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TIME DELAY

IN

NUCLEAR SCATTERING THEORY

by

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A thesis submitted in accordance with the requirements of the Degree of Doctor of Philosophy.

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SUMMARY

A wave packet formulation of non-relativistic scattering is used to discuss the definition and measurement of time delays in collision events. The incident particles are represented by an incoherent mixture of Lorentzian wave packets, while the scatterer is characterized by the Siegert-Humblet-Rosenfeld expansion of the collision matrix. The possibility of determining the time delay depends in general on the spatial localization of the incident particles which for a pure state is given by the energy uncertainty of the beam. For mixed states it is necessary to distinguish classical and quantal effects. The dependence of the shape of the scattered packet, and hence of the cross section and time delay, on the initial beam characteristics is calculated.

In ordinary experiments poor energy resolution is classical in nature whereas the quantal energy uncertainty associated with the finite extension of the incident wave packets is negligible. Consequently the time delay is indeterminate. However, it is shown that when the quantal uncertainty is large, time-dependent scattering experiments, which are complementary to the usual energy-dependent

experiments, are possible in principle. For these experiments the time delay is determinate and a sequential description of the scattering is valid. We suggest time dependent experiments which are within the range of present experimental techniques, and which should provide an experimental confirmation of the theory.

The latter part of the thesis is devoted to a detailed study of the temporal interpretation of the optical model of nuclear reactions. By considering the complementary time-dependent picture, we are able to give a justification of the energy averages introduced in the definition of the model. It is shown that the formal definition is physically meaningful only if, in the context of good time resolution, the particles are scattered in two distinct pulses, that is events with intermediate time delays are insignificant. The coherence of the shape and compound elastic scattering is discussed and correlation functions, which express the relationship between the scattering amplitude and the time spectrum of the scattered particles are defined.

In the final chapter, an experimental method of resolving direct and compound inelastic scattering, based on the parity rule, is proposed. Some illustrative calculations are described briefly.

STATEMENT

This thesis contains no material which has been accepted for the award of any other degree and to the best of my knowledge and belief, contains no material previously published or written by another person except where reference is made in the text of the thesis.

LINDSAY DODD

14/9/64

PREFACE

The research reported in this thesis was carried out during the years 1961-64 in the Department of Mathematical Physics of the University of Adelaide, under the supervision of Dr. I.E. McCarthy.

I wish to thank Dr. McCarthy for his enthusiastic guidance and help. I am grateful to Professor C.A. Hurst for many valuable discussions and Professor H.S. Green for his constant encouragement.

I would like to acknowledge the support of the Commonwealth Government; I held a Commonwealth Post Graduate Award in 1961 and a C.S.I.R.O. Senior Post Graduate Studentship in 1962-63.

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CHAPTER 1

INTRODUCTION.

The aim of this work is to clarify the physical meaning of the averaging procedures used in deriving the optical[1-3] and direct reaction models of nuclear reactions from the formal, exact scattering theory[4-7]. To carry out this aim it is necessary to give a more general discussion of the definition and observation of time delays in collision processes than is customary and in particular to develop a scattering formalism which includes a more realistic description of the incident beam.

From the first, the various phenomenological models of nuclear collisions have rested heavily on the concept of time delay. Bohr's arguments[8,9] for the hypothesis of the independence of formation and decay of the compound nucleus were based on a sequential description of the scattering process. More recently, characteristic delay times have been used in the interpretation of the optical model and direct reactions[10,22]. During the last few years there has been considerable interest in the possibility of measuring delay times in nuclear reactions

in order to distinguish different modes of interaction.[12]

Although the language of delay times, lifetimes and collision times has been both popular and useful, there has been little attempt to relate this semi-classical type of description to the mathematical formalism of the quantum theory of scattering.

The usual discussion of time delay is based on the definition of Wigner and Eisenbud[13]:

$$\Delta t = - i\hbar \frac{d}{dE} \ln S(E) \quad , \qquad (1.1)$$

Δt being the time delay of the scattered particle and $S(E)$ the relevant S-matrix element. This definition has a practical difficulty. According to the relation (1.1) a time delay is defined for a precise energy but it is clear from the uncertainty relation for energy and time that if the energy of the scattering system is definite, such a time delay is **indeterminate.** The definition is incomplete in that it does not indicate the experimental context for which delay times are observable and hence meaningful.

In order to introduce some of the considerations which lead to the view of time delay taken in this thesis, let us consider how the time delay in the one dimensional scattering experiment of figure one would be measured.

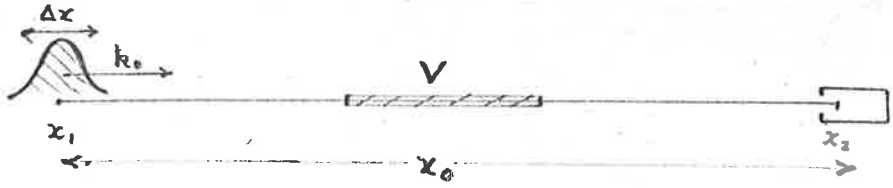

Fig. 1

Classically, a time origin is defined by determining the time that the incident particle passes through a detector at x_1, before interacting with the system represented by V. Another detector is placed at x_2 which measures the time of arrival τ of the particle after the interaction. The experiment is repeated with the interaction removed so that the particle propagates freely between x_1 and x_2. The result of this measurement is τ_0 and the time delay $\tau - \tau_0$.

Quantally, the influence of the measurement at x_1 cannot be made negligible and this determination of the time delay must be modified so that the experiment is consistent with the uncertainty relations.

Suppose the incident particle is prepared at t_0 in a state $|\psi_0\rangle$ which is localized about x_1, having a spread Δx along the x-axis with $\Delta x / x_0 \ll 1$. From the uncertainty relation for momentum and position $(\Delta x \Delta p \sim \hbar)$ the minimum uncertainty in the wave number k_0 is Δk with

$\Delta k \sim \frac{1}{\Delta x}$. We assume here that the state preparation is such that an incident particle can be represented by a single state vector and that the optimum definition of wave number is obtained. Later the consequences of dropping this assumption will be examined. If a scattering experiment is to be feasible $\frac{\Delta k}{k_0}$ must be much smaller than one, so that the energy and momentum of the incident particle are well defined. The time taken for the particle to arrive at x_2 in the case where the interaction is removed is

$$\tau_0 = \frac{m}{\hbar} \frac{x_0 \pm \frac{1}{2}\Delta x}{k_0 \pm \frac{1}{2}\Delta k}$$

with m the mass of the incident particle. Thus the uncertainty in the time taken is

$$\Delta\tau \sim \frac{m x_0}{\hbar k_0}\left(\frac{\Delta x}{x_0} + \frac{\Delta k}{k_0}\right). \qquad (1.2)$$

Any time delay smaller than $\Delta\tau$ is unobservable.

Usually it is assumed that the wave packet state $|\psi_0\rangle$ does not spread during its propagation from x_1 to x_2. This requires

$$\frac{\Delta k}{k_0} << \frac{\Delta x}{x_0}$$

or

$$(\Delta k)^2 = \frac{1}{N}\frac{k_0}{x_0} \qquad N >> 1 \qquad (1.3)$$

with the use of the uncertainty relation. We assume that

this condition of negligible natural spreading is satisfied by the experiments discussed here as it simplifies the calculations in chapters two and three without harming our understanding of the effect of the interaction on the state $|\psi_0\rangle$. With this condition, relation (1.2) can be written as

$$\Delta\tau \sim \frac{\hbar}{2E}\,\frac{k_0}{\Delta k}$$

$$= T_0 (N k_0 x_0)^{\frac{1}{2}} \qquad (1.4)$$

T_0 being the time taken for a classical particle with energy E_0 to travel a distance $\frac{2}{k_0}$. From (1.4) we see that $\Delta\tau$ can be made to assume its minimum value by taking Δk as large as possible. The upper limit on the magnitude of Δk is fixed by the relation (1.3). It would seem possible to achieve any degree of accuracy by making x_0 small enough but it is an advantage to use later the asymptotic form of the scattering state wave function and for this purpose x_0 must be taken much larger than the range of the interaction. Indeed if x_0 is too small, the process can hardly be considered as a scattering event.

A numerical example illustrates the relation (1.4). The scattering of a particle from a nucleus can be divided into two types with respect to a time interval T_c and a

corresponding energy interval Γ ($\sim \frac{\hbar}{T_c}$). If the scattering amplitude as a function of energy varies rapidly over the energy interval Γ, the scattering is said to be compound, if the amplitude varies slowly over this interval, it is called direct. T_c is several order of magnitude greater than T_0. For an incident proton whose energy is 20MeV, $T_0 \sim 10^{-22}$ secs. whereas $T_c \sim 10^{-17}$ secs. With $N = 10^5$ and $x_0 = 1$ metre, $\Delta\tau$ is approximately 10^{-12} secs. Even if the detecting apparatus at x_1 is capable of measuring time to any order of precision the time delay associated with compound scattering by equation (1.1) is **indeterminate,** and certainly on the basis of time measurements cannot be distinguished from the direct scattering. However, in principle by arranging the experiment such that N took its minimum value and by making x_0 small enough, the time delays would be measurable.

In general the localization of the state $|\psi_0\rangle$ determines whether there is in fact any discernible time delay. The state $|\psi_0\rangle$ has a certain spread in energy ΔE, which we call the <u>quantal uncertainty</u>. Normal scattering experiments are characterized by $\Delta E \ll \Gamma$ where Γ is the average width of the fluctuations in energy of the scattering amplitude about the energy of interest. The time delay corresponding to Γ is then unobservable since

$$\Delta\tau \gg \hbar/\Gamma \quad . \tag{1.5}$$

In experiments, real [6], or hypothetical as in the example, for which $\Delta E \ll \Gamma$ and

$$\Delta\tau \ll \hbar/\Gamma \tag{1.6}$$

the time delay is observable.

We regard conditions (1.5) and (1.6) as defining complementary scattering experiments, 'complementary' having the special meaning emphasized by Bohr. A full understanding of the quantum scattering process is based on two complementary classical pictures, one associated with good time resolution, the other with good energy resolution.

Experiments which satisfy conditions intermediate between the limiting cases (1.5) and (1.6) are possible and one of the aims of this thesis is to present a scattering formalism which is sufficiently general to show the relationship between the observable time delay and the quantal uncertainty in a given experiment.

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[6] A real scattering experiment for which condition (1.6) holds has been proposed by the author and I.E. McCarthy and is discussed in chapter 4 and reference (14).

By considering the average time delay in the emergence of a wave packet in a hypothetical time dependent experiment complementary to the usual energy dependent experiment, we can associate a characteristic time delay with the scatterer.

Let us review very briefly the dependence of the usual scattering theories on the description of the state $|\psi_0\rangle$. The time independent treatment[15] in which $|\psi_0\rangle$ is a plane wave represents an idealized physical situation since properties of the beam such as collimation and energy resolution are disregarded. The time dependent treatments fall into two categories, formal theories[16,17] and wave packet methods[18]. The former have the advantage of being easier to apply in complicated situations while the latter have greater appeal to physical insight. Although both have more realistic initial conditions than the stationary treatment, in the form in which they are generally presented they give the same results, since the parameters describing the beam are taken with extreme values. In the formal theories various limits are taken. In the wave packet methods, the initial wave trains are taken much longer than the region of interaction yet much shorter than the

microscopic distance of the scattering centre to the detector; they approach the plane waves of the stationary theory.

Previous presentations, with few exceptions, have been directed toward showing that the sequential description provided by the wave packet method reduces under conditions (1.3) and (1.5) to the stationary picture of ingoing plane waves and outgoing spherical waves at infinity. However, as will be shown in chapters 2 and 3, the wave packet method is not equivalent to the stationary theory and is capable of giving quite different results when the condition (1.6) applies.

Mention should be made of Sasakawa[21], whose ideas are similar to those presented in the early chapters of this thesis. He shows with a different mathematical technique that the cross sections are modified when the finite length of the incident wave packet is taken into account. The work reported here was carried out without the knowledge of Sasakawa's treatment and in the author's opinion is physically more transparent and has greater generality. In his discussion of the optical model, Sasakawa has taken a special form for the collision matrix which obscures the meaning of the averaging procedure.

Also Austern[22] has given a review of general wave packet scattering with applications to nuclear reactions which forms a good introduction to this thesis.

It is tempting to attribute the poor energy resolution of experimental beams to the finite length of the incident wave packets but in actual scattering experiments the energy uncertainty of the incident beam is not accounted for by the quantal uncertainty of the state $|\psi_0\rangle$; the limits on the energy resolution of the incident particles are much wider than those imposed by the uncertainty principle. For example some particles may be moving faster than others because they were accelerated a little more by the accelerator. In fact the beam is not prepared in a single state but must be described by a mixture of wave packet states. The fast and slow beams in principle could be separated by a magnetic spectrometer. This type of uncertainty will be called <u>the classical uncertainty</u> in contrast to the quantal uncertainty which results from the attempt to define the time of arrival of a particle at a point.

Chapters 2 to 5 give in detail the consequences of introducing the classical and quantal uncertainty in the description of the incident beam and are preparatory to the discussion of the optical model in chapter 6.

In chapter 2, the scattering of a wave packet from a potential is considered. The method of Chew and Low[18] leads to the ordinary time independent results for small values of the quantal uncertainty; the shape of the wave packet is undistorted by the potential and there is no discernible time delay. Calculation of the scattering for other values of the quantal uncertainty is facilitated by splitting the scattering amplitude into a part which varies slowly with energy, the amplitude for potential scattering, and a fluctuating part, the amplitude for resonant scattering. A number of different ways of making this separation are possible but for reasons stated later the method of Siegert, Humblet and Rosenfeld[6,32] is chosen. With the aid of a Lorentzian[6] shape for the incident packet and the condition (1.3) of negligible natural spreading, the shape of the outgoing packet, which depends on the energy variation of the

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6 Of course the shape of the wave packet depends on the nature of the state preparation and details could be altered by choosing a different form factor. The artificiality of the Lorentzian shape is compensated by the ease of calculation and the physical transparency of the results, cf. Newton[23] and Sasakawa[10]. Newton discusses the case of an isolated resonance using a Gaussian factor. Sasakawa's results are complicated by the fact that he uses a rectangular factor, which has a sharp cut off, giving rise to transients. The essential results, however, are independent of the precise shape of the packet.

scattering amplitude is easily calculated. When the scattering amplitude has an isolated resonance, the collision function is dominated by a single pole near the energy of interest. For this simple case the scattering cross section is found as a function of the quantal uncertainty, the energy of the incident particle and the resonance parameters. The observable time delay is derived as a function of the quantal uncertainty and this is compared with the definition (1.1). The interference between the potential and resonant scattering is also calculated.

In chapter 3, the discussion is extended to the case of wave packet scattering from many overlapping resonant states, and the cross section and the shape of the scattered packet derived. It is shown that the time delay depends on the spacing of the resonances as well as their widths and the magnitude of the scattered packet depends on the correlation over the residues associated with the poles of the collision function.

A high energy approximation is considered for which the level spacing is much less than the level width. When there is complete correlation between the residues, there is no time delay and the cross section is independent of the quantal uncertainty. Generally the cross section

falls off as the quantal uncertainty is increased, but the rate of decrease depends on the degree of correlation between distant levels.

The classical uncertainty of the beam is introduced in chapter 4. A beam with both classical and quantal uncertainty is described by a collection of wave packets with different mean wave numbers. Using the density matrix formalism[24], we show that the classical uncertainty implies an average of the cross section with respect to energy. It is also shown that the distinction between classical and quantal uncertainty breaks down if the incident wave packets are not spatially separated by the device which produces them. It is decided that the poor resolution of actual scattering experiments is due to classical rather than quantal uncertainty but an experiment is proposed which would test some of the conclusions of chapters 2 and 3.

In chapter 5, we return to the problem of making sequential descriptions in terms of micro-events of experiments for which time intervals are not defined. This type of description which is valid only in the complementary sense, has lead to difficulties in the interpretations of scattering experiments. The question

of whether the amplitudes for scattering processes which have different characteristic times can be added incoherently is answered with the help of the preceding discussion. The compound nucleus provides an instructive example.

Chapter six is devoted to a detailed study of the temporal interpretation of the optical model.

It is well known that the optical model describes some sort of average behaviour of the scattering, the incident beam having a spread in energy, whereas the resonance theories describe the scattering of beams of perfect energy resolution. The single particle type of description associated with the optical model has been shown to be compatible with the resonance theories in a number of theoretical papers[3,4,7,25] by making the correspondence that the scattering amplitude predicted by the optical model be a certain average over energy of the exact scattering amplitude.[6] It is then assumed

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[6] The following arguments are also relevant to inelastic channels. The direct reaction model may be defined by requiring that it give the averages of the appropriate elements of the collision matrix (e.g. reference (7)). In order to keep the formalism as simple as possible, we discuss only elastic channels.

that the potential defined by this procedure is similar to the potential abstracted from experiment.

Our concern will be the explanation of the energy averages used in the formal definition but not the equivalence of the phenomenological and theoretical potentials, the complete justification of which requires a knowledge of the dynamics of the nuclear many-body system. Moreover, we will show that the formal definition of the model leads to a reasonable picture of the part played by the optical potential.

The main question initiating this study was: can the averages with respect to energy used in the definition of the model be introduced at the beginning of the scattering calculation by assuming a more realistic characterization of the beam? In this way the averages which must be added to the ideal resolution theory would arise naturally from the more accurate description of the experimental situation. Specifically, does the energy average of the scattering amplitude follow from the wave packet nature of the beam?

Friedman and Weisskopf[10] have given a partial answer to this question which in some respects is misleading. Their interpretation is based on a description of the scattering using a wave packet formalism and the notion

of time delay. They try to show that the undelayed part of the scattered packet (shape elastic scattering) corresponds to the averaged scattering amplitude. Hayakawa et al.[11] and Namiki[26,27] have extended their approach. A feature common to their arguments is the introduction of short wave packets to describe the progress of the incident particles when they interact with the nucleus.

In section 6.1, we examine the arguments of Friedman and Weisskopf. It is found that their discussion is incomplete, especially for the case of medium energies, where the resonance levels overlap. The attractive idea of identifying the averaging of the scattering amplitude with the finite length of the wave packet is rejected since the beams of actual experiments have negligible quantal uncertainty. The time dependent picture with large quantal uncertainties must be regarded as interpretative and referring to hypothetical rather than real experiments.

The experimental quantities measured are average cross sections. Since an energy average of the cross section corresponds to classical uncertainty, the question of why the scattering amplitude is averaged is considered in section 6.2.

In agreement with the semi-classical picture of the optical model as a model for which the time delay is small, we define the shape elastic scattering as that part of the scattering which is undelayed in a complementary time dependent experiment. A definition of the optical potential which is consistent with the uncertainty principle and which also has a straightforward physical meaning, is that the potential predict the correct shape elastic scattering in a time dependent experiment. Our task then comes down to demonstrating that this physical definition is equivalent to the formal definition in terms of the averaged amplitude.

The formalism developed in the previous chapters leads to the conclusion that these two definitions are equivalent only if certain dynamical assumptions about the cooperative behaviour of the resonances in the scattering amplitude are made. In particular, the fact that the optical potential of the formal theory gives correctly the shape elastic scattering is a result emphasized by Brown[28] that the scattering amplitude when averaged over an energy ΔE, a fraction of the width of a giant resonance, should be a smooth function of energy over several ΔE. This implies that in a complementary time-dependent experiment, the scattered particles can be

separated into two distinct groups; there are no particles which suffer an intermediate time delay.

In section 6.3, we consider the question of the incoherence of the shape elastic and fluctuation scattering. Some arguments which rely on the use of characteristic times are criticised and it is shown that dynamical condition referred to in the previous paragraph must be assumed if the two types of scattering are not to interfere.

In order to make the discussion of the fluctuations in the scattering amplitude more quantitative, correlation functions are introduced. An experiment has been suggested by Eisberg, Yennie and Wilkinson[29] which measures the relative amount of shape elastic and compound elastic scattering. Although this experiment is often cited as measuring a time delay, it actually defines an energy interval which is the reciprocal of the time delay. The information is obtained from the energy spectrum of bremstrahlung from the elastic scattering of charged particles, which is expressed in terms of a correlation function similar to the one which we have defined. We give a time dependent interpretation of their experiment by deriving a very interesting expression for the correlation function of

the scattering amplitude; it is shown that the correlation function for the scattering amplitude is approximately the Fourier cosine transform of the time spectrum in the complementary experiment. This relation allows one to translate statements about energy fluctuations into the language of time delay.

It is difficult to carry further this discussion of time delay and the problem of separating different modes of scattering without making far-reaching assumptions about the structure of the scattering amplitude or more fundamentally about the statistical properties of the positions of the poles of the collision matrix and the residues associated with them. There is a need for experimental techniques which will allow an empirical analysis of the energy correlations in the scattering amplitude to be devised. It is expected that the bremstrahlung experiment will be valuable, and that the correlation function obtained from this experiment may be used to test the assumption emphasized here that multiple scattering processes which have intermediate time delays are negligible.

Finally in chapter seven of this thesis, we suggest a method[30] based on the parity rule[31] of

assessing the relative contribution of the compound nucleus and direct reaction models to inelastic scattering. The results of this experiment should help to answer the question of whether nuclear scattering processes at intermediate energies proceed via two distinct mechanisms.

CHAPTER 2

2.1 GENERAL METHOD

In this section the formalism for the non-relativistic scattering of a wave packet from a potential in three dimensions is introduced. The wave packet describes particles whose position and momentum are determined subject to the limitations imposed by the uncertainty principle. By describing the propagation of the wave packet, we obtain a sequential description of the scattering. It is shown in section 2.2 that this method gives the same results as the time independent theory provided certain conditions are satisfied. The discussion of sections 2.1 and 2.2 is essentially the same as that of Merzbacher[19] which is based on the lecture notes of Chew and Low[18].

Consider the wave packet states $\Psi_{\underline{k}_0}$ which describe particles localized in the region R, macroscopic distance r_0 from the scattering centre (located at the origin) and moving towards the scattering centre with mean momentum $\underline{k}_0$, as in figure 2.

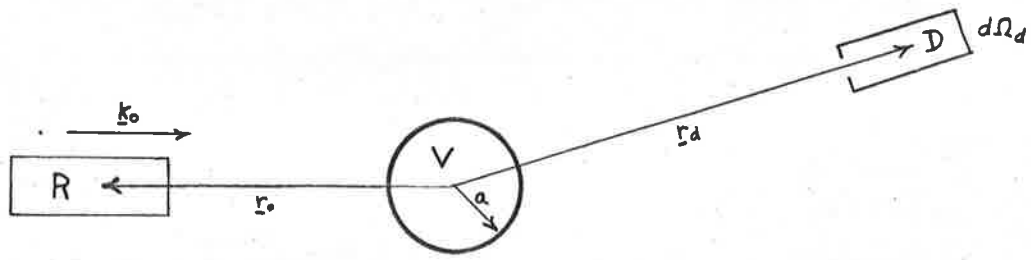


Fig. 2

This is the initial situation. The problem is to find what has happened to the wave packet at a later time when it has passed the scattering centre and reaches the detector. Now the time behaviour of the eigenstates of the total Hamiltonian of the system is known, so that if the wave packet can be expressed as a combination of these states, its temporal behaviour can be derived; the initial state $\Psi_{\underline{k}_0}$ must be expressed as

$$\Psi_{\underline{k}_0} = \sum_i \omega_i(\underline{k}_0)\,\phi_i + \int \omega(\underline{k}_0, \underline{k})\,\xi_{\underline{k}}\,d\underline{k} \qquad (2.1)$$

where the scattering state $\xi_{\underline{k}}$ corresponds to an eigenvalue in the continuous part of the spectrum of eigenvalues of the total Hamiltonian H, and the bound states ϕ_i correspond to the discrete eigenvalues.

The states ϕ_i and $\xi_{\underline{k}}$ satisfy the equations

$$H\phi_i = E_i\phi_i \qquad E_i < 0$$

$$H\xi_{\underline{k}} = E_{\underline{k}}\xi_{\underline{k}} \qquad E_{\underline{k}} > 0 \ . \tag{2.2}$$

where H is given by $H = T+V$. T is the operator for the kinetic energy of the incident particle and V the potential which is assumed to vanish for $|r| > a$. Although the functions $\xi_{\underline{k}}$ are not square integrable they may be normalized such that

$$\int \xi_{\underline{k}}\,\bar{\xi}_{\underline{k}'}\,d\underline{r} = \delta(\underline{k}-\underline{k}') \tag{2.3}$$

and with the states ϕ_i normalized such that

$$\int \phi_i\,\bar{\phi}_j\,d\underline{r} = \delta_{ij} \tag{2.4}$$

they constitute a complete set, which makes the expansion (2.1) permissible.

The coefficients in (2.1) are found with the aid of the orthogonality conditions (2.3) and (2.4),

$$\omega_i(\underline{k}_0) = \int \psi_{\underline{k}_0}\,\bar{\phi}_i\,d\underline{r} \tag{2.5}$$

$$\omega(\underline{k}_0,\underline{k}) = \int \psi_{\underline{k}_0}\,\bar{\xi}_{\underline{k}}\,d\underline{r} \tag{2.6}$$

The time behaviour of the states ϕ_i, $\xi_{\underline{k}}$ is given by

$$\xi_{\underline{k}}(t) = \xi_{\underline{k}}\, e^{-\frac{i}{\hbar}E_{\underline{k}}t} \quad ,$$
$$\phi_i(t) = \phi_i\, e^{-\frac{i}{\hbar}E_i t} \quad . \tag{2.7}$$

The wave packet at time t is then

$$\Psi_{\underline{k}_0}(t) = \sum_i \omega_i(k_0)\,\phi_i\, e^{-\frac{i}{\hbar}E_i t} + \int \omega(\underline{k}_0,\underline{k})\,\xi_{\underline{k}}\, e^{-\frac{i}{\hbar}E_{\underline{k}}t}\, d\underline{k} \tag{2.8}$$

In particular take the initial packet as

$$\Psi_{\underline{k}_0} = (2\pi)^{-\frac{3}{2}} \int \Delta(\underline{k})\, e^{i\underline{k}\cdot(\underline{r}-\underline{r}_0)}\, d\underline{k} \tag{2.9}$$

The weight function $\Delta(\underline{k})$ has mean value $\underline{k}_0$ and is negligible for values

$$k_j > k_{0j} + \delta k_j \quad , \quad k_j < k_{0j} - \delta k_j \quad (j = 1,2,3). \tag{2.10}$$

We call $|\underline{\delta k}|$ the quantal uncertainty; it is a measure of the difference of the physical initial state from the idealized initial state $e^{i\underline{k}_0\cdot\underline{r}}$ of the stationary theory; for small quantal uncertainty the wave packet state tends to a plane wave. If the Fourier transform of equation (2.9) is taken to obtain the wave function in the momentum representation it is

found that the wave packet has spread $\underline{\delta k}$ in momentum space, so that, by the uncertainty relation, it has spread $\underline{\delta r}$ in coordinate space with

$$\delta k_j \, \delta r_j \sim 1 \quad (j = 1,2,3) \, . \tag{2.11}$$

Thus equation (2.9) defines a wave packet which is localized about $\underline{r}_0$ and has mean momentum $\underline{k}_0$.

Next we evaluate the coefficients (2.5) and (2.6). The overlap integral in (2.5) is negligible. The wave packet vanishes outside of the region R and in R the wave function ϕ_i will be extremely small since ϕ_i is a bound state and concentrated in the region of the origin whereas r_0 is a macroscopic distance. The state $\xi_{\underline{k}}$ is

$$\xi_{\underline{k}}(\underline{r}) = (2\pi)^{-\frac{3}{2}} \left[e^{i\underline{k}\cdot\underline{r}} + F(\underline{k},\underline{r}) \right] \tag{2.12}$$

The first term represents an ingoing plane wave and the second tends asymptotically to an outgoing spherical wave; for large r the second term is proportional to e^{ikr}/r . Then

$$\begin{aligned} \omega(\underline{k}_0,\underline{k}) &= \int \psi_{\underline{k}_0} (2\pi)^{-\frac{3}{2}} \left[e^{i\underline{k}\cdot\underline{r}} + F(\underline{k},\underline{r}) \right]^* d\underline{r} \\ &= \frac{1}{(2\pi)^{\frac{3}{2}}} \int \psi_{\underline{k}_0} e^{-i\underline{k}\cdot\underline{r}} d\underline{r} + \frac{1}{(2\pi)^{\frac{3}{2}}} \int \psi_{\underline{k}_0} \overline{F}(\underline{k},\underline{r}) d\underline{r} \end{aligned} \tag{2.13}$$

Consider the plane wave part first,

$$\frac{1}{(2\pi)^3}\iint \Delta(\underline{k}')\, e^{i\underline{k}'.(\underline{r}-\underline{r}_0)}\, d\underline{k}'\, e^{-i\underline{k}.\underline{r}}\, d\underline{r}$$

$$= \Delta(\underline{k})\, e^{-i\underline{k}.\underline{r}_0} \quad . \tag{2.14}$$

We wish to show that the second integral in (2.13) is negligible. If it is assumed that the potential is central then it is well known that $F(\underline{k},\underline{r})$ can be written for large r (taking the plane wave incident along the z axis in spherical polar coordinates) as

$$F(\underline{k},\underline{r}) \sim \sum_{\ell}(2\ell+1)\, i^{\ell} P_{\ell}(\cos\theta)\, A(\ell,k)\, e^{ikr}/r$$

$$\equiv f(k,\theta)\, e^{ikr}/r \tag{2.15}$$

Since $\psi_{\underline{k}_0}$ vanishes outside the region R, the range of the integral may be restricted to R, and since this region is a macroscopic distance from the scatterer, the asymptotic form for $F(\underline{k},\underline{r})$ may be used. From the expression (2.15) it can be seen that $f(k,\theta)$ is a slowly varying function of θ in the region R, R subtending a small angle $\Delta\theta \sim \frac{d}{r_0}$ at the origin where d is the width of the collimating slit. Thus the ingoing spherical wave of $\bar{F}(\underline{k},\underline{r})$ may be represented in R by a plane wave,

$$\bar{F}(\underline{k},\underline{r}) \sim \frac{\bar{f}(k,\pi)}{r_0}\, e^{i\underline{k}.\underline{r}} \quad \text{in } R\,, \tag{2.16}$$

with the propagation vector $\underline{k}$ having the same direc-

tion as $\underline{k}$. Then the integral in question becomes

$$\frac{\bar{f}(k,\pi)}{r_0} \frac{1}{(2\pi)^3} \int \Delta(\underline{k}') e^{-i\underline{k}'\cdot\underline{r}_0} \int_R e^{i(\underline{k}'+\underline{k})\cdot\underline{r}} \, d\underline{r} \, d\underline{k}'$$

The integral over R is negligible unless $\underline{k}'$ lies in the range

$$-k_j - \delta k_j \lesssim k_j' \lesssim -k_j + \delta k_j \qquad (j = 1,2,3)$$

but in this range $\Delta(\underline{k}')$ vanishes since $\underline{k} \parallel \underline{k}_0$ and $|\underline{k}_0| > |\delta \underline{k}|$.

This physical argument differs from the one which uses Green's functions and assumes that the scattering amplitude is a slowly varying function of k. The reader is referred to Merzbacher[19] or Goldberger[34] for the detailed proof.

We see then that at t = 0 the physical wave packet states, consisting of eigenfunctions of the operator T may be expanded in terms of the ideal scattering states of the total Hamiltonian H = T + V, and with insignificant error, the coefficient in the expansion in terms of the eigenstates of T and the coefficients in the expansion in terms of the eigenstates of H are identical.

From (2.8), the wave packet at time t is

$$\psi_{\underline{k}_0}(\underline{r},t) = \int \Delta(\underline{k}) \exp\left(-i\underline{k}.\underline{r}_0 - \frac{i\hbar k^2 t}{2m}\right) \xi_{\underline{k}}(\underline{r})\, d^3k \,. \quad (2.17)$$

The probability of detection at time t is

$$|\psi_{\underline{k}_0}(\underline{r}_d,t)|^2 r_d^2\, d\Omega_d \quad ,$$

$\underline{r}_d$ being the position vector of the detecting apparatus and $d\Omega_d$ the angle subtended by the detector. The total probability of detection is therefore

$$\frac{d\sigma}{d\Omega_d} = v_0 r_d^2 \int_{-\infty}^{+\infty} |\psi_{\underline{k}_0}(\underline{r}_d,t)|^2 dt \quad (2.18)$$

where $\psi_{\underline{k}_0}(\underline{r}_d,t)$ is given by equation (2.17), after appropriate normalization.

2.2 SCATTERING FOR SMALL QUANTAL UNCERTAINTY AND SMOOTH VARIATION OF THE SCATTERING AMPLITUDE.

In this section we wish to show that with the condition that the quantal uncertainty is small compared with the width of the fluctuations in the scattering amplitude with energy, the evaluation of (2.17) leads to the same result for the differential cross section as given by the usual time independent theory. The method due to Low[18] is worth repeating as it shows the assumptions which are implicit in the stationary treatment and provides a basis for the more complete discussion which follows in later sections.

To evaluate the integral (2.17) the first approximation made is to replace the energy term in the exponential by a term depending on k rather than k^2. This means that the spreading of the wave packet during the scattering is assumed to be negligible.

$$
\begin{aligned}
k^2 &= (\,(\underline{k}-\underline{k}_0) + \underline{k}_0)^2 \\
&= (\underline{k}-\underline{k}_0)^2 + k_0^2 + 2\underline{k}_0.(\underline{k}-\underline{k}_0) \\
&= 2\underline{k}.\underline{k}_0 - k_0^2 + (\underline{k}-\underline{k}_0)^2 \\
&\sim 2\underline{k}.\underline{k}_0 - k_0^2
\end{aligned}
$$

In order to drop the $(\underline{k}-\underline{k}_0)^2$ term, we require

$$\frac{(\underline{k}-\underline{k}_0)^2 T\hbar}{2m} << 1$$

with T the time taken for the particle to reach the detector $\left(T = \frac{2r_0 m}{\hbar k_0}\right)$,

i.e.
$$\frac{(\delta k)^2 r_0}{k_0} \ll 1 \quad . \qquad (2.19)$$

Compare this with equation (1.3) of the introduction. This condition can be easily satisfied without destroying the localization of the wave packet which was necessary for the arguments leading to the equation (2.17). As a numerical example consider the allowable spread in wave number for 9 MeV neutrons with r_0 = 10cms.

$$(\delta k)^2 \cdot \frac{10^{14}}{3} \cdot 4.5 = 10^{-2}$$

$$\delta k = 10^{-8} \text{ inverse fermis.}$$

This uncertainty in the initial momentum allows a localization in position of $\delta r \sim \frac{1}{\delta k} = 10^8$ fermis.
The wave packet is large compared with the dimensions of the scattering centre ($\sim$10 fermis) but small compared with the distance r_0. Thus the two conditions

$$\frac{\delta x}{r_0} \ll 1$$

$$\frac{(\delta k)^2}{k_0} \ll 1$$

are not incompatible (for this case), and it is easily seen by taking numerical examples that this is so in general for atomic and nuclear scattering experiments.

With the approximation (2.19) equation (2.17) becomes

$$\Psi_{\underline{k}_0}(\underline{r},t) = \int \Delta(\underline{k}) \exp\{-i\underline{k}.(\underline{r}_0+\underline{v}_0 t)+i\omega_0 t\}\, \xi_{\underline{k}}(r)\, d^3k \tag{2.20}$$

with $v_0 = \frac{\hbar k_0}{m}$, $\hbar\omega_0 = \frac{1}{2} m v_0^2$.

A second basic assumption is necessary:

The scattering amplitude $f(k,\theta)$ is a slowly varying function of k in comparison with the weight function $\Delta(\underline{k})$. In other words the quantal uncertainty must be much less than the average width of the fluctuations, denoted by Γ, of the scattering amplitude at the relevant energy, i.e.

$$\delta \ll \Gamma . \tag{2.21}$$

When this condition is satisfied, the scattering amplitude may be brought outside the integral in (2.20). For large r the form (2.12), with $F(\underline{k},\underline{r})$ given by equation (2.15), may be substituted in (2.20) and by comparison with the initial wave packet (2.9), the wave packet at time t is

$$\Psi_{\underline{k}_0}(\underline{r},t) = \Psi_{\underline{k}_0}(\underline{r}-\underline{v}_0 t, 0)\, e^{i\omega_0 t} + \frac{1}{(2\pi)^{\frac{3}{2}}} \frac{f(k_0,\theta)}{r} e^{i\omega_0 t}$$

$$\times \int \Delta(\underline{k}) \exp\{i[kr - \underline{k}.(\underline{r}_0+\underline{v}_0 t)]\}\, d^3k . \tag{2.22}$$

Furthermore in this approximation

$$kr \sim \underline{k}.\hat{\underline{k}}_o \, r$$

so that

$$\psi_{\underline{k}_o}(\underline{r},t) = \psi_{\underline{k}_o}(\underline{r}-\underline{v}_o t, 0)e^{i\omega_o t} + \frac{f(k_o,\theta)}{r}\psi_{\underline{k}_o}(r\hat{\underline{k}}_o-\underline{v}_o t, 0)e^{i\omega_o t}.$$

The first term represents the original wave packet displaced to the right of the scatterer and more important it propagates as if the scatterer were not present. The second term represents the scattering and is a replica of the original wave packet multiplied by the factor $f(k_o,\theta)/r$. It is important to note that with the above approximations the shape of the initial wave packet is unimportant and moreover, <u>there is no possibility of distinguishing the type of scattering mechanism by observing a time delay in the propagation of the scattered packet</u>. The time of flight of the wave packet is independent of the potential whose only effect is to modify the amplitude of the scattered packet through the function $f(k_o, \theta)$.

The probability of detection is from (2.23) and (2.18) (for the scattered packet only)

$$|f(k_0,\theta_d)|^2 \, d\Omega_d \int_{-\infty}^{\infty} |\Psi_{\underline{k}_0}(\tau \hat{\underline{k}}_0, 0)|^2 d\tau \qquad (2.24)$$

but $\int |\Psi_{\underline{k}_0}(\tau \hat{\underline{k}}_0, 0)|^2 d\tau$ is the number of incident particles per unit area.

Therefore $$\frac{d\sigma}{d\Omega_d} = |f(k_0, \theta_d)|^2 . \qquad (2.25)$$

This result for the differential cross section is identical with that derived by the usual stationary method.

2.3 SCATTERING FOR VARIABLE QUANTAL UNCERTAINTY

It was shown in the last section that with certain reasonable assumptions the wave packet formalism produced the same results as the stationary method. However, in this identification of the two methods, assumption (2.21) was essential. If this condition is not satisfied, the quantal uncertainty being comparable with the fluctuations in the scattering amplitude, the amplitude cannot be taken outside the integration in (2.20) and the integral cannot be evaluated in this way. In this circumstance, it is reasonable to expect that the shape of the scattered wave packet will depend very critically on the exact form of the amplitude and that the time behaviour of the packet will give information about the scattering mechanism, in contrast to the behaviour found in 2.2.

These considerations do not apply to the unscattered wave packet, which is obtained as before and since this wave packet is confined to small forward angles, it can be omitted from the expression for the wave function at the detector.

Our task then is to evaluate

$$\psi_{\underline{k}_0}(\underline{r},t) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int \Delta(\underline{k})\exp\left(-i\underline{k}.\underline{r}_0 - \frac{i\hbar k^2 t}{2m}\right)\frac{e^{ikr}}{r}f(k,\theta_r)\,d^3k$$

$$= \frac{1}{(2\pi)^{\frac{3}{2}}}\int \Delta_\theta(\theta_k)\Delta_k(k)\Delta_\phi(\phi)\exp\left(-ikr_0\cos\theta_k - \frac{ik^2\hbar t}{2m}\right) \times \frac{e^{ikr}}{r}f(k,\theta_r)\,d^3k \ .$$

Suppose that the spread of the angular parts $\Delta_\theta(\theta_k)$ and $\Delta_\phi(\phi_k)$ is much smaller than the spread in $\Delta_k(k)$. This means that the effects of collimation are disregarded and only the finite length of the wave packet is taken into account.

This is not, of course, a necessary restriction but it is reasonable from our point of view; the main concern is the time of arrival of a scattered particle, which is chiefly determined by the finite length of the incident packet, rather than distortion effects due to over-severe collimation.

Then

$$\Delta_\theta(\theta_k)\,\Delta_\phi(\phi_k)\,\Delta_k(k) \approx \frac{\delta(\theta_k-\pi)\,\Delta_k(k)}{2\pi\sin\theta_k\,k^2}$$

and

$$\psi_{\underline{k}_0}(\underline{r},t) = \frac{1}{(2\pi)^{\frac{1}{2}}}\int \Delta_k(k)\exp\left\{i\left(kr_0 - \frac{k^2\hbar t}{2m}\right)\right\}\frac{e^{ikr}}{r}f(k,\theta_r)\,dk \qquad (2.27)$$

Take again the condition of negligible spreading of the wave packet but this time the approximation

$$k \sim \frac{k_0}{2} + \frac{E}{\hbar V_0} \qquad (2.28)$$

is used in the exponential with the aim of changing the variable of integration to energy, yielding

$$\psi_{\underline{k}_0}(r,t) = \frac{1}{(2\pi)^{\frac{1}{2}}} \frac{e^{i\frac{k_0}{2}(r_0+r)}}{r} \int_0^\infty \Delta(E) \exp\left\{\frac{iE}{\hbar}\left(\frac{r_0+r}{V_0} - t\right)\right\} f(k,\theta)\, dE$$

To proceed we need an explicit expression for $f(k,\theta)$ as a function of energy. There are a number of possible 'resonance' expansions of the amplitude, including those of Kapur and Peierls[33], Wigner and Eisenbud[5], Feshbach[7], and Siegert and Humblet[6,32]. The expansion of Humblet and Rosenfeld[6] is used here. Their formalism has the advantage of having parameters which do not depend on the cut-off radius of the potential and the energy of the incident particles. The expansions of Kapur and Peierls, and Feshbach can also be used if it is assumed that the parameters in these expansions vary sufficiently slowly with energy.

A drawback of the method under consideration is that the amplitude must be expanded in partial waves, so that there is no easily stated relation between the resonances in different partial waves, although physically one expects such a relation to exist. Feshbach[36] has given a complex eigenvalue expansion of the scatter-

ing amplitude in which the resonances are associated with the poles of the amplitude, without making an expansion in partial waves. However, this expansion is best fitted for the discussion of isolated resonances because only a limited number of poles (resonances) can be made explicit; all resonances cannot be taken into account since the asymptotic behaviour of the poles for large energy and momenta is uncertain.

For the reason that we will be interested in the region of overlapping resonances, we will make use of the Humblet-Rosenfeld-Siegert expansion (hereafter called the HSR expansion) and consider only one partial wave at the time, thus sidestepping the problem of asymptotic behaviour for large angular momenta. However, it should be emphasized that since the essential consideration is the temporal description of the scattering process, any expansion of the amplitude that shows its energy dependence is appropriate.

The partial wave expansion of the scattering amplitude has been stated in equation (2.15). In the notation of Humblet and Rosenfeld (see equation (1.22) of reference 6)

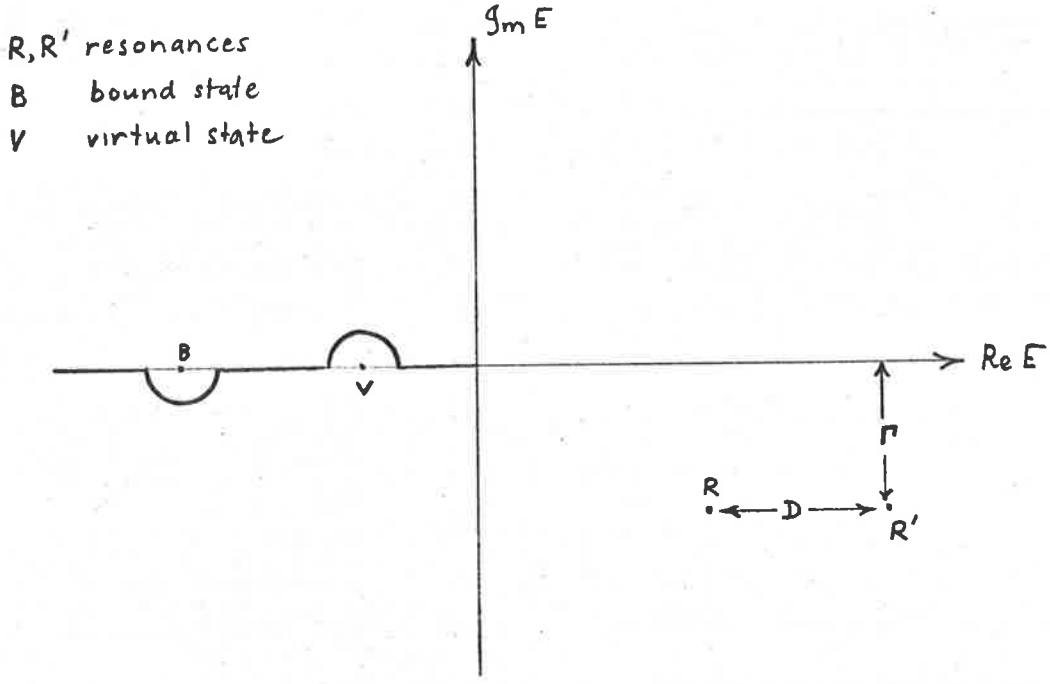
$$f(k,\theta) = \frac{1}{2ik}\sum_{\ell}(2\ell+1)P_{\ell}(\cos\theta)(U_{\ell}-1) \quad . \qquad (2.30)$$

The scattering matrix element U_ℓ is a meromorphic function of k and it is assumed to have only simple poles $k_{\ell n}$. The Mittag-Leffler theorem leads to the following expansion for U_ℓ in the complex energy plane with cut, (c.f. equation (2.15a) of reference 6),

$$U_\ell = 1 + \left(\frac{2mE}{\hbar^2}\right)^{\frac{1}{2}} \left[C_\ell(E) + \sum_n \frac{R_{\ell n}}{E - \epsilon_{\ell n} + \frac{1}{2} i \Gamma_{\ell n}} \right] . \quad (2.31)$$

Here $C_\ell(E)$ is a smoothly varying function of the energy and in the resonance term, $R_{\ell n}$ is complex, $\epsilon_{\ell n}$ and $\Gamma_{\ell n}$ real and $R_{\ell n}$, $\Gamma_{\ell n}$, $\epsilon_{\ell n}$ are independent of energy. Humblet and Rosenfeld show that the poles of U_ℓ in the complex k plane are restricted to the lower half plane and are symmetrical about the imaginary k-axis with the proviso that poles corresponding to bound states lie on the imaginary k-axis in the upper half plane. The corresponding positions of the poles in the complex E-plane with cut from origin to large negative real E are shown in figure 3.

Fig. 3: Singularities of collision function. 2.19



The coefficient $R_{\ell n}$ depends on the residue at the pole at $\epsilon_{\ell n} - \frac{1}{2} i \Gamma_{\ell n} = E_{\ell n}$. The general limits on the positions of the poles as discussed above are a result of a completeness relation; the exact positions $\epsilon_{\ell n} - \frac{1}{2} i \Gamma_{\ell n}$ of the poles, and the residues $R_{\ell n}$ at these poles are determined by the dynamics, that is by the exact form of the Hamiltonian being considered.+

For nuclear scattering at low energies the average spacing between the levels, denoted by D, is much

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+ Nussenzveig[35] has plotted the positions of the poles for the case of the square well.

greater than the average level width, denoted by Γ In this case it is sufficient to consider only one or two terms in the expansion and the scattering may be fitted with a few parameters describing the nearest resonances to the energy in question. At higher energies the poles are closer together and $\Gamma \gg D$. Too many parameters are needed to use the resonance expansion directly for the fitting of experimental data. It is for this reason that models such as the optical model are important to fit gross properties of the overlapping resonances such as the strength function.

With the expansion (2.31), the wave packet (2.29) becomes

$$\Psi_{\underline{k}_0}(\underline{r},t) = K \sum_\ell (2\ell+1) P_\ell(\cos\theta) \int_0^\infty \Delta(E) \exp(iEX) \left(\frac{U_\ell - 1}{2ik} \right) dE$$

$$= K \sum_\ell (2\ell+1) P_\ell(\cos\theta) \int_0^\infty \Delta(E) \exp(iEX) \left[C_\ell(E) + \sum_n \frac{R_{\ell n}}{E - E_{\ell n}} \right] dE \quad (2.32)$$

with $K = \frac{1}{(2\pi)^{\frac{1}{2}}} \cdot \frac{\exp\{i k_0 (r_0 + r)\}}{2ir}$ so that $K\bar{K} = \frac{1}{8\pi r^2}$ (2.33)

and

$$X = \frac{1}{\hbar} \left\{ \frac{r_0 + r}{v_0} - t \right\}. \quad (2.34)$$

We now split the scattered wave packet into a resonant wave packet associated with the resonance terms and a non-resonant packet in the following way,

$$\psi_P(\underline{r},t) = K\sum_\ell (2\ell+1) P_\ell(\cos\theta) \int_0^\infty \Delta(E) \exp(iEX)\, C_\ell(E)\, dE, \qquad (2.35)$$

$$\psi_R(\underline{r},t) = K\sum_\ell (2\ell+1) P_\ell(\cos\theta) \int_0^\infty \Delta(E) \exp(iEX) \sum_n \frac{R_{\ell n}}{E - E_{\ell n}}\, dE, \qquad (2.36)$$

$$\psi_{\underline{k}_0} = \psi_P + \psi_R \quad . \qquad (2.37)$$

Also let us assume a Lorentzian weight factor

$$\Delta(E_0, \delta; E) = \frac{\delta}{\pi\{(E-E_0)^2 + \delta^2\}} \qquad (2.38)$$

with δ, the quantal uncertainty, the spread of $\Delta(E)$.

2.4 EVALUATION OF THE NON-RESONANT SCATTERED PACKET

The non-resonant packet is

$$\psi_p(\underline{r},t) = K\sum_\ell (2\ell+1)P_\ell(\cos\theta)\int_0^\infty \frac{\delta}{\pi(E-E_\delta)(E-\bar{E}_\delta)}\exp(iEX)\,C_\ell(E)\,dE \tag{2.39}$$

with $E_\delta = E_0 - i\delta$, so that $E_\delta - \bar{E}_\delta = 2i\delta$. (2.40)

Now since $C_\ell(E)$ is a slowly varying function of E, the same considerations as given in section (2.2) apply and $C_\ell(E)$ may be brought outside the integral. The integral can now be evaluated by contour integration around infinite semi-circles in the upper and lower halves of the complex energy plane:

$$\psi_p(\underline{r},t) = K\sum_\ell (2\ell+1)P_\ell(\cos\theta)\,C_\ell(E)\exp(i\bar{E}_\delta X)\ , \quad X>0$$

$$\psi_p(\underline{r},t) = K\sum_\ell (2\ell+1)P_\ell(\cos\theta)\,C_\ell(E)\exp(iE_\delta X)\ , \quad X<0. \tag{2.41}$$

From (2.41) the wave packet for potential scattering is centred upon the point X=0, or from the definition (2.34)

$$t = \frac{r_0 + r}{V_0}$$

We conclude that the non-resonant wave packet propagates without delay, since a wave packet propagating freely has velocity V_0.

Now

$$|\Psi_p(\underline{r},t)|^2 = \frac{1}{8\pi^2 r^2} \left|\sum_\ell (2\ell+1) P_\ell(\cos\theta) C_\ell(E)\right|^2 \exp(-2\delta|x|). \qquad (2.42)$$

The time spectrum of $|\Psi_p(\underline{r},t)|^2$ is an exponential rise and fall, exp $(-\delta t/\hbar)$.

The unnormalized probability of detecting a particle is from equation (2.18),

$$\frac{\hbar}{8\pi\delta} \left|\sum_\ell (2\ell+1) P_\ell(\cos\theta) C_\ell(E)\right|^2 d\Omega_d \qquad (2.43)$$

where $d\Omega_d$ is the solid angle subtended by the detector. Equation (2.43) is normalized by dividing by the number of particles incident per unit area, which is

$$\frac{1}{2\pi v_0} \iint \Delta(k) e^{ikz} dk \int \Delta(k') e^{-ik'z} dk' \, dz$$

$$= \frac{1}{v_0} \int \Delta(k) \Delta(k) \, dk$$

$$= \frac{\hbar}{\pi^2} \int \left[\frac{\delta}{(E-E_0)^2+\delta^2}\right]^2 dE$$

$$= \frac{\hbar}{2\pi\delta} . \qquad (2.44)$$

From (2.43) and (2.44)

$$\frac{d\sigma_p}{d\Omega_d} = \frac{1}{4} \left|\sum_\ell (2\ell+1) P_\ell(\cos\theta) C_\ell(E)\right|^2. \qquad (2.45)$$

Comparing (2.45) with (2.30) and (2.31), we see that the differential cross section for potential scattering is independent of the quantal uncertainty δ and identical with that for a normal beam in which δ is negligible. In this evaluation of the non-resonant wave packet, we have merely illustrated the conclusions of section 2.2 with a specific form factor.

2.5 THE RESONANT PACKET FOR AN ISOLATED RESONANCE

We now turn our attention to the resonant wave packet (2.36). If

$$\delta << \Gamma$$

the same discussion and method which was used to evaluate the non-resonant wave packet applies and no difference from the stationary treatment results. On the other hand if δ is comparable or much larger than Γ, we may still evaluate the integral easily, as the HSR expansion is convenient for complex integration. With the notation

$$I_\ell = \int_0^\infty \Delta(E)\, g(E)\, \exp(iEX)\, dX ,$$

$$g(E) = \sum_n \frac{R_{\ell n}}{E - E_{\ell n}} \equiv \sum_n \frac{R_{\ell n}}{E - \epsilon_{\ell n} + \frac{1}{2} i \Gamma_{\ell n}}$$

and

$$\Delta(E) = \frac{\delta}{\pi (E - E_s)(E - \bar{E}_s)}$$

the resonant wave packet is

$$\Psi_R(\underline{r}, t) = K \sum_\ell (2\ell+1) P_\ell(\cos\theta)\, I_\ell \quad . \tag{2.46}$$

The positions of the poles of the integrand of I_ℓ are shown in figure 4.

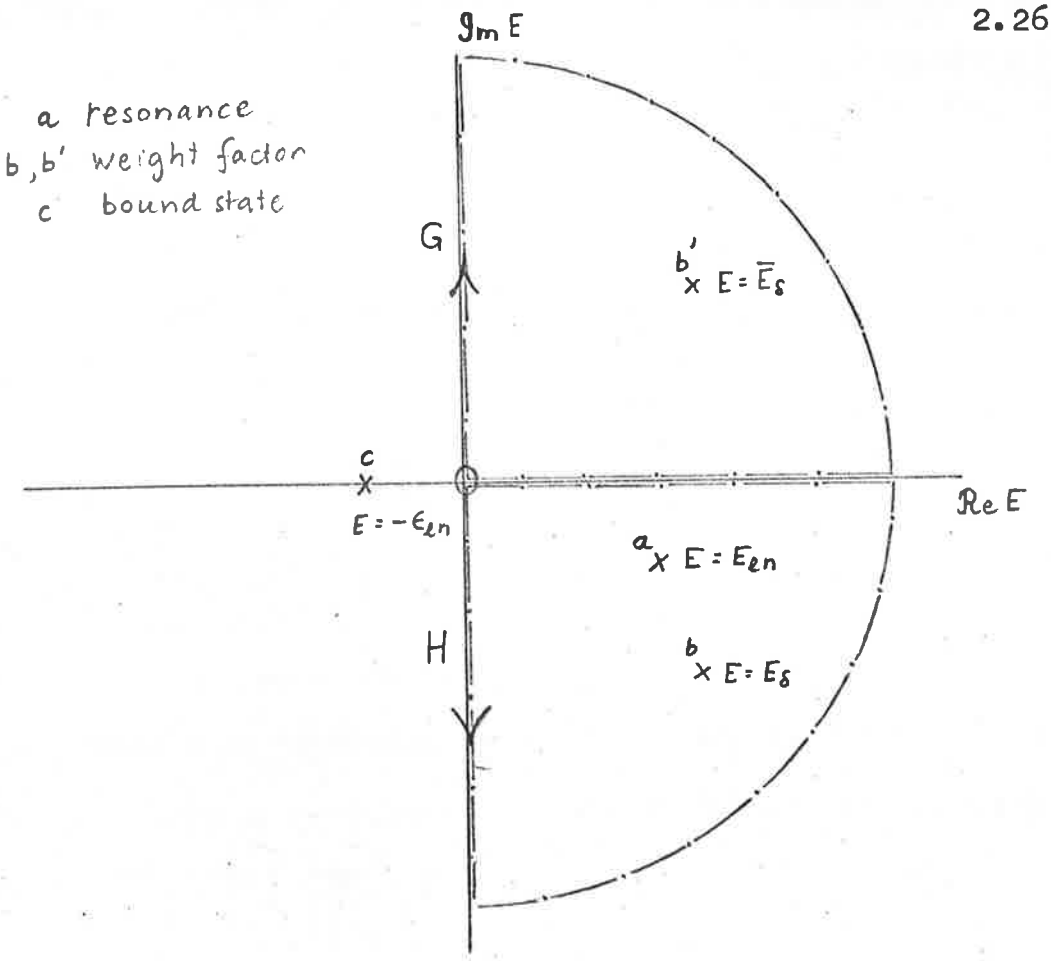

Figure 4: Singularities and Contours for I_ℓ

By complex integration in the upper half plane around the contour shown, for $X > 0$

$$I_\ell = \sum_n \frac{R_{\ell n}}{\bar{E}_s - E_{\ell n}} \exp(i\bar{E}_s X) + \int_G \Delta(E)\, g(E) \exp(iEX)\, dE, \quad (2.47)$$

the contribution from the arc at infinity vanishing for $X > 0$. Similarly, by complex integration around the contour marked in the lower half plane, for $X < 0$

$$I_\ell = \sum_n \frac{R_{\ell n}}{E_\delta - E_{\ell n}} \exp(iE_\delta X) - 2i\delta \sum_n{}' \frac{R_{\ell n}}{(E_n - E_\delta)(E_n - \bar{E}_\delta)} \exp(iE_n X)$$

$$+ \int_H \Delta(E)\, g(E) \exp(iEX)\, dE \,. \tag{2.48}$$

The primed summation includes the proper resonances only, the bound and virtual states do not contribute.

The integrals along G and H are strongly damped by exponential factors when $X \neq 0$, since E is imaginary. In addition the weight factor Δ is small along the imaginary axis. Therefore provided the singularities of g are not near the origin, the integrals G and H are small and may be discarded. One case, however, requires special consideration; a pole representing a bound or virtual state, which lies close to the origin may dominate the scattering. A note about this situation is appended to this chapter.

Generally, we have

$$I_\ell = \sum_n \frac{R_{\ell n}}{\bar{E}_\delta - E_{\ell n}} \exp(i\bar{E}_\delta X) \quad \text{for } X > 0 \,, \tag{2.49}$$

and

$$I_\ell = \sum_n \frac{R_{\ell n}}{E_\delta - E_{\ell n}} \exp(iE_\delta X) - 2i\delta \sum_n \frac{R_{\ell n}}{(E_n - E_\delta)(E_n - \bar{E}_\delta)} \exp(iE_n X)$$

$$\text{for } X < 0 \,. \tag{2.50}$$

where the summations now include only resonant states, since the contribution from virtual and bound states is assumed to be small.

Suppose that the peak energy E_0 of the incident packet is close to the energy $\epsilon_{\ell i}$ of an isolated resonance so that only the ith pole in the summations need be retained. The other resonances can be omitted or put with the amplitude for potential scattering. The s-wave is considered for simplicity of notation.

From equation (2.49), and (2.46), for $X > 0$

$$|\psi_R(\underline{r},t)|^2 = \frac{1}{8\pi r^2} \frac{R_i \bar{R}_i}{(E_i - \bar{E}_i)(\bar{E}_i - E_i)} \exp\{i(\bar{E}_i - E_i)X\} . \tag{2.51}$$

This shows that the leading edge of the scattered packet that is the part for times less than $\frac{r_0 + r}{v_0}$ is propagated with the same shape as it originally had, but, of course, with a different magnitude.

From equation (2.50), for $X < 0$

$$|\psi_R(\underline{r},t)|^2 = \frac{R_i \bar{R}_i}{8\pi r^2} \left[\frac{1}{(E_i - E_i)(\bar{E}_i - \bar{E}_i)} \exp\{i(E_i - \bar{E}_i)X\} \right.$$

$$+ \frac{4\delta^2}{(E_i-E_\delta)(E_i-\bar{E}_\delta)(\bar{E}_i-E_\delta)(\bar{E}_i-\bar{E}_\delta)} \exp\{i(E_i-\bar{E}_i)X\}$$

$$+ \frac{2i\delta}{(E_\delta-E_i)(\bar{E}_i-\bar{E}_\delta)(\bar{E}_i-E_\delta)} \exp\{i(E_\delta-\bar{E}_i)X\}$$

$$- \frac{2i\delta}{(E_\delta-\bar{E}_i)(E_i-E_\delta)(E_i-\bar{E}_\delta)} \exp\{i(E_i-\bar{E}_\delta)X\} \Bigg] .$$

(2.52)

In this expression for the trailing edge of the wave packet there are terms resulting from the pole in the scattering amplitude, which were not present for the potential scattering. The time spectrum is a decaying oscillatory function. In particular if the time width of the incident particle is much less than the decay constant $\hbar/\Gamma_i$ of the resonant state, the time spectrum has a tail falling off like

$$\exp\{i(E_i-\bar{E}_i)X\} = \exp(\Gamma_i X) \qquad \text{for } X < 0 .$$

The other terms which result from the specific form factor die away much more quickly if $\delta \gg \Gamma_i$

This corresponds to exciting the resonance suddenly and

watching it decay with its natural time constant.

The chance of detecting a particle before time T is

$$\hbar r_d^2 d\Omega_d \int_X^\infty |\psi_R(X')|^2 dX'$$

with

$$X = \frac{1}{\hbar}\left\{ \frac{r_0 + r_d}{v_0} - T \right\} .$$

For $T > \frac{r_0 + r_d}{v_0}$ this integral is

$$\frac{\hbar d\Omega_d R_i \bar{R}_i}{8\pi} \left[\frac{1}{2\delta(E_\delta - \bar{E}_i)(\bar{E}_\delta - E_i)} + \frac{\{1 - \exp(2\delta X)\}}{2\delta(E_\delta - E_i)(\bar{E}_\delta - \bar{E}_i)} \right.$$

$$+ \frac{4\delta^2\{1 - \exp(\Gamma_i X)\}}{\Gamma_i(E_i - E_\delta)(E_i - \bar{E}_\delta)(\bar{E}_i - \bar{E}_\delta)(\bar{E}_i - E_\delta)}$$

$$+ \frac{2\delta\{1 - \exp(i(E - E_i)X)\exp((\frac{\Gamma_i}{2} + \delta)X)\}}{(E_\delta - E_i)(\bar{E}_i - \bar{E}_\delta)(\bar{E}_i - E_\delta)(E_\delta - \bar{E}_i)}$$

$$\left. - \frac{2\delta\{1 - \exp(i(E_i - E)X)\exp((\frac{\Gamma_i}{2} + \delta)X)\}}{(\bar{E}_\delta - \bar{E}_i)(E_i - E_\delta)(E_i - \bar{E}_\delta)(\bar{E}_\delta - E_i)} \right]$$

For $\delta \gg \Gamma_i$ we have the familiar result that particles associated with the resonant scattering are delayed with respect to those associated with the potential scattering. In figure 5 the magnitude of the resonant wave packet is plotted for various ratios of δ/Γ_i and compared with the potential wave packet. It is clearly

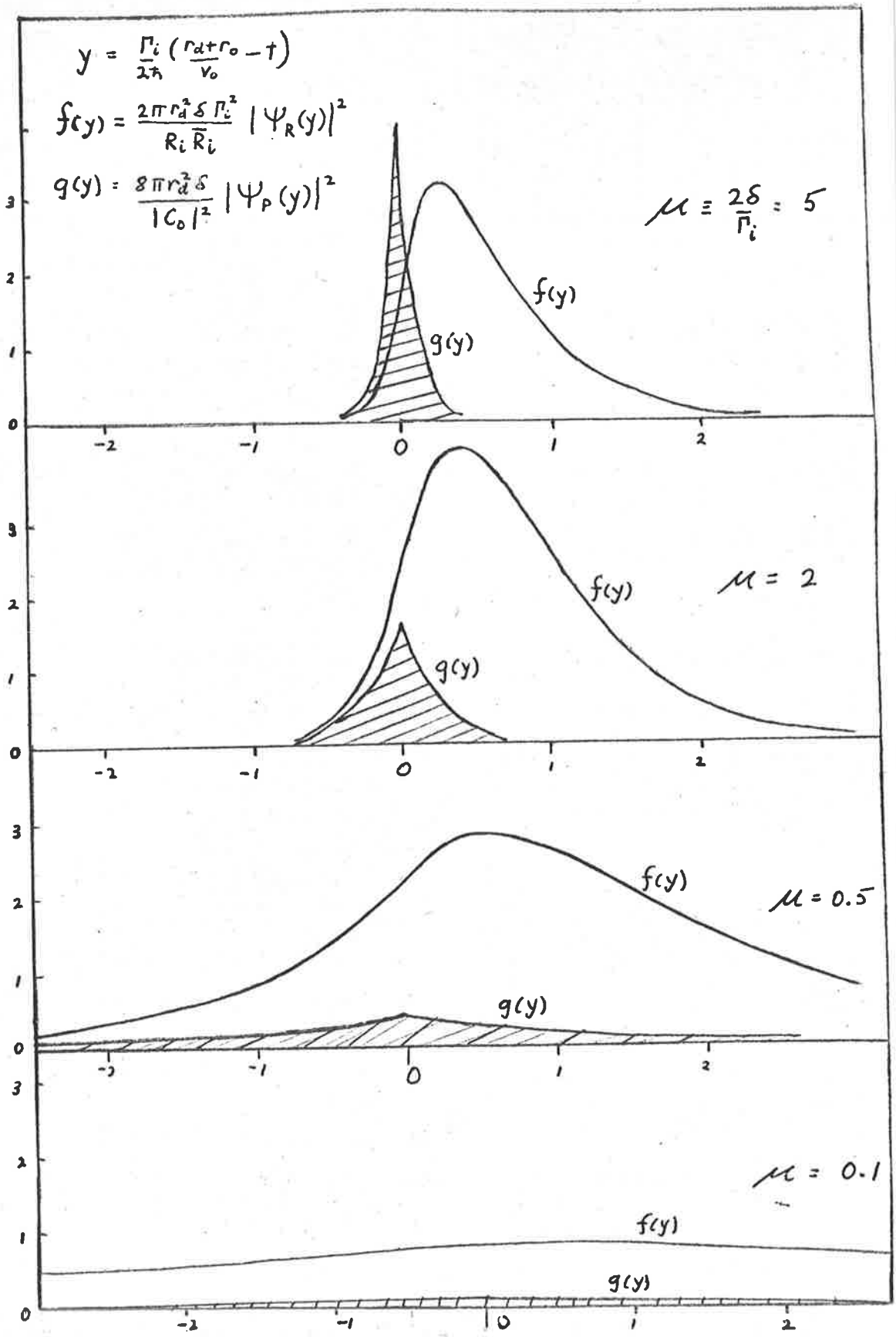

Fig. 5. Shape of wave packet.

seen from this comparison that as δ is made larger, the exponentially decaying tail of the resonant wave packet becomes more significant.

The differential cross section is found by taking $T \to \infty$ in (2.51) and normalizing in the same way as before.

$$\frac{d\sigma_R(\delta)}{d\Omega_d} = \frac{R_i \bar{R}_i}{4} \Bigg[\frac{1}{2\delta(E_\delta - \bar{E}_i)(\bar{E}_\delta - E_i)} + \frac{1}{2\delta(E_\delta - E_i)(\bar{E}_\delta - E_i)}$$

$$+ \frac{4\delta^2}{\Gamma_i (E_i - E_\delta)(E_i - \bar{E}_\delta)(\bar{E}_i - \bar{E}_\delta)(\bar{E}_i - E_\delta)}$$

$$+ \frac{2i\delta\left[\left(\frac{\Gamma_i}{2} + \delta\right) + i(E - \epsilon_i)\right]}{(E_\delta - E_i)(E_i - E_\delta)(E_i - E_\delta)\left\{(E - \epsilon_i)^2 + \left(\frac{\Gamma_i}{2} + \delta\right)^2\right\}}$$

$$+ \frac{2i\delta\left[-\left(\frac{\Gamma_i}{2} + \delta\right) + i(-E + \epsilon_i)\right]}{\left\{(E - \epsilon_i)^2 + \left(\frac{\Gamma_i}{2} + \delta\right)^2\right\}(\bar{E}_\delta - E_i)(E_i - E_\delta)(E_i - \bar{E}_\delta)} \Bigg].$$

i.e.

$$\frac{d\sigma_R(\delta)}{d\Omega_d} = \frac{R_i \bar{R}_i \left[\frac{\Gamma_i}{2}(E - \epsilon_i)^2 + \left(\frac{\Gamma_i}{2} + \delta\right)^2\left(\frac{\Gamma_i}{2} + 2\delta\right)\right]}{2\Gamma_i \left[(E - \epsilon_i)^2 + \left(\delta + \frac{\Gamma_i}{2}\right)^2\right]^2} \tag{2.53}$$

Equation (2.52) shows the dependence of the differential cross section for resonance scattering on the quantal uncertainty. It is important to note that as the quantal uncertainty of the beam is increased the probability of

seeing a particle scattered from the resonance is decreased, the resonance bump in the cross section is broadened and its peak lowered. This fact which arises explicitly from the formalism is suggested by the uncertainty principle. As the measurement of time in the experiment becomes more precise, the detailed energy behaviour is blurred.

Some special cases of (2.52) are:

(i) Perfect resolution, i.e. $\delta = 0$

$$\frac{d\sigma_R}{d\Omega_d} = \frac{R_i \bar{R}_i}{4\left((E-E_i)^2 + \frac{\Gamma_i^2}{4}\right)} \tag{2.54}$$

which agrees with the usual time independent theory.

(ii) Poor resolution, i.e. $\delta \to \infty$

$$\frac{d\sigma_R}{d\Omega_d} \text{ is } O\left(\frac{1}{\delta}\right) \tag{2.55}$$

(iii) Mean energy of the beam equal to the peak energy of the resonance, $E=E_i$,

$$\frac{d\sigma_R}{d\Omega_d} = \frac{R_i \bar{R}_i \left(\frac{\Gamma_i}{2} + 2\delta\right)}{2\Gamma_i \left(\frac{\Gamma_i}{2} + \delta\right)^2} \tag{2.56}$$

The modified shapes of a resonance for different quantal uncertainties are shown in figure 6.

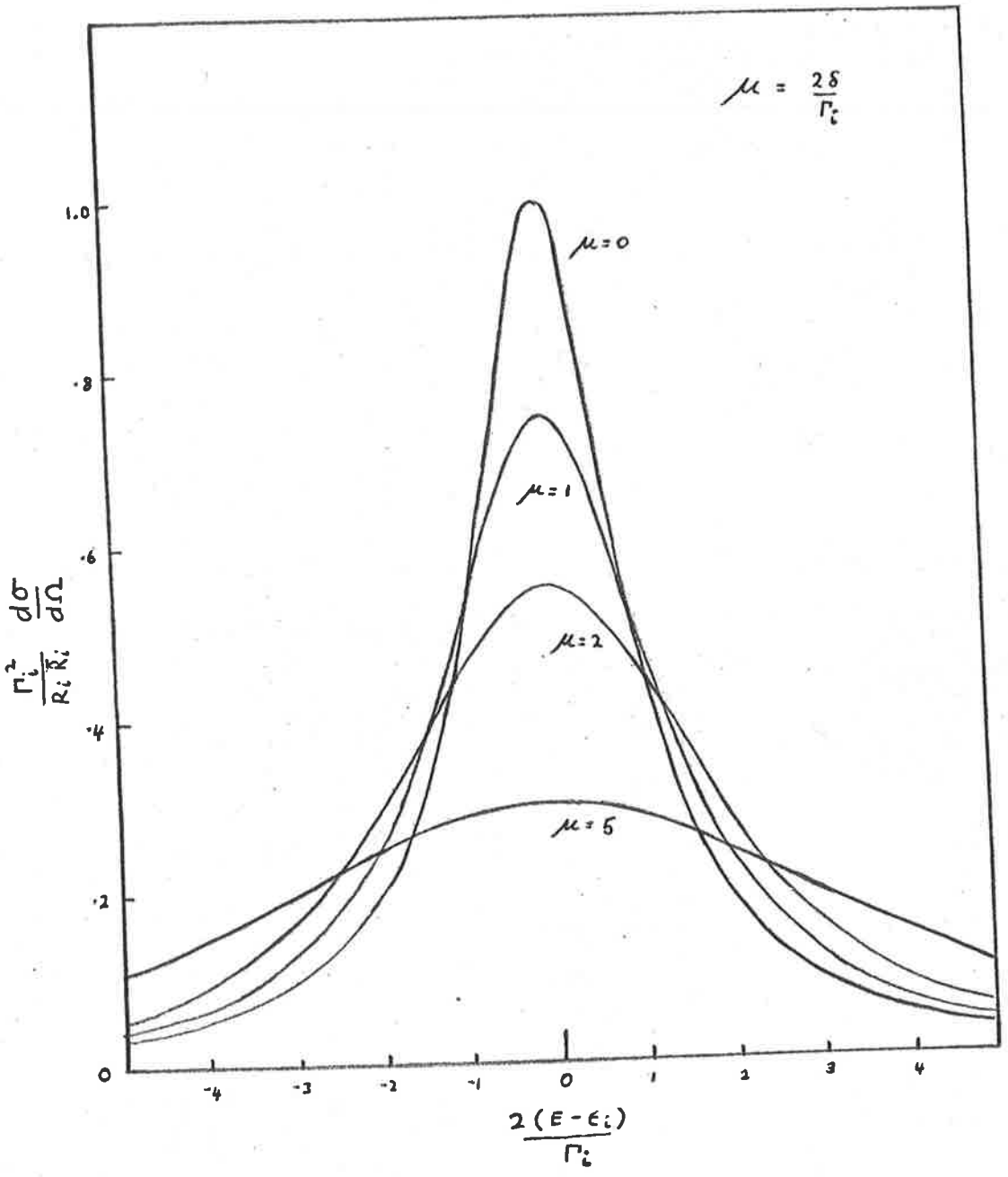

Fig. 6. Cross Section at an Isolated Resonance for various widths of the Incident Wave Packet.

2.6 INTERFERENCE OF THE RESONANT AND POTENTIAL SCATTERING

So far we have ignored the interference between the potential and the resonant scattering. The scattered wave packet was split into two parts, equation (2.37), a wave packet which was transmitted without delay and a wave packet which had an exponential tail and resulted from the isolated resonance level. This division leads to a third term, the interference term, in the differential cross section

$$|\Psi_{\underline{k}_0}(\underline{r},t)|^2 = |\Psi_P(\underline{r},t)|^2 + |\Psi_R(\underline{r},t)|^2 + 2\,\mathcal{R}e\{\overline{\Psi}_P(\underline{r},t)\,\Psi_R(\underline{r},t)\}$$

$$\frac{d\sigma}{d\Omega_d} = \frac{d\sigma_P}{d\Omega_d} + \frac{d\sigma_R}{d\Omega_d} + \frac{d\sigma_I}{d\Omega_d}$$

and

$$\frac{d\sigma_I}{d\Omega_d} = \frac{2\pi\delta r^2}{\hbar}\int_0^\infty 2\,\mathcal{R}e\{\overline{\Psi}_P(\underline{r},t)\,\Psi_R(\underline{r},t)\}\,dt\,. \tag{2.57}$$

Using the expression (2.50) and (2.41) (for the s-wave only) after some manipulation we arrive at

$$\frac{d\sigma_I}{d\Omega_d} = \frac{1}{2}\,\mathcal{R}e\left\{R_i\,\overline{C}_0(E)\,\frac{[(E-\epsilon_i)+i(\frac{\Gamma_i}{2}+2\delta)]}{[(E-\epsilon_i)+i(\frac{\Gamma_i}{2}+\delta)]^2}\right\}, \tag{2.58}$$

which for $\delta \to 0$ is

$$\frac{1}{2} \mathcal{R}e \left\{ \frac{R_i \overline{C}_0(E)}{(E-\epsilon_i) + i\frac{\Gamma_i}{2}} \right\}$$ as required.

On separation into real and imaginary parts (2.58) becomes

$$\frac{1}{2} \mathcal{R}e \left(R_i \overline{C}_0(E) \right) \frac{(E-\epsilon_i)\left[(E-\epsilon_i)^2 + \left(\frac{\Gamma_i}{2}+\delta\right)\left(\frac{\Gamma_i}{2}+3\delta\right)\right]}{\left[(E-\epsilon_i)^2 + \left(\frac{\Gamma_i}{2}+\delta\right)^2\right]}$$

$$+ \frac{1}{2} \mathcal{I}m \left(R_i \overline{C}_0(E) \right) \frac{\left[\frac{\Gamma_i}{2}(E-\epsilon_i)^2 + \left(\frac{\Gamma_i}{2}+\delta\right)^2\left(\frac{\Gamma_i}{2}+2\delta\right)\right.}{\left[(E-\epsilon_i)^2 + \left(\frac{\Gamma_i}{2}+\delta\right)^2\right]} \quad (2.59)$$

The coefficient of $\mathrm{Re}(R_i \overline{C}_0(E))$ is $O(\frac{1}{\delta^2})$ for large δ.

The coefficient of $\mathrm{Im}(R_i \overline{C}_0(E))$ is $O(\frac{1}{\delta})$ for large δ.

It is sometimes thought that the interference between the potential and resonance scattering vanishes when the wave packet treatment is used, since the wave packets for the two processes overlap less as the quantal uncertainty is increased. This is true. However, it is usually overlooked that the wave packet associated with the resonance becomes smaller in magnitude as time is

better defined. A comparison of (2.59) and (2.53) show that the interference and resonance contribution to the scattering fall off at roughly the same rate with increasing δ .

2.7 THE TIME DELAY AND DISCUSSION

The average time of arrival of a scattered particle at the detector is

$$\frac{\int_{-\infty}^{\infty} |\Psi_{\underline{k}_0}(\underline{r},t)|^2 t \, dt}{\int_{-\infty}^{\infty} |\Psi_{\underline{k}_0}(\underline{r},t)|^2 dt}$$

$$= \frac{\int_{-\infty}^{\infty} |\Psi_{\underline{k}_0}(x)|^2 \left(\frac{r_0 + r_d}{v_0} - \hbar x\right) dx}{\int_{-\infty}^{\infty} |\Psi_{\underline{k}_0}(x)|^2 dx}$$

$$= \frac{r_0 + r_d}{v_0} - \hbar \frac{\int_{-\infty}^{\infty} x |\Psi_{\underline{k}_0}(x)|^2 dx}{\int_{-\infty}^{\infty} |\Psi_{\underline{k}_0}(x)|^2 dx} \qquad (2.60)$$

Thus the average time delay for particles scattered from a resonance is

$$\Delta t = -\hbar \frac{\int_{-\infty}^{\infty} |\Psi_R(x)|^2 x \, dx}{\int_{-\infty}^{\infty} |\Psi_R(x)|^2 dx} \qquad (2.61)$$

with $|\Psi_R(x)|^2$ given by equation (2.50). The denominator has been evaluated in section 2.5 and the numerator can be calculated similarly. After some manipulation, we obtain the time delay for an isolated resonance of width Γ_i and energy E_i as

$$\Delta t = -\hbar(\Gamma_i + 2\delta)\left[\frac{1}{(E-E_i)^2 + \left(\frac{\Gamma_i}{2}+\delta\right)^2} - \frac{\Gamma_i^2 + 2\delta\Gamma_i - 4\delta^2}{4\Gamma_i\left(\frac{\Gamma_i}{2}(E-E_i)^2 + \left(2\delta + \frac{\Gamma_i}{2}\right)\left(\frac{\Gamma_i}{2}+\delta\right)^2\right)}\right] . \tag{2.62}$$

Some special cases are

(i) peak energy of the incident packet equal to the energy of the resonance, i.e. $E = E_i$,

$$\Delta t = -\frac{2\hbar}{\Gamma_i}\,\frac{4\delta^2 + 6\delta\Gamma_i + \Gamma_i^2}{8\delta^2 + 6\delta\Gamma_i + \Gamma_i^2} \tag{2.63}$$

(ii) quantal uncertainty large i.e. $\delta \to \infty$,

$$\Delta t = -\frac{\hbar}{\Gamma_i}\left(1 + \frac{\Gamma_i(E-E_i)^2}{4\delta^3}\right)^{-1} \tag{2.64}$$

(iii) quantal uncertainty small $\delta \to 0$,

$$\Delta t = -\frac{\hbar\Gamma_i}{2}\left((E-E_i)^2 + \frac{\Gamma_i^2}{4}\right)^{-1} . \tag{2.65}$$

When the incident and resonant energies are equal, the mean time delay takes values between $-\frac{\hbar}{\Gamma_i}$ and $-\frac{2\hbar}{\Gamma_i}$ the lower limit corresponds to extremely short incident wave packets whereas the upper limit corresponds to wave

packets of infinite length. This shows clearly that a unique time delay cannot be associated with a given scatterer in all possible experiments. The mean time delay depends on the mode of preparation of the beam in addition to the particular scattering mechanism.

For $\delta = 0$ the time delay is $-\frac{2\hbar}{\Gamma_i}$ which is the same as the value resulting from the usual definition (1.1). This time delay would be extremely difficult to observe in practice for the following reason: with the very long wave packet required by the condition $\delta \to 0$ the terms exp $(-\delta|x|)$ in the equations (2.50) mask the exponential decay of the resonance. The arrival times of particles at the dectector vary between wide limits. The time interval $\frac{\hbar}{2\delta}$ defined by these limits is many orders of magnitude greater than the mean time delay.

Since the time delay in the experiment with small quantal uncertainty is essentially unobservable, we prefer to define the characteristic time delay of the scatterer as the time delay given by the complementary experiment for which the quantal uncertainty is large. This seems a sounder procedure, because the time delay defined thus is the value which would result from an experiment designed to give good time resolution. Hence the

characteristic time delay for an isolated resonance is $-\hbar/\Gamma_i$. In general the characteristic time delay for a scatterer is defined by

$$-\lim_{\delta\to 0} \hbar \frac{\int_{-\infty}^{\infty} |\Psi_{\underline{k}_0}(x)|^2 x \, dx}{\int_{-\infty}^{\infty} |\Psi_{\underline{k}_0}(x)|^2 dx} . \tag{2.66}$$

The results of this chapter lead to the following conclusions,

(i) The particles which are scattered from an isolated resonant state of width Γ_i are delayed on the average by $\Delta t = -\hbar/\Gamma_i$ with respect to those particles associated with the potential scattering provided that the condition $\delta >> \Gamma_i$ holds. To an energy fluctuation in the scattering cross section for a beam of perfect energy resolution, we can associate a time delay $\hbar/\Gamma_i$ in the complementary, time dependent experiments of large quantal uncertainty.

(ii) As the quantal uncertainty is increased so that the incident wave packet is better defined with respect to time, the number of particles scattered from the resonance decreases with the result that the cross section loses its detail with respect to energy. This result is expected from the uncertainty principle.

(iii) The interference term in the cross section decreases with increasing definition in time as does the resonance scattering. For large quantal uncertainties only the wave packet for the characterless, smoothly varying background scattering remains and this is propagated without time delay.

2.8 NOTE ON BOUND AND VIRTUAL STATES

It was remarked in section 2.6, that for a bound (or virtual) state of small binding energy, the integrals along the imaginary axis in equations (2.48) and (2.49) may be large and their contribution to the scattered packet cannot be ignored. To estimate this effect, we evaluate the contribution to I_ℓ for $X = 0$.

At $X = 0$,

$$\int_G \Delta(E)\, g(E) \exp(iEX)\, dE = \int_H \Delta(E)\, g(E) \exp(iEX)\, dE$$

$$= i \int_0^\infty \Delta(iE)\, g(iE)\, dE \,, \qquad (2.67)$$

with

$$\Delta(iE) = \frac{\delta}{\pi((iE-E_0)^2+\delta^2)} \,, \qquad g(iE) \approx \frac{R}{\epsilon + iE} \,.$$

R is the strength of the bound state and E its binding energy. The evaluation of the integral (2.67) is straightforward and yields

$$\frac{R}{2\pi\{(E_0+\epsilon)^2+\delta^2\}} \left[\delta \ln\left(\frac{E_0^2+\delta^2}{\epsilon^2}\right) - 2(E_0+\epsilon)\tan^{-1}\left(\frac{\delta}{E_0}\right) \right]$$

which for small ϵ, i.e. $\epsilon \ll E_0$ is

$$-\frac{iR\delta}{\pi(E_0^2+\delta^2)} \ln\left(\frac{\epsilon}{\sqrt{E_0^2+\delta^2}}\right) . \qquad (2.68)$$

As the binding energy of the state approaches zero, the integral (2.67) is logarithmically divergent.

Also we note that for large negative times

$$\int_G \Delta(E)\, g(E)\, \exp(iEX)\, dE \simeq i\Delta(0) \int_0^\infty g(iE) \exp(-EX)\, dE$$

$$= \frac{i\delta}{\pi(E_0^2+\delta^2)} \int_0^\infty \frac{R}{i\eta+\epsilon X} \exp(-\eta)\, d\eta$$

$$= \frac{i\delta R}{\pi(E_0^2+\delta^2)}\, \mathcal{H}(\epsilon X)$$

$$\approx \frac{-i\delta R}{\pi(E_0^2+\delta^2)} \cdot \frac{1}{\epsilon X} \qquad (2.69)$$

Similarly, for large positive times

$$\int_H \Delta(E)\, g(E)\, \exp(iEX)\, dE \simeq \frac{-i\delta R}{\pi(E_0^2+\delta^2)}\, \mathcal{H}(\epsilon X)$$

$$\simeq \frac{i\delta R}{\pi(E_0^2+\delta^2)} \cdot \frac{1}{\epsilon X}$$

The characteristic time duration of the wave packet resulting from the bound state is of the order $\hbar/2\epsilon$.

Therefore for a state which is only loosely bound, the contribution to the emergent wave packet swamps the contribution from the resonances which are close to the peak energy of the incident packet. The long wave length component of the incident packet is strongly reinforced as the incident particle and the scatterer tend to form a bound state.

Actually the choice of the Lonentzian shape for the incident packet overestimates this effect, because the tail of the Lorentzian does not decrease rapidly enough. With a smoothed, rectangular form factor for the incident packet, which then has no low energy component, the integrals along the imaginary axis can certainly be discarded.

CHAPTER 3

SCATTERING FROM MANY LEVELS

3.1 SHAPE OF THE WAVE PACKET

It is necessary to investigate the behaviour of a wave packet which is scattered from a number of overlapping levels. This case is not usually discussed from the point of view of a time dependent theory although it is of considerable relevance to the physical interpretation of the optical model. It is just this region of overlapping levels which motivates the optical model description.

A wave packet scattered from a group of overlapping levels differs in two important characteristics from one scattered from an isolated level.

(i) The shape of the scattered wave packet depends on the spacing and correlation between levels as well as their widths.

(ii) The magnitude of the resonant packet may remain appreciable for large quantal uncertainties, if the residues of distant levels are correlated.

These differences arise from the fact that in some circumstances, which will be discussed, there may

be either constructive or destructive interference between the levels.

The wave packet $\Psi_R(\underline{r},t)$ for $X<0$ (cf. equation (2.50)) which contains the $\exp(+\Gamma X)$ term responsible for the tail of the wave packet and thus the time delay, may be written in a more revealing way. Writing some of the factors with their real and imaginary parts shown explicitly and again only considering the S-wave, we have

$$\Psi_R(\underline{r},t) = K\sum_n \frac{R_n}{E-\epsilon_n+i(\frac{\Gamma_n}{2}-\delta)} \exp(iEX)\exp(\delta X)$$

$$- 2ik\delta \sum_n \frac{\exp(i\epsilon_n X)\exp(+\frac{\Gamma_n}{2}X)\, R_n}{\left[E-\epsilon_n+i(\frac{\Gamma_n}{2}+\delta)\right]\left[E-\epsilon_n+i(\frac{\Gamma_n}{2}-\delta)\right]} . \tag{3.1}$$

That is

$$\Psi_R(\underline{r},t) = K\exp(iEX)\sum_n \frac{R_n}{E-\epsilon_n+i(\frac{\Gamma_n}{2}-\delta)}$$

$$\times\left[\exp(\delta X) - \frac{2i\delta\exp(\frac{\Gamma_n}{2}X)\exp\{i(\epsilon_n-E)X\}}{E-\epsilon_n+i(\frac{\Gamma_n}{2}+\delta)}\right] .$$

First we note that for perfect energy resolution the quantal uncertainty is zero ($\delta=0$), and the wave packet degenerates into the spherical wave of the stationary method, since

$$K\exp(iEX) \sim \exp\left\{i(kr-\frac{Et}{\hbar})\right\}/r \quad . \tag{3.2}$$

In the last chapter we considered the situation with $\Gamma \ll D$, Γ being the mean width of the levels about the energy of interest and D the mean spacing between them. In this case for small quantal uncertainties it is sufficient to omit all but one of the terms in the sum. For larger quantal uncertainties, comparable to D, it is necessary to take more terms to evaluate the cross section but even then the interference between different terms may be neglected, so that the results derived for the isolated level still apply. The cross section is just the sum of the cross sections for each particular resonance.

On the other hand, in the region of overlapping levels the mean width is comparable to or greater than the mean spacing between levels, so that several or many terms make a significant contribution to the sum in equation (3.1) and the total scattered packet may be regarded as a superposition of wave packets, one from each of these levels.

Qualitatively, one can see how the shape and magnitude of the resonant packet depends on the phases of the residues Rn. For good resolution of time, we require a short incident packet, i.e. $\delta \gg \Gamma$. Then all those levels with ϵ_n such that

$$E-\delta \lesssim \epsilon_n \lesssim E+\delta \qquad (3.3)$$

contribute to the sum in equation (3.1). If the residues Rn are uncorrelated, in other words they have random phases, the terms of the sum tend to mutually cancel, and the magnitude of the resultant packet is diminished as the quantal uncertainty is increased. However, if the Rn are correlated the terms add coherently and the magnitude of the scattered packet remains large. But in this case the shape of the wave packet is changed for the following reasons: the tail factor $\exp(-\frac{\Gamma_n}{2}|x|)$ of each of the contributing wave packets is multiplied by a phase factor $\exp\{i(\epsilon_n-E)x\}$. For distant levels the period of the phase factor is much less than the half life of the exponential tail. Thus although distant levels which are correlated may give a large undelayed contribution from the $\exp(-\delta|x|)$ part of the packet, the contribution for large X is small. For correlated levels a larger proportion of the resonant scattering is propagated without time delay.

This behaviour is very important for an understanding of the optical model. In section (6.4) it is shown that in the limiting case when the ratio of

the mean level spacing as the mean level width approaches zero, the resonant packet no longer has an exponential tail characterizing a time delay. It is scattered promptly.

It is well known that in the dispersion theories where the positions of the resonances depend on the radius of interaction, small changes in the boundary conditions produces large changes of position in the high energy resonances, and similarly that small changes of the potential shift the high energy poles by large amounts. The wave packet method confirms the idea that individual high energy poles have no observable consequences, although their average properties are important. Again this conclusion seems obvious from the uncertainty principle. No matter how we decompose the cross section at high energies into 'resonances', if the cross section is a smooth function of energy, the time behaviour of the scattered packet is characterless (c.f. the discussion of the high energy limit of the optical model in chapter 6).

3.2 THE CROSS SECTION

From (2.49) and (2.50), for X>0

$$|\Psi_R(\underline{r},t)|^2 = \frac{\delta^2}{2\pi r^2} \sum_{n,n'} R_n \bar{R}_{n'} \frac{\exp\{i(\bar{E}_\delta - E_\delta)X\}}{(\bar{E}_\delta - E_\delta)(E_\delta - \bar{E}_\delta)(\bar{E}_\delta - E_n)(E_\delta - \bar{E}_{n'})} \tag{3.4}$$

and for X < 0 ,

$$|\Psi_R(\underline{r},t)|^2 = \frac{\delta^2}{2\pi r^2} \sum_{n,n'} R_n \bar{R}_{n'} \Bigg[\frac{\exp\{i(E_\delta - \bar{E}_\delta)X\}}{(E_\delta - \bar{E}_\delta)(\bar{E}_\delta - E_\delta)(\bar{E}_\delta - \bar{E}_{n'})(E_\delta - E_n)}$$

$$+ \frac{\exp\{i(E_\delta - \bar{E}_{n'})X\}}{(E_\delta - \bar{E}_\delta)(E_\delta - E_n)(\bar{E}_{n'} - E_\delta)(\bar{E}_{n'} - \bar{E}_\delta)} + \frac{\exp\{i(E_n - \bar{E}_\delta)X\}}{(\bar{E}_\delta - E_\delta)(\bar{E}_\delta - \bar{E}_{n'})(E_\delta - \bar{E}_\delta)(E_n - E_\delta)}$$

$$+ \frac{\exp\{i(E_n - \bar{E}_{n'})X\}}{(E_n - \bar{E}_\delta)(\bar{E}_{n'} - E_\delta)(\bar{E}_{n'} - \bar{E}_\delta)(E_n - E_\delta)} \Bigg] . \tag{3.5}$$

With the substitutions

$$E_\delta - \bar{E}_\delta = A \ , \quad \bar{E}_\delta - E_n = B \ , \quad E_\delta - \bar{E}_{n'} = C$$

$$E_\delta - E_n = B + A \ , \quad \bar{E}_\delta - \bar{E}_{n'} = C - A$$

$$\bar{E}_n - \bar{E}_{n'} = C - B - A \tag{3.6}$$

equations (3.4) and (3.5) are

$$|\Psi_R(\underline{r},t)|^2 = -\frac{\delta^2}{2\pi r^2} \sum_{n,n'} R_n \bar{R}_{n'} \frac{\exp(-iAX)}{A^2 BC} , \tag{3.7}$$

for $X > 0$

$$|\Psi_R(r,t)|^2 = \frac{\delta^2}{2\pi r^2} \sum_{n,n'} R_n \bar{R}_{n'} \left[\frac{-\exp(iAX)}{A^2(C-A)(B+A)} \right.$$

$$+ \frac{\exp\{i(C-B-A)X\}}{BC(C-A)(B+A)} + \frac{\exp(iCX)}{CA(B+A)(C-A)}$$

$$\left. - \frac{\exp(-iBX)}{BA(C-A)(B+A)} \right] \quad \text{for } X<0 . \qquad (3.8)$$

Then the unnormalized cross section is given by

$$\hbar \int_{-\infty}^{0} |\Psi_R(r,t)|^2 dX + \hbar \int_{0}^{\infty} |\Psi_R(r,t)|^2 dX$$

which is

$$\frac{\delta^2}{2\pi r^2} \frac{\hbar}{i} \sum_{n,n'} R_n \bar{R}_{n'} \left[-\frac{1}{A^3 BC} - \frac{1}{A^2(C-A)(B+A)} \right.$$

$$\left. + \frac{1}{B^2A(C-A)(B+A)} + \frac{1}{C^2A(B+A)(C-A)} + \frac{1}{BC(C-A)(B+A)(C-B-A)} \right]$$

$$= \frac{\delta^2}{2\pi r^2} \frac{\hbar}{i} \sum_{n,n'} R_n \bar{R}_{n'} \frac{\{A(B^2+C^2)+2BC(B-C)\}}{A^3B^2C^2(C-B-A)}$$

since the factors (B+A),(C-A) cancel.

With the same normalization as before,

$$\frac{d\sigma_R}{d\Omega} = \frac{1}{8} \sum_{n,n'} R_n \bar{R}_{n'} \frac{\{A(B^2+C^2)+2BC(B-C)\}}{B^2C^2(B-C+A)} \qquad (3.9)$$

Now $A = -2i\delta$

$$B = \bar{E}_\delta - E_n = E + i\delta - \epsilon_n + \frac{i\Gamma_n}{2} = e_n + i\gamma_n$$

$$C = E_\delta - \bar{E}_{n'} = E - i\delta - \epsilon_{n'} - \frac{i\Gamma_{n'}}{2} = e_{n'} - i\gamma_{n'}$$

$$(B-C+A) = \epsilon_n - \epsilon_{n'} + i\left(\frac{\Gamma_{n'}}{2} + \frac{\Gamma_n}{2}\right) = e_{nn'} + i\gamma_{nn'} \qquad (3.10)$$

if the definitions

$$\gamma_n = \delta + \frac{\Gamma_n}{2}, \qquad e_n = E - \epsilon_n$$

$$e_{nn'} = \epsilon_n - \epsilon_{n'} \qquad \gamma_{nn'} = \frac{\Gamma_{n'} + \Gamma_n}{2} \qquad (3.11)$$

are made.

The real part of $\frac{1}{2}\left\{A(B^2+C^2) + 2BC(B-C)\right\}\overline{B^2C^2}$ is

$$\mathcal{R}_{nn'} = -2\delta\left\{(e_n^2+\gamma_n^2)^2\gamma_{n'}e_{n'} - (e_{n'}^2+\gamma_{n'}^2)e_n\gamma_n\right\}$$
$$+ (e_n^2+\gamma_n^2)(e_{n'}^2+\gamma_{n'}^2)\left\{(e_n^2+\gamma_n^2)e_{n'} - (e_{n'}^2+\gamma_{n'}^2)e_n\right\} \qquad (3.12)$$

and the imaginary part is

$$\mathcal{I}_{nn'} = -\delta\left\{(e_n^2+\gamma_n^2)^2(e_{n'}^2-\gamma_{n'}^2) + (e_{n'}^2+\gamma_{n'}^2)(e_n^2-\gamma_n^2)\right\}$$
$$+ (e_n^2+\gamma_n^2)(e_{n'}^2+\gamma_{n'}^2)\left\{\gamma_{n'}(e_n^2+\gamma_n^2) + \gamma_n(e_{n'}^2+\gamma_{n'}^2)\right\}. \qquad (3.13)$$

So that

$$\frac{d\sigma_R}{d\Omega_d} = \frac{1}{4}\sum_{n,n'} \frac{R_n\bar{R}_{n'}\left\{(\mathcal{R}_{nn'}e_{nn'} + \gamma_{nn'}\mathcal{I}_{nn'}) + i(\mathcal{I}_{nn'}e_{nn'} - \mathcal{R}_{nn'}\gamma_{nn'})\right\}}{(e_n^2+\gamma_n^2)^2(e_{n'}^2+\gamma_{n'}^2)^2(e_{nn'}^2+\gamma_{nn'}^2)}$$

Let $$T_{nn'} = \frac{\mathcal{R}_{nn'}e_{nn'} + \mathcal{I}_{nn'}\gamma_{nn'} + i(\mathcal{I}_{nn'}e_{nn'} - \mathcal{R}_{nn'}\gamma_{nn'})}{(e_n^2+\gamma_n^2)^2(e_{n'}^2+\gamma_{n'}^2)^2(e_{nn'}^2+\gamma_{nn'}^2)}$$

and form $\frac{1}{2}\sum_{n,n'} T_{nn'} R_n \bar{R}_{n'} + \frac{1}{2}\sum_{n,n'} T_{n'n} R_{n'} \bar{R}_n$

$$= \frac{1}{2}\sum_{n,n'}\left\{T_{nn'} R_n \bar{R}_{n'} + T_{n'n} R_{n'} \bar{R}_n\right\} .$$

From inspection of equations (3.12) and (3.13) and the definitions (3.11)

$$\mathcal{R}_{nn'} = -\mathcal{R}_{n'n} , \quad \mathcal{I}_{nn'} = \mathcal{I}_{n'n}$$

$$e_{nn'} = -e_{n'n} , \quad \gamma_{nn'} = \gamma_{n'n}$$

so that $\overline{T}_{nn'} = T_{n'n}$

Consequently the cross section is

$$\frac{d\sigma_R}{d\Omega_\alpha} = \frac{1}{4}\sum_{n,n'} \frac{Re(R_n \bar{R}_{n'})\{\mathcal{R}_{nn'} e_{nn'} + \mathcal{I}_{nn'}\gamma_{nn'}\}}{(e_n^2+\gamma_n^2)^2(e_{n'}^2+\gamma_{n'}^2)^2(e_{nn'}^2+\gamma_{nn'}^2)}$$

$$-\frac{1}{4}\sum_{n,n'} \frac{Im(R_n \bar{R}_{n'})\{\mathcal{I}_{nn'} e_{nn'} - \mathcal{R}_{nn'}\gamma_{nn'}\}}{(e_n^2+\gamma_n^2)^2(e_{n'}^2+\gamma_{n'}^2)^2(e_{nn'}^2+\gamma_{nn'}^2)}$$

(3.14)

If $\Gamma \ll D$ and $\delta \ll D$, the factor $(e_{nn'}^2 + \gamma_{nn'}^2)$ in the denominator of the cross product terms ($n \neq n'$) is much greater than the other factors. Thus in the low energy region where the resonances are isolated, the interference terms in the cross section are negligible, provided the quantal uncertainty is not too large.

Then

$$\frac{d\sigma_R}{d\Omega_d} = \frac{1}{4} \sum_n \frac{R_n \bar{R}_n g_{nn}}{(e_n^2 + \gamma_n^2)^4 \gamma_{nn}}$$

$$= \sum_n \frac{R_n \bar{R}_n \left[e_n^2 (\gamma_n - \delta) + \gamma_n^2 (\gamma_n + \delta) \right]}{2\Gamma_n (e_n^2 + \gamma_n^2)^2} \qquad (3.15)$$

Each term in the sum (3.15) is of the form (2.53). In this approximation the cross section for the resonance scattering from many levels is just the sum of cross sections, one for each resonance.

3.3 HIGH ENERGY APPROXIMATION

It is reasonable to assume here that the neighbouring levels are approximately equidistant and that their widths are nearly equal. First suppose that the residues R_n are as correlated as they can be, in particular $R_n = \rho$ for all n. Then

$$\frac{d\sigma_R}{d\Omega_a} = \frac{\rho\bar{\rho}}{4} \sum_{n,n'} \frac{\mathcal{R}_{nn'} e_{nn'} + \gamma_{nn'} \mathcal{I}_{nn'}}{(e_n^2+\gamma_n^2)^2 (e_{n'}^2+\gamma_{n'}^2)(e_{nn'}^2+\gamma_{nn'}^2)} \tag{3.16}$$

The double sum in (3.16) is only conditionally convergent. This is a result of the unrealistic assumption that the residues are constant. Actually the residues must be such that the series

$$\sum_n \frac{R_n}{\left(e_n^2 + \frac{\Gamma_n^2}{4}\right)^{\frac{1}{2}}}$$ converges, in order that

the Mittag-Leffler expansion be valid.[6]

Therefore we take

$$R_n = \rho \frac{\Delta^2}{\Delta^2 + e_n^2} \tag{3.17}$$

and let $\Delta \to \infty$ at the end of the calculation. Equation (3.16) now becomes, with $\mathcal{R}_{nn'}$ and $\mathcal{I}_{nn'}$ written in terms of e_n and $e_{n'}$,

$$\frac{d\sigma_R}{d\Omega_d} = \frac{\rho\bar{\rho}}{4} \sum_{n,n'} \frac{\Delta^2}{(\Delta^2+e_n^2)} \frac{\Delta^2}{(\Delta^2+e_{n'}^2)(e_{nn'}^2+\gamma_{nn'}^2)}$$

$$\times \left[\frac{e_{nn'}e_{n'}+\gamma_{n'}\gamma_{nn'}}{(e_{n'}^2+\gamma_{n'}^2)} + \frac{\gamma_n\gamma_{nn'}-e_{nn'}e_n}{(e_n^2+\gamma_n^2)} \right.$$

$$\left. \frac{2\delta e_{nn'}e_n\gamma_n-\gamma_{nn'}\delta(e_n^2-\gamma_n^2)}{(e_n^2+\gamma_n^2)^2} - \frac{2\delta\gamma_{n'}e_{n'}e_{nn'}+\gamma_{nn'}\delta(e_n^2-\gamma_{n'}^2)}{(e_{n'}^2+\gamma_{n'}^2)^2} \right].$$

(3.18)

For small level spacing compared with level width, the double sum may be replaced by a double integral and γ_n taken as constant.

$$\gamma_n = \gamma \qquad \gamma_{nn'} = \Gamma$$

$$\sum_n' e_n \rightarrow \frac{1}{D}\int x\,dx$$

With these substitutions the cross section (3.18) becomes

$$\frac{d\sigma_R}{d\Omega_d} = \frac{\rho\bar{\rho}}{4D^2} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \; \frac{\Delta^4}{((x-y)^2+\Gamma^2)(\Delta^2+x^2)(\Delta^2+y^2)}$$

$$\times \left[\frac{-2\delta\gamma y(x-y)-\delta\Gamma(y^2-\gamma^2)}{(y^2+\gamma^2)^2} + \frac{2\delta(x-y)x\gamma-\delta\Gamma(x^2-\gamma^2)}{(x^2+\gamma^2)^2} \right.$$

$$+ \frac{(x-y)\,y + \gamma\Gamma}{(y^2+\gamma^2)} \quad + \quad \frac{\gamma\Gamma - (x-y)x}{(x^2+\gamma^2)} \Bigg] .$$

The interchange of the variables of integration in the second and fourth terms in the square brackets gives

$$\frac{d\sigma_R}{d\Omega_R} = \frac{\rho\bar{\rho}}{2D^2} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \; \frac{\Delta^4}{((x-y)^2+\Gamma^2)(x^2+\Delta^2)(y^2+\Delta^2)}$$

$$\times \left[\frac{-2\delta\gamma y(x-y) + 2\gamma^2\delta\Gamma}{(y^2+\gamma^2)^2} \quad \frac{(x-y)\,y + \Gamma(\gamma-\delta)}{(y^2+\gamma^2)} \right] .$$

(3.19)

By two contour integrations around infinite semi-circles in the upper halves of the x and y-planes.

$$A = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \; \frac{\Delta^4}{((x-y)^2+\Gamma^2)(x^2+\Delta^2)(y^2+\Delta^2)(y^2+\gamma^2)}$$

$$= \frac{\pi^2\Delta^2(\Gamma+2\Delta+\gamma)}{\Gamma\gamma(\Delta+\gamma)(\Gamma+\Delta+\gamma)(\Gamma+2\Delta)}$$

(3.20)

$$B = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \; \frac{\Delta^3 (x-y)\, y}{((x-y)^2+\Gamma^2)(x^2+\Delta^2)(y^2+\Delta^2)(y^2+\gamma^2)}$$

$$= \; - \; \frac{\pi^2 \Delta^3}{(\Delta+\gamma)(\Delta+\gamma+\Gamma)(\Gamma+2\Delta)} \; .$$

(3.21)

And

$$C = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \; \frac{\Delta^4}{((x-y)^2+\Gamma^2)(x^2+\Delta^2)(y^2+\Delta^2)(y^2+\gamma^2)^2}$$

$$= \; - \frac{1}{2\gamma} \frac{\partial}{\partial \gamma} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \; \frac{\Delta^4}{((x-y)^2+\Gamma^2)(x^2+\Delta^2)(y^2+\Delta^2)(y^2+\gamma^2)}$$

$$= \; \frac{\pi^2 \Delta^2 (\Gamma+2\Delta+\gamma)}{2\Gamma\gamma^2(\Delta+\gamma)(\Gamma+\Delta+\gamma)(\Gamma+2\Delta)} \left[-\frac{1}{\gamma} + \frac{1}{\Gamma+2\Delta+\gamma} - \frac{1}{\Delta+\gamma} - \frac{1}{\Delta+\gamma+\Gamma} \right]$$

(3.22)

$$D = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \; \frac{\Delta^4 (x-y)\, x}{((x-y)^2+\Gamma^2)(x^2+\Delta^2)(y^2+\Delta^2)(y^2+\gamma^2)^2}$$

$$= \frac{1}{2\gamma}\frac{\partial}{\partial\gamma}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy\ \frac{\Delta^4(x-y)x}{((x-y)^2+\Gamma^2)(x^2+\Delta^2)(y^2+\Delta^2)(y^2+\gamma^2)}$$

$$= -\frac{\pi^2\ \Delta^3\ (2\Delta+2\gamma+\Gamma)}{2\gamma(\Delta+\gamma)^2(\Delta+\Gamma+\gamma)^2(\Gamma+2\Delta)}\ .$$

(3.23)

Expanding the factors in the above integrals in terms of $\epsilon = \frac{1}{\Delta}$ and retaining only the first power of ϵ, we have

$$A = \frac{\pi^2}{\Gamma\gamma}\left(1-\epsilon\left(\tfrac{3}{2}\gamma+\Gamma\right)\right)$$

$$B = -\frac{\pi^2}{2}\left(1-\epsilon\left(2\gamma+\frac{\Gamma}{2}\right)\right)$$

$$C = \frac{\pi^2}{2\Gamma\gamma^2}\left(1-\epsilon\Gamma\right)$$

$$D = -\frac{\pi^2\epsilon}{2\gamma}$$

and

$$\frac{d\sigma_R}{d\Omega_d} = \frac{\pi^2\rho\bar{\rho}}{2D^2}\left[\Gamma(\gamma-\delta)A+B+2\gamma^2\delta\Gamma C+2\delta\gamma(-D)\right].$$

To first power of ϵ, the cross section is

$$\frac{d\sigma_R}{d\Omega_d} = \frac{\pi^2\rho\bar{\rho}}{4D^2}\left[1+\epsilon(2\delta-\Gamma)\right]. \qquad (3.24)$$

We now compare this cross section with the one which is obtained from the stationary theory by making the same approximations used in deriving (3.24).

$$\frac{d\sigma_R}{d\Omega_d} = \frac{1}{4k^2}|V_0 - 1|^2$$

$$= \frac{1}{4}\sum_{n,n'} \frac{R_n \bar{R}_{n'}}{(e_n + i\gamma_n)(e_{n'} - i\gamma_{n'})}$$

$$= \frac{1}{4}\sum_{n,n'} \mathrm{Re}\left[\frac{R_n \bar{R}_{n'}}{(e_n + i\gamma_n)(e_{n'} - i\gamma_{n'})}\right]$$

and with $R_n = \rho \dfrac{\Delta^2}{\Delta^2 + e_n^2}$ in the limiting case

$$\frac{d\sigma_R}{d\Omega_d} = \frac{\rho\bar{\rho}}{4D^2}\int dx \int dy \frac{\Delta^4 \left(xy + \frac{\Gamma^2}{4}\right)}{\left(x^2 + \frac{\Gamma^2}{4}\right)\left(y^2 + \frac{\Gamma^2}{4}\right)(x^2 + \Delta^2)(y^2 + \Delta^2)}$$

$$= \frac{\pi\rho\bar{\rho}}{4D^2}(1 - \epsilon\Gamma) \tag{3.25}$$

In the limit $\epsilon \to 0$, the results (3.24) and (3.25) are identical.

Under the conditions

(a) complete correlation of phase

(b) level spacing small compared with level widths, it has been shown that the cross section is independent of the quantal uncertainty. In other words the scattering is independent of the shape of the incident wave packet.

More generally, if the residues R_n are correlated in some other way, the cross section may be estimated approximately by placing $\rho = \langle R_n \rangle_\delta$ with $\langle R_n \rangle_\delta$ the average of the residues associated with the energies ϵ_n in the interval

$$E - \delta \lesssim \epsilon_n \lesssim E + \delta \qquad (3.26)$$

The cross section (3.24) then becomes for $\delta > \Gamma$, $D \ll \Gamma$, $\epsilon \to 0$,

$$\frac{d\sigma_R}{d\Omega_d} \sim \frac{\langle R_n \rangle_\delta \overline{\langle R_n \rangle_\delta}}{\langle D \rangle_\delta^2} \qquad (3.27)$$

with $\langle D \rangle_\delta$ the mean spacing of levels in the interval (3.26).

Similarly the equation for perfect resolution (3.25) becomes

$$\frac{d\sigma_R}{d\Omega_d} \sim \frac{\pi^2}{4} \frac{\langle R_n \rangle_{\frac{\Gamma}{2}} \overline{\langle R_n \rangle_{\frac{\Gamma}{2}}}}{\langle D \rangle_{\frac{\Gamma}{2}}^2} \qquad (3.28)$$

It should be noted that in deriving (3.27) the diagonal terms ($n=n'$) of (3.16) are neglected. This can be seen by considering the case of completely random phase for R_n, for which the equation (3.27) gives a vanishing cross section. In this situation the diagonal terms give the major contribution to the cross section.

$$\frac{d\sigma_R}{d\Omega_a} \sim \sum_n \frac{|R_n|^2 \, g_{nn'}}{4\,(e_n^2+\gamma_n^2)\,\Gamma_n}$$

$$= \sum_n \frac{|R_n|^2\left\{(E-\epsilon_n)^2\frac{\Gamma_n}{2} + \left(\frac{\Gamma_n}{2}+\delta\right)^2\left(\frac{\Gamma_n}{2}+2\delta\right)\right\}}{2\Gamma_n\left\{(E-\epsilon_n)^2+\left(\frac{\Gamma_n}{2}+\delta\right)^2\right\}}$$

$$\sim \frac{\langle R_n \bar{R}_n\rangle_\delta}{\langle D\rangle_\delta}\int\left(\frac{1}{x^2+a^2} + \frac{2\delta a}{\frac{\Gamma_n}{2}(x^2+a^2)^2}\right)dx$$

where $a = \frac{\Gamma_n}{2}+\delta$ is assumed to be constant.

Then

$$\frac{d\sigma_R}{d\Omega_a} \sim \frac{\pi\,\langle R_n \bar{R}_n\rangle_\delta}{\Gamma\,\langle D\rangle_\delta} \qquad (3.29)$$

3.4 DISCUSSION

In section 3.1 it was shown that the shape of a wave packet scattered from overlapping resonances depends not only on the level widths but also on the other resonance parameters. It was seen that one could not associate a definite lifetime with a resonant state which overlaps other states. In the region of overlapping levels the co-operative behaviour of the levels and their average properties are more important than their individual parameters.

This is confirmed when the cross sections for are considered. From equations (3.27) and (3.28), we see that the cross section for variable quantal depends on the correlation between levels. As the spread in energy of the incident wave packet is increased, more levels contribute to the averages which determine the cross section. By varying the quantal uncertainty at a fixed energy, the range of the correlations between the levels can be determined. For example if the levels are correlated in energy intervals of width $\sim\Gamma$ and uncorrelated over larger energies, as the time resolution of the beam is increased, there is a sharp drop in the cross section when δ becomes

larger than Γ.

We mention that if beams of suitable characteristics were obtainable for nuclear scattering experiments, in accordance with equation (3.27) the long range correlations between nuclear resonant states responsible for the gross structure of the cross sections could be observed directly, unobscured by the fluctuation cross section which masks the correlations over smaller energy intervals of Γ (equation 3.28). This averaging of the cross section which accompanies good time resolution will be discussed in more detail when the physical interpretation of the optical model is considered.

The cross section for Γ comparable to D have not been discussed in any detail and unfortunately it is difficult to say much about this region without evaluating (3.14) directly.

Nevertheless, the qualitative conclusion made for the case $\Gamma >> D$ apply. The cross section depends on the correlation between levels, since as δ is increased more terms make a significant contribution to (3.14), and in particular the off diagonal terms cancel if the phases of the residues R_n are random. Generally

the cross section falls off as δ is increased, but for correlated levels the rate of decrease is much slower than for uncorrelated levels. This is verified by calculation of (3.14) in particular cases.

CHAPTER 4.

CLASSICAL UNCERTAINTY AND POSSIBLE TIME-DEPENDENT EXPERIMENTS.

4.1 THE MEANING OF CLASSICAL UNCERTAINTY.

So far the possibility of uncertainties due to an incomplete specification of the initial state of the scattering system has not been considered. The initial wave packet of the preceding chapters is a pure state of the quantum mechanical system. Although it is not an eigenstate of the energy operator it is the eigenstate of some dynamical variable. The quantal uncertainty is a measure of the difference of this state from the scattering state of the stationary theory, and when accounted for, leads to a smoothing of the cross section and allows a description of the behaviour of the scattered particle in time.

Now suppose we describe the incident beam as a succession of wave packets of the form (2.9) but with different mean wave numbers $\underline{k}_o$ and different initial vectors $\underline{r}_o$. One such wave packet may be written

$$\psi_{ij}(\underline{r},0) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \Delta(\underline{k}_j, \delta ; \underline{k}) \exp(i\underline{k}\cdot(\underline{r}-\underline{r}_i))\, d^3k \qquad (4.1)$$

The subscript i labels the individual wave packets and j enumerates the various possibilities for the peak wave number $\underline{k}o$. The value of $\underline{r}i$ specifies the starting time of the wave packet at the source. Suppose further that the number of packets of type j is given by n ,

$$n = \Phi_j \cdot N \qquad (4.2)$$

with N the total number of packets.

We wish to know the cross section for a scattering experiment with an incident beam which can be described in this way. We must be careful to distinguish two cases.

Case A. If the $\underline{r}i$ are such that the wave packets do not overlap both before <u>and after</u> scattering, the scattering of each wave packet may be considered as an individual, distinct, event. The total cross section is then just the sum of the cross sections for the scattering of each wave packet.

Case B. The $\underline{r}i$ are such that the wave packets overlap. There will be interference effects between different packets and the simple relation of Case A between the total cross section and the individual cross sections is not valid.

The density matrix formalism provides a convenient mathematical description of these cases. For case A instead of taking wave packets with different $\underline{r}_i$, we use an incoherent superposition of wave packets with the same initial vector $\underline{r}_o$, weighted by the factor Φ_j . The initial density matrix is then

$$\rho(\underline{r},\underline{r}';0) = \sum_j \Phi_j \, \Psi_{oj}(\underline{r},0) \, \overline{\Psi}_{oj}(\underline{r}';0) \tag{4.3}$$

with

$$\Psi_{oj}(\underline{r},0) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \Delta(\underline{k}_j, \delta_j \underline{k}) \, e^{i\underline{k}\cdot(\underline{r}-\underline{r}_o)} \, d^3k \, . \tag{4.4}$$

At time t

$$\rho(\underline{r},\underline{r}',t) = \sum_j \Phi_j \, \Psi_{oj}(\underline{r},t) \, \overline{\Psi}_{oj}(\underline{r}',t) \, . \tag{4.5}$$

The wave packet $\Psi_{oj}(\underline{r},t)$ resulting from the scattering of the wave packet $\Psi_{oj}(\underline{r},0)$ of equation (4.4) is calculated by the methods of chapter 2. The density of particles at the detector is

$$\rho(\underline{r}_d,\underline{r}_d,t) = \sum_j \Phi_j \, \Psi_{oj}(\underline{r}_d,t) \, \overline{\Psi}_{oj}(r_d,t) \, . \tag{4.6}$$

If the packets are normalized as before, the cross section is

$$\frac{d\sigma}{d\Omega_d} = \frac{2\pi\delta \, v_o \, r_d^2}{\hbar} \int_{-\infty}^{\infty} \rho(\underline{r}_d,\underline{r}_d,t) \, dt \, . \tag{4.7}$$

After an interchange of the order of summation and integration, equation (4.7) yields

$$\frac{d\sigma}{d\Omega_d} = \sum_j \Phi_j \frac{2\pi\delta v_0 r_d^2}{\hbar} \int_{-\infty}^{\infty} |\psi_{0j}(\underline{r}_d,t)|^2 dt$$

$$= \sum_j \Phi_j \frac{d\sigma(\underline{k}_j,\delta)}{d\Omega_d} \tag{4.8}$$

where $d\sigma(\underline{k}_j,\delta)/d\Omega_d$ is the differential cross section for the scattering of the wave packet (4.4), with the labels $\underline{k}j$ and δ shown explicitly. For particular values of $\underline{k}j$ and δ this cross section must be calculated by the appropriate method of chapter 2 or 3. Equation (4.8) is the result stated above for case A.

For simplicity suppose that all the incident wave packets have the same direction of propagation and that only the peak energy $Ej = (\hbar \underline{k}_j)^2/2m$ varies. Also, assuming that there are many $\underline{k}j$ which are closely spaced, we make the approximation of replacing the summation over j by an integral so that Φ_j is replaced by $\Phi(E_0)\,dE_0$, then equation (4.8) becomes

$$\frac{d\sigma}{d\Omega_d} = \int \Phi(E_0) \frac{d\sigma(E_0,\delta)}{d\Omega_d} dE_0 \quad . \tag{4.9}$$

The cross section is just the energy average of the cross section for the scattering of a wave packet with peak

energy Eo and quantal uncertainty δ.

If δ is small enough, according to the discussion of section 2.2, the cross section for a particular wave packet is given by the perfect resolution theory. Therefore for beams of negligible quantal uncertainty the cross section is the energy average of the cross sections for scattering events with plane wave and outgoing spherical wave boundary conditions. In other words the cross section for the case under consideration is equal to the one obtained by performing a number of distinct experiments with ideal resolution at different energies and averaging the results with the weighting function $\Phi(E)$. By distinct, we mean physically distinct, that is separate in space or time. It is for this reason that we will call ϕ, the width of the weight function $\Phi(E)$, <u>the classical uncertainty.</u>

4.2 THE DISTINCTION BETWEEN CLASSICAL AND QUANTAL UNCERTAINTY.

It is interesting to ask whether, given the most general density matrix describing the incident particles (for this spinless case), we are able to maintain the distinction between classical and quantal uncertainty. In terms of the plane wave states $\eta(\underline{k})$, the most general density matrix at time $t=0$ is

$$\rho(\underline{r},\underline{r}',0) = \iint a(\underline{k}_0,\underline{k}_0')\,\bar{\eta}(\underline{k}_0')\,\eta(\underline{k}_0)\,d\underline{k}_0\,d\underline{k}_0' \tag{4.10}$$

After writing the density matrix (4.10) in terms of the wave packet states Ψ_{0j} such that

$$\rho(\underline{r},\underline{r}',0) = \sum_j \Phi_j\,\Psi_{0j}\,\bar{\Psi}_{0j} \tag{4.11}$$

we ask whether this diagonalization is unique. If so a definite classical uncertainty ϕ associated with Φ_j and a definite quantal uncertainty δ associated with Δ can be assigned to the incident beam. If not the distinction is meaningless.

In mathematical terms the question is stated: can the hermitian matrix (4.10) be diagonalized by more than one transformation of the type (4.4) of the basis vectors? A necessary condition for the diagonalization to be unique is that the states Ψ_{0j} must be orthogonal like the states $\eta(\underline{k})$.[24]

In fact as they stand the wave packet states are only approximately orthogonal.

$$\int \psi_{oj}\, \bar{\psi}_{oj'}\, d\underline{r}$$

$$= \frac{1}{(2\pi)^3} \iint \Delta(\underline{k})\, \bar{\Delta}(\underline{k}')\, e^{i(\underline{k}-\underline{k}')\cdot\underline{r}_0} \int \eta(\underline{k})\, \bar{\eta}(\underline{k}')\, d\underline{k}\, d\underline{k}'\, d\underline{r}$$

$$= \int \Delta(\underline{k}_j, \delta; \underline{k})\, \bar{\Delta}(\underline{k}_j', \delta; \underline{k})\, d\underline{k} \qquad (4.12)$$

However, it must be remembered that the different initial vectors $\underline{r}_j$ in (4.3) have been suppressed and in fact the wave packet states may not overlap by virtue of having initial vectors which are sufficiently different. Yet is is also obvious that in the matrix (4.11), there must be only a finite number of such wave packet states if they are to be made orthogonal by taking them with different $\underline{r}_j$. But a finite number of Lorentzian packets ψ_{oj} do not form a complete set so that clearly it is not always possible to write the density matrix (4.10) in the form (4.11) in a unique way.*

These considerations confirm the distinction between the cases A and B. For case A we have verified that the quantal and classical uncertainty have definite values (the diagonalization is unique), whereas for

* It is always possible to perform the diagonalization uniquely in terms of the eigenfunctions of ρ. However, these states may not have the required wave packet properties.

case B, ϕ and δ have no such definite meaning.

In case A the wave packets have a definite physical significance; the quantal uncertainty leads to alterations in the usual cross sections. In case B the wave packets may be regarded as mathematical aids to the visualization of the scattering process. For example we can decompose a plane wave into an infinite number of overlapping wave packets and then follow the propagation of one of these packets. Nevertheless this would be without physical significance because the scattering amplitudes for all the packets must be recombined to obtain the observed cross section.

This point is vital to the discussion below. The notion of quantal uncertainty has physical meaning only if we know from an analysis of the experimental production of the beam that the wave packets constituting the beam are quite distinct and do not overlap.

4.3 DESIGN OF EXPERIMENTS.

There are several possible descriptions of the incident beam in a scattering experiment and the methods of calculating the cross sections associated with these descriptions are not equivalent. Furthermore the results derived from these calculations reveal different aspects of the same scatterer. Some kinds of initial conditions lead to information about the behaviour in time whereas others give information about the energy properties of the system. Some possibilities for the initial conditions and the corresponding methods of calculation of the cross section are listed in table 1. The type of boundary condition for a particular experiment is determined by the detailed, physical conditions at hand, and the next step in this discussion is to take up the question of just which description is relevant to actual experiments. When beams of poor energy resolution are used in scattering experiments, we would like to know how much of the energy spread is due to classical and how much is due to quantal uncertainties. How do we describe the state of the particles making up the beam?

These are physical questions which must be answered on the basis of a detailed analysis of the mode of

Table 1.

Particles of definite energy. All particles have the same energy.	monochromatic plane waves	$\phi \ll \Gamma$	$\delta \ll \Gamma$	$\frac{d\sigma}{d\Omega} = \frac{d\sigma(\delta, E)}{d\Omega}, \ \delta \to 0$ or $\frac{d\sigma}{d\Omega} = \lvert f \rvert^2$ $f =$ scattering amplitude of stationary theory
Particles of determinate energy. Energy eigenstate uncertain.	non-coherent plane waves	$\phi \gg \Gamma$	$\delta \ll \Gamma$	$\frac{d\sigma}{d\Omega} = \int \Phi(E) \frac{d\sigma}{d\Omega}(\delta, E)\, dE, \ \delta \to 0$ or $\frac{d\sigma}{d\Omega} = \int \Phi(E) \lvert f \rvert^2 dE$
Particles of indeterminate energy (in wave packet states). All particles in same state.	wave packets with same peak energy.	$\phi \ll \Gamma$	$\delta \gg \Gamma$	$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{d\Omega}(\delta, E)$
Particles of indeterminate energy. Particles in different states.	wave packets with different peak energies	$\phi \gg \Gamma$	$\delta \gg \Gamma$	$\frac{d\sigma}{d\Omega} = \int \frac{d\sigma}{d\Omega}(\delta, E)\, \Phi(E)\, dE$

preparation of the beam. In the author's opinion further analysis of the actual details of preparation of states would be worthwhile. Nevertheless, for the usual scattering experiments, we take the view that poor energy resolution is due almost entirely to classical rather than quantal uncertainty; the incident particles may be assumed to be in eigenstates of energy, although often we are unable to say which eigenstate; the beam consists of an incoherent superposition of wave packets of long duration. We accept this description for the following reasons. If the beam is considered to consist of wave packets of short duration, there is an ambiguity about their position vectors as was explained in section 4.2. Since there is no other information to hand, to avoid bias, all possible initial position vectors must be assumed to be equally probable; the wave packets overlap and have no physical significance. Consider a numerical example. A beam of 9 MeV neutrons has an energy resolution of 100 KeV The length of the wave packet needed to produce a quantal uncertainty of 100 KeV is

$$\sim \frac{\cdot 91\, E^{\frac{1}{2}}}{\Delta E} \approx 30 \text{ fermis} \qquad (4.13)$$

which is very much smaller than the dimensions of the scattering apparatus. It is difficult to see how such a small characteristic length could enter the theory.

The factors which cause less than perfect resolution apart from classical uncertainties can only be due to the finite size of the experimental apparatus and the fact that the beam was switched on some time in the past. Both these limitations involve extremely small quantal uncertainties because they involve macroscopic times and dimensions, e.g. from the relation (4.13), a wave packet for 9 MeV neutrons which is one metre long has a quantal uncertainty of the order of 10^{-14} MeV!

Yet if there is a characteristic rhythm introduced during the production of the beam, the quantal uncertainty may be large and should lead to observable consequences. For example, the particle source might emit particles in regular bursts, or we could imagine some device rapidly opening and shutting the collimating slits so that the very long wave packet of the steady state experiment is chopped into many, short wave packets which do not overlap. For this experi-

ment the quantal uncertainty is roughly $\hbar/\tau$ where τ is the period of the chopper.

To produce appreciable quantal uncertainties for nuclear experiments, the chopping device would have to be very much faster than any mechanical or electronic device in existence at present. An exception will be discussed in the next section. At lower energies, for atomic systems, time-dependent experiments which would require the theory of chapters 2 and 3, are feasible. In particular, an experiment which could easily be modified to test some of the conclusions of chapter 2, is described in the next section.

4.4 POSSIBLE EXPERIMENTAL VERIFICATION.

The necessary condition for a time-dependent scattering experiment is that the experimental definition of time must be accurate in comparison with the characteristic time $\hbar/\Gamma$ of the scattering amplitude. That is we require $\delta > \Gamma$.

Experimental definition of time is at present possible for times as short as about 10^{-10} sec. Typical nuclear values for $\hbar/\Gamma$ are 10^{-15} sec., so experiments with good time resolution cannot be performed with nuclei except in special cases. These cases are metastable states which have life times as long as 10^{-7} sec.

In the experiment of Holland et al.[37,38] which utilizes the Mössbauer effect, the nucleus Fe^{57} was prepared in an excited state by the spontaneous decay of Co^{57}. The Fe^{57} nucleus returns to its ground state by emitting a 137 KeV γ-ray or a 123 KeV γ-ray followed by a 14 KeV γ-ray. The 14 KeV resonant state has a long life-time of 10^{-7} sec. The time of formation of the 14 KeV level was defined by observing the time of emission of the 123 KeV γ-ray (a fast decay). The spectrum of total elapsed time between the excitation of the 14 KeV level and the detection

of the corresponding γ-ray after it had passed through a foil of resonant absorber was then measured. The time spectrum of the final state and the increased width of the absorption line were both observed.

Our interest in the experiment derives from these factors:

(i) time intervals comparable to the lifetime of the resonance are defined and thus the wave packet method is relevant, the condition $\delta \sim \Gamma$ holding rather than the condition $\delta << \Gamma$ of the usual scattering experiment.

(ii) the Mössbauer effect eliminates classical uncertainties which predominate over the quantal uncertainty in the usual scattering experiment.

(iii) the observation of the 123 KeV radiation determines the quantal uncertainty of the system.

We can regard this experiment as a scattering experiment in which the packet incident upon the resonant absorber has a half exponential time spectrum with $\delta = \hbar/\Gamma_s = \frac{\hbar}{10^{-7}}$ sec. Defining the starting time of the excitation of the metastable state puts a quantal uncertainty into the beam of Γ_s. The method of Holland et al. defines the starting time with a minimum tolerance equal to the lifetime of the next highest state in the γ-ray cascade. This is shorter than the experimental time resolution which is itself much shorter than $\hbar/\Gamma_s$.

If the energy amplitude of the incident packet is taken as

$$\Delta(E_0, \Gamma_s ; E) = \frac{i/2\pi}{E - E_0 + i\Gamma_s/2} \tag{4.14}$$

it follows from equation (2.36) that the cross section is

$$\frac{d\sigma}{d\Omega_d} = \frac{R_s^2}{2\Gamma_s} \frac{1}{(E_0 - E_s)^2 + \Gamma_s^2} \tag{4.15}$$

Thus we have the well known result that the line width in the Mossbauer effect is twice the width of the level, assuming all the nuclei in the target foil are capable of absorbing resonantly.

The width of the incident packet in this experiment is fixed. One way of varying the time width of the wave packet would be to vary the resolution with which the starting time is measured. It is possible by this method to decrease δ, but not to increase it.

A better experimental way of varying the wave packet width has been suggested by the author and I.E. McCarthy.[14] A third resonant absorber is introduced between the source and target. The absorber is accelerated in a very short time (10^{-9} sec.) to a speed sufficient to shift the resonance so that the incident beam is no longer absorbed and can hit the target. It is then slowed down again quickly so that the duration of the pulse is of the order of 10^{-9} sec.

This method would produce approximately the exponential wave packet that has been used in the calculations of chapter 2. The broadening of the absorption cross section could be observed by altering the frequency of the pulsing.

It is experimentally just possible to produce the requisite pulsing by using a very thin foil of Fe^{57} as one plate of a parallel plate condenser in a vacuum which is charged first with one sign and then with the opposite sign by a rf pulse. However, the acceleration may be achieved more easily by using a piezo-electric crystal. Another possibility is to use the stark effect to shift the resonance. This requires a tightly bound dielectric crystal containing nuclei with a metastable state.

Wave packet experiments in the atomic energy region may be of interest. An absorber of laser material placed in a laser beam and moved for a short time as suggested above would produce wave packets of laser intensity.

All such experiments observe only the scattering of a wave packet from a single resonance with a trivial angular distribution. Overlapping resonances even in atoms, would probably have widths of the order of a few tenths of an electron volt. Wave packet experiments in this region with $\delta \sim \Gamma$ would require time lengths of

10^{-15} sec. which seem impossible at present. The possibility of doing time dependent electron scattering experiments from atoms is not experimentally remote.

With laser wave packets, it may be possible to observe the changes in angular distribution due to the interference of potential and resonant scattering as the quantal uncertainty is changed. The potential scattering could be obtained by diffraction of laser light around a small crystal of laser material.

CHAPTER 5

COMPLEMENTARITY OF ENERGY AND TIME DESCRIPTIONS

5.1. THE ROLE OF TIME IN THE DESCRIPTION OF SCATTERING.

The arguments already stated make the following, basic points

(i) Various initial conditions in the quantum theory of scattering lead to meaningful descriptions of the scattering process.

(ii) These conditions are not equivalent because the physical answers depend on the choice of initial conditions.

(iii) The particular form of the initial conditions relevant to a given experiment must be decided from an analysis of the mode of preparation of the beam.

In the time dependent experiments of the last chapter either the observation of the 123 kev γ-ray in the case of the experiment by Holland et al., or the pulsing of the beam makes reference to a time origin possible. We emphasize again that in the scattering experiments in which the quantal uncertainty is small (and if our discussion of section 4.3 is correct, this condition applies to the usual scattering experiment) small time intervals are not defined. The poor energy resolution of this type

of experiment results from classical uncertainties, that is an incomplete specification of the state of the incident particles.

It is essential to make a distinction between what can be observed about a system in a given experiment and what can be observed in principle by all theoretically possible experiments. In chapter 2 it was shown that if we wish to consider the behaviour of the system in time, the incident packets must be of short duration. In other words a single energy eigenstate does not provide an adequate description of the scattering. For small quantal uncertainties and wave packets of macroscopic length, the shape and temporal behaviour of the wave packet is independent of the properties of the scatterer. So a consequence of the fact that actual experiments have only small quantal uncertainties is that the temporal properties of the scatterer are not revealed. Rigorously, the sequential description of the interaction of the beam with the scatterer is invalid for these experiments; with wave packets of macroscopic dimensions a sequence of events on the nuclear or atomic scale is not defined.

In spite of this fact many arguments made about nuclear reaction mechanisms in particular depend on a

well defined ordering of events and the use of characteristic times which are very small compared with the time intervals defined in the experiment. For the purpose of forming an intuitive 'picture', this is a harmless enough procedure, if the necessary reservation is made that this type of argument is based on the complementarity of energy and time; the experiment to which the argument in terms of characteristic times applies, is hypothetical and complementary to the actual experiment.

Let us review briefly the usual ways of introducing characteristic times into the scattering process. For definiteness we consider the scattering from an isolated resonance level. If the beam of the actual experiment has nearly ideal energy resolution, there is an energy 'bump' in the cross section. The simplest way of introducing a time interval is to invoke the uncertainty principle[39]. The energy width Γ of the resonance is assumed to result from the finite lifetime of a metastable state. With the aid of the uncertainty relation, the lifetime of this state is defined as

$$\tau = \frac{\hbar}{\Gamma} \tag{5.1}$$

However, in this experiment the energy of the incident and scattered particles is quite definite, so

that it is incorrect to conclude from relation (5.1) that the time τ is actually observable. We suggest the following interpretation: the uncertainty relation (5.1) implies that if the complementary experiment is performed so that very short wave packets are incident on the scatterer, the particles are delayed on the average by τ (c.f. section 2.7). But then the variation of the cross section with energy is structureless. Thus τ and Γ cannot be observed in the same experiment but the relation (5.1) may be tested by measuring τ and Γ in complementary experiments.

Instead of using the relation (5.1), F.T. Smith[40] has made a direct definition of the delay time on collision time of a scattering event by an extension of the definition (1.1) of Wigner and Eisenbud. He introduces the lifetime matrix Q,

$$Q = -i\hbar U^* \frac{dU}{dE} \qquad (5.2)$$

U is the collision matrix. For an isolated resonance the definition (5.2) admits a definite lifetime at a precise energy which is inconsistent with the principles of quantum mechanics. To obviate this difficulty, the observable lifetime is taken as an average value

$$\langle Q \rangle = \int_0^\infty P(E)\, Q(E)\, dE \qquad (5.3)$$

where $P(E)$ is to be determined by the experimental conditions. In our theory the wave packet nature of the beam introduces this average in a natural manner at the beginning of the scattering calculation. As Smith suggests the collision matrix U and the lifetime matrix Q both contain the same information but refer to different experimental contexts. In our language Q is more useful in situations where the quantal uncertainty is large whereas U is of interest when the quantal uncertainty is small. Of course there may be intermediate situations ($\delta \sim \Gamma$) in which neither U nor Q are directly applicable in which case a formalism like that of chapters 2 and 3 should be employed.

Goldberger and Watson[41] have shown how the definition of time delay (1.1) can be used to provide a 'coarse-grained' ordering of scattering events. They show that 'if a complex process involves a sequence of interactions sufficiently separated in time, the S-matrix <u>factors</u> into a product of S-matrices for the separate interactions.' Thus one can order the individual events and the time delays associated with the various stages of the process are additive. This result is based on the replacement of the exact unitary operator $U(t_n, t_{n-1})$ which describes the n-th stage of the interaction, by the S-matrix element

$S = U(\infty, -\infty)$. This means that the time interval $\Delta t_n = t_n - t_{n-1}$ must be such that

$$\frac{\hbar}{\Delta t_n} \ll \Gamma \tag{5.4}$$

where Γ is the characteristic energy fluctuation of the S-matrix element. In our terminology each of the individual stages must be considered as having small quantal uncertainty so that the time ordering is macroscopic.

In contrast our approach is based on the principle of superposition of states for energy and time. By taking a coherent superposition of energy eigenstates, we are able to define and <u>observe</u> time durations of quantum events. The scattering system is not always on the energy shell between scatterings.

Both the present formalism and that of Goldberger and Watson use only the asymptotic form of the scattering state. Thus the experiment suggested in the last chapter provides a test of the superposition principle for energy and time but not the Schrödinger equation, since the time development of the system is not observed in the vicinity of the scatterer.[14]

The above application of the idea of complementarity has been stressed, because mistakes in the interpretation

of scattering experiments are often made through neglecting some of its implications. In particular it does not follow that the time-dependent picture of the complementary experiment can be used to justify approximations in the description of the actual experiments.

As an example of this type of erroneous argument, consider the following: potential and resonant scattering occur on different time scales, therefore they are incoherent and do not interfere, that is if Ψ_P is the wave function for the potential scattering and Ψ_R the wave function for the resonant scattering, the term depending on $2\mathcal{R}e(\Psi_P \bar{\Psi}_R)$ may be dropped from the cross section.

$$\sigma \sim |\Psi_P + \Psi_R|^2 = |\Psi_P|^2 + |\Psi_R|^2 + 2\mathcal{R}e(\Psi_P \bar{\Psi}_R)$$
$$\sim |\Psi_P|^2 + |\Psi_R|^2 \qquad (5.5)$$

Now it is true that in the experiment with large quantal uncertainty, the resonant wave packet has a tail and overlaps less with the potential wave packet as the quantal uncertainty is increased and the incident and potential wave packets become more localized. It does not follow that the interference term can be omitted in the experiment with good energy resolution, as the wave packets Ψ_P and Ψ_R are very long and overlap.

In fact this fallacious argument is not applied to this case of potential and resonant scattering. In practice one calculates the interference between the isolated resonance and the background scattering. Despite this, in a parallel case, the interference of direct and compound scattering in the region of overlapping levels is neglected by many authors on the basis of this same argument. Examples will be given when the optical model is considered.

A second type of misleading argument results from the failure to distinguish between what we have called the classical and quantal uncertainties. If there is ΔE uncertainty in the energy of the incident beam, we cannot deduce that times are defined to within $\hbar/\Delta E$. As we have shown in section 4.2 poor resolution is usually due to classical uncertainty in which case, nothing more is known about time than would be known if the beam consisted of a single energy eigenstate. The uncertainty relation for time and energy specifies the ideal amount of information in an experiment with classical uncertainty energy may be poorly defined and time not at all.

5.2 EXAMPLE: THE COMPOUND NUCLEUS

The criticisms above may be applied to the concept of the compound nucleus in a very instructive way. This example is taken because of its familiarity. The same sort of arguments appear in the next chapter.

The idea of dividing nuclear reactions into two distinct stages was introduced by Bohr[8,9] in 1936 and proved a stimulating picture of nuclear reactions, dominating the theory for some years. Briefly stated the compound nucleus assumption[8] is

> " ..... a collision between a high speed neutron and a heavy nucleus will in the first place result in the formation of a compound system of remarkable stability. The possible later breaking up of this intermediate system by the ejection of a material particle or its passing with the emission of radiation to a final stable state, must in fact be considered as separate competing processes which have no immediate connection with the first stage of the process".

The argument justifying this assumption is often stated in this way[43]:

".... the energy of the incident particle is rapidly shared between all the nucleons of the system and it is not until sufficient energy is again concentrated on one nucleon that the compound nucleus can decay with the omission of that nucleon. This is likely to take a considerable time on a nuclear time scale the characteristic time of a nucleus being the time it takes for a nucleon to travel across the nucleus and this is of the order 10^{-12}cms/10^{9}cms/sec = 10^{-21} sec. Thus the compound nucleus will have 'forgotten' how it was formed by the time it decays."

This description is only valid if the incident particles are represented by extremely short wave packets, small compared with the dimensions of the nucleus ($\delta \gg \Gamma$), so that the above sequential picture refers to the experiment complementary to the actual scattering experiment ($\delta \ll \Gamma$). The basic mistake in the analysis of the scattering mechanism is the association of a definite characteristic time with the nucleus without regard to the experimental context. In actual experiments the characteristic time is the time taken for the wave packet to pass the nucleus, which is much larger than both 'the lifetime' of the compound nucleus and 'the time taken for

a nucleon to cross the nucleus', so that these terms are meaningful only in the complementary sense.

For an isolated resonance the temporal argument gives the same result as the more rigorous argument in terms of energies which relies on the condition that the level width is much less than the level spacing; the cross section factors into the product a cross section for the formation of the compound nucleus and a probability for the decay of this compound nucleus in a particular way (the single level Breit-Wigner formula[39]). However, in the region of overlapping levels, Γ comparable to D, the temporal argument leads to wrong conclusions.

It has been pointed out by Ericson[44] that with long wave packets associated with good energy resolution, the essential part of the wave packet for compound scattering is already being emitted while the incident wave packet is interacting with the nucleus. This is also clear from equation (3.1). If $\delta \ll \Gamma$ the wave packet for the resonant scattering has the same spread in space as the incident wave packet, and is not delayed with respect to the incident packet. For a beam of good energy resolution, in the overlapping resonance region, there is little justification for making the assumption of the independence of formation and decay of the compound nucleus.

Furthermore, even with beams of poor resolution, the temporal arguments only apply if the uncertainty is quantal in nature. According to our analysis the uncertainty is classical, and therefore the temporal argument is invalid.

In his discussion[44] of the compound nucleus, Ericson appears to have confused the part played by classical and quantal uncertainty.[6] He assumes that if the beam has energy spread ΔE, the incident wave packet has time duration $\hbar / \Delta E$, and because of the localization of the incident packet the compound and direct contributions do not interfere.

Now the cross section for the experiment with classical uncertainty is the average over energy of the cross section for ideal resolution (section 4.1). This provides a way in which the compound nucleus assumption may be justified for the average cross sections.

For example Bethe's assumption of random signs[46] allows the <u>average</u> cross section to be factored into two parts for particular values of the angular momentum of the compound nucleus.

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6 In discussion of this point Ericson has stated[45], 'Ordinary beams are classically sufficiently well defined that this should not lead to difficulties. The quantal widths are typically in the kev region which is quite large'.

Let us outline Bethe's arguments in the context of the HSR expansion. The generalization of equation (2.31) which takes account of the possibilities of other collision processes in addition to elastic scattering is given by equation (A1) of Appendix A.

$$U^{(J)}_{c'c} - \delta_{c'c} = \sqrt{k_{c'}k_c}\left[Q^{(J)}_{c'c}(E) - i\sum_\lambda \frac{R^{(J)}_{c'\lambda}\, R^{(J)}_{c\lambda}}{E - E_\lambda}\right.$$

$$E - E_\lambda = E_c - E_{c\lambda} + \tfrac{1}{2}i\Gamma_\lambda = E_{c'} - E_{c'\lambda} + \tfrac{1}{2}i\Gamma_\lambda \qquad (5.6)$$

The subscript c represents the set of quantum numbers which characterize the ingoing channel while c′ labels the outgoing channel.

The average cross section apart from the background scattering is then

$$\langle \sigma_{c'c} \rangle \sim \frac{1}{\epsilon}\int_\epsilon \sum_{\lambda,\mu} \frac{R_{c'\lambda}\, R_{c\lambda}\, \bar{R}_{c'\mu}\, \bar{R}_{c\mu}}{(E - E_\lambda)(E - \bar{E}_\mu)}\, dE \qquad (5.7)$$

which, following the procedure of Bethe[46], is

$$\frac{1}{\epsilon}\sum_{\lambda,\mu\,(\text{in}\,\epsilon)} \frac{R_{c'\lambda}\, R_{c\lambda}\, \bar{R}_{c'\mu}\, \bar{R}_{c\mu}\,(\Gamma_\lambda + \Gamma_\mu)}{(E_{c\mu} - E_{c\lambda})^2 + \frac{1}{4}(\Gamma_\lambda + \Gamma_\mu)^2} \qquad (5.8)$$

with the sums over λ, μ to be taken in the energy interval ϵ, the spread of the classical uncertainty.

If we assume that the phases of $R_{c'\lambda}$ and $R_{c\lambda}$ are uncorrelated, the non-diagonal terms ($\lambda \neq \mu$) may be dropped.

$$\langle \sigma_{c'c} \rangle \sim \frac{1}{\epsilon} \sum_{\lambda (in \epsilon)} \frac{|R_{c'\lambda}|^2 \, |R_{c\lambda}|^2}{\Gamma_\lambda} \tag{5.9}$$

which can be written

$$\langle \sigma_{c'c} \rangle \sim \frac{1}{\epsilon} \sum_{\lambda (in \epsilon)} \frac{q_\lambda^2 \, \Gamma_{c\lambda} \, \Gamma_{c'\lambda}}{\Gamma_\lambda} \tag{5.10}$$

with

$$\sum_c \Gamma_{c\lambda} = \Gamma_\lambda \, . \tag{5.11}$$

With the assumption that the partial widths $\Gamma_{c\lambda}$ are uncorrelated for different channels (5.10) becomes

$$\langle \sigma_{c'c} \rangle \sim \frac{q^2 \Gamma_c}{D} \cdot \frac{\Gamma_{c'}}{\Gamma}$$

$$= \sigma_c \frac{\Gamma_{c'}}{\Gamma} \tag{5.12}$$

with D the average energy spacing between levels, and the suffix λ is dropped, as the quantities $\Gamma_{c'}$ and Γ and q refer to the averages of $\Gamma_{c'\lambda}$, Γ_λ, q_λ in the energy interval ϵ. σ_c is interpreted as the

probability for compound nucleus formation through channel c, and $\frac{\Gamma_{c'}}{\Gamma}$ as the probability for the decay of this nucleus through channel c .

It is clear that characteristics of compound nucleus formation, such as symmetry of angular distributions and independence of formation and decay are only seen in experiments with beams with classical uncertainty. The misleading argument which depends on times, suggests that these characteristics be present at definite energies.

The important point here is that the compound nucleus assumption can only be justified on dynamical grounds. In a sense the introduction of wave packet arguments is specious, since emphasis is shifted from the underlying dynamical assumption of random phases. The wave packet picture complements our understanding of compound nucleus formation but is no substitute for the rigorous argument in terms of energy.

In summary, although the time dependent picture may be helpful in suggesting a model of the scattering interaction, it is necessary to make apparent the corresponding approximations in the experiment with classical uncertainty.

CHAPTER 6

THE INTERPRETATION OF THE OPTICAL MODEL

6.1 DEFINITION OF THE OPTICAL MODEL AND THE PICTURE OF FRIEDMAN AND WEISSKOPF.

In this chapter we use the scattering theory of the preceding chapters to discuss the physical significance of the optical model. In an extension of the argument of Friedman and Weisskopf[10], we give an interpretation of the energy averages of the optical model by considering time dependent experiments complementary to the ordinary energy dependent experiments.

Strictly the results already derived apply to the scattering from a potential, but the generalization to the many-body case is simple. In an appendix it is shown that the results of the preceding chapters hold for the elastically scattered wave packet of the many channel problem.

In the fundamental theoretical papers[3,4,7,25] which relate the optical model to the general theory of nuclear reactions, the optical model is defined by requiring that the complex potential of the model reproduce the average elastic scattering amplitude of the formal, exact, many channel theory:

$$U_\ell^{opt} = \langle U_\ell \rangle \qquad (6.1)$$

where the bracket denotes an energy average defined in this way

$$\langle f(E) \rangle = \int K(E-E') f(E')\, dE' \qquad (6.2)$$

with $K(E-E')$ a suitable weighting function.

Feshbach, Porter and Weisskopf[I] first gave the relationships between the cross sections predicted by the model and the cross sections averaged over energy.

$$\begin{aligned}\langle \sigma_{el} \rangle &= \frac{\pi}{k^2} \Sigma_\ell (2\ell+1) \langle |1-U_\ell|^2 \rangle \\ &= \frac{\pi}{k^2} \Sigma_\ell (2\ell+1) \{ |1-\langle U_\ell \rangle|^2 + \langle |U_\ell|^2 \rangle - |\langle U_\ell \rangle|^2 \} \\ &= \sigma_{el}^{opt} + \frac{\pi}{k^2} \Sigma_\ell (2\ell+1) \{ \langle |U_\ell|^2 \rangle - |\langle U_\ell \rangle|^2 \}\end{aligned} \qquad (6.3a)$$

$$\begin{aligned}\langle \sigma_{re} \rangle &= \frac{\pi}{k^2} \Sigma_\ell (2\ell+1) (1 - \langle |U_\ell|^2 \rangle) \\ &= \frac{\pi}{k^2} \Sigma_\ell (2\ell+1) \{ 1 - |\langle U_\ell \rangle|^2 + |\langle U_\ell \rangle|^2 - \langle |U_\ell|^2 \rangle \} \\ &= \sigma_{re}^{opt} + \frac{\pi}{k^2} \Sigma_\ell (2\ell+1) \{ |\langle U_\ell \rangle|^2 - \langle |U_\ell|^2 \rangle \}\end{aligned} \qquad (6.3b)$$

$$\begin{aligned}\langle \sigma_{tot} \rangle &= \frac{\pi}{k^2} \Sigma_\ell (2\ell+1) (1 - Re \langle U_\ell \rangle) \\ &= \sigma_{tot}^{opt}\end{aligned} \qquad (6.3c)$$

$\langle \sigma_{el} \rangle$, $\langle \sigma_{re} \rangle$, $\langle \sigma_{tot} \rangle$ are the elastic, reaction and total cross sections averaged over energy whereas σ_{el}^{opt}, σ_{re}^{opt}, σ_{tot}^{opt} are the cross sections predicted by the model.

FPW call the cross section for elastic scattering predicted by the optical model, the shape elastic cross section $\sigma_{s.el}$, and the cross section

$$\frac{\pi}{k^2}\sum_{\ell}(2\ell+1)\{\langle|U_\ell|^2\rangle - |\langle U_\ell\rangle|^2\} \qquad (6.4)$$

the compound elastic cross section $\sigma_{c.el}$. The relations (6.3) are expressed more succintly as

$$\langle\sigma_{el}\rangle = \sigma_{s.el} + \sigma_{c.el} \qquad (6.3a')$$

$$\langle\sigma_{re}\rangle = \sigma_{re}^{opt} - \sigma_{c.el} \qquad (6.3b')$$

$$\langle\sigma_{tot}\rangle = \sigma_{tot}^{opt} \qquad (6.3c')$$

Although the names 'compound elastic' and 'shape elastic' have physical associations, it is important to note that the above definitions are entirely formal. Without further assumptions the terminology is arbitrary and the only virtue of the model as defined above is that it yields the average total cross section; the equations (6.3) are merely the consequences of the <u>formal</u> identification of the amplitudes in equation (6.1). To avoid confusion we call the cross section defined by (6.4) the fluctuation cross section and reserve the term 'compound elastic cross section' for the cross section associated with the decay of the compound nucleus.

Friedman and Weisskopf[10] attempted to show that although both terms on the right hand side of equation (6.3a) represent elastic scattering, they are physically quite distinct and describe different types of scattering process. Their argument centres on showing the equivalence of the reaction cross section of the model and the cross section for compound nucleus formation. The nuclear collision is depicted as a one or two stage process. An incident particle may be elastically scattered immediately on reaching the target nucleus, or it may unite with the nucleus forming a long lived compound state. This compound state may then decay in such a way that a particle of the same kind and energy as the incident particle is emitted. This process is called compound elastic scattering. In this picture, which is an extension of Bohr's original compound nucleus idea, the cross sections predicted by the model have a clear physical significance. The cross section for compound elastic scattering is transferred from the average elastic cross section to the reaction cross section. The reaction cross section gives the number of particles absorbed from the incident beam, even though some of these may be re-emitted without change of energy. Thus if the collision process can be considered

as taking place in two stages which are quite distinct in time, a clear interpretation of the model cross sections is possible.

An immediate objection which can be brought against this picture is that it depends on the identification of the compound elastic scattering with the time delayed part of the elastic scattering. We have already discussed in the last chapter the difficulties involved in the description of actual scattering experiments in terms of a sequence of micro-events, but before presenting this objection in detail let us outline the arguments that Friedman and Weisskopf make to justify their interpretation.

Using a resonance expansion for the collision matrix, they show that the identification of the compound elastic cross section with the fluctuation cross section is correct in the low energy region ($\Gamma \ll D$). They also remark that the equivalence holds in the high energey region ($\Gamma \gg D$) for the following reasons: compound elastic scattering is negligible in this energy range because there are so many competing reaction modes and the fluctuation cross section is very small as the high energy amplitudes are smooth functions of energy.

Furthermore, they split the collision matrix into two parts

$$U_\ell = \langle U_\ell \rangle + \{U_\ell - \langle U_\ell \rangle\} \qquad (6.5)$$

and take $\langle |U_\ell - \langle U_\ell \rangle|^2 \rangle = \langle |U_\ell|^2 \rangle - |\langle U_\ell \rangle|^2$. (6.6)

Then by considering the scattering of a suitable wave packet they show that the part of the wave packet corresponding to the average amplitude $\langle U_\ell \rangle$ is propagated immediately, whereas the wave packet corresponding to the fluctuation amplitude $U_\ell - \langle U_\ell \rangle$ has an exponentially decaying tail. In this way the fluctuation cross section is identified with the time delayed part of the scattering, which is just that contribution which one would expect from the decay of the compound nucleus.

The following limitations and difficulties of their argument will be taken up in this chapter.

A. The medium energy case (which is of most interest) has not been discussed. We have already noted in chapter 4 the difficulty of defining the compound nucleus in the region of overlapping resonant states ($\Gamma \sim D$). The nucleus excited to these energies cannot be conceived as being in a definite resonant state as for lower energies of excitation. The state of the system is a superposition of

many resonant states and its lifetime cannot be associated with the width of any particular resonance. The lifetime depends on not just the average width of levels but also the other resonance parameters.

B. The temporal interpretation is based on the correspondence

fluctuation scattering	shape elastic
↕	↕
delayed wave packet	immediately scattered packe

Friedman and Weisskopf show that the wave packet associated with the fluctuation amplitude contains a delayed component, but they do not demonstrate that the part of the fluctuation packet which is undelayed is small. Thus there is an omission in their argument, for if this part is considerable there is a large overlap between the fluctuation and the shape elastic packets and the distinction between the two sorts of scattering is rather arbitrary.

C. The difference between quantal and classical uncertainty is obscured. As indicated in the last chapter actual experiments involve large classical but small quantal uncertainties. On the other hand the wave packet inter-

pretation requires large quantal and small classical uncertainties. Although Friedman and Weisskopf refer to a 'pulsed neutron beam', it is not clear which of their arguments apply to the actual experiments and which apply to the complementary, 'ged anken' experiments. This is evidenced by the frequent misapplication of the temporal arguments to the question of coherence of shape elastic and fluctuation scattering.

The incompleteness of the wave packet interpretation as presented by Friedman and Weisskopf can be seen, if it is recognized that their argument depends only on kinematical factors (the assumption of an expansion of the collision matrix in terms of resonances and an ingoing wave packet) but not on the dynamics of the nuclear system (the particular numbers for the widths and residues of the resonance levels). It does not justify completely the sharp distinction between the model amplitude and the fluctuation amplitude which one expects if the experimental predictions of the optical model are to be relatively independent of the way in which the average in (6.1) is taken.

6.2 THE INTERPRETATION OF THE ENERGY AVERAGES.

In order to give a physical interpretation of the optical model, the mathematical averages of equations (6.1) and (6.3) must be related to the physical processes taking place during the collision.

It is reasonable to expect that the energy averages are connected with the experimental difficulties of obtaining beams of precise energy. In experiments to which the optical model theory can be applied directly there is appreciable uncertainty in the definition of the energy of the incident particles. According to the discussion of chapter 4, this uncertainty derives from two different sources, the incident wave packets are localized to some extent and secondly their peak energies may differ. In ordinary experiments, including those described by the optical model, only the latter is important; the quantal uncertainty resulting from the finite spread of the incident packets is negligible. To obtain the cross section for this situation, it was shown in chapter 4, that the cross section which is valid for monoenergetic beams and which is calculated with the perfect resolution theory, must be averaged over an energy interval equal to the classical uncertainty.

Thus the significance of the averaging procedure on the left of equations (6.3) is quite clear, this averaging of the cross sections is in accordance with the boundary conditions of the experiment.

The meaning of the averaged matrix element in equations (6.1) and (6.5) is more difficult to see. The real part of the collision matrix element, averaged with respect to energy, is connected with the averaged cross section through equation (6.3c). The averaging operation is the same on both sides of the equation. This establishes how the matrix element is to be averaged in equation (6.1). The spread of the average is given by the classical uncertainty. Equation (6.3c) then ensures that the total cross section will be given by the model but does not yield a direct physical interpretation for the averaged matrix element. Similarly the identity (6.5) is merely a formal splitting of the amplitude which is always possible, and has no special physical significance.

The expression for the outgoing wave packet, equation (A.13) of the appendix, may be regarded as the energy average of a quantity which is essentially the collision matrix element. This suggests that the aver-

aged matrix element may have an interpretation in terms of the temporal properties of the scatterer. But the conditions of the actual experiment ($\delta < \Gamma << \phi$) are such that the definition of time is not precise enough to make any time delay distinguishable.

Nevertheless we may consider the complementary experiment with $\delta >> \Gamma$. In this thought experiment we replace the long wave packets, which have different peak energies, of the real experiment with short localized wave packets which have the same peak energy.

If the Lorentz weighting factor is used in the average (6.2), then by contour integration the average of the diagonal entry of the collision matrix (A.13) over an interval 2ϕ equal to the classical uncertainity is

$$\langle U_\ell \rangle_\phi = 1 + \left(\frac{2mE}{\hbar^2}\right)^{\frac{1}{2}} \left[Q_\ell(E) - i \sum_\lambda \frac{R_{\ell\lambda}^2}{E - \epsilon_{\ell\lambda} + i\left(\frac{\Gamma_{\ell\lambda}}{2} + \phi\right)} \right] \tag{6.7}$$

with $\phi >> \Gamma_{\ell\lambda}$

$Q_\ell(E)$ and $E^{\frac{1}{2}}$ are not averaged as they are supposed to be smoothly varying over the interval ϕ. The fluctuation element is then

$$U_\ell - \langle U_\ell \rangle_\phi = -i\left(\frac{2mE}{\hbar^2}\right)^{\frac{1}{2}} \sum_\lambda R_{\ell\lambda}^2 \left[\frac{1}{E - \epsilon_{\ell\lambda} + \frac{1}{2} i \Gamma_{\ell\lambda}} - \frac{1}{E - \epsilon_{\ell\lambda} + i\left(\frac{\Gamma_{\ell\lambda}}{2} + \phi\right)} \right] \tag{6.8}$$

The quantity $\langle U_\ell \rangle_\phi$ varies slowly in energy intervals less than ϕ, so that if we consider only incident wave packets for which $\phi > \delta >> \Gamma_\lambda$, the shape elastic scattering may be evaluated by the method of section 2.4.

For $X > 0$

$$\psi_{s.el.} = K \sum_\ell (2\ell+1) P_\ell(\cos\theta) \{\langle U_\ell \rangle_\phi - 1\} \exp(iEX) \exp(-\delta X) \tag{6.9}$$

and for $X < 0$

$$\psi_{s.el} = K \sum_\ell (2\ell+1) P_\ell(\cos\theta) \{\langle U_\ell \rangle_\phi - 1\} \exp(iEX) \exp(\delta X) \tag{6.10}$$

This wave packet is centered on $X = 0$

$$\text{i.e.} \quad t = \frac{r_0 + r}{v_0}$$

and its spread is 2δ ; the wave packet for the shape elastic scattering is propagated immediately and has the same spread as the incident packet.

Now consider the wave packet for the fluctuation scattering. Since U fluctuates over the interval Γ and $\delta >> \Gamma$, the result of chapter 3 must be used (c.f. equation 3.1). For $X < 0$,

$$\Psi_{fluc} = K \exp(iEX) \sum_{\ell} (2\ell+1) P_{\ell}(\cos\theta) \Big\{ \sum_{\lambda} \frac{R_{\ell\lambda}^2}{E-\epsilon_{\ell\lambda}+i(\frac{\Gamma_{\ell\lambda}}{2}-\delta)}$$

$$\times \left[\exp(\delta X) - \frac{2\delta i \exp(\frac{\Gamma_{\ell\lambda}}{2}X) \exp\{i(\epsilon_{\ell\lambda}-E)X\}}{E-\epsilon_{\ell\lambda}+i(\frac{\Gamma_{\ell\lambda}}{2}+\delta)} \right]$$

$$- \sum_{\lambda} \frac{R_{\ell\lambda}}{E-\epsilon_{\ell\lambda}+i(\frac{\Gamma_{\ell\lambda}}{2}+\phi-\delta)} \left[\exp(\delta X) - \frac{2i\delta \exp[(\frac{\Gamma_{\ell\lambda}}{2}+\phi)X] \exp[i(\epsilon_{\ell\lambda}-E)X]}{E-\epsilon_{\ell\lambda}+i(\frac{\Gamma_{\ell\lambda}}{2}+\phi+\delta)} \right] \Big\}$$

(6.11)

and for $X > 0$

$$\Psi_{fluc} = K \exp(iEX) \sum_{\ell} (2\ell+1) P_{\ell}(\cos\theta) \exp(-\delta X)$$

$$\times \sum_{\lambda} R_{\ell\lambda}^2 \left[\frac{1}{E-\epsilon_{\ell\lambda}+i(\frac{\Gamma_{\ell\lambda}}{2}+\delta)} - \frac{1}{E-\epsilon_{\ell\lambda}+i(\frac{\Gamma_{\ell\lambda}}{2}+\phi+\delta)} \right]$$

(6.12)

The argument of Friedman and Weisskopf is confirmed by an examination of the time dependence of Ψ_{fluc}.

Since

$$\exp(\Gamma X) >> \exp(\delta X) > \exp[(\phi+\frac{\Gamma_{\ell\lambda}}{2})X] \quad \text{for } X << 0$$

a comparison of (6.10) and (6.11) shows that the fluctuation packet includes the delayed part of the scattering.

In chapter 3 we noted that for correlated levels the undelayed part of the resonant scattering is large.

Similarly, although the fluctuation scattering includes the time delayed part of the scattering, it is not clear that it does not also have a large, undelayed component for the case of correlated levels.

To sharpen the distinction between the fluctuation and shape elastic scattering, the magnitude of the fluctuation packet for small X will now be found, thereby giving an estimate of the overlap with the shape elastic packet.

The resonant wave packet contains

$$\sum_\lambda \frac{R_{\ell\lambda}^2}{E-\epsilon_{\ell\lambda}+i\left(\frac{\Gamma_{\ell\lambda}}{2}-\delta\right)}\left[\exp(\delta X)-2i\delta\,\frac{\exp\left(\frac{\Gamma_{\ell\lambda}}{2}X\right)\exp\{i(\epsilon_{\ell\lambda}-E)X\}}{E-\epsilon_{\ell\lambda}+i\left(\frac{\Gamma_{\ell\lambda}}{2}+\delta\right)}\right]$$

$$=\sum_\lambda \frac{R_{\ell\lambda}^2\exp(\delta X)}{E-\epsilon_{\ell\lambda}+i\left(\frac{\Gamma_{\ell\lambda}}{2}+\delta\right)}\left[1-2i\delta\,\frac{\left\{\exp\left[-iX\left(E-\epsilon_{\ell\lambda}+i\frac{\Gamma_{\ell\lambda}}{2}-i\delta\right)\right]-1\right\}}{E-\epsilon_{\ell\lambda}+i\left(\frac{\Gamma_{\ell\lambda}}{2}-\delta\right)}\right] \tag{6.13}$$

which for small negative X is

$$\sum_\lambda \frac{R_{\ell\lambda}^2\exp(\delta X)}{E-\epsilon_{\ell\lambda}+i\left(\frac{\Gamma_{\ell\lambda}}{2}+\delta\right)}\left[1-2\delta X-\delta\left(\frac{\Gamma_{\ell\lambda}}{2}-\delta\right)X^2+i\delta X^2(E-\epsilon_{\ell\lambda})\ldots\right] \tag{6.14}$$

With the aid of this result, Ψ_{fluc} for small negative X is given by

$$K \exp(\delta X) \sum_\ell (2\ell+1) P_\ell(\cos\theta)$$
$$\times \sum_\lambda \left[\frac{R_{\ell\lambda}^2}{E - \epsilon_{\ell\lambda} + i\left(\frac{\Gamma_{\ell\lambda}}{2} + \delta\right)} - \frac{R_{\ell\lambda}^2}{E - \epsilon_{\ell\lambda} + i\left(\frac{\Gamma_{\ell\lambda}}{2} + \delta + \phi\right)} \right]$$
$$\times \left\{ 1 - 2\delta X - \delta\left(\frac{\Gamma_{\ell\lambda}}{2} - \delta\right) X^2 + i\delta X^2 (E - \epsilon_{\ell\lambda}) \cdots \right\}$$

Reinterpreting the terms in square brackets as averages, we have

$$\Psi_{\text{fluc}} = K \exp(\delta X) \sum_\ell P_\ell(\cos\theta)(2\ell+1)\left[\langle U_\ell\rangle_\delta - \langle U_\ell\rangle_{\phi+\delta}\right] \times \left\{ 1 - 2\delta X \cdots \right\} \tag{6.15}$$

and from (6.12), for $X > 0$

$$\Psi_{\text{fluc}} = K \exp(-\delta X) \sum_\ell P_\ell(\cos\theta)(2\ell+1)\left[\langle U_\ell\rangle_\delta - \langle U_\ell\rangle_{\delta+\phi}\right] \tag{6.16}$$

Thus Ψ_{fluc} vanishes for positive time and small negative times if $\langle U_\ell\rangle_\delta - \langle U_\ell\rangle_{\delta+\phi} = 0$. Consequently the wave packets for fluctuation and shape elastic scattering have minimum overlap if the condition

$$\langle U_\ell\rangle_\delta - \langle U_\ell\rangle_{\phi+\delta} = \langle U_\ell - \langle U_\ell\rangle_\phi\rangle_\delta \approx 0 \tag{6.17}$$

holds.

This condition can be regarded as supplementing the purely formal definition of the optical model in

(6.1). It is necessary to stipulate such a condition to make the separation of the amplitudes in equation (6.5) meaningful. The condition (6.17) has been derived on the basis of an argument which depends on time delay but it has another simple meaning. It requires that the collision matrix when averaged over an energy interval ϕ should vary slowly over energy intervals greater than ϕ. This means that the averaged collision matrix element must not depend strongly on the way in which the average is taken; if this condition did not hold, the optical potential would depend markedly on the energy uncertainty of the incident particles which make up the beam. Thus (6.17) supplies a criterion which must be satisfied if the optical potential is to be useful in the description of scattering experiments.

Friedman and Weisskopf assume implicitly an equivalent condition in the relation (6.6). In going from

$$U_\ell = \langle U_\ell \rangle + U_\ell - \langle U_\ell \rangle$$

to $$\langle |U_\ell|^2 \rangle = |\langle U_\ell \rangle|^2 + \langle |U_\ell - \langle U_\ell \rangle|^2 \rangle$$

they omit the interference term

$$2\,\mathrm{Re}\ \langle \langle U_\ell \rangle^* (U_\ell - \langle U_\ell \rangle) \rangle \qquad (6.18)$$

This ensures the equivalence of the fluctuation cross section in equation (6.6).

The approach of this section suggests that if $\langle U_\ell \rangle_\phi$ when considered as a function of ϕ, is weakly dependent on ϕ about $\phi = \phi_0$ compound elastic scattering be defined in the following way: in a thought experiment the incident wave packets of the actual experiment are replaced by wave packets of quantal uncertainty ϕ_0. The compound elastic scattering is then that part of the outgoing wave packet which arrives at the detector after time

$$T = \frac{r_0 + r_d}{v_0} + \frac{\hbar}{\phi_0} \tag{6.19}$$

With this definition it has been shown that compound scattering is equivalent to fluctuation scattering. Also, the averaged matrix element $\langle U_\ell \rangle$ has a direct physical meaning in the context of this hypothetical experiment: if the elastic scattering is only observed for times less than $\frac{r_d + r_0}{v_0} + \frac{\hbar}{\phi_0}$; the cross section is given by

$$\frac{\pi}{k^2} \sum_\ell (2\ell+1) P_\ell(\cos\theta) \, |1 - \langle U_\ell \rangle_{\phi_0}|^2 .$$

Thus, the shape elastic cross section is observed directly. It must be emphasized that this interpretation is only legitimate if condition (6.6) holds.

In general the wave packet for the fluctuation scattering has an appreciable component for $t < T$.

Usually no distinction is made between the fluctuation and the compound elastic scattering, the two terms are regarded as synonymous. This has the difficulty that the compound elastic scattering may have a significant undelayed component in spite of the fact that one would like to associate this type of scattering with the decay of a metastable state of long lifetime.

Thus there are at least two different ways of defining the optical model:

(i) the optical model is the model which predicts the average scattering amplitude, and the compound elastic scattering arises from the fluctuation amplitude..

(ii) the optical model is that model which, in a time dependent experiment, gives the cross section for propagation without time delay and the compound elastic cross section is the remainder.

It is not obvious that these two definitions are equivalent. In fact it has been shown that they are only equivalent if the scattering amplitude satisfies an additional condition. It is just this condition which allows the simple physical interpretation of (ii) to be attributed to the formal definition (i).

6.3 INCOHERENCE OF SHAPE ELASTIC AND COMPOUND ELASTIC SCATTERING.

It is often argued in the following way that shape elastic and compound elastic+ scattering are incoherent[47].

'Feshbach, Porter and Weisskopf have pointed out that the resonance scattering is incoherent with body elastic scattering if the energy spread of the incident beam is sufficiently broad. It can be seen from the uncertainty principle that the interaction time is well defined for the scattering experiment made with a neutron beam which has a large energy spread. Therefore scattering which results from the formation and decay of narrow, long-lived compound states will, however, interfere with neither beam nor body elastic scattering.'

Or more baldly,[48]

'Compound elastic scattering will be incoherent with direct elastic scattering because it occurs so much later'.

The error of this type of argument resides in the misuse of the uncertainty principle. If there is ΔE uncertainty in the energy of the incident beam, the uncertainty principle does not necessarily imply that time

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+ Note that here compound elastic scattering is used in its customary sense and not in the more precise sense of the definition (6.18) advocated here.

intervals are defined to within $\hbar/\Delta E$. The uncertainty relation for energy and time provides an upper limit to the accuracy of the measurement of time intervals, which is achieved if the energy uncertainty of the beam is wholly quantal in origin. On the contrary, according to the considerations of chapters 4 and 5, the uncertainty of the experimental beam is essentially classical and there is no resolution of time in the actual scattering experiment. This is consistent with the choice of the stationary theory to evaluate the scattering amplitude and the use of an energy average over the cross section to represent the effect of the spread in energy of the incident beam. For small quantal uncertainty, that is for incident packets of long duration, the wave packets for shape elastic and compound elastic scattering are of the same extent and there is no possibility of distinguishing the two sorts of scattering by time measurements.

Nevertheless, it might be thought that the argument cited above could be made valid by considering the complementary experiment with wave packets of short duration. Yet even in the experiment with a pulsed beam, we have shown that there is appreciable overlap between the two packets (equations (6.10) and (6.11)) and

that to eliminate this interference the condition (6.17) has to be assumed. Now this condition is equivalent to omitting the interference term (6.18) so that the argument depending on time is circular. The incoherence of compound and shape elastic scattering is an assumption which must be justified in each particular experiment as it implies additional knowledge about the structure of the scattering amplitude which is not contained in the definition (6.1).

It is clear that the condition (6.17) implies that there are no fluctuations of width intermediate between the narrow fluctuations in the scattering amplitude associated with the compound nucleus and the much wider fluctuations associated with the optical model In the context of good time resolution, this is the same as assuming that processes which have delay times intermediate between the lifetime of the compound nucleus and the lifetime of the giant resonances of the optical model, are unimportant. The statement that the scattering of a wave packet, comprising a broad spectrum of energies takes place in two distinct stages is not a consequence of the formal definition of the optical model but is, of course, the condition which makes the definition meaningful.

The question of just what kinds of fluctuations are present in the scattering amplitude can only be answered on the basis of a knowledge of the statistical properties of the positions of the poles $E_{\ell\lambda}$ and their residues $R^2_{\ell\lambda}$ In section 6.5, we will try to make the consideration of the fluctuations more quantitative by introducing correlation functions, but at present we consider a simple example which illustrates the ideas of this and the preceding section.

The following plausible assumptions about the structure of the collision matrix are sometimes made[42,3]:

(i) For medium energies the total widths $\Gamma_{\ell\lambda}$ may be regarded as constant Γ. This assumption is justified on the ground that there are many competing processes (c.f. equation A.4).

(ii) The mean level width Γ is much greater than the level spacing D.

(iii) The residues $R^2_{\ell\lambda}$ have a simple long range correlation; they may be split into two parts

$$R^2_{\ell\lambda} = \langle R^2_{\ell\lambda}\rangle_\phi + \delta R^2_{\ell\lambda} \qquad (6.20)$$

where the average value $\langle R^2_{\ell\lambda}\rangle_\phi$ is slowly varying and the fluctuating part $\delta R^2_{\ell\lambda}$ random over an interval <u>larger than ϕ</u>.

Then the collision function has the form

$$\frac{U_\ell - 1}{2ik} = Q_\ell(E) + \pi \frac{\langle R_{\ell\lambda}^2 \rangle \phi}{D} - i \sum_\lambda \frac{\delta R_{\ell\lambda}^2}{E - E_\lambda} \qquad (6.21)$$

The first two terms comprise the shape elastic scattering amplitude and the third is fluctuation amplitude. U_ℓ has fluctuations of width $\sim \phi$ from the first two terms and fluctuations of width $\sim \Gamma$ resulting from the denominator of the fluctuation term. Γ is often called the coherence energy. It is easily seen that the conditions (6.17) and (6.18) are satisfied and the fluctuation and shape elastic scattering are incoherent. Since the phases $\delta R_{\ell\lambda}^2$ are assumed to be random, the fluctuation cross section can be evaluated as in section 5.2.

If the collision matrix has a more complicated structure than this simple model indicates, the separation of the scattering amplitude into two parts may not be clear cut. For example other sorts of correlations may be present in the $R_{\ell\lambda}^2$. In particular the 'doorway states' of Feshbach[49,50] have an energy width intermediate between the giant resonance of the optical model and the Ericson fluctuations associated with the coherence energy Γ.

Obviously more sophisticated separations of the

scattering amplitudes are possible, the various types of fluctuations being described by a hierarchy of potentials. In the time dependent picture these fluctuations correspond to various groups of particles having time delays intermediate between that of the giant resonance and the compound nucleus. Whether these processes are important for a detailed analysis of experimental information is an open question.

6.4 HIGH ENERGY APPROXIMATION.

The second term in equation (6.13) for the trailing edge of the resonant wave packet is

$$\sum_\lambda \frac{R_{\ell\lambda}^2 \exp(\delta X)}{E-\epsilon_{\ell\lambda}+i\left(\frac{\Gamma_{\ell\lambda}}{2}+\delta\right)} \left[\frac{\exp\left\{-iX\left(E-\epsilon_{\ell\lambda}+i\left(\frac{\Gamma_{\ell\lambda}}{2}-\delta\right)\right)\right\}-1}{E-\epsilon_{\ell\lambda}+i\left(\frac{\Gamma_{\ell\lambda}}{2}-\delta\right)}\right] \qquad (6.23)$$

At high energies the level spacing is much smaller than the mean level width and the level widths may be regarded as constant. Then the sum of (6.23) is written as an integral

$$\int \frac{S(\epsilon)\left[\exp\left\{-iX\left(E-\epsilon+i\left(\frac{\Gamma}{2}-\delta\right)\right)\right\}-1\right]}{\left[E-\epsilon+i\left(\frac{\Gamma}{2}+\delta\right)\right]\left[E-\epsilon+i\left(\frac{\Gamma}{2}-\delta\right)\right]}\, d\epsilon \qquad (6.24)$$

with $S(\epsilon) \equiv \dfrac{R(\epsilon)}{D(\epsilon)}$. $R(\epsilon)$ is distribution function

which approximates the behaviour of the residues and $D(\epsilon)$ is the average level spacing. Now with the assumptions that $S(\epsilon)$ has no singularities in the lower half of the complex ϵ-plane and at the worst tends to a constant at infinity, by integration around a semi-circular contour in the lower half plane the integral is given by the residue at $\epsilon = E + i\left(\frac{\Gamma}{2} - \delta\right)$ which is zero; the integral vanishes and the resonant wave packet for $X < 0$ is

$$K \exp(\delta X) \sum_{\ell} (2\ell+1) P_{\ell}(\cos\theta) \sum_{\lambda} \frac{R_{\ell\lambda}^{2}}{\epsilon - \epsilon_{\ell\lambda} + i\left(\frac{\Gamma_{\ell\lambda}}{2} + \delta\right)} \quad (6.25)$$

Thus the resonant packet is scattered promptly. The fluctuation packet is

$$\Psi_{fluc} = K \exp(-\delta|X|) \sum_{\ell} P_{\ell}(\cos\theta)(2\ell+1) \left[\langle U_{\ell} \rangle_{\delta} - \langle U_{\ell} \rangle_{\delta+\phi} \right] \quad (6.26)$$

If the average scattering amplitude is insensitive to the range of the average, the wave packet for fluctuation scattering vanishes.

At high energies the interpretation of the optical model is particularly simple; the cross section given by the model is identical with that obtained in an experiment with large quantal uncertainty. The cross section for this case has already been evaluated in chapter 3. For the S-wave only

$$\frac{d\sigma_R}{d\Omega_d} = \frac{\pi^2}{4} \frac{\langle R_{D>}^2 \rangle_s \langle R_{D>}^2 \rangle}{D^2} \tag{6.27}$$

whereas for perfect resolution

$$\frac{d\sigma_R}{d\Omega_d} = \frac{\pi^2}{4} \frac{\langle R_{0>}^2 \rangle_{\Gamma_i} \langle R_{0>}^2 \rangle_{\Gamma_i}}{D^2} \tag{6.28}$$

With the choice $\delta = \phi$, equation (6.27) gives the shape elastic cross section.

Hence for closely spaced levels there is, in principle, a method of observing the shape elastic cross section directly; the scattering experiment is performed with a pulsed beam, the time duration of the pulse being equal to $\hbar/\phi$ where ϕ is a fraction of the width of the optical model resonance. Also by varying the frequency of the pulsing the range of the correlation between different levels could be determined.

6.5 CORRELATION FUNCTIONS

We define a correlation function for the scattered packet:

$$C(\tau) = \mathrm{Re}\, r_d^2 \int_{-\infty}^{\infty} \psi^{el}_{\underline{k}_0}(\underline{r}_d, t)\, \overline{\psi}^{el}_{\underline{k}_0}(\underline{r}_d, t+\tau)\, dt \tag{6.29}$$

where $\psi^{el}_{\underline{k}_0}(\underline{r}_d, t)$ is the emergent wave packet corresponding to an incident wave packet which is very well defined in time

How would such a correlation function arise physically? Consider a scattering experiment in which two identical, sharply peaked wave packets separated in space by a distance $v_0\tau$ are incident on the scatterer. In principle this could be achieved by interrupting an incident monoenergetic beam with a shutter, the shutter opening for a time $\frac{\hbar}{\delta} << \tau$, being closed for a period τ and reopening for $\hbar/\delta$. Then at the detector (which should be close to the scattering centre to prevent loss of information by the natural spreading of the packets) the wave function is

$$\psi^{el}_{\underline{k}_0}(\underline{r}_d, t+\tau) + \psi^{el}_{\underline{k}_0}(\underline{r}_d, t) . \tag{6.30}$$

The chance of detecting a particle at time t is

$$|\Psi^{el}_{\underline{k}_0}(\underline{r}_d, t+\tau)|^2 + |\Psi^{el}_{\underline{k}_0}(\underline{r}_d, t)|^2 + 2\mathcal{R}e\, \Psi^{el}_{\underline{k}_0}(\underline{r}_d, t)\, \overline{\Psi}^{el}_{\underline{k}_0}(\underline{r}_d, t+\tau)$$

and the differential cross section is

$$\frac{d\sigma}{d\Omega_d} = v_0 r_d^2 \frac{\int_{-\infty}^{\infty} |\Psi^{el}_{\underline{k}_0}(\underline{r}_d, t)|^2 dt}{\int_{-\infty}^{\infty} |\Psi^{el}_{\underline{k}_0}(r\underline{k}_0, 0)|^2 dt}$$

$$+ v_0 r_d^2 \, \mathcal{R}e \frac{\int_{-\infty}^{\infty} \Psi^{el}_{\underline{k}_0}(\underline{r}_d, t)\, \overline{\Psi^{el}_{\underline{k}_0}(\underline{r}_d, t+\tau)}\, dt}{\int_{-\infty}^{\infty} |\Psi^{el}_{\underline{k}_0}(r\underline{k}_0, 0)|^2 dt} \qquad (6.31)$$

The first term is the cross section which has been derived before. The second, which involves the correlation function (6.29), did not appear in chapter 4, as it was supposed there that the incident packets were so far apart that they did not interact after scattering.

From equation (A.13)

$$\Psi^{el}_{\underline{k}_0}(\underline{r}_d, t) = 2ik \int_0^{\infty} \Delta(E, E_0; \delta) f(E, \Omega_d) \exp(iEx)\, dE \qquad (6.32)$$

with $f(E, \Omega_d)$ denoting the amplitude for elastic scattering.

Then $$r_d^2 \int_{-\infty}^{\infty} \Psi^{el}_{\underline{k}_0}(\underline{r}_d, t)\, \overline{\Psi^{el}_{\underline{k}_0}(\underline{r}_d, t+\tau)}\, dt$$

$$= 4 K\bar{K} \hbar r_d^2 \int_0^\infty dE\, \Delta(E) f(E) \int_0^\infty dE' \Delta(E') \bar{f}(E') \exp\left(i\frac{E'\tau}{\hbar}\right) \int_{-\infty}^{\infty} \exp(i(E-E')x)\, dx$$

$$= \hbar \int_0^\infty |\Delta(E) f(E)|^2 \exp\left(i\frac{E\tau}{\hbar}\right) dE$$

Thus

$$C(\tau) = \hbar \int_0^\infty |\Delta(E) f(E)|^2 \cos\left(\frac{E\tau}{\hbar}\right) dE \qquad (6.33)$$

For good time resolution the function $\Delta(E)$ is very slowly varying compared with $f(E)$ and the relation (6.33) states that the correlation function for the scattered packet is approximately the Fourier cosine transform of the scattering amplitude. This result is analogous to the Wiener-Khintchine theorem in the theory of stochastic processes+. The incident wave packet (signal) has a 'power spectrum' $|\Delta(E)|^2$ and the emergent wave packet (response) has a 'power spectrum' $|\Delta(E) f(E)|^2$ The scattering amplitude $f(E)$ is the 'complex response function' for the linear scattering system. In order to sample a large number of the fluctuations of $f(E)$ the 'bandwith' δ of the incident signal must be large.

+ Porter[49] and Namiki[26] have used the analogy with the theory of stochastic processes to derive a Nyquist-like theorem for nuclear scattering.

Wilkinson, Eisberg and Yennie[29] have suggested that time delays for the region of overlapping levels may be measured by observing the low energy bremsstrahlung which is produced when charged particles are scattered. Yennie and Feshbach[58] have given a quantal derivation of the cross section for bremsstrahlung production. The expression for the cross section contains a term which is proportional to

$$I(\epsilon) = \mathcal{R}e \int \rho(E, E_0; \phi) \overline{f(E)} f(E-\epsilon) dE \tag{6.34}$$

ϕ is the classical uncertainty of the beam which has peak energy E_0, $f(E)$ is the elastic scattering amplitude for the incident particle of energy E and ϵ is the energy of the bremsstrahlung.

In their discussion of the correlation function (6.34) Feshbach and Yennie suppose that so many fluctuations of the scattering amplitude are contained in ϕ that (6.34) may be calculated as a statistical average of $\mathcal{R}e\, \overline{f(E)} f(E-\epsilon)$ over an appropriate probability distribution from which these fluctuations could have been selected, so that

$$I(\epsilon) = \mathcal{R}e \langle \overline{f(E)} f(E-\epsilon) \rangle$$

They introduce an energy interval Γ over which there are substantial correlations between the resonance parameters. Then for $\epsilon > \Gamma$

$$I(\epsilon) = \mathcal{R}e \langle \overline{f(E)} \rangle \langle f(E-\epsilon) \rangle .$$

The shape elastic and the fluctuation cross section may then be obtained by extrapolation as in figure 7b.

Of course the measurement of the coherence energy Γ does not yield the time delay directly and it is interesting to ask whether the correlation function $I(\epsilon)$ has an interpretation in terms of the complementary time dependent experiment. Consider

$$r_d^2 \int |\Psi^{el}_{\underline{k}_0}(\underline{r}_d, t)|^2 \exp\left(\frac{i\epsilon}{\hbar}(t-t_0)\right) dt$$

with $\quad t_0 = \dfrac{r_0 + r_d}{V_0}$

This expression yields

$$4K\bar{K} r_d^2 \hbar \iiint \Delta(E)\, \Delta(E')\, f(E)\, \overline{f(E')} \exp(i(E-E')x) \exp(i\epsilon x)\, dx$$

$$= \hbar \int \Delta(E-\epsilon)\, \Delta(E)\, f(E-\epsilon)\, \overline{f(E)}\, dE$$

That is (6.35)

$$r_d^2 \int_{-\infty}^{\infty} |\Psi^{el}_{\underline{k}_0}(\underline{r}_d, t)|^2 \cos\left[\frac{\epsilon(t-t_0)}{\hbar}\right] dt = \mathcal{R}e\, \hbar \int_0^{\infty} \Delta(E-\epsilon)\, \Delta(E)\, \overline{f(E)}\, f(E-\epsilon)\, dE$$

For very good time resolution, we have $\delta >> \epsilon$ so that $\Delta(E) \approx \Delta(E-\epsilon)$. For the purpose of evaluating the correlation function we may neglect the leading edge of the wave packet. Hence equation (6.35) is approx-

imately

$$r_d^2 \int_0^\infty |\psi^{el}_{\underline{k}_0}(\underline{r}_d,t)|^2 \cos\left(\frac{\epsilon(t-t_0)}{\hbar}\right) dt = \mathrm{Re}\,\hbar \int_{-\infty}^{\infty} \rho(E,E_0;\delta)\, f(E)\, \bar{f}(E-\epsilon)\, dE \qquad (6.36)$$

This equation allows us to translate statements about the fluctuations in the scattering amplitude into the language of time delays.

For the particular model (6.21), neglecting the detailed behaviour about X = 0, we have for the ℓ-th partial wave

$$\psi^{el}_{\underline{k}_0} \sim R^2_{opt} \exp[i(E-E_{opt})X] \exp\left(-\frac{\Gamma_{opt}}{2}X\right) + \sum_\lambda \delta R^2_{\ell\lambda} \exp[i(E-E_{\ell\lambda})X] \exp\left(-\frac{\Gamma}{2}X\right)$$

where we have supposed that the energy dependence of the shape elastic amplitude in the neighbourhood of Eo may be represented by the giant resonance

$$\frac{R^2_{opt}}{E-E_{opt}+\frac{1}{2}i\Gamma_{opt}} \sim Q_\ell(E) + \pi\frac{\langle R^2_{\ell\lambda}\rangle}{D}$$

Then for $X < 0$

$$|\psi^{el}_{\underline{k}_0}|^2 \sim |R^2_{opt}|^2 \exp\left(-\frac{t}{\tau_{opt}}\right) + \sum_\lambda |\delta R^2_{\ell\lambda}|^2 \exp\left(-\frac{t}{\tau_{coh}}\right) \qquad (6.37)$$

with $\tau_{opt} = \frac{\hbar}{\Gamma_{opt}}$, $\tau_{coh} = \frac{\hbar}{\Gamma}$, the cross term vanishing because of the randomness of $\delta R^2_{\ell\lambda}$. The time spectrum is shown in figure 7a. The corresponding correlation function for the complementary experiment follows from equation (6.36),

$$I(\epsilon) = |R^2_{opt}|^2 \frac{\Gamma_{opt}}{\Gamma^2_{opt}+\epsilon^2} + \sum_\lambda |\delta R^2_{\ell\lambda}|^2 \frac{\Gamma}{\Gamma^2+\epsilon^2} \qquad (6.38)$$

and is shown in figure 7b.

The broken line corresponds to processes with delay times intermediate between τ_{opt} and τ_{coh} which are not included by this model.

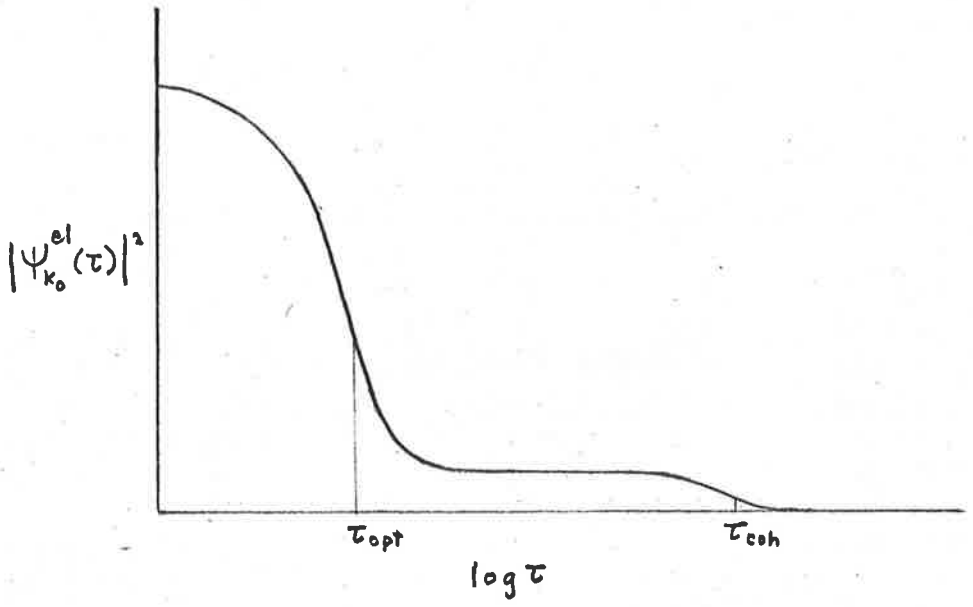

fig. 7a. Time Spectrum

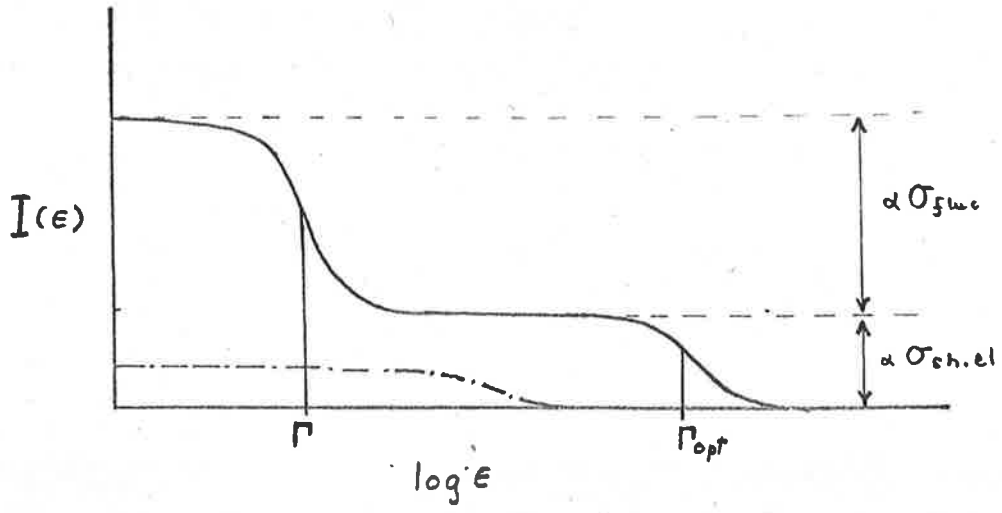

fig. 7b. Amplitude Correlation Function

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CHAPTER 7

THE RESOLUTION OF DIRECT AND COMPOUND INELASTIC SCATTERING.

7.1 INTRODUCTION.

We have shown that shape and compound elastic scattering are two extreme types of scattering process. In the time dependent picture shape elastic scattering occurs as soon as the projectile and the target nucleus begin to interact whereas compound scattering has a comparatively large delay time. In the complementary energy dependent picture, this division of scattering events into two distinct types according to their time delay corresponds to a separation of the amplitude into a part which has a long range energy fluctuation and a part which is supposed to have only short range correlations. Similar considerations apply to processes other than elastic scattering. For example the amplitude for inelastic scattering may be separated into a smoothly varying amplitude for 'direct' processes and an amplitude for compound processes which varies rapidly with energy.

In some experimental work it is possible to assume that either the direct or compound part of the ampli-

tude is dominant. In these cases it is relatively easy to explain the characteristic features of the experimental cross sections in terms of one or other of the simplified reaction models. However, at medium energies, it is expected that both compound and direct processes are important, so that some sort of superposition of the extreme models is necessary. Furthermore processes with intermediate lifetimes may also play a significant part, so that a unified model should perhaps be capable of producing other energy fluctuations in addition to those required for direct and compound processes.

In order to answer these open questions, it is of great importance to be able to separate direct and compound effects experimentally. There has been little systematic work on this problem, because unambiguous methods of analysing the experimental data have not been available+. The time dependent experiments already discussed, if practicable, would

---

+ For example: the angular distribution for a compound process is nearly isotropic or at least symmetrical about 90° in the centre of mass system whereas for direct reactions it was thought that the angular distributions were peaked forward as the Butler theory indicates. However, it has been shown that the distorted wave Born approximation can produce large backward peaks so that a superficial symmetry about 90° does not necessarily indicate compound nucleus formation.

provide a direct way of distinguishing different reaction modes. Unfortunately for nuclear energies, the requisite pulsing and time measurement does not seem possible at present. Other possible techniques include a study of the correlation functions for cross sections, and the bremsstrahlung experiment referred to in the last chapter.

In this chapter we suggest a method[30] based on the parity rule[31] of separating direct and compound effects in inelastic scattering.

In the next section we discuss the general theory and in section 7.3, we describe briefly numerical calculations for a particular example.

7.2 THE PARITY RULE AND GENERAL METHOD

The amplitude for inelastic scattering may be written

$$T_{fi} = \langle T_{fi} \rangle + T_{fi} - \langle T_{fi} \rangle \tag{7.1}$$

where $\langle T_{fi} \rangle$ is the amplitude averaged with respect to energy (c.f. equation (6.1) for elastic scattering). In the distorted wave Born approximation (DWBA) the averaged amplitude $\langle T_{fi} \rangle$ is equal to

$$T_{fi}^{DWB} = \int d^3r \, \chi_f^{(-)*}(\underline{k}_f, \underline{r}) \, \chi_i^{(+)}(\underline{k}_i, \underline{r}) \, \Psi(\underline{r}) \tag{7.2}$$

where $\chi_i^{(+)}(\underline{k}_i, \underline{r})$ and $\chi_f^{(-)*}(\underline{k}_f, \underline{r})$ are the wave functions of the incident and scattered particles with wave vectors $\underline{k}_i$ and $\underline{k}_f$ respectively[25]. The state $\chi_i^{(+)}$ contains a plane wave in the incident channel distorted by the optical potential for the ~~given~~ ground state of the target nucleus. $\chi_f^{(-)*}$ is distorted by the optical potential corresponding to the final excited state of the struck nucleus and is essentially the time reverse of $\chi_f^{(+)}$. $\Psi(\underline{r})$ contains all the properties of the initial and final nuclear states and the interaction;

$$\Psi(r) = \int d^{3N}\xi \; \psi_i(\underline{\xi}) \, v(\underline{r}, \underline{\xi}) \, \psi_f^*(\underline{\xi}) \tag{7.3}$$

Ψ_i and Ψ_f being the initial and final nuclear states and $v(\underline{r}, \underline{\xi})$ the residual interaction between the projectile and the target.

The partial wave expansions of $\chi_i^{(+)}(\underline{k}_i, \underline{r})$ and $\chi_f^{(-)*}(\underline{k}_f, \underline{r})$ are[22]

$$\chi_f^{(-)*}(\underline{k}_f, \underline{r}) = \frac{4\pi}{k_f r} \Sigma_{\ell,m}\, i^{-\ell} e^{i\sigma_\ell} f_\ell(k_f, r)\, Y_\ell^{m*}(\hat{k}_f)\, Y_\ell^{m}(\hat{r})$$

$$\chi_i^{(+)}(\underline{k}_i, \underline{r}) = \frac{4\pi}{k_i r} \Sigma_{\ell,m}\, i^{\ell} e^{i\sigma_\ell} f_\ell(k_i, r)\, Y_\ell^{m}(\hat{k}_i)\, Y_\ell^{*m}(\hat{r})$$

From the properties of the spherical harmonics Y_ℓ^m we have

$$\chi_f^{(-)*}(\underline{k}_f, \underline{r}) = \chi_f^{(+)}(-\underline{k}_f, \underline{r})$$

and

$$\chi_i^{(+)}(\underline{k}_i, -\underline{r}) = \chi_i^{(+)}(-\underline{k}_i, \underline{r})$$

Thus with the transformation $\underline{r} \rightarrow -\underline{r}$ in the integral (7.2),

$$\int d^3r\, \chi_f^{(+)}(-\underline{k}_f, \underline{r})\, \chi_i^{(+)}(\underline{k}_i, \underline{r})\, \Psi(\underline{r}) = \int d^3r\, \chi_f^{(+)}(\underline{k}_f, \underline{r})\, \chi_i^{(+)}(-\underline{k}_i, \underline{r})\, \Psi(-\underline{r}) \tag{7.4}$$

Now if there are no space exchange potentials,

$$\Psi(-\underline{r}) = (-1)^P\, \Psi(\underline{r}) \tag{7.5}$$

where P is the parity change of the nuclear state in the transition $\Psi_i \rightarrow \Psi_f$. Also if the optical potentials for the incident and scattered particles are similar and if $\underline{k}_i \approx \underline{k}_f$,

$$\chi_f^{(+)}(-\underline{k}_f, \underline{r})\, \chi_i^{(+)}(\underline{k}_i, \underline{r}) \approx \chi_f^{(+)}(\underline{k}_f, \underline{r})\, \chi_i^{(+)}(-\underline{k}_i, \underline{r})$$

so that from (7.4) and (7.5), we have

$$T_{fi}^{DWB} = (-1)^P T_{fi}^{DWB} \quad , \quad \underline{k}_i \approx \underline{k}_f \tag{7.6}$$

Thus the parity rule states that in the DWBA for reasonably small Q-values and parity change of the target nucleus, the amplitude for inelastic scattering at forward angles is negligible.

It can also be shown[51] that for even parity changes the amplitude at forward angles is large. This result follows from the general characteristics of the optical model wave functions and is not as strong as the rule for odd parity. Thus by finding whether the differential section is increasing or decreasing at forward angles, the parity change in inelastic scattering can be determined. The reader is referred to the original papers[31,51] for a more complete discussion.

Our interest in the rule derives from the fact that the DWBA amplitude is identical with the average amplitude[25] and thus is the amplitude for direct processes. From the amplitude (7.1) the differential cross section is

$$\frac{d\sigma_{fi}}{d\Omega} \sim |T_{fi}^{DWB}|^2 + |T_{fi} - \langle T_{fi}\rangle|^2 + 2\,\mathcal{R}e\,\{T_{fi}^{DWB\,*}(T_{fi} - \langle T_{fi}\rangle)\} \tag{7.7}$$

where we have omitted kinematical factors. For $T_{fi}^{DWB} = 0$ we have

$$\frac{d\sigma_{fi}}{d\Omega} \sim |T_{fi} - \langle T_{fi} \rangle|^2 , \quad \underline{k}_i \approx \underline{k}_f \tag{7.8}$$

Therefore a direct estimate of the fluctuation scattering may be obtained by looking at forward angles with good angular resolution in experiments which obey the Q-value and parity conditions.

Now if intermediate processes are negligible, the fluctuation and DWBA amplitudes are on the average incoherent, and the interference term disappears from the averaged cross section.

$$\left\langle \frac{d\sigma_{fi}}{d\Omega} \right\rangle = \langle |T_{fi}^{DWB}|^2 \rangle + \langle |T_{fi} - \langle T_{fi} \rangle|^2 \rangle \tag{7.9}$$

In particular, if it is assumed that the relevant collision matrix elements have a structure of the type (6.21), the fluctuation amplitude is a sum of terms of the form

$$\sum_\lambda \frac{\delta R_{c'\lambda} \, \delta R_{c\lambda}}{E - E_\lambda}$$

multiplied by the appropriate spherical harmonics and spin factors. In the region of overlapping levels we may apply the procedure of section 5.2 and the average fluctuation cross section is given by the statistical model (e.g. see discussion of reference 5). The trans-

mission coefficients resulting from $\delta R_{c'\lambda}$ are of course different from the usual ones since the coherent part of the $R_{c\lambda}$ has been removed. Sano[52] gives a useful discussion of this point. It must be stressed that it is an assumption that the fluctuation cross section is describable by the statistical model, since it must be supposed that the residues $\delta R_{c\lambda}$ are random.

To apply the statistical model to the fluctuation amplitude, the classical uncertainty of the beam must be larger than the coherence energy Γ . Thus if Q is the Q-value of the reaction, we require

$$Q > \phi > \Gamma \tag{7.10}$$

To test the assumption that the statistical model describes the second term in (7.9), the variation of $\frac{d\sigma(0°)}{d\Omega}$ with energy should be examined, a smooth variation as in figure 8 indicating that this model is appropriate. For beam widths less than Γ , fluctuations appear in the cross sections.[53] In this case variation of $\frac{d\sigma(0°)}{d\Omega}$ with energy can be used to estimate the coherence energy, the period of the fluctuations giving Γ , and hence the lifetime of the compound system. At forward angles the masking

effect of the direct reactions is eliminated.

Another practical advantage of the parity rule is that it provides a normalization for the DWBA. Since from the symmetry of the statistical model[5],

$$\frac{d\sigma_{stat}(180°)}{d\Omega} = \frac{d\sigma_{stat}(0°)}{d\Omega} = \frac{d\sigma_{exp}(0°)}{d\Omega}$$

Then

$$\frac{d\sigma^{DWB}}{d\Omega}(180°) = \frac{d\sigma_{exp}(180°)}{d\Omega} - \frac{d\sigma_{exp}(0°)}{d\Omega} \quad (7.11)$$

7.3 ILLUSTRATION: INELASTIC SCATTERING OF NEUTRONS ON F^{19}.

As an illustration, we have calculated cross sections for inelastic scattering of neutrons to the first excited state of F^{19}, a $\frac{1}{2}+$ to $\frac{1}{2}-$ transition. The Q-value of 0.11 Mev is very favourable.

The DWBA angular distribution was calculated+ by the method of Kromminga and McCarthy. In this approximation the distorted waves $\chi^{(+)}$ and $\chi^{(-)*}$ of equation (7.2) are represented by simple functions. Further details may be found in reference 54.

The normalization in this approximation is not fixed. In practice, it should be fixed by the experimental results using equation (7.11). Since we have no experimental data, we take the maximum differential cross section as about 20 mb/sr from a consideration of similar experiments. With this normalization, at 10 Mev and 0°, the cross section is 0.003 mb/sr, at 3° it is .02 mb/sr and at 6° about 0.8 mb/sr.

+ A Fortran code for this calculation was provided by Dr. McCarthy and the details of this calculation are not included here. The Fortran code for the statistical model cross section was written by the author and is listed in Appendix C.

The compound part of the cross section was found by a calculation similar to that of McDonald and Douglas.[55] Square well penetration coefficients and Newton's expression[56] for the level density were used. The spin cut-off parameter σ has not been well determined and the value 3 was taken for σ^{-2}. Competing effects such as proton emission were not considered so that the calculated cross sections are an upper limit. The results at 10 Mev give the cross section at 0° between 2 and 4 mb/sr. Some details of this calculation are given in Appendix C.

In the case where condition (7.10) holds, a typical differential cross section obtained by adding incoherently the cross sections for the two models is shown in figure 9.

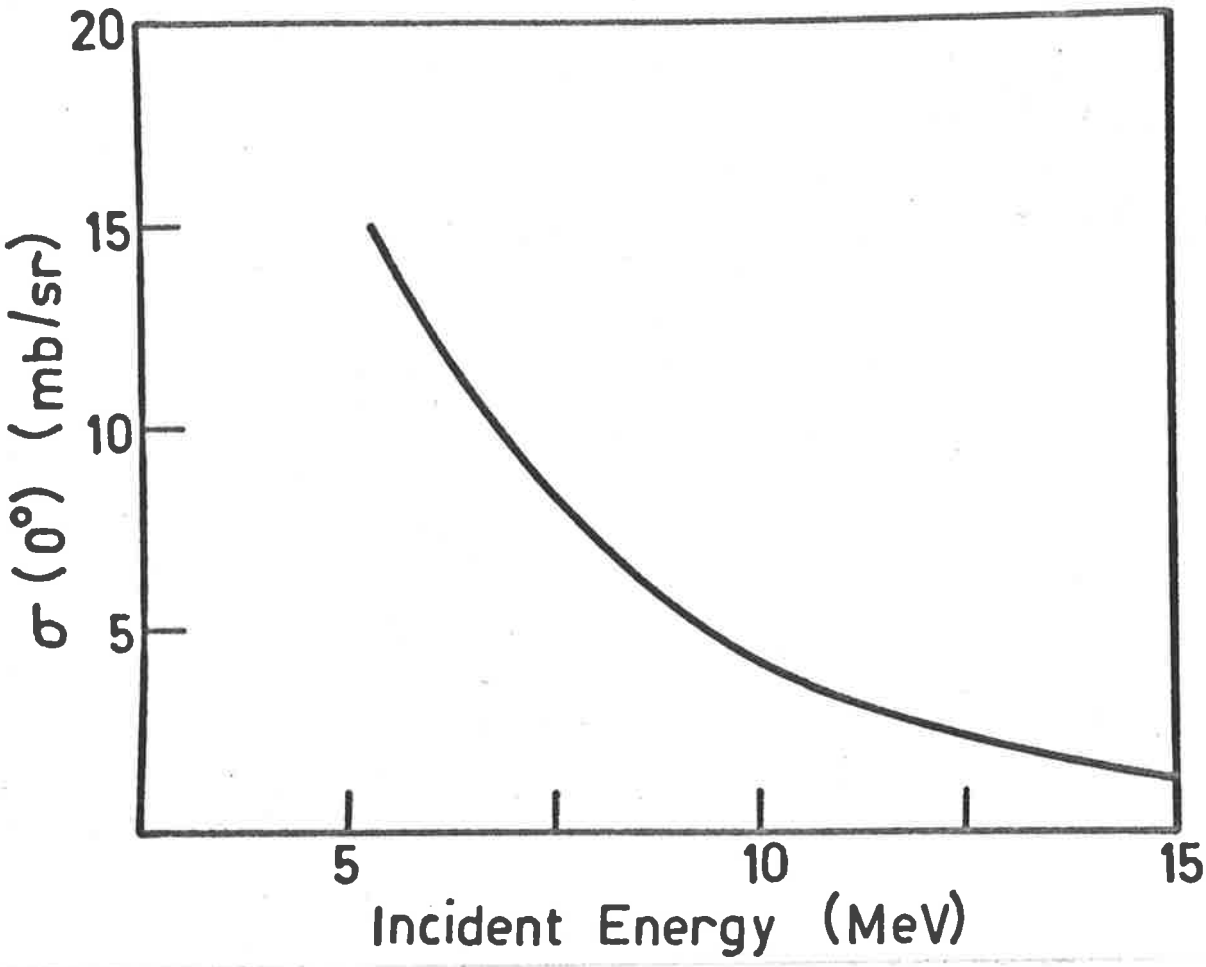

Fig. 8. Variation of $\sigma(0°)$ with energy.

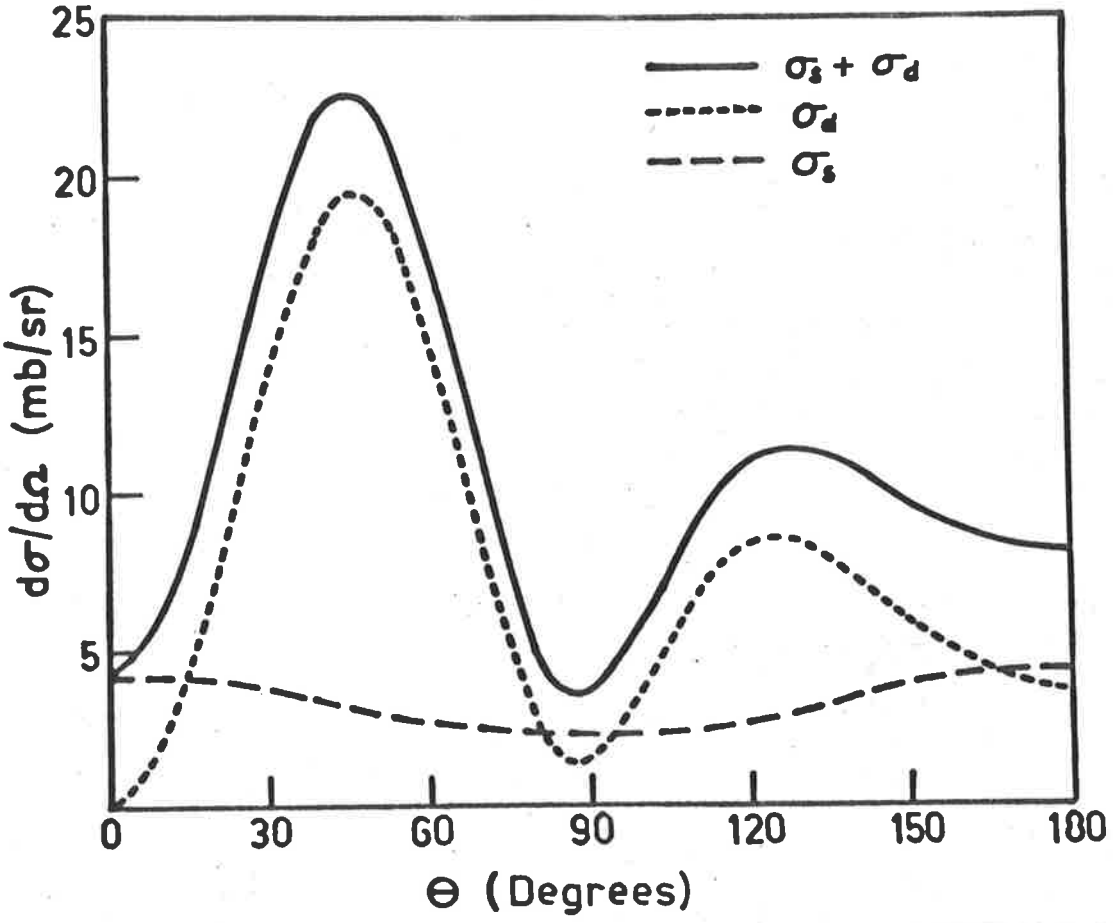

Fig. 9 Resolution of an assumed angular distribution for the reaction $F^{19}(n,n')F^{19*}$ at 10 MeV into compound and direct parts.

APPENDIX A

WAVE PACKET SCATTERING FOR MANY CHANNELS

For the applications to the optical model the wave packet formation of chapters 2 and 3 must be generalized.

In the case of scattering from a local, central potential the collision matrix U_ℓ is diagonal since the only collisions which are allowed energetically are elastic. On the other hand for collisions between systems with internal degrees of freedom, inelastic scattering and reactions are possible processes. The generalization of the single channel collision matrix (2.31) which applies in this multi-channel case is given by equation (10.15b) of reference 6.

$$U_{cc'}^{(J)} - \delta_{cc'} = \sqrt{k_c k_{c'}} \left[Q_{cc'}^{(J)}(E) - i \sum_\lambda \frac{R_{c'\lambda}^{(J)} R_{c\lambda}^{(J)}}{E - E_\lambda} \right] \qquad (A.1)$$

with the resonance denominators

$$E - E_\lambda = E_c^{(J)} - E_{c\lambda}^{(J)} + \tfrac{1}{2} i \Gamma_\lambda^{(J)} = E_{c'}^{(J)} - E_{c'\lambda}^{(J)} + \tfrac{1}{2} i \Gamma_\lambda^{(J)} . \qquad (A.2)$$

J is the quantum number for the total angular momentum of the system, while the subscript c represents the set of other quantum numbers which characterize the

ingoing channel and c labels the outgoing channel. Also from reference 6,

$$|R_{c\lambda}|^2 = \frac{q_\lambda \Gamma_{c\lambda}}{E_{c\lambda}^{\frac{1}{2}}} \quad \text{for} \quad E_{c\lambda} >> \Gamma_\lambda \, . \tag{A.3}$$

with q_λ a constant independent of the channel and the partial widths $\Gamma_{c\lambda}$ are related to the total width Γ_λ by

$$\sum_c \Gamma_{c\lambda} = \Gamma_\lambda \quad . \tag{A.4}$$

In our discussion of the optical model, we need only consider the interactions of spinless particles since the complications introduced by spin are irrelevant. In this case the total angular momentum of the system is equal to the angular momentum of the relative motion of the colliding particles and the expansion (A.1) becomes

$$U^{(\ell)}_{\alpha\alpha'} - \delta_{\alpha\alpha'} = \sqrt{k_\alpha k_{\alpha'}} \left[Q^{(\ell)}_{\alpha\alpha'}(E) - i \sum_\lambda \frac{R^{(\ell)}_{\alpha'\lambda} \, R^{(\ell)}_{\alpha\lambda}}{E - E_\lambda} \right] \tag{A.5}$$

with

$$E - E_\lambda = E^{(\ell)}_\alpha - E^{(\ell)}_{\alpha\lambda} + \tfrac{1}{2} i \Gamma^{(\ell)}_\lambda = E^{(\ell)}_{\alpha'} - E^{(\ell)}_{\alpha'\lambda} + \tfrac{1}{2} i \Gamma^{(\ell)}_\lambda \tag{A.6}$$

α is the set of quantum members associated with the energy eigenstates of the colliding particles whereas α' are the quantum numbers of the products of the collision.

With the help of the collision matrix (A.5), the

wave function which includes all the possible collision processes and which satisfies the boundary conditions appropriate to the scattering problem (incident plane wave and outgoing spherical waves in channel α , and outgoing waves in the other channels) can be expressed asymptotically as

$$\begin{aligned}\xi_\alpha &= \xi_\alpha^{el} + \xi_\alpha^{re} \\ &= \frac{\phi_\alpha}{(2\pi)^{\frac{3}{2}}}\left[e^{i\underline{k}_\alpha\cdot\underline{r}_\alpha} + \frac{1}{2ik_\alpha}\sum_\ell (2\ell+1)P_\ell(\cos\theta_\alpha)(U^{(\ell)}_{\alpha\alpha}-1)\frac{\exp(ik_\alpha r_\alpha)}{r_\alpha}\right] \\ &\quad + \frac{1}{2ik_\alpha(2\pi)^{\frac{3}{2}}}\sum_{\alpha\neq\alpha'}\phi_{\alpha'}\sum_\ell (2\ell+1)P_\ell(\cos\theta_{\alpha'})U^{(\ell)}_{\alpha'\alpha}\frac{\exp(ik_{\alpha'}r_{\alpha'})}{r_{\alpha'}}\end{aligned} \tag{A.7}$$

This is to be compared with the scattering state for the single channel case given by equations (2.12) and (2.15). ξ_α^{el} is almost identical with this state. There is an additional label 'α' to distinguish the entrance channel from the other channels and an extra factor ϕ_α which is the wave function describing the internal states of the colliding systems before interaction. The second part ξ_α^{re} has no counterpart in equation (2.12). It represents outgoing waves in channels other than the entrance channel.

Instead of the incident packet (2.9), we take the wave packet

$$\psi^{(\alpha)}_{\underline{k}_0} = (2\pi)^{-\frac{3}{2}} \int \Delta(\underline{k}_\alpha)\, e^{i\underline{k}_\alpha\cdot(\underline{r}_\alpha - \underline{r}_0)} d\underline{k}_\alpha\, \phi_\alpha \qquad (A.8)$$

By following the procedure of chapter 2, the incident wave packet (A.8) may be expanded in terms of the states $\xi_\alpha(\underline{k}_\alpha, \underline{k}_{\alpha'} \ldots)$. In the expansion corresponding to (2.13) for the expansion coefficient, the additional terms with $\alpha' \neq \alpha$ make no contribution as the integral over the internal coordinates gives zero,

$$\int \phi_\alpha \bar{\phi}_{\alpha'}\, d\tau = \delta_{\alpha\alpha'} \qquad (A.9)$$

Thus the expansion coefficients are unchanged and the outgoing wave packet is given by (2.17) with $\xi_{\underline{k}}$ replaced by (A.7). The elastically scattered part of the emergent wave packet is

$$\psi^{(\alpha)\,el}_{\underline{k}_0} = \frac{1}{(2\pi)^{\frac{1}{2}}} \int \Delta(\underline{k}_\alpha) \exp\left(i\underline{k}_\alpha\cdot\underline{r}_0 - \frac{i\hbar k_\alpha^2 t}{2m_\alpha}\right) \xi^{el}_\alpha\, d\underline{k}_\alpha \qquad (A.10)$$

and that part of the outgoing wave packet corresponding to inelastic scattering and reactions is

$$\psi^{(\alpha)\,re}_{\underline{k}_0} = \frac{1}{(2\pi)^{\frac{1}{2}}} \int \Delta(\underline{k}_\alpha) \exp\left(i\underline{k}_\alpha\cdot\underline{r}_0 - \frac{i\hbar k_\alpha^2 t}{2m_\alpha}\right) \xi^{re}_\alpha\, d\underline{k}_\alpha \qquad (A.11)$$

the wave vectors $\underline{k}_{\alpha'}$ in ξ^{re}_α being written as functions of $\underline{k}_\alpha$.

In the calculation of cross sections it is possible

to consider the wave packets (A.10) and (A.11) independently because the interference term vanishes by virtue of the orthogonality of ϕ_α and $\bar{\phi}_{\alpha'}$. With the method of section 2.3 the elastic wave packet (A.10) reduces to

$$\psi^{(\alpha)el}_{\underline{k}_0} = K \sum_\ell (2\ell+1) P_\ell(\cos\theta_\alpha) \int_0^\infty \Delta(E) \exp(iEX)$$
$$\times \left[Q^{(\ell)}_{\alpha\alpha}(E) - i \sum_\lambda \frac{R^{(\ell)}_{\alpha\lambda} R^{(\ell)}_{\alpha\lambda}}{E - E^{(\ell)}_{\alpha\lambda} + \frac{1}{2} i \Gamma^{(\ell)}_\lambda} \right] \phi_\alpha \, dE .$$

where the expansion (2.31) has been replaced by the more general expansion (A.5) (c.f. equation (2.32)).

Since the outgoing wave packets in the channels with $\alpha' \neq \alpha$ are not of interest in the discussion of the optical model, we omit the label distinguishing the entrance from the other channels. Also since the internal states of the colliding systems does not change during elastic collisions (for this spinless case), we suppress the wave function for the internal state. Then (A.12) may be written

$$\psi^{el}_{\underline{k}_0}(\underline{r},t) = K \sum_\ell (2\ell+1) P_\ell(\cos\theta_\alpha) \int_0^\infty \Delta(E) \exp(iEX)$$
$$\times \left[Q_\ell(E) - i \sum_\lambda \frac{R^2_{\ell\lambda}}{E - \epsilon_{\ell\lambda} + \frac{1}{2} i \Gamma_{\ell\lambda}} \right] dE \qquad \text{(A.13)}$$

with $\quad X = \frac{1}{\hbar} \left\{ \frac{r_0 + r}{v_0} - t \right\}$

and $$K = \frac{1}{(2\pi)^{\frac{1}{2}}} \cdot \frac{\exp\{ i \frac{k_0}{2}(r_0 + r)\}}{2ir}$$

which is formally the same as equation (2.32).

Therefore the results of chapters 2 and 3 are relevant to the elastically scattered wave packet of the many channel problem; the equations have only to be modified by replacing the parameters of the Mittag-Leffler expansion of the collision function by the corresponding parameters of the expansion of the diagonal element of the collision matrix.

One caution is, however, necessary. The branch point at E = 0 in the expansion (2.31) is removed from the integral (2.32) by the factor $\frac{1}{k}$ in the scattering amplitude. The function

$$\frac{U_\ell - 1}{K} = C_\ell(E) + \Sigma_n \frac{R_{\ell n}}{E - E_{\ell n}} \qquad \text{(A.14)}$$

is defined for all finite values of E in the complex E-plane except for the poles at $E_{\ell n}$. However, in

$$\frac{U^{(\ell)}_{\alpha'\alpha} - 1}{K_\alpha} = Q^{(\ell)}_{\alpha'\alpha}(E) - i \Sigma_\lambda \frac{R^{(\ell)}_{\alpha\lambda} R^{(\ell)}_{\alpha'\lambda}}{E - E_\lambda} \qquad \text{(A.15)}$$

the function $Q_{\alpha'\alpha}(E)$ has a number of branch points along the positive real axis in the complex E-plane, each branch point corresponding to a threshold where

another reaction becomes energetically possible.[6] Therefore in the evaluation of the non-resonant wave packet the energy integral of the form (2.39) has contributions from the branch cuts. Yet, if the peak energy of the incident packet is such that the contribution from the nearest threshholds are small, the considerations of section 2.4 apply and the non-resonant wave packet is scattered without time delay.

The temporal behaviour of the wave packet at the threshhold energies can only be evaluated if the explicit form of the background function $\mathcal{D}_{\alpha'\alpha}(E)$ is known.

APPENDIX B

The articles in this appendix were published by the author and I.E. McCarthy during the course of the work.

TIME-DEPENDENT SCATTERING EXPERIMENT

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(Received 10 December 1963)

The principle of superposition of states for position and momentum may be verified, for example, by observing the diffraction pattern built up by single photons scattered from a system whose characteristic space dimension is comparable with the wavelength of the photon.

The verification of the principle for energy and time requires the measurement of very short time intervals on the atomic scale. The time development of the state $\psi(t)$ of a quantum mechanical system with a Hamiltonian H is described by the Schrödinger equation

$$i\hbar d\psi/dt = H\psi. \tag{1}$$

The proposed experiment is a scattering experiment in which only the asymptotic form of the wave function in space and time is used and the development of the wave function from $t=-\infty$ to $t=+\infty$ is given by the S matrix without the use of the Schrödinger equation. Thus the time development of the wave function $\psi(t)$ is not observed in the vicinity of the scattering potential, as would be necessary to verify (1).[1]

The time interval characteristic of the scattering of a particle of energy E in the isolated resonance region by a many-body system is the reciprocal of the width Γ_S of the compound state $|s\rangle$ whose energy ϵ_S is closest to E. The case where more than one compound state contributes appreciably to the scattering amplitude in the vicinity of E will be treated in a subsequent publication.

Most compound nucleus states have lifetimes of the order of 10^{-13} sec. Times of the order of 10^{-10} sec are the smallest that can be experimentally resolved, so it is impossible to do a time-dependent experiment with such a system. Exceptions are the metastable states of nuclei, for example, the 14-keV state of ^{57}Fe which has a lifetime of 10^{-7} sec. Atomic states and particularly laser states of longer lifetime are well known. If a photon is scattered from such a resonance it would be easy to observe the effect of small time changes.

On a nuclear scale a time-dependent experiment has been performed by Holland et al.,[2] who essentially measured the spectrum of total elapsed time between the excitation of the 14-keV level in an ^{57}Fe source and the detection of the corresponding γ ray after it had passed through a resonant absorber. This may be regarded as a scattering experiment in which the incident wave packet has a half-exponential time spectrum of width $\hbar/\Gamma_S$. The time spectra of both the initial and final states must be known separately for a time-dependent scattering experiment.

The proposed experiment consists in following the time development of a wave packet with average energy E (near ϵ_S) and variable time width T (corresponding to an energy width $2\delta=\hbar/T$) which is scattered resonantly from the appropriate nucleus. The theory of the experiment is as follows.

The initial state is a wave packet which can be written as a superposition of the complete set of asymptotic states $\psi_{\vec{k}'}$ of the scattering problem:

$$\xi_{\vec{k}} = \int \omega(\vec{k}', \vec{k})\psi_{\vec{k}'} d^3k', \tag{2}$$

where, if a Hamiltonian H for the problem exists,

$$H\psi_{\vec{k}} = E_{\vec{k}}\psi_{\vec{k}}. \tag{3}$$

The integral in (2) corresponds to a sum over discrete eigenstates and an integral over eigenstates with continuous eigenvalues.

The time development of $\psi_{\vec{k}}$ is given by inte-

grating (1):

$$\psi_{\vec{k}}(t) = \psi_{\vec{k}} \exp(-iE_{\vec{k}}t/\hbar). \tag{4}$$

Substituting this into (2) gives the time development of the wave packet $\xi_{\vec{k}}$.

In the asymptotic region $\psi_{\vec{k}}$ is given for a spherical scatterer by

$$\psi_{\vec{k}}(r) = [e^{i\vec{k}\cdot\vec{r}} + f(k,\theta)e^{ikr}/r], \tag{5}$$

where $f(k,\theta)$ is the S-matrix element for the problem, assuming only one spin state is involved.

The usual theory of the scattering of wave packets[3] gives the time development of $\xi_{\vec{k}}$, in the case where the dimensions of the wave packet are large compared to those of the scatterer and small compared to those of the macroscopic apparatus, as

$$\xi_{\vec{k}}(\vec{r},\vec{r}_0,t) = (2\pi)^{-3/2}\int \Delta(\vec{k}';\vec{k},\delta)\exp(-i\vec{k}'\cdot\vec{r}_0 - iE't/\hbar) \times[\exp(ik'r)/r]f(k',\theta)d^3k', \tag{6}$$

where $\vec{r}_0$ is the position of the center of the wave packet at $t=0$. $\Delta(\vec{k}';\vec{k},\delta)$ is a weight factor which confines the incident wave packet to a time width $T=\hbar/2\delta$ and an angular region in which we are not interested and which may be made very small on the atomic scale by defining the beam by macroscopic slits.

Integrating out the directions of $\vec{k}'$ and substituting $k'=E'/\hbar c$, we have

$$\xi_{\vec{k}}(\vec{r},t) = \frac{(2\pi)^{-1/2}}{\hbar c}\frac{\exp[ik(r_0+r)/2]}{r}\int_0^\infty dE'\Delta(E';E,\delta) \times\exp(iE'X)f(k',\theta), \tag{7}$$

where

$$X = [(r_0+r)/c - t]/\hbar. \tag{8}$$

Near an isolated resonance we can express $f(k',\theta)$ as

$$f(k',\theta) = (2l+1)P_l(\cos\theta)R_s/(E'-\epsilon_s+\tfrac{1}{2}i\Gamma_s), \tag{9}$$

where l is the angular-momentum quantum number of the compound state $|s\rangle$.

We will assume that the incident wave packet can be prepared with a Lorentzian energy dependence which implies an exponential time dependence:

$$\Delta(E';E,\delta) = (\delta/\pi)/[(E'-E)^2+\delta^2]. \tag{10}$$

For photons we are not concerned with the contribution of bound states to the integral in (7), so we can replace the lower limit of integration by $-\infty$.

The time development of the wave packet is now given by

$$\xi_{\vec{k}}(\vec{r},t) = K(2l+1)P_l(\cos\theta)\int_{-\infty}^{\infty}dE'\frac{\delta/\pi}{(E'-E_\delta)(E'-\bar{E}_\delta)} \times\frac{R_s}{E'-E_s}\exp(iE'X), \tag{11}$$

where

$$E_s = \epsilon_s - \tfrac{1}{2}i\Gamma_s,$$

$$E_\delta = E + i\delta,$$

$$\bar{K}K = (8\pi r^2\hbar^2c^2)^{-1}. \tag{12}$$

For $X<0$, contour integration round an infinite semicircle in the lower-half plane gives for the integral in (11)

$$I = \frac{R_s}{E_\delta - E_s}\exp(iE_\delta X) - 2i\delta\frac{R_s}{(E_s-E_\delta)(E_s-\bar{E}_\delta)} \times\exp(iE_sX). \tag{13}$$

The time development of the probability of observing the scattered particle is given by

$$|\xi_{\vec{k}}(\vec{r},t)|^2 = \frac{1}{8\pi}\left[\frac{(2l+1)P_l(\cos\theta)}{r\hbar c}\right]^2\left\{\frac{R_s\bar{R}_s}{(E_\delta-E_s)(\bar{E}_\delta-\bar{E}_s)}\exp(2\delta X) + \frac{4R_s\bar{R}_s\delta^2}{(E_s-E_\delta)(E_s-\bar{E}_\delta)(\bar{E}_s-E_\delta)(\bar{E}_s-\bar{E}_\delta)}\exp(\Gamma_sX) + \left[\frac{2i\delta R_s\bar{R}_s}{(E_\delta-E_s)(\bar{E}_s-\bar{E}_\delta)(E_s-\bar{E}_\delta)}\exp[i(E-i\delta-\epsilon_s-\tfrac{1}{2}i\Gamma_s)X] + \text{complex conjugate}\right]\right\}. \tag{14}$$

For an incident wave packet whose time width T is very small (2δ is large compared with Γ_s), the second term in (14) is most important. The probability of observing the particle decays exponentially in time with a time constant Γ_s.

Exponential decay has been observed by Holland et al., for the 14-keV state of ^{57}Fe. The initial

time for the system was defined by the emission of the 128-keV γ ray resulting from the decay of ^{57}Fe into the 14-keV state. The uncertainty in this definition is small ($\sim 10^{-18}$ sec), so we have the case $2\delta \gg \Gamma_s$. Of course, this case is different from the one under consideration because the 14-keV state was excited in a short time by a different process from resonant absorption. The differential cross section[4] corresponding to (14) is

$$\frac{d\sigma}{d\Omega} = \tfrac{1}{2} R_s \bar{R}_s \left[\frac{(2l+1)P_l(\cos\theta)}{\hbar c}\right]^2 \frac{[\frac{1}{2}\Gamma_s(E-\epsilon_s)^2 + \frac{1}{2}(\Gamma_s+\delta)^2(\frac{1}{2}\Gamma_s+2\delta)]}{\Gamma_s[(E-\epsilon_s)^2+(\delta+\frac{1}{2}\Gamma_s)^2]^2}. \tag{15}$$

The significant thing is that the energy dependence of the differential cross section now has a width $\Gamma_s + 2\delta$. This contrasts with the width Γ_s found in ordinary resonance fluorescence with a nonmonochromatic beam.[5]

For short total elapsed time, which puts an upper limit on the time length of the incident wave packet, Holland et al. found departures from exponential time decay due to the apparently shorter lifetime of the state. They also found increased energy width, but they did not know the time spectra of the initial and final states separately except in the case where they integrated over all time. The incident wave packet is then a half-exponential with lifetime $\hbar/\Gamma_s$. Their experiment was explained in terms of radiation from damped oscillators.[6]

It is proposed that a beam of photons produced with very small energy width by the decay of a state $|s\rangle$ with long lifetime should be modulated into exponential (if possible) pulses of width T which can be varied, but which is much less than $\hbar/\Gamma_s$. The pulses must be separated by a time greater than $\hbar/\Gamma_s$. This can be done, for example, by having another resonant absorber in the beam between the source and the scatterer. The intermediate absorber can be put into a nonresonant condition for a time T, for example, by moving it or by shifting the resonance by the Stark effect with a strong external field applied for a short time.

In the ^{57}Fe case accelerating the absorber fast enough or applying a strong enough field at the nucleus are all on the borderline of present experimental technique. With an appropriate atomic or molecular level it should not be so difficult because the time T can be longer.

We now have a beam consisting of minimum wave packets whose central energy is ϵ_s. We could observe directly the dependence of the absorption spectrum on T [Eq. (15)], or we could observe the time dependence of the probability of detecting a scattered photon [Eq. (14)].

The latter experiment would be a scattering experiment in which purely time rather than energy is resolved. The intrinsic energy uncertainty Γ_s is much smaller than the width 2δ introduced by observing small time intervals T.

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[1]M. L. Goldberger and K. M. Watson, Phys. Rev. 127, 2284 (1962).

[2]R. E. Holland, F. J. Lynch, G. J. Perlow, and S. S. Hanna, Phys. Rev. Letters 4, 181 (1960).

[3]For example, A. Messiah, Quantum Mechanics (North-Holland Publishing Company, Amsterdam, 1961), Chap. 10, Sec. 5.

[4]T. Sasakawa [Suppl. Progr. Theoret. Phys. (Kyoto) 11, 69 (1959)] has derived a physically equivalent result using a different mathematical technique.

[5]For example, W. Heitler, Quantum Theory of Radiation (Clarendon Press, Oxford, 1954), 3rd ed., p. 199.

[6]F. J. Lynch, R. E. Holland, and M. Hamermesh, Phys. Rev. 120, 513 (1960).

Scattering of Energy-Time Wave Packets from Many-Body Systems*

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The wave-packet nature of an experimental beam is discussed. The theory of wave-packet scattering is applied to wave packets with exponential time dependence scattered from resonances in both the isolated-resonance and overlapping-resonance regions. It is shown how a sequential description of scattering depends on some of the resonance parameters in the scattering amplitude. The meaning and usefulness of some experimentally possible wave-packet experiments is discussed.

1. INTRODUCTION

SCATTERING of a particle from a many-body system can be divided into two types with respect to a time interval τ and a corresponding energy interval $\delta(\sim\hbar/\tau)$.[1] If the scattering amplitude as a function of energy varies rapidly over the energy interval δ, the scattering is said to be compound scattering; if the amplitude varies slowly over this interval, we have direct scattering. τ is several orders of magnitude greater than the time it takes the incident particle to traverse a distance the size of the scatterer in free space.

In most actual scattering experiments time is not resolved. The incident beam has an energy spread. The cross section can be regarded as the weighted sum of cross sections for independent experiments, each with definite incident energy. This will be discussed in Sec. 2. It is still possible to define a time delay Δt according to the definition of Wigner,[2]

$$\Delta t = -i\hbar(d/dE)\ln S(E), \tag{1}$$

where $S(E)$ is the S-matrix element for the scattering problem. This time delay may be thought of as the time it takes for the phase of one incident wave to catch up with that of another whose energy differs from it by an infinitesimal amount, and whose phase shift in the scatterer is therefore infinitesimally different.

If the S-matrix element is divided according to some physical prescription into a rapidly and a slowly varying component, this definition is difficult to apply and it certainly does not tell us anything about each component separately. The rapidly varying component corresponds to a large Δt, that is, to particles that spend a long time on the average in the scatterer, and the slowly varying component corresponds to a small Δt, that is to particles that pass rapidly over the scatterer without significant time delay.

An experiment which measures the relative amount of direct and compound scattering has been suggested by Eisberg, Yennie, and Wilkinson.[3] The experiment also defines an energy interval which is the reciprocal of the time delay. The information is obtained from the energy spectrum of bremsstrahlung from elastic scattering of charged particles.

One would like to make a classical picture in which particles are described as being in the scatterer for varying times. To do this one must make a wave-packet argument. It will be shown in Sec. 2 that the physical situation corresponding to a meaningful wave-packet argument is necessarily a time-dependent scattering experiment.

Wave-packet arguments are made for example by Friedman and Weisskopf[4] for the case of shape-elastic and compound-elastic scattering. They show that for nonoverlapping levels (resonances) of the compound system the time delay is the average over the beam energy spread of the time delays for the individual levels. If time delay is defined according to Wigner's definition, this result is obtained at once. It has been done explicitly for example by Goldberger and Watson.[5]

If, however, time delay is defined as the average time delay in the emerging of a wave packet in a hypothetical time-dependent experiment complementary to the usual energy-dependent experiment, then the result is not so clear. The delayed wave packet would be expected to interfere with the immediately scattered wave packet from the shape elastic scattering. It is commonly stated that if the two wave packets are sufficiently short in time they will not interfere.[6] This

* Supported in part by the U. S. Atomic Energy Commission and the Australian Commonwealth Scientific and Industrial Research Organization.

[1] For example H. Feshbach, C. E. Porter, and V. F. Weisskopf, Phys. Rev. 96, 448 (1954).

[2] E. P. Wigner, Phys. Rev. 98, 145 (1955).

[3] R. M. Eisberg, D. R. Yennie, and D. H. Wilkinson, Nucl. Phys. 18, 338 (1960).

[4] F. L. Friedman and V. F. Weisskopf, *Niels Bohr and the Development of Physics* (Pergamon Press, Ltd., London, 1955).

[5] M. L. Goldberger and K. M. Watson, Phys. Rev. 127, 2284 (1962).

[6] For example, H. G. Preston, *Physics of the Nucleus* (Addison Wesley Publishing Co., Inc., Reading, Massachusetts, 1962). R. K. Adair, S. E. Darden, and R. E. Fields, Phys. Rev. 96, 502 (1954).

situation is now extreme hypothetical. It requires very good time definition in the experiment. In an actual experimental situation, time is undefined and the wave packets interfere completely. It may also be asked whether the scattering amplitude, and hence, for example, the angular distribution will not be dependent on the width of the incident wave packet, since, for short wave packets (in time), we might not expect such complete interference as for long wave packets.[7]

The question of what actually happens to a scattered wave packet has been investigated by Sasakawa.[8] He shows that for the scattering of a wave packet from an isolated resonance, the variation of the cross section with energy is characterized not by the width of the resonance but, in addition, by the width of the wave packet. In Sec. 3 we will re-derive this result using a different mathematical technique and Lorentzian wave packets. The Lorentzian shape is one shape for the energy spectrum that can be realized experimentally.

In Sec. 4 we will consider the scattering of wave packets from a system in which the compound levels overlap. This gives new insight into the optical model, which is defined for this purpose as the model for which the time delay in the complementary time-dependent experiment would be small. Our treatment is more general than Sasakawa's treatment of the same situation in a special case.

The usual definition of the optical-model scattering amplitude[9] is that it is the average of the actual scattering amplitude over an energy interval δ. In the actual situation, amplitudes for different energies are independent and it is the cross sections that are experimentally averaged. The meaning of the energy averaging process will be discussed.

In Sec. 5 we will discuss the possibility of performing scattering experiments with time definition which are complementary to the usual ones in which energy but not time is resolved. One such experiment has been suggested by the present authors.[10]

2. DEPENDENCE OF THE SCATTERING CROSS SECTION ON THE INITIAL BEAM CHARACTERISTICS

A beam consists of particles whose energy is defined within certain limits. The time of arrival of particles at a specific point may also be defined within limits by modulating the amplitude of the beam. In general the limits on the time and energy resolution of the beam are wider than those imposed by the uncertainty principle. For example some particles may be moving faster than others because they were accelerated a little more in the accelerator. The fast and slow beams could in principle be separated by a magnetic spectrometer. This type of energy uncertainty will be called classical uncertainty. On the other hand, if the time of arrival of a particle at a point is defined within a time interval τ, we cannot in principle measure its energy with a definition better than $\delta=\hbar/\tau$. This type of uncertainty will be called quantal uncertainty.

A beam with both classical and quantal uncertainty is described by a collection of wave packets with different mean wave numbers $\mathbf{k}_j$ whose position at time t is measured with respect to different initial position vectors $\mathbf{r}_i$. The value of $\mathbf{r}_i$ specifies the starting time of the wave packet at the source. One such wave packet may be written

$$\xi_{ij}(\mathbf{r},t)=(2\pi)^{-\frac{3}{2}}\int\Delta(\mathbf{k}_j,\delta;\mathbf{k})\exp[i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}_i)]d^3k. \quad (2)$$

Suppose that the number of packets of type j is given by n where

$$n=\Phi_j N. \quad (3)$$

N is the total number of wave packets.

We wish to know the cross section for a scattering experiment with an incident beam which can be described in this way. For simplicity we will consider only one spin state. We must be careful to distinguish two cases.

Case A. If the $\mathbf{r}_i$ are such that the wave packets do not overlap both before and after scattering, the scattering of each wave packet may be considered as an individual event. The total cross section is the sum of the cross sections for the scattering of each wave packet.

Case B. The $\mathbf{r}_i$ are such that the wave packets overlap. In this case different wave packets will interfere and the cross sections for individual wave packets do not add incoherently.

The mathematical description of these cases is as follows. For case A we will consider wave packets with the same value $\mathbf{r}_0$ of $\mathbf{r}_i$ and describe the beam by means of a density matrix. The density matrix at time t is

$$\rho(\mathbf{r},\mathbf{r}',t)=\sum_j\Phi_j\xi_{0j}(\mathbf{r},t)\xi_{0j}^*(\mathbf{r}',t). \quad (4)$$

According to the standard theory of wave-packet scattering,[11] ξ_{0j} is given in the case where the space occupied by the wave packet is large compared to the size of the scatterer but small compared to the distance of the detector by

$$\xi_{0j}(\mathbf{r},t)=(2\pi)^{-\frac{3}{2}}\int\Delta(\mathbf{k}_j,\delta;\mathbf{k})\exp(-i\mathbf{k}\cdot\mathbf{r}_0-iEt/\hbar)\times[\exp(ikr)/r]f(k,\Omega)d^3k. \quad (5)$$

[7] I. E. McCarthy, *Proceedings of the International Symposium on Direct Interactions and Nuclear Reaction Mechanisms, Padua, 1962* (Gordon and Breach Publishers, Inc., New York, 1963).

[8] T. Sasakawa, Progr. Theoret. Phys. Suppl. 11, 69 (1959).

[9] For example G. E. Brown, Rev. Mod. Phys. 31, 893 (1959).

[10] L. R. Dodd and I. E. McCarthy, Phys. Rev. Letters 12, 136 (1964).

[11] See, for example, E. Merzbacher, *Quantum Mechanics* (John Wiley & Sons, Inc., New York, 1961). A review of general wave-packet scattering with applications to nuclear reactions has been given by N. Austern in *Selected Topics in Nuclear Theory* (International Atomic Energy Agency, Vienna, 1963).

$f(k,\Omega)$ is the scattering amplitude for scattering of particles into a detector whose angular position is Ω. $\Delta(\mathbf{k}_j,\delta;\mathbf{k})$ is a weight function which confines the wave packet to a volume in momentum space centered at $\mathbf{k}_j$ and of width δ. δ is the *quantal uncertainty*.

We will not be interested in the angular uncertainty in the wave packet, so we will write

$$\Delta(\mathbf{k}_j,\delta;\mathbf{k})=\frac{\delta(\theta_k-\theta_0)\delta(\phi_k-\phi_0)}{2\pi k^2\sin\theta_k}\Delta(k_j,\delta;k), \tag{6}$$

where (θ_k,ϕ_k), (θ_0,ϕ_0) are, respectively, the angular coordinates of $\mathbf{k}$ and $\mathbf{k}_j$, the beam direction.

The probability $P_j(t)$ of detecting a particle at time t at the detector whose position is $\mathbf{r}_d$ is, for the wave packet j,

$$P_j(t)=N_j|\xi_{0j}(\mathbf{r}_d,t)|^2r_d^2d\Omega. \tag{7}$$

This defines the normalization N_j of ξ_{0j}. The differential cross section is

$$(d\sigma/d\Omega)(\mathbf{k}_j,\delta)=v_0N_jr_d^2\int_{-\infty}^{\infty}|\xi_{0j}(\mathbf{r}_d,t)|^2dt, \tag{8}$$

where v_0 is the group velocity of the wave packet.

We may write the differential cross section for the beam described by the density matrix (4) as

$$d\sigma/d\Omega=v_0N_jr_d^2\int_{-\infty}^{\infty}\rho(\mathbf{r}_d,\mathbf{r}_d,t)dt. \tag{9}$$

Substituting from (4) and interchanging the order of summation and integration we have

$$\begin{aligned}d\sigma/d\Omega&=\textstyle\sum_j\Phi_jv_0N_jr_d^2\displaystyle\int_{-\infty}^{\infty}|\xi_{0j}(\mathbf{r}_d,t)|^2dt\\&=\textstyle\sum_j\Phi_jd\sigma(\mathbf{k}_j,\delta)/d\Omega.\end{aligned} \tag{10}$$

This is the result stated above for case A. Since we are only interested in the differences of the peak energies E_j for different wave packets j, we may replace the sum in (10) by an energy integral.

$$\frac{d\sigma}{d\Omega}=\int\Phi(E')\frac{d\sigma}{d\Omega}(E',\delta)dE'. \tag{11}$$

The cross section is the energy average of the cross sections for individual wave packets with different peak energies. In particular, for beams with negligible quantal uncertainty the cross section is the energy average of the cross sections for scattering events with plane wave and outgoing spherical wave boundary conditions. For this reason we will call ϕ, the width of the weight function $\Phi(E)$, the *classical uncertainty*.

We will now consider the more general case B and see under what circumstances the distinction between classical and quantal uncertainty can be maintained.

In terms of plane-wave states $\eta(\mathbf{k})$, the most general density matrix is

$$\rho(\mathbf{r},\mathbf{r}',t)=\int\int a(\mathbf{k},\mathbf{k}')\eta(\mathbf{k}')^*\eta(\mathbf{k})d^3kd^3k'. \tag{12}$$

We wish to write (12) in diagonal form in terms of wave-packet states ξ_{ij}.

$$\rho(\mathbf{r},\mathbf{r}',t)=\textstyle\sum_{ij}\Phi_j\xi_{ij}\xi_{ij}^*. \tag{13}$$

We must know if this diagonalization is unique. If so, we can assign a definite classical weight factor Φ with uncertainty ϕ and a definite quantal weight factor Δ with uncertainty δ. If not, the distinction is meaningless.

In mathematical terms the question is as follows. Can the Hermitian matrix $\rho(\mathbf{r},\mathbf{r}',t)$ in (12) be diagonalized by more than one transformation of the type (2) of the basis vectors? A necessary condition for the diagonalization to be unique[12] is that the states ξ_{ij} must be orthogonal like the states η.

In fact we have

$$\int\xi_{ij}\xi_{i'j'}d^3r=(2\pi)^{-\frac{1}{2}}\int\Delta(\mathbf{k}_j,\delta;\mathbf{k})\Delta(\mathbf{k}_{j'},\delta;\mathbf{k})^*\times\exp[-i\mathbf{k}\cdot(\mathbf{r}_i-\mathbf{r}_{i'})]d^3k. \tag{14}$$

The overlap integral in (14) is approximately zero if the wave packets have sufficiently different starting times given by $\mathbf{r}_i$ and $\mathbf{r}_{i'}$ because of the rapid oscillations of the integrand.

In case A the wave packets have a definite physical significance. The quantal uncertainty leads to alterations in the usual cross sections, as will be shown in the succeeding sections. In case B the wave packets may be regarded as mathematical aids to the visualization of the scattering process. For example we can decompose a plane wave into an infinite number of overlapping wave packets and then follow the propagation of one of these packets. Nevertheless, this would be without physical significance because the scattering amplitudes for all the packets must be recombined to obtain the cross section.

The vital point is that the wave packets both before and after the scattering must be quite distinct in space and time for the quantal uncertainty to have any physical significance.

3. SCATTERING OF A WAVE PACKET FROM A RESONANCE

In the usual theory of scattering it is assumed that $f(k,\Omega)$ in Eq. (5) varies much more slowly with energy than the wave packet amplitude factor $\Delta(\mathbf{k}_j,\delta;\mathbf{k})$; that is, that the quantal uncertainty δ is negligible in com-

[12] U. Fano, Rev. Mod. Phys. 29, 74 (1957).

parison with the width of the energy fluctuations of the scattering amplitude.

We will consider the case where δ is comparable with Γ_s, the width of the compound state $|s\rangle$ of the system comprising the incident particle and the scatterer whose energy ϵ_s is near the central energy E_0 of the wave packet. In the present section we will suppose that only one state $|s\rangle$ contributes to the scattering amplitude at the energies covered by the wave packet.[12a] We will be interested in only one wave packet $j=0$.

Since we are interested in the energy rather than the momentum of the wave packet, it will be convenient to transform the integral in (5) to an energy integral. The transformation is trivial when k is proportional to E as it is for photons ($E=\hbar ck$).

For our discussion of hypothetical time-dependent experiments with finite-mass particles complementary to the usual energy-dependent experiments, we can make the assumption, following Friedman and Weisskopf,[4] that the source and detector are sufficiently close to the scatterer for wave packet spreading to be negligible. We will neglect the final term in the following expansion of E:

$$E=\frac{\hbar^2}{2m}[2\mathbf{k}\cdot\mathbf{k}_0-k_0^2+|\mathbf{k}-\mathbf{k}_0|^2]. \tag{15}$$

Photon wave packets do not spread for practical purposes.

We must now consider a particular form for the energy amplitude factor $\Delta(E_0,\delta;E)$ corresponding to $\Delta(\mathbf{k}_0,\delta;\mathbf{k})$. Wave packets made by electronic means (see Sec. 5) would have a rise and decay function in time that is something like an exponential. We will therefore consider a wave packet that has an exponential rise and decay, with time constant $\hbar/\delta$. We will assume that the beam is switched off as soon as it has attained full strength, so that there is no time for which the beam intensity is constant. The experimental difficulty is in fast switching on and off. Since we want as short a packet in time as possible, this is then the most realistic form. The energy amplitude factor for this form is

$$\Delta(E_0,\delta;E)=\frac{\delta/2\pi}{(E-E_0)^2+\delta^2/4}. \tag{16}$$

One shape of wave packet is already available in nature. This is the photon wave packet from the decay of a level $|s\rangle$ of a many-body system. It has a half-exponential time spectrum. If the moment of excitation of the level is taken as zero time, then the probability of emission of a photon decays exponentially with a time constant $\hbar/\Gamma_s$. In this case we have

$$\Delta(E_0,\Gamma_s;E)=\frac{i/2\pi}{E-E_0+i\Gamma_s/2}. \tag{17}$$

[12a] *Note added in proof.* The decay of a single resonance has been investigated by R. G. Newton, Ann. Phys. (N. Y.) 14, 333 (1961).

We will use the form (16) for most of our discussion.

For the scattering amplitude we will use the expansion of Siegert, Humblet, and Rosenfeld,[13]

$$f[k(E),\theta]=\frac{1}{2i}\sum_l(2l+1)P_l(\cos\theta)\times\left[C_l(E)+\sum_n\frac{R_{ln}}{E-\epsilon_{ln}+\frac{1}{2}i\Gamma_{ln}}\right]. \tag{18}$$

The nonresonant term $C_l(E)$ varies slowly with energy. The splitting off of this term is not unique and can be made according to some physical prescription. Equation (5) now becomes

$$\xi(\mathbf{r},t)=\frac{K}{2\pi}\sum_l(2l+1)P_l(\cos\theta)\int_0^\infty\frac{\delta}{(E-E_\delta)(E-\bar{E}_\delta)}\times\exp(iEX)\left[C_l(E)+\sum_n\frac{R_{ln}}{E-\epsilon_{ln}+i\Gamma_{ln}/2}\right]dE, \tag{19}$$

where

$$X=[(r_0+r)/v_0-t]/\hbar,$$
$$E_\delta=E_0-i\delta/2,$$
$$K\bar{K}=1/8\pi r^2. \tag{20}$$

We have dropped the subscripts on $\xi(\mathbf{r},t)$ because we are now interested only in one particular wave packet.

The integral in (19) is now in a convenient form for contour integration if we make the approximation of extending the lower limit to $-\infty$, thus neglecting the contributions of bound states.

We will first consider the scattering of the wave packet according to the nonresonant scattering amplitude $C_l(E)$ whose variation with energy can be neglected over the energies of the wave packet. Performing the integration in Eq. (19) we find

$$\begin{aligned}\xi(\mathbf{r},t)&=K\sum_l(2l+1)P_l(\cos\theta)C_l(E)\exp(i\bar{E}_\delta X),\quad X>0\\&=K\sum_l(2l+1)P_l(\cos\theta)C_l(E)\times\exp(iE_\delta X),\quad X<0.\end{aligned} \tag{21}$$

The wave packet is centered at $X=0$, i.e., at

$$t=(r_0+r)/v_0. \tag{22}$$

Hence, the nonresonant packet is propagated without time delay. The time spectrum of $|\xi(\mathbf{r},t)|^2$ is an exponential rise and fall, $\exp(-\delta t/\hbar)$.

The differential cross section is, according to Eq. (8),

$$d\sigma/d\Omega=\tfrac{1}{4}|\sum_l(2l+1)P_l(\cos\theta)C_l(E)|^2. \tag{23}$$

Thus, the differential cross section for nonresonant (potential) scattering is independent of the quantal uncertainty δ and identical with that for a normal beam in which δ is negligible.

[13] A. Siegert, Phys. Rev. 56, 750 (1939). J. Humblet and L. Rosenfeld, Nucl. Phys. 26, 529 (1961).

We will now consider the scattering from an isolated resonance state $|s\rangle$. Taking one term of the sum in (18) we have for the integral in (18)

$$I=\int_{-\infty}^{\infty}\frac{\delta/2\pi}{(E-E_\delta)(E-\bar{E}_\delta)}\frac{R_s}{E-E_s}\exp(iEX)dE, \quad (24)$$

where

$$E_s=\epsilon_s-i\Gamma_s/2. \quad (25)$$

Integration round an infinite semicircle in the upper half plane gives

$$I=\frac{R_s}{\bar{E}_\delta-E_s}\exp(i\bar{E}_\delta X), \quad X>0. \quad (26)$$

This means that the leading edge of the wave packet, that is, the part for times greater than $(r_0+r)/v_0$, is propagated with the same shape as it originally had, but, of course, with a different magnitude. The propagation of the trailing edge (the tail) of the wave packet is given by integration round a contour in the lower half plane

$$I=\frac{R_s}{E_\delta-E_s}\exp(iE_\delta X)$$
$$-i\delta\frac{R_s}{(E_s-E_\delta)(E_s-\bar{E}_\delta)}\exp(iE_sX), \quad X<0. \quad (27)$$

The time spectrum given by $|I|^2$ is a decaying oscillatory function. In particular, if the time width of the wave packet is much less than the decay constant of the state $|s\rangle$, the time spectrum of the tail has the shape $\exp(-\Gamma_s t/\hbar)$. This corresponds to exciting the resonance suddenly and watching it decay with its natural time constant.

The differential cross section is

$$\frac{d\sigma}{d\Omega}=\frac{R_s\bar{R}_s[(E-\epsilon_s)^2\Gamma_s/2+(\Gamma_s+\delta)^2(\Gamma_s/2+\delta)/4]}{2\Gamma_s[(E-\epsilon_s)^2+(\Gamma_s+\delta)^2/4]^2}. \quad (28)$$

This reduces to the Breit-Wigner form for very small quantal uncertainty δ. The most significant thing is that the width of the energy spectrum is $\Gamma_s+\delta$. Note also that the differential cross section for scattering from a single resonance is reduced in magnitude for large quantal uncertainty. For large δ, $d\sigma/d\Omega$ is of order $1/\delta$.

In view of the arguments often made about the noninterference of potential and resonance scattering for large δ, it is interesting to consider the interference term. Denote the additional cross section due to the interference of potential and resonant scattering by $d\sigma_I/d\Omega$. It arises from the last term in the splitting of the scattering into potential and resonant parts

$$|\xi(\mathbf{r},t)|^2=|\xi_P(\mathbf{r},t)|^2+|\xi_R(\mathbf{r},t)|^2$$
$$+2\,\mathrm{Re}\,\xi_P(\mathbf{r},t)\xi_R(\mathbf{r},t). \quad (29)$$

Taking only the s state for simplicity of notation,

$$\frac{d\sigma_I}{d\Omega}=\tfrac{1}{2}\left\{R_s\bar{C}_0(E)\frac{(E-\epsilon_s)+i(\Gamma_s/2+\delta)}{[(E-\epsilon_s)+i(\Gamma_s+\delta)^2/4]^2}\right\}, \quad (30)$$

$$=\tfrac{1}{2}\,\mathrm{Re}[R_s\bar{C}_0(E)]\frac{(E-\epsilon_s)(E-\epsilon_s)^2+(\Gamma_s+\delta)(\Gamma_s+3\delta)/4}{[(E-\epsilon_s)^2+(\Gamma_s+\delta)^2/4]^2}$$
$$+\tfrac{1}{2}\,\mathrm{Im}[R_s\bar{C}_0(E)]\frac{(E-\epsilon_s)^2\Gamma_s/2+(\Gamma_s+\delta)^2(\Gamma_s+2\delta)/8}{(E-\epsilon_s)^2+(\Gamma_s+\delta)^2/4}. \quad (31)$$

The coefficient of $\mathrm{Im}[R_s\bar{C}_0(E)]$ is of order $1/\delta$ for large δ. Hence the interference term in the cross section does become smaller for larger δ, that is for better defined wave packets in time, but so also does the resonant scattering term, both being of order $1/\delta$. The potential scattering term is not affected by δ.

4. SCATTERING FROM MANY LEVELS AND THE OPTICAL MODEL

We will now consider the energy region where, for a given channel, the resonances in the scattering amplitude are such that the average width is greater than the average spacing, so that many levels contribute at each energy over the energy spread of the wave packet.

The levels in this region are defined by some plausible model such as has been discussed by Brown[9] for nuclei. Their widths are of the order of 1 eV, so that the corresponding lifetime, 10^{-15} sec, is too short for a practical wave-packet experiment. However, it is interesting to consider a hypothetical wave-packet experiment in order to give a proper quantum mechanical sequential description of the passage of a particle through the scatterer, when we know the S-matrix element for the scattering.

This gives us another way of looking at the optical and direct interaction models. We will restrict ourselves to elastic scattering for simplicity. The optical model has been considered in two ways, each with a different

starting point. The first way starts with a Schrödinger equation. The optical-model Hamiltonian is shown to be an approximation to the many-body Hamiltonian. The second way is independent of any Hamiltonian assumption and starts with the scattering amplitude. The average over some energy interval I of the scattering amplitude S is shown to be capable of being calculated from an optical-model Hamiltonian. The average cross section is split into two terms.

$$\langle\sigma\rangle = |\langle S\rangle|^2 + \langle|S-\langle S\rangle|^2\rangle. \tag{32}$$

The first term is the optical-model or shape elastic cross section. The second term is the fluctuation or compound elastic cross section.

A third, classical, definition of the optical model is sometimes given. The optical-model cross section is the cross section for propagation without time delay while the compound elastic cross section is the remainder.

It is not clear that the last two definitions are equivalent. The splitting in Eq. (32) requires a definition of the averaging procedure. Some authors, for example Brown,[9] have gone to much trouble to discuss the averaging procedure. Brown's argument is essentially a wave-packet argument since it uses an average of the scattering amplitude over an energy interval I. We have seen in Sec. 2 that for a beam without significant time resolution it is the cross sections, not the amplitudes, that are averaged.

Another condition for the validity of this definition of the optical model is required. The average over amplitudes must be nearly equivalent to the average over cross sections. This means that the compound elastic cross section must be very small compared with the optical model cross section.

A quantal statement of the third definition of the optical model must be given by a detailed wave-packet description. It is quite conceivable that the scattering amplitudes for certain problems are such that a large proportion of the cross section is due to time-delayed wave packets. This question will be discussed in detail by one of us (LRD) in a subsequent publication.

At present we will just consider the propagation of the wave packet in general and show how it leads to a large proportion of immediate propagation in the case $\Gamma \gg D$ where Γ is the average level width and D is the average spacing.

Considering only the S wave for simplicity and omitting the potential scattering term, Eq. (19) gives for the time dependence of the trailing edge of the wave packet

$$\xi(\mathbf{r},t) = K\exp(iEX)\sum_n \frac{R_n}{E-\epsilon_n+i(\Gamma_n-\delta)/2}\left\{\exp(\delta X/2) - \frac{i\delta\exp(\Gamma_n X/2)\exp i(\epsilon_n-E)X}{E-\epsilon_n+i(\Gamma_n+\delta)/2}\right\}. \tag{33}$$

The second term in the bracket is the one containing the properties of the scattering amplitude rather than the wave packet. Each level n contributes an exponential tail to the amplitude with a time-delay constant $\hbar/\Gamma_n$. However, this contribution is multiplied by a phase factor $\exp[i(\epsilon_n-E)X]$ which gives a partial cancellation of the tails when the packets from different levels are superposed.

If the level widths Γ_n are much greater than the average spacing D, the factor ϵ_n-E can be large so that the phase factor oscillates rapidly. In this case the phases of the contributions from different levels tend to be random so the tails cancel out giving a large proportion of propagation without time delay.

Thus the sequential description of scattering gives the same result as the usual energy description. The optical model is valid when $\Gamma \gg D$.

The dependence of (33) on the magnitude of the quantal uncertainty δ is also interesting. As δ is increased the magnitude of the contribution from each level to the scattered packet decreases. However more levels contribute significantly to the sum. If the phases of the residues R_n are random the magnitude of the scattered packet will decrease rapidly with increasing δ. If they are correlated, the magnitude will decrease less rapidly. The range of the correlations between levels can be determined in principle by varying δ. This will be discussed in detail in a subsequent publication.

5. POSSIBLE WAVE-PACKET EXPERIMENTS

The condition for a time-dependent scattering experiment is that the experimental definition of time must be accurate in comparison with the characteristic time of the scattering amplitude. That is, we must have δ not much less than Γ.

Experimental definition of time is at present possible for times as short as about 10^{-10} sec. Typical nuclear values for $\hbar/\Gamma$ are 10^{-15} sec, so wave-packet experiments cannot be performed with nuclei except in special cases. These cases are metastable states which can have lifetimes as long as 10^{-7} sec.

One experiment has actually been done using the Mössbauer effect with the 14-keV γ ray from Fe^{57}, which has a decay constant of 10^{-7} sec, by Holland, Lynch, Perlow, and Hanna.[14] The time spectrum of the incident wave packet was defined by using as zero time the time of formation of the 14-keV state, which was defined by the time of emission of the 128-keV γ ray (a fast decay) from the next highest state in the γ-ray cascade from the decay of Co^{57}. The wave packet has an exponential time spectrum with $\delta=\hbar/10^{-7}$ sec. This wave packet was scattered resonantly from an Fe^{57} target. The time spectrum of the final state and the increased width of the absorption line were both

[14] R. E. Holland, F. J. Lynch, G. J. Perlow, and S. S. Hanna, Phys. Rev. Letters 4, 181 (1960); F. J. Lynch, R. E. Holland, and M. Hamermesh, Phys. Rev. 120, 513 (1960).

observed. In this case $\delta=\Gamma_s$. The width of the wave packet in this experiment is of course fixed. By looking at the scattering at times less than 10^{-7} sec, Holland *et al.* were able to observe greater widths, but for these cases the shape of the incident packet was not defined. The spectrum of total elapsed time gives only an upper limit to the time width of the incident wave packet because it is not known if the delay occurred in the source or the scatterer.

The Mössbauer effect is, of course, a wave-packet scattering experiment in which the absorption cross section is measured. If we do not define time but merely observe the cross section we lose the wave-packet property, that is the quantal uncertainty. The cross section is obtained using Eq. (17) to define the energy amplitude factor of the wave packet.

$$\frac{d\sigma}{d\Omega}=\frac{R_s^2}{2\Gamma_s}\frac{1}{(E_0-E_s)^2+\Gamma_s^2}. \tag{34}$$

Thus, we have the well-known result that the linewidth in the Mössbauer effect is twice the width of the level, assuming all the nuclei in the target are capable of absorbing resonantly.

Defining the starting time of the excitation of the metastable state puts a quantal uncertainty into the beam equal to Γ_s. The method of Holland *et al.* defines the starting time with a minimum tolerance equal to the lifetime of the next highest state in the γ-ray cascade. This is shorter than the experimental time resolution, which is itself much shorter than $\hbar/\Gamma_s$.

One way of varying the time width of the wave packet would be to vary the resolution with which the starting time is measured. It is possible by this method to decrease δ, but not to increase it.

A better experimental way of varying the wave-packet width in the Mössbauer effect has been suggested by the present authors. A third resonant absorber is introduced between the source and the target. This absorber is accelerated in a very short time (10^{-9} sec) to a speed sufficient to shift the resonance so that the incident beam is no longer absorbed and can hit the target. It is then slowed down again quickly so that the time duration of the pulse is of the order of 10^{-9} sec. This method would produce approximately the exponential wave packet [Eq. (16)] that we have used in the calculations of Secs. 3 and 4. It is experimentally just possible to achieve the requisite acceleration by using a very thin foil of Fe^{57} as one plate of a parallel plate condenser in a vacuum which is charged first with one sign and then with the opposite sign by an rf pulse. However, the acceleration may be achieved more easily using a piezo-electric crystal. Another possibility is to use the Stark effect to shift the resonance. This requires a tightly bound dielectric crystal containing nuclei with a metastable state.

Wave-packet experiments in the atomic energy region may be interesting. An absorber of laser material placed in a laser beam and moved for a short time as suggested above would produce wave packets of laser intensity. The quantal uncertainty would be much greater than that of a beam from a pulsed laser, whose quantal uncertainty can be no greater than the width of the laser state.

All such experiments observe only the scattering of a wave packet from a single resonance with a trivial angular distribution. Overlapping resonances, even in atoms, would probably have widths of the order of a few tenths of an electron volt. Wave-packet experiments in this region with $\delta\sim\Gamma$ would require time lengths of 10^{-15} sec which seems impossible at present. The possibility of doing time-dependent electron-scattering experiments from atoms is not experimentally remote and would be interesting.

Using laser wave packets, it may be possible to observe the changes in angular distribution due to the interference of potential and resonant scattering as the quantal uncertainty is changed. The potential scattering could be obtained by diffraction of laser light round a small crystal of laser material.

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Dodd, L. R. & McCarthy, I. E. (1962). Resolution of direct and compound inelastic scattering. In *Proceedings of the International Conference on Direct Interactions and Nuclear Reactions Mechanisms, Padua*, (p. 309-311). Gordon and Breach, New York.

NOTE:

This publication is included in the print copy
of the thesis held in the University of Adelaide Library.

APPENDIX C

STATISTICAL MODEL CROSS SECTION.

We use the channel spin representation and the theory as presented by Lane and Thomas[5].

$\underline{I}, \underline{i}$ denote the spins of the target nucleus and incident neutron respectively and the channel spin in the incident channel is

$$\underline{s} = \underline{I} + \underline{i} \qquad (C.1)$$

The total angular momentum is

$$\underline{J} = \underline{s} + \underline{\ell} \qquad (C.2)$$

where $\underline{\ell}$ is the angular momentum of relative motion in the incident channel. The corresponding quantities in the outgoing channel are primed and

$$\underline{J} = \underline{s}' + \underline{\ell}' \qquad (C.3)$$

With the random phase approximation, the differential cross section for inelastic scattering from the nuclear state α to the state α' is

$$d^2\sigma_{\alpha\alpha'} = \frac{1}{(2I+1)(2i+1)} \sum_{ss'} \sum_{J\ell\ell'} (2\ell+1)\, A_J(\ell s|\ell' s'|\theta_{\alpha'})\, |U^{(J)}_{\alpha' s' \ell', \alpha s \ell}|^2 \qquad (C.4)$$

The angular part $A_J(\ell s|\ell' s'|\theta)$ is given by the following combination of Clebsh-Gordon coefficients and spherical harmonics

$$\sum_{m,m'} \left| (s,\ell;m,0|J,m)(s',\ell';m-m',m'|J,m)\, Y_{\ell'}^{m'}(\Omega_{\alpha'}) \right|^2 \tag{C.5}$$

We assume that the collision matrix element $U^{(J)}_{\alpha's'\ell';\alpha s\ell}$ does not depend on the channel spins, and that $|U^{(J)}_{\alpha's'\ell';\alpha s\ell}|^2$ takes the form

$$|U^{(J)}_{\alpha's'\ell';\alpha s\ell}|^2 = \frac{T^J_{\alpha\ell}(E_\alpha)\; T^J_{\alpha'\ell'}(E_{\alpha'})}{\sum_{\alpha''\ell''s''I''} \int_0^{E_{\alpha''max}} dE_{\alpha''}\, T^J_{\alpha''\ell''}(E_{\alpha''})\, \rho(E_{\alpha''}, I'')} \tag{C.6}$$

where the double primed summation is taken over all the possible competing intermediate states and $T^J_{\alpha\ell}$ is a transmission factor. $\rho(E_{\alpha''}, I'')$ is the level density factor.

NOTES ON THE 7090 FORTRAN CODE FOR $n(F^{19}, F^{19*})n'$

The subroutines SIGN, LEGEND, WIGNER, ADIST calculate the angular factor $A_J(\ell, s \mid \ell', s' \mid \theta)$. LEGEND gives the spherical harmonics and WIGNER the Clebsch-Gordon coefficients. The subroutines RIP, HKL, DHKL, PENET produce the transmission factors $T_\ell(E)$. Simple transmission factors for a real square well potential following Blatt and Weisskopf[39] are computed. We assume that the transmission factors do not depend on J or S.

COMET, ADDS, SUMUP, SETUP compute the denominator in (C.6). Simpson's rule is used to perform the energy integration.

SDEN, EDEN give the spin dependence and energy dependence respectively of the level density factor. The spin dependence is taken as

$$\rho(E, I) = \rho(E) \frac{(2I+1)}{2\sigma^2} \exp\left\{ \frac{-(I+\frac{1}{2})^2}{2\sigma^2} \right\} . \qquad (C.7)$$

σ is the spin cut-off parameter.

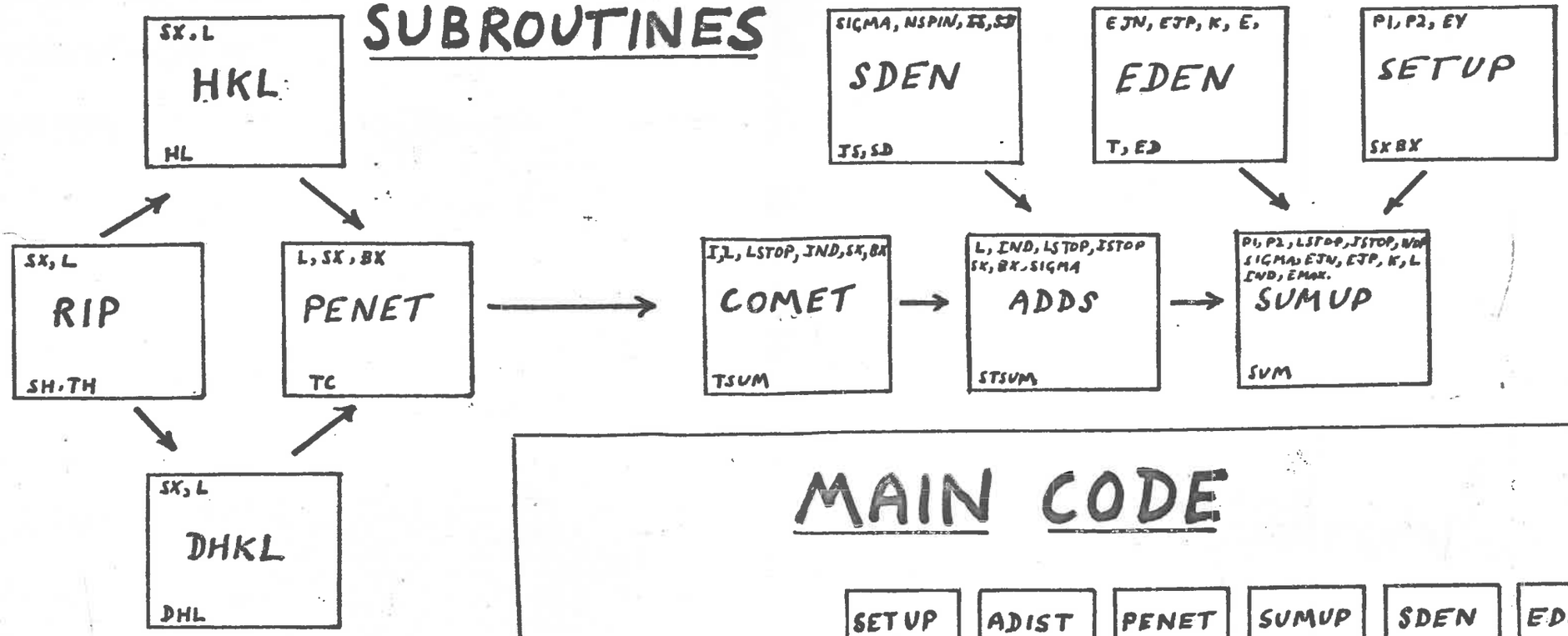
The energy dependence is taken from the work of Newton[56].

For $n(F^{19}, F^{19*})n'$ to the first excited state, the possible channel spins are 0 and 1. This and the angular momentum rules limit greatly the possible

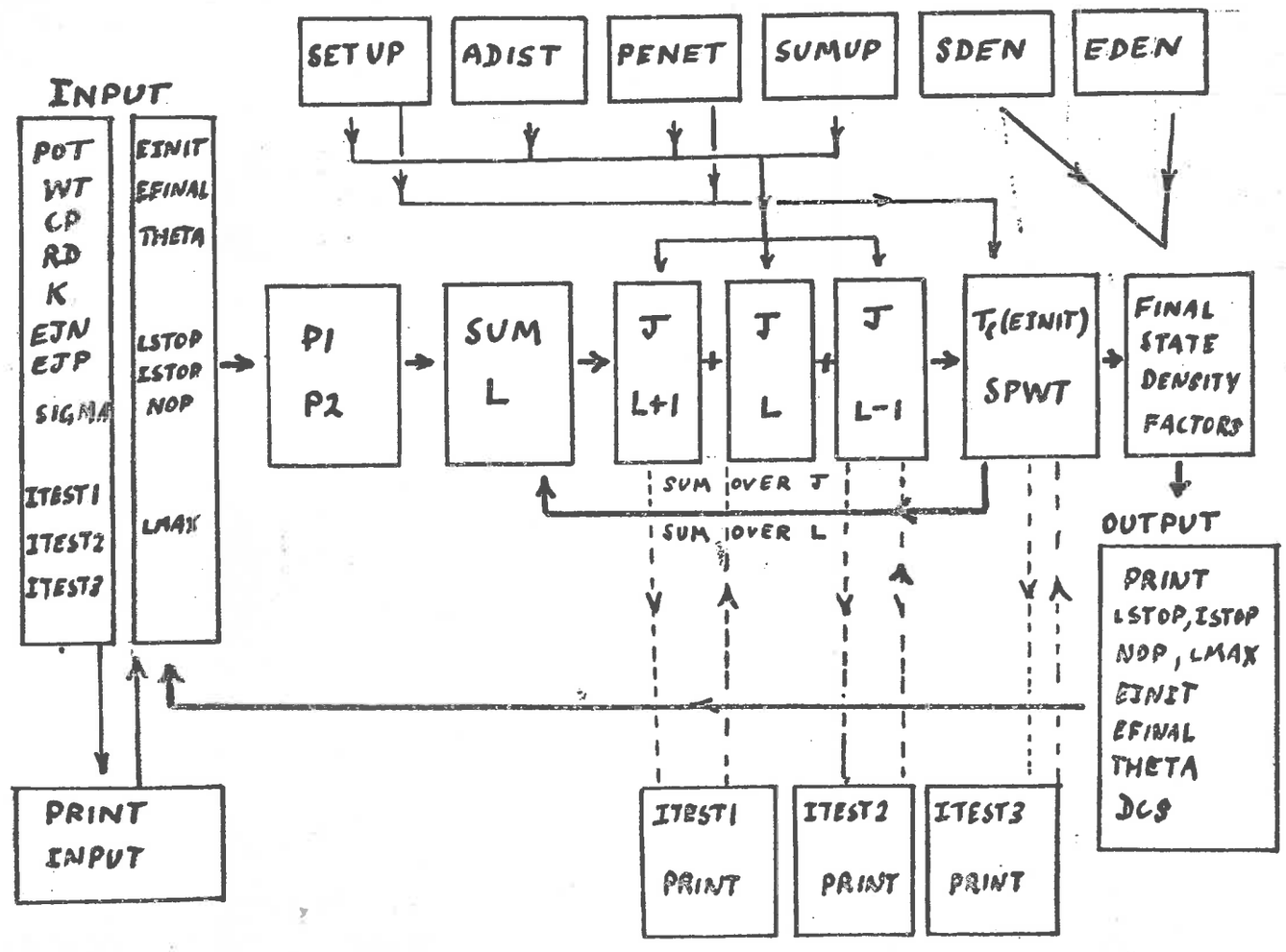
values in the sums over J, ℓ, ℓ', s, s' The MAIN CODE performs two summations which take into account all possibilities.

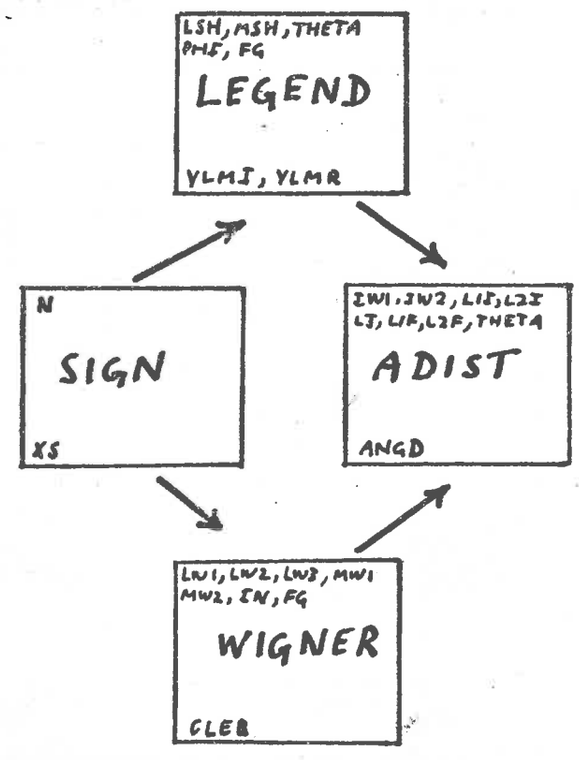
The input parameters for the projectile are EINIT, EFINAL and THETA, being respectively the incident neutron energy, the final neutron energy and the scattering angle. The input parameters for the target are POT, WT, LP, RD, K, EJN, EJP, SIGMA. The first four determine the potential well and the last four the level density. LSTOP, ISTOP, and NOP are parameters specifying limits of the sums over intermediate competing states and LMAX is the number of partial waves taken in the calculation.

SUBROUTINES



MAIN CODE





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```
      SUBROUTINESIGN(N,XS)
      NA=N-(N/2)*2
      IF(NA)2,2,3
    2 XS=1.0
      GOTO4
    3 XS=-1.0
    4 RETURN
      END
      SUBROUTINELEGEND
      COMMONFG,LW1,LW2,LW3,MW1,MW2,IW,LSH,MSH,THETA,PHI,IW1,IW2,L1I,L2I,
     ULJ,L1F,L2F,CLEB,YLMR,YLMI,ANGD
      DIMENSIONFG(100)
      IFDIVIDECHECK100,110
  100 PRINT101
  101 FORMAT(60H DIVIDE CHECK TRIGGER FOUND ON AT START OF LEGEND SUBROU
     1TINE)
  110 ISPILL=0
      JSPILL=0
      AL=LSH
      AM=MSH
      AM=AL+1.-AM
      RPHI=0.01745329252*PHI
      RTHETA=0.01745329252*THETA
      IF(THETA)301,300,301
  300 SN=0.
      CS=1.
      GOTO304
  301 IF(180.-THETA)303,302,303
  302 SN=0.
      CS=-1.
      GOTO304
  303 CS=COSF(RTHETA)
      SN=SINF(RTHETA)
      SNL=LOGF(SN)
      SNLL=AL*SNL
  304 IF(90.0-THETA)1,2,3
    1 ACS=ABSF(CS)
      CSL=LOGF(ACS)
      SOCS=-1.
      CS=ACS
      GOTO4
    2 CSL=1.
      SOCS=0.
      GOTO4
    3 CSL=LOGF(CS)
      SOCS=1.
    4 IF(PHI)5,6,7
    5 PRINT102
  102 FORMAT(49H PHI NEGATIVE IN STATEMENT 4 OF LEGEND SUBROUTINE)
    6 U=0.
      IF(AM)8,9,10
    7 IF(AM)27,28,29
    8 AM=ABSF(AM)
      N1=XFIXF(AM)
      CALLSIGN(N1,R)
      GOTO17
    9 N1=0
      IF(SOCS)11,12,13
   10 N1=XFIXF(AM)
```

```
      R=1.
      GOTO17
   11 CALLSIGN(LSH,R)
      GOTO21
   12 IF(LSH-(LSH/2)*2)14,15,16
   13 R=1.
      GOTO21
   14 PRINT103
  103 FORMAT(41H LSH INCORRECT IN 12 OF LEGEND SUBROUTINE)
      CALLEXIT
   15 LH=LSH/2
      CALLSIGN(LH,R)
      FF=FG(LSH)-AL*FG(2)-2.*FG(LH)
      GOTO58
   16 FF=0.
      R=0.
      GOTO59
   17 IF(SOCS)18,19,20
   18 CALLSIGN(LSH-N1,Z)
      R=R*Z
      GOTO24
   19 GOTO46
   20 GOTO24
   21 IF(THETA)80,23,80
   80 IF(180.0-THETA)22,23,22
   22 GOTO46
   23 FF=0.
      GOTO58
   24 IF(THETA)81,26,81
   81 IF(180.0-THETA)25,26,25
   25 GOTO46
   26 R=0.
      FF=0.
      GOTO59
   27 AM=ABSF(AM)
      CP=COSF(AM*RPHI)
      SP=-SINF(AM*RHPI)
      N1=XFIXF(AM)
      CALLSIGN(N1,Z)
      GOTO30
   28 CP=1.
      SP=0.
      N1=0
      Z=1.
      GOTO30
   29 Z=1.
      N1=XFIXF(AM)
      CP=COSF(AM*RPHI)
      SP=SINF(AM*RPHI)
   30 R=Z*CP
      U=Z*SP
      IF(SOCS)31,32,33
   31 MM=XFIXF(AL-AM)
      CALLSIGN(MM,Z)
      R=Z*R
      U=Z*U
      GOTO34
   32 IF(LSH-(LSH/2)*2)70,71,72
   70 PRINT104
```

```
104 FORMAT(41H LSH INCORRECT IN 32 OF LEGEND SUBROUTINE)
    CALLEXIT
 71 LH=LSH/2
    CALLSIGN(LH,Z)
    FF=FG(LSH)-AL*FG(2)-2.*FG(LH)
    R=Z*R
    U=Z*U
    GOTO58
 72 FF=0.
    R=0.
    U=0.
    GOTO59
 33 GOTO34
 34 IF(THETA)82,36,82
 82 IF(180.0-THETA)35,36,35
 35 GOTO46
 36 IF(LSH+1-MSH)73,74,73
 73 FF=0.
    R=0.
    U=0.
    GOTO59
 74 FF=0.
    GOTO58
 46 IF(LSH-N1-1)47,48,49
 47 FF=SNLL+FG(2*LSH)-AL*FG(2)-FG(LSH)
    GOTO58
 48 IF(SOCS)90,91,90
 90 FF=CSL+FG(2*LSH)+SNL*(AL-1.)-AL*FG(2)-FG(LSH)
    GOTO58
 91 FF=1.
    R=0.
    U=0.
    GOTO58
 49 JL=LSH
    F1=SNLL+FG(2*LSH)-AL*FG(2)-FG(LSH)
    F1=EXPF(F1)
    IF(SOCS)92,93,92
 92 F2=CSL+FG(2*LSH)+SNL*(AL-1.)-AL*FG(2)-FG(LSH)
    F2=EXPF(F2)
    GOTO94
 93 F2=0.
 94 AJL=JL
 50 F3=(2.*(AJL-1.)*CS*F2-SN*F1)/SN/(AL+AJL-1.)/(AL-AJL+2.)
    IFDIVIDECHECK105,106
105 PRINT107
107 FORMAT(52H DIVISOR IS ZERO IN STATEMENT50 OF LEGEND SUBROUTINE)
106 IF(N1+2-JL)51,52,53
 51 AJL=AJL-1.
    JL=JL-1
    F1=F2
    F2=F3
    GOTO50
 52 FF=F3
    AF=ABSF(FF)
    IF(FF)54,55,56
 54 SOF=-1.
    GOTO57
 55 SOF=0.
    GOTO57
```

```
   56 SOF=1.
      GOTO57
   53 PRINT108,JL
  108 FORMAT(48H INCORRECT LOOPING IN 50 OF LEGEND SUBROUTINE E15.5)
      CALLEXIT
   57 FF=LOGF(AF)
      R=R*SOF
      U=U*SOF
   58 N2=LSH-N1+1
      AN2=N2
      N3=LSH+N1
      FF=FF+(LOGF(2.*AL+1.)+FG(N2)-LOGF(AN2)-LOGF(4.*3.14159265)-FG(N3))
     1/2.
   59 YLMR=R*EXPF(FF)
      YLMI=U*EXPF(FF)
      IF(ISPILL)200,201,200
  200 PRINT210,ISPILL
  210 FORMAT(23H UNDERFLOW OCCURRED AT I6,21H IN LEGEND SUBROUTINE)
  201 IF(JSPILL)202,203,202
  202 PRINT220,JSPILL
  220 FORMAT(22H OVERFLOW OCCURRED AT I6,21H IN LEGEND SUBROUTINE)
      CALLEXIT
  203 RETURN
      END
      SUBROUTINEWIGNER
      COMMONFG,LW1,LW2,LW3,MW1,MW2,IW,LSH,MSH,THETA,PHI,IW1,IW2,L1I,L2I,
     ULJ,L1F,L2F,CLEB,YLMR,YLMI,ANGD
      DIMENSIONFG(100)
      ISPILL=0
      JSPILL=0
      AJ=LW1
      AK=LW2
      AL=LW3
      AL2=ABSF(AJ-AK)
      AL3=AJ+AK
      IF(AL2-AL-0.5)1,1,29
    1 IF(AL3-AL+0.5)29,2,2
    2 AM=MW1
      AN=MW2
      GOTO(3,4),IW
    3 GOTO5
    4 AJ=AJ/2.
      AK=AK/2.
      AL=AL/2.
      GOTO5
    5 AM=AJ+1.-AM
      AN=AK+1.-AN
      AAM=ABSF(AM)
      AAN=ABSF(AN)
      FM=AM+AN
      AFM=ABSF(FM)
      IF(AFM-AL-0.5)6,6,29
    6 IF(AAM-AJ-0.5)7,7,29
    7 IF(AAN-AK-0.5)8,8,29
    8 IF(AM)201,202,201
  202 IF(AN)201,203,201
  201 GOTO100
  203 JT=AK+AL+AJ
      JTH=JT/2
```

```
      IF(JT-2*JTH)204,205,204
  204 CLEB=0.
      GOTO30
  205 I1=JT-2*LW1
      I1H=I1/2
      I2=JT-2*LW2
      I2H=I2/2
      I3=JT-2*LW3
      I3H=I3/2
      AI1=I1
      AI2=I2
      AI3=I3
      AI1H=I1H
      AI2H=I2H
      AI3H=I3H
      AJT=JT
      GY=FG(I1+1)+FG(I2+1)+FG(I3+1)-FG(JT)
      Y=(Y+LOGF((2.0*AL+1.0)/(AI1+1.0)/(AI2+1.0)/(AI3+1.0)/(AJT+1.0)))/2
     1.
      Y=Y+FG(JTH)-FG(I1H+1)-FG(I2H+1)-FG(I3H+1)
      Y=Y+LOGF((AI1H+1.)*(AI2H+1.)*(AI3H+1.))
      I6=(LW1+LW2-LW3)/2
      CALLSIGN(I6,XS)
      CLEB=XS*EXPF(Y)
      GOTO30
  100 I1=AL+AJ-AK+0.1
      AI1=I1
      I2=AL-AJ+AK+0.1
      AI2=I2
      I3=AJ+AK-AL+0.1
      AI3=I3
      I4=AL-FM+0.1
      AI4=I4
      I5=AL+FM+0.1
      AI5=I5
      I6=AJ+AK+AL+1.1
      AI6=I6
      I7=AJ-AM+0.1
      AI7=I7
      I8=AJ+AM+0.1
      AI8=I8
      I9=AK-AN+0.1
      AI9=I9
      I10=AK+AN+0.1
      AI10=I10
    9 Y=FG(I1+1)+FG(I2+1)+FG(I3+1)+FG(I4+1)+FG(I5+1)-FG(I6+1)-FG(I7+1)-F
     1G(I8+1)-FG(I9+1)-FG(I10+1)
      Y=Y+LOGF((2.*AL+1.)*(AI6+1.)*(AI7+1.)*(AI8+1.)*(AI9+1.)*(AI10+1.)/
     1(AI1+1.)/(AI2+1.)/(AI3+1.)/(AI4+1.)/(AI5+1.))
      Y=Y/2.
      I11=AK+AL+AM+0.1
      AI11=I11
      NU=XMINOF(I2,I5,I11)
      IF(NU)10,11,12
   10 GOTO29
   11 I15=AJ-AK-FM+0.1
      AI15=I15
      IF(I15)13,14,14
   13 CLEB=0.
```

```
      GOTO30
   14 CALLSIGN(I10,XS)
      YL=FG(I11+1)+FG(I7+1)-FG(I5+1)-FG(I15+1)-FG(I2+1)
      YL=YL+Y+LOGF((AI2+1.)*(AI5+1.)*(AI15+1.)/(AI11+1.)/(AI7+1.))
      CLEB=XS*EXPF(YL)
      GOTO30
   12 I15=AJ-AK-FM
      IF(I15)15,16,16
   15 I15=XABSF(I15)
      AI15=I15
      IF(NU-I15)17,18,19
   17 CLEB=0.
      GOTO30
   18 I16=I10+I15
      CALLSIGN(I16,XS)
      I11=I11-I15
      AI11=I11
      I12=I7+I15
      AI12=I12
      I13=I2-I15
      AI13=I13
      I14=I5-I15
      AI14=I14
      YL=FG(I11+1)+FG(I12+1)-FG(I13+1)-FG(I14+1)-FG(I15+1)
      YL=Y+YL+LOGF((AI13+1.)*(AI14+1.)*(AI15+1.)/(AI11+1.)/(AI12+1.))
      CLEB=XS*EXPF(YL)
      GOTO30
   19 NUMIN=I15+1
      NUMAX=NU
      I16=I10+I15
      CALLSIGN(I16,XS)
      I17=I11-I15
      AI17=I17
      I12=I7+I15
      AI12=I12
      I13=I2-I15
      AI13=I13
      I14=I5-I15
      AI14=I14
      SUML=FG(I17+1)+FG(I12+1)-FG(I13+1)-FG(I14+1)-FG(I15+1)
      SUML=SUML+Y+LOGF((AI13+1.)*(AI14+1.)*(AI15+1.)/(AI17+1.)/(AI12+1.)
     1)
      SUM=XS*EXPF(SUML)
      I15=-I15
      GOTO20
   16 NUMIN=1
      NUMAX=NU
      CALLSIGN(I10,XS)
      AI15=I15
      SUML=FG(I11+1)+FG(I7+1)-FG(I2+1)-FG(I5+1)-FG(I15+1)
      SUML=SUML+Y+LOGF((AI2+1.)*(AI5+1.)*(AI15+1.)/(AI11+1.)/(AI7+1.))
      SUM=XS*EXPF(SUML)
   20 DO21NU=NUMIN,NUMAX
      I21=I11-NU
      AI21=I21
      I12=I7+NU
      AI12=I12
      I13=I2-NU
      AI13=I13
```

```
      I14=I5-NU
      AI14=I14
      I22=NU+I15
      AI22=I22
      I16=I10+NU
      CALLSIGN(I16,XS)
      SUML=FG(I21+1)+FG(I12+1)-FG(I13+1)-FG(I14+1)-FG(I22+1)-FG(NU)
      SUML=SUML+Y+LOGF((AI13+1.)*(AI14+1.)*(AI22+1.)/(AI21+1.)/(AI12+1.)
     1)
      SUMI=XS*EXPF(SUML)
      SUM=SUM+SUMI
   21 CONTINUE
      CLEB=SUM
      GOTO30
   29 CLEB=0.
   30 IF(ISPILL)60,61,60
   60 PRINT160,ISPILL
  160 FORMAT(23H UNDERFLOW OCCURRED AT I6,21H IN WIGNER SUBROUTINE)
   61 IF(JSPILL)62,63,62
   62 PRINT162
  162 FORMAT(22H OVERFLOW OCCURRED AT I6,21H IN WIGNER SUBROUTINE)
      CALLEXIT
   63 RETURN
      END
      SUBROUTINEADIST
      COMMONFG,LW1,LW2,LW3,MW1,MW2,IW,LSH,MSH,THETA,PHI,IW1,IW2,L1I,L2I,
     ULJ,L1F,L2F,CLEB,YLMR,YLMI,ANGD
      DIMENSIONFG(100)
      N1=2*L2I+1
      F=0.0
      DO7M1=1,N1
      IW=IW1
      LW1=L1I
      LW2=L2I
      LW3=LJ
      MW1=L1I+1
      MW2=M1
      CALLWIGNER
      A=CLEB*CLEB
      LW1=L1F
      LW2=L2F
      LW3=LJ
      IW=IW2
      B=0.0
      N2=2*L1F+1
      DO6M2=1,N2
      MW1=M2
      MW2=L2F-L2I+L1F+1+M1-M2
      IF(MW2)3,3,4
    3 E=0.0
      GOTO6
    4 IF(2*L2F+1-MW2)3,5,5
    5 CALLWIGNER
      C=CLEB*CLEB
      LSH=L1F
      MSH=MW1
      PHI=0.0
      CALLLEGEND
      D=YLMR*YLMR+YLMI*YLMI
```

```
      E=C*D
    6 B=E+B
    7 F=A*B+F
      ANGD=F
      RETURN
      END
      SUBROUTINERIP(L2,SX,SH,TH)
      H1=0.0
      H2=-1.0/SX
      H3=H2
      H4=H3/SX
      IF(L2-1)3,4,5
    3 SH=H1
      TH=H2
      GOTO100
    4 SH=H3
      TH=H4
      GOTO100
    5 DO8NH=2,L2
      AN=NH*2-1
      SH=AN*H3/SX-H1
      TH=AN*H4/SX-H2
      IF(NH-L2)7,8,8
    7 H1=H3
      H2=H4
      H3=SH
      H4=TH
    8 CONTINUE
  100 CONTINUE
      RETURN
      END
      SUBROUTINEHKL(L4,SX,HL)
      CALLRIP(L4,SX,SH,TH)
      HL=SH**2+TH**2
      RETURN
      END
      SUBROUTINEDHKL(L3,SX,DHL)
      D1=1.0
      D2=0.0
      D3=1.0/SX
      D4=-1.0+1.0/SX**2
      IF(L3-1)3,4,5
    3 UH=D1
      VH=D2
      GOTO100
    4 UH=D3
      VH=D4
      GOTO100
    5 AH=L3
      CALLRIP(L3-1,SX,SH1,TH1)
      CALLRIP(L3-2,SX,SH2,TH2)
      UH=SX*SH1-AH*(2.0*AH-1.0)*SH1/SX+AH*SH2
      VH=SX*TH1-AH*(2.0*AH-1.0)*TH1/SX+AH*TH2
  100 DHL=UH**2+VH**2
      RETURN
      END
      SUBROUTINEPENET(L,SX,BX,TC)
      IF(SX)5,5,6
    5 TOP=0.0
```

```
      BOT=1.0
      GOTO7
    6 CALLHKL(L,SX,HL)
      CALLDHKL(L,SX,DHL)
      TOP=4.*BX
      BOT=BX*BX*SX*HL+2.*BX+SX*DHL
    7 TC=TOP/BOT
      RETURN
      END
      SUBROUTINESDEN(SIGMA,NSPIN,IS,SD)
      IF(IS-1)3,3,4
    3 ASPIN=NSPIN
      GOTO5
    4 A=NSPIN
      ASPIN=A/2.0
    5 D=2.0*SIGMA*SIGMA
      B=-ASPIN*ASPIN/D
      E=ASPIN+1.0
      C=-E*E/D
      SD=EXPF(B)-EXPF(C)
      RETURN
      END
      SUBROUTINEEDEN(EJN,EJP,K,E,T,ED)
      IF(E-.5)1,1,2
    1 E=.5
    2 A=K
      B=(EJN+EJP+1.0)
      AL=LOGF(A)/3.
      AT=EXPF(AL)
      TOP=4.*F
      BOT=.4982*.4982*B*AT*AT
      C=TOP/BOT
      T=SQRTF(C)
      UT=(2.*E+3.*T)
      BN=SQRTF(2.*EJN+1.)
      BP=SQRTF(2.*EJP+1.)
      TDEN=(AT**5)*BN*BP*UT*UT*EXPF(8.75)
      W=.4982*AT*SQRTF(B*E)
      TNUM=EXPF(W)*10.**6
      ED=TNUM/TDEN
      RETURN
      END
      SUBROUTINESETUP(P1,P2,EY,SX,BX)
      A=P1*EY+P1*P2
      B=P1*EY
      BX=SQRTF(A)
      SX=SQRTF(B)
      RETURN
      END
      SUBROUTINECOMET(I,L,LSTOP,IND,SX,BX,TSUM)
      LSPEC=LSTOP
      IA=I-(I/2)*2
      IF(IND)1,2,3
    3 IF(IA)12,12,13
   13 LMIN=L-I+2
      LMAX=L+I
      GOTO21
   12 LMIN=L-I+1
      LMAX=L+I+1
```

```
      GOTO31
    2 IF(IA)14,14,15
   14 LMIN=L-I+1
      LMAX=L+I-1
      GOTO21
   15 LMIN=L-I
      LMAX=L+I
      GOTO31
    1 IF(IA)16,16,17
   17 LMIN=L-I
      LMAX=L+I-2
      GOTO21
   16 LMIN=L-I-1
      LMAX=L+I-1
      GOTO31
   21 IF(LMIN)22,23,23
   22 LMIN=LMIN+2
      GOTO21
   23 IF(LMAX-LSPEC)24,24,25
   25 LMAX=LSPEC
   24 TSUM=0.0
   27 IF(LMAX-LMIN)100,28,28
   28 LMAX=LMAX+1
      LMIN=LMIN+1
      DO26NI=LMIN,LMAX,2
      N=NI-1
      CALLPENET(N,SX,BX,TC)
   26 TSUM=2.*TC+TSUM
      GOTO100
   31 IF(LMIN)32,33,33
   32 LMIN=LMIN+2
      IF(LMIN)32,35,35
   35 IF(LMAX-LSPEC)36,36,37
   37 LMAX=LSPEC
      GOTO24
   36 CALLPENET(LMAX,SX,BX,TC)
      TSUM=TC
      LMAX=LMAX-2
      GOTO27
   33 IF(LMAX-LSPEC)41,41,42
   42 LMAX=LSPEC
      CALLPENET(LMIN,SX,BX,TC)
      TSUM=TC
      LMIN=LMIN+2
      GOTO27
   41 CALLPENET(LMAX,SX,BX,TC)
      TSUM=TC
      CALLPENET(LMIN,SX,BX,TC)
      TSUM=TC+TSUM
      LMIN=LMIN+2
      LMAX=LMAX-2
      GOTO27
  100 RETURN
      END
      SUBROUTINEADDS(L,IND,LSTOP,ISTOP,SX,BX,SIGMA,STSUM)
      STSUM=0.0
      DO2I=1,ISTOP
      CALLCOMET(I,L,LSTOP,IND,SX,BX,TSUM)
      A=TSUM
```

```
      NSPIN=2*I-1
      CALLSDEN(SIGMA,NSPIN,3,SD)
      B=SD
    2 STSUM=A*B+STSUM
      RETURN
      END
      SUBROUTINESUMUP(P1,P2,LSTOP,ISTOP,NOP,SIGMA,EJN,EJP,K,L,IND,EMAX,S
     1UM)
      Z=NOP
      DELTA=EMAX/(2.*Z)
      CALLSETUP(P1,P2,EMAX,SX,BX)
      CALLADDS(L,IND,LSTOP,ISTOP,SX,BX,SIGMA,STSUM)
      CALLEDEN(EJN,EJP,K,.5,T,ED)
      A=STSUM*ED
      CALLSETUP(P1,P2,0.,SX,BX)
      CALLADDS(L,IND,LSTOP,ISTOP,SX,BX,SIGMA,STSUM)
      CALLEDEN(EJN,EJP,K,EMAX,T,ED)
      B=STSUM*ED
      SUM1=(A+B)*DELTA/3.
      C=2.
      SUM2=0.
      NMAX=2*NOP-1
      DO2M=1,NMAX
      Y=M
      E=DELTA*Y
      CALLSETUP(P1,P2,E,SX,BX)
      CALLADDS(L,IND,LSTOP,ISTOP,SX,BX,SIGMA,STSUM)
      X=EMAX-E
      CALLEDEN(EJN,EJP,K,X,T,ED)
      C=6.-C
    2 SUM2=(C*DELTA*ED*STSUM/3.)+SUM2
      SUM=SUM1+SUM2
      RETURN
      END
      COMMONFG,LW1,LW2,LW3,MW1,MW2,IW,LSH,MSH,THETA,PHI,IW1,IW2,L1I,L2I,
     ULJ,L1F,L2F,CLEB,YLMR,YLMI,ANGD
      DIMENSIONCNT1(50),CNT2(50),CNT3(50)
      DIMENSIONFG(100)
      FG(1)=0.0
      DO7K=2,100
      AK=K
    7 FG(K)=LOGF(AK)+FG(K-1)
      READINPUTTAPE2,2,K,EJN,EJP,CP,WT,RD,POT,SIGMA,ITEST1,ITEST2,ITEST3
      WRITEOUTPUTTAPE 3,55
      WRITEOUTPUTTAPE 3,59,K,EJN,EJP,CP,WT,RD,POT,SIGMA,ITEST1,ITEST2,IT
     1EST3
    4 READINPUTTAPE2,3,LSTOP,ISTOP,NOP,LMAX,EINIT,EFINAL,NTHETA,DELTA
      WRITEOUTPUTTAPE3,500
      WN2=CP/(2.*WT*EINIT)
      P1=2.0*WT*RD*RD/CP
      P2=POT
      LY=LMAX+1
      DO110LZ=1,LY
      M=LZ-1
      CALLSUMUP(P1,P2,LSTOP,ISTOP,NOP,SIGMA,EJN,EJP,K,M,1,EINIT,SUM)
      CNT3(M)=SUM
      WRITEOUTPUTTAPE3,503,CNT3(M)
      CALLSUMUP(P1,P2,LSTOP,ISTOP,NOP,SIGMA,EJN,EJP,K,M,0,EINIT,SUM)
      CNT2(M)=SUM
```

```
      WRITEOUTPUTTAPE3,504,CNT2(M)
      CALLSUMUP(P1,P2,LSTOP,ISTOP,NOP,SIGMA,EJN,EJP,K,M,-1,EINIT,SUM)
      CNT1(M)=SUM
  110 WRITEOUTPUTTAPE3,505,CNT1(M)
      WRITEOUTPUTTAPE3,502
      DO300N=1,NTHETA
      TN=N-1
      THETA=TN*DELTA
      IW1=1
      IW2=1
      SPWT=0.
      LX=LMAX+1
      DO100LI=1,LX
      L=LI-1
      CALLSETUP(P1,P2,EFINAL,SX,BX)
      LPLUS=L+1
      CALLPENET(LPLUS,SX,BX,TC)
      L1I=L
      L2I=1
      LJ=L+1
      L1F=L+1
      L2F=1
      CALLADIST
      A=ANGD
      L1I=L
      L2I=1
      LJ=L+1
      L2F=0
      L1F=L+1
      CALLADIST
      B=ANGD
      S3=(A+B)*TC
      L1I=L
      L2I=1
      LJ=L
      L1F=L+1
      L2F=1
      CALLADIST
      C=ANGD
      L1I=L
      L2I=0
      LJ=L
      L1F=L+1
      L2F=1
      CALLADIST
      D=ANGD
      S2A=(C+D)*TC
      IF(ITEST1)11,11,12
   12 WRITEOUTPUTTAPE 3,50
      WRITEOUTPUTTAPE 3,51,L,TC,A,B,S3,C,D,S2A
   11 LMINUS=L-1
      CALLPENET(LMINUS,SX,BX,TC)
      L1I=L
      L2I=1
      LJ=L
      L1F=L-1
      L2F=1
      CALLADIST
      E=ANGD
```

```
      L1I=L
      L2I=0
      LJ=L
      L1F=L-1
      L2F=1
      CALLADIST
      F=ANGD
      S2B=(E+F)*TC
      S2=S2A+S2B
      L1I=L
      L2I=1
      LJ=L-1
      L1F=L-1
      L2F=1
      CALLADIST
      G=ANGD
      L1I=L
      L2I=1
      LJ=L-1
      L1F=L-1
      L2F=0
      CALLADIST
      H=ANGD
      S1=(H+G)*TC
      TERM3=S3/CNT3(L)
      TERM2=S2/CNT2(L)
      IF(CNT1(L))101,101,102
  101 TERM1=0.
      GOTO103
  102 TERM1=S1/CNT1(L)
  103 TOTAL=TERM1+TERM2+TERM3
      IF(ITEST2)13,13,14
   14 WRITEOUTPUTTAPE 3,52
      WRITEOUTPUTTAPE 3,53,L,TC,E,F,S2B,G,H,S1,S2,TERM1,TERM2,TERM3
   13 CALLSETUP(P1,P2,FINIT,SX,BX)
      CALLPENET(L,SX,BX,TC)
      Z=2*L+1
      SPWT=TC*Z*TOTAL+SPWT
      IF(ITEST3)100,100,15
   15 WRITEOUTPUTTAPE 3,54
      WRITEOUTPUTTAPE 3,56,TC,SPWT
  100 CONTINUE
      DCS1=SPWT*WN2*.7854
      CALLSDEN(SIGMA,1,2,SD)
      E=EINIT-EFINAL
      CALLEDEN(EJN,EJP,K,E,T,ED)
      DCS=DCS1*SD*ED
      WRITEOUTPUTTAPE 3,57
  300 WRITEOUTPUTTAPE 3,58,LSTOP,ISTOP,NOP,LMAX,EINIT,EFINAL,THETA,DCS1,
     1DCS
    2 FORMAT(I3,7E8.4,3I1)
    3 FORMAT(4I3,2E8.4,I3,E8.4)
  502 FORMAT(9H CHECKTWO)
  500 FORMAT(9H CHECKONE)
   50 FORMAT(90H    L   TC           A            B            S3           C
     1           D            S2A          )
   51 FORMAT(1H I3,7E12.4)
   52 FORMAT(116H    L   TC          E           F           S2B        G
     1 H          S1          S2          TERM1       TERM2       TERM3       )
```

```
 53 FORMAT(1H I3,11E10.4)
 54 FORMAT(28H     TC            SPWT          )
503 FORMAT(1H F10.4)
504 FORMAT(1H E10.4)
505 FORMAT(1H E10.4)
 55 FORMAT(51H DIFFERENTIAL CROSS SECTION FOR THE CONTINUUM MODEL/1H /
   11H /111H K     EJN           EJP           CP            WT            RD
   2          POT           SIGMA         ITEST1 ITEST2 ITEST3 )
 56 FORMAT(1H 2E12.4/1H )
 57 FORMAT(89H      LSTOP ISTOP NOP    LMAX  EINIT        EFINAL        THE
   1TA        DCS1          DCS           )
 58 FORMAT(1H 4I6,5E12.4)
 59 FORMAT(1H I3,7E12.4,3I7/1H )
    GOTO4
    END
```