{(4)}$ if

$$\frac{(\theta_2 - 1 + |\lambda_2|)(\phi_2 - 1 - |\mu_2|)}{4(\theta_2\phi_2 - 1)} > \frac{(\theta_2 + 1 - |\lambda_2|)(\theta_2 + 1 - |\mu_2|)}{4(\theta_2\phi_2 - 1)},$$

which holds if and only if $\phi_2 - |\mu_2| > \frac{\theta_2}{|\lambda_2| - 1}$;for, putting $\theta = \theta_2$, $x = \phi_2 - |\mu_2|$ and $y = |\lambda_2| - 1$, the inequality reduces to

$$\frac{x - 1}{x + 1} > \frac{\theta/y - 1}{\theta/y + 1},$$

which is true if and only if $x > \theta/y$, since the function on the left-hand side of the inequality increases with x. We will use this method often, without including all the details. Now for the chain (C), by Table 5.1, $\phi_2 - |\mu_2| > 461 - 29{\cdot}1 > 430$, and $\frac{\theta_2}{|\lambda_2| - 1} < \frac{21{\cdot}1}{0{\cdot}0498} < 425$.

Thus

$$M_2^{(2)} > M_2^{(4)} > \frac{(\theta_2 - 0{\cdot}0499)(\phi_2 - 28{\cdot}0565)}{4(\theta_2\phi_2 - 1)}$$

$$> \frac{(21{\cdot}0364)(433{\cdot}0022)}{38884{\cdot}1} > k,$$

by (3.25) and Lemma 2.7. Similarly

$$M_2^{(3)} = \frac{(19 - (|\lambda_1| - 1)/|\theta_1|)(\phi_2 + 28 + |\tau_3|)}{4(\theta_2\phi_2 - 1)} > \frac{(18\cdot 9)(461\cdot 06 + 28\cdot 05)}{38910}$$

$$> k.$$

It therefore follows that $M_2 > k$.

(ii) Proof that $M_3 > k$

Clearly $M_3^{(2)} > \frac{1}{4}$. Using an analogous method to that in (i), we can show that $M_3^{(4)} > M_3^{(3)}$, if and only if $\frac{|\phi_3|}{1 - |\mu_3|} > \theta_3 - |\lambda_3|$. For the chain (C), Table 5.1 implies that $\theta_3 - |\lambda_3| < 461 - 28 = 433$, and $\frac{|\phi_3|}{1 - |\mu_3|} > \frac{17\cdot 02}{0\cdot 0388} > 438$. Using the appropriate bounds from Table 5.1, we obtain

$$M_3^{(4)} > M_3^{(3)} = \frac{(\theta_3 - 30 + |\sigma_2|)(|\phi_3| + |\tau_4|)}{4(\theta_3|\phi_3| + 1)} > \frac{(431)(17\cdot 0588)}{31385\cdot 9} > k.$$

Similarly

$$M_3^{(1)} = \frac{(\theta_3 + 1 + |\lambda_3|)(|\phi_3| - 1 - |\mu_3|)}{4(\theta_3|\phi_3| + 1)} > \frac{(461 + 29)(17 - 2)}{31352} > k.$$

It therefore follows that $M_3 > k$.

(iii) Proof that $M_4 > k$

Clearly $M_4^{(3)} > \frac{1}{4}$. We may readily show that $M_4^{(2)} > M_4^{(1)}$, if and only if $\frac{\phi_4}{1 + |\mu_4|} > |\theta_4| - |\lambda_4|$. For the chain (C),

$\frac{\phi_4}{1 + |\mu_4|} > \frac{49\cdot 9}{2\cdot 94} > 16\cdot 5 > |\theta_4| - |\lambda_4|$. We have

$$M_4^{(1)} = \frac{(|\theta_4| - 1 - |\lambda_4|)(\phi_4 + 1 + |\mu_4|)}{4(|\theta_4|\phi_4 + 1)} > \frac{(15\cdot 0648)(52\cdot 918)}{34030\cdot 4} > k.$$

Also

$$M_4^{(4)} > \frac{(|\theta_4| - |\sigma_3|)(\phi_4 - 1)}{4(|\theta_4|\phi_4 + 1)} > \frac{(16\cdot 939)(48)}{3336} > k.$$

Thus we have that $M_4 > k$.

(iv) Proof that $M_5 > k$

Clearly $M_5^{(1)} > \frac{1}{4}$. For the chain (C) we have

$$\frac{\theta_5}{|\lambda_5| - 1} > \frac{50}{1\cdot 06} > 47 > \phi_5 - |\mu_5|,$$

which implies, as in (i), that $M_5^{(4)} > M_5^{(2)}$. Thus by (3.25),

$$M_5^{(2)} = \frac{(\theta_5 + 1 + |\sigma_4|)(\phi_5 - 4 - |\tau_6|)}{4(\theta_5\phi_5 - 1)} > \frac{(51\cdot 11)(44\cdot 998)}{4\{(50\cdot 06)(49) - 1\}} > k.$$

Similarly

$$M_5^{(3)} > \frac{(\theta_5 - 3 - |\sigma_4|)(\phi_5 + 2)}{4(\theta_5\phi_5 - 1)} > \frac{(47 - 0\cdot 06)(51\cdot 03)}{4\{(50)(49\cdot 03) - 1\}} > k.$$

Thus we have that $M_5 > k$.

(v) Proof that $M_{2m} > k$, $m \geq 3$

For the purpose of a result in the next chapter, we will show that the inequality (v) remains valid for $a_{2m+1} = -42$ and -44. It follows from (3.25) and the sign pattern of the chain (C), that the same four alternatives occur at each such step, M_{2m}, $m \geq 3$. This is a consequence of the fact that only the order of the products alters whenever the ε-chain is reversed in sign (Theorem 3.12). We will use the following notations:

$$M_{2m}^{(1)} = \frac{1}{4(\theta_{2m}|\phi_{2m}| + 1)} (\theta_{2m} + 1 + |\lambda_{2m}|)(|\phi_{2m}| - 1 - |\mu_{2m}|),$$

$$M_{2m}^{(2)} = \frac{1}{4(\theta_{2m}|\phi_{2m}| + 1)} (\theta_{2m} - 1 + |\lambda_{2m}|)(|\phi_{2m}| + 1 + |\mu_{2m}|),$$

$$M_{2m}^{(3)} = \frac{1}{4(\theta_{2m}|\phi_{2m}| + 1)} (\theta_{2m} - 1 - |\lambda_{2m}|)(|\phi_{2m}| + 1 - |\mu_{2m}|),$$

$$M_{2m}^{(4)} = \frac{1}{4(\theta_{2m}|\phi_{2m}| + 1)} (\theta_{2m} + 1 - |\lambda_{2m}|)(|\phi_{2m}| - 1 + |\mu_{2m}|).$$

Clearly $M_{2m}^{(2)} > \frac{1}{4}$. Now, for $m \geq 3$, we have by Table 5.1, and Lemmas 5.2, 5.3, the following inequalities:

$$48{\cdot}98 < \theta_{2m} < 49{\cdot}024, \quad 42{\cdot}02 < |\phi_{2m}| < 44{\cdot}03,$$

$$2{\cdot}9589 < 3 - |\sigma_5| = |\lambda_6| \le |\lambda_{2m}| < 3 + \frac{3{\cdot}007}{(42)(49)} < 3{\cdot}0015,$$

$$|\mu_{2m}| = |\tau_{2m+1}| = 3/49 = 0{\cdot}0612\ldots$$

Using these bounds and Lemma 2.7, we obtain

$$M_{2m}^{(1)} > \frac{(\theta_{2m} + 3{\cdot}9)(|\phi_{2m}| - 1{\cdot}1)}{4(\theta_{2m}|\phi_{2m}| + 1)} > \frac{(53{\cdot}9)(40{\cdot}9)}{4\left[(50)(42) + 1\right]} > k,$$

$$M_{2m}^{(3)} > \frac{(\theta_{2m} - 4{\cdot}0015)(|\phi_{2m}| + 0{\cdot}938)}{4(\theta_{2m}|\phi_{2m}| + 1)} > \frac{(44{\cdot}9785)(44{\cdot}968)}{4\left[(48{\cdot}98)(44{\cdot}03) + 1\right]} > k,$$

$$M_{2m}^{(4)} > \frac{(\theta_{2m} - 2{\cdot}002)(|\phi_{2m}| - 0{\cdot}939)}{4(\theta_{2m}|\phi_{2m}| + 1)} > \frac{(46{\cdot}978)(41{\cdot}081)}{4\left[(48{\cdot}98)(42{\cdot}02) + 1\right]} > k.$$

Thus, even when $a_{2m+1} = -44$, $M_{2m} > k$, $m \ge 3$.

(vi) <u>Proof that $M_{2m+1} > k$, $m \ge 3$</u>

We again allow $a_{2m+1} = -42$ or -44. For convenience put $2m+1 = r$; then we have, as in (v),

$$M_r^{(1)} = \frac{1}{4(|\theta_r|\phi_r + 1)}(|\theta_r| - 1 + |\lambda_r|)(\phi_r + 1 - |\mu_r|),$$

$$M_r^{(2)} = \frac{1}{4(|\theta_r|\phi_r + 1)}(|\theta_r| + 1 + |\lambda_r|)(\phi_r - 1 + |\mu_r|),$$

$$M_r^{(3)} = \frac{1}{4(|\theta_r|\phi_r + 1)}(|\theta_r| + 1 - |\lambda_r|)(\phi_r - 1 - |\mu_r|),$$

$$M_r^{(4)} = \frac{1}{4(|\theta_r|\phi_r + 1)}(|\theta_r| - 1 - |\lambda_r|)(\phi_r + 1 + |\mu_r|).$$

Clearly $M_r^{(2)} > \frac{1}{4}$. We can readily obtain the following bounds:

$$42{\cdot}02 < |\theta_r| < 44{\cdot}03, \quad 49{\cdot}02 < \phi_r < 49{\cdot}024,$$

$$0{\cdot}0604 < |\lambda_7| \le |\lambda_r| < \frac{1}{49}\left(3 + \frac{0{\cdot}07}{42}\right) < 0{\cdot}0613,$$

$$3{\cdot}0013 < |\mu_r| = 3 + \frac{3}{49|\phi_{r+1}|} < 3{\cdot}0015.$$

Hence it follows that

$$M_r^{(1)} > \frac{(|\theta_r| - 0{\cdot}94)(\phi_r - 2{\cdot}002)}{4(|\theta_r|\phi_r + 1)} > \frac{(41{\cdot}08)(47{\cdot}018)}{4\left[(42{\cdot}02)(49{\cdot}02) + 1\right]} > k,$$

$$M_r^{(3)} > \frac{(|\theta_r| + 0{\cdot}938)(\phi_r - 4{\cdot}002)}{4(|\theta_r|\phi_r + 1)} > \frac{(44{\cdot}968)(45{\cdot}018)}{4\left[(44{\cdot}03)(49{\cdot}02) + 1\right]} > k,$$

$$M_r^{(4)} > \frac{(|\theta_r| - 1{\cdot}062)(\phi_r + 4)}{4(|\theta_r|\phi_r + 1)} > \frac{(40{\cdot}958)(53{\cdot}03)}{4\left[(42{\cdot}02)(49{\cdot}03) + 1\right]} > k.$$

We will now focus our attention on the left-hand side of the chain.

(vii) Proof that $M_0 > k$

By (5.2) and Table 5.1, we have immediately;

$$M_0^{(1)} > \frac{\theta_0(|\phi_0| - 2)}{2(\theta_0|\phi_0| + 1)} > \frac{(1{\cdot}73)(9)}{2\left[(1{\cdot}73)(11) + 1\right]} > k,$$

$$M_0^{(2)} > \frac{(\theta_0 - 1)(|\phi_0| + 2 - |\tau_1|)}{2(\theta_0|\phi_0| + 1)} > \frac{(0{\cdot}73)(13)}{2\left[(1{\cdot}73)(11{\cdot}05) + 1\right]} > k,$$

$$M_0^{(3)} > \frac{|\phi_0| - |\tau_1|}{2(\theta_0|\phi_0| + 1)} > \frac{10{\cdot}95}{2\left[(1{\cdot}74)(11) + 1\right]} > k.$$

Thus $M_0 > k$.

(viii) The case M_{-1}

Clearly $M_{-1}^{(1)} > \frac{1}{2}$. Now for the chain (C),

$\theta_{-1} = [4,4,3,\overline{2,5,5}] > 3{\cdot}72$. Hence

$$\phi_{-1} + |\mu_{-1}| = 2 + \frac{|\mu_0| + 1}{|\phi_0|} > 2 + \frac{1{\cdot}9}{11{\cdot}1} > 2{\cdot}17 > \frac{3{\cdot}72}{1{\cdot}72} > \frac{\theta_{-1}}{\theta_{-1} - 2},$$

which implies by the method used before that $M_{-1}^{(2)} > M_{-1}^{(3)}$.

Since $|\lambda_1| = 1/\theta_0$, the recurrence relations satisfied by the variables, and the sign pattern of the chain (C), imply, by simple substitution, that

$$M_{-1}^{(3)} = \frac{(2 - \theta_0)(3|\phi_0| + 1 + |\mu_0|)}{2(\theta_0|\phi_0| + 1)},$$

$$M_1^{(2)} = \frac{6\theta_0(11 + |\mu_0| - |\phi_0|)}{2(\theta_0|\phi_0| + 1)}.$$

Now, $M_{-1}^{(3)}$ decreases, and $M_1^{(2)}$ increases, as θ_0 increases. The two functions of θ_0 have a common value at the point

$$\bar{\theta}_0 = \frac{2(1 + |\mu_0| + 3|\phi_0|)}{67 + 7|\mu_0| - 3|\phi_0|}.$$

Hence $M_{-1}^{(3)} > M_1^{(2)}$ whenever $\theta_0 < \bar{\theta}_0$. By Lemma 2.7, $\bar{\theta}_0$ increases in $|\phi_0|$, and decreases in $|\mu_0|$; thus, by Table 5.1, for the chain (C),

$$\bar{\theta}_0 > \frac{70{\cdot}1964}{40{\cdot}545} > 1{\cdot}7313 > \theta_0.$$

Thus if $M_1 = k$, as we will later prove, then $M_{-1}^{(3)} > k$.

We will treat the remainder of the chain (C) in a slightly different way, by considering each of the three alternatives in turn.

(ix) <u>Proof that $M_m^{(1)} > k$, $m < -1$</u>

Using (5.3), we find that

$$\left.\begin{aligned} [\overline{5,5,2}] &= \frac{2(4 + \sqrt{10})}{3} = 4{\cdot}77485\ldots \\ [\overline{5,2,5}] &= \frac{7 + 2\sqrt{10}}{3} = 4{\cdot}44151\ldots \end{aligned}\right\} \qquad (5.5)$$

Now, since $|\mu_m| < \phi_m - 1$, then for $a_{m+1} = 2$, 3, or 4, (which implies $\phi_m < 4$),

$$M_m^{(1)} = \frac{\theta_m(\phi_m + 1 - |\mu_m|)}{2(\theta_m\phi_m - 1)} > \frac{\theta_m}{\theta_m\phi_m - 1} > \frac{1}{\phi_m} > \tfrac{1}{4}.$$

When $a_{m+1} = 5$, then <u>either</u> $\phi_m < [5,2,5] < 4{\cdot}45$, $\theta_m < 4{\cdot}45$, <u>or</u> $\phi_m < [5,5,2] < 4{\cdot}78$, $\theta_m < 1{\cdot}8$.

Thus <u>either</u> $M_m^{(1)} > \dfrac{4{\cdot}45}{(4{\cdot}45)^2 - 1} > k$, <u>or</u> $M_m^{(1)} > \dfrac{1{\cdot}8}{(1{\cdot}8)(4{\cdot}78) - 1} > k$.

(x) <u>Proof that $M_m^{(2)} > k,\ m < -1$</u>

By Lemma 4.5, $|\mu_m| = \phi_m - 1 - \alpha_m$, where $\alpha_m = \dfrac{\phi_{-1} - 1 - |\mu_{-1}|}{\phi_{m+1}\phi_{m+2}\cdot\cdots\cdot\phi_{-1}}$.

Now $\{\alpha_m\}$ is a monotone decreasing sequence as $m \to -\infty$; thus we have

$$\alpha_{-2} < \frac{\phi_{-1} - 1}{\phi_{-1}} < \frac{2{\cdot}1 - 1}{2{\cdot}1} < 0{\cdot}6,$$

and for all $m \leq -4$,

$$\alpha_m \leq \alpha_{-4} < \frac{\phi_{-1} - 1}{\phi_{-3}\phi_{-2}\phi_{-1} - 1} = \frac{\phi_{-1} - 1}{15\phi_{-1} - 4} < \frac{1{\cdot}1}{27{\cdot}5} = 0{\cdot}04.$$

Now for $m = -2$ or -3, we have $\theta_m > 2$, $\phi_m > 3$, and so

$$M_m^{(2)} = \frac{(\theta_m - 1)(\phi_m - 1 - \tfrac{1}{2}\alpha_m)}{\theta_m\phi_m - 1} > \frac{1{\cdot}7}{5} > k.$$

For $m \leq -4$, $\theta_m \geq [\overline{2,5,5}] > 1{\cdot}79$, $\phi_m > [2,3,3] > 1{\cdot}62$, and $\alpha_m < 0{\cdot}04$;

thus

$$M_m^{(2)} > \frac{(\theta_m - 1)(\phi_m - 1{\cdot}02)}{\theta_m\phi_m - 1} > \frac{(0{\cdot}79)(0{\cdot}6)}{1{\cdot}9} > k.$$

(xi) <u>Proof that $M_m^{(3)} > k,\ m < -1$</u>

Consider first $m = -2$ or -3; then $|\mu_m| > 2$, $\theta_m < 4$,

$\phi_m < [4,4] = 3{\cdot}75$, implying that

$$M_m^{(3)} > \frac{\phi_m + 3}{2(\theta_m\phi_m - 1)} > \frac{6{\cdot}75}{28} > k.$$

When $m \leq -4$, then $\alpha_m < 0{\cdot}04$, which gives

$$M_m^{(3)} > \frac{\phi_m - 0{\cdot}02}{\theta_m\phi_m - 1}.$$

If $a_m = 5$ and $a_{m+1} = 5$, then $\phi_m < 5$, $\theta_m = [\overline{5,2,5}] < 4{\cdot}45$; thus

$$M_m^{(3)} > \frac{4{\cdot}98}{21{\cdot}25} > k.$$

If $a_m = 5$ and $a_{m+1} = 2$, then $\phi_m < [2,5,5] < 1{\cdot}792$, $\theta_m = [\overline{5,5,2}] < 4{\cdot}775$;

$$M_m^{(3)} > \frac{1{\cdot}772}{7{\cdot}557} > k.$$

If $a_m = 2$, then $\theta_m < 2$, and so $M_m^{(3)} > 1/\theta_m > \tfrac{1}{2} > k$.

There remains only the step M_1 for investigation.

(xii) Proof that $M_1 = k$

Clearly $M_1^{(3)} > k$. Using Table 5.1, we obtain

$$M_1^{(4)} > \frac{(|\theta_1| - 0\cdot5)\phi_1}{4(|\theta_1|\phi_1 + 1)} > \frac{(11)(20\cdot9)}{4\{(11\cdot5)(20\cdot9) + 1\}} > k.$$

Now, for the chain (C), since $|\lambda_1| = 1/\theta_0$,

$$|\theta_1| - |\lambda_1| = 11 > \frac{21}{1\cdot93} > \frac{\phi_1}{1 + |\mu_1|};$$

hence, as for (iii), $M_1^{(1)} > M_1^{(2)}$. Now,

$$M_1^{(2)} = \frac{12(\phi_1 - 1 - |\mu_1|)}{4(|\theta_1|\phi_1 + 1)} = \frac{3(\phi_1 - 2 + |\tau_2|)}{|\theta_1|\phi_1 + 1}.$$

We have $\phi_1 = [21,461,-17,50,\phi_5] = \dfrac{8238730\ \phi_5 - 164581}{392361\ \phi_5 - 7838}$.

Successive substitution of the basic formulae implies:

$$|\tau_2| = \frac{24727\ \phi_5 - 494 + |\mu_5|}{392361\ \phi_5 - 7838}.$$

From (5.3) and (5.4) we have $\phi_5 = \dfrac{147 + \sqrt{21651}}{6}$, $|\mu_5| = \dfrac{3\phi_5}{49}$, and

$|\theta_1| = [11,-\theta_0] = \dfrac{104250 + 2\sqrt{10}}{9005}$. Substituting these values in the expression for $M_1^{(2)}$ above, we obtain from (1.30) and (1.31),

$$M_1^{(2)} = \frac{(3/49)(366458018\ \phi_5 - 7320551)}{(8238730|\theta_1| + 392361)\phi_5 - (164581|\theta_1| + 7838)} = k.$$

We enunciate the following theorem, which we have proved:

THEOREM 5.2. *For the chain (C), $M = k$.*

4. Isolation of the value of a_0

Lemma 5.2 implies that both $|\tau_1| < 0\cdot0668$, and $|\sigma_1| < 0\cdot0668$.

Now, since $|\lambda_1| = |\varepsilon_0| + \dfrac{1 - \theta_0}{\theta_0}$, and $|a_1| - |\varepsilon_0| + 1 > 0$, then

$$0\cdot0668 > |\sigma_1| = \frac{\theta_0(|\varepsilon_0| - 1) + 1}{\theta_0|a_1| + 1} > \frac{|\varepsilon_0| - 1}{|a_1|} > \frac{|\mu_0| - 1\cdot0668}{|\phi_0| + 1/|\phi_1|},$$

implying

$$|\mu_0| < 0{\cdot}0668|\phi_0| + 1{\cdot}14 \ . \tag{5.6}$$

Suppose $\theta_0 > 2{\cdot}3$, then by (5.6), (5.2),

$$M_0^{(3)} < \frac{1{\cdot}0668|\phi_0| + 0{\cdot}14}{2(2{\cdot}3|\phi_0| + 1)} < \frac{1{\cdot}0668}{4{\cdot}6} < k.$$

Thus $\theta_0 < 2{\cdot}3$. If $a_0 = 3$, then $\theta_{-1} = \dfrac{1}{3 - \theta_0} < 1{\cdot}43$; now, by §2, $|\varepsilon_0| \geq 1$, which implies $|\phi_0| \geq 3 + 1/\phi_1 > 2{\cdot}9$, (by Lemma 5.1). (5.6) then implies

$$|\mu_{-1}| = 1 + |\tau_0| < 1{\cdot}0668 + \frac{1{\cdot}14}{|\phi_0|} < 1{\cdot}5.$$

Since $\phi_{-1} > 3$,

$$M_{-1}^{(2)} < \frac{(\theta_{-1} - 1)(\phi_{-1} + 0{\cdot}5)}{2(\theta_{-1}\phi_{-1} - 1)} < \frac{(0{\cdot}43)(3{\cdot}5)}{6{\cdot}58} < k.$$

Thus we have the following result:

THEOREM 5.3. *A critical chain has* $a_0 = 2$.

5. Isolation of the value of a_1

Throughout this section we will employ the following temporary notations. Let $a = |a_1|$, $\varepsilon = |\varepsilon_0|$, and $c = a - \varepsilon$. Note that since a and ε have the same parity, c is even and furthermore, $c \geq 2$. We commence by obtaining a series of lemmas which provide bounds on c.

LEMMA 5.4. $$\frac{c(c + 2)}{c + 1} \geq 4k(|\phi_0| + \frac{1}{\theta_0}) \ .$$

LEMMA 5.5. $$c < |\phi_0| - |\mu_0| + 0{\cdot}063 \ .$$

LEMMA 5.6. *Suppose that* $v = 4k(|\phi_0| + \dfrac{1}{\theta_0})$, *then*

$$c \geq xv,$$

where x is the positive root of $vx^2 - (v - 2)x - 1 = 0;$

$$i.e. \quad x = \frac{\nu - 2 + \sqrt{\nu^2 + 4}}{2\nu} .$$

Note. $x = x(v)$ is an increasing function of v, (since $\frac{dx}{dv} > 0$), and hence we may replace v, in x(v), by any lower bound of $4k(|\phi_0| + \frac{1}{\theta_0})$.

PROOF OF LEMMA 5.4. Taking account of the appropriate signs, we have from (3.25), since $a_1 < 0$, $\varepsilon_0 < 0$,

$$M_1^{(1)} = \frac{1}{4|(\theta_1\phi_1 - 1)|}(|\theta_1| - 1 - |\lambda_1|)|(\phi_1 + 1 - \mu_1)|,$$

$$M_1^{(2)} = \frac{1}{4|(\theta_1\phi_1 - 1)|}(|\theta_1| + 1 - |\lambda_1|)|(\phi_1 - 1 + \mu_1)|.$$

The basic recurrence relations enable us to rewrite these expressions with the variables at the zero step:

$$M_1^{(1)} = \frac{1}{4(\theta_0|\phi_0| + 1)}\theta_0 c|(|\phi_0| - a - |\mu_0| + \varepsilon + 1)|,$$

$$M_1^{(2)} = \frac{1}{4(\theta_0|\phi_0| + 1)}\theta_0(c + 2)|(1 - |\phi_0| + a + |\mu_0| - \varepsilon)|.$$

Now, by (3.27), $\text{sgn}(\phi_1 + 1 - \mu_1) = \text{sgn}(\phi_1 - 1 + \mu_1) = \text{sgn}\,\phi_1$, and since we are assuming both $M_1^{(1)} \geq k$ and $M_1^{(2)} \geq k$, then we have by addition,

$$\frac{1}{\theta_0}\left(\frac{1}{c} + \frac{1}{c + 2}\right) \leq \frac{|(|\phi_0| - a - |\mu_0| + \varepsilon + 1) + (1 - |\phi_0| + a + |\mu_0| - \varepsilon)|}{4k(\theta_0|\phi_0| + 1)} .$$

The result follows immediately.

PROOF OF LEMMA 5.5. Now $|\phi_0| = a + \frac{1}{\phi_1}$ and $|\mu_0| = \varepsilon + \tau_1$; hence

$$c = a - \varepsilon = |\phi_0| - |\mu_0| + \frac{\mu_1 - 1}{\phi_1} .$$

In the case where $\phi_1 > 0$, the result follows from Lemma 5.2. When $\phi_1 < 0$, and $\mu_1 > 0$, then the result holds by Lemma 5.1, since $\theta_1\phi_1 > 0$. In the final case when $\phi_1 < 0$, and $\mu_1 < 0$, then the lemma holds if $(1 + |\mu_1|)/|\phi_1| \leq 1 - 4k < 0{\cdot}063$; if not, then

$$M_1^{(1)} < \frac{|\phi_1| - 1 - |\mu_1|}{4|\phi_1|} < \frac{|\phi_1| - |\phi_1|(1 - 4k)}{4|\phi_1|} = k.$$

This concludes the proof of the lemma.

PROOF OF LEMMA 5.6. From Lemma 5.4 we have,

$$c \geq \frac{c + 1}{c + 2} \cdot v \quad ;$$

Since $c \geq 2$, this inequality holds if and only if

$$c^2 - (v - 2)c - v \geq 0,$$

or

$$c \geq \frac{v - 2 + \sqrt{v^2 + 4}}{2},$$

(the roots of the polynomial in c have opposite sign). The required result follows immediately.

The following three lemmas will reduce the number of possible values that the variable a may take.

LEMMA 5.7. *If* $|\phi_0| > 20$, *then* $|\theta_0| < 1{\cdot}7382$.

LEMMA 5.8. $a \leq 22$.

LEMMA 5.9. *a is odd; furthermore* $11 \leq a \leq 21$, *and if* $\phi_1 < 0$, *then* $17 \leq a \leq 21$.

PROOF OF LEMMA 5.7. Since $\theta_0 < 2$, $v > (0{\cdot}937)(20 + 0{\cdot}5) > 19{\cdot}2$.

Then, by Lemma 5.6,

$$c > \frac{17{\cdot}2 + \sqrt{19{\cdot}2^2 + 4}}{38{\cdot}4} \cdot v > (0{\cdot}89)(|\phi_0| + \tfrac{1}{2}).$$

Combining this with Lemma 5.5, we obtain

$$(0{\cdot}89)(|\phi_0| + \tfrac{1}{2}) < c < |\phi_0| - |\mu_0| + 0{\cdot}063,$$

which implies $|\mu_{-1}| = |\tau_0| < 0{\cdot}11$. Thus if $\theta_{-1} > 3{\cdot}8191$, since $\phi_{-1} > 2$, we have

$$M_{-1}^{(3)} < \frac{\phi_{-1} + 1{\cdot}11}{2(\theta_{-1}\phi_{-1} - 1)} < \frac{3{\cdot}11}{13{\cdot}2764} < k.$$

If $\theta_{-1} < 3{\cdot}8191$, then $\theta_0 < 1{\cdot}7382$.

PROOF OF LEMMA 5.8. When $a \geq 23$, Lemma 5.1 implies that $|\phi_0| > 22{\cdot}937$. By the previous lemma, $\frac{1}{\theta_0} > 0{\cdot}5753$, and so

$$v > 4k(22{\cdot}937 + 0{\cdot}5753) > 22{\cdot}0314.$$

Applying Lemma 5.6 in an analogous way to that in the previous proof, we obtain $x > 0{\cdot}95666$, and so

$$(0{\cdot}8964)(|\phi_0| + 0{\cdot}5753) < c < |\phi_0| - |\mu_0| + 0{\cdot}063,$$

which implies

$$|\mu_0| < 0{\cdot}1036|\phi_0| - 0{\cdot}45.$$

If $\theta_{-1} > 3{\cdot}8123$, then since $\phi_{-1} > 2$, we have

$$M_{-1}^{(3)} < \frac{\phi_{-1} + 1{\cdot}1036}{2(\theta_{-1}\phi_{-1} - 1)} < \frac{3{\cdot}1036}{13{\cdot}2492} < k.$$

If, however, $\theta_{-1} < 3{\cdot}8123$, then $\theta_0 < 1{\cdot}7377$, implying by Lemma 2.7

$$M_0^{(2)} < \frac{(0{\cdot}7377)(1{\cdot}1036|\phi_0| + 0{\cdot}55)}{2(1{\cdot}7377|\phi_0| + 1)} < \frac{(0{\cdot}7377)(1{\cdot}1036)}{3{\cdot}4754} < k.$$

This concludes the proof of the lemma.

PROOF OF LEMMA 5.9. Put $f(c) = \frac{c(c+2)}{c+1}$. Lemma 5.4 implies

$$|\phi_0| < \frac{f(c)}{4k} - \tfrac{1}{2} < U,$$

where U is some convenient upper bound. Tabulating these results for c = 2, 4,...., 20, we obtain:

c	2	4	6	8	10	12	14	16	18	20
f(c)	$\frac{8}{3}$	$\frac{24}{5}$	$\frac{48}{7}$	$\frac{80}{9}$	$\frac{120}{11}$	$\frac{168}{13}$	$\frac{224}{15}$	$\frac{288}{17}$	$\frac{360}{19}$	$\frac{440}{21}$
U	2·35	4·63	6·82	8·99	11·15	13·30	15·44	17·59	19·73	21·87

TABLE 5.2.

Now, whenever a is even, since after §2 $\varepsilon \geq 1$, we have $\varepsilon \geq 2$, $a \geq c + 2$. Lemma 5.1 implies that $|\phi_0| > a - 0{\cdot}063$, and so

$$c + 1{\cdot}937 < |\phi_0| < U.$$

This provides a contradiction for every value of c on Table 5.2, and so excludes the possibility of a being even.

Similarly, when a is odd, $a \geq c + 1$, and so

$$c + 0{\cdot}937 < |\phi_0| < U,$$

which provides a contradiction for all $c \leq 6$, on Table 5.2. Hence a is odd, and $a \geq 9$.

Now if $\phi_1 < 0$, Lemma 5.1 implies that $|\theta_1| > 15{\cdot}8$, since $\theta_1 < 0$. In the case a = 15, we have $\theta_0 < 1{\cdot}25$, and (4.24) implies that

$$M_0^{(2)} < \frac{\theta_0 - 1}{\theta_0} < \frac{0{\cdot}25}{1{\cdot}25} < k.$$

Hence, in this case, $17 \leq a \leq 21$.

When $\phi_1 > 0$, then $c + 1 < |\phi_0| < U$, which is contradicted when c = 8. Thus, $11 \leq a \leq 21$. This completes the proof.

The following lemma enables us to exclude most of the remaining values of a.

<u>*LEMMA 5.10.*</u>

$$M < \frac{(3|\phi_0| + 1 + |\mu_0|)(|\phi_0| + 1 + |\mu_0|)}{2(7|\phi_0|^2 + (3|\mu_0| + 7)|\phi_0| + 2|\mu_0| + 2)}$$

The right-hand side increases with $|\mu_0|$, and decreases with $|\phi_0|$.

<u>*PROOF.*</u> We use a method analogous to that of §3 (viii). The basic recurrence relations imply, after (5.2),

$$M_{-1}^{(3)} = \frac{(2 - \theta_0)(3|\phi_0| + 1 + |\mu_0|)}{2(\theta_0|\phi_0| + 1)},$$

a decreasing function of θ_0. Now $M_0^{(2)}$ is an increasing function of θ_0, and since $M \leq \min\{M_{-1}^{(3)}, M_0^{(2)}\}$, M cannot exceed the common value of the two functions of θ_0, which occurs when

$$\theta_0 = \frac{7|\phi_0| + 3 + 3|\mu_0|}{4|\phi_0| + 2 + 2|\mu_0|}.$$

Thus

$$M \le \frac{(7|\phi_0| + 3 + 3|\mu_0|) - (4|\phi_0| + 2 + 2|\mu_0|)(|\phi_0| + 1 + |\mu_0|)}{2(|\phi_0|(7|\phi_0| + 3 + 3|\mu_0|) + 4|\phi_0| + 2 + 2|\mu_0|)}$$

$$= \frac{(3|\phi_0| + 1 + |\mu_0|)(|\phi_0| + 1 + |\mu_0|)}{2(7|\phi_0|^2 + (3|\mu_0| + 7)|\phi_0| + 2 + 2|\mu_0|)}.$$

It is clear that this function increases with $|\mu_0|$. Make the following abbreviations: $\phi = |\phi_0|$, $\alpha = 1 + |\mu_0|$, $\beta = 3|\mu_0| + 7$, and

$$q(\phi) = \frac{(3\phi + \alpha)(\phi + \alpha)}{7\phi^2 + \beta\phi + 2\alpha}.$$

It is readily verified that

$$\mathrm{sgn}\left(\frac{dq}{d\phi}\right) = -\mathrm{sgn}\left((28\alpha - 3\beta)\phi^2 + \alpha(14\alpha - 12)\phi + \alpha^2(\beta - 8)\right).$$

Now all the coefficients of the powers of ϕ are positive since $\varepsilon \ge 1$, (and so $\beta > 9$), and $\alpha > 1$. Thus $\frac{dq}{d\phi} < 0$, and the result follows.

__*THEOREM 5.4.*__ *Any critical chain has*

$$a_1 = -11, \quad \varepsilon_0 = -1, \ a_2 > 0.$$

__*PROOF.*__ By Table 5.2, it is clear that for all a that remain, $\varepsilon = 1$; for if not, a contradiction is obtained as in the proof of Lemma 5.9.

Suppose that $13 \le a \le 21$. If $\varepsilon_1 = 0$, then $|\mu_1| = |\tau_2| < 0{\cdot}07$; by Lemma 5.1, $|\phi_1| > 2$, and so $|\tau_1| < 0{\cdot}04$. Consequently, if either $\tau_1 < 0$, or $\varepsilon_1 = 0$, then

$$|\mu_0| = 1 + \tau_1 < 1{\cdot}04.$$

Now $|\phi_0| > 13$, (since if a = 13, Lemma 5.9 implies that $\phi_1 > 0$). Substituting these bounds in Lemma 5.10, we find

$$M < \frac{(41{\cdot}04)(15{\cdot}04)}{2637} < k.$$

On the other hand, if $\tau_1 > 0$, and $\varepsilon_1 \ne 0$, we consider three cases:

(i) $\phi_1 < 0$, $(\mu_1 < 0)$ implies

$$M_1^{(1)} < \frac{a-1}{4|\theta_1|} < \frac{a-1}{4(a+0\cdot5)} < \frac{20}{4(21\cdot5)} < k.$$

(ii) $0 < \phi_1 < 10$, $(\mu_1 > 0)$; by Lemma 5.2, $\mu_1 < (0\cdot07)|\phi_1| < 0\cdot7$, contradicting $\varepsilon_1 \neq 0$.

(iii) $\phi_1 > 10$, $(\mu_1 > 0)$ implies, since, by Lemma 5.2, $\mu_1 > 0\cdot93$,

$$M_1^{(1)} = \frac{(a-1)(|\phi_1| + 1 - |\mu_1|)}{4(|\theta_1\phi_1| + 1)} < \frac{(20)(10\cdot07)}{4[(21\cdot5)(10) + 1]} < k.$$

Thus we conclude that $a = 11$, and hence $a_2 > 0$ (lemma 5.9).

6. Isolation of the value of a_2

We commence this section with two lemmas which will enable us immediately to isolate the value of ε_1.

LEMMA 5.11. *We have*

$$1\cdot7165 < \theta_0 < 1\cdot73251,$$

which implies

$$0\cdot5771 < \frac{1}{\theta_0} < 0\cdot5826.$$

Also $\mu_1 < 0$, *and* $\varepsilon_1 \neq 0$.

LEMMA 5.12. $$\varepsilon_1 = -1.$$

PROOF OF LEMMA 5.11. Theorem 5.4 implies that $11 < |\phi_0| < 12$; hence by Lemma 5.2, $\phi_{-1} = 2 + \frac{1}{|\phi_0|} > 2\cdot0833$, and $|\mu_{-1}| = \left|\frac{\mu_0}{\phi_0}\right| < \frac{1\cdot0668}{11} < 0\cdot097$.

If $\theta_{-1} > 3\cdot7384$, then

$$M_{-1}^{(3)} < \frac{2\cdot0833 + 1\cdot097}{2(6\cdot7882)} < k,$$

and so $\theta_0 < 2 - \frac{1}{3\cdot7384} < 1\cdot73251$, and $\frac{1}{\theta_0} > 0\cdot5771$. (5.7)

Now $c = 10$, and using the method of Lemma 5.9, (5.7) implies

$$|\phi_0| < \frac{f(10)}{4k} - \frac{1}{\theta_0} < 11{\cdot}6424 - 0{\cdot}5771 = 11{\cdot}0653,$$

and so $\phi_1 > 15{\cdot}3$. Hence if $\varepsilon_1 \geq 0$, then by Lemma 5.2,

$$M_1^{(1)} < \frac{(10)(\phi_1 + 1{\cdot}0668)}{4(|\theta_1|\phi_1 + 1)} < \frac{163{\cdot}668}{4\big((15{\cdot}3)(11{\cdot}577) + 1\big)} < k.$$

Thus we have $\varepsilon_1 < 0$, which implies

$$|\mu_0| = 1 + \tau_1 < 1. \tag{5.8}$$

If $\theta_0 < 1{\cdot}7165$, then since $|\phi_0| > 11$,

$$M_0^{(2)} < \frac{(\theta_0 - 1)(|\phi_0| + 2)}{2(\theta_0|\phi_0| + 1)} < \frac{(0{\cdot}7165)(13)}{2(19{\cdot}8815)} < k.$$

The lemma is therefore proved.

PROOF OF LEMMA 5.12. If $\varepsilon_1 \neq -1$, then, after (5.8), we have $\varepsilon_1 \leq -2$, which implies $|\mu_1| > 1{\cdot}933$. Now, since $|\lambda_1| = 1/\theta_0$,

$$M_1^{(4)} < \frac{(a - 1 + 2/\theta_0)(\phi_1 - 0{\cdot}933)}{4\big((a + 1/\theta_0)\phi_1 + 1\big)},$$

which, by Lemma 2.7, is an increasing function of $\frac{1}{\theta_0}$. Thus, by the previous lemma, if $\phi_1 < 36$,

$$M_1^{(4)} < \frac{(11{\cdot}166)(\phi_1 - 0{\cdot}933)}{4\big((11{\cdot}5826)\phi_1 + 1\big)} < \frac{(11{\cdot}166)(35{\cdot}067)}{1671{\cdot}8} < k.$$

If, however, $\phi_1 > 36$, then by Theorem 3.9 and Lemma 5.11,

$$M_1 \leq \frac{(|\theta_1| - 1)(\phi_1 + 1)}{4(|\theta_1|\phi_1 + 1)} < \frac{(10{\cdot}59)(37)}{4\big((11{\cdot}59)(36) + 1\big)} < k.$$

This concludes the proof.

THEOREM 5.5. *Any critical chain has*

$$a_2 = 21, \quad |\mu_1| < 1, \quad a_3 > 0, \quad \varepsilon_2 < 0.$$

PROOF. $\varepsilon_1 = -1$, and so a_2 is odd. Consider the two cases:

(i) $|\mu_1| > 1$. We have from Lemma 5.2, that $|\mu_1| < 1{\cdot}0668$.

Now, by Lemma 5.11,

when $\phi_1 > 23{\cdot}5$, $\quad M_1^{(1)} < \dfrac{10(\phi_1 + 2{\cdot}0668)}{4\big((11{\cdot}577)\phi_1 + 1\big)} < \dfrac{255{\cdot}668}{1092{\cdot}2} < k,$

and when $\phi_1 < 21{\cdot}5$, $\quad M_1^{(2)} < \dfrac{12(\phi_1 - 2)}{4\big((11{\cdot}577)\phi_1 + 1\big)} < \dfrac{58{\cdot}5}{249{\cdot}9} < k.$

Hence we have $21{\cdot}5 < \phi_1 < 23{\cdot}5$, which implies, from Lemma 5.1, that $a_2 = 23$. Now, since $|\mu_1| = 1 + \tau_2 > 1$, we have $\dfrac{\mu_2}{\phi_2} > 0$; also $|\lambda_2| > 1$. If $\phi_2 > 0$, then

$$M_2^{(3)} = \frac{(\theta_2 - 1 - |\lambda_2|)(\phi_2 - 1 - \mu_2)}{4(\theta_2\phi_2 - 1)} < \frac{\theta_2 - 2}{4\theta_2} < \frac{21{\cdot}1}{92{\cdot}4} < k.$$

If $\phi_2 < 0$, and $\varepsilon_2 \neq 0$, then since $|\phi_2| > 2$,

$$M_2^{(3)} < \frac{(\theta_2 - 2)(|\phi_2| + 0{\cdot}067)}{4(\theta_2|\phi_2| + 1)} < \frac{(21{\cdot}1)(2{\cdot}067)}{188{\cdot}8} < k.$$

If $\phi_2 < 0$, and $\varepsilon_2 = 0$, then since $\phi_1 > 23$, and

$$|\mu_1| = 1 + |\tau_3|/|\phi_2| < 1{\cdot}04,$$

$$M_1^{(1)} < \frac{(10)(\phi_1 + 2{\cdot}04)}{4(11{\cdot}577\phi_1 + 1)} < \frac{250{\cdot}4}{1069} < k.$$

This completes the exclusion of the case $|\mu_1| > 1$.

(ii) $\underline{|\mu_1| \leq 1}$. Lemma 5.2 implies that $|\mu_1| > 0{\cdot}933$. From Lemma 5.11,

when $\phi_1 > 22{\cdot}6$, $\quad M_1^{(1)} < \dfrac{10(\phi_1 + 2)}{4(11{\cdot}577\phi_1 + 1)} < \dfrac{246}{1050{\cdot}5} < k,$

and when $\phi_1 < 20{\cdot}5$, $\quad M_1^{(2)} < \dfrac{12(\phi_1 - 1{\cdot}933)}{4(11{\cdot}577\phi_1 + 1)} < \dfrac{222{\cdot}9}{953{\cdot}3} < k.$

Thus we have $20{\cdot}5 < \phi_1 < 22{\cdot}6$, which implies, from Lemma 5.1, that $a_2 = 21$.

Suppose then that $a_3 < 0$; it follows that $\tau_2 \leq 0$, and so $\mu_2 \geq 0$. Now $|\lambda_2| = 1 + |\sigma_1| = 1 + \dfrac{1}{11\theta_0 + 1} > 1{\cdot}049$, and $\theta_2 < 21{\cdot}1$; thus when $|\mu_2| + 1 \geq 0\cdot061\ \phi_2$,

$$M_2^{(4)} < \frac{(\theta_2 - 0\cdot049)(0\cdot939|\phi_2|)}{4(\theta_2|\phi_2| + 1)} < \frac{(21\cdot051)(0\cdot939)}{84\cdot4} < k,$$

and when $|\mu_2| + 1 < 0\cdot061\ \phi_2$,

$$M_1^{(2)} \leq \frac{12\left(19 + (1 + |\mu_2|)/|\phi_2|\right)}{4\left((21 + 1/|\phi_2|)|\theta_1| + 1\right)} < \frac{(3)(19\cdot061)}{244\cdot11} < k.$$

We therefore conclude that $a_3 > 0$, and $\mu_2 \leq 0$. If $\varepsilon_2 = 0$, then, as usual, $|\tau_2| < 0\cdot04$. Now $\phi_1 < 21$, and so

$$M_1^{(2)} < \frac{3(\phi_1 - 1\cdot96)}{|\theta_1|\phi_1 + 1} < \frac{(3)(19\cdot04)}{244} < k.$$

The theorem is now complete.

7. The maximal chain for θ_0

We will now examine possible a-chains as $n \to -\infty$. Let us say that a particular chain is *feasible in k, at the point n,* if $M_n \geq k$. For example, the chain (C) of §3 is feasible in k, for all n. The following lemma supplies a new bound on θ_0, under the restrictions imposed on the chain by the previous sections.

LEMMA 5.13. $\theta_0 < 1\cdot73134.$

PROOF. Now, by Theorem 5.5 and Lemma 5.1, $20\cdot9 < \phi_1 < 21$. Thus

$$|\mu_{-1}| = \frac{\phi_1 - |\mu_1|}{11\phi_1 + 1} < \frac{\phi_1 - 0\cdot9332}{11\phi_1 + 1} < \frac{20\cdot0668}{232} < 0\cdot086495,$$

and

$$\phi_{-1} = 2 + \frac{\phi_1}{11\phi_1 + 1} > 2 + \frac{20\cdot9}{230\cdot9} > 2\cdot090515.$$

Now, if $\theta_0 > 1\cdot73134$, then $\theta_{-1} > 3\cdot72217$, which implies that

$$M_{-1}^{(3)} < \frac{3\cdot17701}{2\left((2\cdot090515)(3\cdot72217) - 1\right)} < k.$$

The result therefore follows.

We will now prove two lemmas which will enable us to produce the maximal feasible chain for θ_0.

LEMMA 5.14. *$\theta_0 \le [2,4,4,3,2,\theta_{-5}]$.*

LEMMA 5.15. *$a_n \le 5$, for all $n < 0$.*

PROOF OF LEMMA 5.14. Now (2.15), (2.16) and Lemma 2.7 together imply that the expansion of an irrational to the integer above in semi-regular continued fractions is an increasing function of each partial quotient, independently of what follows. But

$$1{\cdot}73134 = [2,4,4,3,2,19,\ldots.];$$

hence if any of a_{-4}, a_{-3}, a_{-2}, or a_{-1}, increase from these values, then $\theta_0 > 1{\cdot}73134$, contradicting Lemma 5.13.

PROOF OF LEMMA 5.15. The previous lemma implies that $a_n \le 5$, for $-4 \le n \le -1$. However, for any $n < 0$, if $\frac{1}{\theta_n} > 0{\cdot}766$, then by (4.24) and (5.2),

$$M_n^{(2)} < \frac{(\theta_n - 1)(\phi_n - 1)}{\theta_n\phi_n - 1} < \frac{\theta_n - 1}{\theta_n} < k.$$

Thus we have $\frac{1}{\theta_n} < 0{\cdot}766$. Similarly, by symmetry, $\frac{1}{\phi_n} < 0{\cdot}766$, for all $n < 0$.

Now, if $a_n > 5$, then by (4.24),

$$M_n^{(3)} < \frac{\phi_n}{\theta_n\phi_n - 1} = \frac{1}{a_n - \nu_n},$$

where $\nu_n = \frac{1}{\theta_{n-1}} + \frac{1}{\phi_n} < 1{\cdot}532$, and so

$$M_n^{(3)} < \frac{1}{4{\cdot}468} < k.$$

The result now follows.

THEOREM 5.6. *The maximal chain for θ_0 is*

$$[2,4,4,3,\overline{2,5,5}],$$

and this chain is feasible in k for n < -1, when followed by a chain of the form implied by §§2-6.

PROOF. By Lemma 5.15, a_{-5} and a_{-6} cannot exceed 5, and by the argument in Lemma 5.14, if they equal 5, we will have a larger value for θ_0 than when they are replaced by smaller positive integers, independently of the values of a_n, $n \leq -7$. Thus take $a_{-5} = a_{-6} = 5$.

Now $\phi_{-6} > [5,2,3,3] = 57/13$, and if $\theta_{-6} > 4{\cdot}5$, then

$$M_{-6}^{(3)} < \frac{\phi_{-6}}{\theta_{-6}\phi_{-6} - 1} < \frac{114}{487} < k.$$

Thus $\theta_{-6} = [5,\theta_{-7}] < 4{\cdot}5$, which implies $\theta_{-7} < 2$, and so $a_{-7} = 2$.

Using an inductive process, we choose two successive values of a_n to be as large as possible (i.e. 5), which forces the partial quotient of next lower index to be 2; for, if for some $n \leq -2$, we have $a_{3n+1} = a_{3n} = 5$, then $\phi_{3n} > [5,2,3,3]$, of course, implying that $a_{3n-1} = 2$, as above.

Hence the given chain provides an upper bound for θ_0. The chain is also feasible for $n \leq -2$, since the chain (C) is feasible, and an examination of the proof of this (§3 (ix)-(xi)) reveals that we needed no more information about the right-hand part of the chain than we already have proved is a consequence of $M \geq k$.

The importance of this result will be evident later, when we show that the minimum of the critical chain is taken at $M_1^{(2)}$, which is an increasing function of θ_0.

COROLLARY. $$\theta_0 \leq \frac{5195 - 2\sqrt{10}}{2997} = 1{\cdot}7312897\ldots$$

which implies $$\frac{1}{\theta_0} > 0{\cdot}57760405\ldots \qquad (5.9)$$

PROOF. The result follows as for formula (5.3).

8. Isolation of the value of a_3

LEMMA 5.16. *$M_2^{(2)}$ and $M_2^{(4)}$ can be written as functions of θ_0; when θ_0 increases, $M_2^{(2)}$ decreases, while $M_2^{(4)}$ increases.*

PROOF. Now $|\lambda_1| = \frac{1}{\theta_0}$, which implies

$$\frac{\theta_2 - 1 + |\lambda_2|}{\theta_2\phi_2 - 1} = \frac{(\theta_0 + 1)/(11\theta_0 + 1) + 21}{(\theta_0\phi_2)/(11\theta_0 + 1) + 21\phi_2 - 1}.$$

Hence, by Lemma 2.7 and (5.2), $M_2^{(2)}$ decreases as θ_0 increases.

Similarly, putting $\alpha = 21\phi_2 - 1$, we obtain

$$\frac{\theta_2 + 1 - |\lambda_2|}{\theta_2\phi_2 - 1} = \frac{232\theta_0 + 20}{(11\alpha + \phi_2)\theta_0 + \alpha},$$

which by Lemma 2.7 is an increasing function of θ_0, since $3\alpha - 5\phi_2 > 0$. The result for $M_2^{(4)}$ follows immediately from this.

We now prove two lemmas which give bounds for the values of a_3 and $|\tau_2|$.

LEMMA 5.17. $436 < \phi_2 < 470$.

LEMMA 5.18. $0{\cdot}063 < |\tau_2| < 0{\cdot}06316$.

PROOF OF LEMMA 5.17. If $\phi_2 > 470$, then $\phi_1 = 21 - 1/\phi_2 > 20{\cdot}99787$. When $|\mu_2| - 1 \le 0{\cdot}06088\ \phi_2$, then (5.9) implies,

$$M_2^{(1)} = \frac{3\left(19 + (|\mu_2| - 1)/\phi_2\right)}{|\theta_1|\phi_1 + 1} < \frac{(3)(19{\cdot}06088)}{(11{\cdot}57760405)(20{\cdot}99787) + 1} < k.$$

When $|\mu_2| - 1 > 0{\cdot}06088\ \phi_2$, then (5.9) and Lemma 5.16 together imply,

$$M_2^{(4)} < \frac{(21{\cdot}036484)(0{\cdot}93912)(470)}{4\left((21{\cdot}08637)(470) - 1\right)} < k.$$

Thus we have $\phi_2 < 470$.

If $\phi_2 < 436$, then $\phi_1 < 20 \cdot 99771$.

When $|\tau_2| \leq 0 \cdot 06303$, by (5.9),

$$M_1^{(2)} = \frac{3(\phi_1 - 2 + |\tau_2|)}{|\theta_1|\phi_1 + 1} < \frac{(3)(19 \cdot 06074)}{(11 \cdot 57760405)(20 \cdot 99771) + 1} < k.$$

When $|\tau_2| > 0 \cdot 06303$, then by Lemmas 5.11, 5.16,

$$M_2^{(2)} < \frac{(21 \cdot 1367)(0 \cdot 93697\phi_2 - 1)}{4(21 \cdot 08633\phi_2 - 1)} < k.$$

The result is now complete.

PROOF OF LEMMA 5.18. From the previous lemma it follows that

$\phi_1 < 20 \cdot 997873$.

When $|\tau_2| \leq 0 \cdot 063$, (5.9) implies as before,

$$M_1^{(2)} < \frac{(3)(19 \cdot 060873)}{20 \cdot 997873|\theta_1| + 1} < k.$$

When $|\tau_2| \gtrsim 0 \cdot 06316$, Lemmas 5.16, 5.17, imply

$$M_2^{(4)} < \frac{(21 \cdot 036484)\left((436)(0 \cdot 93684) + 1\right)}{4\left((21 \cdot 0863736)(436) - 1\right)} < k.$$

The result follows.

The value of ε_2 can now be isolated by the following rather tedious lemma.

LEMMA 5.19. $\varepsilon_2 = -29$.

PROOF. Suppose that $|\varepsilon_2| \leq 27$, then by Lemmas 5.2, 5.17,

$$\left|\frac{\mu_2}{\phi_2}\right| < \frac{27 \cdot 0668}{436} < 0 \cdot 063.$$

Similarly, when $|\varepsilon_2| \geq 30$,

$$\left|\frac{\mu_2}{\phi_2}\right| > \frac{29 \cdot 93}{470} > 0 \cdot 06316.$$

In both cases we contradict Lemma 5.18.

Let us examine the case $\varepsilon_2 = -28$. When $a_3 \geq 446$, we have as before, $|\tau_2| < \frac{28\cdot0668}{445\cdot93} < 0\cdot063$; when $a_3 \le 442$, then we have $|\tau_2| > \frac{27\cdot93}{442\cdot1} > 0\cdot06316$. Thus, again by the previous lemma, $a_3 = 444$ (since a_3 is even). This case is difficult to exclude; it seems probable that it could provide chains with infimum close to k.

Now we have $\phi_2 < 444\cdot1$, and so $\phi_1 < 20\cdot9977483$. Suppose that $|\tau_2| \ge 0\cdot06307$, then following the method of proof in Lemma 5.17,

$$M_2^{(2)} < \frac{(21\cdot1367)(0\cdot93693\phi_2 - 1)}{4(21\cdot08633\phi_2 - 1)} < k.$$

Thus $|\tau_2| < 0\cdot06307$.

When $|\theta_1| > 11\cdot57763$, $M_1^{(2)} < \frac{(3)(19\cdot0608183)}{(11\cdot57763)(20\cdot9977483) + 1} < k$.

When $|\theta_1| < 11\cdot57763$, then $\theta_0 > 1\cdot73121$. If $|\tau_2| \ge 0\cdot06305$, then since $\phi_2 < 444\cdot1$, by Lemma 5.16,

$$M_2^{(2)} < \frac{(21\cdot13627)(0\cdot93695\phi_2 - 1)}{4(21\cdot08637\phi_2 - 1)} < k.$$

Thus we have $|\tau_2| < 0\cdot06305$. (5.10)

Now since $\theta_3 > 400$, it is easily verified that $|\phi_3| > 12$, by a similar proof to that of Lemma 5.1. Hence $\phi_2 < 444\cdot09$, and whenever $|\mu_2| \ge 28$, $|\tau_2| > \frac{28}{444\cdot09} > 0\cdot06305$, contradicting (5.10).

Thus $|\mu_2| < 28$, and consequently $\frac{\mu_3}{\phi_3} < 0$. Consider the following two cases:

(i) $\phi_3 < 0$ $(\mu_3 > 0)$. When $|\mu_3| + 1 \ge 0\cdot003|\phi_3|$, since $|\lambda_1| = 1/\theta_0$, $\frac{|\lambda_2| - 1}{\theta_2} = \frac{1}{|\theta_0\theta_1\theta_2|} = \alpha$, say, then

$$M_3^{(4)} = \frac{(\theta_3 + 1 - |\lambda_3|)(|\phi_3| - 1 - |\mu_3|)}{4(\theta_3|\phi_3| + 1)} < \frac{(417 + \alpha)(0\cdot997)}{4\theta_3}$$

$$< \frac{(417\cdot003)(0\cdot997)}{(4)(443\cdot95)}$$

$$< k.$$

When $|\mu_3| + 1 < 0{\cdot}003\ |\phi_3|$, then $|\tau_3| < 0{\cdot}003 - 1/|\phi_3|$, and

$$|\tau_2| = \frac{28 - |\tau_3|}{444 + 1/|\phi_3|} > \frac{27{\cdot}997|\phi_3| + 1}{444|\phi_3| + 1} > \frac{27{\cdot}997}{444} > 0{\cdot}06305,$$

thus contradicting (5.10).

(ii) $\underline{\phi_3 > 0\ (\mu_3 < 0)}$. When $|\mu_3| \leq 1{\cdot}03$, then since $\phi_3 > 15$ by Lemma 5.1,

$$M_3^{(3)} < \frac{(415{\cdot}003)(15{\cdot}03)}{4\{(443{\cdot}95)(15) - 1\}} < k.$$

If $|\varepsilon_3| = 1$, then we have $1{\cdot}03 < |\mu_3| < 1{\cdot}07$; if $\phi_3 < 31$,

$|\tau_2| < \dfrac{28\phi_3 - 1}{444\phi_3 - 1} < \dfrac{867}{13763} < 0{\cdot}063$, contradicting Lemma 5.18. If $\phi_3 > 31$,

$$M_3^{(3)} < \frac{(415{\cdot}003)(31{\cdot}07)}{4\{(443{\cdot}95)(31) - 1\}} < k.$$

Thus $|\varepsilon_3| \geq 2$, and $|\mu_3| > 1{\cdot}9332$.

When $\phi_3 < 380$,

$$M_3^{(4)} < \frac{(417{\cdot}003)(\phi_3 - 0{\cdot}93)}{4(443{\cdot}95\phi_3 - 1)} < \frac{(417{\cdot}003)(379{\cdot}07)}{4\{(443{\cdot}95)(380) - 1\}} < k.$$

When $\phi_3 > 380$, suppose that $|\mu_3| - 1 \geq 0{\cdot}003\phi_3$, then

$$M_3^{(4)} < \frac{(417{\cdot}003)(0{\cdot}997)\phi_3}{4(443{\cdot}95\phi_3 - 1)} < \frac{(415{\cdot}752)(380)}{4\{(443{\cdot}95)(380) - 1\}} < k.$$

If, however, $|\mu_3| - 1 < 0{\cdot}003\phi_3$, then $|\tau_3| < 0{\cdot}003 + 1/\phi_3 < 0{\cdot}0057$;

thus $|\tau_2| > \dfrac{27{\cdot}9943}{444} > 0{\cdot}06305$, contradicting (5.10).

The proof of the lemma is now complete, since after excluding the case $\varepsilon_2 = -28$, the only remaining value that it can take is -29.

We can now prove the following theorem which isolates the value of a_3.

<u>*THEOREM 5.7.*</u> *Any critical chain has*

$$\varepsilon_2 = -29, \quad a_3 = 461, \quad a_4 < 0, \quad \mu_3 < 0.$$

PROOF. Suppose that $a_3 \geq 463$, then as usual, $|\tau_2| < \frac{29\cdot07}{462\cdot9} < 0\cdot063$, contradicting Lemma 5.18.

If $a_3 \leq 459$, and $|\mu_2| \geq 29$, then by Lemma 5.1, $|\tau_2| > \frac{29}{459\cdot1} > 0\cdot06316$, also contradicting Lemma 5.18. Thus we have $|\mu_2| < 29$, and hence $\tau_3 < 0$. Clearly if $a_3 \leq 457$, $|\tau_2| > \frac{28\cdot93}{457\cdot1} > 0\cdot06316$. Thus we consider the following two cases for $a_3 = 459$.

(i) $\phi_3 < 0$ $(\mu_3 > 0)$. When $|\tau_3| \geq 0\cdot005$, then since $|\lambda_3| > 28\cdot93$ and $\theta_3 < 460$, we have

$$M_3^{(4)} \leq \frac{(\theta_3 + 1 - |\lambda_3|)(0\cdot995|\phi_3| - 1)}{4(\theta_3|\phi_3| + 1)} < \frac{(460 - 27\cdot93)(0\cdot995)}{(4)(460)} < k.$$

When $|\tau_3| < 0\cdot005$, then $|\tau_2| > \frac{28\cdot995}{459\cdot1} > 0\cdot0631$.

(ii) $\phi_3 > 0$ $(\mu_3 < 0)$. Using the method of proof of Lemmas 5.1, 5.2, we obtain $\theta_3\phi_3 > (436)(15) = 6540$; thus whenever $|\tau_3| \geq 0\cdot06315$, we have by Theorem 3.9,

$$M_3 < \frac{\theta_3\phi_3(0\cdot93685)}{4(\theta_3\phi_3 - 1)} < \frac{(6540)(0\cdot93685)}{(4)(6539)} < k.$$

If, however, $|\tau_3| < 0\cdot06315$, then $|\tau_2| > \frac{28\cdot93685}{459} > 0\cdot063043$.

Thus from cases (i) and (ii), we conclude that if $a_3 = 459$, then $|\tau_2| > 0\cdot063043$; thus by Lemma 5.16, as in the proof of Lemma 5.18,

$$M_2^{(4)} < \frac{(21\cdot036484)\left((458\cdot93)(0\cdot936957) + 1\right)}{4\left((21\cdot0863736)(458\cdot93) - 1\right)} < k.$$

Hence we have that $a_3 = 461$. If, in addition, $|\mu_2| \leq 29$, then $|\tau_2| < \frac{29}{460\cdot93} < 0\cdot063$, contradicting Lemma 5.18. When $|\mu_2| > 29$, $\tau_3 > 0$, and so, if $\phi_3 > 0$, then as before,

$$M_3^{(3)} < \frac{\theta_3 - 1 - |\lambda_3|}{4\theta_3} < \frac{431\cdot003}{(4)(460\cdot95)} < k.$$

The complete result now follows.

9. Isolation of the value of a_4

The previous sections of this chapter indicate that any critical chain is of the form:

$$\begin{array}{r|l} \ldots\ldots 2 & -11,\ 21,\ 461,\ a_4,\ldots\ldots \\ \ldots\ldots 0 & \ -1,\ -1,\ -29,\ \varepsilon_3,\ldots\ldots \end{array}$$

where $a_4 < 0$, and $\varepsilon_3 \leq 0$. We now isolate the pair a_4, ε_3.

THEOREM 5.8. *Any critical chain has*

$$a_4 = -17, \quad \varepsilon_3 = -1, \quad a_5 > 0, \quad \mu_4 < 0.$$

PROOF. Suppose that $|\mu_3| \geq 1$, then if again $\alpha = \dfrac{1}{|\theta_0\theta_1\theta_2|}$, then

$$M_3^{(3)} = \frac{(\theta_3 - 1 - |\lambda_3|)(|\phi_3| + 1 - |\mu_3|)}{4(\theta_3|\phi_3| + 1)} < \frac{(431 + \alpha)}{4\theta_3} < \frac{431\cdot003}{(4)(460\cdot95)} < k.$$

Now, if $\varepsilon_3 = 0$, then as in Lemma 5.19 we have certainly $|\phi_3| > 12$, and so $|\tau_3| < \dfrac{0\cdot07}{12} < 0\cdot006$. Thus $|\tau_2| < \dfrac{29\cdot006}{461} < 0\cdot063$, contradicting Lemma 5.18. It follows that $\varepsilon_3 = -1$, and since $|\mu_3| < 1$, then $\tau_4 < 0$. (5.11)

By Theorem 3.9, if $|\phi_3| < 15\cdot34$, then

$$M_3 < \frac{(461)(14\cdot34)}{4\{(460)(15\cdot34) + 1\}} < k.$$

But if $|\phi_4| < 3$, then as in Lemma 5.1, we have since $|\theta_4| > 15$,

$$M_4 < \frac{(16)(2)}{4\{(15)(3) + 1\}} < k.$$

Thus $a_4 \neq -15$, which implies that $|a_4| \geq 17$ (since a_4 must be odd).

Hence $\phi_2 < 461\cdot07$, and $\phi_1 < 20\cdot9978312$; thus if $|\tau_2| \leq 0\cdot0630211$, then by (5.9),

$$M_1^{(2)} < \frac{(3)(19\cdot0608523)}{(11\cdot57760405)(20\cdot9978312) + 1} < k.$$

When $|\tau_2| > 0{\cdot}0630211$, then

$$29 + \left|\frac{\mu_3}{\phi_3}\right| = |\mu_2| > (0{\cdot}0630211)(461 + 1/|\phi_3|),$$

and so

$$|\mu_3| > 0{\cdot}0630211 + 0{\cdot}0527271|\phi_3|.$$

Suppose that $|\phi_3| > 17{\cdot}1$, then $|\mu_3| > 0{\cdot}964$; consequently, we find, as above,

$$M_3^{(3)} < \frac{(431{\cdot}003)(17{\cdot}1 + 0{\cdot}036)}{4\big((460{\cdot}95)(17{\cdot}1) + 1\big)} < k.$$

Hence $a_4 = -17$. Suppose that $a_5 < 0$, then (5.11) implies

$$\mu_4 > 0. \tag{5.12}$$

When $|\tau_4| \geq 0{\cdot}0405$, since $|\phi_3| < 17$,

$$M_3^{(4)} < \frac{(433{\cdot}003)(|\phi_3| - |\tau_4|)}{4(460{\cdot}95|\phi_3| + 1)} < \frac{(433{\cdot}003)(16{\cdot}9595)}{4\big((460{\cdot}95)(17) + 1\big)} < k.$$

When $|\tau_4| < 0{\cdot}0405$, since $|\theta_4| < 17{\cdot}1$, by (5.12) and Lemma 5.2,

$$M_4^{(1)} = \frac{(|\theta_4| - 1 - |\lambda_4|)(|\phi_4| - 1 + |\mu_4|)}{4(|\theta_4\phi_4| - 1)} < \frac{(17{\cdot}1 - 1{\cdot}93)(1{\cdot}0405)}{(4)(17{\cdot}1)} < k.$$

Thus we have that $a_5 > 0$, and $\mu_4 < 0$, which completes the proof of the theorem.

10. Isolation of the value of a_5

In order to obtain the value of a_5 in any critical chain, we will prove a succession of lemmas, which progressively improve the bounds on the relevant variables.

LEMMA 5.20. $\quad 0{\cdot}03838 < |\tau_4| < 0{\cdot}040406, \quad \varepsilon_4 \neq 0.$

LEMMA 5.21. $\quad \phi_4 < 55.$

LEMMA 5.22. $\quad \varepsilon_4 = -2.$

LEMMA 5.23. $\quad a_5 = 48,\ 50$ *or* $52.$

<u>*PROOF OF LEMMA 5.20.*</u> Put $\alpha = \dfrac{1}{|\theta_0\theta_1\theta_2|}$; now $|\phi_3| > 17$, and $\theta_3 = 461 - \dfrac{1}{\theta_2} > 461 - \dfrac{1}{21\cdot08} > 460\cdot9525$, implying when $|\tau_4| \le 0\cdot03838$,

$$M_3^{(3)} < \frac{(431 + \alpha)(|\phi_3| + |\tau_4|)}{4\big((460\cdot9525)|\phi_3| + 1\big)} < \frac{(431\cdot0024)(17\cdot03838)}{31348\cdot77} < k.$$

Thus $|\tau_4| > 0\cdot03838$. If $\varepsilon_4 = 0$, then by the usual method, $|\tau_4| = \left|\dfrac{\tau_5}{\phi_4}\right| < 0\cdot034$, a contradiction. Hence we have $|\mu_4| > 0\cdot93$.

If $|\phi_4| < 29$, then

$$M_4^{(2)} < \frac{(|\theta_4|+0\cdot07)(\phi_4-1\cdot93)}{4(|\theta_4|\phi_4 + 1)} < \frac{(17\cdot07)(27\cdot07)}{4\big((17)(29) + 1\big)} < k.$$

If $|\phi_4| > 29$, then $|\phi_3| < 17\cdot035$; thus whenever $|\tau_4| \ge 0\cdot040406$,

$$M_3^{(4)} < \frac{(433\cdot0024)(17\cdot035 - 0\cdot040406)}{4\big((460\cdot9525)(17\cdot035) + 1\big)} < k.$$

The result follows.

<u>*PROOF OF LEMMA 5.21.*</u> Now $|\lambda_2| > 1$, which implies

$$|\sigma_3| < \frac{29\ \theta_2 - 1}{461\ \theta_2 - 1} < 0\cdot063;$$

also we have $|\theta_4| < 17\cdot003$, and so if $\phi_4 > 55$, then the previous lemma implies

$$M_4^{(1)} < \frac{(|\theta_4| - 2 + |\sigma_3|)(1\cdot04041\phi_4 + 1)}{4(|\theta_4|\phi_4 + 1)}$$

$$< \frac{(15\cdot066)\big((1\cdot04041)(55) + 1\big)}{4\big((17\cdot003)(55) + 1\big)} < k.$$

The lemma follows.

<u>*PROOF OF LEMMA 5.22.*</u> We have already seen in the proof of Lemma 5.20 that $|\varepsilon_4| \ge 1$, and $|\phi_4| > 29$. Hence if $|\varepsilon_4| = 1$, then $|\tau_4| < \dfrac{1\cdot07}{29} < 0\cdot037$, contradicting Lemma 5.20.

The previous two lemmas imply that

$$|\mu_4| < (0\cdot04041)|\phi_4| < (0\cdot04041)(55) < 2\cdot3.$$

Lemma 5.2 then implies that $\varepsilon_4 = -2$.

<u>*PROOF OF LEMMA 5.23.*</u> If $a_5 \le 46$, then by Lemma 5.1, $|\tau_4| > \frac{1\cdot 93}{46\cdot 1} > 0\cdot 041$. If $a_5 = 54$, then again $|\tau_4| < \frac{2\cdot 0668}{53\cdot 933} < 0\cdot 03833$. In both cases we obtain a contradiction of Lemma 5.20.

Now since a_5 must be even, then by Lemma 5.21, $a_5 = 48, 50$, or 52.

<u>*THEOREM 5.9.*</u> *Any critical chain has*

$$a_5 = 50, \quad |\mu_4| < 2, \text{ and } \quad \varepsilon_5 \neq 0.$$

<u>*PROOF.*</u> Suppose that $|\tau_4| \ge 0\cdot 03945$. Now $|\phi_3| = 17 + \frac{1}{\phi_4} >$ $17\cdot 01919$, and since $|\mu_3| = 1 - |\tau_4| \le 0\cdot 96055$, we have, by Lemma 2.7,

$$|\tau_2| = \frac{29|\phi_3| + |\mu_3|}{461|\phi_3| + 1} < 0\cdot 06302112.$$

We also have $\phi_2 = 461 + \frac{1}{|\phi_3|} < 461\cdot 05876$, implying $\phi_1 < 20\cdot 99783108$; hence by (5.9),

$$M_1^{(2)} < \frac{(3)(19\cdot 0608522)}{20\cdot 99783108|\theta_1| + 1} < k.$$

Thus $|\tau_4| < 0\cdot 03945$. (5.13)

Now $|\theta_4| < 17\cdot 0022$, and since $\theta_2 < 22$, then

$$|\sigma_3| < \frac{29\ \theta_2 - 1}{461\ \theta_2 - 1} < 0\cdot 062815.$$

Suppose that $a_5 = 52$. If $0 < \phi_5 < 20$, then Theorem 3.9 implies that

$$M_5 \le \frac{(\theta_5 - 1)(\phi_5 - 1)}{4(\theta_5\phi_5 - 1)} < \frac{(51\cdot 1)(19)}{4\big((52\cdot 1)(20) - 1\big)} < k.$$

Thus it follows that whatever the sign of ϕ_5, we have $\phi_4 > 51\cdot 95$.

Then,

$$M_4^{(1)} < \frac{(|\theta_4| - 2 + |\sigma_3|)(1\cdot 03945\phi_4 + 1)}{4(|\theta_4|\phi_4 + 1)}$$
$$< \frac{(15\cdot 06502)\big((1\cdot 03945)(51\cdot 95) + 1\big)}{4\big((17\cdot 0022)(51\cdot 95) + 1\big)} < k.$$

Thus $a_5 \neq 52$.

(i) Suppose that $a_5 = 48$, then by the usual method,

$$|\tau_4| > \frac{1\cdot 93}{48\cdot 1} > 0\cdot 04.$$

(ii) Suppose that $a_5 = 50$, and either $|\mu_4| > 2$, or $\varepsilon_5 = 0$, then $|\mu_4| > 1\cdot 993$, since in the latter case $|\tau_5| < 0\cdot 007$, as we have seen before, using Lemmas 5.1, 5.2. Hence

$$|\tau_4| > \frac{1\cdot 993}{50\cdot 1} > 0\cdot 0397.$$

In both cases (i) and (ii), (5.13) is contradicted, and so the theorem is completed.

11. Structure of a critical chain pair, for $n \geq 6$

Reviewing what we have shown so far, we know that any chain which is feasible in k, for all n, must have the following form:

$$\begin{array}{r|l} \ldots\ldots 2 & -11,\ 21,\ 461,\ -17,\ 50,\ldots\ldots \\ \ldots\ldots 0 & -1,\ -1,\ -29,\ -1,\ -2,\ldots\ldots \end{array} \qquad (5.14)$$

We now continue to examine the right-hand side of the chain pair. It is to be expected that there will be unusual behaviour in the chain for small values of $|n|$, for it is here that the change from the homogeneous to the inhomogeneous nature of the problem is reflected. The large and apparently random variations in the chain (5.14) indicate that our expectations are justified. Nevertheless, we would also expect that when the inhomogeneous character of the chain plays a dominant role, (i.e. when $n \to +\infty$), the chain should settle down to some recurring behaviour, as is already suggested will occur for the left-hand chain by Theorem 5.6.

In this section, we will prove that any feasible chain must be of a certain specialised structure. We assume that the chain for θ_0 is held constant. We commence by proving three lemmas which will eventually enable us to obtain the critical chain.

<u>LEMMA 5.24.</u> *If $|\mu_2|$ and $|\tau_2|$ both increase, then $M_1^{(2)}$ increases.*

<u>LEMMA 5.25.</u> *We have $a_6 > 0$.*

<u>LEMMA 5.26.</u> *If $|\mu_5|$ and $|\tau_5|$ both increase, then $M_1^{(2)}$ increases.*

<u>PROOF OF LEMMA 5.24.</u> If ϕ_1 increases, then since $|\tau_2|$ increases,

$M_1^{(2)} = \dfrac{3(\phi_1 - 2 + |\tau_2|)}{|\theta_1|\phi_1 + 1}$ will increase.

If ϕ_1 decreases, then ϕ_2 decreases. Hence

$$M_1^{(2)} = \frac{3(19\phi_2 + |\mu_2| - 1)}{(21|\theta_1| + 1)\phi_2 - |\theta_1|},$$

which is a decreasing function of ϕ_2, will increase.

The lemma then follows.

<u>PROOF OF LEMMA 5.25.</u> Suppose that $a_6 < 0$. Then by Theorem 5.9, $\mu_5 > 0$. Clearly $|\lambda_5| > 2{\cdot}05$, and $\theta_5 < 50{\cdot}1$; thus if $|\tau_5| \geq 0{\cdot}043$,

$$M_5^{(4)} = \frac{(\theta_5 + 1 - |\lambda_5|)(|\phi_5| - 1 - |\mu_5|)}{4(\theta_5|\phi_5| + 1)} < \frac{(49{\cdot}05)(0{\cdot}957)}{(4)(50{\cdot}1)} < k.$$

We have then $|\tau_5| < 0{\cdot}043$, and so $|\tau_4| > \dfrac{1{\cdot}957}{50{\cdot}1} > 0{\cdot}039$. Since $|\phi_3| > 17{\cdot}0199$, then as in Theorem 5.9,

$$|\tau_2| = \frac{29|\phi_3| + |\mu_3|}{461|\phi_3| + 1} < 0{\cdot}063021173.$$

Similarly $|\mu_2| = 29 + |\tau_3| < 29 + \dfrac{0{\cdot}961}{17{\cdot}0199} < 29{\cdot}05647.$

Calculating the values of $|\mu_2|$ and $|\tau_2|$ again for the chain (C) (to sufficient accuracy) we find

$$|\tau_2|_C = 0{\cdot}0630211983\ldots \qquad |\mu_2|_C = 29{\cdot}056475\ldots,$$

where the subscript C denotes the values for the chain (C).

Thus by the previous lemma

$$M_1^{(2)} < \left(M_1^{(2)}\right)_C = k.$$

Consequently, we have $a_6 > 0$.

PROOF OF LEMMA 5.26. By the previous lemma, and (5.14), we obtain

$$|\mu_2| = 29 + \frac{48\phi_5 + |\mu_5| - 1}{851\phi_5 - 17}.$$

Since $\varepsilon_5 \neq 0$, then $|\mu_5| > 0{\cdot}9$, and so Lemma 2.7 implies that $|\mu_2|$ increases when ϕ_5 decreases, since $(48)(17) + (851)(|\mu_5| - 1) > 0$. When ϕ_5 decreases, then ϕ_2 decreases, and so $|\tau_2|$ increases.

Suppose then that ϕ_5 increases. We may write

$$|\mu_2| = 29 + \frac{(48 + |\tau_5|)\phi_5 - 1}{851\phi_5 - 17};$$

thus again by Lemma 2.7, since $851 - (17)(48 + |\tau_5|) > 0$, and $|\tau_5|$ increases, we have $|\mu_2|$ increases. We readily check that $|\mu_3|$ increases while $|\phi_3|$ decreases, which together imply that

$$|\tau_2| = \frac{29|\phi_3| + |\mu_3|}{461|\phi_3| + 1}$$

increases.

The lemma now follows in both cases from Lemma 5.24.

LEMMA 5.27. $|\tau_5| < 0{\cdot}062.$

PROOF. Suppose that $|\tau_5| \geq 0{\cdot}062$. Now $\theta_5 < 50{\cdot}0589$, and

$$|\sigma_4| = \frac{\theta_3 - |\lambda_3|}{17\theta_3 + 1} > \frac{460 - 29}{(17)(460) + 1} > 0{\cdot}0551.$$

Thus if $\phi_5 > 53{\cdot}5$,

$$M_5^{(4)} \leq \frac{(\theta_5 - 1 - |\sigma_4|)(0{\cdot}938\phi_5 + 1)}{4(\theta_5\phi_5 - 1)} < \frac{(49{\cdot}0038)(51{\cdot}183)}{10708} < k.$$

If $\phi_5 < 48{\cdot}1$, since $\theta_5 > 50{\cdot}0588$, and by the above method $|\sigma_4| < 0{\cdot}0552$, we have

$$M_5^{(2)} \le \frac{(\theta_5 + 1 + |\sigma_4|)(0\cdot938\phi_5 - 1)}{4(\theta_5\phi_5 - 1)} < \frac{(51\cdot114)(44\cdot1178)}{9627} < k.$$

Hence, by Lemma 5.1, we have $49 \le a_6 \le 53$, and

$$0\cdot062 \le |\tau_5| < 0\cdot0668,$$

which implies that

$$3\cdot034 < |\mu_5| < 3\cdot6,$$

since $48\cdot937 < \phi_5 < 53\cdot1$. Therefore we have that $|\varepsilon_5| = 3$, and a_6 is odd.

If $a_6 \ge 51$, then $|\mu_5| > (50\cdot9)(0\cdot062) > 3\cdot1$, which is impossible. Then $a_6 = 49$, and $\tau_6 > 0$. Clearly $\varepsilon_6 \ne 0$, else by the usual method $|\mu_5| < 3\cdot01$, a contradiction. Thus $|\mu_6| > 0\cdot93$, and since $\theta_6 < 49$ and $|\phi_6| > 10$, then in all cases we have

$$M_6^{(3)} < \frac{(\theta_6 - 1 - |\lambda_6|)(|\phi_6| + 0\cdot07)}{4(\theta_6|\phi_6| - 1)} < \frac{(45\cdot07)(10\cdot07)}{1956} < k.$$

This completes the lemma, and we can now isolate the values of a_6 and ε_6.

<u>LEMMA 5.28.</u> *Any critical chain has*

$$a_6 = 49, \quad \varepsilon_5 = -3;$$

also $|\mu_5| > 3$, $\varepsilon_6 = 0$, $\mu_6 < 0$, *and* $a_7 < 0$.

<u>PROOF.</u> Suppose $|\tau_5| \le 0\cdot06$; then $|\tau_4| > \frac{1\cdot94}{50} > 0\cdot0388$, and so $|\mu_3| < 0\cdot9612$. Now $|\phi_3| > 17\cdot02$, and hence $|\tau_3| < \frac{0\cdot9612}{17\cdot02} < 0\cdot056475$. Thus with the method and notation of Lemma 5.25,

$$|\tau_2| < \frac{(29)(17\cdot02) + 0\cdot9612}{(461)(17\cdot02) + 1} < 0\cdot063021198 < |\tau_2|_C ,$$

and $|\mu_2| < 29\cdot056475 < |\mu_2|_C$. It follows from Lemma 5.24 that $M_1^{(2)} < k$. Thus we have that

$$0{\cdot}06 < |\tau_5| < 0{\cdot}062. \qquad (5.15)$$

We may now proceed by using the method of the previous lemma.

If $\phi_5 > 59{\cdot}5$, then

$$M_5^{(4)} < \frac{(49{\cdot}0038)(0{\cdot}94\phi_5 + 1)}{4(50{\cdot}0589\phi_5 - 1)} < \frac{2789{\cdot}8}{11910} < k.$$

If $\phi_5 < 43{\cdot}5$, then

$$M_5^{(2)} < \frac{(51{\cdot}114)(0{\cdot}94\phi_5 - 1)}{4(50{\cdot}0588\phi_5 - 1)} < \frac{2039}{8706} < k.$$

Thus, by Lemma 5.1, we have $44 \le a_6 \le 59$, and from (5.15) we obtain $2{\cdot}6 < |\mu_5| < 3{\cdot}7$, implying that $\varepsilon_5 = -3$. Lemma 5.2 then implies that $2{\cdot}933 < |\mu_5| < 3{\cdot}067$. If $a_6 \ge 53$ or $a_6 \le 47$, then we have $|\tau_5| < \frac{3{\cdot}067}{52{\cdot}9} < 0{\cdot}06$ or $|\tau_5| > \frac{2{\cdot}933}{47{\cdot}1} > 0{\cdot}062$, respectively, both contradicting (5.15).

Thus $a_6 = 49$ or 51. Consider the two cases:

(i) $|\mu_5| \le 3$.

If $\phi_5 > 49$, then $|\tau_5| < 3/49 = |\tau_5|_C$, and so by Lemma 5.26 $M_1^{(2)} < k$.

If $\phi_5 < 49$, then $\phi_6 > 0$, and $\mu_6 \le 0$ (since $\tau_6 \le 0$). When $|\tau_6| \le 0{\cdot}018$,

$$M_6^{(3)} \le \frac{(\theta_6 - 4 + |\sigma_5|)(1{\cdot}018\phi_6 - 1)}{4(\theta_6\phi_6 - 1)} < \frac{(45{\cdot}1)(1{\cdot}018)}{(4)(49)} < k.$$

When $|\tau_6| > 0{\cdot}018$, $|\tau_5| < \frac{3 - 0{\cdot}018}{48{\cdot}93} < 0{\cdot}061 < |\tau_5|_C$, which together with $|\mu_5| \le 3$, implies, again by Lemma 5.26, that $M_1^{(2)} < k$.

(ii) $|\mu_5| > 3$.

Now if $\varepsilon_6 \ne 0$, then $|\mu_6| > 0{\cdot}93$, by Lemma 5.2. Since $|\phi_6| > 10$, $\tau_6 > 0$, and $\theta_6 < 51$, we certainly have

$$M_6^{(3)} < \frac{(\theta_6 - 4 + |\sigma_5|)(|\phi_6| + 1 - |\mu_6|)}{4(|\theta_6\phi_6| - 1)} < \frac{(47\cdot07)(10\cdot07)}{4((51)(10) - 1)} < k.$$

Thus we have $\varepsilon_6 = 0$.

When $a_6 = 51$, then $|\tau_6| < 0\cdot007$, and $|\tau_5| < \frac{3\cdot007}{50\cdot93} < 0\cdot06$, contradicting (5.15). Thus $a_6 = 49$, $\varepsilon_6 = 0$, and $|\mu_5| > 3$. If, however, $a_7 > 0$, and hence $\mu_6 > 0$, then

$$M_6^{(3)} < \frac{\theta_6 - 4 + |\sigma_5|}{4\theta_6} < \frac{45\cdot1}{196} < k.$$

The lemma now follows in full.

We will now prove the main theorem of this section. It will fix the structure of any critical chain pair for $n \geq 5$. The proof that the chain (C) considered in §3 is in fact <u>the</u> critical chain is a simple corollary to this result.

<u>THEOREM 5.10.</u> *In any critical chain we have, for $n \geq 3$,*

$$a_{2n} = 49, \quad \varepsilon_{2n-1} = (-1)^n 3, \quad \varepsilon_{2n} = 0,$$

and $a_{2n+1} = -42, -44, or -46.$

<u>Remark.</u> One may also easily exclude the occurrence of $a_{2n+1} = -46$, but as this is not necessary for our purposes, the proof will not be included.

<u>PROOF.</u> We will prove the result inductively.

For the case $n = 3$, Lemma 5.28 implies that $a_7 < 0$, $\mu_6 < 0$, $\varepsilon_6 = 0$, and $a_6 = 49$. In fact, there remains only to reduce the permissible range of values that a_7 may take. We do this by proving that $|\tau_7| > 0\cdot06$.

Suppose that $|\tau_7| \leq 0\cdot06$. Now, since $\theta_4 > 17$, we have

$$|\sigma_5| = \frac{2|\theta_4| + |\lambda_4|}{50|\theta_4| + 1} < \frac{(2)(17) + 0\cdot95}{(50)(17) + 1} < 0\cdot04107,$$

and $\theta_6 < [49,50,-17] < 48\cdot98003$. Thus if $|\phi_6| < 40\cdot085$, since $|\mu_6| \le 0\cdot06$,

$$M_6^{(4)} = \frac{(\theta_6 - 2 + |\sigma_5|)(|\phi_6| - 1 + |\mu_6|)}{4(\theta_6|\phi_6| + 1)} < \frac{(47\cdot0211)(39\cdot145)}{4\{(48\cdot98003)(40\cdot085) + 1\}} < k.$$

Now if $|\phi_6| > 40\cdot085$, and $a_7 = -40$, then $a_8 > 0$ and $\phi_7 < 11\cdot8$. Thus, by Theorem 3.9,

$$M_7 < \frac{(41)(10\cdot8)}{4\{(40)(11\cdot8) + 1\}} < k.$$

Hence we have $|a_7| \ge 42$, and so $|\phi_6| > 41\cdot9$. Thus

$$|\mu_5| = 3 + |\tau_7|/|\phi_6| < 3 + \frac{0\cdot06}{41\cdot9} < 3\cdot00144 < |\mu_5|_C,$$

and

$$|\tau_5| = \frac{3|\phi_6| + |\tau_7|}{49|\phi_6| + 1} \le \frac{3|\phi_6| + 0\cdot06}{49|\phi_6| + 1} < 3/49 = |\tau_5|_C.$$

Consequently, by Lemma 5.26, $M_1^{(2)} < k$. Thus we have

$$|\mu_6| = |\tau_7| > 0\cdot06. \tag{5.16}$$

When $|\phi_6| > 47\cdot9$, we use the above bounds for θ_6 and $|\sigma_5|$ to obtain,

$$M_6^{(3)} < \frac{(45\cdot0211)(47\cdot9 + 0\cdot94)}{4\{(48\cdot98003)(47\cdot9) + 1\}} < k.$$

When $|\phi_6| < 39$, since $|\mu_6| < 0\cdot067$,

$$M_6^{(4)} < \frac{(47\cdot0211)(39 - 0\cdot933)}{4\{(48\cdot98003)(39) + 1\}} < k.$$

Since $\varepsilon_6 = 0$, a_7 is even, and so $40 \le |a_7| \le 46$.

It is more difficult to exclude $a_7 = -40$, than $a_{2n+1} = -40$ for $n \ge 4$, because of the sign of a_5. However, if we can prove $|\tau_7| < 0\cdot06135$, then if $|\phi_6| < 40\cdot03$, (5.17)

$$M_6^{(4)} < \frac{(47\cdot0211)(39\cdot09135)}{4\{(48\cdot98003)(40\cdot03) + 1\}} < k.$$

Later we will see that these conditions must be satisfied.

We will take as our inductive hypothesis the following:

$$\left.\begin{array}{l}\text{For all integral } n \text{ with } 3 \le n \le m, \text{ for an integer} \\ m, \text{ suppose } a_{2n} = 49, \ \varepsilon_{2n-1} = (-1)^n 3, \ \varepsilon_{2n} = 0 \text{ with} \\ (-1)^n \mu_{2n} > 0, \ |\tau_{2n+1}| > 0{\cdot}06, \text{ and } a_{2n+1} = -42, -44, \\ \text{or } -46.\end{array}\right\} \quad (5.18)$$

As in §3 of this chapter, it follows that for all i, $6 \le i \le 2m$, the appropriate products are given in §3 (v), (vi), depending on the parity of i. We will observe the same notations in this section.

Note. Wherever applicable we will use the less stringent condition $|a_{2n+1}| \ge 40$, until we have shown that $|a_7| \ge 42$ (i.e. until the conditions (5.17) are satisfied).

Now if $a_{2m+2} < 0$, since by the inductive hypothesis $|\tau_{2m+1}| > 0{\cdot}06$, $|\theta_{2m+1}| < 46{\cdot}1$, and $(\lambda_{2m+1})(\mu_{2m+1}) < 0$, then

$$M_{2m+1} \le \frac{(|\theta_{2m+1}| - 1 + |\lambda_{2m+1}|)(|\phi_{2m+1}| - 1 - |\mu_{2m+1}|)}{4(|\theta_{2m+1}\phi_{2m+1}| - 1)}$$

$$< \frac{(|\theta_{2m+1}| - 1 + |\sigma_{2m}|)(0{\cdot}94)}{4|\theta_{2m+1}|}$$

$$< \frac{(46{\cdot}1 - 0{\cdot}93)(0{\cdot}94)}{(4)(46{\cdot}1)} < k.$$

Hence we have $a_{2m+2} > 0$, and $(-1)^{m+1}\mu_{2m+1} > 0$, from (5.18).

Now

$$\left.\begin{array}{c} 0{\cdot}06 < |\lambda_7| \le |\lambda_{2m+1}| < \dfrac{3{\cdot}007}{49} < 0{\cdot}0614 \\ 40{\cdot}02 < |\theta_{2m+1}| < 46{\cdot}03 \end{array}\right\}. \quad (5.19)$$

When $\phi_{2m+1} > 59{\cdot}5$, in the notation of §3 (vi), we have

$$M^{(1)}_{2m+1} < \frac{(45{\cdot}0914)(0{\cdot}94\phi_{2m+1} + 1)}{4(46{\cdot}03\phi_{2m+1} + 1)} < \frac{(45{\cdot}0914)(56{\cdot}93)}{10959} < k.$$

When $\phi_{2m+1} < 41{\cdot}5$,

$$M^{(3)}_{2m+1} < \frac{(|\theta_{2m+1}| + 1 - |\sigma_{2m}|)(0\cdot94\phi_{2m+1} - 1)}{4(|\theta_{2m+1}|\phi_{2m+1} + 1)} < \frac{(40\cdot96)(38\cdot01)}{6647} < k.$$

Thus, by Lemma 5.1, we have $42 \le a_{2m+2} \le 59$, and Lemma 5.2 implies that $0\cdot06 < |\tau_{2m+1}| < 0\cdot063$, giving

$$2\cdot5 < (0\cdot06)(41\cdot9) < |\mu_{2m+1}| < (0\cdot063)(59\cdot1) < 3\cdot8.$$

Hence, by what immediately precedes the formula (5.19), we have

$$\mu_{2m+1} = (-1)^{m+1}3.$$

We may now isolate the value of a_{2m+2} by the following series of steps.

(i) If $a_{2m+2} \ge 53$, then $|\tau_{2m+1}| < \frac{3\cdot067}{52\cdot9} < 0\cdot06.$

(ii) If $a_{2m+2} = 51$, then suppose that either $|\mu_{2m+1}| < 3$, or $\varepsilon_{2m+2} = 0$; we have $|\tau_{2m+1}| < \frac{3\cdot007}{50\cdot93} < 0\cdot06.$

Both (i) and (ii) contradict (5.18).

(iii) If $a_{2m+2} = 51$ or 49, then suppose that $|\mu_{2m+1}| > 3$ and $\varepsilon_{2m+2} \ne 0$. Since $\mu_{2m+1} = (-1)^{m+1}(3 + (-1)^m\tau_{2m+2})$, then $(-1)^m\tau_{2m+2} > 0$. It follows that whatever the sign of ϕ_{2m+2}, or whatever the parity of m, we always have

$$M_{2m+2} < \frac{(\theta_{2m+2} - 1 - |\lambda_{2m+2}|)(|\phi_{2m+2}| + 1 - |\mu_{2m+2}|)}{4(\theta_{2m+2}|\phi_{2m+2}| - 1)}$$

Now by Lemmas 5.1, 5.2, we have $|\mu_{2m+2}| > 0\cdot93$, $|\phi_{2m+2}| > 10$, and $|\lambda_{2m+2}| > 3$; thus

$$M_{2m+2} < \frac{(51\cdot1 - 4)(10\cdot07)}{2040} < k.$$

This excludes $a_{2m+2} \ge 51$; and also $a_{2m+2} = 49$, with $|\varepsilon_{2m+2}| \ge 1$ and $|\mu_{2m+1}| > 3$.

Again using the notation of §3 (vi), we have the following:

(iv) If $a_{2m+2} \leq 45$, by the bounds (5.19), since $|\mu_{2m+1}| > 2\cdot 93$,

$$M^{(3)}_{2m+1} < \frac{(41\cdot 02 - 0\cdot 06)(45\cdot 1 - 3\cdot 93)}{4\left((40\cdot 02)(45\cdot 1) + 1\right)} < k.$$

(v) If $a_{2m+2} = 47$ and $\phi_{2m+1} < 47$ $(\phi_{2m+2} > 0)$, suppose that $|\mu_{2m+1}| \geq 2\cdot 948$, then

$$M^{(3)}_{2m+1} < \frac{(40\cdot 96)(47 - 3\cdot 948)}{4\left((40\cdot 02)(47) + 1\right)} < k.$$

Thus we have $|\mu_{2m+1}| < 2\cdot 948$, and by the same argument as in (iii), $(-1)^{m+1}\mu_{2m+2} > 0$, and $|\tau_{2m+2}| > 0\cdot 052$. If $|\varepsilon_{2m+2}| \leq 1$, then $|\mu_{2m+2}| < 1\cdot 07$, and by Lemma 5.1, $\phi_{2m+2} > 15$. Taking, without loss of generality, m to be odd (Theorem 3.12), we obtain since $|\lambda_{2m+2}| > 3$,

$$M^{(2)}_{2m+2} < \frac{(\theta_{2m+2} - 1 - |\lambda_{2m+2}|)(\phi_{2m+2} + 0\cdot 07)}{4(\theta_{2m+2}\phi_{2m+2} - 1)}$$

$$< \frac{(47\cdot 1 - 4)(15\cdot 07)}{4\left((47\cdot 1)(15) - 1\right)}$$

$$< k.$$

If $|\varepsilon_{2m+2}| \geq 2$, then $|\mu_{2m+2}| > 1\cdot 93$. Hence

when $\phi_{2m+2} < 40$, $\quad M^{(1)}_{2m+2} < \frac{(45\cdot 1)(40 - 0\cdot 93)}{4\left((47\cdot 1)(40) - 1\right)} < k,$

and when $\phi_{2m+2} > 40$, we have since $|\tau_{2m+2}| > 0\cdot 052$,

$$M^{(1)}_{2m+2} < \frac{(45\cdot 1)\left((0\cdot 948)(40) + 1\right)}{4\left((47\cdot 1)(40) - 1\right)} < k.$$

This excludes the case when $\phi_{2m+1} < 47$.

(vi) If $a_{2m+2} = 47$ and $\phi_{2m+1} > 47$ $(\phi_{2m+2} < 0)$, suppose that $|\mu_{2m+1}| \geq 2\cdot 978$, then

$$M^{(3)}_{2m+1} < \frac{(40\cdot 96)(47\cdot 1 - 3\cdot 978)}{4\left((40\cdot 02)(47\cdot 1) + 1\right)} < k.$$

Thus we have $|\mu_{2m+1}| < 2\cdot 978$, and since $(-1)^m\mu_{2m+2} > 0$, and

$|\tau_{2m+2}| > 0\cdot022$, then

$$M^{(1)}_{2m+2} = \frac{(\theta_{2m+2} + 1 - |\lambda_{2m+2}|)(|\phi_{2m+2}| - 1 - |\mu_{2m+2}|)}{4(\theta_{2m+2}|\phi_{2m+2}| + 1)}$$

$$< \frac{(45\cdot 1)(0\cdot 978)}{(4)(47\cdot 1)} < k.$$

Thus in (v) and (vi) we have excluded the case $a_{2m+2} = 47$.

(vii) From (iii) then, we conclude that $a_{2m+2} = 49$. Consider the two subcases.

(a) $|\mu_{2m+1}| \le 3$. If $\phi_{2m+1} > 49$, then $|\tau_{2m+1}| < 3/49$. When $\phi_{2m+1} < 49$, $(\phi_{2m+2} > 0)$, then as in (v), $(-1)^{m+1}\mu_{2m+2} \ge 0$.

When $|\tau_{2m+2}| \le 0\cdot 02$, $\quad M^{(2)}_{2m+2} < \dfrac{(45\cdot 1)(1\cdot 02\phi_{2m+2} - 1)}{4\left((49\cdot 1)\phi_{2m+2} - 1\right)} < k$,

and when $|\tau_{2m+2}| > 0\cdot 02$, then $|\tau_{2m+1}| < \dfrac{2\cdot 98}{48\cdot 93} < 3/49$.

Thus if $a_{2m+2} = 49$ and $|\mu_{2m+1}| \le 3$, we have $|\tau_{2m+1}| < 3/49$. (5.20)

(b) $|\mu_{2m+1}| > 3$. This implies that $(-1)^m\tau_{2m+2} > 0$.

If $\phi_{2m+2} > 0$, we have with the previous notation (m odd),

$$M^{(2)}_{2m+2} < \frac{(\theta_{2m+2} - 1 - |\lambda_{2m+2}|)}{4\theta_{2m+2}} < \frac{45\cdot 1}{(4)(49\cdot 1)} < k.$$

Therefore we have after (iii), for the case (b),

$$\phi_{2m+2} < 0, \quad (-1)^{m+1}\mu_{2m+2} > 0, \quad \varepsilon_{2m+2} = 0. \qquad (5.21)$$

As we have seen earlier in this theorem, the fact that $|\theta_{2m+2}| > 41$ implies that $|\phi_{2m+2}| > 11\cdot 8$. Thus $|\tau_{2m+2}| < \dfrac{0\cdot 0668}{11\cdot 8} < 0\cdot 0057$, and it follows that $|\tau_{2m+1}| < \dfrac{3\cdot 0057}{49} < 0\cdot 06135$, since $\phi_{2m+1} > 49$. Thus, in both cases (a) and (b), after (5.20), if $a_{2m+1} = -40$ then $|\phi_{2m}| < 40\cdot 03$, and $|\tau_{2m+1}| < 0\cdot 06135$. Now all these calculations are valid for m = 3, and so the conditions (5.17) are satisfied, implying

$|a_7| \geq 42$. We are now justified in taking $42 \leq |a_7| \leq 46$, in the inductive hypothesis (5.18).

Now for $3 \leq n \leq m$, we have whenever $|\tau_{2n+1}| < 3/49$,

$$|\tau_{2n-1}| = \frac{3 + |\tau_{2n+1}|/|\phi_{2n}|}{49 + 1/|\phi_{2n}|} < 3/49.$$

Thus, if any $|\tau_{2n+1}| < 3/49$, then by a simple inductive argument, $|\tau_5| < 3/49 = |\tau_5|_C$. By (5.20), this is true for the case (vii) (a).

The semi-regular continued fraction expansion

$$|\phi_6| = [|a_7|,-49,|a_9|,-49,\ldots,|a_{2m+1}|,-49,\ldots]$$
$$= (|a_7|,49,|a_9|,49,\ldots,|a_{2m+1}|,49,\ldots)$$

where the latter expression is the ordinary continued fraction expansion; it is a well known result (e.g. [26]) that $|\phi_6|$ is an increasing function of $|a_{2n+1}|$, if $|a_{2r+1}|$ remains fixed for $3 \leq r < n \leq m$. Since by the inductive hypothesis $|a_{2n+1}| \geq 42$, then if for some n, $(3 \leq n \leq m)$, $|a_{2n+1}| > 42$, then $|\phi_6| > |\phi_6|_C = (\overline{42,49})$. Thus if $|\tau_{2m+1}| < 3/49$, then $|\tau_7| < 3/49$, and

$$|\mu_5| = 3 + |\tau_7|/|\phi_6| < 3 + \frac{3}{49|\phi_6|_C} = |\mu_5|_C;$$

also $|\mu_5| < 3/49$, and so by Lemma 5.26, we have $M_1^{(2)} < k$. Thus we may suppose that $a_{2n+1} = 42$, for $3 \leq n \leq m$. (5.22)

We will now prove that if $|\tau_{2n+1}|$ and $|\mu_{2n+1}|$ both increase then so too does $|\mu_{2n-1}|$, whenever $3 \leq n \leq m$. This is easily seen to be true since we have, when $|\tau_{2m+1}| < 3/49$ and hence $|a_{2n}| = 42$,

$$|\mu_{2n-1}| = 3 + \frac{|\mu_{2n+1}|.|\tau_{2n+1}|}{42|\mu_{2n+1}| + |\tau_{2n+1}|},$$

which increases in both $|\tau_{2n+1}|$ and $|\mu_{2n+1}|$.

Suppose that $|\mu_{2m+1}| \leq 3 < |\mu_{2m+1}|_C$, then as we have seen in the case (vii) (a), $|\tau_{2m+1}| < 3/49 = |\tau_{2m+1}|_C$; thus using the above result, together with the fact that $|\tau_{2n+1}| < 3/49$ and $|a_{2n+1}| = 42$, $(3 \le n \le m)$, we obtain $|\mu_5| < |\mu_5|_C$ and $|\tau_5| < |\tau_5|_C$, implying by Lemma 5.26 that $M_1^{(2)} < k$. (5.23)

Thus we have $|\mu_{2m+1}| > 3$, and after (5.21), the inductive hypothesis holds at the step m+1, with the exception of the two conditions $|\tau_{2m+3}| > 0{\cdot}06$ and $42 \le |a_{2m+3}| \le 46$. We now prove that these are also satisfied.

Suppose that $|\tau_{2m+3}| \le 0{\cdot}06$. Then if $|a_{2m+3}| \le 40$, we have $|\phi_{2m+2}| < 40{\cdot}1$; also we have $\theta_{2m+2} < 49{\cdot}1$, $|\lambda_{2m+2}| > 3$, and $|\mu_{2m+2}| < 0{\cdot}07$, implying that, in the notation of §3 (v),

$$M_{2m+2}^{(4)} = \frac{(\theta_{2m+2} + 1 - |\lambda_{2m+2}|)(|\phi_{2m+2}| - 1 + |\mu_{2m+2}|)}{4(\theta_{2m+2}|\phi_{2m+2}| + 1)}$$

$$< \frac{(47{\cdot}1)(39{\cdot}17)}{4\left((49{\cdot}1)(40{\cdot}1) + 1\right)} < k.$$

When $|a_{2m+3}| \ge 42$, then

$$|\mu_{2m+1}| \le 3 + \frac{0{\cdot}06}{|\phi_{2m+2}|} < 3{\cdot}00144 < |\mu_{2m+1}|_C,$$

and

$$|\tau_{2m+1}| \le \frac{3|\phi_{2m+2}| + 0{\cdot}06}{49|\phi_{2m+2}| + 1} < 3/49 = |\tau_{2m+1}|_C.$$

It then follows by an identical argument to (5.23), that $M_1^{(2)} < k$. Thus $|\tau_{2m+3}| > 0{\cdot}06$.

Now we already have $|a_{2m+3}| \ge 42$. If $|\phi_{2m+2}| > 47{\cdot}9$, then

$$M_{2m+2}^{(3)} < \frac{(45{\cdot}1)(47{\cdot}9 + 0{\cdot}94)}{4\left((49{\cdot}1)(47{\cdot}9) + 1\right)} < k.$$

Thus $a_{2m+3} = -42, -44,$ or -46, and all the conditions (5.18) are valid for n = m+1. The theorem is therefore true, by induction.

COROLLARY. *The chain (C) is the critical chain.*

PROOF. Now, we have been holding the chain for θ_0 constant in this section. $M_1^{(2)}$ is clearly an increasing function of θ_0, and so takes its maximum at the largest feasible value of θ_0, which by Theorem 5.6 is $\theta_0 = [2,4,4,3,\overline{2,5,5}]$.

By Lemma 5.3 and Theorem 5.10, we have $|\tau_{2n-1}| = 3/49$, for all $n \geq 3$. Thus any chain which may possibly be critical has $|\tau_5| = |\tau_5|_C$. We have already commented that $|\mu_5|$ is a decreasing function of $|\phi_6|$, and so any chain which has $|a_{2n+1}| > 42$ for some $n \geq 3$, has $|\mu_5| < |\mu_5|_C$, and consequently $M_1^{(2)} < k$.

Thus the chain (C) gives the maximum possible value of $M_1^{(2)}$, for chains feasible in k, for all n. Hence (C) is the critical chain.

The hybrid nature of the problem seems to have been responsible for the length of the proof. Perhaps though, the difficulty of proof is not really surprising. However, it is also of interest to know how the values $M(f;\alpha)$ are distributed in the interval $[0,k)$. This is the subject of investigation in the next chapter.

In conclusion, we are able, after (4.29), to exhibit a critical form

$$\pm \frac{(\theta x + y)(x + \phi y + \alpha)}{\theta\phi - 1},$$

where

$$\theta = \frac{2\sqrt{10} - 5195}{2997}, \quad \phi = \frac{91018391\phi_5 - 1818229}{8238730\phi_5 - 164581},$$

$$\alpha = -\tfrac{1}{2}\left(\phi + \frac{18014063\phi_5 - 359856}{49(8238730\phi_5 - 164581)}\right),$$

with $\phi_5 = (147 + \sqrt{21651})/6$.

The critical value is attained by this mixed form at the point $(x,y) = (-6,0)$. The value k is also taken by all equivalent forms, in the sense of the remark (4.30).

CHAPTER VI

SUBSIDIARY RESULTS FOR THE MIXED FORM PROBLEM

1. Introduction

In Chapter V, we showed that $k = 0{\cdot}23425\ldots$, given by (1.30), is the best possible constant for the mixed form problem, which we formulated as a special degenerate case of the divided cell algorithm for the associated grid. The question raised at the conclusion of the previous chapter was that of the distribution of the infimum values of such forms. We have already noted in the first chapter that the corresponding question for homogeneous forms is not yet completely settled, while the liminf problems in inhomogeneous approximation were considered by Barnes [7].

One might readily imagine that from the structure of the critical chain and Theorem 3.11, k is in fact a point of accumulation of values of $M(f;\alpha)$. For example, we could put in the chain (C), $a_{2n+1} = -44$, for some large n, without effecting the feasibility, in k, of the chain, except at the step M_1, (see Chapter V, §3 (v), (vi)). But Theorem 3.11 ensures us that $M_1^{(2)}$ could be made arbitrarily close to k, by chosing n sufficiently large. In fact, we could construct in a similar way chains with infima arbitrarily close to any k', with $0 \le k' \le k$. Consequently the interval $[0,k]$ would be dense in the values $M(f;\alpha)$. This result follows from the next theorem, the proof of which will constitute the remainder of this chapter.

<u>*THEOREM 6.1.*</u> *For every k', such that $0 \le k' < k$, there exist*

uncountably many binary quadratic forms f, to each of which there corresponds at least one real non-zero number α, with

$$M(f;\alpha) = k'.$$

Note. It will become apparent that the following is really a straight forward extension. There exist uncountably many θ, each for which there correspond uncountably many pairs (ϕ,α), such that

$$\inf_{(x,y)\neq(0,0)} \left|\frac{(\theta x + y)(x + \phi y + \alpha)}{\theta\phi - 1}\right| = k'.$$

2. Construction of the chains (C^*)

We first show that the result holds true when $k' = 0$. Given any integer $s > 0$, we can find an r_s such that for all $r \geq r_s$

$$[(2)_r, x] < 1 + 1/s.$$

Consider those chain pairs which satisfy (4.12), and which have $\epsilon_n = 0$, $n \geq 0$, and

$$\phi_0 = [(2)_{2r_1}, 4, (2)_{2r_2}, 4, \ldots, (2)_{2r_s}, 4, \ldots]$$

Now at the central step of the block $(2)_{2r_s}$ we have, for some m, by Theorem 3.9,

$$M_m \leq \frac{(\theta_m - 1)(\phi_m - 1)}{4(\theta_m\phi_m - 1)} < \frac{(1/s)(1/s)}{4[(1 + 1/s)^2 - 1]} < 1/s.$$

Thus the infimum of such a chain is 0, and there are uncountably many sequences $\{r_s'\}$, with $r_s' \geq r_s$, for all s.

For $k' > 0$, we will construct a chain which is a modification of the critical chain (C). A brief examination of the calculations used to demonstrate the feasibility, in k, for $n \neq 1$, of the chain (C), reveals that only local values of the chain pair were involved.

Similarly the rational bounds on the variables provided by Table 5.1, are uneffected by variations in a_{n+1} and ε_n, provided n is large enough. This is a consequence of Theorem 3.11.

Suppose that we are given some k', with $0 < k' < k$. Since all the complete quotients of the chain (C) are bounded below, Theorem 3.11 ensures that there exists an N such that no matter how we change the chain (C) for $n \geq N$, we have $M_1^{(2)} > k'$.

Define $\omega = [\overline{100}] = 50 + 7\sqrt{51}$, and an irrational (in general) number α by

$$\frac{(\omega - 1)(1 - \alpha)}{4\omega} = k' \tag{6.1}$$

Since $k' < k < 0{\cdot}234255$, we have $0 < \alpha < 1$. If α is irrational, expand $1/\alpha$ as a semi-regular continued fraction to the integer above, and compute the sequence of convergents $\{p_n/q_n\}$ by (3.21). If α is rational, put $p_n/q_n = 1/\alpha$, for all n. By Lemma 2.6, $\{p_n/q_n\}$ converges to $1/\alpha$ from above. Hence

$$p_n/q_n \geq 1/\alpha. \tag{6.2}$$

Now let $\{r_n\}$ $(r_1 > 1)$ be any strictly monotone increasing sequence of positive integers. Consider the chain denoted (C*), which is identical to (C) for all $n \leq N$ (defined above), and for n > N has the form:

$$\begin{array}{cc|ccccccccc} \ldots -42, & 49 & -42, & 100, & 1000, & \begin{bmatrix} 100 \\ 0 \end{bmatrix}_{2r_1-1} & -2mp_1 & \begin{bmatrix} 100 \\ 0 \end{bmatrix}_{2r_2} & -2mp_2 & \begin{bmatrix} 100 \\ 0 \end{bmatrix}_{2r_3} & \ldots \\ \ldots\ 0, & 3 & 0, & -5, & 0, & & 2mq_1 & & 2mq_2 & & \ldots \end{array}$$

$$\begin{array}{ccc} \ldots -2mp_s & \begin{bmatrix} 100 \\ 0 \end{bmatrix}_{2r_{s+1}} & \ldots \\ \ldots\ 2mq_s & & \ldots \end{array} \tag{6.3}$$

where m is an arbitrary positive integer, and the vertical line signifies the point after which the chain differs from (C).

3. Evaluation of the infimum for the chain (C*)

(i) Without loss of generality, take $a_N = 49$, $\epsilon_{N-1} = 3$. All the bounds on the variables for $n < N$, conform to the requirements of Chapter V, §3, implying $M_n > k'$, for $n < N$. Now $|\phi_N|_C > |\phi_N| > 42$, $0{\cdot}0499 < |\mu_N| < |\mu_N|_C$, and $\theta_N > 49{\cdot}023$, implying that $M_N^{(i)}$, $i = 2, 3, 4$, are feasible in k, (see Chapter V). Also

$$M_N^{(1)} > \frac{(\theta_N - 2{\cdot}0015)(|\phi_N| - 0{\cdot}9501)}{4(\theta_N|\phi_N| + 1)} > \frac{(47{\cdot}0215)(41{\cdot}0499)}{4[(49{\cdot}023)(42) + 1]} > k.$$

(ii) At the (N+1)th step, clearly $M_{N+1}^{(3)} > k$, and

$$M_{N+1}^{(1)} > \frac{(|\theta_{N+1}| - 1{\cdot}1)(\phi_{N+1} + 5{\cdot}9)}{4(|\theta_{N+1}|\phi_{N+1} + 1)} > \frac{(40{\cdot}9)(105{\cdot}9)}{(4)(4201)} > k.$$

Now since $\phi_{N+1} > (\phi_{N+1})_C$, and $|\mu_{N+1}| < |\mu_{N+1}|_C$, then we have $M_{N+1}^{(2)} > (M_{N+1}^{(2)})_C > k$. Also $|\theta_{N+1}| > 42$, $\phi_{N+1} > 99$, $|\lambda_{N+1}| > 0{\cdot}06$, and $|\mu_{N+1}| < 5{\cdot}01$, implying

$$M_{N+1}^{(4)} > \frac{(|\theta_{N+1}| - 0{\cdot}94)(\phi_{N+1} - 4{\cdot}01)}{4(|\theta_{N+1}|\phi_{N+1} + 1)} > \frac{(41{\cdot}06)(94{\cdot}99)}{4[(42)(99) + 1]} > k.$$

(iii) Now

$$\mu_{N+2} = \frac{(-1)^{2r_1}}{\phi_{N+3}\cdots\phi_{N+1+2r_1}} \left(\frac{\mu_{N+2+2r_1}}{\phi_{N+2+2r_1}}\right) < 0,$$

and $|\mu_{N+2}| < 0{\cdot}01$, $5 < |\lambda_{N+2}| < 5{\cdot}002$, $\theta_{N+2} > 100$, $\phi_{N+2} > 999$.

Clearly $M_{N+2}^{(1)} > k$,

$$M_{N+2}^{(2)} > \frac{\phi_{N+2} - 1{\cdot}01}{4\phi_{N+2}} > k,$$

$$M_{N+2}^{(3)} > \frac{(\theta_{N+2} - 6{\cdot}002)(\phi_{N+2} - 1)}{4(\theta_{N+2}\phi_{N+2} - 1)} > \frac{(93{\cdot}998)(998)}{4[(100)(999) - 1]} > k,$$

$$M_{N+2}^{(4)} > \frac{(\theta_{N+2} - 4\cdot 002)\phi_{N+2}}{4(\theta_{N+2}\phi_{N+2} - 1)} > \frac{95\cdot 998}{(4)(100)} > k.$$

(iv) Suppose that we examine a step in the chain for which $a_n \geq 100$, $a_{n+1} \geq 100$, and $\varepsilon_{n-1} = \varepsilon_n = 0$, then $|\lambda_n| < 1$ and $|\mu_n| < 1$, implying

$$M_n \geq \frac{(\theta_n - 1 - |\lambda_n|)(\phi_n - 1 - |\mu_n|)}{4(\theta_n\phi_n - 1)} > \frac{(\theta_n - 2)(\phi_n - 2)}{4(\theta_n\phi_n - 1)}$$

$$> \frac{97^2}{4[(99)^2 - 1]} > k.$$

Hence, (i) to (iv) imply that the only places in the chain (C^*) which could possibly be not feasible in k, are M_1, and those steps M_n, M_{n+1} where $a_{n+1} = -2mp_s$, for some natural number s. Let n and s denote such a position in the chain; then by the argument of (iii), we have $\lambda_n < 0$, and $\mu_{n+1} < 0$. We also have, by Lemma 2.6,

$$\theta_n > \omega, \quad \phi_{n+1} > \omega. \tag{6.4}$$

Clearly $M_n^{(1)} > k$. If we apply the methods of §3, Chapter V, we readily obtain that $M_n^{(3)} > M_n^{(2)}$, i.e.

$$\frac{(\theta_n - 1 - |\lambda_n|)(|\phi_n| + 1 + |\mu_n|)}{4(\theta_n|\phi_n| + 1)} > \frac{(\theta_n - 1 + |\lambda_n|)(|\phi_n| + 1 - |\mu_n|)}{4(\theta_n|\phi_n| + 1)},$$

if and only if $\dfrac{\theta_n - 1}{|\lambda_n|} > \dfrac{|\phi_n| + 1}{|\mu_n|}$. Now by the form of a_{n+1} and ε_n, we have that the right-hand side of this latter inequality is uniformly bounded for the particular n under consideration, (the bound being a function of k'). Since $|\lambda_n|$ may be made arbitrarily small by choosing r_1 sufficiently large, a suitable choice of r_1 ensures that this condition is satisfied for all such n.

Similarly, we have that $M_n^{(4)} > M_n^{(2)}$, if and only if

$|\phi_n| - |\mu_n| > \dfrac{|\theta_n|}{1 - |\lambda_n|}$. Now by (6.2), $p_s \geq q_s/\alpha$, implying

$$\begin{aligned} |\phi_n| - |\mu_n| &> (2mp_s + 0{\cdot}01) - (2mq_s + 0{\cdot}001) \\ &> 2m(p_s - q_s) \\ &\geq 2mq_s(1/\alpha - 1), \end{aligned}$$

and hence $|\phi_n| - |\mu_n|$ can be made arbitrarily large (for a fixed k') by choosing m large enough. Since $|\theta_n|/(1 - |\mu_n|)$ is uniformly bounded for the particular n under consideration, we may suppose $M_n^{(4)} > M_n^{(2)}$.

Thus we have $M_n = M_n^{(2)}$ provided both r_1 and m are chosen (as functions of k') to be large enough. Now

$$|\mu_n/\phi_n| < \frac{2mq_s + 0{\cdot}001}{2mp_s + 1/\phi_{n+1}} < q_s/p_s + 0{\cdot}001/|\phi_n|;$$

hence, after (6.1), (6.2) and (6.4), we have

$$\begin{aligned} M_n^{(2)} &> \frac{(\omega - 1)(|\phi_n| + 1 - |\mu_n|)}{4(\omega|\phi_n| + 1)} \\ &> \frac{(\omega - 1)\big((1 - q_s/p_s)|\phi_n| + 0{\cdot}999\big)}{4(\omega|\phi_n| + 1)} \\ &> \frac{(\omega - 1)(1 - q_s/p_s)}{4\omega} \\ &\geq \frac{(\omega - 1)(1 - \alpha)}{4\omega} = k'. \end{aligned}$$

Now at the step M_{n+1}, it is readily verified that the roles of $|\theta_n|$, $|\phi_n|$ and $|\lambda_n|$, $|\mu_n|$ are interchanged (as in Lemma 3.12), and that the same bounds apply for corresponding variables. Thus under the same conditions on r_1 and m, we have

$$M_{n+1} = M_{n+1}^{(3)} > k'.$$

Thus the chain (C^*) is feasible in k', for all n.

Define the set S as follows:

$$S = \{n;\ a_{n+1} = -2mp_s \text{ for some } s\}.$$

Now $r_s \to \infty$ as $s \to \infty$, implying

$$\lim_{\substack{n \to \infty \\ n \text{ in } S}} \theta_n = \omega, \quad \lim_{\substack{n \to \infty \\ n \text{ in } S}} |\lambda_n| = 0, \quad \lim_{\substack{n \to \infty \\ n \text{ in } S}} |\phi_n| = \infty,$$

$$\lim_{\substack{n \to \infty \\ n \text{ in } S}} |\mu_n/\phi_n| = \lim_{i \to \infty} (q_i/p_i) = \alpha.$$

Hence

$$\lim_{\substack{n \to \infty \\ n \text{ in } S}} M_n = \frac{(\omega - 1)(1 - \alpha)}{4\omega} = k'.$$

Consequently, the infimum of the mixed form corresponding to any such chain (C^*) is k'. There are uncountably many forms since $\{r_s\}$ is an arbitrary (except for r_1) increasing sequence, of which there are uncountably many.

The fact that for each (ϕ,α) there exist uncountably many θ, follows from Theorem 3.11, and the fact that $\theta_{-\ell} = [\overline{5,5,2}]$ may be replaced by $\theta_{-\ell} = [4,(3)_{s_n}]_{n=1}^{\infty}$ where ℓ is sufficiently large and $\{s_n\}$ an arbitrary increasing sequence of natural numbers, without effecting the feasibility (in k) of the chain (C^*). I will not give the proof of this, but it follows by straightforward calculations of the type given above.

CHAPTER VII

A RESTRICTED INHOMOGENEOUS MINIMUM OF FORMS

1. Introduction

We will suppose that f is an indefinite binary quadratic form that does not represent zero for integers, and that P is the real point (x_0,y_0). We have already seen in Chapter III, that f and P together define a set of similar grids in the ξ, η-plane. M(f;P) was defined to be the infimum of the products of coordinates at all the grid points. In this chapter we will investigate the infimum taken over those grid points which are in only one of the four possible half-planes defined by the axes.

Let f be an I-reduced form, then we may uniquely denote it, in the usual way, with $|\theta| > 1$, $|\phi| > 1$,

$$f(x,y) = \frac{\pm\Delta}{\theta\phi - 1}(\theta x + y)(x + \phi y); \tag{7.1}$$

then

$$f(x + x_0, y + y_0) = \frac{\pm\Delta}{\theta\phi - 1}(\theta x + y + \xi_0)(x + \phi y + \eta_0), \tag{7.2}$$

where

$$\left.\begin{aligned} \xi_0 &= \theta x_0 + y_0 \\ \eta_0 &= x_0 + \phi y_0 \end{aligned}\right\}. \tag{7.3}$$

We define $M^+(f;P)$ as follows:

$$M^+(f;P) = \inf_{\theta x+y+\xi_0>0} \Delta(\theta x + y + \xi_0)\left|\frac{x + \phi y + \eta_0}{\theta\phi - 1}\right|, \tag{7.4}$$

where the infimum extends over all integers x, y such that $\theta x + y + \xi_0 > 0$.

Similarly we may define $M^-(f;P)$:

$$M^-(f;P) = \inf_{\theta x+y+\xi_0<0} \frac{\Delta|(\theta x + y + \xi_0)(x + \phi y + \eta_0)|}{|\theta\phi - 1|}, \tag{7.5}$$

where the infimum is taken over all integers x, y such that $\theta x + y + \xi_0 < 0$.

Note that Theorem 3.10 implies

$$M(f;P) = \min\{M^+(f;P), M^-(f;P)\} < \tfrac{1}{4}\Delta.$$

Define the form g, after (3.44), by

$$g(x,y) = f\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \frac{\pm\Delta}{\theta\phi - 1}(\phi x + y)(x + \theta y), \tag{7.6}$$

and let Q be the point (y_0, x_0). Thus (7.7)

$$M^+(g;Q) = \inf_{\phi x+y+\eta_0>0} \Delta(\phi x + y + \eta_0)\left|\frac{x + \theta y + \xi_0}{\theta\phi - 1}\right|, \tag{7.8}$$

and

$$M^-(g;Q) = \inf_{\phi x+y+\eta_0<0} \frac{\Delta|(\phi x + y + \eta_0)(x + \theta y + \xi_0)|}{|\theta\phi - 1|}. \tag{7.9}$$

Consider the grid L, given by

$$L: \quad \begin{aligned} \xi &= \beta(\theta x + y + \xi_0) \\ \eta &= \gamma(x + \phi y + \eta_0) \end{aligned}$$

for all integers x, y, where $\beta > 0$, $\gamma > 0$, and $\beta\gamma = \frac{\Delta}{|\theta\phi - 1|}$. Then $M^{\pm}(f;P)$ take an infimum over either the right or left-hand plane of L, while $M^{\pm}(g;Q)$ consider those points in either the upper or lower half-plane.

In order to obtain information which is independent of the particular I-reduced form chosen from an equivalence class, we define, after (7.6) and (7.7):

$$M^*(f;P) = M^*(g;Q) = \max\{M^{\pm}(f;P), M^{\pm}(g;Q)\}, \tag{7.10}$$

and

$$M^*(f) = M^*(g) = \sup_P M^*(f;P), \tag{7.11}$$

where the supremum need only extend over a complete set of grid points incongruent mod 1.

Clearly, if f and h are equivalent I-reduced forms that do not represent zero, then $M^*(f) = M^*(h)$; thus we may define, for any indefinite binary quadratic form g, $M^*(g)$ to be equal to the value $M^*(f)$, where f is any equivalent I-reduced form.

The purpose of this chapter is to investigate the supremum of values taken by $M^*(f)$, and to evaluate this function for a certain sequence of equivalence classes of forms. We will deduce these results from a related problem, solved by Cassels [11] and Descombes [24].

We will re-define the function $k^+(\phi,\alpha)$, on irrationals ϕ, and non-zero real α, such that $\phi x + y + \alpha$ does not represent zero in integers x, y, (see (1.5) and (1.8)). Put

$$k^+(\phi,\alpha) = \liminf_{x \to +\infty} x\|\phi x + \alpha\|, \tag{7.12}$$

and

$$k^+(\phi) = \sup_\alpha k^+(\phi,\alpha). \tag{7.13}$$

Cassels [11] showed that

$$\sup_\phi k^+(\phi) = \frac{27}{28\sqrt{7}},$$

while Descombes [24] proved that there is a decreasing sequence of isolated values for $k^+(\phi,\alpha)$, which approach the limit

$$1/\gamma = \frac{773868 - 28547\sqrt{510}}{366795} = 0{\cdot}352\ldots \tag{7.14}$$

Descombes used the algorithm originally described by Cassels.

It involved the ordinary continued fraction expansion of ϕ, together with an associated sequence of integers, which arose from the inhomogeneity of the problem. By means of yet another modification of the divided cell method previously described, we will re-formulate this problem in terms of semi-regular continued fractions, and then convert Descombes' critical chains into this context. We will then connect the approximation problem with the restricted form problem described above.

2. The critical chains of Descombes

The couples (ϕ,α) and (ϕ',α') are said to be *equivalent* if there exist integers p, q, r, s, a, b, with $ps - qr = \pm 1$, such that

$$\phi' = \frac{p\phi + q}{r\phi + s}, \quad \alpha' = \frac{(ps - qr)\alpha}{r\phi + s} + \frac{a\phi + b}{r\phi + s}, \quad r\phi + s > 0. \tag{7.15}$$

LEMMA 7.1. *If (ϕ,α) and (ϕ',α') are equivalent, then*

$$k^{+}(\phi,\alpha) = k^{+}(\phi',\alpha').$$

The proof is given in [24] (Proposition 3).

We now proceed to quote the results obtained by Descombes. Define the sequence of integers $\{s_n\}$, $n \geq 0$, as follows:

$$s_0 = 0, \quad s_1 = 1, \quad s_{n+1} = 542\, s_n - s_{n-1}. \tag{7.16}$$

We may then define the following sequences of integers based on $\{s_n\}$.

$$\left.\begin{aligned} A_r &= 14\, s_{r+1} - 257\, s_r \\ B_r &= 3\, s_{r+1} + 398\, s_r \\ C_r &= 9\, s_{r+1} - 144\, s_r \\ D_r &= 2\, s_{r+1} + 223\, s_r \end{aligned}\right\} \tag{7.17}$$

$$\left.\begin{aligned} M_{2p} &= s_{p+1} + 257\, s_p, & M_{2p+1} &= 25\, s_{p+1} + 8\, s_p,\\ N_{2p} &= 11\, s_{p+1} + 75\, s_p, & N_{2p+1} &= 189\, s_{p+1} + 2\, s_p,\\ \Delta_{2p} &= 7\, s_{p+1} + 263\, s_p, & \Delta_{2p+1} &= 127\, s_{p+1} + 8\, s_p,\\ \delta_{2p} &= s_{p+1} + 35\, s_p, & \delta_{2p+1} &= 34\, s_{p+1} + 2\, s_p. \end{aligned}\right\} \quad (7.18)$$

Using these sequences, we may define the pair (ψ_r, α_r), and the real number γ_r (for $r \geq -2$), changing the notation of [24] slightly to coincide with (7.12).

$$\left.\begin{aligned} &\psi_{-2} = \frac{7-\sqrt{7}}{14}, \quad \psi_{-1} = \frac{225-\sqrt{510}}{2340}, \quad \psi_0 = \frac{15-\sqrt{110}}{10},\\ &\text{and for } r \geq 1, \quad \psi_r = \frac{\tfrac{1}{2}(A_r - D_r) + \sqrt{\Delta_r\delta_r(\Delta_r\delta_r + 2)}}{C_r}. \end{aligned}\right\} \quad (7.19)$$

$$\left.\begin{aligned} &\alpha_{-2} = 1/14, \quad \alpha_{-1} = 1/90, \quad \alpha_0 = 1/10, \quad \text{and for } r \geq 1,\\ &\alpha_r = (M_r\psi_r + N_r)/2\Delta_r. \end{aligned}\right\} \quad (7.20)$$

$$\left.\begin{aligned} &\gamma_{-2} = \frac{28\sqrt{7}}{27}, \quad \gamma_{-1} = \frac{45\sqrt{510}}{359}, \quad \gamma_0 = \frac{10\sqrt{110}}{37},\\ &\text{and for } r \geq 1, \quad \gamma_r = \frac{8\Delta_r^2\sqrt{\Delta_r\delta_r(\Delta_r\delta_r + 2)}}{C_rN_r^2 + (A_r - D_r)M_rN_r - B_rM_r^2} \end{aligned}\right\} \quad (7.21)$$

We can now enunciate the basic result of [24].

<u>THEOREM 7.1.</u>

(i) $\{\gamma_r\}$ *is an increasing sequence, and if* γ *is given by (7.14)*

$$\lim_{r\to\infty} \gamma_r = \gamma.$$

(ii) For all $r \geq -2$, *we have*

$$k^+(\psi_r, -\alpha_r) = 1/\gamma_r.$$

(iii) If we exclude all couples equivalent (in the sense of (7.15)) to one of $(\psi_r, -\alpha_r)$, *for* $-2 \leq r \leq n$, *then*

$$k^+(\psi, \alpha) < k^+(\psi_n, -\alpha_n) = 1/\gamma_n.$$

(iv) Furthermore, if (ψ,α) *is not equivalent to* $(\psi_r,-\alpha_r)$ *for some r, then*

$$k^+(\psi,\alpha) \le 1/\gamma,$$

and equality holds for uncountably many couples (ψ,α).

The whole of [24] is devoted to the proof of this assertion. It is long and tedious, as might be imagined from the statement of the theorem, and no attempt of proof will be made in this thesis. However, we will deduce several results about $M^*(f)$ from it. We did not include parts (a) and (c) of the theorem ([24], p. 283), because these are really homogeneous results.

For our purposes, the explicit values of $(\psi_r,-\alpha_r)$ given by (7.19) and (7.20) will be of less interest than the algorithmic development of the pair. As in the homogeneous case, since we are dealing with a lim inf problem, only the "tail" of this development will be relevant.

Let β_r' be the tail of the ordinary continued fraction expansion of ψ_r. In the notation of Chapter II, §3, let us denote the ordinary continued fraction blocks as follows:

$$\left.\begin{aligned} A' &= (4,1,1,1) = (4,1_3) \\ B' &= (4,1,1,1,1,1) = (4,1_5) \\ C' &= (3,1,1,1) = (3,1_3) \end{aligned}\right\} . \qquad (7.22)$$

Then, from [24] (pp. 324, 327-330, 351), we may suppose

$$\left.\begin{aligned} \beta_{-2}' = (A_\infty'),\quad \beta_{-1}' = ((B'C')_\infty),\quad \beta_0' = (B_\infty'), \\ \beta_r' = ((A'(B'C')_r)_\infty),\quad \text{for } r \ge 1. \end{aligned}\right\} \qquad (7.23)$$

By Lemma 2.4, the β_r' are reduced for $r \ge -2$, and if

$\beta_r = \beta'_r + 1$, the β_r will be A-reduced. By Lemma 2.12 the A-expansion of β_r will be pure periodic. Define the following semi-regular blocks:

$$A = [6,3], \quad B = [6,3,3], \quad C = [5,3]. \qquad (7.24)$$

Then, by (2.19) after the convention (2.18), the equations (7.23) are valid also when the primes are removed. Thus

$$\left.\begin{array}{c} \beta_{-2} = [A_\infty], \quad \beta_{-1} = [(BC)_\infty], \quad \beta_0 = [B_\infty], \\ \beta_r = [(A(BC)_r)_\infty], \text{ for } r \geq 1. \end{array}\right\} \qquad (7.25)$$

If $\{m_k\}$ is an arbitrary increasing sequence of positive integers, then any irrational ψ, whose ordinary continued fraction expansion tail is given by, say,

$$(A'(B'C')_{m_1}A'(B'C')_{m_2}\ldots.) = (A'(B'C')_{m_k})_{k=1}^{\infty}, \qquad (7.26)$$

together with a corresponding α, has

$$k^+(\psi,\alpha) = 1/\gamma.$$

Clearly there are uncountably many such ψ. The proof of this result may be found in [24] (38, p. 349). The corresponding A-expansion in semi-regular continued fractions, is given by (7.26) with the primes removed from the blocks.

3. Alternative method for calculating $k^+(\phi,\alpha)$

We will now briefly describe a further degenerate case of the divided cell method, used by Barnes [7], to prove the existence of uncountably many pairs (ϕ,α), for which $k^+(\phi,\alpha) = \delta$, for each δ with $0 \leq \delta \leq \frac{1}{4}$.

Clearly we may suppose that, in (7.12),

$$-1 < \alpha < 0, \quad -1 < \phi + \alpha < 0. \qquad (7.27)$$

Consider then the grid L, in the ξ, η-plane.

$$L: \quad \left.\begin{aligned} \xi &= x \\ \eta &= \phi x + y + \alpha \end{aligned}\right\}, \tag{7.28}$$

where x,y take all integral values. Since we are supposing that $\phi x + y + \alpha$ does not represent zero, there are no grid points on the ξ-axis. We readily see (as in [7]) that

$$k^+(\phi,\alpha) = \lim\inf \{|\xi\eta|; (\xi,\eta) \text{ a point of } L, \ \xi > 0\}. \tag{7.29}$$

The cell

$$C_0 = (0,\alpha), \qquad B_0 = (1,\phi + \alpha),$$
$$D_0 = (0,1 + \alpha), \ A_0 = (1,\phi + 1 + \alpha),$$

by (7.27) and (7.28), generates the grid L, which has unit determinant.

Using the formulae (3.9), we can again construct a divided cell S_1, of L, together with an integer pair (h_0,k_0) (as in [7], we may use the convention, if necessary, that an axis may be considered as part of any quadrant that it bounds). Continuing this process for $n \geq 0$, we obtain a sequence of divided cells $\{S_n\}$. However, the algorithm does not apply for $n < 0$, since there are lattice lines parallel to the η-axis. The results of Chapter III apply identically for $n \geq 0$, and by (3.13), the algorithm yields a sequence of pairs $\{a_{n+1},\varepsilon_n\}$, $n \geq 0$, which satisfies the relevant conditions in (3.26). If we denote the vertices of S_n by (3.14), then we have,

$$\left.\begin{aligned} \phi_n &= [a_{n+1},a_{n+2},\dots\dots] \\ \theta_n &= [a_n,a_{n-1},\dots\dots,a_1] \end{aligned}\right\}, \tag{7.30}$$

and

$$\mu_n = 2\eta_n + 1 + \phi_n = \varepsilon_n + \sum_{r=1}^{\infty} \frac{(-1)^r \varepsilon_{n+r}}{\phi_{n+1}\phi_{n+2}\dots\phi_{n+r}}, \quad \}$$

$$\lambda_n = 2\xi_n + \theta_n + 1 = \varepsilon_{n-1} + \sum_{r=1}^{n-1} \frac{(-1)^r \varepsilon_{n-r-1}}{\theta_{n-1}\theta_{n-2}\cdots\theta_{n-r}} + \frac{(-1)^n}{\theta_{n-1}\theta_{n-2}\cdots\theta_1} \quad \}. \quad (7.31)$$

Clearly θ_n and λ_n are rational numbers, and the results are consistent with the convention of Chapter III, whereby h_{-1} and k_{-1} are defined to be infinite, since C_0D_0 is a segment of the η-axis.

Consequently, the form $f_0(x,y) = x(\phi x + y + \alpha)$ is equivalent to each of

$$f_n(x,y) = \frac{\pm 1}{\theta_n\phi_n - 1}(\theta_n x + y + \xi_n)(x + \phi_n y + \eta_n), \quad (7.32)$$

for all $n \geq 0$, if

$$\phi_0 = 1/\phi, \quad \eta_0 = \alpha/\phi. \quad (7.33)$$

By a method similar to those of Chapter III and IV, the converse result may also be seen to be true. Any one-sided chain satisfying (3.26) for $n \geq 0$, corresponds to a sequence of forms (7.32), (7.33).

It is easily proved that the lim inf in (7.29) need only be extended over those grid points in the right hand plane which are vertices of a divided cell of the chain $\{S_n\}$. By the nature of the algorithm, as we have already remarked in Chapter III, A_n is either in the first or third quadrants. Hence we wish to evaluate $|\xi\eta|$ at <u>either</u> A_n and B_n, <u>or</u> C_n and D_n. We have already commented that whenever the slope of A_nD_n is positive (and hence $a_{n+1} > 0$) then A_n and A_{n+1} are in vertically opposite quadrants, while if A_nD_n has negative slope (and hence $a_{n+1} < 0$), then A_n and A_{n+1} are in the same quadrant. Since A_0 is in the first quadrant, it follows from Theorem 3.6 that

$$k^{+}(\phi,\alpha) = \liminf_{n \to \infty} M_n^{+}, \tag{7.34}$$

where

$$M_n^{+} = \begin{cases} \min \{M_n^{(1)}, M_n^{(2)}\}, & \text{if } (-1)^n a_1 a_2 \ldots a_n < 0 \\ \min \{M_n^{(3)}, M_n^{(4)}\}, & \text{if } (-1)^n a_1 a_2 \ldots a_n > 0 \end{cases} \tag{7.35}$$

In (7.35), we will denote the occurrence of the upper alternative by X_n, and the lower alternative by Y_n.

Since the $\liminf_{n \to \infty}$ is required in (7.34), any behaviour of the chain which occurs only a finite number of times, will not effect the value of $k^{+}(\phi,\alpha)$, provided the correct alternative is maintained.

If the rules for deciding which alternative to take at each step of the chain are reversed, in (7.35), (define this value to be M_n^{-}), then we are evaluating $|\xi\eta|$ in the left-hand plane. Put

$$\left.\begin{aligned} k^{-}(\phi,\alpha) = \liminf_{n \to \infty} M_n^{-} &= \liminf_{x \to -\infty} |x(\phi x + y + \alpha)| \\ &= \liminf_{x \to +\infty} x|(\phi x + y - \alpha)| \\ &= k^{+}(\phi,-\alpha) \end{aligned}\right\} . \tag{7.36}$$

Suppose we have two chains which are identical from some point onwards. Say the chain for (ϕ,α) is $\{a_{n+1},\varepsilon_n\}$, and the chain for (ϕ',α') is $\{a'_{n+1},\varepsilon'_n\}$, where

$$\left.\begin{aligned} a_{n+r+1} &= a'_{m+r+1} \\ \varepsilon_{n+r} &= \varepsilon'_{m+r} \end{aligned}\right\} ,$$

for some m and n, and all $r \geq 0$. Then it follows that

$$k^{+}(\phi,\alpha) = \begin{cases} k^{+}(\phi',\alpha'), & \text{if } (-1)^{m+n} a_1 \ldots a_n a'_1 \ldots a'_m > 0 \\ k^{+}(\phi',-\alpha'), & \text{if } (-1)^{m+n} a_1 \ldots a_n a'_1 \ldots a'_m < 0 \end{cases} . \tag{7.37}$$

We will now discuss the application of two of the elementary chain operations mentioned in Theorem 3.12, and their effect on the value of $k^+(\phi,\alpha)$. A prime attached to a variable will signify its value after the operation has been applied.

THEOREM 7.2.

(i) If the sign of the $\{\varepsilon_n\}$ chain is reversed, then

$$k^+(\phi',\alpha') = k^+(\phi,-\alpha) = k^-(\phi,\alpha).$$

(ii) If the signs of the $\{a_n\}$ chain, and alternate members of the $\{\varepsilon_n\}$ chain, are reversed, then

$$k^+(\phi',\alpha') = k^+(\phi,\alpha) \quad \text{or} \quad k^+(\phi,-\alpha).$$

PROOF.

(i) We have $\theta_n = \theta'_n$, $\phi_n = \phi'_n$, $\lambda_n = -\lambda'_n$, $\mu_n = -\mu'_n$. Hence for all $n \geq 0$, the application of this operation interchanges the values $M_n^{(1)}$ and $M_n^{(4)}$, and also $M_n^{(2)}$ and $M_n^{(3)}$. Consequently, although the pairing in (7.35) is maintained, the alternatives X_n and Y_n are interchanged, and the result follows by (7.36).

(ii) We have $\theta_n = -\theta'_n$ and $\phi_n = -\phi'_n$. Suppose we have $\varepsilon_n = \varepsilon'_n$, (either $\varepsilon_r = (-1)^r\varepsilon'_r$ with n even, or $\varepsilon_r = (-1)^{r-1}\varepsilon'_r$ with n odd). Then $\mu_n = \mu'_n$, $\lambda_n = -\lambda'_n$, and it is easily checked, as in (i), that the products within the alternatives X_n and Y_n are interchanged.

If, however, $\varepsilon_n = -\varepsilon'_n$, then we may show that the products within the alternatives X_n and Y_n are preserved. Now, since

$$\text{sgn}\ ((-1)^n a'_1 a'_2 \ldots . a'_n) = (-1)^n \text{sgn}\ ((-1)^n a_1 a_2 \ldots . a_n),$$

it is readily checked that after the application of the operation, the rules (7.35) constantly give either the same alternatives, or the opposite alternatives for the two chains. The result follows from this.

Remarks.

(i) The reversing of a one-sided chain has no real meaning, but later in the restricted form problem, we will see that this operation corresponds to replacing f and P by g and Q in $M^+(f;P)$ (see (7.6) and (7.7)).

(ii) As a consequence of this theorem, if we are investigating both the alternatives of a chain (e.g. the value of $\max\{k^+(\phi,\pm\alpha)\}$), then we may arbitrarily choose the sign of some a_n and ε_n.

4. Solution of the positive approximation problem by semi-regular continued fractions

In this section we will determine the critical semi-regular chains corresponding to those of Descombes in §2. Since the β_r' (for $r \geq -2$) are reduced quadratic irrationals, Theorem 3.15 implies that any rth critical chain must belong to the set of semi-regular expansions that lead forward to one of the following numbers, as complete quotient.

$$\beta_r' = \beta_r - 1, \quad \beta_r' + 1 = \beta_r, \quad \frac{\beta_r'}{1 - \beta_r'} = -\frac{\beta_r - 1}{\beta_r - 2}, \tag{7.38}$$

or their negatives.

We will show that the appropriate semi-regular expansions for the critical chains are those A-expansions of the β_r (or their negatives) indicated in (7.24), (7.25). In order to prove this we will require the following lemma.

LEMMA 7.1. *If one of the following three situations arise,* *(i)* $|a_n| = |a_{n+1}| = 2.$

(ii) $|a_n| = 2, \quad |a_{n+1}| \le 6, \quad a_n a_{n+1} < 0.$

(iii) $|a_n| > 100.$

then, for $i = n$ *or* $n-1$, *we have*

$$\max\{M_i^{\pm}\} < 1/\gamma = 0{\cdot}352\ldots$$

PROOF. By Theorem 7.2, we may suppose without loss of generality, that $a_n > 0$ and $\lambda_n \ge 0$.

(i) When $a_{n+1} = 2$, since $\lambda_n < 1$, $|\mu_n| < 1$, we consider the following cases of (7.35):

(a) $\theta_n < 2$, $\phi_n < 2$. By (3.25) and (3.27),

$$M_n^{\pm} \le \frac{(\theta_n - 1 + |\lambda_n|)(\phi_n - 1 + |\mu_n|)}{4(\theta_n\phi_n - 1)} < \frac{(\theta_n - 1)(\phi_n - 1)}{(\theta_n\phi_n - 1)} < 1/3.$$

(b) $\theta_n > 2$, $\phi_n > 2$. We have

$$M_n^{\pm} < \frac{\theta_n\phi_n}{4(\theta_n\phi_n - 1)} < 1/3.$$

(c) $\theta_n > 2$, $\phi_n < 2$. By a combination of the methods (a), (b),

$$M_n^{\pm} < \frac{\theta_n(\phi_n - 1 + |\mu_n|)}{4(\theta_n\phi_n - 1)} < \frac{\theta_n(\phi_n - 1)}{2(\theta_n\phi_n - 1)} < 1/3.$$

(d) $\theta_n < 2$, $\phi_n > 2$. This follows as in (c).

Consequently, in any critical chain, we have $|\theta_n| > 3/2$, and $|\phi_n| > 3/2$.

When $a_{n+1} = -2$, since $3/2 < \theta_n < 3$, $3/2 < |\phi_n| < 3$, then (3.27) implies,

$$\text{at } X_n: \quad \min\{M_n^{(1)}, M_n^{(2)}\} \le \frac{(\theta_n - 1)(|\phi_n| + 1 + |\mu_n|)}{4(\theta_n|\phi_n| + 1)}$$

$$< \frac{(\theta_n - 1)|\phi_n|}{2(\theta_n|\phi_n| + 1)} < \frac{\theta_n - 1}{2\theta_n} < 1/3.$$

at Y_n: $\min\{M_n^{(3)}, M_n^{(4)}\}$

$$\leq \max\left(\frac{(\theta_n - 1 + \lambda_n)(|\phi_n| + 1 - |\mu_n|)}{4(\theta_n|\phi_n| + 1)}, \frac{(\theta_n + 1 + \lambda_n)(|\phi_n| - 1 - |\mu_n|)}{4(\theta_n|\phi_n| + 1)}\right)$$

$$< \max\left(\frac{\theta_n(|\phi_n| + 1)}{4(\theta_n|\phi_n| + 1)}, \frac{\theta_n(|\phi_n| - 1)}{2(\theta_n|\phi_n| + 1)}\right)$$

$$< \max\left(\frac{3(3/2 + 1)}{4[(3)(3/2) + 1]}, \frac{3}{10}\right) = \frac{15}{44}.$$

(ii) By part (i), we may suppose that $-6 \leq a_{n+1} \leq -3$, and $0 \leq \lambda_n < 1$;

at X_n: as in the proof of Theorem 3.9,

$$\min\{M_n^{(1)}, M_n^{(2)}\} < \frac{(\theta_n - \lambda_n)|\phi_n|}{4(\theta_n|\phi_n| + 1)} < \tfrac{1}{4}.$$

at Y_n: consider the two cases;

<u>when $\mu_n \leq 0$</u>, then since $\theta_n < 3$, $|\phi_n| > 2$, we have

$$M_n^{(3)} = \frac{(\theta_n - 1 + \lambda_n)(|\phi_n| + 1 - |\mu_n|)}{4(\theta_n|\phi_n| + 1)} < \frac{\theta_n(|\phi_n| + 1)}{4(\theta_n|\phi_n| + 1)} < \frac{9}{28};$$

<u>when $\mu_n > 0$</u>, we consider the three subcases;

(I) if $|\phi_n| < 4$, then by (3.27),

$$M_n^{(4)} = \frac{(\theta_n + 1 + \lambda_n)(|\phi_n| - 1 - |\mu_n|)}{4(\theta_n|\phi_n| + 1)} < \frac{\theta_n(|\phi_n| - 1)}{2(\theta_n|\phi_n| + 1)} < \frac{9}{26};$$

(II) if $|\phi_n| > 4$, and $0 \leq \mu_n \leq 1$, then

$$M_n^{(3)} \leq \frac{\theta_n(|\phi_n| + 2)}{4(\theta_n|\phi_n| + 1)} < \frac{9}{26};$$

(III) if $4 < |\phi_n| < 7$, and $\mu_n > 1$, then as in (I),

$$M_n^{(4)} < \frac{\theta_n(|\phi_n| - 2)}{2(\theta_n|\phi_n| + 1)} < \frac{15}{44}.$$

Hence the result (ii) follows in all cases.

(iii) The bound 100 in the enunciation of this lemma is just a convenient number, which could be reduced to 6 with considerably more effort.

<u>When $0 < \lambda_n \le 0{\cdot}39\,\theta_n$</u>, then since we may suppose from (i) that $|\phi_n| > 3/2$, and so $|\theta_n\phi_n| > 150$, we have as in the proof of Theorem 3.9,

$$M_n^{\pm} < \frac{(\theta_n + \lambda_n)|\phi_n|}{4(\theta_n|\phi_n| - 1)} < \frac{(1{\cdot}39)(150)}{(4)(149)} < 1/\gamma.$$

<u>When $\lambda_n > 0{\cdot}39\,\theta_n$</u>, we have, since $\mu_{n-1} > \lambda_n - 2$,

$$\frac{\mu_{n-1}}{\phi_{n-1}} > \frac{\lambda_n\,\theta_n}{\theta_n\phi_{n-1}} - \frac{2}{\phi_{n-1}}$$

$$> \frac{(0{\cdot}39)(a_n - 1) - 2}{a_n + 1} \ge \frac{37}{102} > 0{\cdot}36.$$

Now, since $|\theta_{n-1}| > 3/2$, and $\phi_{n-1} > 100$, we may assert in all cases,

$$M_{n-1}^{\pm} < \frac{(|\theta_{n-1}| + 1 + |\lambda_{n-1}|)(\phi_{n-1} + 1 - \mu_{n-1})}{4(|\theta_{n-1}|\phi_{n-1} - 1)}$$

$$< \frac{|\theta_{n-1}|\big((0{\cdot}64)\phi_{n-1} + 1\big)}{2(|\theta_{n-1}|\phi_{n-1} - 1)}$$

$$< \frac{(1{\cdot}5)(65)}{(2)(149)} < 1/\gamma.$$

We have now concluded the proof of the entire lemma. As a consequence, the situations (i), (ii), and (iii) cannot occur infinitely often in any of the critical chains.

<u>THEOREM 7.3.</u> *The tail of the critical $\{a_n\}$ chains consists of the A-expansions of the β_r, as given by (7.24) and (7.25).*

<u>PROOF.</u> We have already noted that the critical $\{a_n\}$ chains are among those semi-regular chains which lead forward to one of

$$\beta'_r, \quad \beta'_r + 1, \quad \frac{\beta'_r}{1 - \beta'_r}, \quad \text{or their negatives.}$$

Suppose that $\{a_n\}$ is an arbitrary semi-regular chain, which leads forward to $\frac{\beta'_r}{\beta'_r - 1}$, for some r, $r \geq -2$. Now, by (7.22) and (7.23), we have $\beta'_r > 4$, and so

$$\frac{\beta'_r}{\beta'_r - 1} < \frac{4}{3} = [2,2,2].$$

Thus any such chain contains consecutive twos, and so, by Lemma 7.1, the complete quotient $\frac{\beta'_r}{\beta'_r - 1}$, (or its negative), may occur only a finite number of times.

Suppose then, that we have a chain which leads forward to either β'_r or β_r, then we show that only their A-expansions from this point on can be critical. Suppose that we have expanded β'_r or β_r in A.C.F., so that it equals, for some $k > 0$,

$$[a_1,a_2,....,a_k,\alpha],$$

where the a_i will be 3, 5, or 6.

We will investigate the effect of changing the a_k to $a_k - 1$, to give

$$[a_1,a_2,....,a_k - 1,\frac{-\alpha}{\alpha - 1}].$$

Consider the following cases:

(i) $\underline{a_k = 3}$. Equations (7.24) and (7.25) imply that $\alpha > 2$, and hence $\frac{\alpha}{\alpha - 1} < 2$; since $a_k - 1 = 2$, then Lemma 7.1 implies that this change cannot be made infinitely often.

(ii) $\underline{a_k = 5}$. Again by (7.24) and (7.25), $\alpha = [3,6,....]$, and using the notation (2.18) and the transformation (2.19) twice, we obtain

$$\alpha = [3,(2)_0,6,...],$$

and

$$\alpha - 1 = (1,1,4,.....),$$

whereby

$$\frac{1}{\alpha - 1} = (0,1,1,4,....),$$

and

$$\frac{\alpha}{\alpha - 1} = 1 + \frac{1}{\alpha - 1} = [2,3,(2)_3,....].$$

Since the 3 leads without choice to consecutive twos, then this chain segment cannot be part, infinitely often, of a critical chain. However, if we choose the lower alternative and change the 3 to 2, then we will again violate Lemma 7.1.

(iii) $a_k = 6$. If $\alpha = [3,6,....]$, then the result follows exactly as in part (ii). If not, then from (7.24) and (7.25), we readily see that $\alpha = [3,3,a,...]$, where a is either 5 or 6.

Following the method of part (ii), we obtain

$$\frac{1}{\alpha - 1} = (0,1,1,1,1,a - 2,1,....),$$

implying

$$\frac{\alpha}{\alpha - 1} = [2,3,3,(2)_{a-3},3,....].$$

Clearly we cannot leave (infinitely often) the consecutive threes, since they lead to consecutive twos, nor can we change the first 3, without violating Lemma 7.1. However, using the same method, we find that

$$[3,(2)_{a-3},3,...] = [2,-(a - 1),...],$$

which contravenes Lemma 7.1, whether $-(a - 1)$ is changed or not.

Thus we have shown that we cannot deviate from the A-expansion of β_r (or β_r') infinitely often, without implying, for the corresponding ϕ and α,

$$k^+(\phi,\alpha) < 1/\gamma.$$

Consequently the tail of the critical chains must be given by the semi-regular expansions (7.25) (or their negatives). Associated with these a-chains will be a corresponding ε-chain, which we will determine later.

LEMMA 7.2. *For large enough n, the ε-chain for critical chains*
(i) is alternating in sign if $\{a_n\}$ has β_r as its tail,
(ii) has constant sign if $\{a_n\}$ has $-\beta_r$ as its tail.

PROOF. Since, by Theorem 7.2, (ii) follows from (i), we may suppose that $a_n > 0$, for all $n > N$. By the form of the relevant expansions, if n is large enough, we have $\theta_n\phi_n > 4$, and so,as in the proof of Theorem 3.9,

$$\min \{M_n^{\pm}\} < \frac{(\theta_n - |\lambda_n|)\phi_n}{4(\theta_n\phi_n - 1)} < 1/3.$$

Since $a_n > 0$, then the cases X_n and Y_n will alternate with successive values of n; hence, so too will the sign of λ_n, in order to maintain the products containing the factors $(\theta_n \pm 1 + |\lambda_n|)$.

COROLLARY. *The appropriate products, for large enough n, are:*

$$\left.\begin{array}{c} \dfrac{(|\theta_n| + 1 + |\lambda_n|)(|\phi_n| + 1 - |\mu_n|)}{4(|\theta_n\phi_n| - 1)} \\ \text{and} \\ \dfrac{(|\theta_n| - 1 + |\lambda_n|)(|\phi_n| - 1 + |\mu_n|)}{4(|\theta_n\phi_n| - 1)} . \end{array}\right\} \qquad (7.39)$$

This follows immediately from (3.25), (7.35), and the previous lemma.

LEMMA 7.3. *In any critical chain, for n large enough, whenever*

$$a_{n+1} = 3,\ 5,\ \text{or}\ 6,$$

then

$$|\varepsilon_n| = 1,\ 1,\ \text{or}\ 2,\quad \text{respectively.}$$

<u>*PROOF.*</u>

(i) When $a_{n+1} = 3$, the result follows, since ε_n must be odd.

(ii) When $a_{n+1} = 5$, by (7.25) and Theorem 7.3, we have a chain segment

$$[\ldots,6,3,3,5,3,6,\ldots].$$

If $|\varepsilon_n| = 3$, then $|\mu_n| > 3 + 1/\phi_{n+1} > 3\tfrac{1}{4}$, and $|\lambda_n| < 2$, $\theta_n > 2\tfrac{1}{2}$, $\phi_n < 19/4$. Thus, by (7.39),

$$M_n^+ < \frac{(\theta_n + 3)(\phi_n - 2\cdot75)}{4(\theta_n\phi_n - 1)} < \frac{(5\cdot5)(2\cdot5)}{4\{(2\cdot5)(4\cdot75) - 1\}} < 1/\gamma.$$

Consequently, $|\varepsilon_n| = 1$.

(iii) When $a_{n+1} = 6$, then we have a chain segment

$$[\ldots,3,6,3,\ldots].$$

If $|\varepsilon_n| = 4$, then as in the previous case, $|\mu_n| > 4\tfrac{1}{4}$, $|\lambda_n| < 2$, $\phi_n < 23/4$, and $\theta_n > 2\tfrac{1}{2}$, implying

$$M_n^+ < \frac{(5\cdot5)(2\cdot5)}{4\{(2\cdot5)(5\cdot75) - 1\}} < 1/\gamma.$$

If $\varepsilon_n = 0$, then using the method of the proof of Theorem 3.9, we have, since $\theta_{n+1} > 5\cdot5$, $\phi_{n+1} > 2\cdot5$,

$$M_{n+1}^+ < \frac{(\theta_{n+1} + |\lambda_{n+1}|)\phi_{n+1}}{4(\theta_{n+1}\phi_{n+1} - 1)} < \frac{(\theta_{n+1} + 1)\phi_{n+1}}{4(\theta_{n+1}\phi_{n+1} - 1)}$$

$$< \frac{(6\cdot5)(2\cdot5)}{4\{(5\cdot5)(2\cdot5) - 1\}} < 1/\gamma.$$

The lemma now follows in full.

Consequently, if n is large, the ε_n associated with the a_{n+1} in a critical chain is automatically fixed by Lemmas 7.2, 7.3. We may therefore consider the blocks A, B, C, of (7.24), to be blocks of integer pairs. We may now state the following result.

<u>THEOREM 7.4.</u> *Suppose we have a one-sided chain pair which has as its tail the A-expansion of β_r, for some $r \geq -2$, given by (7.24) and (7.25), (or any chain obtained from such a chain pair by applying one of the operations of Theorem 7.2); then for the corresponding ϕ and α, exactly one of $k^+(\phi,\alpha)$ or $k^+(\phi,-\alpha)$ has the value $1/\gamma_r$ (defined in (7.21)), while the other has a value less than ¼.*

This result follows from Lemmas 7.2, 7.3, and Theorems 7.1, 7.3.

5. <u>Supremum of values taken by $M^*(f)$</u>

If f is any I-reduced form given by (7.1), then there corresponds a grid L, as in §1, of this chapter. From (7.4) to (7.9), we obtain:

$$\left.\begin{aligned} M^+(f;P) &= \inf\{|\xi\eta|;\ (\xi,\eta) \text{ a point of } L,\ \xi > 0\} \\ M^-(f;P) &= \inf\{|\xi\eta|;\ (\xi,\eta) \text{ a point of } L,\ \xi < 0\} \\ M^+(g;Q) &= \inf\{|\xi\eta|;\ (\xi,\eta) \text{ a point of } L,\ \eta > 0\} \\ M^-(g;Q) &= \inf\{|\xi\eta|;\ (\xi,\eta) \text{ a point of } L,\ \eta < 0\} \end{aligned}\right\}. \quad (7.40)$$

Because the rules, (3.11), for moving from cell to cell by the algorithm are the same for each modification of the general method, so too the rules for determining which pair of vertices is in the right-hand plane remain unaltered. Thus if A_n is in the first quadrant for some n, then the sign of a_{n+1} completely determines the quadrant of A_{n+1}, and the sign of a_n determines the quadrant of A_{n-1}. Now A_0

is in the first quadrant, and so we may evaluate $M^+(f;P)$ by a similar set of rules to (7.35). It is clear that

$$M^+(f;P) = \inf_n M_n^+(f;P), \tag{7.41}$$

where

$$\left.\begin{array}{l} M_0^+(f;P) = \min\{M_0^{(3)},M_0^{(4)}\}; \\ \text{if } n > 0,\quad M_n^+(f;P) = \begin{cases} \min\{M_n^{(1)},M_n^{(2)}\}, & \text{if } (-1)^n a_1 \cdots a_n < 0 \\ \min\{M_n^{(3)},M_n^{(4)}\}, & \text{if } (-1)^n a_1 \cdots a_n > 0 \end{cases} \\ \text{if } n < 0,\quad M_n^+(f;P) = \begin{cases} \min\{M_n^{(1)},M_n^{(2)}\}, & \text{if } (-1)^n a_0 a_{-1} \cdots a_{n+1} < 0 \\ \min\{M_n^{(3)},M_n^{(4)}\}, & \text{if } (-1)^n a_0 a_{-1} \cdots a_{n+1} > 0 \end{cases} \end{array}\right\} \tag{7.42}$$

Again we will refer to the upper and lower alternatives at the nth step as X_n and Y_n respectively. By consistently reversing the rules (7.42), taking X_0 as starting point (and $M_0^-(f;P) = \min\{M_0^{(1)},M_0^{(2)}\}$) we may calculate $M^-(f;P)$, and consequently $M^*(f;P)$, from (7.8), (7.9), and (7.10). In fact, the chains for g, Q, of (7.6) and (7.7), may be determined from the following lemma.

<u>LEMMA 7.4.</u> *If the doubly infinite chain pair $\{a_{n+1},\varepsilon_n\}$ is reversed about some point (say $n = 0$), then the chain obtained corresponds to the form g, and the point Q, of (7.6) and (7.7).*

<u>PROOF.</u> For a step n, in the original chain, there corresponds a step n' in the resultant chain such that

$$\theta_n = \phi_{n'}, \quad \phi_n = \theta_{n'},$$

$$\lambda_n = \mu_{n'}, \quad \mu_n = \lambda_{n'}.$$

Consequently, the groupings as a whole, of the four products $M_n^{(i)}$, are preserved (for some other value of n), but, however, their order is not.

In particular, the values $M_n^{(2)}$ and $M_n^{(3)}$ are interchanged. Now, as we have already noted in Chapter III, $M_j^{(1)}$, $M_j^{(2)}$, $M_j^{(3)}$ and $M_j^{(4)}$ are derived from C_j, D_j, B_j and A_j, respectively. Thus at the alternative X_n, in the reversed chain, we are evaluating the products of the coordinates at C_n and B_n, vertices of the nth divided cell in the grid associated with the original chain. Similarly, Y_n, corresponds to an evaluation at A_n and D_n. The result may then be checked in all cases, it following from (3.25), (7.8), (7.9), and (7.42).

In §4 of this chapter, certain results were stated, which involved the lim inf of one-sided chains. We will now deduce from these various results on the infimum of two-sided chains. We will use the following obvious extension of the notation used in Chapter II.

$$[_\infty(a_1,a_2,\ldots,a_n),\ldots..]$$

Consider the following chain pairs, where $\{\varepsilon_n\}$ is chosen in accordance with Lemmas 7.2 and 7.3, and the blocks A, B, C, are given in (7.24):

$$\left.\begin{array}{ll} C_{-2}: & [_\infty A_\infty] \\ C_{-1}: & [_\infty(BC)_\infty] \\ C_0: & [_\infty B_\infty] \\ C_r: & [_\infty(A(BC)_r)_\infty] \quad (r \geq 1) \end{array}\right\} \qquad (7.43)$$

THEOREM 7.5. *If the chain pair* C_r $(r \geq -2)$, *corresponds to a form* f_r, *and a point* $P_r = P_r(x_r, y_r)$, *then*

$$\max\{M^{\pm}(f_r;P_r)\} = \Delta/\gamma_r; \qquad (7.44)$$

furthermore, if

$$g_r = f_r\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \text{ and } Q_r = Q_r(y_r, x_r), \qquad (7.45)$$

then

$$\max \{M^{\pm}(g_r;Q_r)\} \leq \Delta/\gamma_r, \tag{7.46}$$

where equality holds if and only if the chain C_r *is symmetrical (identical with its inverse). If equality does not hold, then we may replace the right-hand side of (7.46) by* Δ/γ.

PROOF. It is clear from Theorems 3.11 and 7.4, that if the infimum in the definitions of $M^{\pm}(f_r;P_r)$, were replaced by lim inf, then (7.44) would hold. But the chains C_r are totally periodic, and so there are only a finite number of different values for the $M_n^{(i)}(f_r;P_r)$. Consequently, the infimum will equal the lim inf, and so (7.44) follows.

By Lemma 7.4, the chain pair associated with g_r and Q_r, will be the reverse of the chain C_r. In the case of C_{-2} and C_0 (the only symmetrical C_r), equality will clearly hold in (7.44). But for r = -1 and $r \geq 1$, C_r is not symmetrical, and its reverse provides a new periodic chain; any right-hand half-chain derived from this chain can never be one of the critical chains of Descombes. Thus equality in (7.46) would contradict Theorem 7.4, as would equality with any constant exceeding Δ/γ, on the right-hand side.

COROLLARY. $M^*(f_r;P_r) = \Delta/\gamma_r$.

This is immediate upon (7.10).

THEOREM 7.6. $M^*(f_r) = \Delta/\gamma_r$.

PROOF. We may suppose that $f_r(x,y)$ is given by (7.1) where

$$\phi = \beta_r, \quad \theta = 1/\bar{\phi},$$

$\bar{\phi}$ being the algebraic conjugate of ϕ. Then the chain for f_r and some other point S, incongruent mod 1, must contain a semi-regular expansion of ϕ, as a right-hand chain (together with an associated ε-chain). Now the theory of §4 clearly shows that the A-expansion of ϕ, with its particular ε-chain, must be taken if the infimum of the chain is to be greater than Δ/γ. Hence the lim inf of the chain does not exceed Δ/γ_r, implying

$$M^{\pm}(f_r;S) \leq \Delta/\gamma_r,$$

for all such S. Similarly, $M^{\pm}(g_r;S') \leq \Delta/\gamma_r$. The result follows by (7.11).

We will require the following lemma, which will enable us to construct from a two-sided chain with a certain infimum, a one-sided chain with an arbitrarily close lim inf.

LEMMA 7.5. *If H is a finite set of integer pairs, and $\{a_{n+1},\varepsilon_n\}$ any infinite sequence whose elements are taken from H, then for every integer $j > 0$, there exists a block containing j integer pairs of H, which occurs infinitely many times in the sequence $\{a_{n+1},\varepsilon_n\}$.*

PROOF. The lemma is clearly true for j = 1. Assuming the truth of the assertion for j = k, we have that there exists a block of k members of H which occur consecutively in the given sequence, infinitely often. But each of these blocks can be followed by one of only a finite number of elements of H, one of which must then occur infinitely often. Thus the result holds for j = k + 1.

THEOREM 7.7. *For all forms that do not represent zero, say f, we have*

$$M^*(f) \leq \frac{27\Delta}{28\sqrt{7}}.$$

Suppose that f is not equivalent to the form f_{-2} of Theorem 7.5, then

$$M^*(f) \leq \Delta/\gamma_{-1} = \frac{359\Delta}{45\sqrt{510}},$$

where equality holds for forms equivalent to f_{-1}.

PROOF. Suppose C is any two-sided chain pair which is not identical to C_{-2}. Then C is one of the following three types:

(a) C does not contain the sub-chain $[\ldots A_\infty]$.

(b) $C = [XA_\infty]$, where X is a one-sided chain which does not contain the sub-chain $[_\infty A\ldots]$.

(c) $C = [_\infty AYA_\infty]$, where Y is a finite chain segment not equal to A_n for any positive integer n.

Let f and P correspond to the chain C, and suppose

$$M^+(f;P) = \rho\Delta.$$

Then for all n, we have

$$M_n^+(f;P) = M_n^+ \geq \rho\Delta.$$

Assume, for an appropriate ε, that

$$0 < \varepsilon < \rho - (1/\gamma_{-1}). \tag{7.47}$$

After Lemma 7.1, we have that $|\theta_n|$ and $|\phi_n|$ are bounded in the interval (1·5,101). Thus we may apply Theorem 3.11, and the constant implied by the O notation is independent upon the particular chain segment under consideration. There therefore exists an integer m, with the property that the respective products belonging to the centre of a common chain segment of length 2m from two chain pairs, differ by no more than ε.

Now, in the cases (a) and (b), C must contain some chain segment different from A, which occurs infinitely often. Consequently, by the method of Lemma 7.5, there exists a block, D say, of length 2m, and containing this segment, which occurs in C infinitely often. In the case (c), let $D = A_{2m}$.

Consider the one-sided chain C^*, given by

$$C^*: \qquad [(DZ)_\infty], \qquad (7.48)$$

where in the cases (a) and (b), Z is a chain segment which separates two blocks D, whereas in the case (c), let Z = Y.

Since every step of the chain C^* (far enough along) is the centre of a chain segment of length 2m which also appears in C, then it follows from (7.35), (7.42) and (7.47),that, for some ϕ, α, corresponding to the chain C^*, (suitable Z will preserve the alternatives in D),

$$k^+(\phi,\alpha) \geq \rho - \varepsilon > 1/\gamma_{-1}.$$

But C^* is not one of the critical chains of Theorem 7.4, and so this is a contradiction. The theorem now follows.

6. Other results for $M^*(f)$

The obvious question to be asked now is whether $M^*(f)$ takes only the discrete values Δ/γ_r, $(r \geq -2)$, greater than Δ/γ. In the previous section we have seen that for certain equivalence classes of forms $\{f_r\}$, we have $M^*(f_r) = \Delta/\gamma_r$. But it seems certain that there are doubly infinite chains, other than the C_r of (7.41), for which $M^*(f) > \Delta/\gamma$.

To enable such results to be obtained, it seems as though we would need lemmas of the type Lemme 28 ([24], p. 349), whereby the products at certain "privileged" points of the following chain segments are compared;

$$\ldots A(BC)_j A(BC)_i A \ldots, \quad 0 \leq i < i',$$

$$\ldots A(BC)_{j'} A(BC)_{i'} A \ldots, \quad 1 \leq j < j'.$$

It is probable that the identical results would follow through for the semi-regular algorithm, by the same involved type of argument used in [24] (pp. 326-355). If this were so, then chains of the type

$$[{}_\infty(A(BC)_i)(A(BC)_j)_\infty],$$

for corresponding f and P, would have $M^+(f;P) \leq \Delta/\gamma$, for $i > j$, and $\lim_n \inf M_n^+(f;P) = \Delta/\gamma_j$, for $i < j$. The question as to whether Δ/γ_j is approached from above or below could be settled by detailed, but straight forward analysis of the type needed for Lemme 28 of [24]. Limitations of space prevent such an investigation from being undertaken in this thesis, but it seems reasonable to conjecture that Δ/γ_r are, in fact, the only values exceeding Δ/γ, taken by $M^*(f)$. For example, if $\{r_i\}$ is a finite, strictly increasing sequence of positive integers, with k members, then it would be consistent with [24] to conjecture that the chain

$$[{}_\infty(A(BC)_{r_1})A(BC)_{r_2}\ldots A(BC)_{r_{k-1}}(A(BC)_{r_k})_\infty]$$

has infimum equal to Δ/γ_{r_k}.

CHAPTER VIII

AN UPPER BOUND FOR THE CONSTANT k^+

1. Semi-regular expansions for $\frac{\sqrt{5}+1}{2}$

In this chapter we will demonstrate the method of §3, in the previous chapter, in evaluating $k^+(\phi)$, given by (1.8), where ϕ is equivalent to $\frac{\sqrt{5}+1}{2}$. By (1.15), it is clear that

$$k^+(\phi) \geq k^+.$$

Godwin [30] proved that $k^+ > 0{\cdot}1407\ldots$ We will prove that

$$k^+\left(\frac{\sqrt{5}+1}{2}\right) = \frac{3\sqrt{5}-5}{10} = 0{\cdot}1708\ldots,$$

implying from (1.16), that

$$c^+ \leq k^+ \leq 0{\cdot}1708\ldots,$$

which improves Godwin's bound of $0{\cdot}2114\ldots$

Now since ϕ is reduced, Theorem 3.15 implies that every semi-regular expansion of an equivalent number to ϕ, has as a complete quotient one of

$$\phi = \frac{\sqrt{5}+1}{2}, \quad \psi = \phi + 1 = \frac{\sqrt{5}+3}{2}, \quad \frac{\phi}{1-\phi} = -\frac{\sqrt{5}+3}{2} = -\psi,$$

or their negatives. But

$$\psi = [3,\psi] = [2,-2,-\psi]. \tag{8.1}$$

Consider the blocks

$$\left.\begin{array}{l} A = [3],\ B = [2,-2] \\ C = [-2,2],\ D = [-3] \end{array}\right\} \tag{8.2}$$

Thus any expansion of an equivalent number to ϕ, leads eventually to the following blocks alone:

$$AA,\ AB,\ BC,\ BD,\ CB,\ CA,\ DD,\ DC. \tag{8.3}$$

By the symmetry of the blocks, it is clear that, for large enough n, ϕ_n takes only the values $\pm\frac{\sqrt{5}+3}{2}$ and $\pm\frac{\sqrt{5}+1}{2}$, while θ_n **also takes** values which are arbitrarily close (as $n \to \infty$) to these four numbers.

2. Investigation of certain block sequences

LEMMA 8.1. *The occurrence, infinitely often, of AA, where the two accompanying values of the ε-chain have the same sign, implies for a corresponding α, that*

$$k^{+}\left(\frac{\sqrt{5}+1}{2}, \pm\alpha\right) \le \frac{3\sqrt{5}-5}{10}.$$

PROOF. Suppose $a_n = a_{n+1} = 3$, and without loss of generality, by Theorem 7.2, $\varepsilon_{n-1} = \varepsilon_n = 1$.

By (3.27),

$$|\mu_n - 1| < \frac{\phi_{n+1} - 1}{\phi_{n+1}} = \frac{\sqrt{5}-1}{2}.$$

Similarly

$$\lim_{\text{app. } n} |\lambda_n - 1| \le \frac{\sqrt{5}-1}{2},$$

where app. n means the limit is taken over those appropriate n for which $a_n = a_{n+1} = 3$, and $\varepsilon_{n-1} = \varepsilon_n = 1$. Thus at either alternative

$$\liminf_n M_n^{\pm} \le \frac{[(\sqrt{5}+3)/2 - 2 + (\sqrt{5}-1)/2][(\sqrt{5}+3)/2 + (\sqrt{5}-1)/2]}{(\sqrt{5}+3)^2 - 4}$$

$$= \frac{3\sqrt{5}-5}{10}.$$

COROLLARY. *The same result holds for the block DD, where the associated ε_n are of opposite sign.*

This follows from the lemma, and Theorem 7.2.

LEMMA 8.2. *The occurrence of the block BC, infinitely often, implies that*

$$k^+\left(\frac{\sqrt{5}+1}{2}, \pm\alpha\right) \le \frac{3\sqrt{5}-5}{10}.$$

PROOF. Suppose that $a_n = a_{n+1} = -2$, then we have a chain segment:

$$\begin{matrix} \ldots 2, & -2, & -2, & 2\ldots \\ \ldots 0, & 0, & 0, & 0\ldots \end{matrix}$$

Suppose that there are infinitely many such n, forming the sequence $\{n_k\}$. Then, since $\phi_{n_k+1} = \frac{\sqrt{5}+1}{2}$,

$$|\mu_{n_k}| = |\tau_{n_k+1}| < \frac{\phi_{n_k+1} - 1}{\phi_{n_k+1}} = \frac{3-\sqrt{5}}{2}.$$

Similarly

$$\lim_{k\to\infty} |\lambda_{n_k}| \le \frac{3-\sqrt{5}}{2}$$

Hence it follows that

$$\liminf_{k\to\infty} M^{\pm}_{n_k} < \frac{((\sqrt{5}+3)/2 - 1 + (3-\sqrt{5})/2)^2}{(\sqrt{5}+3)^2 - 4}$$
$$= \frac{3\sqrt{5}-5}{10}.$$

COROLLARY. *The same result holds for the block CB.*

This follows from the lemma, and Theorem 7.2.

LEMMA 8.3. *The occurrence of the block ABD, infinitely often, with $a_n = 2$, and ε_{n-2} and ε_{n+1} of opposite sign, implies*

$$k^+\left(\frac{\sqrt{5}+1}{2}, \pm\alpha\right) \le \frac{3\sqrt{5}-5}{10}.$$

PROOF. We may suppose without loss of generality that $\varepsilon_{n+1} = 1$. Then consider the chain segment:

$$\begin{matrix} \ldots 3, & 2, & -2, & -3\ldots \\ \ldots -1, & 0, & 0, & 1\ldots \end{matrix}$$

Now, $\mu_n > 0$, $0 < \lambda_n < \theta_n - 1$, and $\phi_n = -\frac{\sqrt{5}+1}{2}$. Again let such n form the sequence $\{n_k\}$. Hence

$$\text{at } X_{n_k}: \quad \liminf_{k\to\infty} M^{\pm}_{n_k} \le \liminf_{k\to\infty} \frac{(\theta_{n_k} - 1 - \lambda_{n_k})(|\phi_{n_k}| + 1 - \mu_{n_k})}{4(\theta_{n_k}|\phi_{n_k}| + 1)}$$

$$< \frac{(\sqrt{5} - 1)(\sqrt{5} + 3)}{8\sqrt{5}(\sqrt{5} + 1)}$$

$$= \frac{1}{4\sqrt{5}} .$$

$$\text{at } Y_{n_k}: \quad \liminf_{k\to\infty} M^{\pm}_{n_k} \le \liminf_{k\to\infty} \frac{(\theta_{n_k} + 1 + \lambda_{n_k})(|\phi_{n_k}| - 1 - \mu_{n_k})}{4(\theta_{n_k}|\phi_{n_k}| + 1)}$$

$$< \frac{(\sqrt{5} + 1)(\sqrt{5} - 1)}{4\sqrt{5}(\sqrt{5} + 1)}$$

$$= \frac{5 - \sqrt{5}}{20} .$$

<u>COROLLARY.</u> *The following chain segments cannot occur infinitely often:*

$$\left.\begin{matrix} \ldots 3, & 2, & -2, & -3\ldots \\ \ldots -1, & 0, & 0, & 1\ldots \end{matrix}\right\}$$

$$\left.\begin{matrix} \ldots 3, & 2, & -2, & -3\ldots \\ \ldots 1, & 0, & 0, & -1\ldots \end{matrix}\right\}$$

$$\left.\begin{matrix} \ldots -3, & -2, & 2, & 3\ldots \\ \ldots 1, & 0, & 0, & 1\ldots \end{matrix}\right\} \qquad (8.4)$$

$$\left.\begin{matrix} \ldots -3, & -2, & 2, & 3\ldots \\ \ldots -1, & 0, & 0, & -1\ldots \end{matrix}\right\}$$

This follows from the lemma, and Theorem 7.2.

<u>LEMMA 8.4.</u> *The only possible chains for which*

$$k^+\left(\frac{\sqrt{5} + 1}{2}, \alpha\right) > \frac{3\sqrt{5} - 5}{10}$$

are those whose tail is

$$[A_{r_1} BD_{r_2} CA_{r_3} BD_{r_4} C\ldots\ldots], \qquad (8.5)$$

where $\{r_i\}$ *is a sequence of positive integers, and the* ε*-chain is* *chosen so that none of (8.4) occur infinitely often, and Lemma 8.1 is not contravened.*

PROOF. Clearly we cannot have $[A_\infty]$ or $[D_\infty]$, since, by Lemma 8.1 and its corollary, this would contradict the conditions (3.26), for large n. The result now follows from (8.3) and Lemmas 8.2 and 8.3.

3. Evaluation of $k^+\left(\frac{\sqrt{5}+1}{2}\right)$

THEOREM 8.1. *Suppose the sequence* $\{r_i\}$ *in (8.5) tends to infinity, then for the corresponding* α,

$$k^+\left(\frac{\sqrt{5}+1}{2},\alpha\right) = \frac{3\sqrt{5}-5}{10}.$$

PROOF. Suppose that i is arbitrarily large, and $a_n = 2$, say, then consider the chain segment

$$\ldots CA_{r_i}BD_{r_{i+1}}\ldots,$$

which consists of the following integers, putting $v = r_i+1$,

$$\begin{array}{l}\ldots\ 3,\ldots 3\ \ 2,\ -2,\ -3,\ldots -3\ldots \\ \ldots(-1)^v,\ldots 1,\ 0,\ \ 0,\ \ 1,\ldots\ 1\ldots\end{array} \tag{8.6}$$

As usual, let $\{n_k\}$ be the sequence of such steps in the chain (i.e. where $|a_{n_k}| = |a_{n_k+1}| = 2$). It is clear from the sign pattern of the chain that at each such step, the two appropriate products will correspond. In the chain segment (8.6), we have $\lambda_n < 0$, $\mu_n > 0$. Hence at the sequence of alternatives corresponding to the choice Y_n:

$$\liminf_{k\to\infty} M^{\pm}_{n_k} \le \liminf_{k\to\infty} \frac{(|\theta_{n_k}|+1)(|\phi_{n_k}|-1)}{4(|\theta_{n_k}\phi_{n_k}|+1)} \le \frac{1}{4\sqrt{5}}.$$

Thus it is clear that the appropriate alternative at each step is the one which involves the factors $(|\theta_r| \pm 1 + |\lambda_r|)$.

Now, since $r_i \to \infty$, it is readily checked that

$$\lim_{k \to \infty} |\mu_{n_k}| = \lim_{k \to \infty} (|\phi_{n_k}| - 1) = \frac{\sqrt{5} - 1}{2},$$

and

$$\lim_{k \to \infty} |\lambda_{n_k}| = \frac{\sqrt{5} - 1}{2}.$$

It may be shown, as in Chapter V, that $M_n^{(1)} > M_n^{(2)}$, if and only if $\theta_n + |\lambda_n| < \frac{|\phi_n|}{1 - |\mu_n|}$; but we have

$$\lim_{k \to \infty} (|\theta_{n_k}| + |\lambda_{n_k}|)(1 - |\mu_{n_k}|)/|\phi_{n_k}| = 5 - 2\sqrt{5} < 1.$$

Thus

$$\lim_{k \to \infty} M_{n_k}^{+} = \lim_{k \to \infty} \frac{(|\theta_{n_k}| - 1 + |\lambda_{n_k}|)(|\phi_{n_k}| + 1 - |\mu_{n_k}|)}{4(|\theta_{n_k}\phi_{n_k}| + 1)}$$
$$= \frac{3\sqrt{5} - 5}{10}.$$

Now, re-defining n, let us suppose that $a_n = a_{n+1} = 3$, and $\varepsilon_{n-1} = -\varepsilon_n = 1$. Then Y_n is the relevant alternative, and so by (3.27),

$$M_n^{(4)} > \frac{\theta_n + 1 + \lambda_n}{2(\theta_n\phi_n - 1)} > \frac{1}{2\phi_n} > \frac{3\sqrt{5} - 5}{10}.$$

Now ε_n is either followed by $\varepsilon_{n+1} = 1$, in which case $|\mu_n| > 1$, or by two consecutive zeros, whence

$$|\mu_n| = 1 - |\tau_{n+1}| = 1 - \left|\frac{\mu_{n+2}}{\phi_{n+1}\phi_{n+2}}\right| > 1 - \frac{2(\sqrt{5} - 1)}{(\sqrt{5} + 3)(\sqrt{5} + 1)} = 1 - \frac{7 - 3\sqrt{5}}{2}.$$

A similar result holds, in the limit, for $|\lambda_n|$. If $\{n_k\}$ is the sequence of such n, then

$$\liminf_{k \to \infty} M_{n_k}^{(3)} \geq \liminf_{k \to \infty} \frac{\left(\theta_{n_k} - (7 - 3\sqrt{5})/2\right)\left(\phi_{n_k} - (7 - 3\sqrt{5})/2\right)}{4(\theta_{n_k}\phi_{n_k} - 1)}$$
$$= \frac{4(\sqrt{5} - 1)^2}{\sqrt{5}(\sqrt{5} + 1)} > \frac{3\sqrt{5} - 5}{10}.$$

By Theorem 7.2, this also covers the cases when $a_n = a_{n+1} = -3$.

Let us now consider the step where $a_{n-1} = -a_n = 2$, $a_{n+1} = 3$, and $\varepsilon_n = 1$. As before, $M_n^{(1)} < M_n^{(2)}$, if and only if

$|\theta_n| + |\lambda_n| < \dfrac{|\phi_n|}{\mu_n - 1}$. For the appropriate sequence $\{n_k\}$, we have

$$\lim_{k \to \infty} (|\theta_{n_k}| + |\lambda_{n_k}|)(|\mu_{n_k}| - 1)/|\phi_{n_k}| = \frac{3(\sqrt{5} - 1)}{\sqrt{5} + 3} < 1.$$

Also

$$\lim_{k \to \infty} \frac{(|\theta_{n_k}| - 1 + |\lambda_{n_k}|)(|\phi_{n_k}| - 1 + |\mu_{n_k}|)}{4(|\theta_{n_k}\phi_{n_k}| - 1)} = \frac{2(\sqrt{5} + 1)}{\sqrt{5}(\sqrt{5} + 1)^2} > \frac{3\sqrt{5} - 5}{10}.$$

By Theorem 7.2, there only remains to investigate the steps for which $a_n = 3$, $a_{n+1} = 2$, with $\varepsilon_{n-1} = 1$. Clearly, for the appropriate $\{n_k\}$,

$$\lim_{k \to \infty} M^+_{n_k} = \lim_{k \to \infty} \frac{(|\theta_{n_k}| - 1 + |\lambda_{n_k}|)(|\phi_{n_k}| - 1 - |\mu_{n_k}|)}{4(|\theta_{n_k}\phi_{n_k}| - 1)}$$

$$= \frac{(\sqrt{5} + 1)(\sqrt{5} - 1)}{\sqrt{5}(\sqrt{5} + 1)^2} = \frac{3\sqrt{5} - 5}{10}.$$

The complete result now follows.

THEOREM 8.2. *Suppose that in (8.5) $\{r_i\} \not\to \infty$, then for the corresponding α,*

$$k^+\left(\frac{\sqrt{5} + 1}{2}, \pm\alpha\right) \leq \frac{3\sqrt{5} - 5}{10}.$$

PROOF. Let $\{n_k\}$ be the sequence of n for which $|a_n| = |a_{n+1}| = 2$.

(i) Suppose that $\{r_i\}$ is eventually constant, then

$$|\mu_{n_k}| = \zeta < \frac{\sqrt{5} - 1}{2}, \text{ and } \lim_{k \to \infty} |\lambda_{n_k}| = \zeta.$$

If $\phi = \dfrac{\sqrt{5} + 1}{2}$, then

$$\lim_{k \to \infty} M^+_{n_k} = \frac{\phi^2 - (1 - \zeta)^2}{4(\phi^2 + 1)} < \frac{3\sqrt{5} - 5}{2}.$$

(ii) If $\{r_i\}$ is not constant then there are infinitely many j, for which $r_j < r_{j+1}$. Let $a_n = 2$, in the segment

$$\ldots(3)_{r_j}, 2, -2, (-3)_{r_{j+1}}\ldots$$

Clearly

$$|\lambda_n| < |\mu_n| < \frac{\sqrt{5} - 1}{2},$$

and the result holds by the method of (i).

<u>*COROLLARY*</u>.

$$k^+\left(\frac{\sqrt{5} + 1}{2}\right) = \frac{3\sqrt{5} - 5}{10}.$$

<u>*PROOF*</u>. This follows immediately from Lemma 8.4, and Theorems 8.1, and 8.2.

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