WICK'S RELATIVISTIC TWO-BODY EQUATION FOR BOUND STATES.

by

L. H. D. Reeves B.Sc (Tasmania)

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Department of Mathematical Physics
University of Adelaide
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PREFACE.

This thesis contains the result of research done in the Department of Mathematical Physics, University of Adelaide during 1959 - 62. The word was supervised by Dr. C. A. Hurst in 1959, 1960 and 1962 and by Professor H. S. Green in 1961. During the first three years the author held a Commonwealth Postgraduate Scholarship.

The calculations described in Chapter 4 were done on the I.B.M. 7090 computer at Salisbury, South Australia through the University of Adelaide Computing Centre to whom thanks are due. The author is most sincerely grateful to Dr. C. A. Hurst and Professor H. S. Green for constant assistance and encouragement in the work.

CONTENTS.

SUMMARY.

Wick's equation is the special case of the Bethe-Salpeter equation in which all the particles are scalar bosons. It simplifies if the meson carrying the interaction has zero mass. This thesis is principally concerned to investigate the solutions of Wick's equation using a suitable coordinate system in which the equation separates.

In the first two chapters the salient features of previous work on the subject are discussed. Because of its importance for the rest of the thesis, Wick's analysis is given in some detail. In the third chapter the necessary coordinate system is introduced. It is shown that in fact the equation is not strictly separable in the usual Minkowski metric but that after Wick's analytic continuation to the imaginary time axis, when the metric becomes Euclidean, separation of the variables is straightforward. The problem of determining the eigenvalues of the equation reduces to the solution of a Heun's differential equation with certain boundary conditions. The analytic properties of the solutions in momentum space are shown to be consistent with those required by Wick.

In the first part of Chapter 4 approximate expressions for the eigenvalues are obtained in the two limits - the energy of the bound state very small and the binding energy very small. It is found that solutions can be classified as normal or abnormal according as the coupling constant does or does not tend to zero when the beinding energy tends to zero. In the second part of Chapter 4 the results of the numerical calculation of exact eigenvalues are compared with the approximate eigenvalues. The approximate eigenvalues for the normal solutions give good results only

in the extreme nonrelativistic region.

In Chapter 5 a detailed comparison is made of these separable solutions and the solutions of Wick and Cutkosky. They are shown to be completely equivalent.

Chapter 6 considers the corresponding equation in the instantaneous interaction approximation. The equation is not solved but an indication of the eigenvalues is given and it is suggested that the approximation is good only in the extreme nonrelativistic region.

In Chapter 7 a normalization condition differing from that previously used is developed by a method like that used for one-particle wave equations. As the solutions of Wick's equation can be normalized by the customary method and the new condition is weaker, it cannot be used to exclude any solutions.

Finally, in Chapter 8 a few additional comments and conclusions are given.

STATEMENT.

This thesis contains no material which has been accepted for the award of any other degree and as far as I know contains no material previously published or written by any other person, except where reference is made.

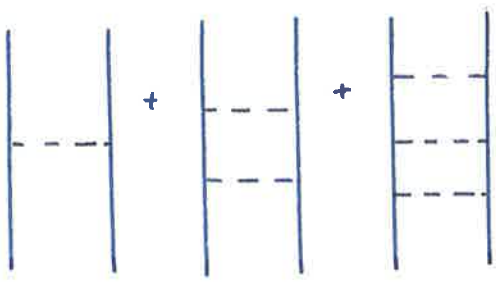
HUGH REEVES.
29/10/62

Chapter 1.

INTRODUCTION.

Wick's equation, which will be studied in this thesis, is a special case of the Bethe-Salpeter equation; hence we begin by considering this equation.

A crude but very simple derivation of the equation can be given as follows (12). Just as the probability amplitude for a particle being at a space-time point x is given by a wave function $\psi(x)$ so there will be a probability amplitude $\chi(x_1, x_2)$ for one particle to be at the point x_1 and another at the point x_2. If the particles (which may generally be referred to as nucleons) interact via some other particle referred to as a meson then the processes shown in the following diagrams will occur.



etc.

Suppose that the probability amplitude for the two particles being at x_1 and x_2 having exchanged n mesons is $\chi_n(x_1, x_2)$. Then clearly

$$\chi_n(x_1, x_2) = g^2 \int S(y_1 - x_1)\, S(y_2 - x_2)\, \Delta(y_1 - y_2)\, \chi_{n-1}(y_1, y_2)\, dy_1\, dy_2$$

according to the ordinary Feynman rules, where $S(y_1 - x_1)$ is the propagator for the nucleons multiplied by suitable vertex factors, $\Delta(y_1 - y_2)$ is the propagator for the mesons, and g is a coupling constant. Then if $\chi(x_1, x_2) = \sum_{n=0}^{\infty} \chi_n(x_1\, x_2)$ is the total amplitude it will satisfy the equation

$$\chi(x_1,x_2) = \chi_0(x_1,x_2) + g^2 \int S(y_1-x_1)\, S(y_2-x_2)\, \Delta(y_1-y_2)\, \chi(y_1,y_2)\, dy_1\, dy_2$$

where χ_0 is the amplitude for no interactions. For a bound state in which an infinite time is available for interactions to take place χ_0 will be negligible so the homogeneous equation is obtained:

$$\chi(x_1,x_2) = g^2 \int S(y_1-x_1)\, S(y_2-x_2)\, \Delta(y_1-y_2)\, \chi(y_1,y_2)\, dy_1\, dy_2 \qquad (1.1)$$

An alternative form of this equation is given by noting that

$$D_{x1}\, S(y_1 - x_1) = \delta(y_1 - x_1)$$

where D_{x1} is some differential operator. Hence

$$D_{x1}\, D_{x2}\, \chi(x_1\, x_2) = g^2\, \Delta(x_1 - x_2)\, \chi(x_1\, x_2) \qquad (1.2)$$

This is the Bethe-Salpeter equation in the ladder approximation and was first given by Nambu (13).

The equations (1.1) and (1.2) show two characteristic features of the Bethe-Salpeter equation. First it is obviously relativistically covariant in contrast to previous attempts to deal with bound states. This aesthetic gain is balanced by the practical difficulties of interpreting a wave function which depends on two independent times. Secondly, the equations include the effects due to a large number of graphs even though no graphs with crossed meson lines are present.

A derivation of the exact covariant two body equation was given by Bethe and Salpeter (1) and more rigorously by Gell-Mann and Low. The two body propagator or Green's function is introduced

$$K(x_1,\, x_2;\, x_3\, x_4) = (\Omega\, ,\, T\, \psi(x_1)\, \varphi(x_2)\, \overline{\psi}(x_3)\, \overline{\varphi}(x_4)\, \Omega\,) \qquad (1.3)$$

where $\psi(x_1)$ and $\varphi(x_2)$ are the Heisenberg operators of the nucleons,

T is the usual time ordering symbol and Ω is the true vacuum. This propagator is then written in terms of the interaction representation and the following integral equation found:

$$K(1,\ 2;\ 3,\ 4) = S'(1,\ 3)\ S'(2,\ 4) - \int dx_5 dx_6 dx_7 dx_8\ S'(1,\ 5)\ S'(2,6)\ G(5,\ 6;\ 7,\ 8)\ K(7,\ 8;\ 3,\ 4) \tag{1.4}$$

where $S'(1,\ 3)$ is the one body propagator

$$S'(x,\ y) = (\Omega,\ T\ \psi(x)\ \bar{\psi}(y)\Omega)$$

and the kernel $G(5,\ 6;\ 7,\ 8)$ includes all the interactions. No closed expression can be given for G; it is a power series in the coupling constant with lowest order term

$$G(5,\ 6;\ 7,\ 8) = g^2\ \Delta(x_5 - x_6)\ \delta(x_5 - x_7)\ \delta(x_6 - x_8).$$

corresponding to the ladder approximation.

Now the two body system considered will have a number of states Ψ_n corresponding to various values of the total energy - momentum, angular momentum, etc. and as these states form a complete set, the propagator can be written

$$\begin{aligned} K(1,\ 2;\ 3,\ 4) &= \sum_n (\Omega,\ T\ \psi(x_1)\ \varphi(x_2)\ \Psi_n)(\Psi_n,\ T\ \bar{\psi}(x_3)\ \bar{\varphi}(x_4)\Omega) \\ &= \sum_n \chi_n(1,\ 2)\ \bar{\chi}_n(3,\ 4) \end{aligned} \tag{1.5}$$

The χ_n are the two-body wave functions for the state Ψ_n and satisfy the equation

$$\chi_n(1,\ 2) = -\int dx_3\ dx_4\ dx_5\ dx_6\ S'(1,\ 3)\ S'(2,\ 4)\ G(3,\ 4;\ 5,\ 6)\ \chi_n(5,\ 6) \tag{1.6}$$

If the nucleons are fermions and the mesons are scalar bosons with scalar coupling then in the ladder approximation $S'(1,\ 3)$ must be replaced by $S(1,\ 3)$ where

$$S(1,3) = \frac{i}{(2\pi)^4}\int \frac{e^{-ip(x_1-x_3)}}{\gamma p - M}\,dp$$

and $G(3,4;5,6) = 4\pi g^2 \; \Delta(x_3 - x_4) \; \delta(x_3 - x_5) \; \delta(x_4 - x_6)$

where $\Delta(x_3 - x_4) = \frac{i}{(2\pi)^4} \int \frac{e^{-ik(x_3 - x_4)}}{k^2 - \mu^2} \, dk$

(M is the mass of the nucleons, μ that of the bosons, and each has a small negative imaginary part; k^2 is $k_o^2 - \underline{k}^2$).

On the other hand if all the particles are scalar bosons then the equation in the ladder approximation becomes

$$\chi_n(1,2) = -g^2 \int dx_3 \, dx_4 \; \Delta^M(x_1 - x_3) \; \Delta^M(x_2 - x_4) \; \Delta^\mu(x_3 - x_4) \; \chi_n(3,4).$$

Now Ψ_n is an eigenstate of the displacement operator and therefore if $X = \frac{1}{2}(x_1 + x_2)$, $x = x_1 - x_2$, $\chi_n(1,2)$ must have the form $e^{-iPX}\varphi(x)$ where P is the total energy-momentum. If $\psi(p)$ is the Fourier transform of $\varphi(x)$, so that p is the relative energy-momentum, the equations for the ladder approximation take the simpler form

$$\left[\gamma\left(\tfrac{P}{2} + p\right) - M\right]\left[\gamma\left(\tfrac{P}{2} - p\right) - M\right]\psi(p) = \frac{ig^2}{4\pi^3} \int \frac{\psi(k)\, dk}{(p-k)^2 - \mu^2} \qquad (1.7)$$

for the fermion case and

$$\left[\left(p + \tfrac{P}{2}\right)^2 - M^2\right]\left[\left(p - \tfrac{P}{2}\right)^2 - M^2\right]\psi(p) = \frac{i\lambda}{\pi^2} \int \frac{\psi(k)\, dk}{(p-k)^2 - \mu^2} \qquad (1.8)$$

for the scalar case where in the second equation $\left(\frac{g}{4\pi}\right)^2$ has been replaced by λ. It is this last equation which is known as Wick's equation, especially if $\mu = 0$, though it was first derived by Hayashi and Munakata (15).

These equations differ from the ordinary schrodinger equation in momentum space by the appearance of the relative energy p_o. In practice, it is convenient to replace the four vector P by $(2E, 0, 0, 0)$ so that the total momentum is zero. The binding energy is then $2M - 2E$.

The Bethe-Salpeter equation was first used in calculations of the deuteron coupling constant by Bethe and Salpeter (1), calculation of corrections to the fine structure of hydrogen by Salpeter (16), and calculation of the hyperfine structure of positronium by Karplus and Klein (17). In all these papers the instantaneous interaction approximation is adopted, that is, the term $(p - k)^2$ which appears in the denominator of (1.7) and (1.8) is replaced by $-(\underline{p} - \underline{k})^2$. This implies that the meson travels with infinite velocity or that the motion of the nucleons while the meson is travelling is neglected, and hence is only valid in the non relativistic region.

A first attempt at a covariant solution of the B - S equation was given by Goldstein (2). He assumed that the ladder approximation was an exact equation and that therefore the coupling constant could have any value. Since the equation (1.7) for the fermion case appeared to be too difficult to solve in general, he considered the limiting case $E = 0$ and also put μ equal to zero. As is usual he regarded E as fixed and g^2 the eigenvalue to be determined. He was then concerned with the fact that the kernel of the equation (1.7) is highly singular. This can be seen roughly by noting that the right hand side of (1.7) is approximately p^{-2} for large p; hence $\psi(p) \simeq p^{-4}$ for large p, which is not sufficient to ensure convergence of the integral on the right hand side. This problem does not arise for (1.8) as in it $\psi(p) \simeq p^{-6}$ and the integral does converge. It had been suggested by Hayashi and Munakata (15) that the equation (1.7) would have no solutions. Goldstein found that in fact this is not the case but that solutions exist for all values of g^2. He applied a normalization condition which excluded values of $\frac{g^2}{4\pi}$ less than $\frac{3}{16}$. He then attempted to further reduce the spectrum by imposing a cutoff at high momenta but as pointed out by Green (3) the modified equation has no solution. Mandelstam (25), using a different normalization condition, found that acceptable solutions had $\frac{g^2}{4\pi} < \frac{1}{4}$, and conjectured that the eigenvalue spectrum would be discrete.

As will be shown presently, Wick's equation (1.8) is readily solved when $E = 0$ and the eigenvalues are

$$\frac{\lambda}{M} = n(n + 1), \qquad n = 1, 2, --- .$$

Sugano and Munakata (22) have considered the intermediate problem in which one nucleon is a fermion, the other a scalar boson and again the interaction is via a scalar massless boson. They also have no difficulty in solving their equation for $E = 0$ and find the eigenvalues

$$\frac{\lambda}{M} = \tfrac{1}{2}n(n + 1) \qquad n = 2, 3, ---$$

A different approach to the Bethe-Salpeter equation was given by Schwinger (18). He introduced an external source, which was merely a quantity to be varied and finally set equal to zero. An infinite set of coupled equations was easily obtained for a set of Green's functions such as (1.3). However in order to solve these equations the set had to be terminated somewhere and ordinary perturbation methods used.

The approach of Matthews and Salam (19), developed by Zimmermann (20) and Nishijima (21) was somewhat similar except that they considered sets of wave functions rather than Green's functions. If the Heisenberg state vector of the two particle system, Ψ_n, were known then so would all the properties of the system but of course it is rather inaccessible. However if $\psi(x)$ is the Heisenberg field operator of the nucleons (for simplicity, supposed to be the same) and $A(x)$ that of the mesons then the set of wave functions

$$u(x_1,x_2) = (\Omega, T\,\psi(x_1)\,\psi(x_2)\,\Psi_n)$$
$$v(x_1,x_2;y) = (\Omega, T\,\psi(x_1)\,\psi(x_2)\,A(y)\,\Psi_n) \qquad (1.9)$$
$$v(x_1,x_2;y_1,y_2) = (\Omega, T\,\psi(x_1)\,\psi(x_2)\,A(y_1)\,A(y_2)\,\Psi_n) - \Delta(y_1-y_2)\,u(x_1,x_2)$$

etc.

can be regarded as components of the vector Ψ_n. In the last equation of (1.9) the second term on the right hand side is inserted so that $v(x_1, x_2; y_1, y_2)$ is zero in the interaction free case. This is equivalent to using normal ordering in the first term, rather than time ordering, provided that normal ordering is suitably defined, (see (19)). Obviously $u(x_1, x_2)$ is to be understood as the probability amplitude for finding just the two nucleons, $v\,(x_1, x_2; y)$ that for finding two nucleons and one meson, and so on.

Being Heisenberg operators, $\psi(x)$ and $A(x)$ are related by the field equations, for example

$$(\Box + M^2)\,\psi = g\,\psi\,A$$
$$\Box A = g\,\psi^2 \qquad (1.10)$$

if ψ is a scalar particle with mass M and A is a massless scalar. Correspondingly

$$(\Box_1 + M^2)\,u(x_1,x_2) = g\,v(x_1,x_2;x_1) \qquad (1.11)$$
$$(\Box_1 + M^2)\,v(x_1,x_2;y) = g\,v(x_1,x_2;x_1,y) + g\,\Delta(y-x_1)\,u(x_1,x_2)$$

etc.

where $\Box_1$ means the D'Alembertian operator in terms of x_1, and the commutation relations for $\psi(x)$ have been used. These two equations are sufficient to give the ladder approximation in which the probability of two mesons being present simultaneously is neglected, that is, $v(x_1, x_2; x_1\, y) = 0$. Eliminating $v(x_1, x_2; y)$ we get

$$(\Box_1 + M^2)(\Box_2 + M^2)\, u\,(x_1, x_2) = g^2\, \Delta(x_1 - x_2)\, u\,(x_1, x_2) \qquad (1.12)$$

which is just the differential form of Wick's equation (1.8) in coordinate space.

The infinite set of coupled equations (1.11) can be replaced by a set of integral equations and Zimmermann (20) has shown that they are equivalent to the Bethe-Salpeter equation (1.6). He has also discussed the renormalization of this set.

We now consider the researches of Wick (10) and, because of their importance for what follows, give his results in some detail.

The starting point is the function

$$u(x_1, x_2) = (\Omega\,,\, T\,\psi(x_1)\,\psi(x_2)\,\Psi_n)$$

but as the centre of mass motion can be separated out, it is more convenient to take

$$\chi(x) = e^{iP_n X}\,(\Omega,\, T\,\psi(x_1)\,\psi(x_2)\,\Psi_n)$$

where x is the relative coordinate $x_1 - x_2$ and P_n is the total energy-momentum of the state Ψ_n. If the relative time, t, is positive then $\chi(x)$ can be written

$$\chi(x) = e^{iP_n X} \sum_\alpha (\Omega,\, \psi(x_1)\, \Omega_\alpha)(\Omega_\alpha,\, \psi(x_2)\, \Psi_n) \qquad (1.13)$$

The states Ω_α which have to be included may contain mesons and nucleon-antinucleon pairs but the result of adding all the nucleons and subtracting all the antinucleons will be just one nucleon. All such states satisfy the condition

$$P^2 \geqslant M^2 \tag{1.14}$$

This essentially physical assumption is called by Wick the stability condition. On the other hand, by the hypothesis of a bound state

$$P_{no} = 2M - B < 2M$$

(P_{no} is the energy component of P_n and B is the binding energy).

Now since P_α is an eigenvalue of the displacement operator

$$(\Omega, \psi(x_1)\,\Omega_\alpha) = e^{-i P_\alpha x_1}(\Omega, \psi(0)\,\Omega_\alpha)$$

so

$$\chi(x) = e^{iP_n X}\sum_\alpha e^{-iP_\alpha x_1}(\Omega,\psi(0)\,\Omega_\alpha)(\Omega_\alpha,\psi(0)\,\Psi_n)e^{-i(P_n - P_\alpha)x_2}$$

$$= \sum_\alpha e^{-ix(P_\alpha - \frac{1}{2}P_n)}(\Omega,\psi(0)\,\Omega_\alpha)(\Omega_\alpha,\psi(0)\,\Psi_n)$$

From the preceding conditions on P_{no} and $P_{\alpha o}$ we have

$$P_{\alpha o} - \tfrac{1}{2}P_{no} \geqslant \sqrt{M^2 + \underline{P}_\alpha^2} - M + \tfrac{1}{2}B > 0$$

so that, expressing the sum as an integral

$$\chi(x) = \int d\underline{p} \int_{\omega_{min}}^{\infty} f(\underline{p}, \omega)\, e^{-i\omega t} e^{i\underline{p}\cdot\underline{x}}\, d\omega \tag{1.15}$$

where $\omega_{min} = \sqrt{M^2 + (\underline{p} + \frac{1}{2}\underline{P})^2} - M + \frac{1}{2}B. > 0$

Thus for t positive $\chi(x)$ contains only positive frequencies. Similarly if t is negative $\chi(x)$ contains only negative frequencies. If t is regarded as a complex variable then (1.15) shows that $\chi(x)$ can be continued downwards from the positive real axis of t and upwards from the negative real axis. Furthermore $\chi(x) \to 0$ as $t \to \infty$ in any direction away from the real axis - a boundary condition on $\chi(x)$.

Now if $\chi(x) = \chi_1 + \chi_2$ where $\chi_1 = 0$ if $t < 0$ and $\chi_2 = 0$ if $t > 0$ then the Fourier transform of χ_1 is

$$\Phi_1(\underline{p}, p_0) = \frac{1}{2\pi i}\int_{\omega_{min}}^{\infty} \frac{f(\underline{p},\omega)}{\omega - p_0 - i\epsilon}\, d\omega \tag{1.16}$$

Hence $\Phi_1(\underline{p}, p_o)$ is an analytic function in the complex p_o plane with a cut along the real axis from ω min to infinity. It can be continued from the cut in the positive, anticlockwise direction. Similarly $\Phi_2(\underline{p}, p_o)$ has a cut along the negative real axis from which it can be continued in the same direction and $\Phi(p) = \Phi_1 + \Phi_2$ has the two cuts but can be continued from the upper to the lower half plane through the gap between the cuts.

The threshold implied in (1.16) seems rather peculiar. It corresponds to one of the particles having free-particle energy-momentum - either $p_1^2 = M^2$ or $p_2^2 = M^2$. Equation (1.16) shows (see reference (30) chapter 4) that near the threshold $\Phi(p) = -\frac{f(\underline{p}, \omega\text{min})}{2\pi i}\log(p_o - \omega\text{min})$.

These analytic properties of $\Phi(p)$ which are independent of the ladder approximation were then used by Wick on the equation (1.8)

$$(p^2 - 2Ep_0 - c^2)(p^2 + 2Ep_0 - c^2)\,\Phi(p) = \frac{i\lambda}{\pi^2}\int \frac{\Phi(k)\,dk}{(p-k)^2 + i\varepsilon}$$

where we have put $P = (2E, 0, 0, 0)$ and $c^2 = M^2 - E^2$. He assumed that $\Phi(k) \to 0$ at least like k_o^{-2} when $k_o \to \infty$ in any direction and then rotated the path of integration from the real k_o axis to the imaginary k_o axis. Moving the point p_o up to the imaginary axis gives the equation

$$(p^2 - 2iEp_4 + c^2)(p^2 + 2iEp_4 + c^2)\,\Phi(p) = \frac{\lambda}{\pi^2}\int \frac{\Phi(k)\,dk}{(p-k)^2}$$

where now $p^2 = p_4^2 + \underline{p}^2$. This equation is solved by writing

$$\Phi(p) = \int_{-1}^{1} \frac{g(z)\,dz}{(p^2 + 2izEp_4 + c^2)^3}$$

(this is a special case, the generalizations were given by Cutkosky (11) and will be mentioned in Chapter 5). It is found that $g(z)$ satisfies the equation

$$g''(z) + \frac{\lambda}{(1 - z^2)(c^2 + E^2 z^2)}\, g(z) = 0$$

with the boundary conditions $g(\pm 1) = 0$. In the limit as $E \to 1$, the binding energy tends to zero., two types of solution were found. It could happen that the eigenvalue condition is

$$\lambda = \frac{2}{\pi} \sqrt{1 - E^2}$$

which is just what one would expect in the non relativistic limit so this is called the normal solution. However, other abnormal solutions were found for which $\lambda \to \frac{1}{4}$ as $E \to 1$. Thus although Wick had clarified the significance of the relative time by getting a boundary condition on the wave function as $t \to \infty$ he had introduced a new problem into the study of the Bethe-Salpeter equation.

Chapter 2.

RECENT WORK ON WICK'S EQUATION.

Before going on to describe the recent work on Wick's equation it is convenient to mention briefly some of the questions raised by the Bethe-Salpeter equation. Wick's equation corresponds to no physical situation but it does provide a soluble example of the B-S equation.

1. The ladder approximation is certainly good only for small values of the coupling constant but there is no certainty that it is good even then. For lack of any useful alternative, it has to be assumed that the expansion of the $G(x_1, x_2; x_3, x_4)$ in powers of the coupling constant is either a convergent or at least an asymptotic series. Possibly the recent dispersion relation methods of field theory, valid for low energies rather than for weak couplings, may lead to a covariant equation for bound states in which the binding energy is small, but they have not done so yet.

Otherwise, some information independent of the ladder approximation can be gained by the study of the analytic properties of the wave function, based on general principles of field theory. Such a study is obviously not a substitute for solving the equation but, as Wick found, it is of great assistance in finding the solution. Until the ladder approximation can be circumvented it is necessary to make as much use as possible of the resulting equation; one may hope that an understanding of its solutions may point the way to the next step.

2. While relative space coordinates are a familiar enough idea, the appearance of a relative time is new and its significance obscure. Bertocchi, Fubini, Stroffolini and Tonin (23) have stressed that the

B-S equation is a relativistic counterpart of the stationary, rather than the time-dependent, Schrodinger equation. Green and Biswas (24) remarked that associated with the relative time an additional quantum number will occur, and hence each non relativistic state might correspond to a whole set of relativistic states. They suggested that this new quantum number might be applied to the classification of elementary particles. Wick found a boundary condition on the wave function as the relative time tends to infinity. He also found the additional quantum number but the new states it characterizes are such that one would rather look for reasons to eliminate them, than to use them. Indeed, the abnormal solutions were the most surprising result of Wick's work and it is important to know if they occur in any B.-S. equation or only in Wick's equation.

3. It was known that the equation had obviously improper solutions when the coupling constant is zero. For example in coordinate space Wick's equation becomes

$$((\Box + c^2)^2 + 4E^2 \frac{\partial^2}{\partial t^2}) \chi(x) = 0$$

which can be solved by $\chi(x) = e^{ikx}$ where k satisfies the condition $k_0 = E + \sqrt{M^2 + \underline{k}^2}$ for example, and E is arbitrary. However such a solution is eliminated by Wick's boundary condition on χ.

4. The equation is mathematically unfamiliar as it is a singular integral equation. Wick's analytic continuation partly overcomes this but a method giving the solution directly from the integral equation in the conventional metric would be very interesting. Possibly there would then be solutions without the correct analyticity properties. Further, Goldstein's result suggests that in the more highly singular fermion case there may be either no solutions or a continuum of solutions.

5. Finally, to use the solutions for anything more than the calculation

of the energy eigenvalue they must be normalized.

Since the work of Wick and Cutkosky some light has been shed on some of these questions. We consider first the normalization problem.

Nishijima (21) pointed out that two associated problems have to be solved. The first is the normalization of the Bethe-Salpeter wave functions and the second is the determination of the expectation values of a given observable corresponding to a given bound state. He dealt with the second problem by introducing sets of wave functions as discussed in the previous chapter and then showed that the expectation value of an observable could be expressed in terms of these functions. Since the functions are not independent of one another the expectation value could be given in terms of the simplest one, the ordinary two-nucleon wave function. Then by choosing a particular observable whose expectation values are known to be certain numbers a normalization condition for the wave function is found. For example if $J_i(x)$ is the charge-current density operator then $(\Psi_a, J_i(x)\,\Psi_b)$ is the expectation value of the current at the point x and the integral of this must be $2\,\delta_{ab}$ (if both nucleons are charged).

Mandelstam (25) uses a similar idea but by taking advantage of the fact that propagators become singular for values of the energy-momentum corresponding to a bound state his results are simpler. He finds that in the ladder approximation for two nucleons the charge current density is given by

$$(\Psi_a, J_i(x)\,\Psi_b) = \int dx_1\, \bar{\chi}_a(x, x_1)\, \gamma_i^{(1)} \left(i\,\gamma^{(2)} \frac{\partial}{\partial x_1} + iM\right) \chi_b(x, x_1)$$

and the normalization condition is then

$$\int (\Psi_a, J_0(x)\,\Psi_b)\, d^3x = \delta(\underline{P}_a - \underline{P}_b)\ \delta_{ab}$$

Klein and Zemach (28), using an equivalent method found the same result except that they symmetrised it with respect to the two particles since Mandelstam was only looking for the charge-current density due to one particle. They also pointed out that the orthogonality part of the result follows directly from the equation for $\chi(x_1, x_2)$. For if the equation for $\varphi_E(x) = e^{iE(t_1 + t_2)}\chi(x_1, x_2)$

is

$$(\gamma^{(1)}p_1 + M)(\gamma^{(2)}p_2 + M)\varphi_E(x) = \int K(x,x')\varphi_E(x')\,dx'$$

and that for the adjoint $\bar{\varphi}_E(x)$ is

$$\bar{\varphi}_{E'}(x)(\gamma^{(1)}p_1 + M)(\gamma^{(2)}p_2 + M) = \int \bar{\varphi}_{E'}(x')\,K(x',x)\,dx'$$

then multiplying the first equation by $\varphi_{E'}(x)$ and the second by $\varphi_E(x)$, integrating over x and subtracting gives for $E \neq E'$

$$(E-E')\int \bar{\varphi}_{E'}(x)\left[\gamma_0^{(1)}(\gamma^{(2)}p_2 + M) + (\gamma^{(1)}p_1 + M)\gamma_0^{(2)}\right]\varphi_E(x)\,dx = 0$$

It is understood here, as in any such integral that both solutions have been obtained for the same coupling constant.

The corresponding orthogonality result, given by Scarf (35) for the scalar nucleon case is in momentum space

$$(E^2 - E'^2)\int dp\,\bar{\varphi}_{E'}(p)\,(c^2 + \underline{p}^2 - p_4^2)\,\varphi_E(p) = 0$$

In this integral p_4 is the imaginary relative energy so $\underline{p}^2 - p_4^2$ is not the length of the relative energy-momentum vector. As might be expected the normalization integral found by the method of Nishijima or Mandelstam is just the same integral for $E = E'$ set equal to one.

Allcock (26) found a normalization condition by a different method. Instead of looking for some observable whose expectation value is known, he analysed the propagator in the vicinity of its singularities and obtained an expression for the scalar product (Ψ_a, Ψ_b) in terms of the wave function. However it was shown by Allcock and Hooton (27) that the resulting condition is the same as that of Mandelstam.

This normalization has been criticised by Green (29). No doubt the charge-current density must be conserved but it seems in principle objectionable to have to appeal to a particular observable like this to establish a normalization. On the other hand the derivation of Allcock while more generally convincing is not at all intuitively obvious. There is another, more serious objection. In the derivation of the ladder approximation it is necessary to consider the possibility of one meson being present together with the two nucleons, yet the normalization in this approximation only includes the two nucleon state. Higher approximations would allow for mesons but the normalization would always have one meson less than the number allowed for in setting up the B-S equation. This has been stressed by Mandelstam. Green therefore considers first ordinary one particle wave equations such as the Dirac equation or the Klein-Gordon to define a probability density for the creation of a particle. Then in complete analogy probability densities are defined for the creation of a two nucleon state or of a two nucleon plus one meson state. The integrated sum of these must clearly be put equal to one - a normalization condition which is the same as Allcock's if only the two nucleon state is considered but differs if both states are included. The detailed application of this method to Wick's equation is worked out in Chapter 7.

The Mandelstam normalization was made use of by Scarf and Umezawa (33) who looked for reasons to exclude the abnormal solutions. They adapted Wick's equation by using the Sakata-Taketani formalism and hence had to consider a wave function $\Phi(x)$ together with its first and second time derivatives. They then find an analytical difference between the normal and the abnormal solutions; in the limit $E \to 1$ the normal solutions have $\frac{\partial^2 \Phi}{\partial x_0^2}$ well behaved but the abnormal solutions have it tending to infinity. Nevertheless the normalization condition makes $\frac{\partial^2 \Phi}{\partial x_0^2}$ bounded in both cases. They then turn to the propagator

$$K(1, 2; 3, 4) = \sum_i \chi_i(1, 2)\, \overline{\chi}_i(3, 4)$$

where $\chi_i(1, 2)$ contains Φ and its first two derivatives and has been normalized. The sum is over all states with a particular value of λ; if $\lambda > \frac{1}{4}$ it will include both normal and abnormal states, if $\lambda < \frac{1}{4}$ only normal states. Hence $K(1, 2; 3, 4)$ will apparently be discontinuous and probably unbounded at $\lambda = \frac{1}{4}$. They state that no such difficulty occurs in the ordinary Wick equation as then the sum is over the solutions $\Phi(x)$, and the normalized abnormal solutions are zero at $E = 1$. Scarf and Umezawa finally consider a condition used by Gell-Mann and Low in deriving the Bethe-Salpeter equation. However it is not clear that this condition should be applied at $\lambda = \frac{1}{4}$, when $E = 1$ for all the abnormal solutions.

Ohnuki, Takao and Umezawa (34) look for other reasons to exclude the abnormal solutions of the B-S equation. They first suggest that in the corresponding scattering problem abnormal solutions will not occur. They then consider adiabatic variation of λ in the vicinity of $\lambda = \frac{1}{4}$. As λ decreases through the value $\frac{1}{4}$, the abnormal bound state will disappear and there is no scattering state to come into existence. Hence the abnormal solutions are physically meaningless. However the argument is obscure and not very convincing.

The same authors also introduce the static model, that is, they consider the Bethe-Salpeter equation for fixed nucleons. In this case the eigenvalues of the Hamiltonian are known. They find that the B-S eigenvalue spectrum is just the same as that of Wick's equation - a normal solution for which $\lambda \to 0$ as $E \to 1$ and a number of abnormal solutions for all of which $\lambda \to \frac{1}{4}$ as $E \to 1$. Only the normal solution has the eigenvalue predicted by the Hamiltonian so that in this case the abnormal solutions are definitely spurious. In other words, while all eigenstates of the Hamiltonian are solutions of the B-S equation, the converse is not true, at least in this example. However, no obvious mathematical condition can be applied which would exclude the abnormal solutions if the eigenvalues of the Schrodinger equation had not been known.

Mugibayashi (36) has extended the same model in another direction. The complete B-S equation can be obtained, as the expansion of the kernel in powers of the coupling constant contains only the ladder approximation and one more term. It had been hoped that the abnormal solutions might be characteristic of the ladder approximation and would disappear in higher approximations. Mugibayashi seems to find that all solutions now have $\lambda \to 0$ as $E \to 1$ so to this extent the hope is justified. However only one solution has the correct form of the eigenvalue as predicted by the Schrodinger equation with the Hamiltonian and so the remaining solutions must be spurious.

For the purpose of this thesis an important discovery was that of Green (4) who found that Wick's equation is separable in a suitable co-ordinate system and hence that the integral transform used by Wick and Cutkosky is unnecessary.

The problem of solutions in the case that the nucleons are fermions has been largely solved by Green and Biswas (24) and Biswas (37). They impose a condition that the wave function and its first derivatives should be finite and continuous everywhere and then find that this leads to an eigenvalue condition. The condition becomes meaningless when $E = 0$, explaining why Goldstein got no solutions, but can be applied for any other value of E, at least if E is near zero.

Other recent work on Wick's equation may be mentioned briefly. Okubo and Feldman (38) considered Wick's equation for a nucleon-antinucleon pair, including an annihilation term. It turns out that the extra term affects only S-states. In order to solve the equation for small binding energy, that is, $E \simeq 1$, they are obliged to make the approximation $E \to \infty$, which seems to make the results open to question. They also have to deal with integrals like $\int_0^s (1-z)^x z^{-\theta} dz$ in which θ is close to an integer. However perhaps these integrals could be circumvented. Despite these features the results are reasonable; the eigenvalues agree with those for Wick's normal solutions and the method leaves open the possibility of abnormal solutions. Watanabe (39) looked for differences in the analytic properties of the normal and abnormal solutions which might be used to show that the abnormal solutions are not eigenstates of the Hamiltonian. However, he found no differences. Vosko (32) used a variational method to calculate eigenvalues of Wick's equation even when the mass of the meson is non-zero. The method would be suitable for isolated eigenvalues but excessively laborious for a number of them. His results suggest that the instantaneous interaction approximation is not good, particularly if the meson has mass.

Bertocchi, Fubini, Stroffolini and Tonin (23) have applied the spectral method of Martin, used in potential theory, to Wick's equation.

They stress that their results are incomplete as they can only consider the space-like region. For this reason they cannot make use of Wick's boundary condition on the wave function which is a causality condition and requires solutions in the whole of space time. Nevertheless, they do find that in the non-relativistic limit with a non-zero meson mass, the solution is just that of a Schrodinger equation with Yukawa potential.

We finally mention two recent applications of the Bethe-Salpeter equation. First Baumann, Freund and Thirring (40) have looked for the possibility that pions are nucleon-antinucleon bound states and that photons are electron-positron bound states. Their equations differ from any considered previously in that instead of the meson having zero or small mass it is much heavier than the nucleons. With the coupling constant given by the universal Fermi interaction the pion-nucleon coupling constant turns out to have the right order of magnitude. Secondly, in his fundamental studies on axiomatic field theory, Symanzik is led to a Bethe-Salpeter equation in the elimination of the one-particle structure of Green's functions.

Chapter 3.

SEPARABLE SOLUTIONS.

As was shown in the Introduction, Wick's equation describing two scalar particles interacting via scalar, massless mesons can be written

$$\left\{ (p^2 - c^2)^2 - 4E^2 p_0^2 \right\} \Phi(p) = \Psi(p) = \frac{i\lambda}{\pi^2} \int \frac{\Phi(k)\, d^4k}{(p-k)^2 + i\varepsilon} \tag{3.1}$$

In this equation $c^2 = M^2 - E^2$, $p^2 = p_0^2 - \underline{p}^2$ and the limit as tends to zero is to be taken. Thus this is the equation in the Minkowski metric.

The kernel of the integrand in (3.1) is the Feynman causal function of course, so this integral equation may be converted to a differential equation by means of the D'Alembertian operator, giving

$$\Box \Psi(p) = 4\lambda \Phi(p) \tag{3.2}$$

It was noticed by Green (4) that in bipolar coordinates this equation is separable. The transformation is as follows:

$$p_0 = \frac{c \sin\alpha}{\cos\alpha - \cos\beta}, \qquad p_s = \frac{c \sin\beta}{\cos\alpha - \cos\beta} \tag{3.3}$$

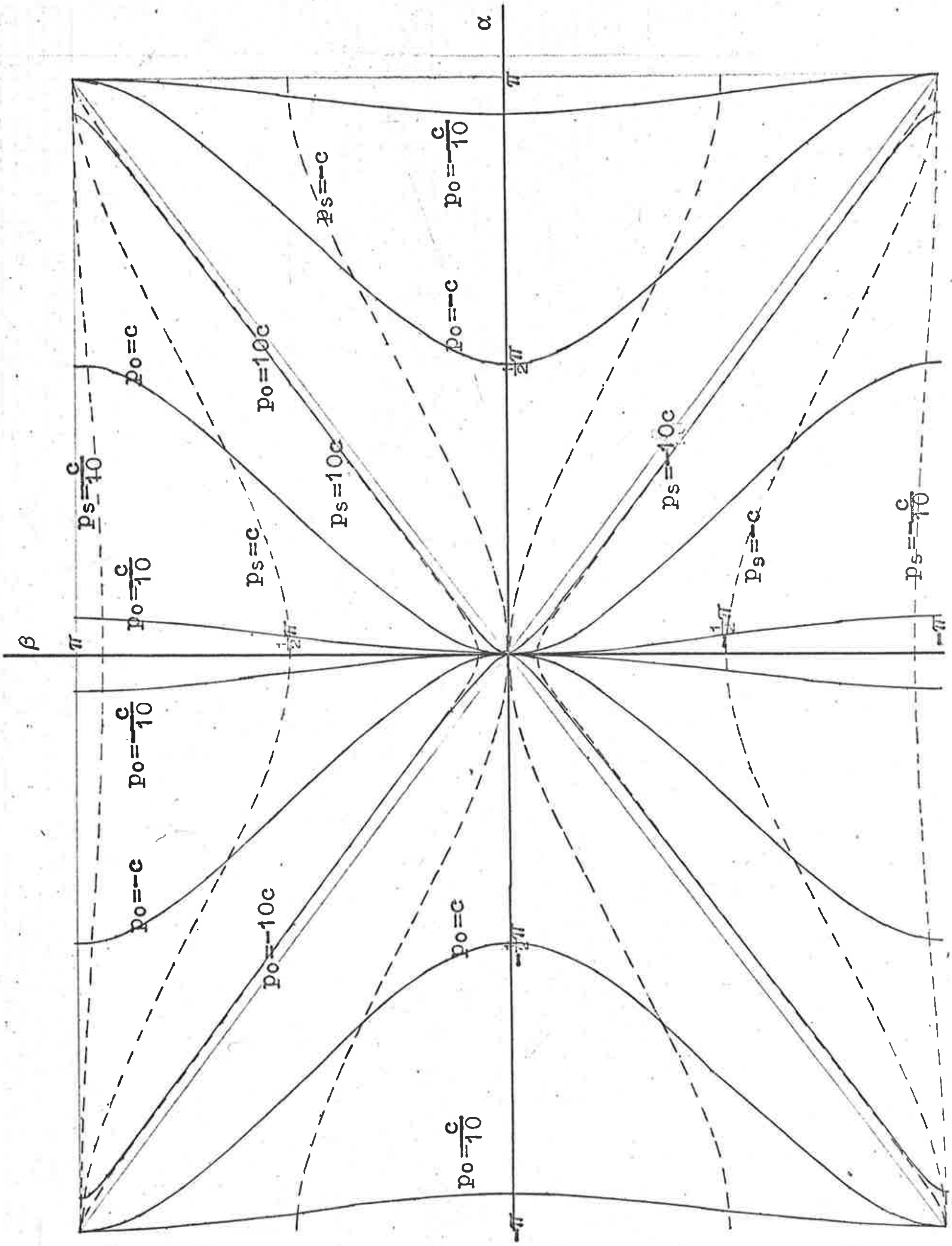
$$p_1 = p_s \sin\theta \cos\varphi, \quad p_2 = p_s \sin\theta \sin\varphi, \quad p_3 = p_s \cos\theta$$

Conversely

$$\sin\alpha = \frac{2cp_0}{\left[(p^2 - c^2)^2 + 4c^2 p_0^2\right]^{\frac{1}{2}}}, \quad \cos\alpha = \frac{c^2 - p^2}{\left[(p^2-c^2)^2+4c^2 p_0^2\right]^{\frac{1}{2}}} \tag{3.4}$$

$$\sin\beta = \frac{2cp_s}{\left[(p^2 - c^2)^2 + 4c^2 p_0^2\right]^{\frac{1}{2}}}, \quad \cos\beta = \frac{-(p^2 + c^2)}{\left[(p^2-c^2)^2+4c^2 p_0^2\right]^{\frac{1}{2}}}$$

and the Jacobian of the transformation is $\dfrac{c^4 \sin^2\beta \ \sin\theta}{(\cos\alpha - \cos\beta)^4}$



The Transformation $(p_o, p_s) \to (\alpha, \beta)$

The range of values will be taken as $0 \leq \varphi < 2\pi$, $0 \leq \theta < \pi$, $0 < \beta \leq \pi$, $|\alpha| < \beta$ which differs from Green's choice but seems from the graph of the transformation to be the appropriate range to give positive values of p_s and all values of p_θ.

In the new variables the integral equation becomes

$$\Psi = \frac{4c^2(M^2\cos^2\alpha - E^2)}{(\cos\alpha - \cos\beta)^2}\,\Phi$$

$$h_s\Psi = \frac{ic^2\lambda \sin\beta}{2\pi^2}\int \sin\theta' d\theta' d\varphi' \int_0^\pi d\beta' \sin\beta' \tag{3.5}$$

$$\times \int_{-\beta'}^{\beta'} \frac{d\alpha' \quad h_s\Phi}{(\cos\alpha' - \cos\beta')^2\left[\cos\Omega - \cos(\alpha-\alpha') + i\epsilon(\cos\alpha - \cos\beta)(\cos\alpha' - \cos\beta')\right]}$$

where $\cos\Omega = \cos\beta\cos\beta' + \sin\beta\sin\beta'(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi'))$ is the angle between the two four dimensional unit vectors whose polar coordinates are (β,θ,φ) and $(\beta',\theta',\varphi')$. The differential equation becomes

$$\frac{(\cos\alpha - \cos\beta)^2}{c^2}\left[\frac{\partial^2}{\partial\alpha^2} - \frac{\partial^2}{\partial\beta^2} + \frac{l(l+1)}{\sin^2\beta}\right](h_s\Psi) = 4\lambda h_s\Phi \tag{3.6}$$

This differential equation, with the relation (3.5) between Ψ and Φ suggests that Ψ may be separated in the form:

$$\Psi = \frac{1}{h_s} f(\alpha)\, g(\beta)\, Y_l^m(\theta,\varphi) \tag{3.7}$$

where $Y_l^m(\theta,\varphi)$ is a spherical harmonic. Then f (α) and g (β) will satisfy the equations

$$\frac{d^2 g}{d\beta^2} + \left(n^2 - \frac{l(l+1)}{\sin^2\beta}\right)g = 0 \tag{3.8}$$

$$\frac{d^2f}{d\alpha^2} + \left(n^2 - \frac{\lambda}{M^2\cos^2\alpha - E^2}\right) f = 0 \qquad (3.9)$$

These differential equations need to be supplemented by boundary conditions if they are to be completely equivalent to the integral equation (3.5). Despite considerable work, it has not been possible to get satisfactory boundary conditions but drawing on later results the following comments may be made:

(a) Although there is a separation of variables in the differential equation, this is not the case in the integral equation where the α-integration is over the range $(-\beta, \beta)$

(b) The important part of the integral (3.5) is

$$\int_0^{\pi} d\beta' \sin\beta'\, g(\beta') \int_{-\beta'}^{\beta'} \frac{d\alpha'\, f(\alpha')}{(M^2\cos^2\alpha' - E^2)(\cos\Omega - \cos(\alpha-\alpha') + i\varepsilon(\cos\alpha' - \cos\beta'))}$$

It will be shown later that near the singularity $\cos^2\alpha = \frac{E^2}{M^2}$, $f(\alpha)$ has the form

$$f(\alpha) \simeq A(M^2\cos^2\alpha - E^2)\log(\cos^2\alpha - E^2/M^2) + B$$

Thus $f(\alpha)/M^2\cos^2\alpha - E^2$ possesses a pole at $\cos\alpha = \pm E/M$ and in passing through the singularity $f(\alpha)$ develops an imaginary part.

(c) It is not clear which of the α and β integrations should be performed first or whether they can be interchanged, but assuming that they can, part of the integral can be written:

$$\int_{|\alpha'|}^{\pi} \sin\beta'\, d\beta' \int_0^{\pi} \sin\theta'\, d\theta' \int_0^{2\pi} d\varphi' \frac{g(\beta')\, Y_m^{\ell}(\theta';\varphi')}{\cos\Omega - \cos(\alpha-\alpha') + i\varepsilon(\cos\alpha' - \cos\beta')}$$

If the β-integration were over the range $(0 - \pi)$ it would be possible to apply the theorem of Hecke (5), (6) and then

$\frac{1}{\sin\beta}\, g(\beta)\; Y_{\ell}^{m}(\theta, \varphi)$ would be a spherical harmonic on a hypersphere in four dimensions. Unfortunately it does not seem possible to find an analogue of this theorem when the integration is only over a sector of the sphere. Attempts to extend the range of integration by assuming various properties for $g(\beta)$ meet with the difficulty that so long as $|\alpha'| < \beta' < \pi$, $\cos\alpha' > \cos\beta'$ and hence the coefficient of $i\varepsilon$ is positive. Outside this range the coefficient changes sign so the kernel of the integral is no longer a function of the four dimensional angle only.

(d) The usual method of obtaining boundary conditions is to substitute the differential equation into the integral equation and then integrate by parts. If that is done in this case one has

$$f(\alpha)\, g(\beta)\, Y_\ell^m(\theta,\varphi) = \frac{i \sin\beta}{8\pi^2} \int_0^\pi d\beta' \int \sin\theta' d\theta' d\varphi' \int_{-\beta'}^{\beta'} d\alpha'\, \mathcal{L}(\alpha',\beta')$$

$$\times \left[\frac{\partial^2}{\partial\alpha'^2} - \frac{\partial^2}{\partial\beta'^2} + \frac{\ell(\ell+1)}{\sin^2\beta'}\right] f(\alpha')\, g(\beta')\, Y_\ell^m(\theta',\varphi')$$

where

$$\mathcal{L}(\alpha',\beta') = \frac{\sin\beta'}{\cos\Omega - \cos(\alpha-\alpha') + i\varepsilon(\cos\alpha' - \cos\beta')}$$

Performing the first integration gives

$$\int_{-\beta'}^{\beta'} d\alpha'\, \mathcal{L}(\alpha',\beta') \frac{d^2 f}{d\alpha'^2} = \left[\mathcal{L}(\alpha',\beta') \frac{df}{d\alpha'}\right]_{-\beta'}^{\beta'} - \int_{-\beta'}^{\beta'} d\alpha' \frac{\partial\mathcal{L}}{\partial\alpha'} \frac{df}{d\alpha'}$$

$$= \mathcal{L}(\beta',\beta') \frac{df(\beta')}{d\beta'} + \mathcal{L}(-\beta',\beta') \frac{df(-\beta')}{d\beta'}$$

$$- \int_{-\beta'}^{\beta'} d\alpha' \frac{\partial\mathcal{L}}{\partial\alpha'} \frac{df}{d\alpha'}$$

Substituting the boundary terms back into the integral gives i. e.

$$\int_0^\pi d\beta' \int \sin\theta' d\theta' d\varphi'\, \mathcal{L}(\beta',\beta') \frac{df(\beta')}{d\beta'}\, g(\beta')\, Y_\ell^m(\theta',\varphi')$$

But this integral is not well defined as $\mathcal{L}(\beta', \beta')$ no longer contains a term $i\epsilon$ in the denominator. Hence this method is not able to produce suitable boundary conditions.

The problem of boundary conditions for the equation in the Minkowski metric can be avoided by using Wick's results to go over to a Euclidean metric. The rotation from the real to the imaginary axis of p_0, putting $p_0 = ip_4$, corresponds exactly to a rotation from the real to the imaginary axis of α and we shall put $\alpha = ia$. But whereas the limits of α are $-\beta$ and β, the limits of a are $-\infty$ and ∞. The details of this transformation are given in appendix A to this chapter, but the final result is the integral equation

$$H(a,\beta,\theta,\varphi) = \frac{\lambda}{8\pi^2}\int_{-\infty}^{\infty} da' \int_0^{\pi} \sin^2\beta' \, d\beta' \int_0^{\pi} \sin\theta' \, d\theta'$$

$$\times \int_0^{2\pi} \frac{d\varphi' \quad H(a',\beta',\theta',\varphi')}{(\epsilon^2\cosh^2 a' + E^2\sinh^2 a')(\cosh(a-a') - \cos\Omega)} \qquad (3.10)$$

where $H = \frac{ks}{\sin\beta}\Psi$ and again $\cos\Omega$ is the angle between two unit vectors whose polar coordinates are (β,θ,φ) and $(\beta',\theta',\varphi')$

In this equation there is no difficulty in writing H as a product of functions of one variable and then getting the differential equation and boundary conditions satisfied by each function. However it is simpler to use the theorem of Hecke (6), (7), and write

$$H(a,\beta,\theta,\varphi) = f(a)\, S_{n-1,\ell,m}(\beta,\theta,\varphi) \qquad (3.11)$$

where

$$S_{n-1,\ell,m}(\beta,\theta,\varphi) = \sin^{\ell}\beta \; C^{\ell+1}_{n-\ell-1}(\cos\beta)\; Y^m_{\ell}(\theta,\varphi)$$

is a spherical harmonic of degree $n - 1$ in four dimensions and $n \geqslant \ell + 1$.

Then

$$\int_0^{\pi} \sin^2\beta' \, d\beta' \int_0^{\pi} \sin\theta' \, d\theta' \int_0^{2\pi} d\varphi' \, \frac{S_{n-1,\ell,m}(\beta',\theta',\varphi')}{\cosh(a-a') - \cos\Omega}$$

$$= L_n \, S_{n-1,\ell,m}(\beta,\theta,\varphi)$$

with $$L_n = \frac{4\pi}{n} \int_{-1}^{1} dx \, \sqrt{1-x^2} \, \frac{C^1_{n-1}(x)}{\cosh(a-a') - x}$$ where

$C^1_{n-1}(x)$ is a Gegenbauer polynomial. The denominator may be expanded in powers of $e^{a-a'}$ or $e^{a'-a}$, whichever is less than one, and the orthogonality relation for Gegenbauer polynomials used to give

$$L_n = \frac{4\pi^2}{n} e^{-n|a-a'|}$$

Thus we finally get the integral equation for f(a),

$$f(a) = \frac{\lambda}{2n} \int_{-\infty}^{a} da' \, \frac{f(a')\, e^{n(a'-a)}}{M^2\cosh^2 a' - E^2} + \frac{\lambda}{2n} \int_a^{\infty} da' \, \frac{f(a')\, e^{n(a-a')}}{M^2\cosh^2 a' - E^2} \qquad (3.12)$$

This integral equation is equivalent to the differential equation

$$\frac{d^2 f}{da^2} - \left(n^2 - \frac{\lambda}{M^2 \cosh^2 a - E^2}\right) f = 0 \qquad (3.13)$$

together with the boundary conditions

$$\lim_{a \to -\infty} e^{na} \left(\frac{df}{da} - n\, f(a) \right) = 0$$

$$\lim_{a \to \infty} e^{-na} \left(\frac{df}{da} + n\, f(a) \right) = 0 \qquad (3.14)$$

A second form of these equations can be obtained by the substitution $q = \tanh a$. Then

$$f(q) = \frac{\lambda}{2n} \int_{-1}^{q} dq' \, \frac{f(q')}{c^2 + E^2 q'^2} \left(\frac{1+q'}{1-q'}\right)^{\frac{n}{2}} \left(\frac{1-q}{1+q}\right)^{\frac{n}{2}}$$

$$+ \frac{\lambda}{2n} \int_{q}^{1} dq' \, \frac{f(q')}{c^2 + E^2 q'^2} \left(\frac{1-q'}{1+q'}\right)^{\frac{n}{2}} \left(\frac{1+q}{1-q}\right)^{\frac{n}{2}} \qquad (3.15)$$

$$(1 - q^2)\, f''(q) - 2qf'(q) - \frac{n^2 f(q)}{1 - q^2} + \frac{\lambda f(q)}{c^2 + E^2 q^2} = 0 \qquad (3.16)$$

and the boundary conditions are

$$\lim_{q\to -1} (1 + q)^{\frac{n}{2}} \left[(1 - q^2)\, f'(q) - n\, f(q)\right] = 0$$

$$\lim_{q\to 1} (1 - q)^{\frac{n}{2}} \left[(1 - q^2)\, f'(q) + n\, f(q)\right] = 0 \qquad (3.17)$$

Putting $f(q) = (1 - q^2)^{-\frac{n}{2}} g_n(q)$ gives Cutkosky's integral and differential equations.

A third form of the equations is obtained by noting that from the form of the equations for $f(a)$, it must be either an even or an odd function. The substitution $z = \cosh^2 a$ leads to different integral equations for the even and odd functions so we put

$S(a) = f(a)$ if f is even in a,

and $T(a) = f(a)$ if f is odd in a, and

also for convenience we replace λ by $M^2\lambda$ and put

$$e = E^2/M^2 \qquad (3.18)$$

Then

$$S(z) = \frac{\lambda}{4n}\left\{\int_2^\infty \left(\frac{\sqrt{z'} - \sqrt{z'-1}}{\sqrt{z} - \sqrt{z-1}}\right)^n \frac{S(z')\, dz'}{(z'-e)\sqrt{z'}\sqrt{z'-1}} + \int_1^z \left(\frac{\sqrt{z'} + \sqrt{z'-1}}{\sqrt{z} + \sqrt{z-1}}\right)^n \frac{S(z')\, dz'}{(z'-e)\sqrt{z'}\sqrt{z'-1}} + \int_1^\infty \left(\frac{\sqrt{z'} - \sqrt{z'-1}}{\sqrt{z} + \sqrt{z-1}}\right)^n \frac{S(z')\, dz'}{(z'-e)\sqrt{z'}\sqrt{z'-1}}\right\} \qquad (3.19)$$

$$T(z) = \frac{\lambda}{4n}\left\{\int_z^\infty \left(\frac{\sqrt{z'} - \sqrt{z'-1}}{\sqrt{z} - \sqrt{z-1}}\right)^n \frac{T(z')\, dz'}{(z'-e)\sqrt{z'}\sqrt{z'-1}} + \int_1^z \left(\frac{\sqrt{z'} + \sqrt{z'-1}}{\sqrt{z} + \sqrt{z-1}}\right)^n \frac{T(z')\, dz'}{(z'-e)\sqrt{z'}\sqrt{z'-1}} - \int_1^\infty \left(\frac{\sqrt{z'} - \sqrt{z'-1}}{\sqrt{z} + \sqrt{z-1}}\right)^n \frac{T(z')\, dz'}{(z'-e)\sqrt{z'}\sqrt{z'-1}}\right\} \qquad (3.20)$$

Both $S(z)$ and $T(z)$ satisfy the equation

$$z(z-1)(z-e)\,S''(z) + \tfrac{1}{2}(z-1)(z-e)\,S'(z) + \tfrac{1}{2}(z-e)z\,S'(z) - \tfrac{n^2}{4}(z-e)\,S(z) + \tfrac{\lambda}{4}\,S(z) = 0 \tag{3.21}$$

and the boundary condition

$$\lim_{z \to \infty} z^{-\frac{n}{2}} \left(n\,S(z) + 2z\,S'(z) \right) = 0 \tag{3.22}$$

but in addition S satisfies

$$\lim_{z \to 1} \sqrt{z-1}\; S'(z) = 0$$

while T satisfies (3.23)

$$T(1) = 0$$

We now consider the differential equation (3.21) with its boundary conditions. In the first place, it is an equation of the Sturm-Liouville (7) type. For given values of n and e, the boundary conditions determine an infinite, discrete, increasing set of eigenvalues for λ denoted by λ_k. The integer k will be taken to range from zero upwards and k, n, l, m constitute a set of four quantum numbers specifying the state of the system. Corresponding to each eigenvalue will be an eigenfunction and even values of k will lead to eigenfunctions $S_k(z)$ while odd values of k will correspond to $T_k(z)$. Different eigenfunctions will be orthogonal in the sense:

$$\int_1^{\infty} \frac{dz}{(z-e)\sqrt{z}\sqrt{z-1}}\; S_k(z)\; S_\ell(z) = 0 \qquad k \neq \ell \tag{3.24}$$

where one or both of the $S(z)$ may be replaced by $T(z)$. As e increases any given λ_k must decrease.

In the second place, (3.21) is an example of Heun's equation, that is, it is an equation with four regular singularities, the points $z = 0, e, 1, \infty$. In the notation of Erdelyi (8) we have

$$\alpha = \frac{n}{2}, \ \beta = -\frac{n}{2}, \ \gamma = \frac{1}{2}, \ \delta = \frac{1}{2}, \ \epsilon = \frac{1}{2}, \ h = e + \frac{\lambda}{n^2}$$

The function whose various branches are the solutions of the equation is partly specified by the Riemannian scheme:

$$P \begin{Bmatrix} 0 & e & 1 & \infty & \\ 0 & 0 & 0 & -\frac{n}{2} & z \\ \frac{1}{2} & 1 & \frac{1}{2} & \frac{n}{2} & \end{Bmatrix} \qquad (3.25)$$

The boundary conditions (3.23) show that the required solutions have the exponent $\frac{n}{2}$ at the singularity $z = \infty$, and at $z = 1$ they have exponent 0 in the case of $S_k(z)$ and $\frac{1}{2}$ in the case of $T_k(z)$. In general, solutions of Heun's equation which have a particular exponent at one singularity will be a mixture of solutions with each exponent at any other singularity but for particular values of λ solutions called Heun functions may be found with given exponents at two singularities, as required by our boundary conditions. In this way the eigenvalue condition arises.

In passing, it may be pointed out that z ranges from 1 to ∞ because of the change from a Minkowski to a Euclidean metric which made the substitution $z = \cosh^2 a$ appropriate. Had the substitution $z = \cos^2\alpha$ been made in the equation (3.9) for $f(\alpha)$ in the Minkowski metric, precisely the same equation (3.21) would have been derived, but with z ranging from 0 to 1. It seems reasonable to suppose that boundary conditions at $z = 1$ and $z = \infty$ being given, it must be possible to find a boundary condition at $z = 0$. In this rather roundabout way the boundary conditions appropriate to the Minkowski metric could be found. Of course, it is likely that a Heun function with specified exponents at two singularities will contain a mixture of exponents at a third, so the boundary condition at $z = 0$ will be more complicated than those given in (3.22) and (3.23). Not enough seems to be known about Heun's equation to settle this question.

To get solutions to the differential equation (3.21) the simplest method is to use power series for $S(z)$ and $T(z)$. Putting

$$S(z) = \sum_{\nu=0}^{\infty} a_\nu z^{-\nu-\frac{1}{2}n} \tag{3.26}$$

$$T(z) = (z-1)^{\frac{1}{2}} \sum_{\nu=0}^{\infty} b_\nu z^{-\nu-\frac{1}{2}(n+1)} \tag{3.27}$$

into (3.21) gives the recurrence relations

$$\left.\begin{aligned} M_0 a_1 + L_0 a_0 &= 0 \\ M_\nu a_{\nu+1} + L_\nu a_\nu + K_\nu a_{\nu-1} &= 0 \\ \text{where}\quad M_\nu &= (\nu+1)(\nu+n+1) \\ L_\nu &= \frac{\lambda}{4} - e\nu(\nu+n) - (\nu+\tfrac{n}{2})(\nu+\tfrac{n+1}{2}) \\ K_\nu &= e(\nu+\tfrac{n-1}{2})(\nu+\tfrac{n}{2}-1) \end{aligned}\right\} \tag{3.28}$$

and

$$\left.\begin{aligned} M_0 b_1 + L'_0 b_0 &= 0 \\ M_\nu b_{\nu+1} + L'_\nu b_\nu + K'_\nu b_{\nu-1} &= 0 \\ \text{where}\quad L'_\nu &= \frac{\lambda}{4} - e\nu(\nu+n) - (\nu+\tfrac{n+1}{2})(\nu+\tfrac{n}{2}+1) \\ K'_\nu &= e(\nu+\tfrac{n-1}{2})(\nu+\tfrac{n}{2}) \end{aligned}\right\} \tag{3.29}$$

Now these solutions clearly have the correct behaviour at $z = \infty$. To investigate the behaviour at $z = 1$, we must consider the convergence of the two power series and make use of a theorem of Poincare, (Milne-

Thompson (9) Chapter 17) which states that if $t = \lim_{\nu\to\infty} \frac{a_{\nu+1}}{a_\nu}$

then t is equal to one of the zeros of the equation

$$M\,t^2 + L\,t + K = 0$$

where $M = \lim_{\nu\to\infty} M_\nu/\nu^2$ etc., provided that the moduli of the zeros be different.

In the present case the equation is

$$t^2 - (e+1)\,t + e = 0 \qquad \text{so } t = e \text{ or } 1.$$

With the help of a further theorem of Perron it is shown in appendix B to this chapter that in general $t = 1$ but that if the continued fraction

$$\frac{L_0}{M_0} - \cfrac{K_1/M_1}{\dfrac{L_1}{M_1} - \cfrac{K_2/M_2}{\dfrac{L_2}{M_2} - \cfrac{K_3/M_3}{\dfrac{L_3}{M_3} - \ddots}}} = 0 \tag{3.30}$$

then $t = e$. In the general case the power series (3.26)

$$S(z) = \sum_{\nu=0}^{\infty} a_\nu z^{-\nu-\frac{n}{2}}$$ converges if $z > 1$ but the behaviour at $z = 1$ requires further consideration. If we refine the previous result by putting

$$\frac{a_{\nu+1}}{a_\nu} = t + \frac{s}{\nu} \qquad \text{for large } \nu \text{ and using}$$

$$\frac{M_\nu}{\nu^2} = 1 + \frac{n+2}{\nu}$$

$$\frac{L_\nu}{\nu^2} = -e - 1 - \frac{ne}{\nu} - \frac{n+\frac{1}{2}}{\nu}$$

$$\frac{K_\nu}{\nu^2} = e + \frac{e(n+3/2)}{\nu}$$

we find

$$S = \frac{2t^2 - \frac{1}{2}t - \frac{3}{2}e}{2t - e - 1} = -3/2 \quad \text{for } t = 1$$

Hence $a_\nu \simeq k\,\nu^{-3/2}$ where k is a constant. Thus $S(z)$ is convergent for $z = 1$ but $\frac{ds}{dz}$ is not and the boundary condition that $S(z)$ should have exponent 0 at $z = 1$ is not satisfied.

In the particular case that (3.30) is satisfied, $t = e$ and the power series converges for $z > e$ so that both $S(z)$ and $\frac{ds}{dz}$ are bounded at $z = 1$ and the boundary condition is satisfied. The equation (3.30) is a transcendental equation for λ, the eigenvalue condition. The recurrence relation (3.29) may be treated in the same way and leads to the corresponding eigenvalue condition.

Before going on, in the next chapter, to discuss the numerical solution of the recurrence relations (3.28) and (3.29) it is of interest to consider further the analytic properties of $S(z)$ and $T(z)$ and their implications for the properties of the Bethe-Salpeter wave function $\Phi(p)$.

In the complex z-plane, $S(z)$ can only have singularities at $z = 0, 1, e, \infty$. In the vicinity of $z = \infty$ we already have

$$S(z) \simeq z^{-\frac{n}{2}} \left(a_o + a_1 z^{-1} + \text{---}\right)$$

In the vicinity of $z = 1$:

$$S(z) = c_o + c_1 (z-1) + c_2 (z-1)^2 + \text{---}$$

At $z = e$ the exponent difference is 1 so a logarithmic branch point occurs:

$$S(z) = d_o + d_1(z-e) + \text{---} + \left[d_o' + d_1' (z-e) + \text{---}\right](z-e) \log (z-e)$$

And at $z = 0$

$$S(z) = e_o + e_1 z + \text{---} + \sqrt{z}\,(e_o' + e_1' z + \text{---})$$

Now $z = \cosh^2 a$, $S(z) = f(a)$ and

$$\Psi = \frac{1}{\rho s} Y_\ell^m(\theta,\varphi) \sin^{\ell+1}\beta \; C^{\ell+1}_{n-\ell-1}(\cos\beta)\; f(a)$$

In terms of p_o with $p^2 = p_o^2 - p_s^2$, $c^2 = M^2(1-e) = M^2 - E^2$

$$\cosh a = \frac{c^2 - p^2}{\left[(p^2-c^2)^2 + 4c^2p_o^2\right]^{\frac{1}{2}}} \qquad \sinh a = \frac{2icp_o}{\left[(p^2-c^2)^2 + 4c^2p_o^2\right]^{\frac{1}{2}}}$$

$$\cos\beta = \frac{-(p^2+c^2)}{\left[(p^2-c^2)^2 + 4c^2p_o^2\right]^{\frac{1}{2}}} \qquad \sin\beta = \frac{2cp_s}{\left[(p^2-c^2)^2 + 4c^2p_o^2\right]^{\frac{1}{2}}}$$

so that $z = \infty$ implies $(p^2 - c^2)^2 + 4c^2p_o^2 = 0$ or $p_o = \pm p_s \pm ic$,

$z = 1$ implies $p_o = 0$

$z = 0$ implies $p^2 - c^2 = 0$ or $p_o = \pm\sqrt{p_s^2 + c^2}$

and $$z - e = \frac{c^2}{M^2}\,\frac{(p^2-c^2)^2 - 4E^2p_o^2}{(p^2-c^2)^2 + 4c^2p_o^2}$$ so

$z = e$ implies $p_o = \pm E \pm \sqrt{M^2 + p_s^2}$

The Gegenbauer polynomial $C^{\ell+1}_{n-\ell-1}(\cos\beta)$ is a polynomial in $\cos\beta$ containing only even powers if $n-\ell-1$ is even and only odd powers if $n-\ell-1$ is odd. In either case the highest power is $\cos^{n-\ell-1}$, so $C^{\ell+1}_{n-\ell-1}(\cos\beta)$ can have singularities only if $\cos\beta = \infty$, that is, if $p_o = \pm p_s \pm ic$. The product $\sin^{\ell+1} C^{\ell+1}_{n-\ell-1}(\cos\beta)$ will behave like $\left[(p^2-c^2)^2 + 4c^2p_o^2\right]^{-\frac{n}{2}}$ near these points but $f(a)$ behaves like $\cosh^{-n}a$, that is like $\left[(p^2-c^2)^2 + 4c^2p_o^2\right]^{\frac{n}{2}}$, so Ψ is bounded at $p_o = \pm p_s \pm ic$. Furthermore, as the later terms of $f(a)$ are smaller by powers of $\cosh^2 a$, and the later terms of $C^{\ell+1}_{n-\ell-1}$, differ by powers of $\cos^2\beta$, $\Psi(p)$ cannot have a branch point at $p_o = \pm p_s \pm ic$.

When $z = 1$, $S(z)$ has no singularity so $\Psi(p)$ will have no singularity when $p_o = 0$. When $z = 0$, $S(z)$ has a branch point which is removed by the substitution $z = \cosh^2 a$ so $\Psi(p)$ has no singularity at $p_o = \pm\sqrt{p_s^2 + c^2}$. Finally, at $z = e$, $S(z)$ has a logarithmic singularity which implies that $\Psi(p)$ has logarithmic singularities at the four points $p_o = \pm E \pm \sqrt{M^2 + p_s^2}$ and these are the only singularities possessed by $\Psi(p)$ in the complex p_o-plane. If $\Psi(p)$ is permitted to go round these singularities it passes to a different branch of the function $S(z)$, which will not satisfy the boundary conditions, so cuts are required from $p_o = E + \sqrt{p_s^2 + M^2}$ to $p_o = -E + \sqrt{p_s^2 + M^2}$ and from $p_o = E - \sqrt{p_s^2 + M^2}$ to $p_o = -E - \sqrt{p_s^2 + M^2}$. If M^2 is considered to have an infinitesimal negative imaginary part the first cut will lie below the real p_o axis and the second above it. The Bethe-Salpeter wave function $\Phi(p)$ is related $\Psi(p)$ by

$$\Phi(p) = \frac{\Psi(p)}{(p^2-c^2)^2 - 4E^2 p_o^2}$$

so its analytic properties differ from $\Psi(p)$ only in that at each of the four branch points it possesses a first order pole and an unbounded logarithmic branch point. Thus the function $\Phi(p)$ is consistent with Wick's conditions that the wave function should be analytic everywhere in the p_o-plane with the exception of cuts from which it may be continued in an anticlockwise direction. The fact that the cuts here extend for a finite distance only, instead of to $\pm\infty$, is probably due to the use of the ladder approximation.

Chapter 3 - Appendix A.

CHANGE FROM MINKOWSKI TO EUCLIDEAN METRIC.

We consider the integral

$$\int \Phi(k_0, \underline{k})\, \ell(k_0, \underline{k})\, dk_0\, d^3k \tag{A3.1}$$

where $\quad \ell^{-1}(k_0, \underline{k}) = (p-k)^2 + i\epsilon$

Then as a first step to the transformation (3.3) we replace k_0 by $\alpha + ia$ by means of the equation

$$(k^2 - c^2) \sin(\alpha + ia) + 2k_0 c \cos(\alpha + ia) = 0 \tag{A3.2}$$

which is what one gets if β is eliminated from equations (3.3). The range $(-\infty, \infty)$ of k_0 corresponds to the range $(-\pi, \pi)$ of α with $a = 0$ and we shall complete the contour in the (α, a) plane as shown in fig (1), by means of a path from $(\pi, 0)$ to (π, ∞) then to $(0, \infty)$ and then to $(0, -\infty)$ and so on. The path has to be indented at the points where $\sinh a = {}^{c}/k_s$.

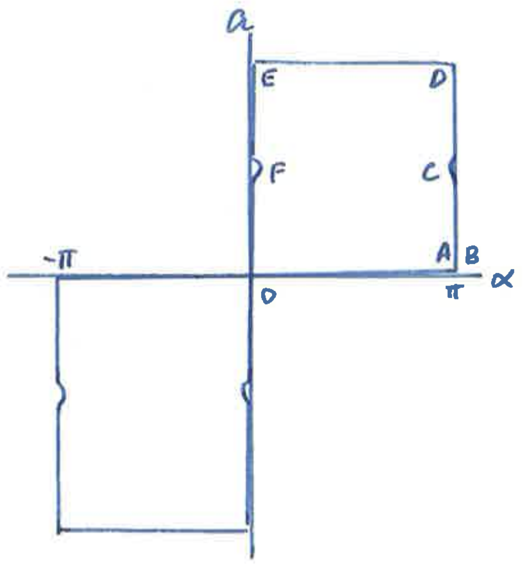


Fig 1.

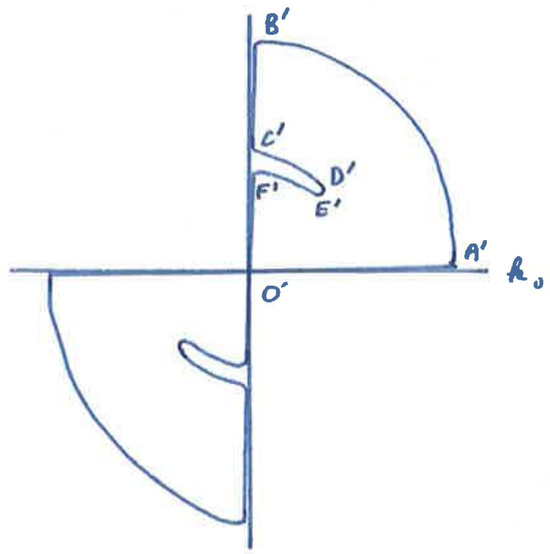


Fig 2.

This path corresponds to the path in the complex k_o plane shown in fig (2) and corresponding points have been labelled. The indentation C' D' E' F' arises essentially because the transformation from k_o to α + ia is singular at the points $k_o = \pm k_s \pm ic$.

The equation (A3.2) may be solved for k_c as follows:

Along O A $$k_o = \frac{-c\cos\alpha + \sqrt{k_s^2\sin^2\alpha + c^2}}{\sin}$$

Along A C $$k_o = \frac{i\,c\cosh a + i\sqrt{c^2 - k_s^2\sinh^2 a}}{\sinh a}$$

Along C D $$k_o = \frac{i\,c\cosh a + \sqrt{k_s^2\sinh^2 a - c^2}}{\sinh a}$$

Along E F $$k_o = \frac{i\,c\cosh a - \sqrt{k_s^2\sinh^2 a - c^2}}{\sinh a}$$

Along F O $$k_o = \frac{i\,c\cosh a - i\sqrt{c^2 - k_s^2\sinh a}}{\sinh a}$$

Considering, for brevity, only the quadrant $\alpha \geq 0$, $a \geq 0$, we have from Wick's results that Φ is analytic inside the contour O A C D E F O. Hence

$$0 = \int \sin\theta\, d\theta\, d\phi \int_0^\infty k_s^2\, dk_s \left\{ \int_0^\pi d\alpha\, \frac{\sqrt{k_s^2\sin^2\alpha + c^2} - c\cos\alpha}{\sin^2\alpha\sqrt{k_s^2\sin^2\alpha + c^2}}\, \Phi(k_s,\alpha)\, \mathcal{L}(k_s,\alpha) \right.$$

$$\left. - i\int_0^{\operatorname{arsinh}\frac{c}{k_s}} da\, \frac{\sqrt{c^2 - k_s^2\sinh^2 a} + c\cosh a}{\sinh^2 a\sqrt{c^2 - k_s^2\sinh^2 a}}\, \Phi(k_s,\pi+ia)\, \mathcal{L}(k_s,\pi+ia) \right.$$

$$-i\int_{\operatorname{arsinh}\frac{c}{k_s}}^{\infty} da\, \frac{\sqrt{k_s^2\sinh^2 a - c^2}+ic\cosh a}{\sinh^2 a\sqrt{k_s^2\sinh^2 a - c^2}}\, \Phi(k_s,\pi+ia)\, \mathcal{L}^+(k_s,\pi+ia)$$

$$-i\int_{\infty}^{\operatorname{arsinh} c/k_s} da\, \frac{\sqrt{k_s^2\sinh^2 a - c^2}+ic\cosh a}{\sinh^2 a\sqrt{k_s^2\sinh^2 a - c^2}}\, \Phi(k_s,ia)\, \mathcal{L}^-(k_s,ia)$$

$$-i\int_{\operatorname{arsinh} c/k_s}^{0} da\, \frac{\sqrt{c^2-k_s^2\sinh^2 a}-c\cosh a}{\sinh^2 a\sqrt{c^2-k_s^2\sinh^2 a}}\, \Phi(k_s,ia)\, \mathcal{L}(k_s,ia) \qquad \text{(A3.3)}$$

where

$$\left[\mathcal{L}(k_s,\alpha)\right]^{-1} = p^2 + 2p_s k_s\cos\omega + \frac{c^2\cos^2\alpha + c^2 - 2c\cos\alpha\sqrt{k_s^2\sin^2\alpha + c^2}}{\sin^2\alpha}$$

$$+2p_0\frac{c\cos\alpha - \sqrt{k_s^2\sin^2\alpha + c^2}}{\sin\alpha} + i\epsilon$$

$$\left[\mathcal{L}^{\pm}(k_s,\alpha)\right]^{-1} = p^2 + 2p_s k_s\cos\omega + \frac{c^2\cos^2\alpha + c^2 \pm 2ic\cos\alpha\sqrt{-c^2-k_s^2\sin^2\alpha}}{\sin^2\alpha}$$

$$+2p_0\frac{c\cos\alpha \pm i\sqrt{-c^2-k_s^2\sin^2\alpha}}{\sin\alpha}$$

Now in the third and fourth integrals $\mathcal{L}^+(k_s,\pi+ia) = \mathcal{L}^-(k_s, ia)$ and as we are considering the paths C D and E F, $\Phi(k_s, \pi+ia)$ is calculated at the same point in the k_0 plane as $\Phi(k_s, ia)$. Hence the sum of the two integrals is zero.

In the second and fifth integrals we first interchange the order of integration to give

$$-i\int_{0}^{\infty} da \int_{0}^{\frac{c}{\sinh a}} k_s^2\, dk_s\, \frac{\sqrt{c^2-k_s^2\sinh^2 a}+c\cosh a}{\sinh^2 a\sqrt{c^2-k_s^2\sinh^2 a}}\, \Phi(k_s,\pi+ia)\, \mathcal{L}(k_s,\pi+ia)$$

$$+ i\int_0^\infty da \int_0^{\frac{c}{\sinh a}} k_s^2\, dk_s \frac{\sqrt{c^2 - k_s^2 \sinh^2 a} - c\cosh a}{\sinh^2 a \sqrt{c^2 - k_s^2\sinh^2 a}}\, \Phi(k_s, ia)\, \ell(k_s, ia)$$

Now put $k_s = \dfrac{c \sin\beta}{\cosh a - \cos\beta}$ noting that in order to give the desired expression for k_o, along the path B C we must have $0 \leq \beta \leq \arccos \dfrac{1}{\cosh a}$ while along the path F 0 we must have $\arccos \dfrac{1}{\cosh a} \leq \beta \leq \pi$. Then the two integrals above give

$$-ic^3\int_0^\infty da \int_0^{\arccos\frac{1}{\cosh a}} d\beta \frac{\sin^2\beta}{(\cosh a - \cos\beta)^4}\, \Phi(\beta, \pi + ia)\, \ell(\beta, a)$$

$$-ic^3\int_0^\infty da \int_{\arccos\frac{1}{\cosh a}}^{\pi} d\beta \frac{\sin^2\beta}{(\cosh a - \cos\beta)^4}\, \Phi(\beta, ia)\, \ell(\beta, a)$$

where in both cases

$$[\ell(\beta, a)]^{-1} = \mu^2 + \frac{2\mu_s c \sin\beta\cos\omega}{\cosh a - \cos\beta} - \frac{c^2(\cosh a + \cos\beta)}{\cosh a - \cos\beta} - \frac{2i\mu_o c \sinh a}{\cosh a - \cos\beta}$$

Noting from fig (2) that in the k_o plane the paths B' C' and F' O' are adjacent we may regard $\Phi(\beta, ia)$ as the continuation of $\Phi(\beta, \pi + ia)$ for $\beta >$ arccos $\dfrac{1}{\cosh a}$ and simply denote it by $\Phi(\beta, a)$. Then putting $k_s = \dfrac{c \sin\beta}{\cos\alpha - \cos\beta}$ in the first integral of (A3.3) we finally have

$$0 = c^3\int_{-\pi}^{\pi} d\alpha \int_{|\alpha|}^{\pi} d\beta \frac{\sin^2\beta}{(\cos\alpha - \cos\beta)^4} \frac{\Phi(\beta,\alpha)}{\left(\mu^2 + \frac{2\mu_s c\sin\beta\cos\omega - 2\mu_o c\sin\alpha - c^2(\cos\alpha + \cos\beta)}{\cos\alpha - \cos\beta} + i\epsilon\right)}$$

$$-ic^3\int_{-\infty}^{\infty} da \int_0^{\pi} d\beta \frac{\sin^2\beta}{(\cosh a - \cos\beta)^4} \frac{\Phi(\beta, a)}{\left(\mu^2 + \frac{2\mu_s c\sin\beta\cos\omega - 2i\mu_o c\sinh a - c^2(\cosh a + \cos\beta)}{\cosh a - \cos\beta}\right)}$$

The integral equation is now

$$\Psi(p) = -\frac{\lambda c^3}{\pi^2}\int \sin\theta\, d\theta\, d\varphi \int_{-\infty}^{\infty} da \int_0^{\pi} d\beta\ \frac{\sin^2\beta}{(\cosh a - \cos\beta)^3} \times$$

$$\frac{\Phi(\beta, a)}{p^2(\cosh a - \cos\beta) + 2p_s c \sin\beta\cos\omega - 2ip_0 c \sinh a - c^2(\cosh a + \cos\beta)}$$

and rotating p_0 to the imaginary axis, that is,

putting $p_0 = \frac{i\ c\ \sinh a'}{\cosh a' - \cos\beta'}$, $p_s = \frac{c\ \sin\beta'}{\cosh a' - \cos\beta'}$ we get

$$p_s \Psi(\beta', a') = \frac{c^2\lambda \sin\beta'}{8\pi^2}\int \sin\theta\, d\theta\, d\varphi \int_{-\infty}^{\infty} da \int_0^{\pi} d\beta\ \frac{\sin\beta \quad k_s \Phi(\beta, a)}{(\cosh a - \cos\beta)^2\left[\cosh(a-a') - \cos\Omega\right)}$$

with $\Psi = \frac{4\ c^2\ (M^2\ \cosh^2 a - E^2)}{(\cosh a - \cos\beta\ )^2}\ \Phi$

$$p_s \Psi(\beta', a') = \frac{\lambda \sin\beta'}{8\pi^2}\int \sin\theta\, d\theta\, d\varphi \int_{-\infty}^{\infty} da \int_0^{\pi} d\beta\ \frac{\sin\beta \quad k_s \Psi(\beta, a)}{(M^2\cosh^2 a - E^2)(\cosh(a-a') - \cos\Omega)}$$

or putting $H(a, \beta, \theta, \varphi) = \frac{p_s\ \Psi(\beta, a)}{\sin\beta}$

$$H(a', \beta', \theta', \varphi')$$

$$= \frac{\lambda}{8\pi^2}\int_{-\infty}^{\infty} da \int_0^{\pi} \sin^2\beta\, d\beta \int \sin\theta\ \frac{d\theta\, d\varphi \quad H(a, \beta, \theta, \varphi)}{(M^2\cosh^2 a - E^2)(\cosh(a-a') - \cos\Omega)}$$

Chapter 3 - Appendix B.

THE RECURRENCE RELATION.

We have the recurrence relation (3.28)

$$M_o a_1 + L_o a_o = 0 \tag{B3.1}$$

$$M_\nu a_{\nu+1} + L_\nu a_\nu + K a_{\nu-1} = 0 \tag{B3.2}$$

The second equation considered by itself, is a second order difference equation and in general has two independent solutions. The theorem of Perron (9) goes beyond that of Poincare in stating that there exists a solution a_ν^1 such that $\lim\limits_{\nu\to\infty} \dfrac{a^1_{\nu+1}}{a^1_\nu} = 1$ and there exists a second solution a_ν^2 such that $\lim\limits_{\nu\to\infty} \dfrac{a^2_{\nu+1}}{a^2_\nu} = e$. Now consider the quantity

$$X_n^\nu = \frac{a^1_{\nu+n}\, a^2_\nu - a^2_{\nu+n}\, a^1_\nu}{a^1_{\nu+n}\, a^2_{\nu-1} - a^2_{\nu+n}\, a^1_{\nu-1}}$$

Use of (B3.2) shows that

$$X_n^\nu = \frac{-K_\nu/M_\nu}{\frac{L_\nu}{M_\nu} + X^{\nu+1}_{n-1}}$$

$$= \cfrac{-K_\nu/M_\nu}{\frac{L_\nu}{M_\nu} - \cfrac{K_{\nu+1}/M_{\nu+1}}{\frac{L_{\nu+1}}{M_{\nu+1}} + X^{\nu+2}_{n-2}}}$$

The fraction may be extended until $X_o^{\nu+n}$ is reached. But $X_o^{\nu+n} = 0$.

Thus

$$X_n^\nu = \cfrac{-K_\nu/M_\nu}{\frac{L_\nu}{M_\nu} - \cfrac{+K_{\nu+1}/M_{\nu+1}}{\frac{L_{\nu+1}}{M_{\nu+1}} - \cdots \cdots \cfrac{}{\frac{K_{\nu+n-1}}{L_{\nu+n-1}}}}}$$

Now we have

$$\lim_{\nu\to\infty} \frac{a^2_{\nu+1}}{a^2_{\nu}} \frac{a^1_{\nu}}{a^1_{\nu+1}} = c$$

$$\text{or} \quad \lim_{\nu\to\infty} \frac{a^2_{\nu+1}}{a^1_{\nu+1}} \Bigg/ \frac{a^2_{\nu}}{a^1_{\nu}} = c$$

$$\text{Hence} \quad \lim_{\nu\to\infty} \frac{a^2_{\nu}}{a^1_{\nu}} = 0$$

$$\text{Thus} \quad \lim_{n\to\infty} x^{\nu}_n = \lim_{n\to\infty} \frac{a^2_{\nu} - \dfrac{a^2_{\nu+n}\, a^1_{\nu}}{a^1_{\nu+n}}}{a^2_{\nu-1} - \dfrac{a^2_{\nu+n}}{a^1_{\nu+n}}\, a^1_{\nu-1}} = \frac{a^2_{\nu}}{a^2_{\nu-1}}$$

and in particular for $\nu = 1$

$$\frac{a^2_1}{a^2_0} = \cfrac{-K_1/M_1}{\cfrac{L_1}{M_1} - \cfrac{K_2/M_2}{\cfrac{L_2}{M_2} - \cdots}}$$

But for our solution $\dfrac{a_1}{a_0} = -\dfrac{L_0}{M_0}$ from (B3.1). Hence if our solution

is to be the second solution, with $\lim_{\nu\to\infty} \dfrac{a_{\nu+1}}{a_{\nu}} = c$, we must have

$$\frac{L_0}{M_0} = \cfrac{K_1/M_1}{\cfrac{L_1}{M_1} - \cfrac{K_2/M_2}{\cfrac{L_2}{M_2} - \cdots}}$$

Chapter 4.

CALCULATION OF EIGENVALUES.

In the calculation of eigenvalues the simplest case to consider is $e = 0$. The continued fraction (3.30) may in general be rewritten

$$\frac{L_\nu}{M_\nu} = \cfrac{K_\nu/M_\nu}{\frac{L_{\nu-1}}{M_{\nu-1}} - \cfrac{K_{\nu-1}/M_{\nu-1}}{\frac{L_{\nu-2}}{M_{\nu-2}} - \cfrac{K_{\nu-2}/M_{\nu-2}}{\ddots \; \overline{\;L_0/M_0\;}}}} + \cfrac{K_{\nu+1}/M_{\nu+1}}{\frac{L_{\nu+1}}{M_{\nu+1}} - \cfrac{K_{\nu+2}/M_{\nu+2}}{\ddots}} \tag{4.1}$$

Then if $e = 0$, $K_\nu = 0$ and so $L_\nu = 0$, that is

$$\frac{\lambda}{4} = \left(\nu + \frac{n}{2}\right)\left(\nu + \frac{n+1}{2}\right) \qquad \nu = 0, 1, \cdots$$

The corresponding relation derived from the second continued fraction, with L'_ν and K'_ν, is

$$\frac{\lambda}{4} = \left(\nu + \frac{n+1}{2}\right)\left(\nu + \frac{n+2}{2}\right) \qquad \nu = 0, 1, \cdots$$

Hence the full set of eigenvalues for $e = 0$ is

$$\lambda_k = (k + n)\ (k + n + 1) \tag{4.2}$$

where k is the quantum number introduced in the last chapter. For values of e near zero we may put $\dfrac{\lambda}{4} = \displaystyle\sum_{m=0}^{\infty} x_m e^m$ and, expanding the continued fractions in (4.1) up to any power of e, x_m can be found for any value of m. For example if $K_\nu = e\,J_{\nu-1}$ defines $J_{\nu-1}$, then

$$x_o = J_\nu$$

$$x_1 = J_\nu \frac{M_{\nu-1}}{J_\nu - J_{\nu-1}} + \frac{M_\nu}{J_\nu - J_{\nu+1}}$$

Unfortunately this region of small e is not very interesting. Such enormous values of the coupling constant would make nonsense of the ladder approximation. The more important values of e are those close to one, which correspond to small binding energy, for if B is the binding energy, then

$$\begin{aligned} B &= 2(M - E) \\ &= 2M\left(1 - \frac{E}{M}\right) \\ &\simeq M(1 - e) \qquad \text{for } \frac{E}{M} \simeq 1. \end{aligned} \tag{4.3}$$

It might be thought that as the differential equation (3.21) shows a degree of symmetry between the points 0 and 1, the limit $e \to 1$ should be equally easy to deal with as the limit $e \to 0$. For example, the transformation $z = 1 - x$ gives the same differential equation except that e is replaced by $1 - e$. However no solution in power series of x^{-1} is possible for the boundary conditions would have to be applied at $x = \infty$ and $x = 0$ and no such power series could be convergent at both points. There are 24 homographic substitutions of the type $x = \frac{Az + B}{Cz + D}$ which lead to a Heun's equation with singularities at $x = 0, 1, \infty$ and a fourth point, but detailed consideration shows that in every case, either the two singularities where the boundary conditions are applied are not adjacent to one another, or else the continued fraction corresponding to (4.1) does not simplify for $e \to 1$. Clearly the boundary conditions destroy the apparent symmetry between the points 0 and 1.

Erdelyi (8) has suggested expanding $S(z)$ in terms of hypergeometric functions instead of in a power series, and thus taking into account the behaviour at some of the singularities more directly. For example a convenient series in our case would be

$$S(z) = z^{-\frac{n}{2}} \sum_{0}^{\infty} c_\nu (-)^\nu \frac{\Gamma(\nu+n+\frac{1}{2})}{\Gamma(\nu+1)} F(\nu+n+\tfrac{1}{2}, -\nu; n+1; z^{-1}) \tag{4.4}$$

and the hypergeometric function can be transformed to give

$$S(z) = z^{-\frac{n}{2}} \sum_{0}^{\infty} c_\nu (-)^\nu \frac{\Gamma(\nu+n+\frac{1}{2})\,\Gamma(n+1)\,\Gamma(\frac{1}{2})}{\Gamma(\nu+1)\,\Gamma(\frac{1}{2}-\nu)\,\Gamma(n+\nu+1)} F(\nu+n+\tfrac{1}{2}, -\nu; \tfrac{1}{2}; \tfrac{z-1}{z})$$

so it can be seen immediately that $S(z)$ has the correct behaviour both at $z = 1$ and $z = \infty$, provided the series are convergent at those points. Substituting (4.4) into the differential equation leads to a three term recurrence relation, similar to the previous one, but with more complicated coefficients, and the convergence condition leads to another continued fraction. Once again, the limit $e \to 0$ is easy to deal with and of course leads to the same eigenvalues as before. Also as before, no homographic transformation or use of hypergeometric functions different from those in (4.4) leads to simple results for the limit $e \to 1$.

In order to get eigenvalues of λ for e near one a method very similar to Wick's may be used. The differential equation for $S(z)$ is

$$z(z-1)(z-e)\frac{d^2S}{dz^2} + (z-\tfrac{1}{2})(z-e)\frac{dS}{dz} - \frac{n^2}{4}(z-e)S + \frac{\lambda}{4}S = 0$$

If $e \simeq 1$ then for large values of z this equation may be replaced by

$$z(z-1)^2\frac{d^2S}{dz^2} + (z-\tfrac{1}{2})(z-1)\frac{dS}{dz} - \frac{n^2}{4}(z-1)S + \frac{\lambda}{4}S = 0$$

which reduces to a hypergeometric equation. The solution with the correct boundary condition at $z = \infty$ is

$$S_\ell(z) = (z-1)^{\frac{1}{4}+\frac{1}{2}\rho}\, z^{-\frac{n+\rho}{2}-\frac{1}{4}}\, F\left(\tfrac{3}{4}+\tfrac{n+\rho}{2},\ \tfrac{1}{4}+\tfrac{n+\rho}{2};\ n+1;\ z^{-1}\right)$$

where $\rho^2 = \frac{1}{4} - \lambda$

For values of z near $z = 1$, first put

$$\frac{z - e}{1 - e} = \ell$$

so that the differential equation becomes

$$[\ell(1-e)+e]\,\ell(\ell-1)\frac{d^2S}{d\ell^2} + \tfrac{1}{2}(\ell-1)\ell(1-e)\frac{dS}{d\ell} + \tfrac{1}{2}(\ell(1-e)+e)\frac{dS}{d\ell} - \frac{n^2}{4}\ell(1-e)S + \frac{\lambda}{4}S = 0$$

and then neglect terms containing $1 - e$, giving

$$\ell(\ell-1)\frac{d^2S}{d\ell^2} + \tfrac{1}{2}\ell\frac{dS}{d\ell} + \frac{\lambda}{4}S = 0$$

The solution to this equation which has the correct boundary condition at $z = 1$, that is, at $\ell = 1$, is

$$S_s = \ell\, F\left(\tfrac{3}{4}+\tfrac{\ell}{2},\ \tfrac{3}{4}-\tfrac{\ell}{2};\ \tfrac{1}{2};\ 1-\ell\right)$$

The continuation formulae for the hypergeometric function show that for values of z near $z = 1$ (neglecting multiplicative constants)

$$S_\ell = \frac{\Gamma(-\rho)\, z^{\frac{1}{2}}\, (z-1)^{\frac{1}{4}+\frac{1}{2}\rho}}{\Gamma\left(\frac{1+2n-2\rho}{4}\right)\Gamma\left(\frac{3+2n-2\rho}{4}\right)}\, F\left(\frac{3+2n+2\rho}{4},\ \frac{3-2n+2\rho}{4};\ 1+\rho;\ 1-z\right)$$
$$+ \frac{\Gamma(\rho)\, (z-1)^{\frac{1}{4}-\frac{1}{2}\rho}}{\Gamma\left(\frac{1+2n+2\rho}{4}\right)\Gamma\left(\frac{3+2n+2\rho}{4}\right)}\, F\left(\frac{1+2n-2\rho}{4},\ \frac{1-2n-2\rho}{4};\ 1-\rho;\ 1-z\right)$$

which for $z - 1$ very small implies,

$$S_\ell \simeq (z-1)^{\frac{1}{4}}\left\{\frac{\Gamma(-\rho)(z-1)^{\frac{1}{2}\rho}}{\Gamma\left(\frac{2n-2\rho+1}{4}\right)\Gamma\left(\frac{2n-2\rho+3}{4}\right)} + \frac{\Gamma(\rho)\,(z-1)^{-\frac{1}{2}\rho}}{\Gamma\left(\frac{2n+2\rho+3}{4}\right)\Gamma\left(\frac{2n+2\rho+1}{4}\right)}\right\}$$

Similarly for large ℓ

$$S_s = \frac{\Gamma(-\rho)\, \ell^{\frac{1}{4}-\frac{1}{2}\rho}}{\Gamma(\frac{3}{4}-\frac{1}{2}\rho)\Gamma(-\frac{1}{4}-\frac{1}{2}\rho)} F(\tfrac{3}{4}+\tfrac{1}{2}\rho, -\tfrac{1}{4}+\tfrac{1}{2}\rho ; 1+\rho ; \ell^{-1})$$

$$+ \frac{\Gamma(\rho)\, \ell^{\frac{1}{4}+\frac{1}{2}\rho}}{\Gamma(\frac{3}{4}+\frac{1}{2}\rho)\Gamma(-\frac{1}{4}+\frac{1}{2}\rho)} F(\tfrac{3}{4}-\tfrac{1}{2}\rho, -\tfrac{1}{4}-\tfrac{1}{2}\rho ; 1-\rho ; \ell^{-1})$$

and since $\ell = \frac{z - e}{1 - e} = \frac{z - 1}{1 - e} + 1 \simeq \frac{z - 1}{1 - e}$ for large ℓ

$$S_s \simeq (z-1)^{\frac{1}{4}} \left\{ \frac{\Gamma(-\rho)}{\Gamma(\frac{3}{4}-\frac{1}{2}\rho)\Gamma(-\frac{1}{4}-\frac{1}{2}\rho)} \left(\frac{z-1}{1-e}\right)^{-\frac{1}{2}\rho} + \frac{\Gamma(\rho)}{\Gamma(\frac{3}{4}+\frac{1}{2}\rho)\Gamma(-\frac{1}{4}+\frac{1}{2}\rho)} \left(\frac{z-1}{1-e}\right)^{\frac{1}{2}\rho} \right\}$$

Now we want to be able to join smoothly these two solutions for some value of z such as $z = 1 + \sqrt{1 - e}$ where $z - 1$ is very small but $\frac{z - e}{1 - e}$ is very large. First suppose $\lambda < \frac{1}{4}$. Then $\rho = \sqrt{\frac{1}{4} - \lambda}$ is real and $0 < \rho < \frac{1}{2}$ and the dominant part of S_ℓ is

$$S_\ell \simeq (z - 1)^{\frac{1}{4} - \frac{1}{2}\rho}$$ since $z - 1$ is small. However $\left(\frac{z - 1}{1 - e}\right)$ is large so the dominant part of S_s is

$$S_s \simeq (z - 1)^{\frac{1}{4} + \frac{1}{2}\rho}$$

Thus S_ℓ and S_s cannot join smoothly. This is true for λ not too close to zero, but if λ is very small, then $\rho \simeq \frac{1}{2}$ and the term $\Gamma(-\frac{1}{4} + \frac{1}{2}\rho)$ in S_s becomes very large. Hence there is reason to think that $\lambda \simeq 0$ may be an eigenvalue for $e \simeq 1$ but this method gives no way of finding the exact relation between λ and e.

Next suppose $\lambda > \frac{1}{4}$ so that ρ is imaginary, say $\rho = i\sigma$.

Then
$$S_\ell \simeq (z - 1)^{\frac{1}{4}} \cos\left[\tfrac{1}{2}\sigma \log (z - 1) + \theta\right]$$

where

$$\frac{\Gamma(-i\sigma)}{\Gamma(\frac{2n+1}{4} - \frac{i\sigma}{2})\Gamma(\frac{2n+3}{4} - \frac{i\sigma}{2})} = r_1 e^{i\theta}$$

and

$$S_s = (z - 1)^{\frac{1}{4}} \cos\left[\tfrac{1}{2}\sigma \log (z - 1) - \tfrac{1}{2}\sigma \log (1 - e) + \varphi\right]$$

where

$$\frac{\Gamma(i\sigma)}{\Gamma(\frac{3}{4}+\frac{i\sigma}{2})\Gamma(-\frac{1}{4}+\frac{i\sigma}{2})} = r_2 e^{i\varphi}$$

Hence S_ℓ and S_s can join smoothly provided that

$$\tfrac{1}{2}\sigma \log(1-e) = m\pi + \varphi - \theta \tag{4.5}$$

the same result as was found in Chapter 2.

The function $T(z)$ can be treated in exactly the same way. The part $T_\ell(z)$ for large values of z will be the same as $S_\ell(z)$ but the part $T_s(z)$ will have a different form as it satisfies a different boundary condition. We finally get

$$T_s(z) = (z-1)^{\frac{1}{4}}\left\{\frac{\Gamma(\rho)}{\Gamma(\frac{5}{4}+\frac{1}{2}\rho)\Gamma(\frac{1}{4}+\frac{1}{2}\rho)}\left(\frac{z-1}{1-e}\right)^{\frac{1}{2}\rho} + \frac{\Gamma(-\rho)}{\Gamma(\frac{5}{4}-\frac{1}{2}\rho)\Gamma(\frac{1}{4}-\frac{1}{2}\rho)}\left(\frac{z-1}{1-e}\right)^{-\frac{1}{2}\rho}\right\}$$

If ρ is real

$T_s(z) \simeq (z-1)^{\frac{1}{4}+\frac{1}{2}\rho}$ so the functions T_s and T_ℓ cannot join smoothly and in the present case this remains true even if $\rho \simeq \frac{1}{2}$.

If $\rho = i\sigma$ is imaginery, then

$$T_s = (z-1)^{\frac{1}{4}} \cos\left[\tfrac{1}{2}\sigma\log(z-1) - \tfrac{1}{2}\sigma\log(1-e) + \varphi'\right]$$

where

$$\frac{\Gamma(i\sigma)}{\Gamma(\frac{5}{4}+\frac{i\sigma}{2})\Gamma(\frac{1}{4}+\frac{i\sigma}{2})} = r_2' e^{i\varphi'}$$

and the eigenvalue condition is

$$\tfrac{1}{2}\sigma \log(1-e) = m\pi + \varphi' - \theta \tag{4.6}$$

The case $\lambda \to 0$ as $e \to 1$ remains to be considered. The simplest procedure seems to be to transform to the equation (3.16) by means of the relation

$z = \dfrac{1}{1 - q^2}$ which gives for the differential equation

$$(1 - q^2)\, f''(q) - 2\, q\, f'(q) - \frac{n^2 f(q)}{1 - q^2} + \frac{f(q)}{1 - e + e\, q^2} = 0$$

As pointed out in Chapter 3 if $f(q) = (1 - q^2)^{-\frac{n}{2}}\, g_n(q)$ then $g_n(q)$ is Wick's function g_n. Hence we can use Wick's procedure of replacing

$$\frac{1}{1 - e + eq^2} \quad \text{by} \quad \frac{\pi}{\sqrt{1-e}}\, \delta(q)$$

This replacement can be justified either by noting that

$$\frac{1}{1-e+eq^2} = \frac{1}{2i\sqrt{1-e}}\left[\frac{1}{\sqrt{e}\,q - i\sqrt{1-e}} - \frac{1}{\sqrt{e}\,q + i\sqrt{1-e}}\right]$$

$$= \frac{1}{2i\sqrt{1-e}}\left[\frac{1}{\sqrt{e}\,q} + \pi i\, \delta(\sqrt{e}\,q) - \frac{1}{\sqrt{e}\,q} + \pi i\, \delta(\sqrt{e}\,q)\right] \quad \text{for very small } 1 - e$$

$$= \frac{\pi\, \delta(q)}{\sqrt{e}\,\sqrt{1-e}}$$

or by noting that the Fourier transform of $\dfrac{1}{1 - e + eq^2}$ is $\dfrac{\pi}{\sqrt{e}\,\sqrt{1-e}}\, e^{-\sqrt{\frac{1-e}{e}}\,|k|}$ which is approximated by $\dfrac{\pi}{\sqrt{e}\,\sqrt{1 - e}}$ and the transform inverted to give $\dfrac{\pi\, \delta(q)}{\sqrt{e}\,\sqrt{1 - e}}$. Substituting this into the integral equation gives

$$f(q) = \frac{\lambda \pi}{2n\sqrt{e}\,\sqrt{1-e}}\, f(0) \left(\frac{1 \mp q}{1 \pm q}\right)^{\frac{n}{2}} \qquad \text{according as } q \gtrless 0$$

Putting $q = 0$ gives

$$\frac{\lambda\pi}{2n\,\sqrt{e}\,\sqrt{1 - e}} = 1 \quad \text{or}$$

$$\lambda = \frac{2n}{\pi}\sqrt{e}\,\sqrt{1 - e} \qquad (4.7)$$

It does not seem to be easy to improve this approximation.

This last relation (4.7) can be written

$$B = \frac{\lambda^2 \pi^2}{4n^2}$$ where B is the binding energy and as is well known this is just the non-relativistic Balmer formula for the energy of a bound state. Solutions which have this property, that is, that as $\lambda \to 0$ the Balmer formula is obtained, are called normal solutions, while those for which $\lambda \to \frac{1}{4}$ as $e \to 1$ are called abnormal solutions and have no analogue in non-relativistic quantum mechanics. It was pointed out by Wick, and will be verified presently, that for the normal solutions the quantum number k is zero, while for the abnormal solutions $k > 0$.

Numerical Calculations.

In order to verify the limiting cases just considered and to obtain eigenvalues for intermediate values of e, a series of numerical calculations were undertaken on an I.B.M. 7090 computer. The continued fraction was not used but instead the equivalent procedure of successively calculating the coefficients of the power series by means of the recurrence relation. Values of λ and e were selected and some large value of ν, then putting $a_\nu = 1$ and $a_{\nu+1} = e$, $a_{\nu-1}$ could be found from

$$a_{\nu-1} = -\frac{1}{K_\nu}\left(M_\nu a_{\nu+1} + L_\nu a_\nu\right).$$

Given a_ν and $a_{\nu-1}$, $a_{\nu-2}$ could be found and so on until a_1 and a_o were determined. If the quantity

$$B = M_o a_1 + L_o a_o$$

was zero, λ was an eigenvalue. If not either λ or e, say λ, was varied in successive steps until B changed sign. Finally some new value of e was selected and the whole process repeated.

The critical point in the process is the choice of the initial value of ν which should be as small as possible in order to reduce the time required for the calculations. For sufficiently large values of ν, there is no doubt that

$$\frac{a_{\nu+1}}{a_{\nu}} = e$$

and we already know that

$$\frac{a_{\nu+1}}{a_{\nu}} = e\left(1 - \frac{2}{\nu}\right)$$

is a better approximation. A method of further improvement is suggested by Milne-Thompson (9), Chapter 14, that of expanding a in factorial series i.e. put

$$a_{\nu} = \frac{e^{\nu}}{(\nu+\frac{m}{2})(\nu+\frac{m}{2}+1)}\left\{ g_0 + \frac{g_1}{\nu+\frac{m}{2}+2} + \frac{g_2}{(\nu+\frac{m}{2}+2)(\nu+\frac{m}{2}+3)} + \frac{g_3}{(\nu+\frac{m}{2}+2)(\nu+\frac{m}{2}+3)(\nu+\frac{m}{2}+4)} + \cdots \right\} \tag{4.8}$$

Then it can be shown that the coefficients g_s satisfy the recurrence relation

$$(1 - e)\, g_1 + g_0\left[\frac{\lambda}{4} + \tfrac{1}{2}(2e - 1)\right] = 0$$

$$s(1-e)g_s + g_{s-1}\left(\frac{\lambda}{4} + s(s-\tfrac{1}{2}) - (1-e)s(2s-1)\right] + es\left(\frac{m}{2}+s-1\right)\left(\frac{m}{2}-s+1\right)g_{s-2} = 0$$

In its turn g_s can be expanded in the form

$$g_s = g_0 \left\{ \frac{A_s}{(1 - e)^s} + \frac{B_s}{(1 - e)^{s-1}} + \cdots \right\}$$

and A_s can be evaluated to give

$$A_s = \frac{(-)^s}{\lfloor s}\left(\frac{3}{4} - \frac{\sigma}{2}\right)\left(\frac{3}{4} + \frac{\sigma}{2}\right)\left(\frac{7}{4} - \frac{\sigma}{2}\right)\left(\frac{7}{4} + \frac{\sigma}{2}\right) \cdots \left(s - \frac{1}{4} - \frac{\sigma}{2}\right)\left(s - \frac{1}{4} + \frac{\sigma}{2}\right)$$

Thus it would appear that for small values of $1 - e$ the dominant part of a_ν is

$$a_\nu = \frac{e^\nu g_0}{(\nu+\frac{n}{2})(\nu+\frac{n}{2}+1)}\left[1 - \frac{(\frac{3}{4}-\frac{\xi}{2})(\frac{3}{4}+\frac{\xi}{2})}{(\nu+\frac{n}{2}+2)(1-e)} + \frac{(\frac{3}{4}-\frac{\xi}{2})(\frac{7}{4}-\frac{\xi}{2})(\frac{3}{4}+\frac{\xi}{2})(\frac{7}{4}+\frac{\xi}{2})}{(\nu+\frac{n}{2}+2)(\nu+\frac{n}{2}+3)\,\underline{|2}\,(1-e)^2} + \cdots\right]$$

where the bracketed part is a hypergeometric series

$F(\frac{3}{4}-\frac{\xi}{2}, \frac{3}{4}+\frac{\xi}{2}; \nu+\frac{n}{2}+2; \frac{-1}{1-e})$. This series could be rewritten as the sum of two series $F(\ , \ ; \ ; 1 - e)$ and hence an approximation for a_ν obtained for small values of $1 - e$. However, the term contain B_s cannot be neglected. It is impossible to evaluate B_s exactly but approximately

$$B_s \simeq \frac{A_s}{\underline{|s-1}}$$

When $s(1 - e) > 1$, $\dfrac{B_s}{(1-e)^{s-1}} > \dfrac{A_s}{(1-e)^s}$. More careful considerations suggest that the series (4.8) diverges.

There is a different and more lengthy procedure for getting a series of this general type for a_ν and the result is

$$a_\nu = \frac{e^\nu}{\nu(\nu+n)}\left[g_0 + \frac{g_1}{\theta(\nu+\frac{n}{2})+2} + \frac{g_2}{(\theta(\nu+\frac{n}{2})+2)(\theta(\nu+\frac{n}{2})+3)} + \cdots\right]$$

where $0 < \theta < (1 - e)$ but it is of little use as no general form can be found for the g_s.

Thus we shall simply use the previous results that the recurrence relation has two solutions a^1_ν and a^2_ν where

$$\frac{a^1_{\nu+1}}{a^1_\nu} \simeq 1 - \frac{3/2}{\nu} \qquad \text{or} \qquad a^1_\nu \simeq A_1 \nu^{-3/2}$$

$$\frac{a^2_{\nu+1}}{a^2_\nu} \simeq e(1 - \frac{2}{\nu}) \qquad \text{or} \qquad a^2_\nu = A_2 e^\nu \nu^{-2}$$

The assumption that $\dfrac{a_{\nu+1}}{a_\nu} = e$ for some large but finite value of ν, implies that a_ν is a linear combination of a_ν^1 and a_ν^2 i.e.

$$a_\nu = pa_\nu^1 + qa_\nu^2$$

$$\therefore \quad e = \frac{a_{\nu+1}}{a_\nu} = \frac{pa_{\nu+1}^1 + qa_{\nu+1}^2}{pa_\nu^1 + qa_\nu^2}$$

Using the above expressions for a_ν^1 and a_ν^2 gives

$$p = \frac{2q\, e^{\nu+1}}{\frac{A_1}{A_2}\, \nu^t \left[(1-e)\nu - 3/2 \right]}$$

Hence the condition that p should be negligible is $(1 - e)\nu > 3/2$.

In practice, values of ν were selected so that $(1 - e)\nu > 6$.

The first program run was no. 18 for which a flow sheet and the FORTRAN statements are given at the end of this thesis. The recurrence relation (3.28) was modified by introducing $d_\nu = e^{-\nu} a_\nu$ in order to have numbers which would be more nearly constant. Values of e from 0.1 to 0.82 in steps of 0.06 were selected and the initial value of ν was 100 throughout. For each value of e, λ was increased in steps of 0.5 from 0.001 to 18.001 and the quantity

$$B = d_0 \left[\frac{\lambda}{4} - \frac{n}{2} \left(\frac{n+1}{2} \right) \right] + e d_1 (n+1)$$

calculated. A change of sign of B indicated that an eigenvalue of λ had been passed so λ was then reduced by 0.5 to the previous value and increased in steps of 0.1 in order to find a more accurate eigenvalue. Only the first recurrence relation, derived from S(z), was used.

The second program, no. 21, extended the range of e from 0.82 up to 0.997 and also used the second recurrence relation derived from T(z), to get a second set of eigenvalues corresponding to odd values of the quantum number k.

A third and a fourth program, which are not listed here, were used to take e from $1.0 - 4 \times 10^{-3}$ up to $1.0 - 4 \times 10^{-4}$. However, for these small values of $1 - e$, the initial value of ν had to be 15,000, which implies some 500,000 computations in going from d_ν down to d_0. As the I.B.M. 7090 computer operates with about eight decimal digits and always rounds off numbers in the same direction, it is not surprising that the eigenvalues appeared to be in error when $1 - e$ was less than 10^{-3}.

Hence a fifth program was written - no. 180 for which a flow sheet and the FORTRAN statements are given also at the back of this thesis. Program 180 used the double precision facility of the computer. As this facility operates relatively slowly only a few isolated eigenvalues could be calculated, for values of $1 - e$ down to 2.8×10^{-4}. The recurrence relation was further modified by introducing

$$f_\nu = (\nu + n)\, d_\nu ,$$

and the program made as flexible as possible in order to permit calculations for either recurrence relation, any value of n, any value of e, and any eigenvalue.

The precise numerical values obtained for the eigenvalues are of no great significance so the results are not tabulated; instead they are presented in the following graphs. In the first, figure 1, λ is plotted against e over the full range of e for various values of n. Comparison of this graph with Cutkowsky's (11) figure 1 indicates full agreement. From this graph it is obvious that if $k = 0$, $\lambda \to 0$ as $e \to 1$.

The succeeding graphs show the region $e \simeq 1$ in more detail and compare the results obtained in the calculations with the approximate eigenvalues given earlier in this chapter. For the normal solutions we

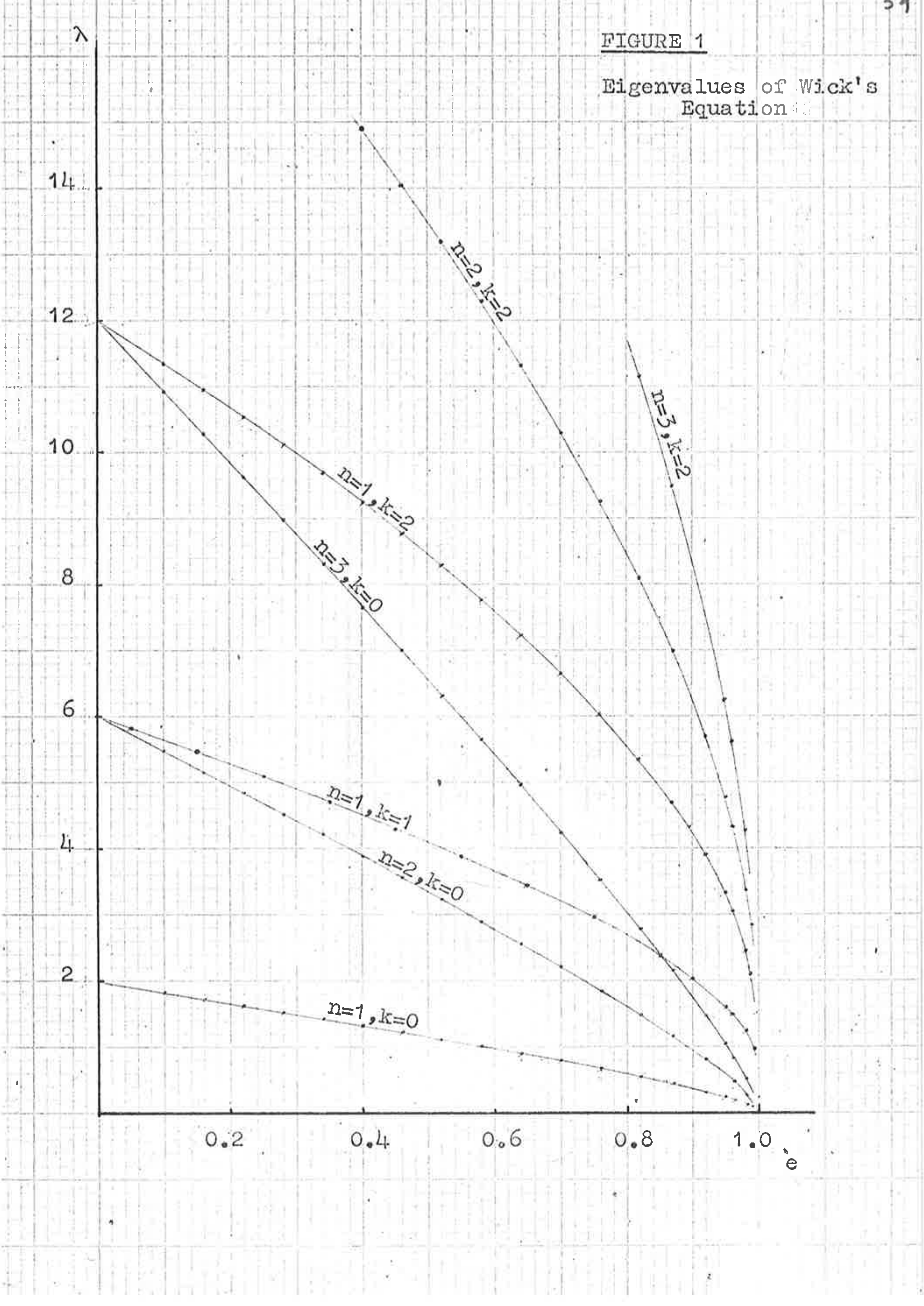


FIGURE 1
Eigenvalues of Wick's Equation
λ
14
12
10
8
6
4
2
0.2
0.4
0.6
0.8
1.0
e
n=2,k=2
n=3,k=2
n=1,k=2
n=3,k=0
n=1,k=1
n=2,k=0
n=1,k=0

have the approximate solution

$$\lambda = \frac{2n}{\pi} \sqrt{e} \sqrt{1 - e}$$

$$\text{or } \lambda = \frac{2n}{\pi} \sqrt{1 - e} \quad \text{for } e \simeq 1$$

so in figure 2, λ is plotted against $\sqrt{1 - e}$. Only the results for $n = 1$ are given; those for other values of n did not differ significantly. The line $\lambda = \frac{2}{\pi}\sqrt{1 - e}$ is also shown on the graph and as can be seen the plotted points approach this line as $e \rightarrow 1$. However if $\sqrt{1 - e} = 0.05$, which is in the non-relativistic region, the value of λ given by the approximate formula is about 15% less than the true value. Probably, only if $\sqrt{1 - e} < 0.01$, i.e. $e > 0.9999$ would the approximate formula give an error of less than 5% and such values are extremely non-relativistic. Inserting the factor $\sqrt{e}$ into the approximate formula to give $\lambda = \frac{2}{\pi}\sqrt{e}\sqrt{1 - e}$ would make the errors very slightly greater.

For the abnormal solutions the approximate formulae (4.5) and (4.6) give, very roughly

$$\tfrac{1}{2}\,\sigma \log 1 - e = m\pi \qquad \text{where} \quad \sigma = \sqrt{\lambda - \tfrac{1}{4}}$$

so that for purposes of comparison the numerical results have been plotted with λ and $\sigma \log (1 - e)$ as coordinates. The first abnormal solution, with quantum number k equal to one, is derived from the second recurrence relation, corresponding to $T(z)$. Its eigenvalues are plotted in Figure 3. The eigenvalues of the second and fourth abnormal solutions, with $k = 2$ and $k = 4$, derived the first recurrence relation are shown in Figures 4 and 5. In each case the values $n = 1$ and $n = 3$ have been chosen.

The approximate eigenvalues from the first recurrence relation,

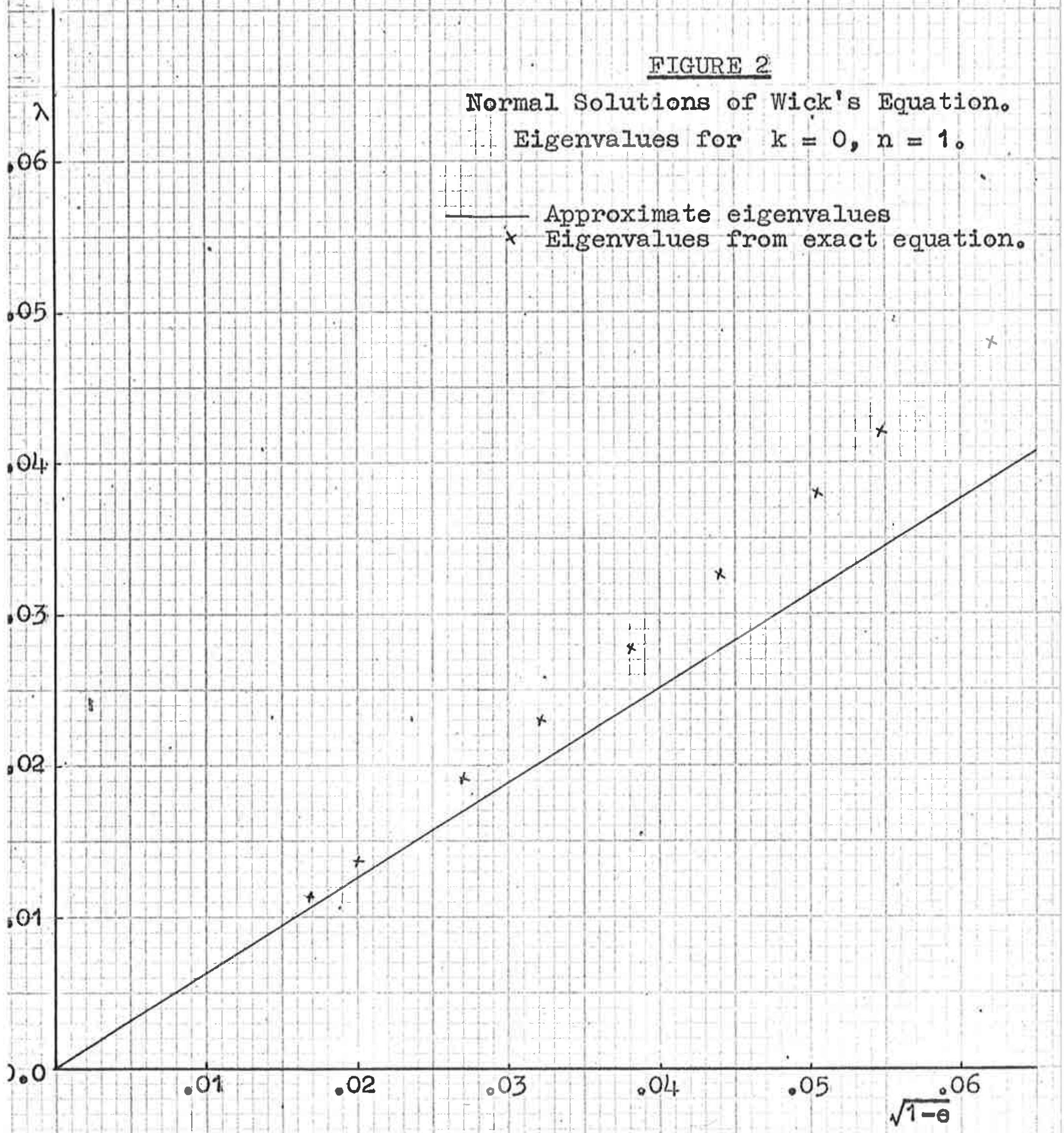


FIGURE 2
Normal Solutions of Wick's Equation.
Eigenvalues for k = 0, n = 1.
Approximate eigenvalues
Eigenvalues from exact equation.
λ
.06
.05
.04
.03
.02
.01
0.0
.01
.02
.03
.04
.05
.06
√1-e

FIGURE 3

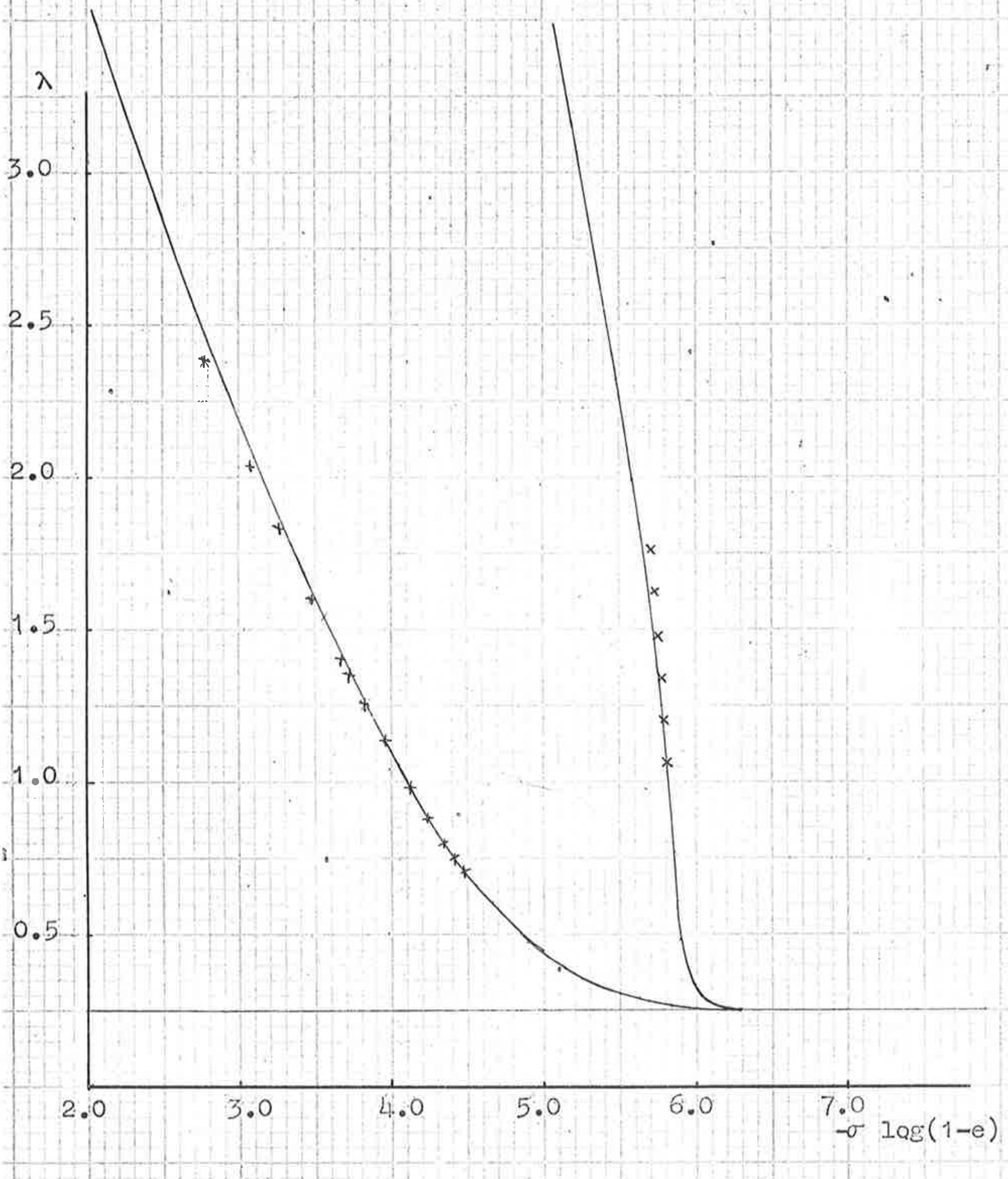
Abnormal solutions of Wick's Equation

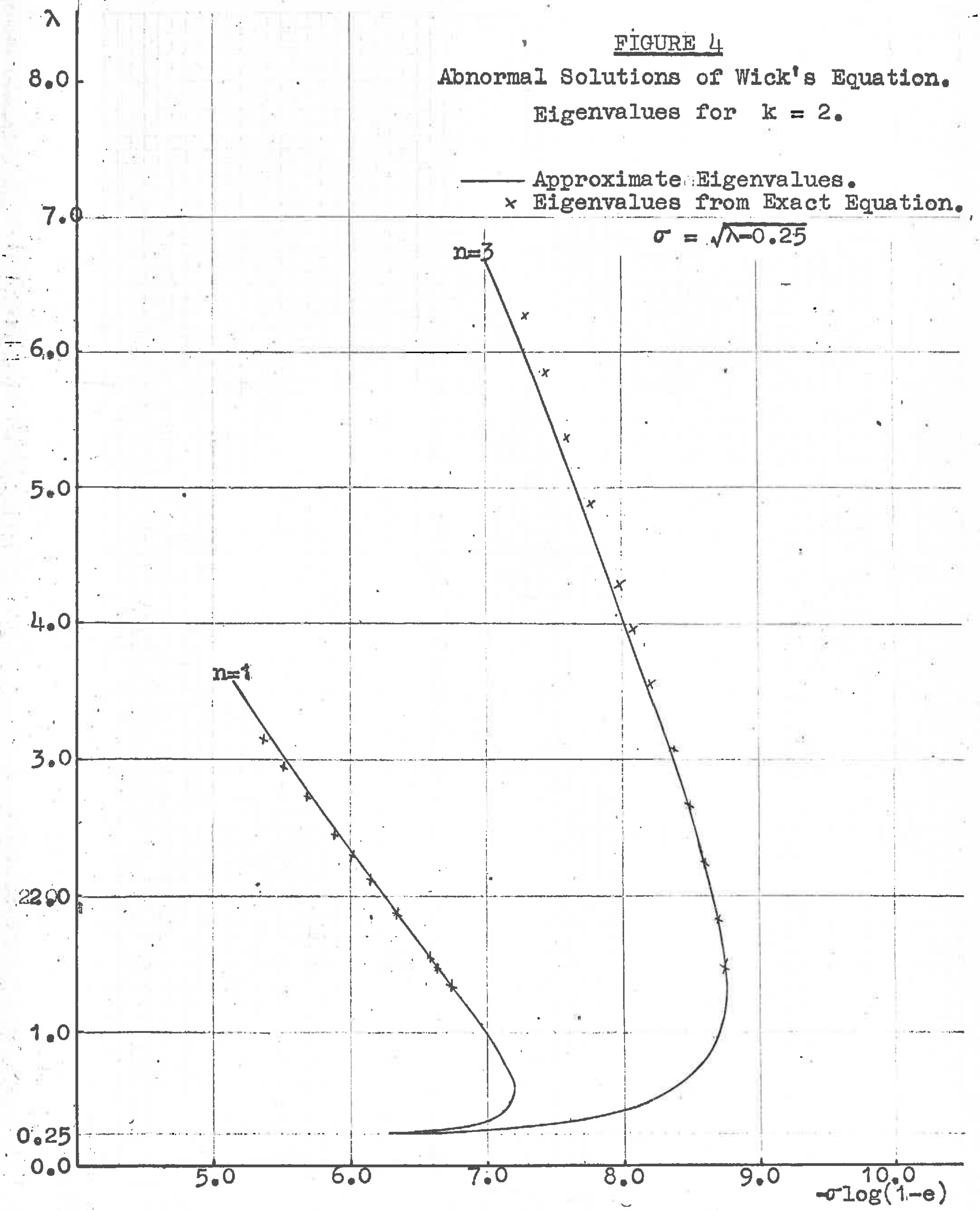
Eigenvalues for k=1

—— Approximate Eigenvalues

× Eigenvalues from Exact Equation

$\sigma = \sqrt{\lambda - 0.25}$




λ
8.0
7.0
6.0
5.0
4.0
3.0
2.0
1.0
0.25
0.0
FIGURE 4
Abnormal Solutions of Wick's Equation.
Eigenvalues for k = 2.
Approximate Eigenvalues.
× Eigenvalues from Exact Equation.
σ = √λ-0.25
n=3
n=1
5.0
6.0
7.0
8.0
9.0
10.0
-σlog(1-e)

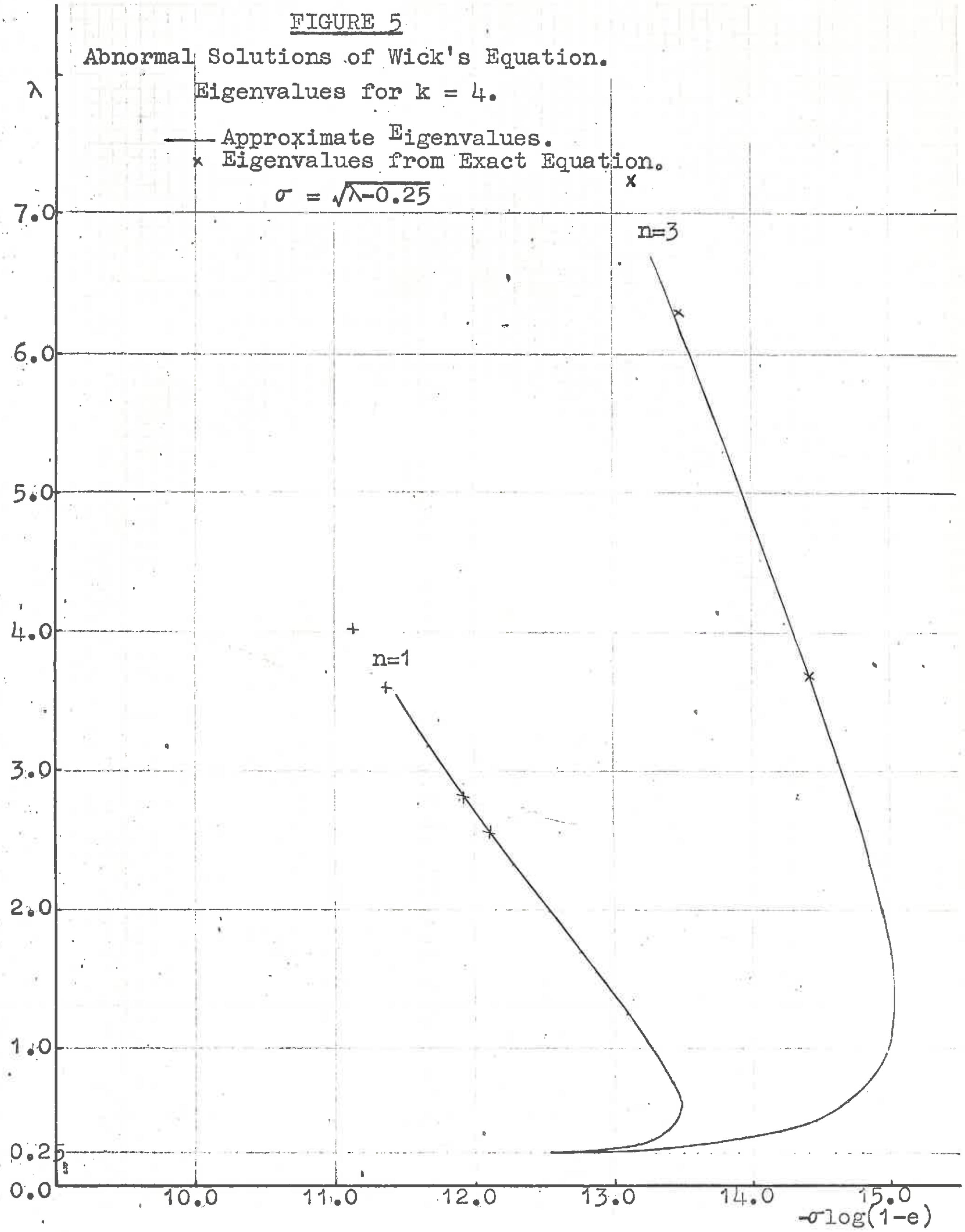


FIGURE 5
Abnormal Solutions of Wick's Equation.
Eigenvalues for k = 4.
Approximate Eigenvalues.
× Eigenvalues from Exact Equation.
σ = √λ-0.25
λ
7.0
6.0
5.0
4.0
3.0
2.0
1.0
0.25
0.0
10.0
11.0
12.0
13.0
14.0
15.0
-σlog(1-e)
n=1
n=3

for even value of k, are given by (4.5)

$$\tfrac{1}{2}\,\sigma \log (1 - e) = m\pi + \varphi - \theta$$

where φ and θ have been defined earlier. After some manipulation this expression can be rewritten

$$-\,\sigma \log (1 - e) = k\pi + Q_n$$

where $Q_n = 2 \operatorname{artan} 2\sigma - 2\sigma \log 2 + 2 \arg \Gamma(n + \tfrac{1}{2} + i\sigma)$

$$+ 4 \arg \Gamma\left(\tfrac{3}{4} + \frac{i\sigma}{2}\right) - 4 \arg \Gamma(1 + i\sigma)$$

and $-\sigma \log (1 - e)$ has been written since $\log 1 - e$ is negative. The fact that the coefficient of π is just k has been determined by comparison with the exact results. Similarly, for the odd values of k, the approximate formula (4.6) is equivalent to

$$-\,\sigma \log (1 - e) = k\pi + Q_n + 4 \operatorname{artan} e^{-\pi\sigma}$$

(In the term $e^{-\pi\sigma}$, e is the base of natural logarithms.) These lines have also been plotted in figures 3, 4 and 5.

It may be noticed that $Q_n \to 0$ as $\sigma \to 0$ but $\operatorname{artan} e^{-\pi\sigma} \to \frac{\pi}{4}$ as $\sigma \to 0$. Thus for all values of k, $\sigma \log (1 - e)$ tends to a multiple of 2π, contrary to the statement of Cutkosky (11), equation (B.6). However, even for small values of λ, Q_n and $\operatorname{artan} e^{-\pi\sigma}$ are not small.

As can be easily seen from the graphs the approximate formula gives remarkably good results. For $e > 0.99$, that is, in the non relativistic region, there is no apparent error in the formula and even if e is 0.95 the error is small.

Thus it may be concluded that the approximate formulae are valid as $e \to 1$ and, in particular, there is no reason to doubt that as $e \to 1$, $\lambda \to \frac{1}{4}$ for the abnormal solutions.

Chapter 5.

THE RELATION WITH WICK'S SOLUTIONS.

Cutkosky (11) states that his solutions of Wick's equation form a complete set. If this is true then there must be some relation between his solutions and those found in Chapter 3. Cutkosky's solutions are denoted $\varphi_n^{\ell m}(p)$ and for purposes of finding the relation it is convenient to introduce $\psi_n^{\ell m}(p)$ defined by

$$\psi_n^{\ell m}(p) = \left[(p^2 + c^2)^2 + 4E^2 p_4^2\right] \varphi_n^{\ell m}(p)$$

where Wick's analytic continuation has been used so that $p^2 = p_4^2 + \underline{p}^2$, and the space is Euclidean. The solutions found in Chapter 3 may here be denoted by

$$\Psi_{kn}^{\ell m} = \frac{1}{p_s} \sin^{\ell+1}\beta \quad f_{kn}(a)\, C_{n-\ell-1}^{\ell+1}(\cos\beta)\, Y_\ell^m(\theta,\varphi)$$

where $\sin^\ell\beta \quad C_{n-\ell-1}^{\ell+1}(\cos\beta)\, Y_\ell^m(\theta,\varphi)$ is a four-dimensional spherical harmonic. A comparison of the eigenvalues, for example for $E = 0$, shows that n is the same quantity in each case and that Wick's quantum number κ is the same as the k used here.

A little consideration shows that required relation is very simple; $\Psi_{kn}^{\ell m}$ and $\psi_n^{\ell m}$ must be multiples of one another. For the functions $f_{kn}(a)$ appear to form a complete set and certainly the spherical harmonics do, hence $\psi_n^{\ell m}$ must be a linear combination of the $\Psi_{kn}^{\ell m}$. But the eigenvalues of λ are functions of n and k so any linear combination of the $\Psi_{kn}^{\ell m}$ is a state without a single definite value for λ. Since Cutkosky's solutions do have definite values for λ they must each correspond to just one solution $\Psi_{kn}^{\ell m}$.

The foregoing argument seems convincing but it is of interest to obtain the same result by an analysis of the functions $\psi_n^{\ell m}(p)$. Cutkosky begins by introducing functions

$$\varphi_n^{\ell m}(p, z) = \frac{p_s^{\ell}\, Y_\ell^m(\theta,\varphi)}{(p^2 + 2\, i\, z\, E\, p_4 + M^2 - E^2)^{n+2}}$$

Correspondingly we have

$$\psi_n^{\ell m}(p, z) = \frac{p_s^{\ell}\, Y_\ell^m(\theta,\varphi)\left[(p^2 + c^2)^2 + 4 E^2 p_4^2\right]}{(p^2 + 2\, i\, z\, E\, p_4 + c^2)^{n+2}}$$

where $c^2 = M^2 - E^2$; changing to the (a, β) coordinate system gives

$$\psi_n^{\ell m}(a,\beta,z) = \frac{Y_\ell^m(\theta,\varphi)\sin^{\ell+1}\beta\,(\cosh a - \cos\beta)^{n-\ell-1}\,(c^2\cosh^2 a + E^2\sinh^2 a)}{p_s\, 2^n c^{n-\ell-1}\,(c\cosh a + iEz\sinh a)^{n+2}}$$

Cutkosky then defines his function $\varphi_n^{\ell m}(p)$ as

$$\varphi_n^{\ell m}(p) = \sum_{k=0}^{n-\ell-1}\int_{-1}^{1} g_n^k(z)\,\varphi_{n-k}^{\ell m}(p, z)\,dz$$

where the functions $g_n^k(z)$ are to be determined. For the functions $\psi_n^{\ell m}$ this implies

$$\psi_n^{\ell m}(a,\beta) = \frac{Y_\ell^m(\theta,\varphi)\sin^{\ell+1}\beta}{p_s}\sum_{k=0}^{n-\ell-1}(\cosh a - \cos\beta)^{n-k-\ell-1}A_n^k(a) \tag{5.1}$$

where

$$A_n^k(a) = \int_{-1}^{1}\frac{dz\; g_n^k(z)\,(c^2\cosh^2 a + E^2\sinh^2 a)}{2^{n-k}\,c^{n-k-\ell-1}\,(c\cosh a + iEz\sinh a)^{n-k+2}} \tag{5.2}$$

Now if these functions $\psi_n^{\ell m}(a,\beta)$ are to be solutions of Wick's equation they must satisfy a certain integral equation, or equivalently, a certain differential equation together with boundary conditions. The boundary conditions turn out to be satisfied without further consideration and the differential equation is (see equation (3.6))

$$\left(\frac{\partial^2}{\partial a^2} + \frac{\partial^2}{\partial \beta^2} - \frac{\ell(\ell+1)}{\sin^2\beta} + \frac{\lambda}{M^2\cosh^2 a - E^2}\right)(p_s\,\psi_n^{\ell m}) = 0$$

Substituting the expression (5.1) for $\psi_n^{\ell m}$ into this equation and slightly rearranging the terms gives

$$\sum_{k=0}^{n-\ell-1} (\cosh a - \cos\beta)^{n-k-\ell-1} \Big[\frac{d^2 A_n^k}{da^2} + \frac{\lambda A_n^k}{M^2\cosh^2 a - E^2} - (n-k)^2 A_n^k$$
$$+ 2(n-k-\ell)\sinh a \frac{dA_n^{k-1}}{da} + 2(n-k+1)(n-k-\ell)\cosh a\, A_n^{k-1} \Big] = 0$$

where $A_n^{k-1}(a) \equiv 0$ for $k = 0$. Since the expressions $(\cosh a - \cos\beta)^{n-k-\ell-1}$ are linearly independent we have

$$\frac{d^2 A_n^o}{d a^2} + \frac{\lambda}{M^2 \cosh^2 a - E^2} A_n^o - n^2 A_n^o = 0 \qquad (5.3)$$

$$\frac{d^2 A_n^k}{da^2} + \frac{\lambda}{M^2\cosh^2 a - E^2} A_n^k - (n-k)^2 A_n^k = -2(n-k-\ell)\sinh a \frac{dA_n^{k-1}}{da}$$
$$-2(n-k-\ell)(n-k+1)\cosh a\, A_n^{k-1} \qquad (5.4)$$

It may be shown that these equations are equivalent to Cutkosky's equations for his functions $g_n^k(z)$ since the $A_n^k(a)$ are transforms of the $g_n^k(z)$. The first equation (5.3), determines A_n^o and the eigenvalues of λ. Since it is identical with (3.13), the equation for $f_{kn}(a)$, obviously the eigenvalues will be identical with those found in Chapter 4. Then the equations (5.4) enable the $A_n^k(a)$ to be determined recursively. However, this would be a difficult procedure and it will now be shown that owing to the particular structure of (5.4) it is unnecessary; $A_n^o(a)$ is the only function needed.

First, set $q = n - \ell - 1$; then (5.1) becomes

$$\psi_n^{\ell m} = \frac{1}{\sqrt{s}} Y_\ell^m(\beta,\varphi) \sin^{\ell+1}\beta \sum_{k=0}^{q} (\cosh a - \cos\beta)^{q-k} A_n^k(a)$$

As we are interested in comparing $\psi_n^{\ell m}$ with the separated solutions $\Psi_{kn}^{\ell m}$, we expand the binomial factor and then use the fact that a power of $\cos\beta$ can be expressed as a sum of Gegenbauer polynomials:

$$\frac{1}{\lfloor j} \cos^j\beta = \sum_{m=0}^{[\frac{1}{2}j]} a_m^j \, C_{j-2m}^{\sigma}(\cos\beta)$$

where $[\frac{1}{2}j]$ means the largest integer less than or equal to $\frac{1}{2}j$, σ is arbitrary, and

$$a_m^j = \frac{(\sigma+j-2m)\ \Gamma(\sigma)}{2^j \lfloor m \ \Gamma(\sigma+j+1-m)}$$

After some rearrangement of the series we get

$$\psi_n^{\ell m} = \frac{1}{\sqrt{s}} Y_\ell^m(\theta,\varphi) \sin^{\ell+1}\beta \sum_{j=0}^{q} (-)^j \frac{\Gamma(\sigma)}{\Gamma(\sigma+j)\, 2^j} \ C_j^{\sigma}(\cos\beta)\ D_q^j(a)$$

where

$$D_q^j(a) = \sum_{k=0}^{q-j} \frac{\lfloor j+k}{\lfloor k} \cosh^k a \ A_n^{q-j-k}(a)\, F(-\tfrac{1}{2}k, \tfrac{1}{2}(1-k); \sigma+j+1; \cosh^{-2} a)$$

Then, use of the recurrence relation (5.4) for the $A_n^{q-j-k}(a)$ shows that if $\sigma = \ell+1$.

$$\frac{d^2 D_q^j(a)}{da^2} + \frac{\lambda}{M^2\cosh^2 a - E^2} D_q^j(a) - (n+j-q)^2 D_q^j(a) = 0$$

Now this is just the equation (5.3) for A_n^o with n^2 replaced by $(n+j-q)^2$. Hence solving this equation would lead to an eigenvalue condition for λ in which $(n+j-q)$ will appear. But λ has already been determined from (5.3) by the same condition in which n appears. Thus λ will not satisfy the new condition. Hence the equation for $D_q^j(a)$ has no non-trivial solution unless $j = q$, or

$$D_q^j = 0 \qquad j \neq q$$

Therefore we are left with

$$\psi_n^{\ell m} = \frac{1}{\sqrt{s}} Y_\ell^m(\theta,\varphi) \sin^{\ell+1}\beta \ \frac{(-)^{n-\ell-1}\ \Gamma(\ell+1)\,\Gamma(n-\ell)}{2^{n-\ell-1}\ \Gamma(n)} \ C_{n-\ell-1}^{\ell+1}(\cos\beta)\, A_n^o(a)$$

which is a multiple of $\psi_{kn}^{\ell m}$. It is now clear that Wick's method and the present method of solving Wick's equation are completely equivalent.

Chapter 6.

THE INSTANTANEOUS INTERACTION APPROXIMATION.

We commence with the equation

$$\left[(p^2 - c^2)^2 - 4E^2p_o^{\ 2}\right] \Phi(p) = \frac{i\lambda}{\pi^2}\int \frac{\Phi(k)\, d^4k}{(p-k)^2 + i\varepsilon}$$

As has been pointed out in the Introduction, the instantaneous interaction approximation consists of replacing the term $(p - k)^2$ by $-(\underline{p} - \underline{k})^2$. This approximation has been made in all the applications of the Bethe-Salpeter equation to physical situations and although it is undoubtedly valid in the non relativistic limit, the errors which it introduces are difficult to estimate without an exact solution. It is therefore of interest to make the same approximation in Wick's equation and to compare the resulting eigenvalues with the exact eigenvalues previously calculated. If the errors are small there would be more reason to hope that the approximation introduces small errors into the B.-S. equation for two fermions.

The approximated equation

$$\left[(p^2 - c^2)^2 - 4E^2p_o^{\ 2}\right] \Phi(p) = \frac{-i\lambda}{\pi^2}\int \frac{\Phi(k)\, d^4k}{(\underline{p} - \underline{k})^2} \tag{6.1}$$

takes on a rather simple form in coordinate space

$$\left[(\Box + c^2)^2 + 4E^2\frac{\partial^2}{\partial t^2}\right] u(x) = \frac{-4i\pi\lambda}{r}\,\delta(t)\, u(x) \tag{6.2}$$

where $\Box = \frac{\partial^2}{\partial t^2} - \nabla^2$. However, the equation (6.1) seems to be easier to reduce further. Since the right hand side is independent of p_o, the p_o dependence of Φ must be given by the substitution

$$\Phi(p) = \frac{v(\underline{p})}{(c^2 - p^2)^2 - 4E^2p_o^2} \tag{6.2}$$

Putting this in (6.1) and integrating over k_o gives

$$v\,(\underline{p}) = \frac{\lambda}{2\pi} \int \frac{d^3k \quad v\,(\underline{k})}{(c^2 + \underline{k}^2)\,\sqrt{M^2 + k^2}\,(\underline{k} - \underline{p})^2} \tag{6.3}$$

For the remainder of this chapter we shall put $M = 1$, so that $c^2 = 1 - E^2$ and $p = |\underline{p}|$. Hecke's theorem (5), (6) can be applied to (6.3) if we put

$$v\,(\underline{p}) = \frac{1}{p}\, S(p)\; Y_\ell^m\,(\theta, \varphi) \tag{6.4}$$

Then $S(p)$ satisfies a one dimensional integral equation

$$S(p) = \lambda \int_0^\infty \frac{S\,(k)\,dk}{(c^2 + k^2)\,\sqrt{1 + k^2}}\; Q_\ell\left(\frac{p^2 + k^2}{2pk}\right) \tag{6.5}$$

where Q_ℓ is a Legendre function of the second kind.

Surprisingly, this equation seems to be more difficult to solve than the full Wick's equation. Goldstein (2) has considered a similar equation, without the factor $c^2 + k^2$ and finds $\lambda = \frac{1}{2\pi}$ (when $\ell = 0$) as the single eigenvalue, but his method is difficult to generalise. The following procedure suggested by the techniques of Muskhelishvili (30), and some work of Heins and MacCamy (31), gives a strong hint that the eigenvalues are given by $\frac{\lambda\pi}{2EC} = n$ where n is an integer.

For simplicity we consider only s - states. Then (6.5) becomes

$$S(p) = \lambda \int_0^\infty \frac{S\,(k)\,dk}{(c^2 + k^2)\,\sqrt{1 + k^2}}\; \log\left|\frac{p + k}{p - k}\right| \tag{6.6}$$

Now make the transformation $p = \sqrt{\frac{1 - x}{x}}$ so that

$$1 + p^2 = \frac{1}{x}, \quad c^2 + p^2 = \frac{1 - E^2 x}{x},$$

$$S(x) = \frac{\lambda}{2} \int_0^1 \frac{S(t)\,dt}{(1 - E^2 t)\,\sqrt{1 - t}}\; \log\left|\frac{\sqrt{t}\,\sqrt{1-x} + \sqrt{x}\,\sqrt{1-t}}{\sqrt{t}\,\sqrt{1-x} - \sqrt{x}\,\sqrt{1-t}}\right|$$

We define a new function $w(z)$ of the complex variable z by the equation

$$w(z) = \frac{\lambda}{2} \int_0^1 \frac{S(t)dt}{(1-E^2t)\sqrt{1-t}} \log \frac{\sqrt{t}\left(\frac{z-1}{z}\right)^{\frac{1}{2}} - i\sqrt{1-t}}{\sqrt{t}\left(\frac{z-1}{z}\right)^{\frac{1}{2}} + i\sqrt{1-t}} \qquad (6.7)$$

The branch of $\left(\frac{z-1}{z}\right)^{\frac{1}{2}}$ is specified by stating that it is to be real and positive for z anywhere on the real axis except the segment $(0, 1)$. If $0 < x < 1$, ε small and $z = x + i\varepsilon$, then $\left(\frac{z-1}{z}\right)^{\frac{1}{2}} = i\sqrt{\frac{1-x}{x}}$ and if if $z = x - i\varepsilon$, $\left(\frac{z-1}{z}\right)^{\frac{1}{2}} = -i\sqrt{\frac{1-x}{x}}$. Clearly $w(z)$ is regular and single-valued in the whole plane cut along the real axis from 0 to 1. Let $w^+(x)$, $w^-(x)$ be the limiting values of $w(z)$ on the upper and lower sides of the cut respectively. An analysis of the logarithm in the vicinity of the cut shows that

$$w^+(x) = \frac{\lambda}{2} \int_0^1 \frac{S(t)\,dt}{(1-E^2t)\sqrt{1-t}} \log \left| \frac{\sqrt{1-t}\sqrt{x} - \sqrt{t}\sqrt{1-x}}{\sqrt{1-t}\sqrt{x} + \sqrt{t}\sqrt{1-x}} \right| - \frac{\lambda\pi i}{2} \int_0^x \frac{S(t)\,dt}{(1-E^2t)\sqrt{1-t}}$$

$$w^-(x) = \frac{\lambda}{2} \int_0^1 \frac{S(t)\,dt}{(1-E^2t)\sqrt{1-t}} \log \left| \frac{\sqrt{1-t}\sqrt{x} + \sqrt{t}\sqrt{1-x}}{\sqrt{1-t}\sqrt{x} - \sqrt{t}\sqrt{1-x}} \right| - \frac{\lambda\pi i}{2} \int_0^x \frac{S(t)\,dt}{(1-E^2t)\sqrt{1-t}}$$

Hence

$$w^+(x) + w^-(x) = -\lambda\pi i \int_0^x \frac{S(t)\,dt}{(1-E^2t)\sqrt{1-t}}$$

$$w^+(x) - w^-(x) = -\lambda \int_0^1 \frac{S(t)\,dt}{(1-E^2t)\sqrt{1-t}} \log \left| \frac{\sqrt{1-t}\sqrt{x} + \sqrt{t}\sqrt{1-x}}{\sqrt{1-t}\sqrt{x} - \sqrt{t}\sqrt{1-x}} \right|$$

$$= -2S(x)$$

according to the integral equation for $S(x)$. Therefore

$$w^+(x) + w^-(x) = \frac{\lambda\pi i}{2} \int_0^x \frac{dt}{(1-E^2t)\sqrt{1-t}} \left(w^+(t) - w^-(t)\right)$$

or

$$\frac{dw^+(x)}{dx} + \frac{dw^-(x)}{dx} = \frac{\lambda\pi i}{2(1-E^2x)\sqrt{1-x}} \left(w^+(x) - w^-(x)\right) \qquad (6.8)$$

The solution of the integral equation has been reduced to the problem of finding a function analytic in the cut plane whose boundary values on the cut satisfy this differential equation (and a boundary condition that $w^+(0) + w^-(0) = 0$). Once this function has been found, $S(x)$ is given by its discontinuity across the cut

$$S(x) = -\tfrac{1}{2}\left(w^+(x) - w^-(x)\right).$$

The equation (6.8) can be rewritten

$$i\sqrt{1-x}\,\frac{dw^+}{dx} + \frac{\lambda\pi}{2(1-E^2x)}\,w^+ = -i\sqrt{1-x}\,\frac{dw^-}{dx} + \frac{\lambda\pi}{2(1-E^2x)}\,w^-$$

Now $i\sqrt{1-x}$ is simply the boundary value of $\sqrt{z-1}$ on the upper side of the cut and $-i\sqrt{1-x}$ is the boundary value of $\sqrt{z-1}$ on the lower side. Furthermore, we obviously can write

$$\frac{dw^+}{dx} = \left(\frac{dw}{dz}\right)^+$$ while $(1-E^2x)^{-1}$ is regular on and near the cut for $E^2 < 1$, so the above equation becomes

$$\left[\sqrt{z-1}\,\frac{dw}{dz} + \frac{\lambda\pi}{2(1-E^2z)}\,w\right]^+ = \left[\sqrt{z-1}\,\frac{dw}{dz} + \frac{\lambda\pi}{2(1-E^2z)}\,w\right]^-$$

Thus the function $\sqrt{z-1}\,\frac{dw}{dz} + \frac{\lambda\pi}{2(1-E^2z)}\,w$ has the same value on each side of the cut, i.e. it is analytic on the cut. Note, however, that this equality applies on the cut, the segment $(0, 1)$, but not along the negative real axis where $w(z)$ is analytic but of course $\sqrt{z-1}$ has a cut. We can therefore write

$$\sqrt{z-1}\,\frac{dw}{dz} + \frac{\lambda\pi\, w}{2(1-E^2z)} = P(z) \tag{6.9}$$

where $P(z)$ is cut along the entire negative real axis. In addition $P(z)$ must have a pole at $z = E^{-2}$. This differential equation can be solved and gives

$$w(z) = \left(\frac{c - E\sqrt{z-1}}{c + E\sqrt{z-1}}\right)^{\frac{\lambda\pi}{2Ec}} \int_{z_0}^{z} \frac{P(z')}{\sqrt{z'-1}} \left(\frac{c + E\sqrt{z'-1}}{c - E\sqrt{z'-1}}\right)^{\frac{\lambda\pi}{2Ec}} dz' \tag{6.10}$$

where z_0 is a constant to be determined from the boundary condition.

The function $P(z)$ is probably not unique but is severely restricted by the condition that the integral (6.10), when multiplied by $\left(\frac{c-E\sqrt{z-1}}{c+E\sqrt{z-1}}\right)^{\frac{\lambda\pi}{2Ec}}$ gives a function analytic along the negative real axis.

If we put $a = \frac{\lambda\pi}{2Ec}$ and

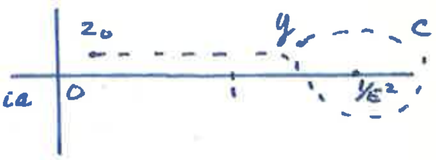
$$g(z) = \left(\frac{c+E\sqrt{z-1}}{c-E\sqrt{z-1}}\right)^{a} \quad \text{then (6.10) becomes}$$

$$w(z) = \frac{1}{g(z)} \int_{z_0}^{z} \frac{P(z')\,g(z')}{\sqrt{z'-1}}\,dz'.$$

Now at the point $z = E^{-2}$, $g(z)$ has in general a branch point and $P(z)$ has a pole, but $w(z)$ must be regular. Suppose that y is a point near E^{-2} and let w_1 be the value of w at y; then

$$w_1 = \frac{1}{g(y)} \int_{z_0}^{y} \frac{P(z')\,g(z')}{\sqrt{z'-1}}\,dz'$$

Let the point y make a circuit round the point E^{-2}. Then $g(y)$ will become $g(y)e^{-2\pi ia}$ and the new value of $w(z)$ will be



$$w_2 = \frac{e^{2\pi ia}}{g(y)}\left[\int_{z_0}^{y} \frac{P(z')\,g(z')\,dz'}{\sqrt{z'-1}} + \int_{c} \frac{P(z')\,g(z')\,dz'}{\sqrt{z'-1}}\right]$$

But $w(z)$ is regular in the vicinity of E^{-2} so $w_1 = w_2$, that is

$$\int_{z_0}^{y} \frac{P(z')\,g(z')\,dz'}{\sqrt{z'-1}} = e^{2\pi ia}\int_{z_0}^{y} \frac{P(z')\,g(z')\,dz'}{\sqrt{z'-1}} + e^{2\pi ia}\int_{c} \frac{P(z')\,g(z')\,dz'}{\sqrt{z'-1}}$$

or

$$\int_{c} \frac{P(z')\,g(z')\,dz'}{\sqrt{z'-1}} = (e^{2\pi ia}-1)\int_{z_0}^{y} \frac{P(z')\,g(z')\,dz'}{\sqrt{z'-1}} \qquad (6.11)$$

The simplest way to satisfy this equation is to let a be an integer. Certainly these may not be the only possibilities, nor are they necessarily correct, for there may be no function $P(z)$ which gives w(z) the correct properties. However, confidence in these eigenvalues

$$\lambda = \frac{2}{\pi} E\, c\, n$$

is increased by noting that this is just the result obtained in Chapter 4, and previously by Wick, for the non-relativistic limit of the normal solutions of the full Wick's equation. As was observed there, these eigenvalues are a good approximation only in the extreme non relativistic limit. For example if $E^2 = 0.9975$ the error is about 15% and if $E^2 = 0.99$ it reaches 35%. Vosko (32) found that if the meson which carries the interaction has non-zero mass then the error in the eigenvalue obtained from the corresponding Schrodinger equation is even larger for $E^2 = 0.99$, about 40%. If the present method could be made to give the instantaneous interaction eigenvalues with certainty, there would probably be no difficulty in extending it to the general case of non-zero meson mass.

Chapter 7.

THE NORMALIZATION CONDITION.

In this chapter a normalization condition for Wick's equation will be derived along the lines suggested by Green (29).

As was described in the Introduction, a set of wave functions can be defined

$$\begin{aligned} u(x_1, x_2) &= (\Omega, T\psi(x_1)\psi(x_2)\Psi) \\ v(x_1, x_2; y) &= (\Omega, T\psi(x_1)\psi(x_2)\varphi(y)\Psi) \end{aligned} \tag{7.1}$$

etc., corresponding to the presence of two nucleons and no mesons, one meson, etc. Their complex conjugates are

$$\begin{aligned} \bar{u}(x_1, x_2) &= (\Psi, \bar{T}\psi(x_1)\psi(x_2)\Omega) \\ \bar{v}(x_1, x_2; y) &= (\Psi, \bar{T}\psi(x_1)\psi(x_2)\varphi(y)\Omega) \end{aligned} \tag{7.2}$$

For simplicity, the nucleons are supposed neutral and identical so that u and v are even functions under an interchange of x_1 and x_2. For the ladder approximation, only these two functions are considered, and they satisfy the equations

$$\begin{aligned} (\Box_1 + M^2)\, u(x_1, x_2) &= g\, v(x_1, x_2; x_1) \\ (\Box_2 + M^2)\, v(x_1, x_2; y) &= g\, D(x_2 - y)\, u(x_1\, x_2) \end{aligned} \tag{7.3}$$

where $\quad D(x) = \frac{i}{(2\pi)^4} \int \frac{e^{ikx}}{k^2 + i\varepsilon}\, dk.$

Going to centre of mass coordinates, $\frac{1}{2}(x_1 + x_2) = X$, $x_1 - x_2 = x$, $y - X = z$ these equations become

$$\begin{aligned} K_1\, u(X, x) &= g\, v(X, x; \tfrac{1}{2}x) \\ K_2\, v(X, x; z) &= g\, D(\tfrac{1}{2}x + z)\, u(X, x) \end{aligned} \tag{7.4}$$

where $K_1 = \frac{1}{4}\Box_X + \Box_x + \nabla_X \nabla_x + M^2$

$$K_2 = \tfrac{1}{4}\Box_X + \Box_x + \tfrac{1}{4}\Box_z - \nabla_X \nabla_x - \tfrac{1}{2}\nabla_z \nabla_X + \nabla_x \nabla_z + M^2$$

and similarly

$$K_1 \bar{u}(X, x) = g\,\bar{v}(X, x; \tfrac{1}{2}x)$$

$$K_2 \bar{v}(X, x; z) = g\,\bar{D}\,(\tfrac{1}{2}x + z)\,\bar{u}\,(X, x)$$

Now the general idea of the present method is to regard these equations as a set of equations for two classical fields. If this is legitimate then they result from the variation of a Lagrangian and from the Lagrangian an expression for a current can be derived. This current will necessarily be conserved by virtue of the field equations and the integral of it over a spacelike surface must be a constant. Giving a value to the constant normalises the wave functions.

The meson propagator $D(\tfrac{1}{2}x + z)$ must first be eliminated so the equations (7.3) become

$$\begin{aligned}
K_1\, u(X, x) &= g\, v(X, x; \tfrac{1}{2}x) \\
\Box_z K_2\, v(X, x; z) &= igu(X,x)\ \delta(\tfrac{1}{2}x + z) \\
K_1\, \bar{u}(X, x) &= g\, \bar{v}(X, x; \tfrac{1}{2}x) \\
\Box_z K_2\, \bar{v}(X, x; z) &= -ig\bar{u}(X,x)\ \delta(\tfrac{1}{2}x + z).
\end{aligned} \qquad (7.5)$$

Unfortunately the factor i occurring in two of these equations seems to make it impossible to find a Lagrangian which they could have derived from. This difficulty is overcome by changing to the imaginary time axis. Wick has shown that $u(X,x)$ can be continued into the complex t-plane and from the analytic properties of $u(X,x)$ those of $v(X,x;z)$ can be deduced by using the equations (7.5). It may be noted that the properties of $u(X,x)$ are independent of the ladder approximation while the properties of $v(X, x; z)$ found in this way are not necessarily valid in general. The rotation to the imaginary time axis is most conveniently carried out in momentum space. Suppose that $U(P, p)$ and $V(P, p, q)$ are the Fourier transforms of u and v then the first two equations of (7.5) become

$$(M^2 - \tfrac{1}{4}P^2 - p^2 - Pp)\, U(P,p) = \frac{g}{2\pi}\int dq\, V(P, \tfrac{p+q}{2}, p-q)$$

$$(M^2 - \tfrac{1}{4}P^2 - p^2 + Pp)\, V(P, \tfrac{p+q}{2}, p-q) = \frac{ig}{(2\pi)^3}\,\frac{U(P,q)}{(p-q)^2 + i\varepsilon} \tag{7.6}$$

Since $U(P,q)$ has cuts from the zeros of $(M^2 - \frac{1}{4}P^2 - p^2 \pm Pp)$ to $\pm$, the second of these equations shows that $V(P, \frac{p+q}{2}, p-q)$ has the same cuts together with two poles at the zero of $(p-q)^2 + i\varepsilon$. Hence V can be continued in the same direction as U. After taking both p_o and q_o to the imaginary axis and putting $p_o = ip_4$, $q_o = iq_4$ the equations (7.6) become

$$(M^2 - \tfrac{1}{4}P^2 + p^2 - ip_4P_o + \underline{P}.\underline{p})\, U(P, ip_4, \underline{p})$$
$$= \frac{ig}{2\pi}\int dq\, V(P, \frac{i(p_4+q_4)}{2}, \frac{\underline{p}+\underline{q}}{2}, i(p_4-q_4), \underline{p}-\underline{q})$$

and

$$(M^2 - \tfrac{1}{4}P^2 + p^2 + iP_op_4 - \underline{P}.\underline{p})\, V(P, \frac{i(p_4+q_4)}{2}, \frac{\underline{p}+\underline{q}}{2}, i(p_4-q_4), \underline{p}-\underline{q})$$
$$= \frac{-ig}{(2\pi)^3}\,\frac{U(P, ip_4, \underline{p})}{(p-q)^2}$$

where $p^2 = p_4^2 + \underline{p}^2$.

A similar procedure can be carried out for the second pair of equations (7.5) but the rotation in the complex p_o plane must be in the opposite direction so that the substitutions $p_o = -ip_4$, $q_o = -iq_4$ must be made. We can then make a second Fourier transformation

$$U(P, ip_4, \underline{p}) = \frac{1}{4\pi^2}\int u(x_4, \underline{x})\, e^{iPX} e^{-ipx}\, dX\, dx$$

$$\bar{U}(P, -ip_4, \underline{p}) = \frac{1}{4\pi^2}\int \bar{u}\,(x_4, \underline{x})\, e^{-iPX}\, e^{ipx}\, dX\, dx$$

etc. and finally get

$$
\begin{aligned}
K_1' \, u(x) &= i \, g \, v\,(x, \tfrac{x}{2}) \\
\Box_z K_2' \, v(x, z) &= i \, g \, u(x) \, \delta\,(z + \tfrac{x}{2})
\end{aligned}
\qquad (7.7)
$$

etc. where the centre of mass coordinate has been suppressed and

$$
\begin{aligned}
K_1' &= M^2 + \tfrac{1}{4}\Box_X - i \frac{\partial^2}{\partial X_0 \partial x_4} - \nabla_{\underline{X}} \cdot \nabla_{\underline{x}} - \Box_x \\
&= M^2 + \tfrac{1}{4}\Box_X + i \frac{\partial}{\partial X_0}\left(\frac{\partial}{\partial x_4} + \frac{1}{2}\frac{\partial}{\partial z_4}\right) + \nabla_{\underline{X}} \cdot \left(\nabla_{\underline{x}} + \tfrac{1}{2}\nabla_{\underline{z}}\right) - \left(\nabla_{\underline{x}} + \tfrac{1}{2}\nabla_{\underline{z}}\right)^2
\end{aligned}
$$

Finally, since $u(x) = u(-x)$ and $v\,(x, z) = v(-x, z)$

$$
\begin{aligned}
K_1' \, u(x) &= i \, g \, v\,(x, \tfrac{1}{2}x) \\
\Box_z K_3' \, v(x, z) &= i \, g \, u\,(x) \, \delta\,(z - \tfrac{1}{2}x) \\
\bar{K}_1' \, \bar{u}(x) &= -i \, g \, \bar{v}(x, \tfrac{1}{2}x) \\
\Box_z \, K_3' \, \bar{v}\,(x, z) &= -i \, g \, \bar{u}\,(x) \, \delta\,(z - \tfrac{1}{2}x)
\end{aligned}
\qquad (7.8)
$$

where K_3' is derived from K_2' by changing the sign of x_4 and $\underline{x}$ in K_2'.

Now this set of equations can be derived from a Lagrangian density $\mathcal{L}$ which turns out to be a function of $u(x_4, \underline{x})$ and $\bar{u}(-x_4, \underline{x})$ and similarly for v so that $\mathcal{L}$ satisfies

$$\bar{\mathcal{L}}\,(-x_4, -z_4) = \mathcal{L}\,(x_4, z_4)$$

a condition obviously related to the fact that x_4 and z_4 are implicitly multiplied by a factor i. As $\mathcal{L}$ is rather lengthy it will not be given here in detail but the general form for the Lagrangian L is

$$
\begin{aligned}
L = \int dX\,dx \Big\{ \mathcal{L}_u(X, x) + \int dz \Big(&\mathcal{L}_v(X, x, z) + ig\, u(x)\, \bar{v}(-x_4, \underline{x}, -z_4, \underline{z}) \\
&- ig\, \bar{u}(-x_4, \underline{x})\, v(x, z) \Big) \Big\}
\end{aligned}
\qquad (7.9)
$$

If we then find δL, the change in L due to independent variations of u, $\bar{u}$, v and $\bar{v}$ and then put $\delta\bar{u} = -i\bar{u}$, $\delta u = iu$, $\delta\bar{v} = -i\bar{v}$, $\delta v = iv$ and $\delta L = 0$ we get

$$0 = \int dX \left(\frac{\partial}{\partial X_0} J_0 - \nabla_{\underline{X}} \cdot \underline{J} \right)$$

where

$$J_0 = i \int dx \left(\tfrac{1}{4} \bar{u}(-x_4) \frac{\partial u(x_4)}{\partial X_0} - \tfrac{1}{4} u \frac{\partial \bar{u}}{\partial X_0} - \tfrac{i}{2} \bar{u} \frac{\partial u}{\partial x_4} + \tfrac{i}{2} u \frac{\partial \bar{u}}{\partial x_4} \right)$$

$$- i \int dx\, dz \left(\tfrac{1}{4} \bar{v} \Box_2 \frac{\partial v}{\partial X_0} - \tfrac{1}{4} v \Box_2 \frac{\partial \bar{v}}{\partial X_0} - \tfrac{i}{2} \bar{v} \Box_2 \left(\frac{\partial}{\partial x_4} - \tfrac{1}{2} \frac{\partial}{\partial z_4} \right) v \right. \qquad (7.10)$$

$$\left. + \tfrac{i}{2} v \Box_2 \left(\frac{\partial}{\partial x_4} - \tfrac{1}{2} \frac{\partial}{\partial z_4} \right) \bar{v} \right)$$

and similarly for $\underline{J}$. In each term of this expression it must be understood that the arguments of u are x_4, $\underline{x}$ and of $\bar{u}$ are $-x_4$, $\underline{x}$ and similarly those of v are x_4, $\underline{x}$, z_4, $\underline{z}$ but those of $\bar{v}$ are $-x_4$, $\underline{x}$, $-z_4$, $\underline{z}$.

Since J is a vector which is conserved it corresponds to some physical quantity and is most readily interpreted as the density and flux of the bound state as a whole. Since the system must be somewhere the normalization condition is

$$\int J_0 \, d^3 X = 1 \qquad (7.11)$$

or $\int J_\mu \, d\sigma_\mu = 1$ for a general spacelike surface.

For a particular solution of the field equations

$$u(X, x) = e^{-iPX} f(x) = \frac{e^{-iPX}}{4\pi^2} \int dp \, e^{ipx} \, \psi(ip_4, \underline{p})$$

and
$$v(X, x, z) = \frac{e^{-iPX}}{(2\pi)^4} \int dp \, dq \, e^{ipx} e^{iqz} \, \theta(ip_4, iq_4, \underline{p}, \underline{q})$$

the equation (7.11) becomes, in momentum space

$$\frac{1}{V} = \int dp \, (E + ip_4) \, \bar{\psi}(ip_4, \underline{p}) \, \psi(ip_4, \underline{p})$$

$$+ \int dp \, dq \, (E + ip_4 - \tfrac{i}{2} q_4) \, q^2 \, \bar{\theta}(ip_4, iq_4, \underline{p}, \underline{q}) \, \theta(ip_4, iq_4, \underline{p}, \underline{q})$$

where V is the integration volume, $\underline{P} = 0$ and $P_o = 2E$.

In the first integral, the term containing p_4 is zero since ψ is even in p_4, and in the second the functions θ and $\bar{\theta}$ can be eliminated by means of the field equations. Then the integration over p can be performed to give

$$\frac{1}{V} = E \int dp\ (1 + \lambda I(p))\ \bar{\psi}(ip_4, \underline{p})\ \psi(ip_4, \underline{p}) \tag{7.12}$$

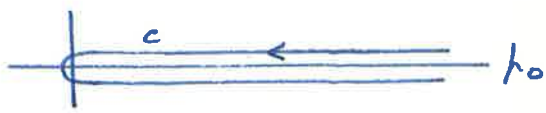
where $\lambda = (\frac{g}{4\pi})^2$ is the coupling constant used in the previous chapters and

$$I(p) = \frac{1}{(p^2 - E^2)^2 + 4E^2 p_4^2} \left\{ \left(\frac{p^2 - E^2}{2} + p_4^2\right) \log \frac{p^2 + c^2 + 4E^2 p_4^2}{M^4} + \frac{p_4}{E}(3E^2 - p_4^2) \arctan \frac{2Ep_4}{p^2 + c^2} \right\} \tag{7.13}$$

A similar expression can be obtained for the normalisation integral in the Minkowski metric by using the analytic continuation of ψ to give

$$\frac{1}{V} = -i \int_c (1 + \lambda I'(p))\ \bar{\psi}(p)\, \psi(p)\ dp$$

The function $I'(p)$ is rather complicated and the p_o-integration must be over the contour shown.



For application of this normalization to the solutions found in Chapter 3, allowance must be made for the way the argument p_4 has been introduced. If $\Phi(p_4, p)$ is the Bethe-Salpeter wave function in the Euclidean metric then

$$\psi(ip_4, \underline{p}) = \Phi(p_4, \underline{p})$$

and

$$\bar{\psi}(ip_4, \underline{p}) = \bar{\Phi}(-p_4, \underline{p})$$

and if $\psi(p_4, \underline{p}) = \dfrac{\Phi}{(p^2 + c^2)^2 + 4E^2 p_4^2}$

then the integral becomes

$$\frac{1}{V} = E \int dp \, (1 + \lambda I(p) \, \frac{\psi(p_4, \underline{p}) \, \overline{\psi}(-p_4, \underline{p})}{\left[(p^2 + c^2)^2 + 4E^2 p_4^2\right]^2} \qquad (7.14)$$

If the variables are transformed to the (a, β , θ , φ) used earlier and the solutions of Chapter 3 inserted in (7.14) the final result is rather complicated. Since the nucleons are assumed identical, only the solutions whose quantum number k is even are to be included, because the others do not satisfy the condition $\psi(p) = \psi(-p)$. From the first term in (7.14) the essential integral is, if $z = \cosh^2 a$,

$$\int_1^\infty \sqrt{\frac{z}{z-1}} \, (S(z))^2 \, \frac{dz}{(z-e)^2}$$

Since $S(z) \simeq z^{-\frac{n}{2}}$ as $z \to \infty$ and $S(z)$ is a constant as $z \to 1$, this integral is convergent. The second term in (7.14), containing the factor $I(p)$ cannot be reduced to an integral over a single variable but it seems clear that it improves the convergence. Thus the present solutions are normalizable according to this normalization condition.

If the ladder approximation were not made the equation (7.12) would be replaced by

$$\frac{1}{V} = E \int dp \, (1 + \lambda I_1(p) + \lambda^2 I_2(p) + \cdots) \overline{\psi} \psi$$

The normalization could break down if the power series in the coupling constant were not convergent or if one of the later terms were divergent. Presumably the first difficulty will only occur if the full Bethe-Salpeter kernel is not convergent, and since the first factor $I_1(p)$

improves the convergence, it is reasonable to suppose the later terms will not make it any worse.

The normalization given by Nishijima (21) is

$$\frac{1}{V} = 4i\pi\lambda \int dp\, dq \frac{(p_0 + E)\ \overline{\Phi}(p)\ \Phi(q)}{((p+E)^2 - M^2)((p-q)^2 + i\varepsilon)}$$

which, when the integral equation for $\Phi(p)$ is used leads to

$$\frac{1}{V} = 4\pi^3 \int dp\ (p_0 + E) \left[(p - E)^2 - M^2 \right] \overline{\Phi}(p)\ \Phi(p)$$

Presumably this expression should be symmetrised in p_0 and it then becomes

$$\frac{1}{V} = 4\pi^3 E \int dp\ (p_0^2 + \underline{p}^2 + c^2)\ \overline{\Phi}(p)\ \Phi(p)$$

This may not be quite correct as no account has been taken of the analytic properties of $\Phi(p)$. However it obviously is quite different from (7.12) and does not even correspond to the first term of (7.12). Since Nishijima's condition is more stringent than (7.12) and he finds that the solutions of Wick's equation are normalizable it is not surprising that they can also be normalised by the present method.

It is interesting to see what the normalisation condition gives in a non relativistic approximation. As has been shown in a previous chapter in the instantaneous interaction approximation, the wave function $\Phi(p)$ can be replaced by

$$\Phi(p) = \frac{v\ (\underline{p})}{(\underline{p}^2 + c^2)^2 + 4E^2 p_4^2}$$

and if this is inserted into the first term of the equation (7.14) (since λ must certainly be small if this approximation is used) one gets

$$\frac{1}{V} = \frac{\pi E}{16} \int \frac{d^3p\ \bar{v}(\underline{p})\ v(\underline{p})\ (4M^2 + c^2 + 5p^2)}{(M^2 + p^2)^{3/2}\ (p^2 + c^2)^3}$$

where now $p = |\underline{p}|$. On the other hand various authors (1), (32) suggest that the analogue in three dimensions of $\Phi(p)$ is

$$\varphi(\underline{p}) = \int \Phi(p)\, dp_4$$

Hence

$$\varphi(\underline{p}) = \frac{\pi\, v(\underline{p})}{2\sqrt{M^2 + p^2}\,(p^2 + c^2)}$$

Thus the normalisation condition is

$$\frac{1}{V} = \frac{E}{4\pi} \int d^3p\ \bar{\varphi}(\underline{p})\, \varphi(\underline{p})\ \frac{4M^2 + c^2 + 5p^2}{\sqrt{M^2+p^2}\,(p^2+c^2)} \qquad (7.15)$$

Although this differs considerably from the usual normalisation of non-relativistic wave mechanics it must be remembered first that φ does not satisfy a Schrodinger equation. On the other hand Nishijima's normalization in this approximation gives

$$\frac{1}{V} = 4\pi^2 E \int d^3p\ \bar{\varphi}(\underline{p})\, \varphi(\underline{p}) \sqrt{M^2 + p^2}$$

which is certainly simpler but equally differs from the usual non-relativistic normalization.

Chapter 8.

CONCLUSION.

The work described in the preceding chapters has not led to any major revision of accepted ideas on Wick's equation so that few general conclusions are in order. Some of the principal features of each chapter are summarized here and some additional comments made.

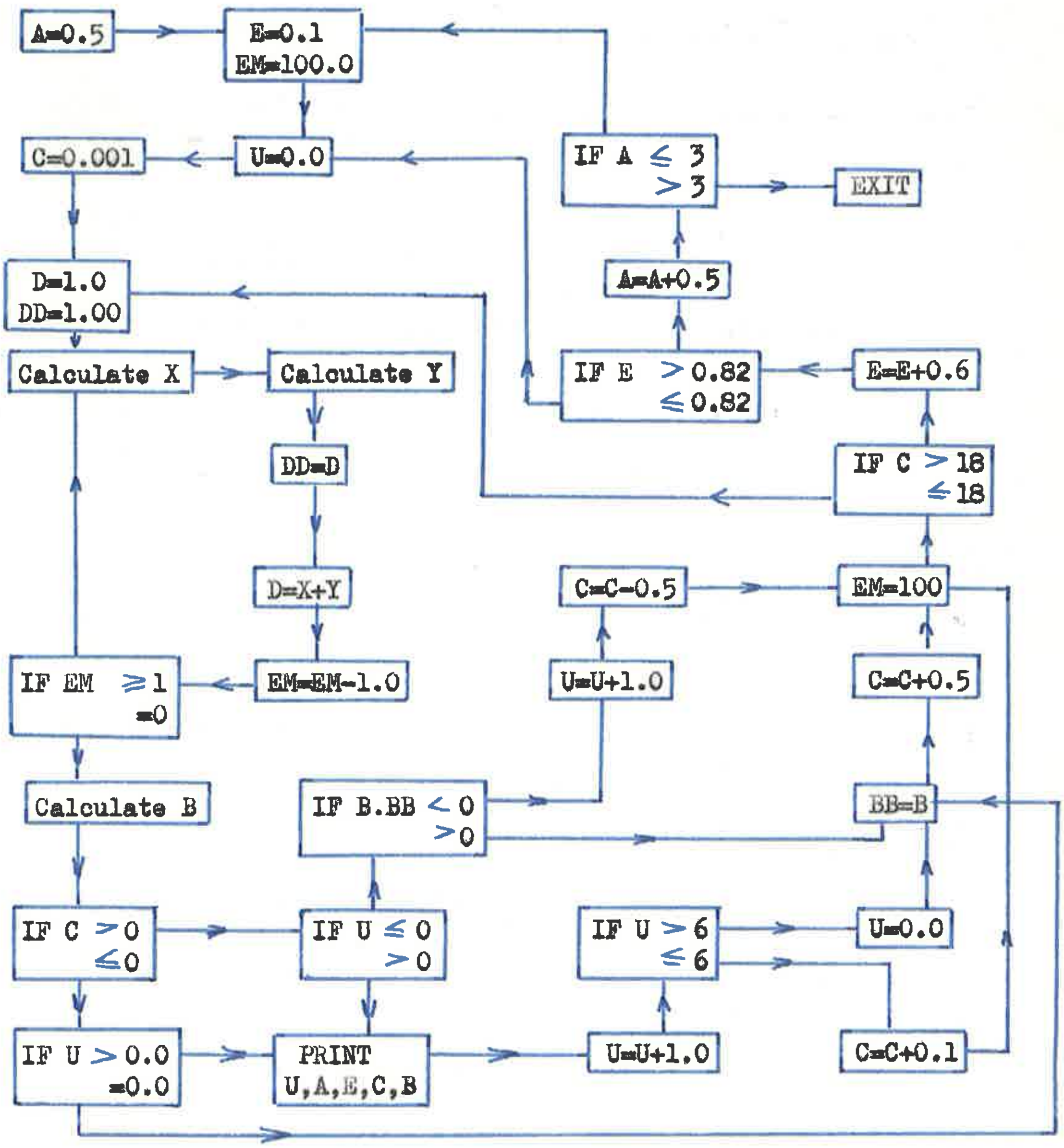
In Chapter 3 it was found that the analytic properties of the solutions in the complex p_0 plane are simpler than those found by Wick for the general Bethe-Salpeter wave function. The same discovery was made by Watanabe (39), and it is difficult to avoid concluding that it is a consequence of the ladder approximation.

Just because of this analytic structure, it was very difficult to solve the integral equation in the Minkowski metric so the imaginary relative energy was introduced. Similarly in deriving the normalization condition in Chapter 7 the change to imaginary relative time was essential. This seems to be only a matter of convenience but might have deeper significance. At least it may be desirable in all bound state problems.

The normalization of Chapter 7 differs considerably from that previously proposed by Mandelstam and Allcock. However, as emphasized by Green (29), the exact normalization may not matter very much; since the total charge is presumably a constant of the motion, Mandelstam's condition must still be applicable. Whereas in non relativistic wave mechanics, the normalization condition is used to determine the eigenvalues, in the present theory they are given directly since an integral equation is used. Similarly Green and Biswas (24) found that simple regularity conditions were sufficient to produce eigenvalues of the two-fermion problem. The one situation in which the exact form of the normalization could be important is in the elimination of unwanted solutions.

The abnormal solutions remain one of the intractable problems of Wick's equation. While it must be remembered that the ladder approximation is not to be trusted for $\lambda \geq \frac{1}{4}$ it would still be preferable to find reason to eliminate them. It might be possible to make various perturbations to the potential $\Delta(x_1 - x_2)$ which would still allow the separation of variables and lead to a reasonably simple differential equation in one variable. If the perturbation was not too unrealistic and if all the eigenvalues then tended to zero a part of the problem would then be removed. However in one sense it would be worse in view of the studies of Mugibayashi (36) on the static model where it was shown that the abnormal solutions do not correspond to eigenstates of the Hamiltonian. The singular behaviour of the propagator at $\lambda = \frac{1}{4}$ due to the confluence of all the abnormal solutions does not seem important at the present stage; most likely it is a feature of the ladder approximation and would not occur in higher approximations.

PROGRAM 18 - FLOW SHEET.



Text Symbols	$\frac{1}{2}n$	e	ν	λ	d_ν	$d_{\nu+1}$
FORTRAN Symbols	A	E	EM	C	D	DD

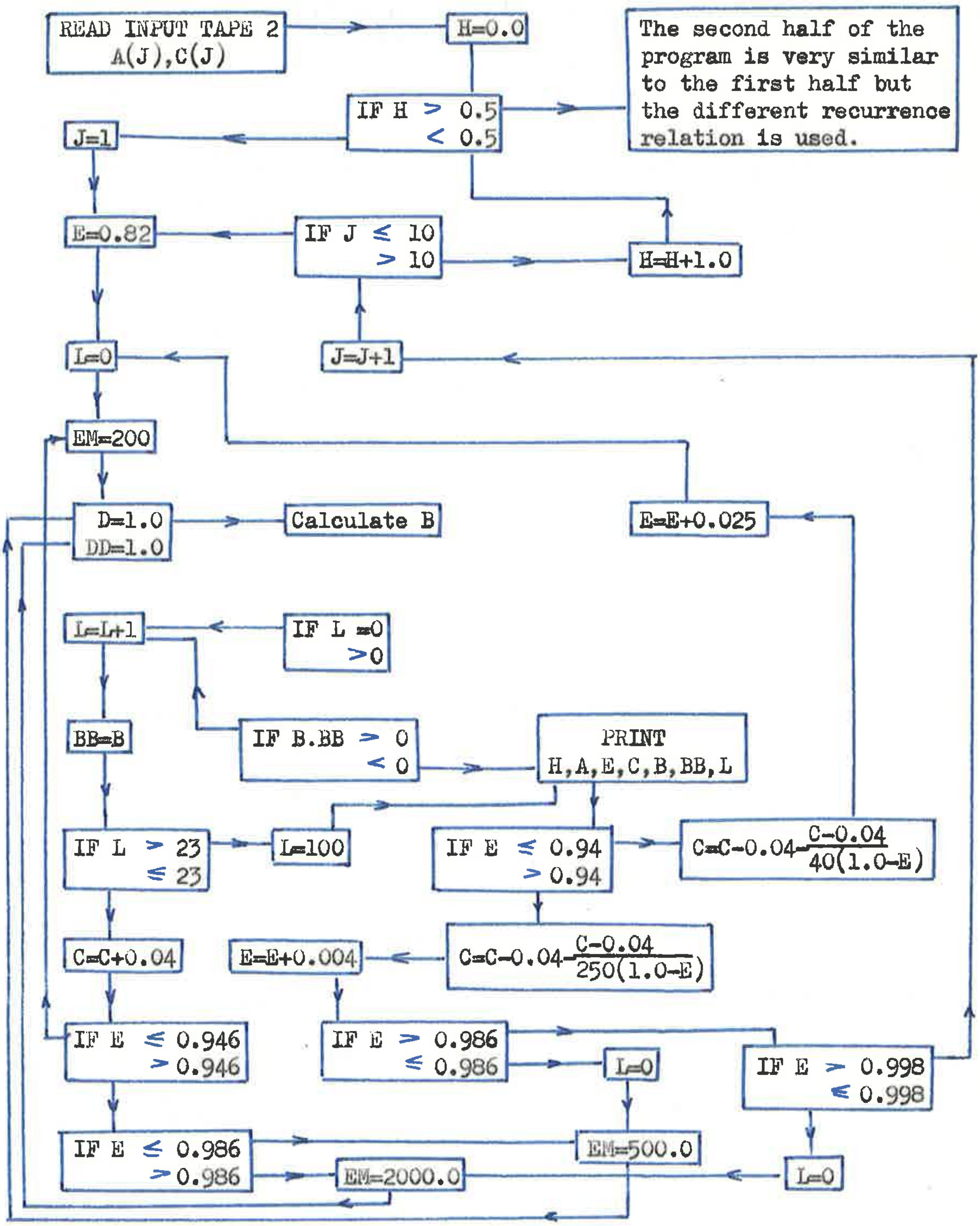
$$B = d_0 \left[\frac{\lambda}{4} - \frac{n}{2}\left(\frac{n+1}{2}\right)\right] + e d_1 (n+1)$$

```
*E5        18 UNIVERSITY ADELAIDE L H D REEVES                                   4
*          EXEC TIME 4 MIN
C          EIGENVALUES OF WICKS EQUATION (H=0)
*          XEQ
*          CARDS COLUMN
  2        A=0.5
  3        E=0.1
  4        EM=100.0
  5        U=0.0
  6        C=0.001
  7        D=1.0
  8        DD=1.0
  9        X=D*((EM+A)*(EM+A+0.5)+EM*E*(EM+2.0*A)-0.25*C)/((EM+A-0.5)*(EM+A-
         1 1.0))
           Y=-E*DD*((EM+1.0)*(EM+2.0*A+1.0))/((EM+A-0.5)*(EM+A-1.0))
 10        DD=D
           D=X+Y
           EM=EM-1.0
 11        IF(EM)14,14,9
 14        B=D*(0.25*C-A*(A+0.5))+E*DD*(2.0*A+1.0)
 17        IF(C-0.001)18,18,44
 18        IF(U)19,19,38
 19        BB=B
 21        C=C+0.5
 22        EM=100.0
 23        IF(C-18.0)7,7,24
 24        E=E+0.06
 33        IF(E-0.82)5,5,34
 34        A=A+0.5
 35        IF(A-3.0)3,3,53
 38        PRINT52,(U,A,E,C,B)
           U=U+1.0
 39        IF(U-6.0)40,40,43
 40        CONTINUE
 41        C=C+0.1
           GOTO22
 43        U=0
           GOTO19
 44        IF(U)45,45,38
 45        IF(B*BB)46,46,49
 46        U=U+1.0
 47        C=C-0.5
           GOTO22
 49        IF(E-0.16)51,50,51
 50        PRINT52,(U,A,E,C,B)
           GOTO19
 51        IF(E-0.76)19,50,19
 52        FORMAT(F4.0,F5.1,F8.3,F8.3,E15.6)
           FREQUENCY17(0,1,40),18(0,10,1),23(40,1,1),33(40,1,1),
         1 35(6,1,1),39(6,1,1),44(0,2,1)
 53        CALLEXIT
```

90

```
END(0,0,1,0,0)
```

PROGRAM 21 - FLOW SHEET



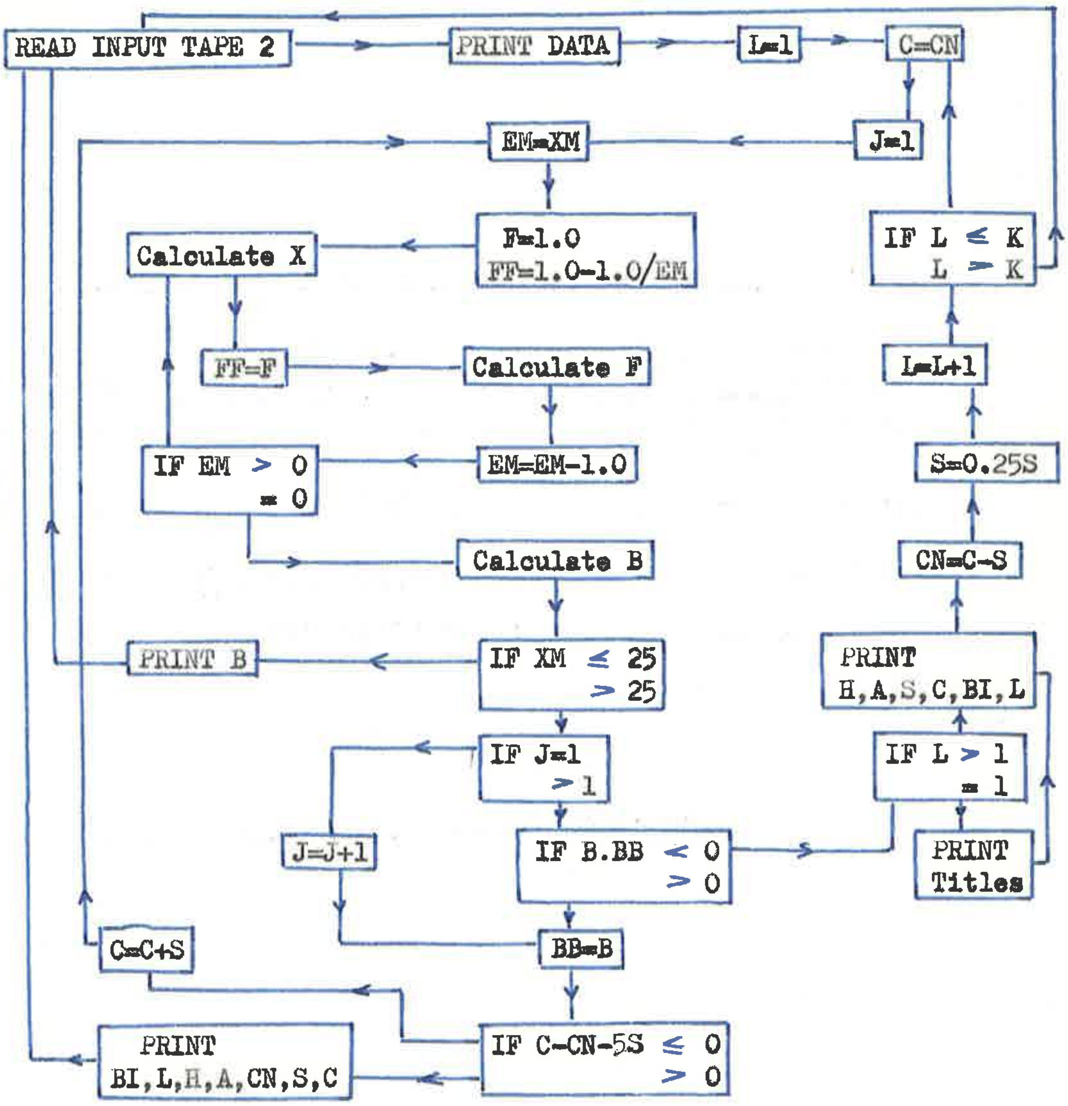
The correspondence of symbols is as before.

```
*E5         21 UNIVERSITY ADELAIDE LHD REEVES                                  3
*           EXEC TIME 10 MIN
C           EIGENVALUES OF WICKS EQUATION
*           XEQ
*           CARDS COLUMN
            DIMENSIONA(19),C(19)
            READINPUTTAPE2,3,(A(J),C(J),J=1,19)
  3         FORMAT(6(F5.1,F7.2))
            H=0.0
  1         IF(H-0.5)2,2,4
  2         J=1
  5         E=0.82
  6         L=0
  7         EM=200.0
 21         D=1.0
            DD=1.0
  8         X=D*((EM+A(J))*(EM+A(J)+0.5)+EM*E*(EM+2.0*A(J))-0.25*C(J))
     1      /((EM+A(J)-0.5)*(EM+A(J)-1.0))
            Y=-E*DD*((EM+1.0)*(EM+2.0*A(J)+1.0))/((EM+A(J)-0.5)*(EM+A(J)
     1      -1.0))
            DD=D
            D=X+Y
            EM=EM-1.0
 53         IF(EM-0.5)9,8,8
  9         B=D*(0.25*C(J)-A(J)*(A(J)+0.5))+E*DD*(2.0*A(J)+1.0)
 54         IF(L)10,10,11
 10         L=L+1
            BB=B
 55         IF(L-23)12,12,13
 12         C(J)=C(J)+0.04
            IF(E-0.946)7,7,14
 11         IF(B*BB)15,15,10
 13         L=100
 15         PRINT16,(H,A(J),E,C(J),B,BB,L)
 16         FORMAT(F4.0,F5.1,F7.3,F9.5,2E13.4,I4)
 56         IF(E-0.94)17,17,18
 17         C(J)=C(J)-0.04-(0.025*(C(J)-0.04)/(1.0-E))
            E=E+0.025
            GOTO6
 18         C(J)=C(J)-0.04-(0.004*(C(J)-0.04)/(1.0-E))
            E=E+0.004
 57         IF(E-0.986)19,19,20
 19         L=0
 25         EM=500.0
            GOTO21
 20         IF(E-0.998)22,22,23
 22         L=0
 24         EM=2000.0
            GOTO21
 14         IF(E-0.986)25,25,24
 23         J=J+1
```

```
58      IF(J-10)5,5,26
26      H=H+1.0
        GOTO1
 4      E=0.05
27      L=0
28      IF(E-0.86)29,29,30
29      EM=100.0
31      D=1.0
        DD=1.0
32      X=D*((EM+A(J)+0.5)*(EM+A(J)+1.0)+E*EM*(EM+2.0*A(J))-0.25*C(J))
     1  /((EM+A(J)-0.5)*(EM+A(J)))
        Y=-E*DD*(EM+1.0)(EM+2.0*A(J)+1.0)/((EM+A(J)-0.5)*(EM+A(J)))
        DD=D
        D=X+Y
        EM=EM-1.0
59      IF(EM-0.5)33,32,32
33      B=D*(0.25*C(J)-(A(J)+1.0)*(A(J)+0.5))+E*DD*(2.0*A(J)+1.0)
        IF(L)34,34,35
34      L=L+1
        BB=B
        IF(L-23)36,36,37
36      C(J)=C(J)+0.04
        GOTO28
35      IF(B*BB)38,38,34
38      PRINT39,(H,A(J),E,C(J),B,BB,L)
39      FORMAT(F4.0,F5.1,F7.4,F9.5,2E13.4,I4)
40      IF(E-0.84)41,41,42
41      C(J)=C(J)-0.04-(0.1*(C(J)-0.04)/(1.0-E))
        E=E+0.1
        GOTO27
42      IF(E-0.94)43,43,44
43      C(J)=C(J)-0.04-(0.025*(C(J)-0.04)/(1.0-E))
        E=E+0.025
        GOTO27
44      C(J)=C(J)-0.04-(0.0035*(C(J)-0.04)/(1.0-E))
        E=E+0.0035
45      IF(E-0.996)27,27,46
37      L=100
        GOTO38
30      IF(E-0.951)47,47,48
47      EM=200.0
        GOTO31
48      IF(E-0.986)49,49,50
49      EM=500.0
        GOTO31
50      EM=2000.0
        GOTO31
46      J=J+1
```

```
51        IF(J-19)4,4,52
52        CALLEXIT
          FREQUENCY1(1,0,1),53(1,0,500),54(0,1,2),55(100,1,0),
        1 11(1,0,2),56(3,0,2),57(9,0,1),20(20,0,1),14(9,0,1),
        2 58(10,1,0),28(1,0,1),59(1,0,500),35(1,0,3),40(2,0,3),
        3 42(3,0,2),45(25,0,1),30(2,0,1),48(10,0,1),51(10,1,0)
          END(1,0,1,0,0)
*         DATA
 0.5       0.52  0.5    5.33  0.5  13.70  1.0    1.44  1.0    8.06  1.5    2.75
 1.5      11.12  2.0    4.42  2.0  14.55  2.5    6.46  0.5    5.79  0.5   19.42
 0.5      40.86  1.0   11.57  1.0  29.11  1.5   19.24  1.5   40.69  2.0   28.81
 2.5      40.29
```

PROGRAM 180 - FLOW SHEET



Text Symbols	$\frac{1}{2}n$	$(1-e)$	ν	$\lambda/4$	f_ν	$f_{\nu+1}$
FORTRAN Symbols	A	BI	EM	C	F	FF

$$B = n\,e\,f_1 + \left(\frac{\lambda}{4} - \left(\frac{n}{2} + H\right)\left(\frac{n+1}{2} + H\right)\right) f_o$$

```
*E5         180 UNIVERSITY ADELAIDE L H D REEVES
*E5         180 UNIVERSITY ADELAIDE L H D REEVES
*           EXEC TIME 5 MIN
*           CARDS COLUMN
C           DOUBLE PRECIS CHECK
            PRINT1,
      1     FORMAT(36H DOUBLE PRECIS CHECK (C=0.25*LAMBDA))
            PRINT4,
      4     FORMAT(56H K  H   A            BI           XM         CN          S
     1      )
      2     READINPUTTAPE2,3,K,H,A,BI,XM,CN,S
      3     FORMAT(I2,F4.1,F4.1,E16.7,E13.3,F9.6,2PF7.4)
            PRINT5,K,H,A,BI,XM,CN,S
      5     FORMAT(I2,F4.1,F4.1,E16.7,E13.3,F9.6,2PF7.4,5H DATA)
            L=1
      6     C=CN
            J=1
D     7     EM=XM
D           F=1.0
D           FF=1.0-1.0/EM
D     8     X=(1.0-BI)*(EM*F-(EM+1.0)*FF)+F*((EM+A+H)*(EM+A+H+0.5)-C)/(EM+2.0*
     1      A)
D           FF=F
D           F=(EM+2.0*A-1.0)*X/((EM+A+H-0.5)*(EM+A+H-1.0))
D           EM=EM-1.0
            IF(EM-0.1)9,9,8
      9     B=2.0*A*(1.0-BI)*FF+F*(C-(A+H)*(A+H+0.5))
            IF(XM-25.0)22,22,23
     22     PRINT24,B
     24     FORMAT(9H        B=,E14.5)
            GOTO2
     23     IF(J-1)11,11,10
     10     IF(B*BB)16,16,12
     11     J=J+1
     12     BB=B
            IF(C-CN-5.0*S-1.0E-7)13,14,14
     13     C=C+S
            GOTO7
     14     PRINT15,(BI,L,H,A,CN,S,C)
     15     FORMAT(E16.6,I3,F5.1,F5.1,F9.6,F9.6,3H C=,F9.6)
            GOTO2
     16     IF(L-1)17,17,19
     17     PRINT18,
     18     FORMAT(1H050H    H      A       S         C              BI         L)
     19     PRINT20,(H,A,S,C,BI,L)
     20     FORMAT(1XF8.1,F5.1,F9.6,F9.6,E17.6,I3)
            CN=C-S
            S=0.25*S
            L=L+1
            IF(L-K)6,6,2
            END
```

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