INTERIOR GRADIENT BOUNDS FOR NON-UNIFORMLY ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS OF DIVERGENCE FORM

by

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CONTENTS

SUMMARY

Let Ω be a domain in R^n. The equation

$$\sum_{i,j=1}^{n} A_{ij}(x,u,\nabla u)\frac{\partial^2 u}{\partial x_i \partial x_j} = B(x,u,\nabla u) ,$$

where $B(x,z,p)$, $A_{ij}(x,z,p)$, $i,j=1,\ldots,n$, are given functions of the variables $(x,z,p) \in \Omega\times R\times R^n$, and where

$$0< \sum_{i,j=1}^{n} A_{ij}(x,z,p)\xi_i\xi_j \ , \xi \in R^n-\{0\}, (x,z,p) \in \Omega\times R\times R^n ,$$

is called a quasilinear elliptic equation on Ω.

We will here be concerned only with quasilinear elliptic equations which can be written in divergence form - i.e. equations of the form

$$(1) \quad \sum_{i=1}^{n}\frac{d}{dx_i} A_i(x,u,\nabla u) \left(\equiv \sum_{i,j=1}^{n} A_{ip_j}(x,u,\nabla u)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} A_{iz}(x,u,\nabla u)\frac{\partial u}{\partial x_i} + \sum_{i=1}^{n} A_{ix_i}(x,u,\nabla u)\right) = B(x,u,\nabla u) ,$$

where the vector function $A(x,z,p) = (A_1(x,z,p),\ldots,A_n(x,z,p))$ is such that

$$0< \sum_{i,j=1}^{n} A_{ip_j}(x,u,\nabla u)\ \xi_i\xi_j , \xi \in R^n-\{0\}, \ (x,z,p) \in \Omega\times R\times R^n .$$

Note that (1) can be written in the integral form

$$\sum_{i=1}^{n} \int_\Omega A_i(x,u,\nabla u)\ \frac{\partial \zeta}{\partial x_i}\ dx = - \int_\Omega B(x,u,\nabla u)\ \zeta\ dx , \quad \zeta \in Lip_c(\Omega) ,$$

where $Lip_c(\Omega)$ is the set of all Lipschitz functions with compact support contained in Ω.

In case there exist positive continuous functions $c(z)$, $c'(z)$, defined for all $z \in R$ and such that

$$c(z)\,|\xi|^2 \leq \sum_{i,j=1}^{n} A_{i p_j}(x,z,p)\xi_i\xi_j \leq c'(z)\,|\xi|^2 ,$$

for all $\xi \in R^n$ and $(x,z,p) \in \Omega\times R\times R^n$,

then (1) is said to be uniformly elliptic .

If however the maximum and minimum eigenvalues of the matrix $\frac{1}{2}(A_{i p_j}(x,z,p) + A_{j p_i}(x,z,p))$, denoted $\Lambda(x,z,p)$ and $\lambda(x,z,p)$ respectively , are such that the quotient

$$\Lambda(x,z,p)/\ \lambda(x,z,p)$$

is not bounded as $|p| \to \infty$, then (1) is said to be non-uniformly elliptic . An example of a non-uniformly elliptic divergence form equation of classical interest is the minimal surface equation

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_i} / \sqrt{1 + |\nabla u|^2}\right) = 0 .$$

For this example we have

$$A_i(x,z,p) = p_i/\sqrt{1 + |p|^2}$$

and

$$\lambda(x,z,p) = (1+ |p|^2)^{-3/2},\ \Lambda(x,z,p)=(1+ |p|^2)^{-1/2},$$

so that

$$\Lambda(x,z,p)/\ \lambda(x,z,p) = 1 + |p|^2.$$

This thesis will be concerned with non-uniformly elliptic equations. In particular , we will be concerned with proving the existence of local a-priori interior gradient bounds for sufficiently smooth solutions to equations of the form (1). That is , for suitable equations of the form (1) we will establish the existence of a real-valued function $\Gamma(\rho,M)$, defined for $\rho>0$, $M>0$, such that any sufficiently smooth solution u of (1) with $|u|\leq M$ must satisfy

$$\sup_{\Omega'} |\nabla u| \leq \Gamma(\rho,M)$$

for all subdomains $\Omega' \subset \Omega$ which are such that the distance between Ω' and $\partial\Omega$ is no less than ρ.

It is to be emphasised that $\Gamma(\rho,M)$ does not depend on the particular solution u , but that $\Gamma(\rho,M)$ may not be bounded as $\rho \to 0$ or as $M \to \infty$. In fact it can be arranged that Γ does not depend explicitly on the functions A , B of (1) ; rather it can be arranged that Γ depends on certain structural quantities which are invariant under appropriate changes in A and B .

It is not difficult to demonstrate the importance of such local gradient bounds. For example , many of the results in [1] can be extended to the case when the boundary data is merely continuous (rather than C^2) provided a local gradient bound is known .

Generally speaking , one can think of a local gradient bound as locally reducing a non-uniformly elliptic equation to a uniformly elliptic one , thus making it possible to use the theory of uniformly elliptic equations to study the solutions to non-uniformly elliptic equations .

The gradient bounds obtained in this thesis generalize the work of Bombieri-de Giorgi-Miranda [3] and Ladyzhenskaya-Ural'tseva [4],Part II . The main results appear in Chapter 2 . The techniques used are generally modifications and extensions of those used by the above authors . The main analytic tool is the Sobolev-type inequality derived in Chapter 1 . Chapter 3 consists of extensions of the results of Chapter 2 to other equations , under the assumption that an estimate of Hölder continuity of the solutions is already known . In Chapter 4 refined gradient estimates for a certain sub-class of those equations dealt with in Chapter 2 are obtained .

This thesis contains no material which has been accepted for the award of any other degree or diploma in any University. To the best of my knowledge and belief, the thesis contains no material previously published or written by any other person, except where due reference is made in the text of the thesis.

(L.M. Simon)

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CHAPTER 1

A SOBOLEV-TYPE INEQUALITY ON C^2 SUBMANIFOLDS OF R^n

In [2] a Sobolev-type inequality on non-parametric C^2 minimal hypersurfaces was established. The result was later used by de-Giorgi (see [3]) to obtain an interior gradient bound for solutions to the minimal hypersurface equation. In Part II of [4] Ladyzhenskaya and Ural'tseva generalized these results to non-parametric hypersurfaces which represented solutions of certain class of divergence form quasilinear elliptic partial differential equations.

The Sobolev-type inequalities in both [2] and [4] were obtained using the theory of integral currents, and in particular an isoperimetric inequality, established by Federer and Fleming.

In 1970 J. H. Michael produced a simple test function proof of the inequality in [2] which led to the results in the Appendix on minimal hypersurfaces. It was during the course of an attempt to generalize this proof that the inequality to be presented in this chapter was discovered. The inequality, given in Section 3, will be used in Chapter 2 to give gradient bounds for the solutions of a large class of divergence form quasilinear elliptic equations, including all those treated in Part II of [4].

1. C^2 Submanifolds of R^n

This section is devoted to a discussion of some basic definitions and results concerning C^2 submanifolds of R^n. Except for the material concerning the sum of squares of principal curvatures, the discussion here essentially follows that in [5].

M will denote an m-dimensional C^2 submanifold of $R^n (1 \leq m \leq n)$. By this we mean that $M \subset R^n$ and that for each point $x_0 \in M$ there is an open set $U \subset R^m$ together with a one-one C^2 map $x: U \to R^n$ such that $x_0 \in V \cap M \subset x(U) \subset M$ for some open set $V \subset R^n$ and such that $\frac{\partial x(u)}{\partial u_i}$, $i = 1, \ldots, m$, are linearly independent for each $u = (u_1, \ldots, u_m) \in U$. Such a map x will be called a representation for M near x_0.

The m-dimensional subspace of R^n spanned by $\frac{\partial x(u_0)}{\partial u_1}, \ldots, \frac{\partial x(u_0)}{\partial u_m}$ is called the tangent space of M at $x_0 = x(u_0)$, denoted by $T_{x_0}(M)$. $T_{x_0}(M)$ is clearly independent of the particular map x used to represent M near x_0.

The orthogonal complement of $T_{x_0}(M)$, taken in R^n, is called the normal space to M at x_0, and is denoted by $N_{x_0}(M)$.

Relative to the map $x: U \to R^n$ representing M near x_0, let g_{ij}, $i,j = 1,\ldots,m$, and g be defined on U by

$$g_{ij} = \frac{\partial x}{\partial u_i} \cdot \frac{\partial x}{\partial u_j} = \sum_{\ell=1}^{n} \frac{\partial x_\ell}{\partial u_i} \frac{\partial x_\ell}{\partial u_j}, \tag{1.1}$$

and

$$g = \det (g_{ij}). \tag{1.2}$$

It is easily checked that (g_{ij}) is a positive definite symmetric matrix, so that $g > 0$. Also, for any continuous function χ defined on M with spt (χ) (the support of χ) a compact subset of $x(U)$, we have

$$(1.3) \qquad \int_U \chi(x(u)) \sqrt{g(u)}\, du_1 \ldots du_m = \int_M \chi\, dH_m ,$$

where H_m denotes m-dimensional Hausdorff measure in R^n.

Using the notation $(g^{ij}) = (g_{ij})^{-1}$ let the $n \times n$ matrix $(\tilde{g}^{ij}(x_0))$ be defined by

$$(1.4) \quad \tilde{g}^{ij}(x_0) = \sum_{r,s=1}^{m} \frac{\partial x_i}{\partial u_r}(u_0) \frac{\partial x_j}{\partial u_s}(u_0)\, g^{rs}(u_0), i,j = 1,\ldots,n.$$

Then $(\tilde{g}^{ij}(x_0))$ is independent of the particular map $x: U \to R^n$ used to represent M near x_0. In fact, the matrix $(\tilde{g}^{ij}(x_0))$ represents the linear transformation of R^n into itself which leaves invariant all the vectors in the tangent space $T_{x_0}(M)$ and which takes to zero all the vectors in $N_{x_0}(M)$. That is, $(\tilde{g}^{ij}(x_0))$ is the matrix of the projection taking R^n onto the tangent space $T_{x_0}(M)$, and $(\delta_{ij} - \tilde{g}^{ij}(x_0))$ is the matrix of the projection taking R^n onto $N_{x_0}(M)$. Verification of these facts is left to the reader.

Now let $v = \sum_{i=1}^{m} a_i \frac{\partial x(u_0)}{\partial u_i}$ be a unit vector in $T_{x_0}(M)$. Then the vector $\omega = (\omega_1,\ldots,\omega_n)$ defined by

$$(1.5) \quad \omega_r = \sum_{s=1}^{n} \sum_{i,j=1}^{m} a_i a_j \left(\delta_{rs} - \tilde{g}^{rs}(x_0)\right) \frac{\partial^2 x_s(u_0)}{\partial u_i \partial u_j}, r=1,\ldots,n,$$

is called the normal curvature vector to M at x_0 in the direction v, and is independent of the particular map $x: U \to R^n$ used to represent M near x_0. Geometrically, if $y(s)$

is a C^2 curve in M with s the parameter of arc-length, with $y(0) = x_0$, and with $y'(0) = v$, then ω is the normal component of the vector $\frac{d^2y(0)}{ds^2}$ (see [5]).

We now define non-negative quantities $\mathscr{C}^2(x_0)$ and $\mathscr{C}^2(x_0,\xi)$, where $\xi \in N_{x_0}(M)$, by

$$\mathscr{C}^2(x_0) = \sum_{i,j=1}^{n} \sum_{r,r',s,s'=1}^{m} (\delta_{ij} - \tilde{g}^{ij}(x_0))\, g^{rr'}(u_0) g^{ss'}(u_0) \frac{\partial^2 x_i(u_0)}{\partial u_r \partial u_s} \frac{\partial^2 x_j(u_0)}{\partial u_{r'} \partial u_{s'}},$$

(1.6)

$$\mathscr{C}^2(x_0,\xi) = \sum_{r,r',s,s'=1}^{m} g^{rr'}(u_0) g^{ss'}(u_0) \frac{\partial^2 x(u_0)}{\partial u_r \partial u_s} \cdot \xi \, \frac{\partial^2 x(u_0)}{\partial u_{r'} \partial u_{s'}} \cdot \xi .$$

It is easily checked that $\mathscr{C}^2(x_0)$ and $\mathscr{C}^2(x_0,\xi)$ are independent of the particular map $x: U \to R^n$ used to represent M near x_0. Also, when $m = n$ we clearly have $\mathscr{C}^2(x_0) = \mathscr{C}^2(x_0,\xi) = 0$. To give a geometric interpretation of $\mathscr{C}^2(x_0)$ and $\mathscr{C}^2(x_0,\xi)$ when $n > m$ and $|\xi| = 1$, suppose $a_{ij}, i,j = 1,\dots,m$ are such that the vectors $v_r = \sum_{i=1}^{m} a_{ri} \frac{\partial x(u_0)}{\partial u_i}$, $r = 1,\dots,m$, form an orthonormal basis for $T_{x_0}(M)$, so that $\sum_{i,j=1}^{m} a_{ri} a_{sj} \frac{\partial x(u_0)}{\partial u_i} \cdot \frac{\partial x(u_0)}{\partial u_j} = \delta_{rs}, r,s = 1,\dots,m$. Then multiplying by $b_{j's} a_{ri'}$, where $(b_{ij}) = (a_{ij})^{-1}$, and summing over r and s, we get $\sum_{i,r=1}^{m} a_{ri'} a_{ri} \frac{\partial x(u_0)}{\partial u_i} \cdot \frac{\partial x(u_0)}{\partial u_{j'}} = \delta_{i'j'}$, which shows that

(1.7) $$g^{ij}(u_0) = \sum_{r=1}^{m} a_{ri} a_{rj}, i,j = 1,\dots,m,$$

so that defining

$$\eta_{rs} = \sum_{i,j=1}^{m} a_{ri} a_{sj} \frac{\partial^2 x(u_0)}{\partial u_i \partial u_j} \cdot \xi, r,s = 1,\ldots,m,$$

and using (1.7) in (1.6), we get

$$\mathscr{C}^2(x_0,\xi) = \sum_{r,s=1}^{m} \eta^2_{rs} . \tag{1.8}$$

Now let (p_{ij}) be an m x m orthogonal matrix which diagonalizes (η_{rs}) to $(\tilde{\eta}_{rs})$, so that $\tilde{\eta}_{rs} = \sum_{r',s',i,j=1}^{m} p_{rr'} a_{r'i} \, p_{ss'} a_{s'j} \frac{\partial^2 x(u_0)}{\partial u_i \partial u_j} \cdot \xi = 0, r \neq s$. Then (1.8) gives

$$\mathscr{C}^2(x_0,\xi) = \sum_{r=1}^{m} \tilde{\eta}^2_{rr} . \tag{1.9}$$

But, by (1.5), η_{rr} is the ξ-component of the normal curvature vector of M at x_0 in the direction v_r, and $\tilde{\eta}_{rr}$ is the ξ-component of the normal curvature vector in the direction $\tilde{v}_r = \sum_{r',i=1}^{m} p_{rr'} a_{r'i} \frac{\partial x}{\partial u_i}, r = 1,\ldots,m$. Thus, since $\mathscr{C}^2(x_0,\xi) = \sum_{r=1}^{m} \tilde{\eta}^2_{rr} = \sum_{r,s=1}^{m} \eta^2_{rs} \geq \sum_{r=1}^{m} \eta^2_{rr}$, we have the geometric interpretation that $\mathscr{C}^2(x_0,\xi)$ is the maximum value attainable by taking the sum of squares of the ξ-components of normal curvature vectors of M at x_0 in the directions specified by an orthonormal basis for $T_{x_0}(M)$.

The directions specified by a basis which, like the basis $\{\tilde{v}_1,\ldots,\tilde{v}_m\}$, gives this maximum value, are called principal directions of M at x_0 relative to ξ. The $\tilde{\eta}_{rr}, r = 1,\ldots,m$, (which are eigenvalues of the matrix (η_{rs}) and also roots of the equation $\det\left(\frac{\partial^2 x(u_0)}{\partial u_i \partial u_j} - \lambda g_{ij}(u_0)\right) = 0$)

are called the principal curvatures of M at x_0 relative to ξ.

Thus (1.9) says that, for $|\xi| = 1$, $\mathscr{C}^2(x_0,\xi)$ is the sum of squares of principal curvatures of M at x_0 relative to ξ. Also, we can deduce from (1.8) that if $\xi^{(m+1)},\ldots,\xi^{(n)}$ form an orthonormal basis for $N_{x_0}(M)$, then

$$(1.10) \qquad \mathscr{C}^2(x_0) = \sum_{i=m+1}^{n} \mathscr{C}^2(x_0,\xi^{(i)}) .$$

In the case when $m = n - 1$ a formula like (1.6) for the sum of squares of principal curvatures is given by Miranda in [6].

It will also be necessary subsequently to refer to the normal mean curvature vector of M at x_0, denoted $\mathscr{H}(x_0) = (\mathscr{H}_1(x_0),\ldots,\mathscr{H}_n(x_0))$. This is defined by

$$(1.11) \quad \mathscr{H}_r(x_0) = \sum_{s=1}^{n} \sum_{i,j=1}^{m} (\delta_{rs} - \tilde{g}^{rs}(x_0))\, g^{ij}(u_0)\, \frac{\partial^2 x_s}{\partial u_i \partial u_j}(u_0), r = 1,\ldots,n.$$

Note that by (1.7) we can write

$$(1.12) \quad \mathscr{H}_r(x_0) = \sum_{s=1}^{n} \sum_{i,j,r'=1}^{m} (\delta_{rs} - \tilde{g}^{rs}(x_0))\, a_{r'i} a_{r'j} \frac{\partial^2 x_s}{\partial u_i \partial u_j}(u_0), r = 1,\ldots,n,$$

so that, by (1.5), $\mathscr{H}(x_0)$ is the sum of normal curvature vectors of M at x_0 in the directions specified by any orthonormal basis of $T_{x_0}(M)$.

Using (1.8),(1.10), and (1.12) it can be quite easily proved that

(1.13) $$|\mathcal{H}(x_0)|^2 \leq m\, \mathscr{E}^2(x_0).$$

The components $\mathcal{H}_r$ of $\mathcal{H}$ are also given by the identity

(1.14) $$\mathcal{H}_r(x) = \sum_{i,j=1}^{m} \frac{1}{\sqrt{g}} \frac{\partial}{\partial u_j} \left(\sqrt{g}\, \frac{\partial x_r}{\partial u_i}\, g^{ij}\right), r = 1,\ldots,n,$$

on U ([5], Theorem (2.4)). To prove this, first note that because $g^{rs}g_{si} = \delta_{ri}$, we have $\frac{\partial}{\partial u_j}(g^{rs}g_{si}) = 0, r,i = 1,\ldots,m$. Thus $g_{si}\frac{\partial}{\partial u_j}(g^{rs}) = -g^{rs}\frac{\partial}{\partial u_j}(g_{si})$ and hence, multiplying by $g^{i\ell}$ and summing over i, we have

(1.15) $$\frac{\partial}{\partial u_j}(g^{r\ell}) = -\sum_{i,s=1}^{m} g^{i\ell} g^{rs} \frac{\partial g_{si}}{\partial u_j}.$$

Also,

(1.16) $$\frac{\partial}{\partial u_j}\sqrt{g} = \tfrac{1}{2}\sqrt{g} \sum_{s,l=1}^{m} g^{s\ell} \frac{\partial g_{s\ell}}{\partial u_j},$$

which clearly follows from the known result that if $b_{ij}(t), i,j = 1,\ldots,m$ are differentiable functions of $t \in R$, $b = \det(b_{ij}) \neq 0$, and $(b^{ij}) = (b_{ij})^{-1}$, then $\frac{db}{dt} = b \sum_{s,\ell=1}^{m} b^{s\ell} \frac{d}{dt} b_{s\ell}$. Then, by (1.15) and (1.16),

$$\frac{\partial}{\partial u_j}\left(\sqrt{g}\,\frac{\partial x_r}{\partial u_i}\, g^{ij}\right) = \tfrac{1}{2}\sqrt{g} \sum_{i,j,\ell,s=1}^{m} g^{s\ell} \frac{\partial g_{s\ell}}{\partial u_j} \frac{\partial x_r}{\partial u_i} g^{ij} + \sqrt{g}\sum_{i,j=1}^{m} g^{ij} \frac{\partial^2 x_r}{\partial u_i \partial u_j} - \sqrt{g} \sum_{i,j,\ell,s=1}^{m} \frac{\partial x_r}{\partial u_i} g^{\ell j} g^{is} \frac{\partial g_{s\ell}}{\partial u_j},$$

and, using the identity $$\frac{\partial g_{s\ell}}{\partial u_j} = \sum_{q=1}^{n} \left(\frac{\partial x_q}{\partial u_s} \frac{\partial^2 x_q}{\partial u_\ell \partial u_j} + \frac{\partial^2 x_q}{\partial u_s \partial u_j} \frac{\partial x_q}{\partial u_\ell}\right),$$

it follows that

$$\frac{\partial}{\partial u_j}\left(\sqrt{g}\,\frac{\partial x_r}{\partial u_i}\,g^{ij}\right) =$$

$$\sqrt{g}\sum_{i,j=1}^{m}\left(g^{ij}\frac{\partial^2 x_r}{\partial u_i\partial u_j} - \sum_{q=1}^{n}\sum_{\ell,s=1}^{m}\frac{\partial x_r}{\partial u_i}\frac{\partial x_q}{\partial u_s}g^{is}g^{\ell j}\frac{\partial^2 x_q}{\partial u_\ell \partial u_j}\right).$$

Thus the required identity is established since, by (1.4),

$$\sum_{i,s=1}^{m}\frac{\partial x_r}{\partial u_i}\frac{\partial x_q}{\partial u_s}g^{is} = \tilde{g}^{qr}(x).$$

A real valued function h with domain M will be called a $C'(M)$ function if, for each local representation $x\colon U \to R^n$ of M, the function $\bar{h}$ defined on U by $\bar{h}(u) = h(x(u))$ has continuous partial derivatives. Thus $\tilde{g}^{ij} \in C'(M), i,j = 1, \ldots, m$.

For $h \in C'(M)$, $\delta h = (\delta_1 h, \ldots, \delta_n h)$ will denote the continuous vector function on M defined by

$$(1.17)\quad \delta_i h(x_0) = \sum_{r,s=1}^{m}\frac{\partial x_i(u_0)}{\partial u_r}\,g^{rs}(u_0)\,\frac{\partial \bar{h}(u_0)}{\partial u_s}, i = 1,\ldots,n,$$

where $x\colon U \to R^n$ is a representation for M near $x_0 = x(u_0)$, and $\bar{h}$ is defined by $\bar{h}(u) = h(x(u))$. It is easily checked that $\delta h(x_0)$ defined by (1.17) is independent of the particular representation for M near x_0 and that the dependence of δh on x_0 is continuous. We note that on U

$$|\delta h(x)|^2 = \sum_{r,r',s,s'=1}^{m}\sum_{i=1}^{n}\frac{\partial x_i}{\partial u_r}\frac{\partial x_i}{\partial u_{r'}}g^{rs}g^{r's'}\frac{\partial \bar{h}}{\partial u_s}\frac{\partial \bar{h}}{\partial u_{s'}},$$

so that, by (1.1),

$$(1.18)\qquad |\delta h(x)|^2 = \sum_{s,s'=1}^{m} g^{ss'}\frac{\partial \bar{h}}{\partial u_s}\frac{\partial \bar{h}}{\partial u_{s'}}.$$

If V is an open subset of R^n containing M and if $h \in C'(M)$ is the restriction to M of a function $\tilde{h}$ which has continuous partial derivatives $\frac{\partial \tilde{h}(\xi)}{\partial \xi_j}$, $\xi = (\xi_1, \ldots, \xi_n) \in V$, then it follows from (1.4) that

$$\delta_i h(x_0) = \sum_{j=1}^{n} \tilde{g}^{ij}(x_0) \frac{\partial \tilde{h}}{\partial \xi_j}, i = 1, \ldots, n.$$

Thus $\delta h(x_0)$ is the projection of the gradient vector $(\frac{\partial \tilde{h}(x_0)}{\partial \xi_1}, \ldots, \frac{\partial \tilde{h}(x_0)}{\partial \xi_n})$ onto $T_{x_0}(M)$, and

$$|\delta h(x_0)|^2 = \sum_{i,j=1}^{n} \tilde{g}^{ij}(x_0) \frac{\partial \tilde{h}(x_0)}{\partial \xi_i} \frac{\partial \tilde{h}(x_0)}{\partial \xi_j}.$$

We will subsequently make use of the fact that if k is positive integer with $1 \leq k \leq n$ and if $\tilde{g}^{kk}(x_0) \neq 0$ then it is possible to arrange for a representation $x: U \to R^n$ of M near x_0 which is such that for some integer ℓ, $1 \leq \ell \leq m$,

(1.19) $\qquad x_k(u) = u_\ell$ for all $u = (u_1, \ldots, u_m) \in U$.

To show that this can be done, let $\hat{x} : \hat{U} \to R^n$ be any representation for M near x_0 and let $\hat{u}_0 \in \hat{U}$ be such that $\hat{x}(\hat{u}_0) = x_0$. Then, because the $n \times m$ matrix $(\frac{\partial \hat{x}_i(u)}{\partial u_j})$, $u = (u_1, \ldots, u_m) \in \hat{U}$, has rank m, and because $(\frac{\partial \hat{x}_k(\hat{u}_0)}{\partial u_1}, \ldots, \frac{\partial \hat{x}_k(\hat{u}_0)}{\partial u_m}) \neq 0$ (since by (1.4) $\sum_{r,s=1}^{m} \frac{\partial \hat{x}_k(\hat{u}_0)}{\partial u_r} \frac{\partial \hat{x}_k(\hat{u}_0)}{\partial u_s} g^{rs}(\hat{u}_0) = \tilde{g}^{kk}(x_0) \neq 0$) it follows that there must exist integers $i_1, \ldots, i_m$ with $1 \leq i_1 < \ldots < i_m \leq n$,

$i_\ell = k$ for some ℓ, and such that the Jacobian matrix

$$\frac{\partial(\hat{x}_{i_1},\ldots,\hat{x}_{i_m})}{\partial(u_1,\ \ldots,u_m)}$$

is non-singular in some neighbourhood of $\hat{u}_0$. Hence the transformation $(\hat{x}_{i_1},\ldots,\hat{x}_{i_m}) : \hat{U} \to R^m$ has a local inverse χ which is a C^2 function defined on some neighbourhood U of the point $u_0 = (\hat{x}_{i_1}(\hat{u}_0),\ldots,\hat{x}_{i_m}(\hat{u}_0)) \in R^m$. Then the composite function $x = \hat{x}_0\chi : U \to R^n$ is a representation for M near x_0 with the desired property (1.19), where ℓ is such that $i_\ell = k$.

If we now define subsets $M_{k,t}$ of M for $k=1,\ldots,n$, $t \in R$, by

(1.20) $M_{k,t} = \{\xi = (\xi_1,\ldots,\xi_n) \in M;\ \tilde{g}^{kk}(\xi) \neq 0,\ \xi_k = t\}$,

then, when $m \geq 2$, we have either $M_{k,t} = \emptyset$ or that $M_{k,t}$ is an $(m-1)$-dimensional submanifold of R^n. In fact, if $m \geq 2$ and $x_0 \in M_{k,t}$, and if $x : U \to R^n$ is a representation for M near x_0 with the property (1.19), then the restriction of x to $U_t = \{u \in U;\ u_\ell = t\}$ provides a representation for $M_{k,t}$ near x_0. It follows that if χ is a continuous function on M with $\mathrm{spt}(\chi)$ a compact subset of $x(U) \cap \{\xi \in M;\ \tilde{g}^{kk}(\xi) \neq 0\}$, then by (1.3)

(1.21) $$\int_{M_{k,t}} \chi\ dH_{m-1} = \int_{U_t} \sqrt{g_{k,t}(u)}\ \chi(x(u))du_1\ldots du_{\ell-1}du_{\ell+1}\ldots du_m,$$

where $g_{k,t}$ is calculated, in accordance with (1.1)-(1.2), by taking the cofactor of $g_{\ell\ell}$ in the $m\times m$ matrix $(g_{ij}) = \left(\frac{\partial x}{\partial u_i}\cdot\frac{\partial x}{\partial u_j}\right)$. Then, by the adjoint formula for the inverse of a matrix, we must have $g^{\ell\ell} = g_{k,t}/g$. But, using $\frac{\partial x_k(u)}{\partial u_r} = \delta_{\ell r}, r=1,\ldots,m$, in (1.4), we also have $g^{\ell\ell}(u) = \tilde{g}^{kk}(x(u))$, $u\in U$. Hence $g_{k,t}(u) = g(u)\tilde{g}^{kk}(x(u))$, $u\in U_t$, and (1.21) becomes

$$(1.22)\quad \int_{M_{k,t}} \chi\, dH_{m-1} =$$

$$\int_{U_t} \sqrt{\tilde{g}^{kk}(x(u))}\,\sqrt{g(u)}\,\chi(x(u))\, du_1\ldots du_{\ell-1}\, du_{\ell+1}\ldots du_m.$$

Integrating with respect to t and using (1.3) this gives

$$(1.23)\qquad \int_{-\infty}^{\infty}\int_{M_{k,t}} \chi\, dH_{m-1}\, dt = \int_M \sqrt{\tilde{g}^{kk}}\,\chi\, dH_m.$$

By considering a suitable partition of unity for M we see that (1.23) is valid when $\operatorname{spt}(\chi)$ is an arbitrary compact subset of $\{\xi\in M;\ \tilde{g}^{kk}(\xi)\neq 0\}$.

Subsequently, for the manifold $M_{k,t}$ (when $m\geq 2$ and $M_{k,t}\neq 0$), $(\tilde{g}^{ij}_{k,t})$, $\mathscr{C}^2_{k,t}$, and $\delta^{k,t} = (\delta^{k,t}_1,\ldots,\delta^{k,t}_n)$ will denote the quantities corresponding to $(\tilde{g}^{ij})$, $\mathscr{C}^2$, and $\delta = (\delta_1,\ldots,\delta_n)$ respectively defined above for M. Note that if $h\in C'(M)$ then the restriction $h^{k,t}$ of h to $M_{k,t}$ is a $C'(M_{k,t})$ function, so that $\delta^{k,t}_i(h^{k,t})$ makes sense. We will always write $\delta_i^{k,t}(h)$ rather than $\delta_i^{k,t}(h^{k,t})$.

2. Preliminary Results

The main inequality in the next section will be proved by induction on m, the dimension of M. The results of this section will enable us to carry out the inductive step of the proof, and also to establish the main inequality in the case $m=1$.

First we will establish the inequalities

$$(2.1) \qquad |\delta\sqrt{\tilde{g}^{kk}}\,(x_0)|^2 \leq \mathscr{C}^2(x_0),$$

$$(2.2) \qquad \tilde{g}^{kk}(x_0)\,\mathscr{C}^2_{k,t}(x_0) \leq (n-m)\,\mathscr{C}^2(x_0),$$

$$(2.3) \qquad |\delta^{k,t}h(x_0)| \leq |\delta h(x_0)|,\ h \in C'(M),$$

(2.1) being valid for $m \geq 1$ and $k = 1,\ldots,n$ at all points x_0 of the submanifold $M' = \{\xi \in M;\ \tilde{g}^{kk}(\xi) > 0\}$, and (2.2), (2.3) being valid for $m \geq 2$ at all points $x_0 \in M_{k,t}$, $k=1,\ldots,n$, $t \in R$.

Proof of (2.1):

In the proof, let us use the notation $x_{\ell,i} = \dfrac{\partial x_\ell}{\partial u_i}$, $x_{\ell,ij} = \dfrac{\partial^2 x_\ell}{\partial u_i \partial u_j}$, $\ell=1,\ldots,n$, $i,j=1,\ldots,m$, where $x: U \to R^n$ is a representation for M near x_0. By (1.15)

$$\frac{\partial g^{rs}}{\partial u_i} = -\sum_{\ell,j=1}^{m} g^{s\ell} g^{rj}\frac{\partial g_{j\ell}}{\partial u_i}$$

$$= -\sum_{q=1}^{n}\sum_{\ell,j=1}^{m} g^{s\ell} g^{rj}(x_{q,\ell}x_{q,ij} + x_{q,j}x_{q,\ell i}),$$

and hence, by (1.4), it follows that

$$\frac{\partial \tilde{g}^{kk}(x)}{\partial u_i} = -2 \sum_{q=1}^{n} \sum_{r,s,\ell,j=1}^{m} g^{s\ell} g^{rj} x_{q,ij} x_{q,\ell} x_{k,r} x_{k,s} +$$

$$+ 2 \sum_{r,s=1}^{m} g^{rs} x_{k,ri} x_{k,s}.$$

Using (1.4) this gives

$$\frac{\partial \tilde{g}^{kk}(x)}{\partial u_i} = 2 \sum_{q=1}^{n} \sum_{r,s=1}^{m} (\delta_{kq} - \tilde{g}^{kq}(x)) g^{rs} x_{q,si} x_{k,r},$$

so that, by (1.18),

$$|\delta \tilde{g}^{kk}(x)|^2 = 4 \sum_{r',r,s,s',i,j=1}^{m} g^{ij} g^{rs} g^{r's'} \lambda^k_{is} x_{k,r} \lambda^k_{js} x_{k,r'},$$

where

$$\lambda^k_{ij} = \sum_{q=1}^{n} (\delta_{kq} - \tilde{g}^{kq}(x)) x_{q,ij} \quad , \; i,j=1,\ldots,m, \; k=1,\ldots,n.$$

Hence, since we can write $g^{ij} = \sum_{p=1}^{m} a_{pi} \; a_{pj}$, $i,j=1,\ldots,n$, for a suitable $m \times m$ matrix (a_{ij}) (see (1.7)), we have

$$|\delta \tilde{g}^{kk}(x)|^2 = 4 \sum_{p=1}^{m} \Big(\sum_{r,s,i=1}^{m} g^{rs} a_{pi} \lambda^k_{is} x_{k,r} \Big)^2.$$

Using the Cauchy inequality $\Big(\sum_{r,s=1}^{m} g^{rs} \xi_r \eta_s \Big)^2$

$$\leq \Big(\sum_{r,s=1}^{m} g^{rs} \xi_r \xi_s \Big) \Big(\sum_{r,s=1}^{m} g^{rs} \eta_r \eta_s \Big),$$

this gives

$$|\delta\tilde{g}^{kk}(x)|^2 \leq 4\left(\sum_{r,s=1}^{m} g^{rs}x_{k,r}x_{k,s}\right)\cdot$$

$$\cdot\left(\sum_{p=1}^{m}\sum_{r,s=1}^{m} g^{rs}\left(\sum_{l=1}^{m} a_{pl}\lambda^k_{ls}\right)\left(\sum_{l'=1}^{m} a_{pl'}\lambda^k_{l'r}\right)\right)$$

$$\leq 4\left(\sum_{r,s=1}^{m} g^{rs}x_{k,r}x_{k,s}\right)\left(\sum_{p,k'=1}^{m}\sum_{r,s=1}^{m} g^{rs}\left(\sum_{l=1}^{m} a_{pl}\lambda^{k'}_{ls}\right)\cdot\right.$$

$$\left.\cdot\left(\sum_{l'=1}^{m} a_{pl'}\lambda^{k'}_{l'r}\right)\right),$$

which, by (1.4) and (1.7),

$$= 4\tilde{g}^{kk}(x)\sum_{k',q,q'=1}^{n}\sum_{r,s,l,l'=1}^{m}(\delta_{k'q}-\tilde{g}^{k'q}(x))(\delta_{k'q'}-\tilde{g}^{k'q'}(x))\cdot$$

$$\cdot g^{rs}g^{ll'}x_{q,ls}x_{q',l'r}.$$

Thus, since $\sum_{k'=1}^{n}(\delta_{k'q}-\tilde{g}^{k'q}(x))(\delta_{k'q'}-\tilde{g}^{k'q'}(x)) =$

$= \delta_{qq'}-\tilde{g}^{qq'}(x)$, we have

$$|\delta\tilde{g}^{kk}|^2 \leq 4\tilde{g}^{kk}\mathscr{C}^2.$$

(2.1) now follows by noting that $2\delta\sqrt{\tilde{g}^{kk}} = \dfrac{\delta\tilde{g}^{kk}}{\sqrt{\tilde{g}^{kk}}}$ on

$M' = \{\xi \in M;\ \tilde{g}^{kk}(\xi) \neq 0\}$.

Proof of (2.2):

In the case $m=n$, it is not difficult to show that $\mathscr{C}^2_{k,t}(x_0) = 0$, and hence (2.2) is satisfied in this case.

In the case $n>m$, let $x_0 \in M_{k,t}$ and let $x:U \to R^n$ be a representation for M near $x_0 = x(u_0)$ with $x_k(u) = u_\ell$ as in (1.19). Let

$$v_r = \sum_{i=1}^{m} a_{ri} \frac{\partial x(u_0)}{\partial u_i}, \; r = 1,\ldots,m,$$

form an orthonormal basis for $T_{x_0}(M)$ such that $v_1,\ldots,v_{m-1}$ form an orthonormal basis for $T_{x_0}(M_{k,t})$ and such that $a_{r\ell} = 0$, $r = 1,\ldots,m-1$. Note that such a basis can be found, because $\frac{\partial x(u_0)}{\partial u_i} \in T_{x_0}(M_{k,t})$, $i \neq \ell$. Define $e_k = (\delta_{k1},\ldots,\delta_{kn})$, so that $|e_k| = 1$ and $e_k \in N_{x_0}(M_{k,t})$, and let $\xi^{(m+1)},\ldots,\xi^{(n)}$ be such that e_k, $\xi^{(m+1)},\ldots,\xi^{(n)}$ form an orthonormal basis for $N_{x_0}(M_{k,t})$.

Then, defining

$$\omega_{rs} = \sum_{i,j=1}^{m} a_{ri} a_{sj} \frac{\partial^2 x(u_0)}{\partial u_i \partial u_j}, \; r,s = 1,\ldots,m,$$

we have, using (1.8) and (1.10) together with the fact that $e_k \cdot \omega_{rs} = 0$ for $r,s = 1,\ldots,m$,

$$\mathscr{C}^2_{k,t}(x_0) = \sum_{i=m+1}^{n} \sum_{r,s=1}^{m-1} (\omega_{rs} \cdot \xi^{(i)})^2 \leq \sum_{i=m+1}^{n} \sum_{r,s=1}^{m} (\omega_{rs} \cdot \xi^{(i)})^2.$$

But, by (1.8) and (1.10),

$$\sum_{r,s=1}^{m} (\omega_{rs} \cdot \eta)^2 = \mathscr{C}^2(x_0,\eta) \leq \mathscr{C}^2(x_0)$$

for each $\eta \in N_{x_0}(M)$ with $|\eta|=1$, so the required result will clearly be established if we can show that for each

$\xi^{(i)}$, $i = m+1,\ldots,n$, there is an $\eta^{(1)} \in N_{x_0}(M)$ with $|\eta^{(1)}| = 1$ and with

$$(\omega_{rs}\cdot\xi^{(1)})^2 \leqslant \frac{1}{\tilde{g}^{kk}(x_0)} (\omega_{rs}\cdot\eta^{(1)})^2, \quad r,s = 1,\ldots,m.$$

To construct such $\eta^{(1)}$, first let

$$e_k' = \frac{1}{\sqrt{\tilde{g}^{kk}(x_0)}} \left(\tilde{g}^{k1}(x_0),\ldots,\tilde{g}^{kn}(x_0)\right),$$

so that $|e_k'| = 1$, and define $\eta^{(1)} = \tilde{\eta}^{(1)}/|\tilde{\eta}^{(1)}|$, where

$$(2.4) \qquad \tilde{\eta}^{(1)} = (e_k\cdot e_k')\xi^{(1)} - (\xi^{(1)}\cdot e_k')e_k.$$

Then, because $\xi^{(1)}$ and e_k are orthogonal unit vectors, we have

$$|\tilde{\eta}^{(1)}|^2 = (e_k\cdot e_k')^2 + (\xi^{(1)}\cdot e_k')^2 \leqslant |e_k'|^2 = 1.$$

Also, since $\tilde{\eta}^{(1)}$ is orthogonal to $v_1,\ldots,v_{m-1}$, and e_k', which form a basis for $T_{x_0}(M)$, we must have $\tilde{\eta}^{(1)} \in N_{x_0}(M)$.

It follows that $\eta^{(1)}$ has the required property, because by (2.4) (again using the fact that $e_k\cdot\omega_{rs} = 0, r,s=1,\ldots,m$)

$$(\omega_{rs}\cdot\eta^{(1)}) = \frac{(e_k\cdot e_k')(\omega_{rs}\cdot\xi^{(1)})}{|\tilde{\eta}^{(1)}|},$$

and $e_k\cdot e_k' = \sqrt{\tilde{g}^{kk}(x_0)}$.

Proof of (2.3):

Let $x_0 \in M_{k,t}$ and let $x : V \to R^n$ be a representation for M near $x_0 = x(u_0)$ with $x_k(u) = u_\ell$ as in (1.19). Let $\overline{h}(u) = h(x(u))$, $u \in U$. Since the rank of the $m \times m$ matrix $\left(\frac{\partial x_i(u_0)}{\partial u_j}\right)$ is m, we can find $(\lambda_1, \ldots, \lambda_n) \in R^n$ such that

$$\frac{\partial \overline{h}(u_0)}{\partial u_s} = \sum_{r=1}^{n} \frac{\partial x_r(u_0)}{\partial u_s} \lambda_r,$$

$s = 1, \ldots, m$, so that substituting this in (1.17) and using (1.3) we have

$$\delta_i h(x_0) = \sum_{j=1}^{n} \tilde{g}^{ij}(x_0)\lambda_j, \quad i=1,\ldots,n.$$

Similarly, since the restriction of x to

$$U_t = \{u \in U;\ u_\ell = t\}$$

provides a representation for $M_{k,t}$ near x_0, we have

$$\delta_i^{k,t} h(x_0) = \sum_{j=1}^{n} \tilde{g}_{k,t}^{ij}(x_0)\lambda_j, \quad i=1,\ldots,n.$$

But the vectors $\left(\sum_{j=1}^{n} \tilde{g}^{1j}(x_0)\lambda_j, \ldots, \sum_{j=1}^{n} \tilde{g}^{nj}(x_0)\lambda_j\right)$ and

$\left(\sum_{j=1}^{n} \tilde{g}_{k,t}^{1j}(x_0)\lambda_j, \ldots, \sum_{j=1}^{n} \tilde{g}_{k,t}^{nj}(x_0)\lambda_j\right)$ are the projections of the vector $(\lambda_1, \ldots, \lambda_n)$ onto $T_{x_0}(M)$ and its subspace $T_{x_0}(M_{k,t})$ respectively. Hence (2.3) follows.

The following lemma, which gives an integral identity for $C'(M)$ functions having compact support in M, is proved by using the identity (1.14).

(2.5) Lemma

Suppose $h \in C'(M)$ is such that $\mathrm{spt}(h)$ is a compact subset of M. Then, for $k=1,\ldots,n$ and $t \in R$,

$$\int_{M_{k,t}} \sqrt{\tilde{g}^{kk}h}\, dH_{m-1} = \int_{M^{+}_{k,t}} (\delta_k h + h\, H_k)\, dH_m,$$

where

$$M^{+}_{k,t} = \{\xi = (\xi_1,\ldots,\xi_n) \in M;\ \xi_k > t\}.$$

Remarks. (i) If $m=1$ and if $\chi \in C'(M_{k,t})$ has compact support in $M_{k,t}$ then $\int_{M_{k,t}} \chi\, dH_{m-1}$ means $\sum_{\xi \in M_{k,t}} \chi(\xi)$.

Note that this sum is either empty (when it is interpreted as zero) or has at most a finite number of non-zero terms.

(ii) If it is only true that $\chi \in C'(M_{k,t})$ is the uniform limit of a sequence $\{\chi_r\}$ of $C'(M_{k,t})$ functions with compact support in $M_{k,t}$ (e.g. $\chi = \sqrt{\tilde{g}^{kk}h}$ as in the lemma), then $\int_{M_{k,t}} \chi\, dH_{m-1}$ means

$$\lim_{r\to\infty} \int_{M_{k,t}} \chi_r\, dH_{m-1}.$$

Proof of Lemma (2.5):

Let $x : U \to R^n$ be a local representation for M such that $x_k(u) = u_\ell$, $u = (u_1, \ldots, u_m) \in U$, as in (1.19), and define $\bar{h}(u) = h(x(u))$, $u \in U$. First we will suppose that $\operatorname{spt}(h) \subset x(U)$, so that $\operatorname{spt}(\bar{h}) \subset U$. Define $U_t = \{u=(u_1,\ldots,u_m) \in U; u_\ell = t\}$, $U_t^+ = \{u=(u_1,\ldots,u_m) \in U; u_\ell > t\}$, $t \in R$. Note that when $m=1$ either $U_t = \emptyset$ or else consists of the single point t.

Using (1.3) with $M_{k,t}^+$ in place of M gives

$$\int_{M_{k,t}^+} \delta_k h \, dH_m = \int_{U_t^+} \sqrt{g} \sum_{r,s=1}^{m} \left(\frac{\partial x_k}{\partial u_r} g^{rs} \frac{\partial \bar{h}}{\partial u_s} \right) du_1 \ldots du_m .$$

Then integrating by parts with respect to u_s and using (1.14) we have, for $U_t = \emptyset$ and $m \geq 1$,

$$\int_{M_{k,t}^+} \delta_k h \, dH_m = -\int_{M_{k,t}^+} h \, H_k \, dH_m ,$$

while for $U_t \neq \emptyset$,

$$\int_{M_{k,t}^+} \delta_k h \, dH_m =$$

$$\begin{cases} \displaystyle\int_{U_t} \sum_{r=1}^{m} \frac{\partial x_k}{\partial u_r} g^{r\ell} \sqrt{g}\, \bar{h} \, du_1 \ldots du_{\ell-1} du_{\ell+1} \ldots du_m - \int_{M_{k,t}^+} h \, H_k \, dH_m , & m > 1, \\ \displaystyle\frac{dx_k}{du}(t) \sqrt{g^{11}}(t) \, \bar{h}(t) - \int_{M_{k,t}^+} h \, H_k \, dH_1 , & m = 1, \end{cases}$$

where we have used the fact that in the case $m=1$ (when $u=u_1$) $(g^{ij}) = g^{11} = \frac{1}{g_{11}} = \frac{1}{g}$. Now because $\frac{\partial x_k}{\partial u_r} = \delta_{\ell r}$, $r=1,\ldots,m$, $m \geq 1$, we have, by (1.3),

$$\frac{\partial x_k}{\partial u_r} g^{r\ell} = g^{\ell\ell} = \tilde{g}^{kk}$$

(and $\frac{dx_k}{du}\sqrt{g^{11}} = \sqrt{\tilde{g}^{kk}}$ in the case $m=1$). Then the required result is established in the case $m=1$, and in the case $m>1$ the result follows from (1.22) because $\sqrt{g^{kk}}\, h$ is the uniform limit of a sequence of continuous functions with compact support on $x(U)\cap\{\xi \in M;\ \tilde{g}^{kk}(\xi) \neq 0\}$.

When $\operatorname{spt}(h)$ is an arbitrary compact subset of M the lemma is proved by taking a suitable partition of unity.

(2.6) <u>Corollary</u>

If h is as in Lemma (2.5), then

$$\int_{M_{k,t}} \tilde{g}^{kk}|h|\,dH_{m-1} \leq \int_M (\tilde{g}^{kk}|\delta h| + (1+\sqrt{m}\sqrt{\tilde{g}^{kk}})|h|\mathscr{C})\,dH_m,\ m>1, k=1,\ldots,n,$$

$$\sup_{\xi\in M}|h(\xi)| \leq \int_M (|\delta h| + 2n|h|\mathscr{C})\,dH_1,\ m=1.$$

Proof:

Suppose $\varepsilon > 0$ and replace h in Lemma (2.5) by $\left(\sqrt{\tilde{g}^{kk}}|h|\right)^{1+\varepsilon}$. This gives

$$(2.7)\quad \int_{M_{k,t}} (\tilde{g}^{kk})^{1+\frac{\varepsilon}{2}} |h|^{1+\varepsilon} dH_m =$$

$$\int_{M_{k,t}^{\ddagger}} (\delta_k(\sqrt{\tilde{g}^{kk}}|h|)^{1+\varepsilon} + \left(\sqrt{\tilde{g}^{kk}}|h|\right)^{1+\varepsilon} H_k) dH_m .$$

Now at points where $\sqrt{\tilde{g}^{kk}}\ h \neq 0$

$$\delta_k\left(\sqrt{\tilde{g}^{kk}}\ h\right)^{1+\varepsilon} = (1+\varepsilon)\left(\sqrt{\tilde{g}^{kk}}|h|\right)^{\varepsilon}\left\{\sqrt{\tilde{g}^{kk}}\ \delta_k|h| + |h|\,\delta_k\sqrt{\tilde{g}^{kk}}\right\},$$

and, as described in the proof of (2.3), we can find $\lambda_1,\ldots,\lambda_n$ such that $\delta_i|h| = \sum_{j=1}^{n}\tilde{g}^{ij}\lambda_j$, $i=1,\ldots,n$. Note that then $|\delta h| = |\delta|h|| = \sum_{i,j=1}^{n}\tilde{g}^{ij}\lambda_i\lambda_j$, so that writing

$\delta_k|h| = \sum_{i,j=1}^{n}\delta_{ki}\tilde{g}^{ij}\lambda_j$, and using the Cauchy inequality

$\left|\sum_{i,j=1}^{n}\tilde{g}^{ij}\xi_i\eta_j\right| \leq \{\sum_{i,j=1}^{n}\tilde{g}^{ij}\xi_i\xi_j\}^{\frac{1}{2}}\{\sum_{i,j=1}^{n}\tilde{g}^{ij}\eta_i\eta_j\}^{\frac{1}{2}}$, we have

$$|\delta_k|h|| = |\delta_k h| \leq \sqrt{\tilde{g}^{kk}}\sqrt{\sum_{i,j=1}^{n}\tilde{g}^{ij}\lambda_i\lambda_j} = \sqrt{\tilde{g}^{kk}}\ |\delta h|$$

at all points where $h \neq 0$.

Hence, using (2.1) and letting $\varepsilon \to 0$, it follows from (2.7) that

$$\int_{M_{k,t}} \tilde{g}^{kk}|h|\,dH_{m-1} \leq \int_{M_{k,t}^{\ddagger}} (\tilde{g}^{kk}|\delta h| + |h|(\mathscr{C} + \sqrt{\tilde{g}^{kk}}\ |H|)dH_m .$$

Then by (1.13)

$$\int_{M_{k,t}} \tilde{g}^{kk} |h| \, dH_{m-1} \leq \int_M (\tilde{g}^{kk} |\delta h| + (1+\sqrt{m} \sqrt{\tilde{g}^{kk}}) \mathscr{C} \, dH_m ,$$

which proves the required result in the case $m>1$, while in the case $m=1$ (see the remark (i) after Lemma (2.5)) we get

$$\sup_{\xi \in M} \tilde{g}^{kk}(\xi) |h(\xi)| \leq \int_M (\tilde{g}^{kk} |\delta h| + (1 + \sqrt{\tilde{g}^{kk}}) |h| \mathscr{C} dH_1 .$$

Then the required result is also established in the case $m=1$ by summing over k and using $\sum_{k=1}^{n} \tilde{g}^{kk} = 1$ together with $(1 + \sqrt{\tilde{g}^{kk}}) \leq 2$.

3. The Main Inequality

The main inequality will now be stated.

(3.1) Theorem

Suppose $h \in C'(M)$ is such that spt(h) is a compact subset of M. Then

$$\left\{ \int_M |h|^{\frac{mp}{m-p}} dH_m \right\}^{\frac{m-p}{mp}} \leq \left\{ \int_M |\delta h|^p dH_m \right\}^{\frac{1}{p}} + n^m \left\{ \int_M (|h| \mathscr{C})^p dH_m \right\}^{\frac{1}{p}}, m>1, \quad 1 \leq p < m,$$

$$\sup_{\xi \in M} |h(\xi)| \leq \int_M |\delta h| \, dH_1 + 2n \int_M |h| \mathscr{C} dH_1 , \quad m=1.$$

Proof:

The proof of the theorem in the case $p=1$ is by mathematical induction on m, the dimension of M. The case $p>1$ is easily deduced from this case by replacing h by $h^{p(\frac{m-1}{m-p})}$ and using Hölder's inequality.

Let $P_r, r=1,2,\ldots$ be the proposition that the theorem holds (with $p=1$) in the case $m=r$ for all $n\geq r$. P_1 has already been established in (2.6).

Then suppose $m\geq 2$, and for $k=1,\ldots,n$ and $\varepsilon > 0$ let $\psi_{k,\varepsilon} = (\max\{(\sqrt{\tilde{g}^{kk}} - \varepsilon),0\})^{1+\varepsilon}$. Then $\psi_{k,\varepsilon}$ is a $C'(M)$ function with compact support in M and $\operatorname{spt}(\psi_{k,\varepsilon})\cap M_{k,t}$ is compact for each $t\in R$. Hence if P_{m-1} holds we have, since $M_{k,t}$ is either empty or an $(m-1)$-dimensional submanifold of R^n,

$$(3.2)\quad \left\{\int_{M_{k,t}} (\psi_{k,\varepsilon}|h|)^{\frac{m-1}{m-2}} dH_{m-1}\right\}^{\frac{m-2}{m-1}} \leq \int_{M_{k,t}} (|\delta^{k,t}(\psi_{k,\varepsilon}h)| + c\psi_{k,\varepsilon}|h|\mathscr{C}_{k,t})\, dH_{m-1}, m>2,$$

$$\sup_{M_{k,t}} \psi_{k,\varepsilon}|h| \leq \int_{M_{k,t}} (|\delta^{k,t}(\psi_{k,\varepsilon}h)| + c\psi_{k,\varepsilon}|h|\mathscr{C}_{k,t})\, dH_1,\ m=2,$$

where $c = n^{m-1}$, $m>2$, and $c = 2n$, $m=2$.

Now, using (2.1) and (2.3),

$$|\delta^{k,t}(\psi_{k,\varepsilon}h)| \leq |\delta(\psi_{k,\varepsilon}h)| \leq \psi_{k,\varepsilon}|\delta h| + |h||\delta\psi_{k,\varepsilon}| =$$

$$= \psi_{k,\varepsilon}|\delta h| + (1+\varepsilon)\psi_{k,\varepsilon}^{\frac{\varepsilon}{1+\varepsilon}}|h||\delta\sqrt{\tilde{g}^{kk}}| \leq$$

$$\leq \psi_{k,\varepsilon}|\delta h| + (1+\varepsilon)\psi_{k,\varepsilon}^{\frac{\varepsilon}{1+\varepsilon}}|h|\mathscr{C},$$

so that, by using (2.2) and letting $\varepsilon \to 0$, (3.2) gives

$$\left\{\int_{M_{k,t}} \left(\sqrt{\tilde{g}^{kk}}\,|h|\right)^{\frac{m-1}{m-2}} dH_{m-1}\right\}^{\frac{m-2}{m-1}}$$

$$\leqslant \int_{M_{k,t}} \left(\sqrt{\tilde{g}^{kk}}\,|\delta h| + (c\sqrt{n-m}+1)\,|h|\mathscr{C}\right) dH_{m-1}, m>2,$$

(3.3)

$$\sup_{M_{k,t}} \sqrt{\tilde{g}^{kk}}\ |h| \leqslant \int_{M_{k,t}} \left(\sqrt{\tilde{g}^{kk}}\,|\delta h| + (c\sqrt{n-m}+1)\,|h|\mathscr{C}\right) dH_{m-1}, \quad m=2.$$

Also, by (2.6),

(3.4) $$\int_{M_{k,t}} \tilde{g}^{kk}|h|\,dH_{m-1} \leqslant \int_{M} \left(\tilde{g}^{kk}|\delta h| + (1+\sqrt{m}\sqrt{\tilde{g}^{kk}})\,|h|\mathscr{C}\right) dH_m, m\geqslant 2.$$

But, using the Hölder inequality, we have

$$\int_{M_{k,t}} \left(\sqrt{\tilde{g}^{kk}}\right)^{\frac{m+1}{m-1}} |h|^{\frac{m}{m-1}}\, dH_{m-1} \leqslant$$

$$\leqslant \left\{\int_{M_{k,t}} \left(\sqrt{\tilde{g}^{kk}}\,|h|\right)^{\frac{m-1}{m-2}} dH_{m-1}\right\}^{\frac{m-2}{m-1}} \left\{\int_{M_{k,t}} \tilde{g}^{kk}|h|\,dH_{m-1}\right\}^{\frac{1}{m-1}}, m>2,$$

(3.5)

$$\int_{M_{k,t}} \left(\sqrt{\tilde{g}_{kk}}\right)^{3} |h|^{2} dH_1 \leqslant$$

$$\sup_{\xi \in M_{k,t}} \left(\sqrt{\tilde{g}^{kk}(\xi)}\,|h(\xi)|\right) \int_{M_{k,t}} \tilde{g}^{kk}|h|\,dH_1, m=2,$$

and, using (3.3) and (3.4), it follows that

$$\int_{M_{k,t}} \frac{(\tilde{g}^{kk}|h|)^{\frac{m}{m-1}}}{\sqrt{\tilde{g}^{kk}}}\, dH_{m-1} \leq$$

$$\left\{\int_{M_{k,t}} \left(\sqrt{\tilde{g}^{kk}}\ |\delta h| + (c\sqrt{n-m}+1)\,|h|\mathscr{C}\right) dH_{m-1}\right\} \times$$

$$\times \left\{\int_M \left(\tilde{g}^{kk}|\delta h| + \left(1+\sqrt{m}\sqrt{\tilde{g}^{kk}}\right)|h|\mathscr{C}\right) dH_m\right\}^{\frac{1}{m-1}},\quad m\geq 2.$$

Thus, integrating with respect to t and using (1.23),

$$\int_M \left(\tilde{g}^{kk}|h|\right)^{\frac{m}{m-1}} dH_m \leq \left\{\int_M \left(\tilde{g}^{kk}|\delta h| + \varphi_k |h|\mathscr{C}\right) dH_m\right\}^{\frac{m}{m-1}},\quad m\geq 2,$$

where

$$\varphi_k = \max\{(c\sqrt{n-m}+1)\sqrt{\tilde{g}^{kk}}\},\ 1+\sqrt{m}\sqrt{\tilde{g}^{kk}}\},\quad k=1,\ldots,n.$$

Hence, since

$$\left\{\int_M \left(\sum_{k=1}^n \tilde{g}^{kk}|h|\right)^{\frac{m}{m-1}} dH_m\right\}^{\frac{m-1}{m}} \leq \sum_{k=1}^n \left\{\int_M \left(\tilde{g}^{kk}|h|\right)^{\frac{m}{m-1}} dH_m\right\}^{\frac{m-1}{m}},$$

the required result is obtained by summing over k, using $\sum_{k=1}^n \tilde{g}^{kk}=m$, and noting that $\dfrac{\sum_{k=1}^n \varphi_k}{m} \leq n^m$. Clearly the constant n^m can be improved by giving a more careful argument.

In the case when M is an m-dimensional non-parametric hypersurface in R^{m+1} given by the global representation $\tilde{x} = (\tilde{x}_1,\dots,\tilde{x}_{m+1}) : \Omega \to R^{m+1}$, where $\tilde{x}_i(x) = x_i$, $i=1,\dots,m$, and $\tilde{x}_{m+1}(x) = f(x)$ for $x = (x_1,\dots,x_m) \in \Omega$, with Ω an open subset of R^n and $f \in C^2(\Omega)$, direct calculation using this representation gives

$$g_{ij} = \delta_{ij} + \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}, \quad i,j=1,\dots,m,$$

$$g = 1 + \sum_{i=1}^{m}\left(\frac{\partial f}{\partial x_i}\right)^2,$$

$$g^{ij} = \tilde{g}^{ij} = \delta_{ij} - \left(\frac{\partial f}{\partial x_i}\cdot\frac{\partial f}{\partial x_j}\right) \Big/ \left(1 + \sum_{i=1}^{m}\left(\frac{\partial f}{\partial x_i}\right)^2\right), \quad i,j=1,\dots,m,$$

$$\tilde{g}^{i\,m+1} = \tilde{g}^{m+1\,i} = -\frac{\partial f}{\partial x_i} \Big/ \left(1 + \sum_{i=1}^{m}\left(\frac{\partial f}{\partial x_i}\right)^2\right), \quad i=1,\dots,m,$$

$$\tilde{g}^{m+1\,m+1} = 1 - 1/\left(1 + \sum_{i=1}^{m}\left(\frac{\partial f}{\partial x_i}\right)^2\right),$$

$$\mathscr{C}^2 = \frac{1}{1 + \sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}\right)^2} \sum_{i,j,r,s=1}^{m} g^{ij} g^{rs} \frac{\partial^2 f}{\partial x_i \partial x_r}\frac{\partial^2 f}{\partial x_j \partial x_s}.$$

Also, if $h \in C'(\Omega)$ then h^* defined on M by $h^*(x_1,\dots,x_{m+1}) = h(x_1,\dots,x_m)$, $(x_1,\dots,x_{m+1}) \in M$, is a $C'(M)$ function and

$$\delta_i h^* = \sum_{j=1}^{m} g^{ij}\frac{\partial h}{\partial x_j},$$

$$|\delta h^*|^2 = \sum_{i,j=1}^{m} g^{ij}\frac{\partial h}{\partial x_i}\frac{\partial h}{\partial x_j}.$$

Then (3.1) gives, for $m \geq 2$, $1 \leq p < m$,

$$(3.6)\quad \left\{\int_\Omega |h|^{\frac{mp}{m-p}} \sqrt{1 + \sum_{i=1}^m \left(\frac{\partial f}{\partial x_i}\right)^2}\, dx\right\}^{\frac{m-p}{mp}}$$

$$\leq \left\{\int_\Omega \left(\sum_{i,j=1}^m g^{ij} \frac{\partial h}{\partial x_i} \frac{\partial h}{\partial x_j}\right)^{p/2} \sqrt{1 + \sum_{i=1}^m \left(\frac{\partial f}{\partial x_i}\right)^2}\, dx\right\}^{1/p} +$$

$$+(m+1)^m \left\{\int_\Omega \left[\left(1 + \sum_{i=1}^m \left(\frac{\partial f}{\partial x_i}\right)^2\right)^{-1} \sum_{i,j,r,s=1}^m g^{ij} g^{rs} \frac{\partial^2 f}{\partial x_i \partial x_r} \frac{\partial^2 f}{\partial x_j \partial x_s}\right]^{p/2} \cdot \right.$$

$$\left. \cdot \sqrt{1 + \sum_{i=1}^m \left(\frac{\partial f}{\partial x_i}\right)^2}\, dx\right\}^{1/p} .$$

CHAPTER 2

INTERIOR GRADIENT BOUNDS FOR DIVERGENCE-FORM ELLIPTIC EQUATIONS

As mentioned in Chapter 1, in [3] Bombieri, de-Giorgi, and Miranda were able to derive an a-priori interior gradient bound for C^3 solutions of the minimal surface equation with n independent variables, $n \geq 3$, thus extending the result previously established only for $n = 2$. Their method was to use test function arguments ([3] p.261 and pp. 263-4) together with the Sobolev-type inequality on the minimal surface given in Lemma 1 of [3].

Since the test-function arguments in [3] generalized without much difficulty to many other non-uniformly elliptic equations, it was apparent that interior gradient bounds could be obtained for these other equations provided the appropriate analogues to the Sobolev-type inequality could be established. Recently Ladyzhenskaya and Ural'tseva [4] obtained such inequalities (Lemma 1 in Part II of [4]) for a class of divergence form elliptic equations including the minimal surface equation. They were thus able to obtain the required gradient bounds for this class of equations.

In Section 3 of this chapter the general Sobolev-type inequality established in Chapter 1 will be used, together with the appropriate test function arguments (established in Section 2), to obtain interior gradient bounds for a very large class of non-uniformly elliptic divergence form equations, including all those mentioned above. The main results appear in Theorems (3.1) - (3.4).

Some particular applications of the results are discussed in Section 4.

1. Notation and Structural Conditions

In this and subsequent sections, Ω will be a bounded open subset of $R^n (n \geq 2)$ with boundary $\partial\Omega$ and closure $\bar{\Omega}$. $x = (x_1, \ldots, x_n)$, z, and $p = (p_1, \ldots, p_n)$ will denote points in Ω, R, and R^n respectively. $K_\rho(x_0)$, for $\rho > 0$ and $x_0 \in \Omega$, will denote the (closed) sphere of radius ρ and centre x_0. Lip(Ω) will denote the set of real valued functions ζ on Ω such that, for some constant $k > 0$, $|\zeta(x)-\zeta(x')| \leq k|x-x'|$ for all $x, x' \in \bar{\Omega}$. $W_m^\ell(\Omega)$, ℓ, m positive integers, will denote the Sobolev space consisting of all functions in $L_m(\Omega)$ having generalized partial derivatives of orders $1, \ldots, \ell$ lying in $L_m(\Omega)$ (see [7]), $W_m^{\ell, loc}(\Omega)$ the functions which are in $W_m^\ell(\Omega')$ for each open set Ω' with closure contained in Ω, and $W_{m,0}^\ell(\Omega)$ the set of $W_m^\ell(\Omega)$ functions having compact support in Ω. For $\zeta \in W_1^{1,loc}(\Omega)$, $\nabla\zeta$ and $\zeta_{,i}$, $i = 1, \ldots, n$, will be defined by $\nabla\zeta = (\zeta_{,1}, \ldots, \zeta_{,n}) = \left(\frac{\partial\zeta}{\partial x_1}, \ldots, \frac{\partial\zeta}{\partial x_n}\right)$. For $\zeta \in W_1^{2,loc}(\Omega)$, $\zeta_{,ij}$ will denote $\frac{\partial^2\zeta}{\partial x_i \partial x_j}$, $i, j = 1, \ldots, n$.

Unless otherwise stated u will denote a $\mathrm{Lip}(\Omega) \cap W_2^{2,loc}(\Omega)$ function with $\sup_{x\in\Omega}|u| \leq M$, M constant, which satisfies almost everywhere in Ω the equation

$$(1.1) \qquad \frac{d}{dx_i} A_i(x,u,\nabla u) + B(x,u,\nabla u) = 0,$$

where $A(x,z,p) = (A_1(x,z,p),\ldots,A_n(x,z,p))$ and $B(x,z,p)$ are respectively vector and scalar functions of $(x,z,p) \in \Omega \times R \times R^n$ such that A is locally Lipschitz (with partial derivatives denoted A_{x_i}, A_z, A_{p_i}, $i=1,\ldots,n$) and B is locally bounded and measurable.

In (1.1), and in what follows, repeated indices indicate summation from 1 to n. Of course, in (1.1), $\frac{d}{dx_i} A_i(x,u,\nabla u)$ should be taken to mean $\frac{\partial \psi_i(x)}{\partial x_i}$ where ψ_i is defined by $\psi_i(x) = A_i(x,u(x),\nabla u(x))$, $x \in \Omega$, $i = 1,\ldots,n$.

The weak form of (1.1) is

$$(1.1)' \quad \int_\Omega A_i(x,u,\nabla u)\zeta_{,i}\,dx = \int_\Omega B(x,u,\nabla u)\zeta\,dx, \quad \zeta \in W_{1,0}^1(\Omega).$$

It will be assumed that (1.1) is elliptic, at least for large enough values of $|\nabla u|$. That is, letting $\lambda(x,z,p)$ denote the minimum eigenvalue of the symmetric matrix $\frac{1}{2}(A_{ip_j}(x,z,p) + A_{jp_i}(x,z,p))$, conditions will be imposed to ensure that $\lambda(x,u,\nabla u) > 0$ for large enough values of $|\nabla u|$. It is perhaps worth pointing out that if $u \in \mathrm{Lip}(\Omega)$, if (1.1)$'$ is satisfied, if $\lambda(x,u,\nabla u) \geq c > 0$

on Ω, c constant, and if B is locally Lipschitz on $\Omega \times R \times R^n$, then u will automatically be in $W_2^{2,loc}(\Omega)$ and will satisfy (1.1) almost everywhere in Ω. (A proof of this can be based on the argument in [7], pp 273-6, but note that the argument is simplified by the assumption $u \in Lip(\Omega)$.)

The precise structural conditions for the functions A and B will be formulated in terms of quantities $f, w, v, \chi, \varphi, \psi, \sigma, \tau_0, \mu$, and n^*, where f denotes an arbitrary positive continuous function on $[1,\infty)$ (for many equations it suffices to make a choice for f of the form $f(t) = t^{\ell}$, ℓ constant - specific examples are discussed in Section 4), w is defined on $[1,\infty)$ by

$$w(t) = \begin{cases} 1 - \ln\left\{\left(\int_t^\infty \frac{ds}{f(s)}\right) \Big/ \left(\int_1^\infty \frac{ds}{f(s)}\right)\right\} & \text{when } \int_1^\infty \frac{ds}{f(s)} < \infty \\ 1 + \int_1^t \frac{ds}{f(s)} & \text{when } \int_1^\infty \frac{ds}{f(s)} = \infty, \end{cases}$$

$v = \sqrt{1+|p|^2}$, $\chi, \varphi, \psi, \sigma$ denote positive $C'(0,\infty)$ functions to be further specified later, $\tau_0 \geqslant 1$ and $\mu \geqslant 0$ denote arbitrary constants, $n^* = n$ for $n > 2$, and n^* denotes an arbitrary constant greater than 2 for $n = 2$.

Then it will subsequently be assumed that various combinations of the following conditions hold.

$$(1.3)\quad v\left\{\chi(w(v))\right\}^{1-2/n^*}\left\{1 + \frac{w(v)}{vw'(v)} + \left(\frac{w(v)\chi'(w(v))}{\chi(w(v))}\right)^2\right\} g^{ij}\xi_i\xi_j \leq f(v)A_{ip_j}(x,z,p)\xi_i\xi_j$$

for all $\xi \in R^n$, $x \in \Omega$, $|z| \leq M$, and $|p| \geq \tau_0$, where

$$g^{ij} = \delta_{ij} - (p_i p_j)/(1+|p|^2),\ i,j = 1,\ldots,n.$$

$$(1.4)\quad |f(v)A_{ip_j}(x,z,p)\eta_i\xi_j| \leq f(v)A_{ip_j}(x,z,p)\xi_i\xi_j + \varphi(w(v))p.A(x,z,p)\sum_{i=1}^{n}\eta_i^2$$

for all $\xi,\eta \in R^n$, $x \in \Omega$, $|z| \leq M$, and $|p| \geq \tau_0$.

$$(1.5)\quad F(v)\{v|A_z(x,z,p)|+|A_x(x,z,p)|+|B(x,z,p)|\} \leq \{\lambda(x,z,p)\}^{\frac{1}{2}}\{\psi(w(v))p.A(x,z,p)\}^{\frac{1}{2}},$$

for all $x \in \Omega$, $|z| \leq M$, and $|p| \geq \tau_0$, where

$$F(v) = \begin{cases} \left\{\sqrt{f(v)}\displaystyle\int_1^v \frac{ds}{f(s)}\right\}^{-1} + \left\{v\displaystyle\int_1^v \frac{ds}{f(s)}\right\}^{-\frac{1}{2}} & \text{when } \displaystyle\int_1^\infty \frac{ds}{f(s)} = \infty \\ \dfrac{1}{\sqrt{w(v)}}\left(\left\{\sqrt{f(v)}\displaystyle\int_v^\infty \frac{ds}{f(s)}\right\}^{-1} + \left\{v\displaystyle\int_v^\infty \frac{ds}{f(s)}\right\}^{-\frac{1}{2}}\right) & \text{when } \displaystyle\int_1^\infty \frac{ds}{f(s)} < \infty, \end{cases}$$

and $\lambda(x,z,p)$ is the minimum eigenvalue of $\frac{1}{2}(A_{ip_j}(x,z,p)+A_{jp_i}(x,z,p))$.

(1.6) $|A(x,z,p)| \leqslant \sigma(w(v))p.A(x,z,p)$, $|B(x,z,p)| \leqslant \mu p.A(x,z,p)$

for all $x \in \Omega$, $|z| \leqslant M$, and $|p| \geqslant \tau_0$.

(1.7) $|\xi.A(x,z,p)|^2 \leqslant \mu^2 p.A(x,z,p)\ f(v)\ A_{ip_j}(x,z,p)\xi_i\xi_j$

for all $\xi \in R^n$, $x \in \Omega$, $|z| \leqslant M$ and $|p| \geqslant \tau_0$.

It will become apparent that the condition (1.3) is imposed in order that appropriate use can be made of the Sobolev-type inequality established in Chapter 1. We will in fact here use only the following consequence of the inequality (3.6) of Chapter 1:

(1.8) $$\left\{\int_\Omega \eta^{2n^*/(n^*-2)} v\,dx\right\}^{\frac{n^*-2}{n^*}} \leqslant c(n^*)\left\{\int_{\Omega\cap \mathrm{spt}(\eta)} v\,dx\right\}^{(1-n/n^*)} \cdot \int_\Omega \left(\eta^2 \frac{1}{v^2} g^{ij}g^{rs}u_{,ir}u_{,js} + g^{ij}\eta_{,i}\eta_{,j}\right) v\,dx,$$

where η is an arbitrary bounded $W^1_{2,0}(\Omega)$ function, n^* is as in (1.3), $c(n^*)$ is a constant determined by n^*, $v = \sqrt{1+|\nabla u|^2}$, and $g^{ij} = \delta_{ij} - (u_{,i}u_{,j})/(1+|\nabla u|^2)$, $i,j=1,\ldots,n$.

In the case when $u,\eta \in C^2(\Omega)$ (1.8) is obtained by using $p=2$ in equality (3.6) of chapter 1 when $n > 2$, and by using $p = \frac{n^*}{n^*-1}$ together with Hölder's inequality when $n=2$. Generally, when u,η are respectively $\mathrm{Lip}(\Omega) \cap W^{2,loc}_2(\Omega)$ and bounded $W^1_{2,0}(\Omega)$ functions, (1.8) is obtained by taking suitable $C^2(\Omega)$ approximating sequences for u and η.

If in (1.4) we replace ξ by $\sqrt{\varepsilon}\xi$ and η by $\frac{1}{\sqrt{\varepsilon}}\eta$ $(\varepsilon > 0)$ then (1.4) becomes

$$(1.4)' \quad |f(v)A_{i p_j}(x,z,p)\eta_i\xi_j| \leq \varepsilon\, f(v)A_{i p_j}(x,z,p)\xi_i\xi_j + \frac{\varphi(w(v))}{\varepsilon}\, p.A(x,z,p)\sum_{i=1}^{n}\eta_i^2.$$

Choosing $\varepsilon = \frac{1}{2}$ and $\xi = \eta$, (1.4)′ gives

$$(1.9) \quad f(v)A_{i p_j}(x,z,p)\xi_i\xi_j \leq 4\varphi(w(v))p.A(x,z,p)\sum_{i=1}^{n}\xi_i^2$$

for all $(\xi_1,\ldots,\xi_n) \in R^n$, so that $4\varphi(w(v))p.A(x,z,p)$ is not smaller than the maximum eigenvalue of the symmetric matrix $\frac{1}{2}f(v)(A_{i p_j}(x,z,p) + A_{j p_i}(x,z,p))$, and hence certainly

$$(1.10) \quad f(v)\lambda(x,z,p) \leq 4\varphi(w(v))p.A(x,z,p).$$

If the functions B, $A_{i z}$, $A_{i x_j}$, $A_{i p_j}$, $i,j=1,\ldots,n$, are locally Lipschitz on $\Omega \times R \times R^n$, then in subsequent sections it will always be possible to use the following condition instead of (1.5):

$$(1.5)'\begin{cases} \frac{f(v)w'(v)}{w(v)}\Big|\frac{p_\ell}{v} A_{1x_\ell x_1}(x,z,p) + \frac{p_\ell p_1}{v} A_{1x_\ell z}(x,z,p) + \\ \frac{p_\ell}{v}B_{x_\ell}(x,z,p) + \frac{v^2-1}{v}\Big\{A_{1zx_1}(x,z,p)+p_1A_{1zz}(x,z,p) + \\ B_z(x,z,p)\Big\}\Big| \leq \psi(w(v))p.A(x,z,p). \\ \Big\{(f(v))^{\frac{1}{2}} + \left(\frac{f(v)w'(v)v}{w(v)}\right)^{\frac{1}{2}}\Big\} \sum_{1,k=1}^{n}\Big|\frac{p_\ell}{v}A_{1x_\ell p_k}(x,z,p) + \\ \frac{p_1}{v}B_{p_k}(x,z,p) + \frac{v^2-1}{v} A_{1zp_k}(x,z,p)\Big| \\ \leq \{\lambda(x,z,p)\}^{\frac{1}{2}}\{\psi(w(v))p.A(x,z,p)\}^{\frac{1}{2}}. \end{cases}$$

The condition (1.7) is implied by the more convenient condition

$$(1.7)' \quad |A(x,z,p)|^2 \leq \mu^2 p.A(x,z,p)f(v)\lambda(x,z,p).$$

Since φ, ψ, σ are assumed positive, each of the conditions (1.4), (1.5), and (1.6) requires $p.A(x,z,p) \geq 0$ for $x \in \Omega$, $|z| \leq M$, $|p| \geq \tau_0$. In fact, if (1.3) holds in addition to any one of these conditions, it follows that

$$(1.11) \quad p.A(x,z,p) \geq \frac{c}{2}\sqrt{1 + |p|^2},$$

$$c = \inf\left\{\frac{\chi^{1-2/n^*}(w(\sqrt{1+|p|^2}))}{f(\sqrt{1+|p|^2})\sqrt{1+|p|^2}}; \tau_0 \leq |p| \leq \tau_0+1\right\},$$

for $x \in \Omega$, $|z| \leq M$, $|p| \geq \tau_0+1$. This is easily seen by integrating the identity

$$\frac{d}{dt}p\cdot A(x,z,tp) = A_{ip_j}(x,z,tp)p_ip_j$$

between $t = \tau_0/|p|$ and $t = 1$ (c.f.[8]). For then, using (1.3) together with the inequality $g^{ij}\xi_i\xi_j \geq \frac{1}{1+|p|^2}|\xi|^2$, valid for $\xi = (\xi_1,\ldots,\xi_n) \in R^n$, we have, for $|p| \geq \tau_0+1$,

$$p\cdot A(x,z,p) - p\cdot A(x,z,\frac{\tau_0 p}{|p|}) \geq$$

$$\left(\int_{\tau_0/|p|}^{1} \frac{\chi^{1-2/n^*}(w(\sqrt{1+t^2|p|^2}))}{f(\sqrt{1+t^2|p|^2})\sqrt{1+t^2|p|^2}}dt\right)|p|^2$$

$$\geq \left(\int_{\tau_0/|p|}^{(\tau_0+1)/|p|} \frac{\chi^{1-2/n^*}(w(\sqrt{1+t^2|p|^2}))dt}{f(\sqrt{1+t^2|p|^2})\sqrt{1+t^2|p|^2}}\right)|p|^2,$$

which, by the definition of c,

$$\geq c\left(\frac{\tau_0+1}{|p|} - \frac{\tau_0}{|p|}\right)|p|^2 = c|p|.$$

(1.10) follows since

$$c|p| \geq \frac{c}{2}\sqrt{1+|p|^2} \quad \text{and} \quad p\cdot A(x,z,\tau_0\frac{p}{|p|}) \geq 0.$$

Subsequently we will always assume conditions (1.3) and (1.5), hence (1.11) can also be assumed to hold throughout. Then it follows from (1.8) that

$$(1.8)' \quad \left\{\int_{\Omega} \eta^{2n^*/(n^*-2)} v \, dx\right\}^{\frac{n^*-2}{n^*}} \leq c'\left\{\int_{\Omega\cap \mathrm{spt}(\eta)} (1+|\nabla u . A|)dx\right\}^{1-n/n^*} .$$

$$\cdot \int_{\Omega} (\eta^2 \frac{1}{v^2} g^{ij} g^{rs} u_{,ir} u_{,js} + g^{ij}\eta_{,i}\eta_{,j}) v dx,$$

where c' is a constant depending on τ_0, n^*, and the constant c of (1.11). Of course, this modification of (1.8) is only of interest in the case $n=2$, since for $n>2$ we have $n^*=n$ and the factor

$$\left\{\int_{\Omega\cap \mathrm{spt}(\eta)} v dx\right\}^{(1-n/n^*)}$$

can be omitted from (1.8).

2. Integral Inequalities

In this section, using test function arguments, some integral inequalities to be used in the next section are established.

The first of these inequalities gives a bound for $\int_{K_\rho(x_0)} |\nabla u . A(x,u,\nabla u)| dx$ provided $K_{4\rho}(x_0) \subset \Omega$. The test function used in the proof of this bound will also be important (in a modified form) in Chapters 3 and 4.

(2.1) Lemma

Suppose $K_{4\rho}(x_0) \subset \Omega$ and suppose that (1.6) holds with $\sigma(t) \to 0$ as $t \to \infty$. Let $\tau_* \geq \tau_0$ be a constant such that $(1+4\mu M)\frac{M}{\rho}\sigma(w(t)) \leq \frac{1}{2}$ for all $t \geq \sqrt{1+\tau_*^2}$, and let Δ be a constant such that

$$(1+4\mu M)\, 2^{8\mu M}\left(\frac{M}{\rho}|A(x,z,p)|+M|B(x,z,p)|+|p.A(x,z,p)|\right) \leq \Delta$$

for all $x \in \Omega$, $|z| \leq M$, and $|p| \leq \tau_*$. Then

$$\int_{K_\rho(x_0)} |\nabla u.A(x,u,\nabla u)|dx \leq c_n \Delta \rho^n,$$

where c_n is a constant depending only on n.

Proof:

Define $\zeta = \max\{u+3M-Mr/\rho, 0\}$, where $r = |x-x_0|$, and let $k = 1+4\mu M$. Using (1.1)′ with ζ^k in place of ζ we have

$$k\int_E \zeta^{k-1}\nabla u.A(x,u,\nabla u)dx = k\frac{M}{\rho}\int_E \zeta^{k-1}\nabla r.A(x,u,\nabla u)dx$$

$$+ \int_E \zeta^k B(x,u,\nabla u)dx,$$

where $E = \{x \in \Omega; \zeta(x) \neq 0\}$.

Then defining $E_\tau = \{x \in E;\ |\nabla u| \geq \tau\}$ and noting that $|\zeta| < 4M$ and that $|\nabla r.A| \leq |\nabla r||A| = |A|$, it follows that

$$k\int_{E_\tau} \zeta^{k-1}\nabla u.A\,dx \leq k\frac{M}{\rho}\int_{E_\tau} \zeta^{k-1}|A|dx + 4M\int_{E_\tau} \zeta^{k-1}|B|dx$$

$$+ \int_{E-E_\tau} \zeta^{k-1}\left(k\frac{M}{\rho}|A| + 4M|B| + k|\nabla u.A|\right)dx,$$

where of course A and B are abbreviations for $A(x,u,\nabla u)$ and $B(x,u,\nabla u)$ respectively. Then, using (1.6) together with the fact that $k\frac{M}{\rho}\sigma(w(\sqrt{1+|\nabla u|^2})) \leq \frac{1}{2}$ on E_τ, this gives

$$k\int_{E_\tau} \zeta^{k-1}\nabla u.A dx \leq (\tfrac{1}{2}+4\mu M)\int_{E_\tau} \zeta^{k-1}\nabla u.A dx$$

$$+ \int_{E-E_\tau}\zeta^{k-1}(k\frac{M}{\rho}|A| + 4M|B| + k|\nabla u.A|)dx.$$

It clearly follows that

$$\int_{E_\tau} \zeta^{k-1}\nabla u.A dx \leq 8k\int_{E-E_\tau}\zeta^{k-1}\left(\frac{M}{\rho}|A| + M|B| + |\nabla u.A|\right)dx,$$

and, adding $\int_{E-E_\tau} \zeta^{k-1}|\nabla u.A|dx$ to each side of this inequality and using the definition of Δ,

$$\int_E\zeta^{k-1}|\nabla u.A|dx \leq 9k\int_{E-E_\tau}\zeta^{k-1}\left(\frac{M}{\rho}|A| + M|B| + |\nabla u.A|\right)dx$$

$$\leq 9.2^{-8\mu M}\Delta\int_{E-E_\tau} \zeta^{k-1}dx \leq 9.2^{-8\mu M}\Delta\int_E\zeta^{k-1}dx.$$

Then the proof is completed by using the fact that $K_\rho(x_0) \subset E \subset K_{4\rho}(x_0)$ together with the estimates $\zeta \geq M$ on $K_\rho(x_0)$ and $\zeta \leq 4M$ on $K_{4\rho}(x_0)$.

The following two lemmas are proved using an argument which is a modification of that in [3], p.263.

(2.2) <u>Lemma</u>

Suppose $K_\rho(x_0) \subset \Omega$, let δ be a positive constant, let h be a positive $C'(0,\infty)$ function, and suppose that (1.6) and (1.7) hold with $\sigma(w) \leq \mu/h^\delta(w)$. Then for $q \geq 1$, $q' \geq 1$, and $\tau \geq \tau_0$

$$\int_{K_{\rho,\tau}} (\rho-r)^{2q} h_\tau^{2q'} \nabla u . A(x,u,\nabla u)dx$$

$$\leq 32.2^{2\mu m\rho}(\mu m\rho)^2\Big\{q^2\int_{K_{\rho,\tau}} (\rho-r)^{2q-2}h_\tau^{2q'-2\delta}\nabla u . A(x,u,\nabla u)dx +$$

$$(q')^2\int_{K_{\rho,\tau}}(\rho-\tau)^{2q}h_\tau^{2q'-2}f(v)A_{ip}(x,u,\nabla u)h_{\tau,i}h_{\tau,j}dx\Big\},$$

where

$$v = \sqrt{1+|\nabla u|^2}$$

$$h_\tau = \max\{h(w(v))-h(w(\sqrt{1+\tau^2})),0\},$$

$$r = |x-x_0|,$$

$$K_{\rho,\tau} = K_\rho(x_0) \cap \mathrm{spt}(h_\tau),$$

and

$$m_\rho = \sup_{x\in K_\rho(x_0)} u(x) - \inf_{x\in K_\rho(x_0)} u(x).$$

(2.3) Lemma

Let $\bar{v}$ be a continuous function on $\bar{\Omega}$ such that $\sqrt{1+|\nabla u|^2} \leq \bar{v}$ on Ω, let h be a positive $C'(0,\infty)$ function, and suppose (1.7) holds. Then for $\tau > \max\{\sup_{\partial\Omega} \bar{v}, \tau_0\}$ and for $q \geq 1$

$$\int_{\Omega_\tau} h_\tau^{2q} \nabla u . A(x,u,\nabla u)\, dx$$

$$\leq 16.2^{2\mu M}(\mu M)^2 q^2 \int_{\Omega_\tau} h_\tau^{2q-2} f(v) A_{ip_j}(x,u,\nabla u) h_{\tau,i} h_{\tau,j}\, dx,$$

where v and h_τ are as in Lemma (2.2) and $\Omega_\tau = \Omega \cap \operatorname{spt}(h_\tau)$.

Proof of Lemma (2.2):

Write $\zeta = \eta^k h_\tau^{2q} (\max\{(\rho-r),0\})^{2q}$, where $\tau \geq \tau_0$, $\eta = m_\rho + u - \inf_{K_\rho(x_0)} u$, and $k = 1+2\mu m_\rho$. Then by (1.1)$'$, using $\nabla\zeta = k\eta^{k-1}\nabla u(\rho-r)^{2q}h_\tau^{2q} + \eta^k\{2q'(\rho-r)^{2q}h^{2q'-1}\nabla h_\tau - 2q(\rho-r)^{2q-1}h_\tau^{2q'}\nabla r\}$ on $K_\rho(x_0)$, we have after some rearrangement of terms

$$k\int_{K_{\rho,\tau}} \eta^{k-1}(\rho-r)^{2q}h_\tau^{2q'}\nabla u.A\, dx$$

$$= -(2q')\int_{K_{\rho,\tau}} \left\{\eta^{\frac{k}{2}+\frac{1}{2}}(\rho-r)^q h_\tau^{q'-1} \frac{\nabla h_\tau . A}{\sqrt{\nabla u . A}}\right\}\left\{\eta^{\frac{k}{2}-\frac{1}{2}}(\rho-r)^q h_\tau^{q'}\sqrt{\nabla u . A}\right\}dx$$

$$+ 2q\int_{K_{\rho,\tau}} \left\{\eta^{\frac{k}{2}+\frac{1}{2}}(\rho-r)^{q-1}h_\tau^{q'} \frac{\nabla r.A}{\sqrt{\nabla u.A}}\right\}\left\{\eta^{\frac{k}{2}-\frac{1}{2}}(\rho-r)^q h_\tau^{q'}\sqrt{\nabla u.A}\right\}dx$$

$$+ \int_{K_{\rho,\tau}} \eta^k(\rho-r)^{2q}h_\tau^{2q'}Bdx.$$

Then using the inequality $|ab| \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2$ (valid for $\varepsilon > 0$ and $a,b \in R$) on each of the first two integrals on the right gives

$$k \int_{K_{\rho,\tau}} \eta^{k-1}(\rho-r)^{2q}h_\tau^{2q'}\nabla u.Adx$$

$$\leq \frac{(q')^2}{\varepsilon} \int_{K_{\rho,\tau}} \eta^{k+1}(\rho-r)^q h_\tau^{2q'-2} \frac{(\nabla h_\tau .A)^2}{\nabla u.A}dx +$$

$$+ \frac{q^2}{\varepsilon} \int_{K_{\rho,\tau}} \eta^{k+1}(\rho-r)^{2q-2}h_\tau^{2q'}\frac{(\nabla r.A)^2}{\nabla u.A}dx +$$

$$2\varepsilon \int_{K_{\rho,\tau}} \eta^{k-1}(\rho-r)^{2q}h_\tau^{2q'}\nabla u.Adx + \int_{K_{\rho,\tau}} \eta^k(\rho-r)^{2q}h_\tau^{2q'}|B|dx.$$

Now using (1.7) and (1.6) together with the inequalities $|\nabla r.A|^2 \leq |\nabla r|^2|A|^2 = |A|^2$ and $|\eta| \leq 2m_\rho$ it follows that

$$k \int_{K_{\rho,\tau}} \eta^{k-1}(\rho-r)^{2q} h_\tau^{2q'} \nabla u . A dx$$

$$\leq \frac{\mu^2(q')^2}{\varepsilon} \int_{K_{\rho,\tau}} \eta^{k+1}(\rho-r)^{2q} h_\tau^{2q'-2} f(v) A_{ip_j}(x,u,\nabla u) h_{\tau,i} h_{\tau,j} dx +$$

$$+ \frac{\mu^2 q^2}{\varepsilon} \int_{K_{\rho,\tau}} \eta^{k+1}(\rho-r)^{2q-2} h_\tau^{2q'-2\delta} \nabla u . A dx +$$

$$(2\varepsilon+2\mu m_\rho) \int_{K_{\rho,\tau}} \eta^{k-1}(\rho-r)^{2q} h_\tau^{2q'} \nabla u . A dx.$$

Then the required result is established by choosing $\varepsilon = \frac{1}{4}$ and noting that $m_\rho \leq \eta \leq 2m_\rho$.

Lemma (2.3) is proved in the same way as Lemma (2.2), except that we start with $\zeta = \tilde{\eta}^k h_\tau^{2q}, \tilde{\eta} = M+u-\inf_\Omega u$, instead of $\zeta = \eta^k h_\tau^{2q'}(\max\{(\rho-r),0\})^{2q}$ as in the proof above.

Note that this can be done since h_τ is a bounded $W^1_{2,0}(\Omega)$ function when $\tau > \max\left\{\sup_{\partial\Omega} \bar{v}, \tau_0\right\}$.

The following three lemmas are proved by what is basically a standard test function argument (see e.g. [3], [4], and [7] for similar arguments).

(2.4) Lemma

Suppose $K_\rho(x_0) \subset \Omega$, let h be a positive $C^2(0,\infty)$ function with $h', h'' \geqslant 0$, and suppose that φ and ψ are any positive $C'(0,\infty)$ functions such that either (1.4) and (1.5) or (1.4) and (1.5)′ hold. Then for $q, q' \geqslant 1$ and for $\tau \geqslant \tau_0$

$$\int_{K_{\rho,\tau}} (\rho-r)^{2q} h_\tau^{2q'-1} \frac{f(v)h'(w(v))w'(v)}{v} g^{st} A_{i p_j}(x,u,\nabla u) u_{,is} u_{,jt} dx$$

$$+ q' \int_{K_{\rho,\tau}} (\rho-r)^{2q} h_\tau^{2q'-2} f(v) A_{i p_j}(x,u,\nabla u) h_{\tau,i} h_{\tau,j} dx$$

$$\leqslant c(q+q') \int_{K_{\rho,\tau}} (\rho-r)^{2q} h_\tau^{2q'-2} \psi(w(v)) \{w^2(v)h''(w(v))h_\tau +$$

$$w(v)h'(w(v))h_\tau + (w(v)h'(w(v)))^2\} \nabla u \cdot A(x,u,\nabla u) dx$$

$$+ c' q(1+q/q') \int_{K_{\rho,\tau}} (\rho-r)^{2q-2} h_\tau^{2q'} \varphi(w(v)) \nabla u \cdot A(x,u,\nabla u) dx,$$

where

$$v = \sqrt{1+|\nabla u|^2}$$

$$h_\tau = \max\{h(w(v)) - h(w(\sqrt{1+\tau^2})), 0\},$$

$$g^{ij} = \delta_{ij} - (u_{,i} u_{,j})/(1+|\nabla u|^2), \quad i,j = 1,\ldots,n,$$

$$r = |x - x_0|$$

$$K_{\rho,\tau} = K_\rho(x_0) \cap \mathrm{spt}(h_\tau),$$

and c and c' are constants depending only on n.

(2.5) Lemma

Let h be a positive $C^2(0,\infty)$ function with $h', h'' \geq 0$, and suppose that ψ is any positive $C'(0,\infty)$ function such that either (1.5) or (1.5)$'$ holds. Let $\overline{v}$ be a continuous function on $\overline{\Omega}$ such that $\sqrt{1+|\nabla u|^2} \leq \overline{v}$ on Ω. Then, for $\tau > \max\{\sup_{\partial\Omega} \overline{v}, \tau_0\}$ and $q \geq 1$,

$$\int_{\Omega_\tau} h_\tau^{2q-1} \frac{f(v)w'(v)}{v} h'(w(v)) g^{st} A_{i p_j}(x,u,\nabla u) u_{,is} u_{,jt} dx$$

$$+ q\int_{\Omega_\tau} h_\tau^{2q-2} f(v) A_{i p_j}(x,u,\nabla u) h_{\tau,i} h_{\tau,j} dx$$

$$\leq cq\int_{\Omega_\tau} h_\tau^{2q-2}\psi(w(v))\{w^2(v)h''(w(v))h_\tau + w(v)h'(w(v))h_\tau +$$

$$(w(v)h'(w(v)))^2\}\nabla u.A(x,u,\nabla u)dx,$$

where the notation is as in Lemma (2.4) and $\Omega_\tau = \Omega \cap \mathrm{spt}(h_\tau)$.

(2.6) Lemma

Suppose $K_\rho(x_0) \subset \Omega$, $\int_1^\infty \frac{ds}{f(s)} < \infty$, and suppose that φ, ψ are any positive $C'(0,\infty)$ functions such that either (1.4) and (1.5) or (1.4) and (1.5)$'$ hold. Then, for $q \geq 1$, $q' \geq 2$, and $\tau \geq \tau_0$,

$$\int_{K_{\rho,\tau}} (\rho-r)^{2q} w_\tau^{2q'-2} \frac{f(v)w'(v)}{v} g^{st} A_{i p_j}(x,u,\nabla u) u_{,is} u_{,jt} dx$$

$$+ \int_{K_{\rho,\tau}} (\rho-r)^{2q} w_\tau^{2q'-2} f(v) A_{i p_j}(x,u,\nabla u) w_{\tau,i} w_{\tau,j} dx$$

$$\leq c \int_{K_{\rho,\tau}} (\rho-r)^{2q} w_\tau^{2q'-2} \psi(w(v)) w(v) \nabla u . A(x,u,\nabla u) dx$$

$$+ c'(q^2+\rho^2 q'^2) \int_{K_{\rho,\tau}} (\rho-r)^{2q-2} w_\tau^{2q'-4} (\psi(w(v)) w(v) +$$

$$\varphi(w(v)) w_\tau^2) \nabla u . A(x,u,\nabla u) dx,$$

where $w_\tau = \max\{w(v)-w(\sqrt{1+\tau^2}),0\}$ and the other notation is as in Lemma (2.4).

Proof of Lemma (2.4):

We will first give the proof assuming (1.5) holds.

If $\zeta \in W^2_{1,o}(\Omega)$ we can replace ζ in (1.1)$'$ by $\zeta_{,\ell}$, $\ell = 1,\ldots,n$.

Then, integrating by parts with respect to x_ℓ, (1.1)$'$ gives

$$(2.7) \quad \int_\Omega \frac{d}{dx_\ell}(A_i(x,u,\nabla u)\zeta_{,i} dx = -\int_\Omega B(x,u,\nabla u)\zeta_{,\ell} dx, \quad \ell=1,\ldots,n,$$

and by considering approximating sequences in $C^2(\Omega)$ we see that this holds for all $\zeta \in W^1_{2,o}(\Omega)$. Hence we can replace ζ in (2.7) by $f(v)h'(w(v))w'(v) \frac{u_{,\ell}}{v} \eta$, where $\eta \in W^1_{2,o}(\Omega)$ will be chosen later. Now, for $i,\ell=1,\ldots,n$, using the fact that

$$\frac{d}{dt}(f(t)w'(t)) = \begin{cases} f(t)(w'(t))^2, & \int_1^\infty \frac{ds}{f(s)} < \infty \\ 0, & \int_1^\infty \frac{ds}{f(s)} = \infty, \end{cases}$$

we have

$$\frac{\partial}{\partial x_1}(f(v)h'(w(v))w'(v)\ \frac{u_{,\ell}}{v}\eta) =$$

$$\{\theta f(v)(w'(v))^2h'(w(v))+f(v)(w'(v))^2h''(w(v))\}\ \frac{u_{,\ell}}{v}\eta v_{,1}$$

$$+f(v)h'(w(v))w'(v)\ \frac{g^{\ell s}}{v}u_{,s1}\eta+f(v)h'(w(v))w'(v)\ \frac{u_{,\ell}}{v}\eta_{,1},$$

where

$$\theta = \begin{cases} 0, & \int_1^\infty \frac{ds}{f(s)} = \infty \\ 1, & \int_1^\infty \frac{ds}{f(s)} < \infty, \end{cases}$$

and where we have used $\frac{\partial}{\partial x_1}\left(\frac{u_{,\ell}}{v}\right) = \frac{g^{\ell s}}{v}\ u_{,s1}$. Also we have

$$\frac{d}{dx_\ell}(A_1(x,u,\nabla u)) = A_{1p_j}(x,u,\nabla u)u_{,j\ell}+A_{1z}(x,u,\nabla u)u_{,\ell} +$$

$$A_{1x_\ell}(x,u,\nabla u),$$

so that after some rearrangement and summing over ℓ (2.7) gives

$$\iint_{\Omega}\Big\{\frac{fh'w'}{v}A_{i p_j}g^{\ell s}u_{,j\ell}u_{,is}\eta+f(w')^2(\theta h'+h'')A_{ip_j}u_{,j\ell}\frac{u_{,\ell}}{v}v_{,i}\eta+$$

$$(2.8)\qquad fh'w'A_{ip_j}u_{,j\ell}\frac{u_{,\ell}}{v}\eta_{,i}\Big\}dx =$$

$$-\int_{\Omega}C_{i\ell}\Big\{f(w')^2(\theta h'+h'')\frac{u_{,\ell}}{v}v_{,i}\eta+fh'w'g^{\ell s}\frac{u_{,si}}{v}\eta+$$

$$+fh'w'\frac{u_{,\ell}}{v}\eta_{,i}\Big\}dx,$$

where

$$C_{i\ell} = A_{ix_\ell}(x,u,\nabla u) + u_{,\ell}A_{iz}(x,u,\nabla u)+\delta_{i\ell}B(x,u,\nabla u),$$

$$i,\ell = 1,\ldots,n,$$

and where we have written f, h', h'', w', and A_{ip_j} as abbreviations for $f(v)$, $h'(w(v))$, $h''(w(v))$, $w'(v)$, and $A_{ip_j}(x,u,\nabla u)$ respectively. Using the identities $u_{,j\ell}\frac{u_{,\ell}}{v} = v_{,j}$, $(w(v))_{,i} = w'(v)v_{,i}$, $h_{\tau,i} = h'(w(v))w'(v)v_{,i}$ on $\mathrm{spt}(h_\tau)$, and the Cauchy inequality

$$|C_{i\ell}D_{i\ell}| \leqslant \sqrt{\sum_{i,\ell=1}^{n}C_{i\ell}^2}\ \sqrt{\sum_{i,\ell=1}^{n}D_{i\ell}^2}$$

(valid for arbitrary real matrices $(C_{i\ell}),(D_{i\ell})$), together with the fact that

$$(2.9)\quad g^{\ell s}g^{\ell t}\xi_s\xi_t = g^{st}\xi_s\xi_t - \frac{1}{v^2}\Big(\frac{u_{,s}}{v}\xi_s\Big)^2 \leqslant g^{st}\xi_s\xi_t,$$

$$(\xi_1,\ldots,\xi_n)\in R^n,$$

it follows that

$$(2.10)\quad \iint_\Omega \Big\{ \frac{fh'w'}{v} A_{ip_j} g^{\ell s} u_{,j\ell} u_{,is} \eta + f(\theta h' + h'') A_{ip_j} (w(v))_{,i} \cdot$$

$$\cdot\ (w(v))_{,j} \eta + f A_{ip_j} h_{\tau,i} \cdot \eta_{,i} \Big\} dx$$

$$\leq \int_\Omega |C| \Big\{ fw'(\theta h' + h'') |\nabla w(v)| \eta + \frac{fh'w'}{v} \sqrt{g^{st} u_{,si} u_{,ti}}\ \eta +$$

$$fh'w' |\nabla \eta| \Big\} dx,$$

where $|C| = \sqrt{\sum_{i,\ell=1}^{n} C_{i\ell}^2}$. Denoting the right side of this inequality by R we have, upon substituting $\eta = h_\tau^{2q'-1}(\max\{(\rho - r), 0\})^{2q}$ and using the inequality $|\nabla\eta| \leq 2q(\rho-r)^{2q-1}h_\tau^{2q'-1} + (2q'-1)(\rho-r)^{2q}h_\tau^{2q'-2}|\nabla h_\tau|$ on $K_\rho(x_0)$,

$$R \leq \int_{K_\rho(x_0)} |C| \{ fw'(\theta h' + h'')(\rho-r)^{2q} h_\tau^{2q'-1} |\nabla w(v)| +$$

$$+ (2q'-1) fh'w' (\rho-r)^{2q} h_\tau^{2q'-2} |\nabla h_\tau|$$

$$+ \frac{fh'w'}{v} \sqrt{g^{st} u_{,si} u_{,ti}}\ (\rho-r)^{2q} h_\tau^{2q'-1} +$$

$$+ 2q fh'w' (\rho-r)^{2q-1} h_\tau^{2q'-1} \} dx.$$

After some rearrangement of terms this can be written

$$R \leq \int_{K_{\rho,\tau}} \left\{(\rho-r)^{2q}h_\tau^{2q'-1}(\theta h'+h'')f\lambda|\nabla w(v)|^2\right\}^{\frac{1}{2}}\left\{(\rho-r)^{2q}h_\tau^{2q'-1}(\theta h'+ +h'')f(w')^2|C|^2/\lambda\right\}^{\frac{1}{2}}dx$$

$$+ \int_{K_{\rho,\tau}} \left\{(2q'-1)(\rho-r)^{2q}h_\tau^{2q'-2}f\lambda|\nabla h_\tau|^2\right\}^{\frac{1}{2}}\left\{(2q'-1)(\rho-r)^{2q}h_\tau^{2q'-2} \cdot f(h'w')^2|C|^2/\lambda\right\}^{\frac{1}{2}}dx$$

(2.11)

$$+ \int_{K_{\rho,\tau}} \left\{(\rho-r)^{2q}h_\tau^{2q'-1}\frac{fh'w'}{v}\lambda g^{st}u_{,s1}u_{,t1}\right\}^{\frac{1}{2}}\left\{(\rho-r)^{2q}h_\tau^{2q'-1} \cdot \frac{fh'w'}{v}|C|^2/\lambda\right\}^{\frac{1}{2}}dx$$

$$+ \int_{K_{\rho,\tau}} \left\{2q(\rho-r)^{2q-2}h_\tau^{2q'}f\lambda\right\}^{\frac{1}{2}}\left\{2q(\rho-r)^{2q}h_\tau^{2q'-2}f(h'w')^2|C|^2/\lambda\right\}^{\frac{1}{2}}dx,$$

where $\lambda = \lambda(x,u,\nabla u)$. Then, using the inequality $|ab| \leq \varepsilon a^2 + \frac{1}{4\varepsilon}b^2$ (valid for $\varepsilon > 0$ and $a,b \in R$) in each of the integrals and noting that

$$\lambda g^{st}u_{,s1}u_{,t1} \leq A_{1p_j}g^{st}u_{,is}u_{,jt}$$

(which is easily seen by writing $g^{st} = a_{sk}a_{tk}, s,t=1,\ldots,n$, for a suitable real $n\times n$ matrix (a_{ij})), it follows that

$$R \leqslant \varepsilon \int_{K_{\rho,\tau}} (\rho-r)^{2q} h_\tau^{2q'-1} (\theta h' + h'') f A_{i p_j} (w(v))_{,i} (w(v))_{,j} dx +$$

$$+ \varepsilon(2q'-1) \int_{K_{\rho,\tau}} (\rho-r)^{2q} h_\tau^{2q'-2} f A_{i p_j} h_{\tau,i} h_{\tau,j} dx$$

(2.12)

$$+ \varepsilon \int_{K_{\rho,\tau}} (\rho-r)^{2q} h_\tau^{2q'-1} \frac{fh'w'}{v} A_{i p_j} g^{st} u_{,is} u_{,jt} dx +$$

$$+ \frac{1}{4\varepsilon} \int_{K_{\rho,\tau}} (\rho-\tau)^{2q} h_\tau^{2q'-2} f (w')^2 \{\theta h' + h'') h_\tau +$$

$$+ (2q+2q'-1)(h')^2 \} |C|^2/\lambda dx$$

$$+ \frac{1}{4\varepsilon} \int_{K_{\rho,\tau}} (\rho-r)^{2q} h_\tau^{2q'-1} \frac{fh'w'}{v} |C|^2/\lambda dx +$$

$$2q\varepsilon \int_{K_{\rho,\tau}} (\rho-r)^{2q-2} h_\tau^{2q'} f\lambda \, dx.$$

Now denoting the left hand side of (2.10) by L we have

$$(2.13) \quad L = \int_{K_{\rho,\tau}} (\rho-r)^{2q} h_\tau^{2q'-1} \Big\{ \frac{fh'w'}{v} A_{i p_j} g^{\ell s} u_{,j\ell} u_{,is} +$$

$$f(\theta h' + h'') A_{i p_j} (w(v)_{,i} (w(v))_{,j} \Big\} dx$$

$$+(2q'-1)\int_{K_{\rho,\tau}}(\rho-\tau)^{2q}h_\tau^{2q'-2}fA_{ip_j}h_{\tau,i}h_{\tau,j}dx\ -$$

$$-2q\int_{K_{\rho,\tau}}(\rho-r)^{2q-1}h_\tau^{2q'-1}fA_{ip_j}h_{\tau,j}r_{,i}dx,$$

and, using the inequality (1.4)′ with $\varepsilon(2q'-1)/(2q)$ in place of ε, $h_{\tau,i}/h_\tau$ in place of ξ_i, and with $r,i/(\rho-r)$ in place of η,i, we have

$$(2.14)\quad -2q\int_{K_{\rho,\tau}}(\rho-r)^{2q-1}h_\tau^{2q'-1}fA_{ip_j}h_{\tau,i}r_{,j}dx \geq$$

$$-\varepsilon(2q'-1)\int_{K_{\rho,\tau}}(\rho-r)^{2q}h_\tau^{2q'-2}fA_{ip_j}h_{\tau,i}h_{\tau,j}dx$$

$$-\frac{4q^2}{\varepsilon(2q'-1)}\int_{K_{\rho,\tau}}(\rho-r)^{2q-2}h_\tau^{2q'}\varphi(w(v))\nabla u.Adx.$$

Using this estimate in (2.13), and using (2.12), it follows from (2.10) that

$$(1-\varepsilon)\int_{K_{\rho,\tau}}(\rho-\tau)^{2q}h_\tau^{2q'-1}\left\{\frac{fh'w'}{v}A_{ip_j}g^{\ell s}u_{,j\ell}u_{,is}\ +\right.$$

$$\left.+f(\theta h'+h'')A_{ip}\ (w(v))_{,i}(w(v))_{,j}\right\}dx$$

$$+(2q'-1)(1-2\varepsilon)\int_{K_{\rho,\tau}}(\rho-r)^{2q}h_\tau^{2q'-2}fA_{ip_j}h_{\tau,i}h_{\tau,j}dx \leqslant$$

$$\frac{1}{4\varepsilon}\int_{K_{\rho,\tau}}(\rho-r)^{2q}h_\tau^{2q'-2}f(w')^2\Big\{(\theta h'+h'')h_\tau+(2q+2q'-1)(h')^2\Big\}\cdot$$

$$|C|^2/\lambda dx+\frac{1}{4\varepsilon}\int_{K_{\rho,\tau}}(\rho-r)^{2q}h_\tau^{2q'-1}\frac{fh'w'}{v}|C|^2/\lambda dx$$

$$+\Big(8q\varepsilon+\frac{4q^2}{\varepsilon(2q'-1)}\Big)\int_{K_{\rho,\tau}}(\rho-r)^{2q-2}h_\tau^{2q'}\varphi(w(v))\nabla u\cdot A dx,$$

where (1.10) has been used to estimate the last term in (2.12). Then the required inequality follows by choosing $\varepsilon=\frac{1}{4}$ and noting that (1.5) implies

$$\max\Big\{\frac{f(w')^2}{w^{2-\theta}},\ \frac{fw'}{vw}\Big\}\frac{|C|^2}{\lambda}\leqslant c\,\psi(w(v))\nabla u\cdot A(x,u,\nabla u),$$

where c depends only on n.

In the case when (1.5)′ is assumed instead of (1.5), first note that the right hand side of (2.8) is just

$-\int_\Omega C_{i\ell}\frac{\partial}{\partial x_i}\Big(fw'h'\frac{u_{,\ell}}{v}\eta\Big)dx$, which upon integration by parts can be written

$$\int_\Omega fh'w'\frac{u_{,\ell}}{v}\eta\Big\{A_{ix_\ell x_i}+u_{,i}A_{ix_\ell z}+u_{,\ell}A_{izx_i}+u_{,\ell}u_{,i}A_{izz}+B_{x_\ell}+$$

$$+u_{,\ell}B_z\Big\}dx$$

$$+ \int_{\Omega} fh' w' \frac{u_{,\ell}}{v} \eta \left\{ A_{i x_\ell p_k} + u_{,\ell} A_{i z p_k} + \delta_{i\ell} B_{p_k} \right\} u_{,ki} dx.$$

Defining

$$C = \frac{u_{,\ell}}{v} \left\{ A_{i x_\ell x_i} + u_{,i} A_{i x_\ell z} + u_{,\ell} A_{i z x_i} + u_{,\ell} u_{,i} A_{i z z} + B_{x_\ell} + \right.$$

$$\left. + u_{,\ell} B_z \right\},$$

and

$$D_{ik} = \frac{u_{,\ell}}{v} \left\{ A_{i x_\ell p_k} + u_{,\ell} A_{i z p_k} + \delta_{i\ell} B_{p_k} \right\}, \quad i,k = 1,\ldots,n,$$

and using the identity $u_{,ki} = g^{ks} u_{,si} - \frac{u_{,k}}{v} v_{,i}$,

$$i,k = 1,\ldots,n,$$

this can be written

$$\int_{\Omega} fh' w' \eta C dx + \int_{\Omega} fh' w' \eta D_{ik} g^{ks} u_{,si} dx - \int_{\Omega} fh' w' \eta D_{ik} \frac{u_{,k}}{v} v_{,i} dx,$$

so that, using the inequality $|D_{ik} E_{ik}| \leq \sqrt{\sum_{i,k=1}^{n} D_{ik}^2} \sqrt{\sum_{i,k=1}^{n} E_{ik}^2}$ (valid for arbitrary real $n \times n$ matrices $(D_{ik}),(E_{ik})$) together with (2.9), it follows from (2.8) that

$$(2.10)' \quad L \leq \int_{\Omega} \eta \left\{ fh' w' |C| + fh' w' |D| \sqrt{g^{st} u_{,si} u_{,ti}} + \right.$$

$$\left. + f|D| \, |\nabla h(w(v))| \right\} dx,$$

where L is the left hand side of (2.10) and

$$|D| = \sqrt{\sum_{i,k=1}^{n} D_{ik}^2} \; .$$

Denoting the right hand side of (2.10)′ by R and choosing $\eta = h_\tau^{2q'-1}(\max\{(\rho-r),0\})^{2q}$, after some rearrangement we can write

$$(2.11) \quad R = \int_{K_{\rho,\tau}} \left\{(\rho-r)^{2q} h_\tau^{2q'-1} \frac{fh'w'}{v} \lambda g^{st} u_{,si} u_{,ti}\right\}^{\frac{1}{2}} \cdot$$

$$\left\{(\rho-r)^{2q} h_\tau^{2q'-1} fh'w'v|D|^2/\lambda\right\}^{\frac{1}{2}} dx$$

$$+ \int_{K_{\rho,\tau}} \left\{(\rho-r)^{2q} h_\tau^{2q'-2} f\lambda |\nabla h_\tau|^2\right\}^{\frac{1}{2}} \left\{(\rho-r)^{2q} h_\tau^{2q'} f|D|^2/\lambda\right\}^{\frac{1}{2}} dx +$$

$$\int_{K_{\rho,\tau}} (\rho-r)^{2q} h_\tau^{2q'-1} fh'w' |C| dx.$$

Using the inequality $ab \leqslant \varepsilon a^2 + \frac{1}{4\varepsilon} b^2$ in each of the first two integrals here gives

$$(2.12)' \quad R \leqslant \varepsilon \int_{K_{\rho,\tau}} (\rho-r)^{2q} h_\tau^{2q'-1} \frac{fh'w'}{v} A_{ip_j} g^{st} u_{,si} u_{,tj} dx +$$

$$\varepsilon \int_{K_{\rho,\tau}} (\rho-r)^{2q} h_\tau^{2q'-2} f A_{ip_j} h_{\tau,i} h_{\tau,j} dx$$

$$+ \frac{1}{4\varepsilon} \int_{K_{\rho,\tau}} (\rho-r)^{2q} h_\tau^{2q'-1} (fh'w'v + fh_\tau) |D|^2/\lambda dx +$$

$$\int_{K_{\rho,\tau}} (\rho-r)^{2q} h_\tau^{2q'-1} f h' w' |C| dx.$$

The remainder of the argument is similar to that given above.

Lemma (2.5) can be proved by using $\eta = h_\tau^{2q'-1}$ instead of $\eta = h_\tau^{2q'-1}(\max\{(\rho-r),0\})^{2q}$ and following the above arguments exactly, taking advantage of the simplifications due to the absence of the factor $(\max\{(\rho-r),0\})^{2q}$ in η.

Lemma (2.6) can now also be easily proved by using essentially the same arguments as above, with $h(w) \equiv w$, $\eta = w_\tau^{2q'-2}(\max\{(\rho-r),0\})^{2q}$ instead of $\eta = h_\tau^{2q'-1}(\max\{(\rho-r),0\})^{2q}$ $(w_\tau = \max\{w(v) - w(\sqrt{1+\tau^2}),0\})$, and subject to the following modifications:

(i) in place of (2.14) use the inequality

$$-2q\int_{K_{\rho,\tau}} (\rho-r)^{2q-1} w_\tau^{2q'-2} f A_{ip_j} w_{\tau,j} r_{,i} dx \geq$$

$$-\varepsilon\int_{K_{\rho,\tau}} (\rho-r)^{2q} w_\tau^{2q'-2} f A_{ip_j} w_{\tau,i} w_{\tau,j} dx$$

$$-\frac{4q^2}{\varepsilon}\int_{K_{\rho,\tau}} (\rho-r)^{2q-2} w_\tau^{2q'-2} \varphi(w(v)) \nabla u.A\, dx,$$

(ii) in the case when condition (1.5) is assumed (2.11) is replaced by

$$R \leqslant \int_{K_{\rho,\tau}} \left\{(\rho-r)^{2q} w_\tau^{2q'-2} f\lambda |\nabla w_\tau|^2\right\}^{\frac{1}{2}} \left\{(\rho-r)^{2q} w_\tau^{2q-2} f (w')^2 |C|^2/\lambda\right\}^{\frac{1}{2}} dx$$

$$+ \int\!\!\int_{K_{\rho,\tau}} \left\{(\rho-r)^{2q} w_\tau^{2q'-2} f\lambda |\nabla w_\tau|^2\right\}^{\frac{1}{2}} \left\{(2q'-2)^2 \rho^2 (\rho-r)^{2q-2} w_\tau^{2q'-4} f (w')^2 \cdot |C|^2/\lambda\right\}^{\frac{1}{2}} dx$$

$$+ \int_{K_{\rho,\tau}} \left\{(\rho-\tau)^{2q} w_\tau^{2q'-2} \frac{fw'}{v} \lambda g^{st} u_{,s1} u_{,t1}\right\}^{\frac{1}{2}} \left\{(\rho-r)^{2q} w_\tau^{2q-2} \frac{fw'}{v} \cdot |C|^2/\lambda\right\}^{\frac{1}{2}} dx$$

$$+ \int\!\!\int_{K_{\rho,\tau}} \left\{4q^2 (\rho-r)^{2q-2} w_\tau^{2q'-2} f\lambda\right\}^{\frac{1}{2}} \left\{(\rho-r)^{2q} w_\tau^{2q'-2} f (w')^2 |C|^2/\lambda\right\}^{\frac{1}{2}} dx$$

(where we have used $(\rho-r)^{2q} \leqslant \rho^2(\rho-r)^{2q-2}$ in the second integral), and

(iii) in the case when condition (1.5)′ is assumed (2.11)′ is replaced by

$$R = \int\!\!\int_{K_{\rho,\tau}} \left\{(\rho-r)^{2q} w_\tau^{2q'-2} \frac{fw'}{v} \lambda g^{st} u_{,s1} u_{,t1}\right\}^{\frac{1}{2}} \left\{(\rho-r)^{2q} w_\tau^{2q'-2} fw' v \cdot |D|^2/\lambda\right\}^{\frac{1}{2}} dx$$

$$+ \int\!\!\int_{K_{\rho,\tau}} \left\{(\rho-r)^{2q} w_\tau^{2q'-2} f\lambda |\nabla w_\tau|^2\right\}^{\frac{1}{2}} \left\{(\rho-r)^{2q} w_\tau^{2q'-2} f |D|^2/\lambda\right\}^{\frac{1}{2}} dx +$$

$$\int_{K_{\rho,\tau}} (\rho-r)^{2q} w_\tau^{2q'-2} fw' |C| \, dx.$$

3. The Gradient Bounds

The following theorems represent the main results of this chapter. Some specific applications of the theorems will be discussed in Section 4. In all the theorems condition (1.5)′ can be used in place of condition (1.5).

In the first theorem condition (1.7) is not required to hold, and for this reason the theorem is especially applicable to equations of the form (1.1) when $p.A(x,z,p)$ has rapid growth as $|p| \to \infty$ (see example (4.1)).

(3.1) Theorem

Suppose $K_{4\rho}(x_0) \subset \Omega$, let δ be a positive constant, and suppose (1.3) - (1.6) hold with

$$v\chi(w(v)) \geq \frac{1}{\mu}\, p.A(x,z,p)$$

for all $x \in \Omega$, $|z| \leq M$, and $|p| \geq \tau_0$,

with $\varphi(w) \equiv \mu/w^{2\delta}$, and with $\psi(t), \sigma(t) \to 0$ as $t \to \infty$. Then the estimate

$$\text{ess sup}|\nabla u| \leq \Gamma$$

holds, where the essential sup is taken over $K_{\rho/4}(x_0)$ and where Γ is a constant determined by n^*, f, ψ, δ, μ, τ_0, and the quantity Δ of Lemma (2.1).

The proof of this theorem will be given later.

The next Theorem shows that, in the case when condition (1.7) is assumed, condition (1.3) can be considerably relaxed.

(3.2) <u>Theorem</u>

(a) Suppose $K_{4\rho}(x_0) \subset \Omega$, let k, δ be constants such that $k \geq 0$ and $\delta > 0$, and suppose (1.3) - (1.7) hold with

$$v\chi(w(v)) \geq \frac{1}{\mu}(w(v))^{-k} p.A(x,z,p)$$

for all $x \in \Omega$, $|z| \leq M$, and $|p| \geq \tau_0$,

with $\varphi(w) \equiv \mu/w^{2\delta}$, $\sigma(w) \equiv \mu/w^{\delta}$, and with $\psi(t) \to 0$ as $t \to \infty$. Then the estimate

$$\text{ess sup}|\nabla u| \leq \Gamma$$

holds, where the essential sup is taken over $K_{\rho/4}(x_0)$ and where Γ is a constant determined by $n^*, f, \psi, k, \delta, \mu, \tau_0$, M, ρ, and the quantity Δ of Lemma (2.1).

(b) Suppose the hypotheses are as in (a) except that $\psi \equiv \mu$. Then the estimate

$$\text{ess sup } |\nabla u| \leq \Gamma$$

holds, where the essential sup is taken over $K_{\rho/4}(x_0)$ and where Γ is determined by the same quantities as in (a), together with the modulus of continuity of u on Ω.*

(c) Suppose the hypotheses are as in (a) except that $\psi(w) \equiv \mu/w^{2\delta}$ and

---

* By the modulus of continuity of u on Ω we mean the real valued function ω defined for all $\varepsilon > 0$ by $\omega(\varepsilon) = \sup\{t \leq 1; |u(x)-u(x')| \leq \varepsilon$ whenever $x, x' \in \Omega$ and $|x-x'| \leq t\}$.

$$v\chi(w(v)) \geq \frac{1}{\mu} e^{-\theta(v)(w(v))^{\delta}} p.A(x,z,p)$$

for all $x \in \Omega$, $|z| \leq M$, and $|p| \geq \tau_0$,

where θ is a non-negative continuous function on $(0,\infty)$ such that $\theta(t) \to 0$ as $t \to \infty$. Then the estimate

$$\text{ess sup } |\nabla u| \leq \Gamma$$

holds, where the essential sup is taken over $K_{\rho/4}(x_0)$ and where Γ is determined by the same quantities as in (a), together with the function θ.

The proof is given later.

In the case when $\int_1^{\infty} \frac{ds}{f(s)} < \infty$ the condition on φ can be relaxed to an extent that enables equations like the minimal surface equation and those equations in part II of [4] to be treated:

(3.3) Theorem

(a) Suppose $K_{4\rho}(x_0) \subset \Omega$, $\int_1^{\infty} \frac{ds}{f(s)} < \infty$, let k,δ be constants such that $k \geq 0$ and $0 < \delta \leq 1$, and suppose that (1.3) - (1.7) hold with

$$v\chi(w(v)) \geq \frac{1}{\mu}(w(v))^{-k} p.A(x,z,p)$$

for all $x \in \Omega$, $|z| \leq M$, and $|p| \geq \tau_0$,

with $\varphi \equiv \mu$, $\sigma(w) \equiv \mu/w^{\delta}$, and with $\frac{\psi(w(t))}{w(t)} \to 0$ as $t \to \infty$. Then the estimate

$$\text{ess sup } |\nabla u| \leq \Gamma$$

holds, where the essential sup is taken over $K_{\rho/4}(x_0)$ and where Γ is a constant determined by $n^*, f, \psi, k, \delta, \mu, \tau_0$, M, ρ, and the quantity Δ of Lemma (2.1).

(b) Suppose the hypotheses are as in (a) except that $\frac{\psi(w)}{w} \equiv \mu$. Then the estimate

$$\text{ess sup } |\nabla u| \leq \Gamma$$

holds, where the essential sup is taken over $K_{\rho/4}(x_0)$ and where Γ is determined by the same quantities as in (a), together with the modulus of continuity of u on Ω.

(c) Suppose the hypotheses are as in (a) except that $\frac{\psi(w)}{w} \equiv \mu/w^{2\delta}$ and

$$v\chi(w(v)) \geq \frac{1}{\mu} e^{-\theta(v)(w(v))^{\delta/2}} p.A(x,z,p)$$

$$\text{for all } x \in \Omega, \ |z| \leq M, \ \text{and } |p| \geq \tau_0,$$

where θ is a non-negative continuous function on $(0,\infty)$ such that $\theta(t) \to 0$ as $t \to \infty$. Then the estimate

$$\text{ess sup } |\nabla u| \leq \Gamma$$

holds, where the essential sup is taken over $K_{\rho/4}(x_0)$ and where Γ is determined by the same quantities as in (a), together with the function θ.

The proof is given later.

Of course each of the local gradient bounds obtained in the theorems above implies a gradient bound on arbitrary interior subregions $\Omega' \subset \Omega$ which are such that $\overline{\Omega}' \subset \Omega$. For example, from Theorem (3.1) we can deduce that if $\inf\{|x'-x|; x' \in \Omega', x \in \partial\Omega\} \geq \rho$, $\rho > 0$ constant, and if (1.3) - (1.6) hold with $v\chi(w(v)) \geq \frac{1}{\mu} p.A(x,z,p)$, $\varphi(w) = \mu/w^{2\delta}$, and with $\psi(t)$, $\sigma(t) \to 0$ as $t \to \infty$, then

$$\operatorname{ess\,sup}_{\Omega'} |\nabla u| \leq \Gamma,$$

where Γ is determined by $n^*, f, \psi, \delta, \mu, \tau_0, \rho$, and the quantity Δ of Lemma (2.1).

When a suitable bound for the gradient of u on $\partial\Omega$ is known, condition (1.4) can be dropped altogether and a global estimate for $|\nabla u|$ is obtained:

(3.4) <u>Theorem</u>

(a) Let $\bar{v}$ be a continuous function on $\overline{\Omega}$ such that $\sqrt{1+|\nabla u|^2} \leq \bar{v}$ on Ω, let $k \geq 0$ be a constant and suppose (1.3), (1.5), (1.6), and (1.7) hold with

$$v\chi(w(v)) \geq \frac{1}{\mu} (w(v))^{-k} p.A(x,z,p)$$
$$\text{for all } x \in \Omega, \ |z| \leq M, \text{ and } |p| \geq \tau_0,$$

with σ arbitrary, and with $\psi(t) \to 0$ as $t \to \infty$. Then the estimate

$$\operatorname{ess\,sup}_{\Omega} |\nabla u| \leq \Gamma$$

holds, where Γ is a constant determined by n^{*}, f, ψ, τ_0, $\sup_{\partial\Omega} \bar{v}, k, \mu$, $p.A(x,z,p)$ and the measure of Ω.

(b) Suppose the hypotheses are as om (a), except that $\psi(w) \equiv \mu/w^{2\delta}$ and

$$v\chi(w(v)) \geqslant \frac{1}{\mu} e^{-\theta(v)(w(v))^{\delta}} p.A(x,z,p)$$

for all $x \in \Omega$, $|z| \leqslant M$, and $|p| \geqslant \tau_0$,

where δ is a positive constant and θ is a non-negative continuous function on $(0,\infty)$ such that $\theta(t) \to 0$ as $t \to \infty$. Then the estimate

$$\operatorname{ess\,sup}_{\Omega} |\nabla u| \leqslant \Gamma$$

holds, where Γ is a constant determined by the same quantities as in (a), together with the function θ and the constant δ.

Before giving the proofs of these four theorems we need to establish the following lemma, which is proved by an iterative argument like that in Theorem 1 of §4 of [9], making use of the inequality (1.8)′ and Lemmas (2.4) and (2.5).

(3.5) <u>Lemma</u>

(a) Suppose $K_{\rho}(x_0) \subset \Omega$, let $\delta, \alpha_i, \beta_i$, $i=1,\ldots,4$, be constants such that $\delta, \beta_i > 0$ and $\alpha_i \geqslant 0$, $i=1,\ldots,4$, let h be a $C^2(0,\infty)$ function with $h \geqslant 1$, $h' \geqslant 1$, $h'' \geqslant 0$, and $w^2h''(w)h(w) + (wh'(w))^2 \leqslant \beta_1 h^{2+\alpha_1}(w)$, and suppose (1.3) - (1.5) hold with

$$v\chi(w(v)) \geq \beta_2(h(w(v)))^{-\alpha_2}\left(\frac{h(w(v))}{w(v)}\right)^{(2n^*)/(n^*-2)} p.A(x,z,p)$$

for all $x \in \Omega$, $|z| \leq M$, and $|p| \geq \tau_0$,

with $\varphi(w) \leq \beta_3 h^{\alpha_3 - 2\delta}(w)$, and with $\psi(w) \leq \beta_4 h^{\alpha_4}(w)$. Then for $\tau \geq \tau_0$ and for any constant $\alpha_0 > 0$

$$\operatorname*{ess\,sup}_{K_\rho(x_0)} (\rho-r)h_\tau^\delta \leq \gamma+\gamma'\left\{\int_{K_{\rho,\tau}}(1+h_\tau)^m(1+(\rho-r)h_\tau^\delta)^{2\alpha_0/\delta}\nabla u.A(x,u,\nabla u)dx\right\}^d$$

where $r, h_\tau, K_{\rho,\tau}$ are as in Lemma (2.4),

$m = \left(\frac{n^*}{2} - 1\right)\alpha_2 + \frac{n^*}{2}\max\{\alpha_3, \alpha_1+\alpha_4\}$, γ, γ' are constants determined by $f, n^*, m, \alpha_0, \rho, \beta_1, \ldots, \beta_4, \delta, \tau, \alpha_2$ and h, and where d is a constant determined by n^* and α_0.

(b) Let $\bar{v}$ be a continuous function on $\bar{\Omega}$ which is such that $\sqrt{1+|\nabla u|^2} \leq \bar{v}$ on Ω, let $\alpha_i\beta_i, i=1.2.3$ be constants such that $\alpha_i \geq 0$ and $\beta_i > 0, i = 1,2,3$, let h be a $C^2(0,\infty)$ function with $h \geq 1$, $h' \geq 1$, $h'' \geq 0$ and with $w^2h'(w)h(w) + (wh'(w))^2 \leq \beta_1(h(w))^{2+\alpha_1}$, and suppose that (1.3) and (1.5) hold with

$$v\chi(w(v)) \geq \beta_2 h^{-\alpha_2}(w(v))\left(\frac{h(w(v))}{w(v)}\right)^{(2n^*)/(n^*-2)} p.A(x,z,p)$$

for all $x \in \Omega$, $|z| \leq M$, and $|p| \geq \tau_0$,

and with $\psi(w) \leq \beta_3 h^{\alpha_3}(w)$. Then for $\tau > \max\{\tau_0, \sup_{\partial\Omega} \bar{v}\}$, and for any constant $\alpha_0 > 0$,

$$\operatorname*{ess\,sup}_{\Omega} h_\tau \leq \gamma + \gamma'\left\{\int_{\Omega_\tau}(1+h_\tau)^{m+2\alpha_0}\nabla u.A(x,u,\nabla u)dx\right\}^d$$

where $m = \left(\frac{n^*}{2} - 1\right)\alpha_2 + \frac{n^*}{2}(\alpha_1+\alpha_3)$, h_τ is as in Lemma (2.4), $\Omega_\tau = \Omega \cap \mathrm{spt}(h_\tau)$, γ,γ' are determined by $f, n^*, m, \alpha_0, \alpha_2$ $\beta_1, \beta_2, \beta_3, \tau$, and h, and where d is determined by n^* and α_0.

Proof of Lemma (3.5)(a):

By adding the inequalities obtained from Lemma (2.4) with $q' = q\delta$ and $q' = q\delta+\ell$, where $q \geq \max\{1, 1/\delta\}$ and $2\ell = (1-2/n^*)(m+\alpha_2)$, and using the estimates

$$c_1\{h(w(v))\}^{2\ell} \leq 1+h_\tau^{2\ell} \leq 2\{h(w(v))\}^{2\ell},$$

where $c_1 = \{h(1+w(\sqrt{1+\tau^2}))\}^{-2\ell}$, * it follows that

$$(3.6)\quad \int_{K_{\rho,\tau}} (\rho-r)^{2q} h_\tau^{2q\delta-1} h^{2\ell} \frac{fh'w'}{v} g^{st} A_{ip_j} u_{,is} u_{,jt} dx$$

$$+q\int_{K_{\rho,\tau}} (\rho-r)^{2q} h_\tau^{2q\delta-2} h^{2\ell} f A_{ip_j} h_{\tau,i} h_{\tau,j} dx$$

$$\leq c_2 q\int_{K_{\rho,\tau}} (\rho-r)^{2q} h_\tau^{2q\delta-2} h^{2\ell} \psi(w(v))\{w^2h''h_\tau + wh'h_\tau + (wh')^2\}\nabla u.\cdot A dx$$

$$+c_2 q\int_{K_{\rho,\tau}} (\rho-r)^{2q-2} h_\tau^{2q\delta} h^{2\ell} \varphi(w(v)) \nabla u.A dx,$$

where c_2 depends on c_1 and the constants c, c' of

---

* Here we have used the inequality $h(1+c)-h(c) \geq 1$, valid for all $c > 0$, which follows from the fact that $h' \geq 1$.

Lemma (2.4), and where we have used h, h', h'', and w as abbreviations for $h(w(v))$, $h'(w(v))$, $h''(w(v))$, and $w(v)$ respectively. Now it follows from (1.3) that

$$f\, A_{ip_j} h_{\tau,i} h_{\tau,j} \geq v\{\chi(w(v))\}^{1-2/n^*}\left\{1 + \left(\frac{w(v)\chi'(w(v))}{\chi(w(v))}\right)^2\right\} \cdot g^{ij} h_{\tau,i} h_{\tau,j},$$

and, since we can write $g^{st} = a_{sk}a_{tk}$ (so that $A_{ip_j} g^{st} u_{,is} u_{,jt} = \sum_{k=1}^{n} (A_{ip_j} \xi_{ki} \xi_{kj})$, where $\xi_{ki} = a_{sk}\, u_{,is}, i,k = 1,\ldots,n)$ for a suitable $n \times n$ matrix (a_{ij}), we also have by (1.3) that

$$\frac{fw'h'}{v} g^{st} A_{ip_j} u_{,is} u_{,jt} \geq v\{\chi(w(v))\}^{1-2/n^*} \frac{wh'}{v^2} g^{ij} g^{st} u_{,is} u_{,jt}.$$

Then using $wh' \geq h_\tau$ * in this last inequality, and noting that $h_{\tau,i} = (w(v))_{,i} h'$ on $\operatorname{spt}(h_\tau)$, it follows that

$$(3.7)\quad L \geq \int_{K_{\rho,\tau}} [(\rho-r)^{2q} h_\tau^{2q\delta-2} h^{2\ell} v\{\chi(w(v))\}^{1-2/n^*} \cdot \left\{1 + \left(\frac{w(v)\chi'(w(v))}{\chi(w(v))}\right)^2\right\} \cdot (h')^2 g^{ij} (w(v))_{,i} (w(v))_{,j}]\,dx + \int_{K_{\rho,\tau}} (\rho-r)^{2q} h_\tau^{2q\delta} h^{2\ell} v\chi^{1-2/n^*}(w(v)) \frac{1}{v^2} g^{ij} g^{st} u_{,is} u_{,it}\,dx,$$

---

* Here we have used the inequality $th'(t) \geq h(t) - h(0)$, valid for all $t \in R$, which follows from the fact that $h'' \geq 0$.

where L is the left hand side of inequality (3.6).

Now, using $h \geq 1$, $h' \geq 1$, it is easily checked that

$$\left\{\left[\frac{d}{dt}\left\{t h^{\ell-1}(t)\chi^{\frac{1}{2}-1/n^*}(t)(h(t)-h(w(\sqrt{1+\tau^2})))^{q\delta}\right]_{t=w(v)}\right\}^2$$

$$\leq 2(q\delta+\ell+1)^2\{\chi(w(v))\}^{1-2/n^*}\left\{1+\left(\frac{w(v)\chi'(w(v))}{\chi(w(v))}\right)^2\right\}\cdot$$

$$\cdot\left\{h'(w(v))\right\}^2 h_\tau^{2q\delta-2}h^{2\ell}(w(v))$$

on $spt(h_\tau)$. Hence using this inequality in (3.7) and using the inequality $h^{2\ell} \geq w^2h^{2\ell-2}$ on the second term on the right of (3.7), we have

$$(3.8)\quad L \geq \tfrac{1}{2}(q\delta+\ell+1)^{-2}\iint\limits_{K_{\rho,\tau}}\left\{((\rho-r)^qH)^2\,\frac{1}{v^2}g^{ij}g^{rs}u_{,is}u_{,jt}\,+\right.$$
$$\left.+(\rho-r)^{2q}g^{ij}H_{,i}H_{,j}\right\}v\,dx,$$

where $H = h_\tau^{q\delta}wh^{\ell-1}\{\chi(w(v))\}^{\frac{1}{2}-1/n^*}$.

Also, since g^{ij} is a positive definite symmetric matrix, we have

$$g^{ij}((\rho-r)^qH)_{,i}((\rho-r)^qH)_{,j} \leq 2q^2H^2(\rho-r)^{2q-2}g^{ij}r_{,i}r_{,i}$$
$$+\,2(\rho-r)^{2q}g^{ij}H_{,i}H_{,j},$$

and, by (1.3) and (1.9),

$$v\chi^{1-2/n^*}(w(v))\;g^{ij}r_{,i}r_{,j} \leq f(v)A_{ipj}r_{,i}r_{,j}$$
$$\leq 4\varphi(w(v))\nabla u.A|\nabla r|^2$$
$$= 4\varphi(w(v))\nabla u.A$$

so that

$$(\rho-r)^{2q}g^{ij}H_{,i}H_{,j} \geq \tfrac{1}{2}g^{ij}((\rho-r)^{q}H)_{,i}((\rho-r)^{q}H)_{,j}$$

$$-4q^2(\rho-r)^{2q-2}h_T^{2q\delta}w^2h^{2\ell-2}\varphi(w(v))\,\frac{\nabla u.A}{v}$$

Hence, using this last inequality, it follows from (3.8) that

$$L \geq \frac{1}{4}(q\delta+\ell+1)^{-2}\int_{K_{\rho,T}}(\eta^2\frac{1}{v^2}\, g^{ij}g^{st}u_{,is}u_{,jt}+g^{ij}\eta_{,i}\eta_{,j})\,v\,dx$$

$$-\frac{2q^2}{(q\delta+\ell+1)^2}\int_{K_{\rho,T}}(\rho-r)^{2q-2}h_T^{2q\delta}w^2h^{2\ell-2}\varphi(w(v))\nabla u.A\,dx$$

where $\eta = (\rho-r)^{q}H$.

Then, using this estimate for L together with the inequality (1.8)$'$, it follows from (3.6) that

(3.9) $$\left\{\int_{K_{\rho,T}}\eta^{2n^*/(n^*-2)}\,v\,dx\right\}^{1-2/n^*} \leq c_3q^3\left\{\int_{K_{\rho,T}}\left(1+\nabla u.A\right)dx\right\}^{1-n/n^*}.$$

$$\cdot\left\{\int_{K_{\rho,T}}[(\rho-r)^{2q}h_T^{2q\delta-2}h^{2\ell}\psi(w(v))(w^2h''h_T+wh'h_T+(wh')^2)\right.$$

$$\left.+(\rho-r)^{2q-2}h_T^{2q\delta}h^{2\ell-2}(h^2+w^2)\varphi(w(v))]\nabla u.A\,dx\right\},$$

where c_3 depends on ℓ, $h^{\ell}(1+w(\sqrt{1+T^2}))$, n^*, δ, and the constant c' of (1.8)$'$.

Now the stated conditions on h give

$$w^2h''h_\tau+wh'h_\tau+(wh')^2 \leqslant \beta_1h^{2+\alpha_1}+\sqrt{\beta_1}h^{1+\alpha_1/2}h_\tau \leqslant 2(\beta_1+1)h^{2+\alpha_1}.$$

Also, $h^2h_\tau^{2q\delta-2} = \{h_\tau+h(w(\sqrt{1+\tau^2}))\}^2h_\tau^{2q\delta-2} \leqslant 2h_\tau^{2q\delta}$

$+ 2\{h(w(\sqrt{1+\tau^2}))\}^2h_\tau^{2q\delta-2}$ on $K_{\rho,\tau}$, so that we can estimate

$$(\rho-r)^{2q}h^2h_\tau^{2q\delta-2} \leqslant 2((\rho-r)h_\tau^\delta)^{2q}+2\{h(w(\sqrt{1+\tau^2}))\}^2\rho^{2/\delta}.$$

$$.((\rho-r)h_\tau^\delta)^{2q-2/\delta} \text{ on } K_{\rho,\tau}.$$

Then using these estimates together with the given estimate $v\chi(w(v)) \geqslant \beta_2\, \nabla u.A\, h^{-\alpha_2}\left(\frac{h}{w}\right)^{2n^*/(n^*-2)}$, (3.9) gives

$$\left\{\int_{K_{\rho,\tau}}\left((\rho-r)h_\tau^\delta\right)^{2qn^*/(n^*-2)} h^{2\ell(n^*/(n^*-2))-\alpha_2}\nabla u.Adx\right\}^{(n^*-2)/n^*}$$

$$\leqslant c_4q^3\left\{\int_{K_{\rho,\tau}}(1+\nabla u.A)dx\right\}^{(1-n/n^*)}\times$$

(3.10)

$$\times\left\{\int_{K_{\rho,\tau}}\left((\rho-r)h_\tau^\delta\right)^{2q}\psi(w(v))h^{2\ell+\alpha_1}+\right.$$

$$+ c_5\left((\rho-r)h_\tau^\delta\right)^{2q-2/\delta}\psi(w(v))h^{2\ell+\alpha_1}+$$

$$\left.+\left((\rho-r)h_\tau^\delta\right)^{2q-2}\varphi(w(v))h^{2\ell+2\delta}]\nabla u.Adx\right\},$$

where c_4 depends on c_3, β_1, and β_2, and where

$$c_5 = \left(\rho h^\delta (w(\sqrt{1+\tau^2}))\right)^{2/\delta}.$$

From the given conditions on ψ and φ and the estimate $\int_{K_{\rho,\tau}} 1+\nabla u.A dx \leqslant \int_{K_\rho(x_0)} dx + \int_{K_{\rho,\tau}} \nabla u.A dx$ it follows that

$$\left\{\int_{K_{\rho,\tau}} \left((\rho-r)h_\tau^\delta\right)^{2qn^*/(n^*-2)} h^m \nabla u.A dx\right\}^{(n^*-2)/n^*}$$

$$\leqslant c_6 q^3 \left\{1+\int_{K_{\rho,\tau}} \nabla u.A dx\right\}^{(1-n/n^*)} \times$$

(3.11)

$$\times \left\{\int_{K_{\rho,\tau}} \left[\left((\rho-r)h_\tau^\delta\right)^{2q} + \left((\rho-r)h_\tau^\delta\right)^{2q-2/\delta} + \left((\rho-r)h_\tau^\delta\right)^{2q-2}\right]. .h^m \nabla u.A dx,\right.$$

where c_6 depends $n^*, \rho, c_4, c_5, \beta_3$, and β_4. Noting $\left((\rho-r)h_\tau^\delta\right)^{2q-2/\delta} + \left((\rho-r)h_\tau^\delta\right)^{2q-2} \leqslant 2\left((\rho-r)h_\tau^\delta\right)^{2q} + 2$, this gives

(3.12) $$\left\{\int_{K_{\rho,\tau}} \left((\rho-r)h_\tau^\delta\right)^{2q\kappa} h^m \nabla u.A dx\right\}^{\frac{1}{\kappa}} \leqslant 3c_6 q^3 \left\{1+\int_{K_{\rho,\tau}} \nabla u.A dx\right\}^{(1-n/n^*)} \times$$

$$\times \left\{\int_{K_{\rho,\tau}} \left((\rho-r)h_\tau^\delta\right)^{2q} h^m \nabla u.A dx + \int_{K_{\rho,\tau}} h^m \nabla u.A dx\right\},$$

where $\kappa = n^*/(n^*-2)$. Now using Hölder's and Young's inequalities ([7]p.40), we have

$$\int_{K_{\rho,\tau}} \left((\rho-r)h_\tau^\delta\right)^{2q} h^m \nabla u.A dx$$

$$= \int_{K_{\rho,\tau}} \left((\rho-r)h_\tau^\delta\right)^{2q\alpha} ((\rho-r)h_\tau^\delta)^{2q(1-\alpha)} h^m \nabla u.A dx$$

$$\leq \left\{\int_{K_{\rho,\tau}} \left((\rho-r)h_\tau^\delta\right)^{2q\kappa} h^m \nabla u.A dx\right\}^{\alpha/\kappa} .$$

$$\cdot\left\{\int_{K_{\rho,\tau}} \left((\rho-r)h_\tau^\delta\right)^{2q\kappa(1-\alpha)/(\kappa-\alpha)} h^m \nabla u.A dx\right\}^{(1-\alpha)/\kappa}$$

$$\leq \varepsilon\left\{\int_{K_{\rho,\tau}} \left((\rho-r)h_\tau^\delta\right)^{2q\kappa} h^m \nabla u.A dx\right\}^{1/\kappa} +$$

$$(1-\alpha)\left(\frac{\alpha}{\varepsilon}\right)^{\alpha/(1-\alpha)} \int_{K_{\rho,\tau}} \left((\rho-r)h_\tau^\delta\right)^{2q\kappa(1-\alpha)/(\kappa-\alpha)} h^m \nabla u.A dx\Bigg\}^{\frac{\kappa-\alpha}{\kappa(1-\alpha)}}$$

for arbitrary constants $\alpha \in (0,1)$ and $\varepsilon > 0$. For the remainder of the argument let us assume $\alpha_0 \in (0,1)$. This clearly involves no loss of generality. Then choosing $\varepsilon = \frac{1}{2}\left(3c_6 q^3\left\{1+\int_{K_{\rho,\tau}} \nabla u.A dx\right\}^{1-n/n^*}\right)^{-1}$ and α such that

$\frac{\kappa(1-\alpha)}{\kappa-\alpha} = \alpha'$, where $\alpha' = \min\{\alpha_0, \alpha_0/\delta\}$, it follows from (3.12) and (3.13) that

$$(3.14)\quad \Big\{\int_{K_{\rho,\tau}}\Big((\rho-r)h_\tau^\delta\Big)^{2q\kappa} h^m \nabla u.Adx\Big\}^{\frac{1}{\kappa}} \leqslant Hq^\theta\Big\{\Big[\int_{K_{\rho,\tau}}\Big((\rho-r)h_\tau^\delta\Big)^{2q\alpha'} h^m \cdot$$

$$\cdot \nabla u.Adx\Big]^{1/\alpha'} + \int_{K_{\rho,\tau}} h^m\nabla u.Adx\Big\},$$

where $H = c_7\Big\{1 + \int_{K_{\rho,\tau}}\nabla u.Adx\Big\}^{(n^*-n)(\kappa-\alpha')/(2\alpha'\kappa)}$, c_7 depending on c_6 and α_0, and where θ depends on n^* and α'. (3.14) clearly implies

$$\Big\{\int_{K_{\rho,\tau}}\Big((\rho-r)h_\tau^\delta\Big)^{2q\kappa} h^m\nabla u.Adx\Big\}^{\alpha'/\kappa} \leqslant H^{\alpha'} q^{\theta\alpha'}\Big\{\int_{K_{\rho,\tau}}\Big((\rho-r)h_\tau^\delta\Big)^{2q\alpha'} h^m \cdot$$

$$\cdot \nabla u.Adx + \Big[\int_{K_{\rho,\tau}} h^m\nabla u.Adx\Big]^{\alpha'}\Big\}$$

$$\leqslant \tilde{H}q^{\theta\alpha'}\Big\{1 + \int_{K_{\rho,\tau}}\Big((\rho-r)h_\tau^\delta\Big)^{2q\alpha'} h^m\nabla u.Adx\Big\},$$

where $\tilde{H} = cH^{\alpha'}\Big\{1 + \Big(\int_{K_{\rho,\tau}} h^m\nabla u.Adx\Big)^{\alpha'}\Big\}$. Thus, defining $I(s) =$

$$= 1 + \int_{K_{\rho,\tau}} \left((\rho - r)h_\tau^\delta\right)^{2\alpha' s} h^m \nabla u.A dx, \; s \geq 0,$$ we have

$$(I(\kappa' q))^{1/\kappa'} \leq (1 + \tilde{H}) q^{\theta\alpha'} I(q), \; q \geq 1/\delta,$$

where $\kappa' = \kappa/\alpha'$. Writing $\delta' = \min\{\delta, 1\}$ and using this inequality repeatedly with $q = 1/\delta'$ $\kappa'/\delta', \ldots, (\kappa')^\nu/\delta'$ (ν a positive integer), we obtain

$$(I((\kappa')^\nu/\delta'))^{1/(\kappa')^\nu} \leq$$

$$c_q^{\;1 + \frac{1}{\kappa'} + \cdots + \left(\frac{1}{\kappa'}\right)^{\nu-1}} (\kappa')^{1 + 2\left(\frac{1}{\kappa'}\right) + \cdots + \nu\left(\frac{1}{\kappa'}\right)^{\nu-1}} I(1/\delta'),$$

where $c_9 = (1+c_8)(1/\delta')^{\theta\alpha'}$. Now $\sum_{\nu=0}^{\infty} \left(\frac{1}{\kappa'}\right)^\nu = \frac{\kappa'}{\kappa'-1}$ and

$\sum_{\nu=0}^{\infty} \nu\left(\frac{1}{\kappa'}\right)^{\nu-1} = \left(\frac{\kappa'}{\kappa'-1}\right)^2$, hence we have

$$(3.15) \quad \limsup_{\nu\to\infty} [I((\kappa')^\nu/\delta')]^{1/(\kappa')^\nu} \leq$$

$$c_9^{\;\kappa'/(\kappa'-1)} (\kappa')^{(\kappa'/(\kappa'-1))^2} I(1/\delta').$$

Then the required inequality follows from (3.15) by using the known fact that

$$\limsup \left\{\int_\Omega \gamma^s dx\right\}^{\frac{1}{s}} = \text{essential} \sup_\Omega \gamma$$

for any non-negative measurable function γ on Ω, together with the fact that $\nabla u.A(x,u,\nabla u) \geq c > 0$ for

$|\nabla u| \geq \tau' > \tau_0$, where c depends on χ, f, n^*, τ', and τ_0. (This last fact is established by the same argument as that used to prove inequality (1.11), except that τ' is used instead of τ_0+1.)

Lemma (3.5)(b) can be proved in exactly the same way as above, except that one starts the argument by using Lemma (2.5) instead of Lemma (2.4). The details of the proof are somewhat simplified due to the absence of the factor $(\rho-r)^{2q}$.

The remainder of this section is devoted to the proofs of Theorems (3.1) - (3.4).

Proof of Theorem (3.1):

From the hypotheses of the theorem it is easily checked that we can use Lemma (3.5)(a) with $h(w) \equiv w$, $\alpha_i = 0$, $i = 1,\ldots,4$, and with $\beta_i = \sup_{t\in R} \psi(t)$. Then $m=0$ and the lemma gives, with $\alpha_0=1$,

$$\operatorname*{ess\,sup}_{K_\rho(x_0)}(\rho-r)w_\tau \leq \gamma + \gamma'\left\{\int_{K_{\rho,\tau}}\left(1+(\rho-r)w_\tau^\delta\right)^{2/\delta}\nabla u.Adx\right\}^d,$$

and hence the required bound will be obtained if we can appropriately estimate $\int_{K_{\rho,\tau}}\left(1 + (\rho-r)w_\tau^\delta\right)^{2/\delta}\nabla u.Adx$.

To do this, first note that from inequality (3.10) with $h(w) \equiv w$, $\varphi(w) \equiv \mu w^{-2\delta}$, and $\ell = \alpha_1 = \alpha_2 = 0$, we have

$$\left\{\int_{K_{\rho,\tau}}\left((\rho-r)w_\tau^\delta\right)^{2\kappa q}\nabla u.A dx\right\}^{\frac{1}{\kappa}} \leq \mu c_4 q^2\left\{\int_{K_{\rho,\tau}} 1+\nabla u.A dx\right\}^{(1-n/n^*)} \times$$

$$\left\{\int_{K_{\rho,\tau}}\psi(w(v))((\rho-r)w_\tau^\delta)^{2q}\nabla u.A dx+\int_{K_{\rho,\tau}}[c_5((\rho-r)w_\tau^\delta)^{2q-2/\delta}\psi(w(v))+\right.$$

$$\left.\mu((\rho-r)w_\tau)^{2q-2}]\nabla u.A dx\right\},$$

where $\kappa = \frac{n^*}{n^*-2}$, and since by Hölder's inequality

$$\int_{K_{\rho,\tau}}\left((\rho-r)w_\tau^\delta\right)^{2q}\nabla u.A dx \leq \left\{\int_{K_{\rho,\tau}}\left((\rho-r)w_\tau^\delta\right)^{2q\kappa}\nabla u.A dx\right\}^{\frac{1}{\kappa}}\left\{\int_{K_{\rho,\tau}}\nabla u.A dx\right\}^{1-\frac{1}{\kappa}},$$

this gives

$$\int_{K_{\rho,\tau}}\left((\rho-r)w_\tau^\delta\right)^{2q}\nabla u.A dx \leq H\int_{K_{\rho,\tau}}\psi(w(v))((\rho-r)w_\tau^\delta)^{2q}\nabla u.A dx +$$

$$H(c_5+\mu)\int_{K_{\rho,\tau}}[\left((\rho-r)w_\tau^\delta\right)^{2q-2/\delta}+((\rho-r)w_\tau^\delta)^{2q-2}]\nabla u.A dx,$$

where $H = \mu c_4 q^2\left\{\int_{K_{\rho,\tau}} 1+\nabla u.A dx\right\}^{2-1/\kappa-n/n^*} \leq \tilde{H}$,

$$\tilde{H} = \mu c_4 q^2\left\{\int_{K_\rho(x_0)} 1+|\nabla u.A|\, dx\right\}^{2-1/\kappa-n/n^*}.$$

Then choosing τ large enough to ensure $\mathfrak{H}\psi(w(v)) < \frac{1}{2}$ on $K_{\rho,\tau}$ (the choice of τ can be made to depend only on the quantity Δ of Lemma (2.1), q, n^*, μ, and δ), this gives

$$\int_{K_{\rho,\tau}} \left((\rho-r)w_\tau^\delta\right)^{2q} \nabla u.A dx \leq c_6 \int_{K_{\rho,\tau}} [\left((\rho-r)w_\tau^\delta\right)^{2q-2/\delta} + \left((\rho-r)w_\tau^\delta\right)^{2q-2}] \nabla u.A dx$$

where c_6 depends on $\Delta, q, n^*, \mu, \delta, c_4$, and c_5.

But now, by Young's inequality

$$(|ab| \leq \varepsilon |a|^{\frac{1}{\alpha}} + (1-\alpha)\left(\frac{\alpha}{\varepsilon}\right)^{\frac{\alpha}{1-\alpha}} |b|^{\frac{1}{1-\alpha}},$$ valid for $a, b \in R$, $\alpha \in (0,1)$, and $\varepsilon > 0$) with $\varepsilon = \frac{1}{4}$ and $\alpha = \frac{2q-2/\delta}{2q}$ *, we have

$$((\rho-r)w_\tau^\delta)^{2q-2/\delta} c_6 \leq \frac{1}{4}((\rho-r)w_\tau^\delta)^{2q} + c$$

and

$$((\rho-r)w_\tau^\delta)^{2q-2} c_6 \leq \frac{1}{4}((\rho-r)w_\tau^\delta)^{2q} + c',$$

where c depends on c_6, q, and δ, and c' depends on c_6 and q. It follows that

---

* The case $2q-2/\delta = 0$ is trivially handled without Young's inequality.

$$\int_{K_{\rho,\tau}} \left((\rho-r)w_\tau^\delta\right)^{2q} \nabla u.A dx \leq 2(c+c') \int_{K_{\rho,\tau}} \nabla u.A\, dx.$$

Clearly this gives

$$\int_{K_{\rho,\tau}} \left(1+(\rho-r)w_\tau^\delta\right)^{2q} \nabla u.A dx \leq c_7 \int_{K_{\rho,\tau}} \nabla u.A dx \leq c_7 \int_{K_\rho(x_0)} |\nabla u.A|\, dx,$$

where c_7 depends on q, c, and c'. Then using Lemma (2.1) we have an estimate for $\int_{K_{\rho,\tau}} \left(1+(\rho-r)w_\tau^\delta\right)^{2q} \nabla u.A dx$ for any $q \geq \max\{1, 1/\delta\}$. Hence certainly we have an appropriate bound for $\int_{K_{\rho,\tau}} (1+(\rho-r)w_\tau^\delta)^{2/\delta} \nabla u.A dx$ and the proof of Theorem (3.1) is complete.

Proof of Theorem (3.2)(a):

From the hypotheses of the theorem we can use Lemma (3.5)(a) with $h(w) \equiv w$, $\alpha_2 = k$, $\alpha_1 = \alpha_3 = \alpha_4 = 0$, $\alpha_0 = 1$, and with $\rho/2$ in place of ρ. Then

$$\operatorname*{ess\,sup}_{K_{\rho/2}(x_0)} (\rho/2-r)w_\tau^\delta \leq \gamma + \gamma' \left\{ \int_{K_{\rho/2,\tau}} (1+w_\tau)^{\frac{kn^*}{2}+2} (1+(\rho/2-r)w_\tau^\delta)^{2/\delta} . \nabla u.A dx \right\}^d,$$

and since $\rho/2-r \geq \rho/4$ on $K_{\rho/4}(x_0)$ this gives

$$\operatorname{ess}\sup_{K_{\rho/4}(x_0)} w_\tau^\delta \leq \tilde{\gamma}\left\{1+\left(\int_{K_{\rho/2,\tau}}\left(1+w_\tau\right)^{\frac{kn^*}{2}+4}\nabla u.Adx\right)^d\right\}$$

where $\tilde{\gamma}$ depends on $\gamma,\gamma',d,\delta,$ and ρ. Thus the proof will be complete if we can appropriately estimate

$$\int_{K_{\rho/2,\tau}}\left(1+w_\tau\right)^{\frac{kn^*+4}{2}}\nabla u.Adx.$$

To do this, first note that by Lemma (2.4) with $h(w) \equiv w$ we have, using $g^{st}A_{ip_j}u_{,is}u_{,jt} \geq 0$,

$$q'\int_{K_{\rho,\tau}}(\rho-r)^{2q}w_\tau^{2q-2}fA_{ip_j}w_{\tau,i}w_{\tau,j}dx \leq$$

$$\leq 2c(q+q')\int_{K_{\rho,\tau}}(\rho-r)^{2q}w_\tau^{2q'-2}w^2\psi(u(v))\nabla u.Adx +$$

$$+c'q(1-q/q')\int_{K_{\rho,\tau}}(\rho-r)^{2q-2}w_\tau^{2q'}\varphi(w(v))\nabla u.Adx.$$

Now since $w^2 = (w_\tau+w(\sqrt{1+\tau^2}))^2 \leq 2w_\tau^2 + 2w^2(\sqrt{1+\tau^2}$ on $K_{\rho,\tau}$, we have

$$(\rho-r)^{2q}w_\tau^{2q'-2}w^2 \geq 2(\rho-r)^{2q}w_\tau^{2q'}+2w^2(\sqrt{1+\tau^2})(\rho-r)^{2q}w_\tau^{2q'-2},$$

so that

$$\int_{K_{\rho,\tau}} (\rho-r)^{2q} w_\tau^{2q'-2} f A_{ipj} w_{\tau,i} w_{\tau,j} dx \leq$$

(3.16)

$$\leq 4c\left(1 + \frac{q}{q'}\right)\Big\{\int_{K_{\rho,\tau}} (\rho-r)^{2q} w_\tau^{2q'} \psi(w(v))\nabla u, A dx +$$

$$+ c_1 \int_{K_{\rho,\tau}} (\rho-r)^{2q} w_\tau^{2q-2} \psi(w(v))\nabla u.A dx\Big\}$$

$$+c'(q/q')(1-q/q')\int_{K_{\rho,\tau}} (\rho-r)^{2q-2} w_\tau^{2q} \varphi(w(v))\nabla u.A dx,$$

where $c_1 = w^2(\sqrt{1+\tau^2})$.

Setting $q' = q\delta$, $q \geq \max\{1, 1/\delta\}$, and using the estimate

$$(\rho-r)^{2q} w_\tau^{2q\delta-2} \leq \rho^{2/\delta}((\rho-r) w_\tau^\delta)^{2q-2/\delta}$$

together with the given identity $\varphi(w) \equiv \mu w^{-2\delta}$, this gives

$$\int_{K_{\rho,\tau}} (\rho-r)^{2q} w_\tau^{2q\delta-2} f A_{ip} w_{\tau,i} w_{\tau,j} dx \leq$$

$$4c(1+\tfrac{1}{\delta})\int_{K_{\rho,\tau}} \left((\rho-r) w_\tau^\delta\right)^{2q} 4(w(v))\nabla u.A dx +$$

(3.17)

$$+ c_2 \int_{K_{\rho,\tau}} \Big\{\left((\rho-r) w_\tau^\delta\right)^{2q-2/\delta} \psi(w(v)) + \left((\rho-r) w_\tau^\delta\right)^{2q-2}\Big\}\nabla u.A dx,$$

where c_2 depends on n, μ, c_1, δ, and ρ.

Next, by Lemma (2.2) with $h(w) \equiv w$ and $q' = q\delta$, we have

$$\int_{K_{\rho,\tau}} \left((\rho-r)w_\tau^\delta\right)^{2q} \nabla u.A\,dx \leq c_3 q^2 \int_{K_{\rho,\tau}} \left((\rho-r)w_\tau^\delta\right)^{2q-2} \nabla u.A\,dx +$$

(3.18)

$$+ c_3 \delta^2 q^2 \int_{K_{\rho,\tau}} (\rho-r)^{2q} w_\tau^{2q\delta-2} f A_{ipj} w_{\tau,i} w_{\tau,j}\,dx,$$

where $c_3 = 32.2^{2\mu m\rho}(\mu m\rho)^2$.

Combining (3.17) and (3.18) it follows that

$$(3.19)\quad \int_{K_{\rho,\tau}} \left((\rho-r)w_\tau^\delta\right)^{2q} \nabla u.A\,dx \leq 8cc_3 q^2 \delta^2 \int_{K_{\rho,\tau}} \left((\rho-r)w_\tau^\delta\right)^{2q} \cdot$$

$$\cdot \psi(w(v)) \nabla u.A\,dx +$$

$$+ c_4 \int_{K_{\rho,\tau}} \left\{ \left((\rho-r)w_\tau^\delta\right)^{2q-2/\delta} \psi(w(v)) + \left((\rho-r)w_\tau^\delta\right)^{2q-2} \right\} \nabla u.A\,dx,$$

where $c_4 = c_3 q^2 (1 + c_2 \delta^2)$.

Then choosing τ large enough to ensure $8cc_3 q^2 \delta^2 \psi(w(v)) < \frac{1}{2}$ on $K_{\rho,\tau}$, we obtain

$$(3.20) \quad \int_{K_{\rho,\tau}} \left((\rho-r)w_\tau^\delta\right)^{2q} \nabla u.A dx \leq$$

$$c_5 \int_{K_{\rho,\tau}} \left\{ \left((\rho-r)w_\tau^\delta\right)^{2q-2/\delta} + \left((\rho-r)w_\tau^\delta\right)^{2q-2} \right\} \nabla u.A dx,$$

where c_5 depends on c_4 and ψ.

Now, using Young's inequality as in the proof of Theorem (3.1), we have

$$c_5\left((\rho-r)w_\tau^\delta\right)^{2q-2/\delta} \leq \frac{1}{4}\left((\rho-r)w_\tau^\delta\right)^{2q} + c_6,$$

$$c_5\left((\rho-r)w_\tau^\delta\right)^{2q-2} \leq \frac{1}{4}\left((\rho-r)w_\tau^\delta\right)^{2q} + c_6',$$

where c_6 depends on c_5, q, and δ, and c_6' depends on c_5 and q. Hence (3.20) gives

$$\int_{K_{\rho,\tau}} \left((\rho-r)w_\tau^\delta\right)^{2q} \nabla u.A dx \leq 2(c_6+c_6') \int_{K_{\rho,\tau}} \nabla u.A dx,$$

and, since $\rho-r \geq \rho/2$ on $K_{\rho/2}(x_0)$, it follows that

$$(3.21) \quad \int_{K_{\rho/2,\tau}} (1+w_\tau)^{2q\delta} \nabla u.A dx \leq c_7 \int_{K_{\rho,\tau}} \nabla u.A dx, \quad q \geq \max(1,1/\delta)$$

where c_7 depends on $c_6+c_6', q, \delta,$ and ρ. Thus choosing q such that $2q\delta \geq \frac{kn^*}{2} + 4$ and using Lemma (2.1), Theorem (3.2)(a) is established.

To prove Theorem (3.2)(b), first note that if we replace ρ by $\tilde{\rho}$ in (3.19), where $\tilde{\rho} \leq \rho/2$ is small enough to ensure $8\mu c\tilde{c}_3 q^2 \delta^2 < \frac{1}{2}$ ($\tilde{c}_3 = 32.2^{2\mu m\tilde{\rho}}(\mu m\tilde{\rho})^2$), then we still obtain (3.20) (with $\tilde{\rho}$ in place of ρ), and hence

$$(3.21)' \quad \int_{K_{\tilde{\rho}/2,\tau}} (1+w_\tau^\delta)^{2q\delta} \nabla u.Adx \leq \tilde{c}_7 \int_{K_{\tilde{\rho},\tau}} \nabla u.Adx, q \geq \max\{1, 1/\delta\},$$

where $\tilde{c}_7$ depends on $n, \mu, w(\sqrt{1+\tau^2}), \psi, q, \delta, \rho,$ and $\tilde{c}_3 (= 32.2^{2\mu m\tilde{\rho}}(\mu m\tilde{\rho})^2)$.

Then repeating the remainder of the argument gives

$$(3.22) \qquad \text{ess sup} |\nabla u| \leq \Gamma,$$

where essential sup is taken over $K_{\tilde{\rho}/4}(x_0)$ and where Γ depends on $n^*, f, k, \delta, \mu, \tau_0, M, \tilde{\rho}$ and the quantity Δ of Lemma (2.1).

Now $\tilde{\rho}$ can clearly be chosen to depend only on $\rho, q, \mu, n, \delta,$ and the modulus of continuity of u on Ω. In particular (provided that $K_{4\rho}(x_0) \subset \Omega$) $\tilde{\rho}$ can be chosen independent of x_0, so that, since $K_{4\tilde{\rho}}(x_0') \subset \Omega$ for any $x_0' \in K_{\rho/4}(x_0)$ (because we have stipulated $\tilde{\rho} \leq \rho/2$), the required estimate for $|\nabla u|$ on $K_{\rho/4}(x_0)$ is obtained.

Proof of Theorem (3.2)(c):

To prove (c) first define h by $h(w) \equiv e^{\varepsilon w^\delta}$, where $\varepsilon > 0$ is a constant, and let $\tau_1 \geq \tau_0$ be large enough to ensure $\theta(t) \leq \varepsilon$, $h'(w(t)) \geq 1$, and

$h''(w(t)) \geq 0$ for $t \geq \sqrt{1+\tau_1^2}$. Then we can use Lemma (3.5) (a) * with $\tau=\tau_1$, $\alpha_0=\alpha_1=1$, $\alpha_2 = \frac{2n^*}{n^*-2} + 1$, $\alpha_3=2\delta$, $\alpha_4=0$, with a suitable choice of $\beta_1,\ldots,\beta_4$, and with $\rho/2$ in place of ρ. Then the lemma gives

$$\operatorname{ess}\sup_{K_{\rho/2}(x_0)} (\rho/2-r)h^{\delta}_{\tau_1} \leq \gamma+\gamma'\left\{\int_{K_{\rho/2,\tau_1}} (1+h_{\tau_1})^m(1+(\rho/2-r)h^{\delta}_{\tau_1})^{2/\delta} \cdot \nabla u.Adx\right\}^d,$$

where m depends on n^* and δ. Then since $\rho/2-r \geq \rho/4$ on $K_{\rho/4}(x_0)$ it follows that

$$\operatorname{ess}\sup_{K_{\rho/4}(x_0)} h^{\delta}_{\tau_1} \leq \tilde{\gamma}\left(1+\left\{\int_{K_{\rho/2}} (1+h_{\tau_1})^{m+2}\nabla u.Adx\right\}^d\right),$$

where $\tilde{\gamma}$ depends on $\gamma,\gamma',\rho,d,$ and δ.

Then since ε can be chosen as near to zero as we please, and since m does not depend on ε, it is clear that the required bound will be obtained if we can show that $\int_{K_{\rho/2,\tau_1}} e^{\varepsilon w^{\delta}_{\tau_1}}\nabla u.Adx$ is appropriately bounded for some constant $\varepsilon>0$. We will in fact obtain a bound for $\int_{K_{\rho/2,\tau_0}} e^{\varepsilon w^{\delta}_{\tau_0}}\nabla u.Adx$ for some suitable $\varepsilon > 0$.

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* An examination of the proof of Lemma (3.5)(a) shows that it is really only necessary for $h' \geq 1$ and $h'' \geq 0$ on the interval $[w(\sqrt{1+\tau^2}),\infty)$, and we are using this fact here.

To show that such a bound exists, first note that, by (3.19) with $\psi \equiv \mu/w^{2\delta}$ and $\tau=\tau_0$, we have

$$\int_{K_{\rho,\tau_0}}\Big((\rho-r)w^{\delta}_{\tau_0}\Big)^{2q}\nabla u.Adx \leq 8cc_3q^2\delta^2\mu\int_{K_{\rho,\tau_0}}(\rho-r)^{2q}(w^{\delta}_{\tau_0})^{2q-2}\nabla u.Adx +$$

$$c_4\mu\int_{K_{\rho,\tau_0}}\Big((\rho-r)w^{\delta}_{\tau}\Big)^{2q-2/\delta}\nabla u.Adx +$$

$$c_4\int_{K_{\rho,\tau_0}}\Big((\rho-r)w^{\delta}_{\tau_0}\Big)^{2q-2}\nabla u.Adx, \quad q \geq \max\{1,1/\delta\}.$$

Then using $(\rho-r)^{2q} \leq \rho^2(\rho-r)^{2q-2}$ in the first term of the right hand side of this inequality, we obtain

$$(3.23)\quad \int_{K_{\rho,\tau_0}}\Big((\rho-r)w^{\delta}_{\tau_0}\Big)^{2q}\nabla u.Adx \leq$$

$$\leq d_1\int_{K_{\rho,\tau_0}}\Big\{q^2\Big((\rho-r)w^{\delta}_{\tau_0}\Big)^{2q-2}+\Big((\rho-r)w^{\delta}_{\tau_0}\Big)^{2q-2/\delta}\Big\}\nabla u.Adx,$$

where d_1 depends on $\rho, c_4, \mu,$ and cc_3.

Now by Young's inequality

$$d_1q^2\Big((\rho-r)w^{\delta}_{\tau_0}\Big)^{2q-2} \leq \frac{1}{4}\Big((\rho-r)w^{\delta}_{\tau_0}\Big)^{2q} + \frac{1}{q}(4d_1q^2)^q\Big(\frac{q-1}{q}\Big)^q,$$

and

$$d_1\Big((\rho-r)w^{\delta}_{\tau_0}\Big)^{2q-2/\delta} \leq \frac{1}{4}\Big((\rho-r)w^{\delta}_{\tau_0}\Big)^{2q}+\frac{1}{(q\delta)}(4d_1)^{q\delta}\Big(\frac{q-\frac{1}{\delta}}{q}\Big)^{q\delta}.$$

Also, we have the inequality $q^{2q} \leqslant (2q)!\left(\frac{e}{2}\right)^{2q} (e=\exp(1))$ *, valid for all positive integers q. Then using these inequalities it follows from (3.23) that

$$\int_{K_{\rho,\tau_0}} \left((\rho-r)w^{\delta}_{\tau_0}\right)^{2q} \nabla u.Adx \leqslant (d_2)^{2q}(2q)!\int_{K_{\rho,\tau_0}} \nabla u.Adx$$

for all integral $q \geqslant \max\{1,1/\delta\}$, where d_2 depends on d_1 ,δ, and ρ. Now after rearrangement, and noting that $(\rho-r) \geqslant \rho/2$ on $K_{\rho/2,\tau_0}$, we have

$$(3.24) \quad \int_{K_{\rho/2,\tau_0}} \frac{(d_3 w^{\delta}_{\tau_0})^{2q}}{(2q)!} \nabla u.Adx \leqslant (\tfrac{1}{2})^{2q}\int_{K_{\rho,\tau_0}} \nabla u.Adx$$

where $d_3 = \rho/(4d_2)$. Then letting ν be the least positive integer $\geqslant \max\{1,1/\delta\}$ and using the fact that $\sum_{q=\nu}^{\infty} \frac{t^{2q}}{(2q)!} \geqslant ce^t-d$ for all real t, where $c>0$ and d are constants depending on δ, we obtain the estimate

$$\int_{K_{\rho/2,\tau_0}} e^{d_3 w^{\delta}_{\tau_0}} \nabla u.Adx \leqslant \frac{d+1}{c}\int_{K_{\rho,\tau_0}} \nabla u.Adx \leqslant \frac{d+1}{c} \int_{K_{\rho}(x_0)} |\nabla u.A|dx.$$

Then using Lemma (2.1) this clearly yields an appropriate estimate for

---

* Because $(q^q/q!)((q-1)!/(q-1)^{q-1})=\{q/(q-1)\}^{q-1}\uparrow e$, and hence $q^q/q!\leqslant e(q-1)^{q-1}/(q-1)!$, and this inequality can be iterated to give $q^q/q!\leqslant e^q$.

$$\int_{K_{\rho/2,\tau_0}} e^{\varepsilon w_\tau^\delta} \circ \nabla u . A dx$$

for some constant $\varepsilon>0$ depending on $n, w, \tau_0, \mu, \mu m\rho, q, \delta$, and ρ.

Proof of Theorem (3.3)(a):

By Lemma (3.5)(a) with $h(w) \equiv w$, $\alpha_0=1$, $\alpha_1=0$, $\alpha_2=k$, $\alpha_3=2\delta$, $\alpha_4=1$, $\beta_1,\ldots,\beta_4$ depending on μ and ψ, and with $\rho/2$ in place of ρ, we obtain

$$\operatorname*{ess\,sup}_{K_{\rho/2}(x_0)} \left((\rho/2-r)w_\tau^\delta\right) \leqslant \gamma+\gamma'\left\{\int_{K_{\rho/2,\tau}} (1+w_\tau)^m (1+(\rho/2-r)w_\tau^\delta)^{2/\delta} . \nabla u . A dx\right\}^d,$$

where m depends on n^*, k and δ, and d depends on n^*. Then, since $\rho/2-r \geqslant \rho/4$ on $K_{\rho/4}(x_0)$, this gives

$$\operatorname*{ess\,sup}_{K_{\rho/4}(x_0)} w_\tau^\delta \leqslant \tilde{\gamma}\left(1+\left\{\int_{K_{\rho/2,\tau}} (1+w_\tau)^{m+2}\nabla u . A dx\right\}^d\right),$$

where $\tilde{\gamma}$ depends on $\gamma, \gamma', \rho, \delta$, and d,

and hence the required bound will be obtained if we can appropriately estimate $\int_{K_{\rho/2,\tau}} (1+w_\tau)^{m+2}\nabla u . A dx$.

To do this, note first that by Lemma (2.6) with $\varphi \equiv \mu$ and $q' = q\delta$, $q \geqslant 1/\delta$,

$$\int_{K_{\rho,\tau}} (\rho-r)^{2q} w_\tau^{2q\delta-2} f A_{ip_j} w_{\tau,i} w_{\tau,j} dx$$

$$\leq c \int_{K_{\rho,\tau}} (\rho-r)^{2q} w_\tau^{2q\delta-2} \psi(w) w \nabla u.Adx +$$

$$+(c_1+\mu)c' q^2(1+\rho^2\delta^2) \int_{K_{\rho,\tau}} (\rho-r)^{2q-2} w_\tau^{2q\delta-4} w^2 \nabla u.Adx,$$

where c_1 is a constant such that $\frac{\psi(t)}{t} \leq c_1$ for all $t \in (0,\infty)$.

Now since $w^2=(w_\tau+w(\sqrt{1+\tau^2}))^2 \leq 2w_\tau^2+2w^2(\sqrt{1+\tau^2}$ on $K_{\rho,\tau}$, we have, after some rearrangement and after using the inequality $\rho-r \leq \rho$ (c.f. the relevant part of the proof of Theorem (3.2)(a)),

$$\int_{K_{\rho,\tau}} (\rho-r)^{2q} w_\tau^{2q\delta-2} f A_{ip_j} w_{\tau,i} w_{\tau,j} dx$$

$$\leq d_1 \int_{K_{\rho,\tau}} \left((\rho-r)w_\tau^\delta\right)^{2q} \frac{\psi(w)}{w} \nabla u.Adx +$$

$$d_2 q^2 \int_{K_{\rho,\tau}} \left\{\left((\rho-r)w_\tau^\delta\right)^{2q-2/\delta} + \left((\rho-r)w_\tau^\delta\right)^{2q-4/\delta}\right\} \nabla u.Adx,$$

where d_1 depends on c, and d_2 depends on c, c', ρ, δ, c_1, μ, and $w(\sqrt{1+\tau^2})$. Note that here we have used $0<\delta\leq 1$.

Combining this inequality with (3.18) we obtain

$$(3.19)\quad \int\limits_{K_{\rho,\tau}} \left((\rho-r)w_\tau^\delta\right)^{2q} \nabla u.A dx \leq c_3\delta^2 d_1 q^2 \int\limits_{K_{\rho,\tau}} \left((\rho-r)w_\tau^\delta\right)^{2q} \frac{\psi(w)}{w} \nabla u.A dx +$$

$$+c_3(\delta^2 d_2+1)q^4 \iint\limits_{K_{\rho,\tau}} \Big\{ \left((\rho-r)w_\tau^\delta\right)^{2q-2} + \left((\rho-r)w_\tau^\delta\right)^{2q-2/\delta} + + \left((\rho-r)w_\tau^\delta\right)^{2q-4/\delta} \Big\} \nabla u.A dx.$$

The remainder of the argument follows the relevant part of the proof of Theorem (3.2)(a).

The modifications of the above argument which are necessary to prove Theorem (3.3)(b) are identical to the corresponding modifications in the case of Theorem (3.2).

Proof of Theorem (3.3)(c):

The argument is similar to the proof of Theorem (3.2)(c), but subject to the following modifications:

(i) Lemma (3.5)(a) is applied to the function $h(w) \equiv e^{\varepsilon w^{\delta/2}}$, with τ_1 chosen as before and with α_1, β_1 chosen as before, except that $\alpha_4 = 1$. Then the problem is reduced to estimating the integral $\int\limits_{K_{\rho/2,\tau_1}} e^{\varepsilon w_{\tau_1}^{\delta/2}} \nabla u.A dx$ for some $\varepsilon > 0$.

(ii) The next part of the argument is started by using (3.19)′, with $\tau=\tau_0$, instead of (3.19). Then using $\psi(w)/w \equiv \mu w^{-2\delta}$ and

$$\left((\rho-r)w_{\tau_0}^\delta\right)^{2q-2/\delta} + \left((\rho-r)w_{\tau_0}^\delta\right)^{2q-4/\delta} \leq 2\left((\rho-r)w_{\tau_0}^\delta\right)^{2q-2} + 2$$

(because $\delta<1$), we have

$$\int_{K_{\rho,\tau_0}}\left((\rho-r)w_{\tau_0}^{\delta}\right)^{2q}\nabla u.A dx \leq d_3 q^4\int_{K_{\rho,\tau_0}}\left\{\left((\rho-r)w_{\tau_0}^{\delta}\right)^{2q-2}+1\right\}\nabla u.A dx,$$

where d_3 depends on μ and on d_1, d_2, and c_3 of (3.19)';

(iii) after an application of Young's inequality and the use of the inequality $q^{4q} \leq (4q)!\left(\frac{e}{4}\right)^{4q}$ (instead of $q^{2q} \leq (2q)!\left(\frac{e}{2}\right)^{2q}$ as before as in the case of Theorem (3.2) (c)), we obtain

$$(3.23)' \quad \int_{K_{\rho/2,\tau_0}}\frac{(d_4 w_{\tau_0}^{\delta/2})^{4q}}{(4q)!}\nabla u.A dx \leq \left(\tfrac{1}{2}\right)^{4q}\int_{K_{\rho,\tau_0}}\nabla u.A dx, \quad q \geq 1/\delta,$$

where $d_4 > 0$ depends on d_3 and ρ. Then this yields a bound for $\int_{K_{\rho/2,\tau_0}} e^{d_4 w_{\tau_0}^{\delta/2}}\nabla u.A dx$ by summing over integral $q \geq 1/\delta$ and using Lemma (2.1) as in the proof of Theorem (3.2)(c).

The proofs of Theorem (3.4)(a) and Theorem (3.4)(b) are similar to the proofs of Theorem (3.2)(a) and Theorem (3.2)(c) respectively, except that Lemmas (2.3), (2.5), and (3.5)(b) are used instead of Lemmas (2.2), (2.4), and (3.5) (a) respectively. The only complication is that we now need an estimate for $\int_{\Omega_\tau}\nabla u.A(x,u,\nabla u)dx$ at steps of the argument corresponding to those steps in the proofs of Theorem (3.2)(a) and Theorem (3.2)(c) where Lemma (2.1) was used.

$\int_{\Omega_\tau} \nabla u.A dx$ can in fact be quite easily estimated.

First, we have the inequality

$$(3.25)\quad \int_{\Omega_\tau} w_\tau^{2q} \nabla u.A dx \leq d_1 \int_{\Omega_\tau} w_\tau^{2q} \psi(w(v)) \nabla u.A dx$$

$$+ d_2 \int w_\tau^{2q-2} \psi(w(v)) \nabla u.A dx, \; \tau > \max\left\{\sup_{\partial\Omega} \bar{v}, \tau_0\right\},$$

where d_1 depends on μ, M and q and d_2 depends on μ,M,w,τ, and q. This inequality is derived in a similar way to (3.19) except that Lemma (2.3) and Lemma (2.5) are used instead of Lemma (2.2) and Lemma (2.4) respectively.

Now since we are given $\psi(t) \to 0$ as $t \to \infty$ in each part of Theorem (3.4), we can find $\tau' > \max\left\{\sup_{\partial\Omega} \bar{v}, \tau_0\right\}$ such that $(1+d_1)\psi(w(v)) \leq \frac{1}{2}$ on $\Omega_{\tau'}$. Then it follows from (3.25) (with $q=2$ and $\tau=\tau'$) that

$$(3.26)\quad \int_{\Omega_{\tau'}} w_\tau^2 \nabla u.A dx \leq d_2 \int_{\Omega_{\tau'}} \nabla u.A dx.$$

Now take $\tau'' > \tau'$ such that $w(\sqrt{1+(\tau'')^2}) - w(\sqrt{1+(\tau')^2}) > 1 + d_2$. Then it follows from (3.26) that

$$\int_{\Omega_{\tau''}} \nabla u.A dx + \int_{\Omega_{\tau'}-\Omega_{\tau''}} w_\tau^2 \nabla u.A dx \leqslant d_2 \int_{\Omega_{\tau'}-\Omega_{\tau''}} \nabla u.A dx,$$

so that

$$\int_{\Omega_{\tau''}} \nabla u.A dx \leqslant d_2 \int_{\Omega_{\tau'}-\Omega_{\tau''}} \nabla u.A dx .$$

Then letting $\tau_1 = \max\left\{\sup_{\partial\Omega} \bar{v}, \tau_0\right\}$ and adding $\int_{\Omega_{\tau_1}-\Omega_{\tau''}} \nabla u.A dx$ to each side of this last inequality gives

$$\int_{\Omega_{\tau_1}} \nabla u.A dx \leqslant (1+d_2)\int_{\Omega_{\tau_1}-\Omega_{\tau''}} \nabla u.A dx$$

$$\leqslant (1+d_2)\sup\{p.A(x,z,p); x\in\Omega, |z| \leqslant M, \tau_1\leqslant|p|\leqslant\tau''\}.\mathrm{meas}(\Omega),$$

which is a suitable estimate.

4. Some Examples

(4.1) As an example where Theorem (3.1) applies, consider the equation

$$\frac{d}{dx_1}(\exp(|\nabla u|^2)u_{,1}) = B(x,u,\nabla u),$$

which in the case $B \equiv 0$ is the Euler equation for the integral $\int_\Omega \exp(|\nabla u|^2)dx$.

It is not too difficult to check that the conditions of Theorem (3.1) are satisfied, provided

$|B(x,z,p)| \leq \mu(1+|p|^2)\exp(|p|^2)$, with $f(v) = \frac{1}{v^2}$, $\delta = \frac{1}{3}$, $\tau_0 = 1$ and with χ such that $\chi\left(\frac{v^3}{3}\right) = v \exp v^2$.

We note that for this and for many examples where $p.A(x,z,p)$ has rapid growth as $|p| \to \infty$, it would suffice to use the ordinary Sobolev inequality instead of the inequality (1.8).

(4.2) The mean curvature equation

$$\frac{\partial}{\partial x_i}(u_{,i}/\sqrt{1+|\nabla u|^2}) = \mathcal{H}(x,u,\nabla u)$$

can be treated, provided $\mathcal{H}$ is measurable on $\Omega \times R \times R^n$ and $|\mathcal{H}(x,z,p)| \leq \frac{\mu}{\sqrt{1+|p|^2}}$ by using Theorem (3.3)(a) with $f(v) = v^2$, $\delta=1$, $\tau_0=1$, $\chi \equiv 1$, and $k = 1$.

More generally, let us consider the class of equations

$$\frac{\partial}{\partial x_i}(\Psi(v)u_{,i}) = B(x,u,\nabla u), \quad v = \sqrt{1+|\nabla u|^2},$$

where Ψ is a positive $C'(0,\infty)$ function such that for some constant c

$$-1 \leq \frac{t\Psi'(t)}{\Psi(t)} \leq c \quad \text{for all} \quad t \in (1,\infty).$$

Note that this condition on Ψ is equivalent to the requirements that $\Psi(t)/t^c$ is a non-increasing function and $t\Psi(t)$ is non-decreasing, $t \geq 1$.

Then provided either $|B(x,z,p)| \leq \mu\Psi(v)$ or

$$v^{-1}|B_x(x,z,p)| + |B_z(x,z,p)| + v|B_p(x,z,p)| \leq \mu\Psi(v), \quad (v = \sqrt{1+|p|^2}),$$

Theorem (3.3)(a) applies with $f(v) = v^2$, $k=\delta=\tau_0=1$, and $\chi(w(v)) \equiv v\Psi(v)$. This can be checked by writing

$$A_i(x,z,p) = \Psi(v)p_i, \quad i=1,\ldots,n, \quad v = \sqrt{1+|p|^2},$$

and noting that

$$A_{i\,p_j}(x,z,p) = \Psi\left\{\delta_{ij} + \frac{p_i}{v}\frac{p_j}{v}\left(v\,\frac{\Psi'(v)}{\Psi(v)}\right)\right\}, \quad i,j = 1,\ldots,n,$$

$$p_iA_i(x,z,p) = |p|^2\Psi(v),$$

and

$$|A(x,z,p)| = |p|\Psi(v).$$

We note here that the equations in part II of [4] can also be handled by Theorem (3.3)(a), provided we take $f(v) \equiv v^2$, $k=\delta=1$, $\chi \equiv 1$, and τ_0 large enough.

(4.3) Finally, as an example to which Theorem (3.2)(a) is applicable consider the equation

$$\frac{\partial}{\partial x_i}\Phi(p_i) = 0,$$

where Φ is a $C'(R)$ function such that

$$\Phi'(t) = \frac{\ln^2(1+t^2)}{1+t^2}, \quad t \in R, \quad \text{and} \quad \Phi(0) = 0.$$

Clearly Φ is a bounded, odd, increasing function and $t\Phi(t) \geq \Phi(1)|t|$, for $|t| > 1$. Thus it follows

$$p_i\Phi(p_i) \geqslant \Phi(1)|p|,$$

and writing $A_i(p) = \Phi(p_i)$, $i=1,\ldots,n$, we have

$$A_{i\,p_j}(p) = \delta_{ij}\Phi'(p_i), \quad i,j=1,\ldots,n.$$

From these facts it can be checked that the hypotheses of Theorem (3.3)(a) hold with $f = \frac{v}{\ln^2 v}$ (so that $w = 1 + \frac{1}{3}\ln^3(v)$), $\tau_0 = 1$, $\chi \equiv 1$, $\delta = \frac{1}{3}$, and $k = 1$.

CHAPTER 3

GRADIENT BOUNDS FOR HÖLDER CONTINUOUS SOLUTIONS OF DIVERGENCE-FORM ELLIPTIC EQUATIONS

In this chapter another Sobolev-type inequality is derived. This time (c.f. Chapter 1) the inequality will be obtained on non-parametric hypersurfaces in R^{n+1} which correspond to solutions u of suitable elliptic equations of the form $\frac{d}{dx_i} A_i(x,u,\nabla u) + B(x,u,\nabla u) = 0$ for which estimates of Hölder continuity are already known.

The proof of the Sobolev-type inequality is given in Section 1, and in Section 2 we discuss how, using the techniques of Chapter 2, the inequality can be used to derive a gradient bound for u.

The structural conditions on the functions $A=(A_1,\dots,A_n)$ and B are such that we can treat many interesting non-uniformly elliptic equations to which the results of Chapter 2 are not applicable.

1. A Sobolev-type Inequality

Throughout this chapter u will denote a $\mathrm{Lip}(\Omega) \cap W_2^2(\Omega)$ function with $|u| \leq M$, M constant, satisfying almost everywhere in Ω the equation

$$\frac{d}{dx_i} A_i(x,u,\nabla u) + B(x,u,\nabla u) = 0,$$

where $A(x,z,p) = (A_1(x,z,p),\dots,A_n(x,z,p))$ and $B(x,z,p)$ are respectively vector and scalar functions of

$(x,z,p) \in \Omega \times R \times R^n$ such that A is locally Lipschitz (with partial derivatives denoted A_{x_i}, A_z, A_{p_i}, $i=1,\ldots,n$) and B is locally bounded and measurable.

Also, we assume $H > 0$ and $\alpha \in (0,1)$ are fixed constants such that $|u(x)-u(x')| \leq H|x-x'|^\alpha$ for all $x,x' \in \Omega$. (u is then said to be Hölder continuous with exponent α and constant H.)

The structural conditions on the functions A and B will be as in Chapter 2, except that in place of the conditions (1.3) and (1.6) of Chapter 2 we need the following two conditions, which are formulated in terms of an arbitrary constant $\alpha' \in (0,\alpha)$ and a positive $C^2(0,\infty)$ function Λ, where $\Lambda \geq 1$, $\Lambda' \geq 0$, $\Lambda'' \geq 0$, and $\Lambda(t)/t$ is non-decreasing for $t \geq 1$. The other quantities f, w, χ, μ, and τ_0 appearing in the formulation of the conditions are as in Chapter 2.

(1.1) $$\frac{\{\chi(w(v))\}^{(n-1-\alpha')/(n-1+\alpha')}}{v}\left\{1+\left(\frac{w(v)\chi'(w(v))}{\chi(w(v))}\right)^2\right\}|\xi|^2 \leq f(v)A_{i\,p_j}(x,z,p)\xi_i\xi_j$$

for all $\xi \in R^n$, $x \in \Omega$, $|z| \leq M$, and $|p| \geq \tau_0$.

(1.2) $$\Lambda(v) \leq \mu p.A(x,z,p) + \mu$$

$$|A(x,z,p)| \leq \mu\Lambda(v)/v,$$

$$|B(x,z,p)| \leq \mu\Lambda(v),$$

for all $(x,z,p) \in \Omega \times [-M,M] \times R^n$.

The proof of the Sobolev-type inequality will be based on the following Lemma, which gives a bound for

$$\int_{K_\rho(x_0)} \sqrt{1+|\nabla u|^2}\,dx.$$

(1.3) <u>Lemma</u>

Suppose $K_{4\rho_0}(x_0) \subset \Omega (\rho_0 > 0)$ and suppose that (1.2) holds. Then

$$\int_{K_\rho(x_0)} \sqrt{1+|\nabla u|^2}\,dx \leqslant \Delta\rho^{n-1+\alpha}$$

for all $\rho \leqslant \rho_0$, where Δ depends on $n, H, \alpha, \mu, \rho_0$, and Λ.

<u>Proof</u>

The first part of the argument is similar to the proof of Lemma (2.1) of Chapter 2. We start by defining $\zeta = \max\{u - \inf_{K_{4\rho}(x_0)} u + 3m - mr/\rho, 0\}$, where $r = |x-x_0|$, $m = \sup_{K_{4\rho}(x_0)} u - \inf_{K_{4\rho}(x_0)} u$, and where $\rho \leqslant \rho_0$ satisfies $4H\mu^2(4\rho)^\alpha \leqslant \frac{1}{2}$. Note that since $m \leqslant H(4\rho)^\alpha$ is given, this means that $4m\mu^2 \leqslant \frac{1}{2}$.

Now, using $\nabla\zeta = \nabla u - (m/\rho)\nabla r$ on $\mathrm{spt}(\zeta)$, (1.1)′ of Chapter 2 gives

$$\int_E \nabla u . A\,dx = (m/\rho)\int_E \nabla r . A\,dx + \int_E \zeta B\,dx,$$

where $E = \mathrm{spt}(\zeta)$. Then since $|\zeta| \leqslant 4m$ (1.2) gives

$$\int_E (\Lambda(v)-\mu)dx \leqslant \mu^2(m/\rho)\int_E \Lambda(v)/v dx + 4\mu^2 m\int_E \Lambda(v)dx, \quad v = \sqrt{1+|\nabla u|^2},$$

so that, using $\int_E dx \leqslant \omega_n(4\rho)^n$ where ω_n = volume of the unit sphere in R^n, we have

$$\int_E \Lambda(v)dx \leqslant \mu^2(m/\rho)\int_E \Lambda(v)/v dx + 4\mu^2 m\int_E \Lambda(v)dx + \mu\ \omega_n(4\rho)^n.$$

Since $4\mu^2 m \leqslant \frac{1}{2}$ this gives

$$(1.4)\quad \int_E \Lambda(v)dx \leqslant 2\mu^2(m/\rho)\int_E \Lambda(v)/v dx + 2\mu\ \omega_n(4\rho)^n.$$

Now, writing $E_\tau = \{x \in E; |\nabla u| \geqslant \tau\}$, we have

$$\int_E \Lambda(v)/v dx = \int_{E_\tau} \Lambda(v)/v dx + \int_{E-E_\tau} \Lambda(v)/v dx,$$

and since $\Lambda(t)/t$ is non-decreasing for $t \geqslant 1$ it follows that

$$\int_E \Lambda(v)/v dx \leqslant \frac{1}{\tau}\int_{E_\tau} \Lambda(v)dx + \frac{\Lambda(\tau)}{\tau}\int_{E-E_\tau} dx$$

$$\leqslant \frac{1}{\tau}\int_E \Lambda(v)dx + \omega_n \frac{\Lambda(\tau)}{\tau}(4\rho)^n$$

for all $\tau \geqslant 1$. Then, using this inequality in (1.4) and choosing $\tau = 1+4\mu^2 m/\rho$, we have

$$\int_E \Lambda(v)dx \leq \tfrac{1}{2}\int_E \Lambda(v)dx + (\tfrac{1}{2}\Lambda(\tau)+2\mu)\omega_n(4\rho)^n,$$

so that

$$(1.5)\quad \int_E \Lambda(v)dx \leq (4^n\Lambda(\tau)+4^{n+1}\mu)\omega_n\rho^n.$$

Now, since $\Lambda \geq 1$ and $\Lambda'' \geq 0$,

$$4^n\Lambda(\tau) + 4^{n+1}\mu \leq (4^n+4^{n+1}\mu)\Lambda(\tau),$$

and

$$(4^n+4^{n+1}\mu)\Lambda(\tau) \leq \Lambda([4^n+4^{n+1}\mu]\tau+c'),$$

where c' is such that $\Lambda(c') = (4^n+4^{n+1}\mu)\Lambda(0)$ *, we obtain

$$\int_E \Lambda(v)dx \leq \omega_n\rho^n\Lambda([4^n+4^{n+1}\mu]\tau+c').$$

Then using $K_\rho(x_0) \subset E$, together with Jensen's inequality ([10]p.21) we have

$$\Lambda\left(\left(\int_{K_\rho(x_0)} v\ dx\right) \Big/ (\omega_n\rho^n)\right) \leq \Lambda([4^n+4^{n+1}\mu]\tau+c')$$

so that

$$\int_{K_\rho(x_0)} v\ dx \leq \omega_n(4^n+4^{n+1}\mu)\tau\rho^n + c'\omega_n\rho^n,$$

---

* Here we have used the fact that $\Lambda(ct+c')-c\Lambda(t)$, where $c \geq 1$ and c' is such that $\Lambda(c') = c\Lambda(0)$, is an increasing function for $t > 0$ with value zero at $t = 0$.

and hence, because of our choice of τ,

$$\int_{K_\rho(x_0)} v\,dx \leqslant 4\mu^2\omega_n(4^n+4^{n+1}\mu)m\rho^{n-1}+c''\omega_n\rho^n,$$

$c'' = (4^n+4^{n+1}\mu) + c'$.

Thus, since $m \leqslant H(4\rho)^\alpha$, the required result is established for all ρ such that $\rho \leqslant \rho_0$ and $4H\mu^2(4\rho)^\alpha \leqslant \frac{1}{2}$. The required result for all $\rho \leqslant \rho_0$ then follows.

(1.6) Corollary

If the hypotheses are as in Lemma (1.3) and if $\alpha' \in (0,\alpha)$, then

$$\int_{K_\rho(x_0)} \sqrt{1+|\nabla u|^2}\ |x-x_0|^{1-n-\alpha'}dx \leqslant c\ \rho^{\alpha-\alpha'}$$

where c depends on $n,H,\alpha,\alpha',\mu,\rho_0$, and Λ.

Proof:

The result is a direct consequence of the inequality in Lemma (1.3), using Lemma (4.3) on p.59 of [7].

The Sobolev-type inequality will now be stated and proved. The proof was motivated by the proof of the Sobolev-type inequality on minimal hypersurfaces given by J.H. Michael (see the Appendix).

(1.7) Lemma

Suppose $K_{4\rho}(x_0) \subset \Omega$, and let $h \geqslant 0$ be a Lipschitz function with $spt(h) \subset K_{\rho/2}(x_0)$. Then for all $\alpha' \in (0,\alpha)$

$$\left\{\int_\Omega h^\kappa v\,dx\right\}^{1/\kappa} \leqslant c'\int_\Omega\left(\frac{|\nabla h|}{v}\right)^p v\,dx, \qquad 1 \leqslant p < \frac{n-1}{\alpha'} + 1,$$

where

$$\kappa = \frac{p(n-1+\alpha')}{(n-1)-\alpha'(p-1)}$$

and where c' depends on ρ, α, α' and the constant c of Corollary (1.6).

<u>Proof</u>

We first prove the lemma in the case $p=1$. The case $1 < p < \frac{n-1}{\alpha'} + 1$ is easily deduced from this case by using $h^{p(n-1)/(n-1-\alpha' p+\alpha')}$ in place of h and by using Hölder's inequality.

We start with the well known identity (see e.g. [7], p.61)

$$h(\xi) = \frac{1}{n\omega_n}\int_{K_{\rho/2}(x_0)} \frac{\partial h}{\partial x_i}\,\frac{x_i-\xi_i}{|x-\xi|^n}\,dx,$$

which holds for almost all $\xi \in K_{\rho/2}(x_0)$, and where ω_n = volume of unit sphere in R^n. Using Cauchy's inequality, this gives

$$h(\xi) \leqslant \frac{1}{n\omega_n}\int_{K_{\rho/2}(x_0)} |\nabla h|\,|x-\xi|^{1-n}dx.$$

Multiplying by $(h(\xi))^{\kappa-1}v(\xi)$, integrating over $\xi \in K_{\rho/2}(x_0)$, and interchanging the order of integration on the right, this gives

$$\int_{K_{\rho/2}(x_0)} h^{\kappa}(\xi)v(\xi)d\xi \leq \frac{1}{n\omega_n}\int_{K_{\rho/2}(x_0)} |\nabla h| \left\{\int_{K_{\rho/2}(x_0)} h^{\kappa-1}(\xi)|x-\xi|^{1-n}v(\xi)d\xi\right\}dx.$$

Using Holder's inequality this gives

$$\int_{K_{\rho/2}(x_0)} h^{\kappa}(\xi)v(\xi)d\xi \leq \frac{1}{n\omega_n}\int_{K_{\rho/2}(x_0)} [|\nabla h| \left\{\int_{K_{\rho/2}(x_0)} h^{\kappa}(\xi)v(\xi)d\xi\right\}^{1-1/\kappa}.$$

$$\left\{\int_{K_{\rho/2}(x_0)} |x-\xi|^{(1-n)\kappa}v(\xi)d\xi\right\}^{1/\kappa}]dx,$$

and, noting that $K_{\rho/2}(x_0) \subset K_{\rho}(x)$ for each $x \in K_{\rho/2}(x_0)$, it follows that

$$(1.8)\quad \left\{\int_{K_{\rho/2}(x_0)} h^{\kappa}(\xi)v(\xi)d\xi\right\}^{1/\kappa} \leq \frac{1}{n\omega_n}\int_{K_{\rho/2}(x_0)} [|\nabla h|\left\{\int_{K_{\rho}(x)}|x-\xi|^{(1-n)\kappa}.\right.$$

$$\left. .v(\xi)d\xi\right\}^{1/\kappa}]dx.$$

Now by Corollary (1.6)

$$\int_{K_{\rho}(x)} |x-\xi|^{(1-n)\kappa}v(\xi)d\xi \leq c\ \rho^{\alpha-\alpha'},$$

so that the required inequality follows from (1.8).

2. Applications to Gradient Bounds

The purpose of this section is to demonstrate that theorems similar to (3.1)-(3.4) of Chapter 2 hold when conditions (1.1) and (1.2) of this chapter are used in place of conditions (1.3) and (1.6) of Chapter 2.

We will in fact only consider the following two theorems, which correspond to Theorems (3.2)(b) and (3.3)(b) of Chapter 2.

(2.1) Theorem

Suppose $K_{4\rho}(x_0) \subset \Omega$, let k, δ be constants such that $k \geq 0$ and $\delta > 0$, suppose $w^{\delta}(t) \leq \mu t$ for all real $t \geq 1$, and suppose conditions (1.2),(1.3) of this chapter together with conditions (1.4), (1.5), and (1.7) of Chapter 2 hold with $v\chi(w(v)) \equiv \frac{1}{\mu}(w(v))^{-k}\Lambda(v)$, $\varphi(w) \equiv \mu/w^{2\delta}$, and with $\psi \equiv \mu$. Then the estimate

$$\text{ess} \quad \sup|\nabla u| \leq \Gamma$$

holds, where the supremum is taken over all $x \in K_{\rho/4}(x_0)$ and where Γ is a constant determined by $n, H, \alpha, \alpha', f, \Lambda, k, \delta$, and ρ.

(2.2) Theorem

Suppose $K_{4\rho}(x_0) \subset \Omega$, $\int_1^{\infty} \frac{ds}{f(s)} < \infty$, let k, δ be constants such that $k \geq 0$ and $0 < \delta \leq 1$, suppose $w^{\delta}(t) \leq \mu t$ for all real $t \geq 1$, and suppose conditions (1.2), (1.3) of this chapter together with conditions (1.4), (1.5), (1.7) of Chapter 2 hold with $v\chi(w(v)) \equiv \frac{1}{\mu}(w(v))^{-k}\Lambda(v)$, $\varphi(w) \equiv \mu$, and with $\frac{\psi(w)}{w} \equiv \mu$. Then the estimate

$$\text{ess} \quad \sup|\nabla u| \leq \Gamma$$

holds, where the supremum is taken over all $x \in K_{\rho/4}(x_0)$ and where Γ is a constant determined by $n, H, \alpha, \alpha', f, \Lambda, k$, δ, and ρ.

Proof of Theorem (2.1):

It is not difficult to check that all the hypotheses of Theorem (3.2)(b) of Chapter 2 are satisfied except for the condition (1.3). In fact, the conditions (1.4), (1.5), and (1.7) of Chapter 2 with $\varphi(w) \equiv \mu/w^{2\delta}$ and $\psi \equiv \mu$ are also given as hypotheses there, and the condition (1.6) of Chapter 2 with $\sigma(w) \equiv \tilde{\mu}/w^{\delta}$ ($\tilde{\mu}$ depending on μ and Λ) is clearly a consequence of condition (1.2) and the given inequality $w^{\delta}(t) \leq \mu t$.

Thus we can repeat that part of the proof of Theorem (3.2)(b) which leads to the inequality (3.21)$'$, since for this part of the proof condition (1.3) was not needed. That is, we have

$$(2.3) \quad \int_{K_{\tilde{\rho}/2,\tau}} (1+w_\tau)^{2q\delta} \nabla u . A dx \leq c \int_{K_{\tilde{\rho},\tau}} \nabla u . A dx, \quad q \geq \max\{1, 1/\delta\},$$

where c depends on n, f, Λ, k, μ, τ_0, and δ, and where $\tilde{\rho}$ depends on n, ρ, q, μ, δ and the modulus of continuity of u on Ω. Thus $\tilde{\rho}$ can be chosen to depend on n, ρ, q, δ, H, and α.

Then since we can appropriately estimate $\int_{K_{\tilde{\rho},\tau}} \nabla u . A dx$ (by Lemma (2.1) of Chapter 2), we see from (2.3) that the required gradient bound for u will be established if we can estimate $\sup_{K_{\tilde{\rho}/4}(x_0)} |\nabla u|$ in terms of

$$\int_{K_{\tilde{\rho}/2,\tau}} (1+w_\tau)^{2q\delta} \nabla u.A dx, \quad \text{for some } q \geq 1.$$

Such an estimate will be a consequence of the following lemma, which is proved using the Sobolev-type inequality of Section 1, together with the conditions (1.1), and Lemma (2.4) of Chapter 2.

2.4 Lemma

Suppose $K_\rho(x_0) \subset \Omega$, let $\delta, \alpha_i, \beta_i$, $i=1,2,3$, be constants such that $\delta, \beta_i > 0$ and $\alpha_i \geq 0$, $i=1,2,3$, and suppose (1.1) of this chapter together with (1.4) and (1.5) of Chapter 2 hold with $v\chi(w(v)) \geq \beta_1 (w(v))^{-\alpha_1}$, $\varphi(w) \leq \beta_2 w^{\alpha_2 - 2\delta}$, and with $\psi(w) \leq \beta_3 w^{\alpha_3}$. Then for each constant $\alpha_0 > 0$

$$\operatorname{ess}\sup_{K_\rho(x_0)} (\rho - r) w_\tau^\delta \leq \gamma + \gamma' \left\{ \int_{K_{\rho,\tau}} (1+w_\tau)^m \left(1 + (\rho - r) w_\tau^\delta\right)^{2\alpha_0/\delta} \nabla u.A \right\}^d$$

where $r, K_{\rho,\tau}$ are as in Lemma (2.4) of Chapter 2, w_τ is as above, m depends on α', n, and α_1, γ, γ' are constants determined by $f, n, \alpha', \alpha_0, \alpha_i, \beta_i$, $i=1,2,3, \delta, \tau$, and where d is determined by n, α', and α_0.

Proof:

The same proof as for Lemma (3.5)(a) can be used, except that g^{ij} is replaced by $\frac{\delta_{ij}}{v^2}$ and $\frac{n^*}{n^*-2}$ is replaced by $\frac{n-1+\alpha'}{n-1-\alpha'}$.

We will not prove Theorem (2.2) since the modifications of the relevant theorem of Chapter 2 are of the same type as in the case of Theorem (2.1).

CHAPTER 4

REFINED GRADIENT ESTIMATES FOR A CLASS OF EQUATIONS

In [3] the interior gradient bound for C^2 solution of the minimal hypersurface equation is obtained in particularly precise form. In fact it is shown that if u is a non-negative solution to the minimal surface equation on $K_\rho(x_0)$ then

$$|\nabla u(x_0)| \leqslant c_1 \exp(c_2 u(x_0)/\rho).$$

In this chapter it will be shown that analogous results can be obtained for a class of equations including some of those equations discussed in part II of [4].

The derivation of gradient bounds in the form $|\nabla u(x_0)| \leqslant \Psi(u(x_0)/\rho)$ is of more than aesthetic interest; the Bernstein-type result discussed in [3] demonstrates this.

The main result of this chapter is Theorem (3.1). The technique of proof is based on that used in [3].

1. Notation and Structural Conditions

Ω will denote an open subset of R^n, $n \geqslant 2$. It is not assumed that Ω is bounded.

u will denote a locally C', $W_2^{2,loc}(\Omega)$ function which satisfies almost everywhere in Ω the equation

$$\frac{d}{dx_i} A_i(x,u,\nabla u) + B(x,u,\nabla u) = 0,$$

where $A(x,z,p) = (A_1(x,z,p),\ldots,A_n(x,z,p))$ and $B(x,z,p)$ are respectively vector and scalar functions defined for $(x,z,p) \in \Omega \times R \times R^n$ and such that A has locally Lipschitz first partial derivatives and B is locally Lipschitz.

f will denote a positive continuous function on $(0,\infty)$ such that $\int_1^\infty \frac{ds}{f(s)} < \infty$ and w will be defined on $[1,\infty)$ by

$$w(t) = 1 - \ln\left\{\left(\int_t^\infty \frac{ds}{f(s)}\right)\Big/\left(\int_1^\infty \frac{ds}{f(s)}\right)\right\}.$$

Let x_0 be a fixed point in Ω, let τ_0,μ be positive constants, let $v = \sqrt{1+|\nabla u|^2}$, and suppose that the following conditions are satisfied for all ρ such that $K_\rho(x_0) \subset \Omega$.

(1.1)
$$\rho B(x,u,\nabla u) \leq \mu,$$
$$v \leq \mu\nabla u.A(x,u,\nabla u) + \mu,$$
$$|A(x,u,\nabla u)| \leq \mu,$$

for all $x \in K_\rho(x_0)$.

(1.2) $$|f(v)A_{i p_j}(x,u,\nabla u)\eta_i\xi_j| \leq fA_{i p_j}(x,u,\nabla u)\xi_i\xi_j + \mu \min\{v|\eta|^2, fA_{i p_j}(x,u,\nabla u)\eta_i\eta_j\}$$

for all $\xi,\eta \in R^n$ and those $x \in K_\rho(x_0)$ where $w(v(x)) \geq \tau_0$.

(1.3) $$|f(v)A_{i p_j}(x,u,\nabla u)u_{,i}\xi_j| \leq fA_{i p_j}(x,u,\nabla u)\xi_i\xi_j+\mu v$$

for all $\xi \in R^n$ and those $x \in K_\rho(x_0)$ where $w(v(x)) \geq \tau_0$.

(1.4) $\frac{1}{\mu} v g^{ij} \xi_i \xi_j \leq f(v) \min\{1, w'(v)\ v\} A_{ip_j}(x,u,\nabla u)\xi_i\xi_j$

for all $\xi \in R^n$ and $x \in K_\rho(x_0)$ such that $w(v(x)) \geq \tau_0$, where

$$g^{ij} = \delta_{ij} - (u_{,i}u_{,j})/(1+|\nabla u|^2),\ i,j = 1,\dots,n.$$

(1.5) $$|\xi.A(x,u,\nabla u)|^2 \leq \mu^2 vf(v) A_{ip_j}(x,u,\nabla u)\xi_i\xi_j$$

for all $\xi \in R^n$ and $x \in K_\rho(x_0)$ such that $w(v(x)) \geq \tau_0$.

(1.6) $$\{f(v) + vf(v)w''(v)\}^{\frac{1}{2}}\rho|D| \leq \mu\{\lambda(x,u,\nabla u)\}^{\frac{1}{2}}\{v\}^{\frac{1}{2}}$$

$$\{f(v)w'(v)\}^{\frac{1}{2}}\rho^2|C| \leq \mu v$$

for all $x \in K_\rho(x_0)$ such that $w(v(x)) \geq \tau_0$, where C and $|D|$ are as in inequality (2.10)′ of Chapter 2, and where $\lambda(x,u,\nabla u)$ is the minimum eigenvalue of the symmetric matrix $\frac{1}{2}(A_{ip_j}(x,u,\nabla u)+A_{jp_i}(x,u,\nabla u))$.

We should mention that the first part of condition (1.1) can be replaced by the inequality $-\rho B(x,u,\nabla u) \leq \mu$.

2. Preliminary Results

Subsequently, for $\lambda \in R$ and $x' \in K_\rho(x_0)$, we let

$$Q_\rho(\lambda) = \{x \in K_\rho(x_0);\ \ |u(x) - \lambda| \leq \rho\},$$

and

$$S_{\rho,\tau}(x') = \{x \in K_\rho(x');\ [|x-x'|^2 + (u(x) - u(x'))^2]^{\frac{1}{2}} \leq \rho \text{ and } w(v(x)) > \tau\}.$$

Also, for τ such that $\tau < w(v(x_0))$,

(2.1) $$m_{\rho,\tau} = \rho + \sup(u - u(x_0)),$$

where the supremum is taken over the component of the set $\{x \in K_\rho(x_0)\ ;\ w(v(x)) > \tau\}$ which contains x_0.

(2.2) Lemma

Suppose $K_{2\rho}(x_0) \subset \Omega$ and suppose (1.1) is satisfied. Then for each $\lambda \in R$

$$\int_{Q_\rho(\lambda)} v dx \leq c\rho^n ,$$

where c is a constant determined by μ and τ_0.

(2.3) Lemma

Suppose $K_{2\rho}(x_0) \subset \Omega$, $x' \in K_\rho(x_0)$, suppose (1.1) - (1.6) hold, and suppose $\tau_* \leq w(v(x'))-1$. Then

$$c'\rho^n \leq \int_{S_{\rho,\tau_*}(x')} v\, dx,$$

where c' is a positive constant determined by n and μ

Proof of Lemma (2.2):

Let γ be the Lipschitz function defined on R such that $\gamma(s)=1$ if $s > \rho$, $\gamma(s) = 0$ if $s < -\rho$, and $\gamma(s)$ increases linearly for s between $-\rho$ and ρ. Define

$$\varphi(x) = \int_{-\rho}^{\rho} \gamma(u(x)- \lambda -t)dt.$$

Clearly φ is a locally Lipschitz function on Ω.

Now using (1.1)' of Chapter 2 with $\zeta(x) = \varphi(x) \max\{2\rho-r,0\}$, $r = |x-x_0|$, we have

$$\int_{K_{2\rho}(x_0)} A_1 u_{,1} \int_{-\rho}^{\rho} \gamma'(u-\lambda-t)dt(2\rho-r)dx =$$

$$= \int_{K_{2\rho}(x_0)} \varphi A_1 r_{,1} dx + \int_{K_{2\rho}(x_0)} (2\rho-r)\varphi B dx\,,$$

where we have used

$$\frac{\partial}{\partial x_1}\int_{-\rho}^{\rho} \gamma(u-\lambda-t)dt = \int_{-\rho}^{\rho} \frac{\partial}{\partial x_1}\gamma(u-\lambda-t)dt.$$

Then using

$|A_1 r_{,1}| \leqslant |A||\nabla r| = |A|$ and $\gamma'(u-\lambda-t) = -\frac{d}{dt}\gamma(u-\lambda-t)$,

together with (1.1), it follows that

$$\int_{K_{2\rho}(x_0)} (v-\mu)\{\gamma(u-\lambda+\rho)-\gamma(u-\lambda-\rho)\}(2\rho-r)dx$$

$$\leqslant 2\mu^2 \int_{K_{2\rho}(x_0)} \varphi\, dx,$$

and the proof is completed by observing that $\gamma(u-\lambda+\rho) - \gamma(u-\lambda-\rho) \geqslant \frac{1}{2}$ when $|u-\lambda| \leqslant \rho$ and that $|\varphi| \leqslant 2\rho$ on $K_{2\rho}(x_0)$.

The proof of Lemma (2.3) is somewhat more difficult.

Proof of Lemma (2.3):

Let $\varphi = 1-\tilde{r}/\rho$, $\tilde{r} = \{|x-x'|^2+(u(x)-u(x'))^2\}^{\frac{1}{2}}$, and let $h_\tau = \max\{e^{-\tau}-e^{-w(v)},0\}$ $(v = \sqrt{1+|\nabla u|^2})$ for $\tau \geqslant \tau_0$.

Then, using the inequality (3.6) of Chapter 1 with $p=1$, $m=n$, and $\eta = h_\tau \varphi$ we have

$$\left\{\int_{S_{\rho,\tau}(x')} (h_\tau\varphi)^{\frac{n}{n-1}} v\, dx\right\}^{\frac{n-1}{n}} \leq c\int_{S_{\rho,\tau}(x')} \left\{g^{ij}(h_\tau\varphi)_{,i}(h_\tau\varphi)_{,j}\right\}^{\frac{1}{2}} v dx +$$

$$c\int_{S_{\rho,\tau}(x')} h_\tau\varphi\{g^{ij}g^{st}u_{,is}u_{,jt}\}^{\frac{1}{2}}dx,$$

where c depends on n.

Using the Hölder inequality on each term of the right hand side of this inequality gives

$$(2.4)\quad \left\{\int_{S_{\rho,\tau}(x')} (h_\tau\varphi)^{\frac{n}{n-1}} v dx\right\}^{\frac{n-1}{n}} \leq 2c\left\{\int_{S_{\rho,\tau}(x')} v dx\right\}^{\frac{1}{2}} \cdot$$

$$\cdot\left\{\int_{S_{\rho,\tau}(x')} g^{ij}(h_\tau\varphi)_{,i}(h_\tau\varphi)_{,j} v dx + \int_{S_{\rho,\tau}(x')} (h_\tau\varphi)^2 g^{ij}g^{st}u_{,is}u_{,jt}\frac{1}{v}dx\right\}^{\frac{1}{2}}.$$

Now estimating terms on the right of (2.4) we first have

$$(2.5)\quad g^{ij}(h_\tau\varphi)_{,i}(h_\tau\varphi)_{,j} \leq 2\varphi^2 g^{ij}h_{\tau,i}h_{\tau,j} + 2h_\tau^2 g^{ij}\varphi_{,i}\varphi_{,j}.$$

Next, by condition (1.4)

$$(2.6)\quad (h_\tau\varphi)^2 g^{ij}g^{st}u_{,is}u_{,jt} \leq (h_\tau\varphi)^2\, fw'\cdot A_{ip_j}\, g^{st}u_{,is}u_{,jt}.$$

Now if we use (2.10)′ of Chapter 2 with $\eta = (h_\tau \varphi)^2$ and $h(w) \equiv w$ we will obtain

$$\iint_\Omega \left\{ (h_\tau\varphi)^2 \frac{fw'}{v} A_{i\,p_j} g^{st} u_{,is} u_{,jt} + f A_{i\,p_j} w_{\tau,i} (h_\tau\varphi)^2_{,i} \right\} dx$$

$$\overset{(2.7)}{\leqslant} \int_\Omega (h_\tau\varphi)^2 \{ fw' |C| + fw' |D| \sqrt{g^{st} u_{,si} u_{,ti}} + f|D|\,|\nabla w| \} dx,$$

and using the given condition (1.6) we have

$$fw' |D| \sqrt{g^{st} u_{,si} u_{,ti}} \leqslant \mu \left\{ \frac{fw'}{v} \lambda g^{st} u_{,si} u_{,ti} \right\}^{\frac{1}{2}} \left\{ v \right\}^{\frac{1}{2}},$$

so that, using the Cauchy inequality $|ab| \leqslant \varepsilon a^2 + \frac{1}{4\varepsilon} b^2$ and noting that $\lambda g^{st} u_{,si} u_{,ti} \leqslant A_{i\,p_j} g^{st} u_{,si} u_{,tj}$, we have

$$fw' |D| \sqrt{g^{st} u_{,si} u_{,ti}} \leqslant \varepsilon \frac{fw'}{v} A_{i\,p_j} g^{st} u_{,si} u_{,tj} + \frac{c_1 v}{\rho^2},$$

where c_1 depends on μ and ε.

Also, again by (1.6) we have

$$fw' |C| \leqslant \mu \frac{v}{\rho^2},$$

and

$$f|D|\,|\nabla w| \leqslant \mu \sqrt{f\lambda |\nabla w|}\ \frac{\sqrt{v}}{\rho}$$

$$\leqslant \varepsilon f A_{i\,p_j} w_{\tau,i} w_{\tau,j} + \frac{cv}{\rho^2},$$

where c depends on μ and ε.

Next, we have clearly that

$$fA_{ip_j}w_{\tau,j}(h_\tau^2\varphi^2)_{,i} = 2\varphi^2h_\tau e^{-w}fA_{ip_j}w_{\tau,i}w_{\tau,j}$$

$$+ 2\varphi h_\tau^2 fA_{ip_j}w_{\tau,i}\varphi_{,j}$$

and, using condition (1.2), this gives

$$fA_{ip_j}w_{\tau,i}(h_\tau\varphi^2)_{,i} \geq -2\varphi^2h_\tau(h_\tau-e^{-w})fA_{ip_j}w_{\tau,i}w_{\tau,j}$$
$$-2\mu h_\tau^2 fA_{ip_j}\varphi_{,i}\varphi_{,j}$$

Thus collecting together all these estimates and choosing $\varepsilon=\frac{1}{2}$, it follows from (2.7) that

(2.8) $$\int_{S_{\rho,\tau}(x')} (h_\tau\varphi)^2 \frac{fw'}{v} A_{ip_j}g^{st}u_{,is}u_{,jt}dx$$

$$\leq c_2e^{-2\tau}\int_{S_{\rho,\tau}(x')} \varphi^2 fA_{ip_j}w_{\tau,i}w_{\tau,j}dx + \frac{c_3e^{-2\tau}}{\rho^2}\int_{S_{\rho,\tau}(x')} v\,dx +$$

$$+ c_4e^{-2\tau}\int_{S_{\rho,\tau}} fA_{ip_j}\varphi_{,i}\varphi_{,j}dx,$$

where we have used $h_\tau \leq e^{-\tau}$ and $e^{-w} \leq e^{-\tau}$ on $S_{\rho,\tau}(x')$, and where c_2,c_3,c_4 depend on μ.

Now it follows from (1.2) and (1.3) that

$$fA_{ip_j}\varphi_{,i}\varphi_{,j} \leq \frac{c_5}{\rho^2}v,$$

where c_5 depends on μ. Then using this inequality in (2.8), and using the inequality

(2.9) $g^{ij}(h_\tau\varphi)_{,i}(h_\tau\varphi)_{,j} \leqslant 2\mu e^{-2\tau}\varphi^2 \frac{f}{v} A_{ip_j} w_{\tau,i}w_{\tau,j} + 2\mu c_5 \frac{e^{-2\tau}}{\rho^2}$,

(which is proved using (1.4), (1.2), and (1.3)) it follows from (2.4) that

$$(2.10)\quad \left\{\int_{S_{\rho,\tau}(x')} (h_\tau\varphi)^{\frac{n}{n-1}} v\,dx\right\}^{\frac{n-1}{n}} \leqslant c_6 e^{-\tau}\left\{\int_{S_{\rho,\tau}(x')} v\,dx\right\}^{\frac{1}{2}} \cdot \left\{\int_{S_{\rho,\tau}(x')} (\varphi^2 f A_{ip}\, w_{\tau,i}w_{\tau,j} + \frac{1}{\rho^2}v)\,dx\right\}^{\frac{1}{2}},$$

where c_6 depends on μ and n.

By considering the inequality (2.10)′ of Chapter 2 with $\eta = \varphi^2$ and $h(w) \equiv w$ together with the condition (1.6), it is not difficult to prove

$$\int_{S_{\rho,\tau}(x')} \varphi^2 f A_{ip}\, w_{\tau,i}w_{\tau,j}\,dx \leqslant \frac{c_7}{\rho^2}\int_{S_{\rho,\tau}(x')} v\,dx,$$

where c_7 depends on μ.

Thus (2.10) finally gives

$$(2.11)\quad \left\{\int_{S_{\rho,\tau}(x')} (h_\tau\varphi)^{\frac{n}{n-1}} v\,dx\right\}^{\frac{n-1}{n}} \leqslant c_8 \frac{e^{-\tau}}{\rho}\int_{S_{\rho,\tau}(x')} v\,dx.$$

Now, noting that $\varphi \geqslant \frac{1}{2}$ on $S_{\rho/2,\tau}(x')$, and that $h_\tau \geqslant e^{-\tau} - e^{-\tau'}$ on $S_{\rho/2,\tau'}(x')$, (2.11) gives

$$(2.12)\quad \{1 - e^{\tau}/e^{\tau'}\}\{I_{\rho/2,\tau'}\}^{\frac{1}{\kappa}} \leqslant \frac{2c_8}{\rho} I_{\rho,\tau}$$

for $\tau' > \tau$, where $\kappa = \frac{n-1}{n}$, $I_{\rho,\tau} = \int_{S_{\rho,\tau}(x')} v\,dx$.

The remainder of the argument is the same as that in [3] pp 261-2: We choose a sequence $\{t_\nu\}$ such that $t_0 = e^{\tau_*}$ and such that $t_\nu = t_{\nu-1} + \frac{e^{w(v(x'))} - e^{\tau_*}}{2^\nu}$, $\nu \geq 1$, and use inequality (2.12), with $\tau = \tau_{\nu-1} = e^{t_{\nu-1}}$ and $\tau' = \tau_\nu = e^{t_\nu}$, and with $\rho/2^\nu$ and $\rho/2^{\nu+1}$ in place of ρ and $\rho/2$ respectively.

This leads to the required result by iteration.

Next we have a result analogous to inequality (3.20) of Chapter 2.

(2.13) <u>Lemma</u>

Suppose $K_\rho(x_0) \subset \Omega$, suppose $w(v(x_0)) > \tau \geq \tau_0$, and let $\zeta = \max\{m_{\rho,\tau} + u(x_0) - u - 2m_{\rho,\tau} r/\rho, 0\}$, where $r = |x - x_0|$ and where $m_{\rho,\tau}$ is as in (2.1). Let $w_\tau = \max\{w(v) - \tau, 0\}$ and let E_τ be the component of the set $\{x \in K_\rho(x_0); \zeta w_\tau^2 > 0\}$ containing x_0. Then, if (1.1)-(1.6) hold and $\frac{w^2(t)}{t} \leq \mu$, $t \geq 1$,

$$\int_{E_\tau} w_\tau^2 v\,dx \leq c\left(\frac{m_{\rho,\tau}}{\rho}\right)^2 \int_{E_\tau} v\,dx,$$

where c depends on μ and τ_0.

Proof:

It is not difficult to check that γ_ε defined by $\gamma_\varepsilon = \max\left\{w_\tau^2\zeta - \varepsilon, 0\right\}$ on E_τ and $\gamma_\varepsilon = 0$ on $K_\rho(x_0) - E_\tau$ is a $W^1_{2,0}(K_\rho(x_0))$ function. Hence using

$$\frac{\partial}{\partial x_i}(\gamma_\varepsilon) = 2w_\tau w_{\tau,i}\zeta - w_\tau^2 u_{,i} - w_\tau^2 \frac{m_{\rho,\tau}}{\rho} r_{,i} \quad \text{on } \operatorname{spt}(\gamma_\varepsilon) \subset E_\tau$$

and using (1.1)′ of Chapter 2 with γ_ε in place of ζ, we have, after letting $\varepsilon \to 0$,

$$\int_{E_\tau} w_\tau^2 \nabla u . A dx = -2\int_{E_\tau} w_\tau A_i w_{\tau,i} \zeta dx + \frac{2m_{\rho,\tau}}{\rho}\int_{E_\tau} w_\tau^2 A_i r_{,i} dx$$

$$+ \int_{E_\tau} \zeta w_\tau^2 B dx.$$

Then, using conditions (1.1) and (1.5) and rearranging as in the proof of Lemma (2.2) of Chapter 2,

$$(2.14) \quad \int_{E_\tau} w_\tau^2 v dx < c\int_{E_\tau} f A_{i p_j} w_{\tau,i} w_{\tau,j} \zeta^2 dx + c\left(\frac{m_{\rho,\tau}}{\rho}\right)^2 \int_{E_\tau} w_\tau^2 dx$$

$$c\left(\frac{m_{\rho,\tau}}{\rho}\right)\int_{E_\tau} w_\tau^2 dx \quad ,$$

where c depends on μ.

Now the term $\int_{E_\tau} f A_{i p_j} w_{\tau,i} w_{\tau,j} \zeta^2 dx$ can be estimated by inequality (2.10)′ of Chapter 2.

In fact, using (2.10)′ of Chapter 2 with $h(w) \equiv w$ and with ζ^2 in place of η, and using the condition (1.6), it follows that

$$\int_{E_\tau} f A_{i p_j} w_{\tau,i} w_{\tau,j} \zeta^2 dx \leqslant c(\mu) \left(\frac{m_{\rho,\tau}}{\rho}\right)^2 \int_{E_\tau} v dx.$$

The details of the calculation required to prove this inequality are nearly identical to the corresponding calculations that were made in Chapter 2, except that terms like $A_{ip_j} w_{\tau,i} \zeta_{,j} = -A_{ip_j} w_{\tau,i} u_{,j} - \frac{2m_{\rho,\tau}}{\rho} A_{ip_j} w_{\tau,i} r_{,j}$

are handled by using (1.2) as well as (1.3).

Then since $\frac{w^2(t)}{t} \leqslant \mu$ the required inequality follows from (2.14).

3. <u>The Gradient Bound</u>

We will now state and prove the main result.

(3.1) <u>Theorem</u>

Suppose $K_\rho(x_0) \subset \Omega$ and suppose conditions (1.1) - (1.6) hold, together with the condition $\frac{w^2(t)}{t} \leqslant \mu, t \geqslant 1$. Then

$$w(v(x_0)) \leqslant c\left(\frac{m_\rho}{\rho}\right),$$

where $m_\rho = \sup_{x \in K_\rho(x_0)} u(x_0) - u + \rho$ and c depends on n, μ, and τ_0.

Proof:

By Lemma (2.13) we have for $\tau_0 \leqslant \tau < w(v(x_0))$

(3.2) $$\int_{E_\tau} w_\tau^2 \, vdx \leqslant c\left(\frac{m_{\rho,\tau}}{\rho}\right)^2 \int_{E_\tau} vdx.$$

Now suppose τ' is such that $\tau+2 < \tau' < w(v(x_0))$. Then clearly from (3.2), since $w_\tau \geqslant (\tau' - \tau - 1) \geqslant \frac{1}{2}(\tau' - \tau)$ on $E_{\tau'-1}$,

(3.3) $$(\tau' - \tau)^2 \int_{E_{\tau'-1}} vdx \leqslant 4c\left(\frac{m_{\rho,\tau}}{\rho}\right)^2 \int_{E_\tau} vdx.$$

Now from the definition of E_τ it is not difficult to check that the oscillation of u on E_τ

is no greater than $2m_{\rho,\tau}$. Then it follows that we can find real numbers $\lambda_1, \ldots, \lambda_\nu$, where ν is the least positive integer greater than $\frac{m_{\rho,\tau}}{\rho}$, such that $E_\tau \subset \bigcup_{i=1}^{\nu} Q_\rho(\lambda_i)$. But then by Lemma (2.2)

$$(3.4) \int_{E_\tau} v\,dx \leqslant \sum_{i=1}^{\nu} \int_{Q_\rho(\lambda_i)} v\,dx \leqslant c\left(1 + \frac{m_{\rho,\tau}}{\rho}\right) \rho^n \leqslant 2c\left(\frac{m_{\rho,\tau}}{\rho}\right) \rho^n ,$$

where c depends on μ and n.

Now it follows from the definition of $m_{\rho/4,\tau'}$ that $u - u(x_0)$ takes all values between 0 and $m_{\rho/4,\tau'} - \rho/4$ on the component of the set

$$S = \{x \in K_{\rho/4}(x_0) \; ; \; u(x) \leqslant u(x_0) \text{ and } w(v(x)) > \tau''\}$$

which contains x_0. Then in addition to x_0 we must be able to choose ν' other points, $x_1, \ldots, x_{\nu'}$, ν'' = largest integer $< \frac{m_{\rho/4,\tau'}}{(\rho/2)} - \frac{1}{2}$, in the set S which are such that

$$u(x_1) = u(x_0) - \rho/2$$

and

$$u(x_i) = u(x_{i-1}) - \rho/2, i = 2, \ldots, \nu'.$$

(When $\frac{m_{\rho/4,\tau'}}{(\rho/2)} - \frac{1}{2} \leqslant 1$ this statement should be interpreted as meaning that we choose only the single point x_0 from S.)

It is easily checked that $S_{\rho/4,\tau'-1}(x_i) \subset E_{\tau'-1}$, $i = 0, 1, \ldots, \nu'$, and hence, using Lemma (2.3),

$$(3.5) \int_{E_{\tau'-1}} v\,dx \geqslant \sum_{i=0}^{\nu'} \int_{S_{\rho/4,\tau'-1}(x_i)} v\,dx \geqslant c\left(\frac{m_{\rho/4,\tau'}}{(\rho/2)}\right) = \frac{1}{2}\, c\left(\frac{m_{\rho/4,\tau'}}{(\rho/4)}\right) ,$$

for some constant $c > 0$ depending on μ and n.

Thus using (3.4) and (3.5) in (3.3) we have

$$(3.6) \qquad (\tau' - \tau)^2 \frac{m_{\rho/4,\tau'}}{(\rho/4)} \leq c\left(\frac{m_{\rho,\tau}}{\rho}\right)^3$$

for all τ, τ' such that $\tau \geq \tau_0$ and $\tau + 2 < \tau' < w(v(x_0))$, where c depends on μ, n, and τ_0. But since $m_{\rho,\tau} \geq m_{\rho/4,\tau'}$ and ρ, (3.6) trivially holds with $c = 16$ when $0 < \tau' - \tau \leq 2$.

Thus we can assume (3.6) for all τ, τ' such that $\tau_0 \leq \tau < \tau' < w(v(x_0))$. The required inequality is now established from (3.6) by using the same iterative argument as in [3].

APPENDIX

A RESULT CONCERNING SUBHARMONIC FUNCTIONS, AND A SOBOLEV-TYPE INEQUALITY ON MINIMAL SUBMANIFOLDS OF R^n.

Throughout this appendix the notation will be the same as that in Chapter 1, except that here M will denote an m-dimensional minimal C^2 submanifold of R^n. By this we mean that $\mathcal{H} \equiv 0$, where $\mathcal{H}$ is the mean curvature of M defined in (1.11) of Chapter 1.

The material here originates from the work of J.H. Michael who, using a method which was essentially equivalent to that used here, proved a special case of the result concerning subharmonic functions (Theorem 1) and the Sobolev inequality (Theorem 2) on non-parametric minimal hypersurfaces.

The Sobolev-type inequality obtained here is of the same type as the one used in [3] to derive interior gradient bounds for solutions to the minimal surface equation. It is interesting to note that the corollary to Theorem 1 can be used instead of the Sobolev-type inequality in the derivation of such a gradient bound. (This follows from the fact that if u is a solution to the minimal surface equation and if $v = \sqrt{1+|\nabla u|^2}$, then $\ln v$ is subharmonic on the corresponding minimal hypersurface.)

We will need the formulae

(1) $$\int_M \delta_k h \, dH_m = 0, \quad k = 1,\ldots,n,$$

valid for $C'(M)$ functions h with $\operatorname{spt}(h) \cap M$ compact. (1) is easily established from Lemma (2.5) of Chapter 1 by noting that $H_k \equiv 0$ and by choosing t small enough to ensure $M_{k,t}^{+} \cap \operatorname{spt}(h) = M \cap \operatorname{spt}(h)$ (so that $M_{k,t} \cap \operatorname{spt}(h) = \emptyset$). The identity (1) is also used in [3] and [6].

By replacing h by $\varphi\psi$, where φ and ψ are $C'(M)$ functions at least one of which has compact support contained in M, it follows from (1) that

(2) $$\int_M \varphi \delta_k \psi \, dH_m = -\int_M \psi \delta_k \varphi \, dH_m, \quad k = 1,\ldots,n.$$

A function $\chi \in C'(M)$ will be called subharmonic if

(3) $$\sum_{k=1}^{n} \int_M (\delta_k \varphi)(\delta_k \chi) \, dH_m \leqslant 0$$

for each non-negative function $\varphi \in C'(M)$ with $\operatorname{spt}(\varphi) \cap M$ compact. Note that if $\delta_k \chi \in C'(M)$, $k = 1,\ldots,n$, then this gives (by (2))

$$\sum_{k=1}^{n} \int_M \varphi \delta_k \delta_k \chi \, dH_m \geqslant 0.$$

Thus $C^2(M)$ functions χ are subharmonic if and only if

$$\sum_{k=1}^{n} \delta_k \delta_k \chi \geqslant 0.$$

The operator $\sum_{k=1}^{n} \delta_k \delta_k$ is called the Laplace-Beltrami operator on M.

It will be convenient to write (2) and (3) in terms of the projection matrix $(\tilde{g}^{ij})$ of Chapter 1. To do this, recall from Chapter 1 that if V is an open subset of R^n with $M \subset V$ and if $h \in C'(M)$ is the restriction to M of a $C'(V)$ function $\tilde{h}$, with partial derivatives $\frac{\partial \tilde{h}(x)}{\partial x_i}$, $x = (x_1,\ldots,x_n) \in V$, $i = 1,\ldots,n$, then

(4) $$\delta_i h = \sum_{j=1}^{n} \tilde{g}^{ij} \frac{\partial \tilde{h}}{\partial x_j} \quad \text{on } M,\ i = 1,\ldots,n.$$

Thus, letting $\tilde{\varphi},\tilde{\psi}$, be C^2 functions defined on some open set containing M, and letting φ,ψ be their ($C^2(M)$) restrictions to M, (2) gives

(2)′ $$\sum_{j=1}^{n} \int_M \varphi \tilde{g}^{ij} \frac{\partial \tilde{\psi}}{\partial x_j}\, dH_m = -\sum_{j=1}^{n} \int_M \psi \tilde{g}^{ij} \frac{\partial \tilde{\varphi}}{\partial x_j}\, dH_m,\ i = 1,\ldots,n,$$

and (3) becomes (with $\tilde{\chi}$ a C^2 function defined on an open set containing M such that $\tilde{\chi} = \chi$ on M)

$$\sum_{i,j,k=1}^{n} \int_M \tilde{g}^{kj}\tilde{g}^{ki} \frac{\partial \tilde{\varphi}}{\partial x_i} \frac{\partial \tilde{\chi}}{\partial x_j}\, dH_m \leq 0 \quad \text{on } M, \text{ which, since}$$

$$\sum_{k=1}^{n} \tilde{g}^{kj}\tilde{g}^{ki} = \tilde{g}^{ij},\ i,j = 1,\ldots,n, \quad \text{gives}$$

(3) $$\sum_{i,j=1}^{n} \int_M \tilde{g}^{ij} \frac{\partial \tilde{\varphi}}{\partial x_i} \frac{\partial \tilde{\chi}}{\partial x_j}\, dH_m \leq 0.$$

Then using (2)′ with $\psi = \chi$ and with $\frac{\partial \tilde{\varphi}}{\partial x_i}$ in place of $\tilde{\varphi}$, this gives

(5) $$\sum_{i,j=1}^{n} \int_M \chi \tilde{g}^{ij} \frac{\partial^2 \tilde{\varphi}}{\partial x_i \partial x_j} \, dH_m \geq 0.$$

Incidentally we note that if $\tilde{\chi} \in C^2$, by using (2′), (4), and (2) we have

$$\sum_{i,j=1}^{n} \int_M \varphi \tilde{g}^{ij} \frac{\partial^2 \tilde{\chi}}{\partial x_i \partial x_j} \, dH_m = - \sum_{i,j=1}^{m} \int_M \tilde{g}^{ij} \frac{\partial \tilde{\varphi}}{\partial x_i} \frac{\partial \tilde{\chi}}{\partial x_j} \, dH_m$$

$$= - \sum_{i,j,k=1}^{n} \int_M \tilde{g}^{ki} \tilde{g}^{kj} \frac{\partial \tilde{\varphi}}{\partial x_i} \frac{\partial \tilde{\chi}}{\partial x_j} \, dH_m = - \sum_{k=1}^{n} \int_M (\delta_k \varphi)(\delta_k \chi) \, dH_m$$

$$= \sum_{k=1}^{n} \int_M \varphi \, \delta_k \delta_k \chi \, dH_m$$

for all $\varphi \in C'(M)$ with $\operatorname{spt}(\varphi) \cap M$ compact, so that the Laplace-Beltrami operator $\sum_{k=1}^{n} \delta_k \delta_k$ satisfies

$$\sum_{k=1}^{n} \delta_k \delta_k \chi = \sum_{i,j=1}^{n} \tilde{g}^{ij} \frac{\partial^2 \tilde{\chi}}{\partial x_i \partial x_j} .$$

We can now state the first theorem.

<u>Theorem 1</u>.

Let $x_0 \in M$, suppose $K_{\rho_0}(x_0) \cap M$ $(\rho_0 > 0)$ is a compact subset of R^n, and suppose $\chi \in C'(M)$ is a non-negative subharmonic function on M. Then the function

$$\varphi(\rho) = \left(\int_{K_\rho(x_0) \cap M} \chi \, dH_m \right) \Big/ \rho^m$$

is a non-decreasing function of ρ for $0 < \rho < \rho_0$.

Proof:

Note first that (5) must clearly hold even if the first partial derivatives $\frac{\partial\tilde{\varphi}}{\partial x_i}$ of $\tilde{\varphi}$ are Lipschitz functions (instead of $C'(M)$ functions).

Then consider the function $\tilde{\varphi}$ defined by

$$\tilde{\varphi}(x) = \mu(r),$$

where $r = |x-x_0|$ and μ is such that $\mu(r) = 0$ for $r > \rho$ and $\mu'(r) = -r\,\gamma_\varepsilon(r)$ for γ_ε, $0 < \varepsilon < \rho$, satisfying $\gamma_\varepsilon(r) = 1$ when $r \leq \rho - \varepsilon$, $\gamma_\varepsilon(r) = 0$ when $r \geq \rho$, and decreasing linearly between $r = \rho-\varepsilon$ and $r=\rho$. Thus $\gamma'_\varepsilon = -1/\varepsilon$, $\rho - \varepsilon < r < \rho$, and $\gamma'_\varepsilon(r) = 0$ for $r < \rho - \varepsilon$ or $r > \rho$, and clearly $\tilde{\varphi}$ has first derivatives which are Lipschitz functions.

Substituting our choice of $\tilde{\varphi}$ in (5), and carrying out the required differentiation, we obtain

$$\sum_{i=1}^{n}\int_M \chi\,\tilde{g}^{ii}\gamma_\varepsilon(r)\,dH_m + \sum_{i,j=1}^{n}\int_M \chi\,\tilde{g}^{ij}\,\frac{(x_i-x_{0i})}{r}\,\frac{(x_j-x_{0j})}{r}\,(r\gamma'_\varepsilon(r))\,dH_m \leq 0.$$

Then using $\sum_{i=1}^{n}\tilde{g}^{ii} = m$ and

$$0 \leq \sum_{i,j=1}^{n}\tilde{g}^{ij}\,\frac{(x_i-x_{0i})}{r}\,\frac{(x_j-x_{0j})}{r} \leq 1$$

(which hold because $(\tilde{g}^{ij})$ is the matrix of the projection of R^n onto the tangent space $T_{x_0}(M)$), and using the stated facts concerning γ_ε, it follows that

$$m\int_{K_{\rho-\varepsilon}(x_0)\cap M} \chi \, dH_m - \frac{1}{\varepsilon}\int_{\Lambda_{\rho-\varepsilon,\rho}} \chi \, dH_m \leq 0,$$

where $\Lambda_{\rho-\varepsilon,\rho} = (K_\rho(x_0)-K_{\rho-\varepsilon}(x_0))\cap M$, so that

(6) $$m\psi(\rho-\varepsilon) - \frac{1}{\varepsilon}(\psi(\rho)-\psi(\rho-\varepsilon)) \leq 0,$$

where $\psi(\rho) = \int_{K_\rho(x_0)\cap M} \chi \, dH_m$.

Now $\psi(\rho)$ is an increasing function of ρ, and hence differentiable for almost all ρ, so that letting $\varepsilon \to 0$, (6) gives

$$m\psi(\rho) - \psi'(\rho) \leq 0$$

for almost all $\rho \in (0,\rho_0)$. But this last inequality can be written

$$\frac{d}{d\rho}\left(\frac{\psi(\rho)}{\rho^m}\right) \geq 0$$

for almost all ρ, hence the required result is established (because $\varphi(\rho) = \frac{\psi(\rho)}{\rho^m}$).

<u>Corollary</u>

If the hypotheses are as in Theorem 1 then

$$\chi(x_0) \leq (\omega_m\rho^m)^{-1}\int_{K_{\rho_0}(x_0)\cap M} \chi \, dH_m$$

where ω_m is the volume of the unit sphere in R^m.

Proof:

From the theorem we have

$$(\omega_m \rho^m)^{-1} \int_{K_\rho(x_o) \cap M} \chi \, dH_m \leqslant (\omega_m \rho^m)^{-1} \int_{K_{\rho_o}(x_o) \cap M} \chi \, dH_m$$

for all $\rho \in (0, \rho_o)$. Then the required result is obtained by letting $\rho \to 0$.

We now proceed to the statement and proof of the Sobolev-type inequality.

<u>Theorem 2</u>

Suppose $K_{2\rho_o}(x_o) \cap M$ is a compact subset of R^n and suppose h is a $C'(M)$ function such that $spt(h) \subset K_{\rho_o}(x_o) \cap M$. Then, defining $S_\rho(x_o) = K_\rho(x_o) \cap M$,

$$\left\{ \int_{S_{\rho_o}(x_o)} |h|^{m/(m-1)} dH_m \right\}^{(m-1)/m} \leqslant c \rho_o^{-m} \left\{ \int_{S_{2\rho_o}(x_o)} dH_m \right\}^{(m-1)/m} \int_{S_{\rho_o}(x_o)} |\delta h| \, dH_m ,$$

where c depends only on m, n.

Proof:

We can assume $h \geqslant 0$, otherwise replace h by the Lipschitz function $|h|$. We start the proof with an application of (1). In terms of $\tilde{g}^{ij}$ (1) gives

$$(1)' \qquad \sum_{j=1}^{n} \int_M \tilde{g}^{ij} \frac{\partial \tilde{\varphi}}{\partial x_j} \, dH_m = 0, \quad i = 1, \ldots, n,$$

for any C' function $\tilde{\varphi}$ defined in some open set containing M. Clearly (1)′ must also hold if $\tilde{\varphi}$ is only a Lipschitz function. Then let us substitute $(x_i-\xi_i)\gamma_\varepsilon(|x-\xi|)\tilde{h}$ for $\tilde{\varphi}$, where $\tilde{h}$ is a C' function on an open set containing M and such that $\tilde{h} = h$ on M, where γ_ε $(\varepsilon>0)$ is such that $\gamma_\varepsilon(r) = r^{-m}$ for $r \geq \varepsilon$ and $\gamma_\varepsilon(r) = \varepsilon^{-m}$ for $r \leq \varepsilon$, and where ξ is any point in M.

Then carrying out the relevant differentiation we have

$$(7) \quad \sum_{i=1}^{n} \int_M \tilde{g}^{ii}\gamma_\varepsilon(|x-\xi|)h \, dH_m + \sum_{i,j=1}^{n} \int_M \tilde{g}^{ij} \frac{(x_i-\xi_i)}{|x-\xi|} \frac{(x_j-\xi_j)}{|x-\xi|} \cdot |x-\xi|\gamma_\varepsilon'(|x-\xi|)h\,dH_m$$

$$= -\sum_{i,j=1}^{n} \int \tilde{g}^{ij} \frac{x_i-\xi_i}{|x-\xi|} \frac{\partial \tilde{h}}{\partial x_j} |x-\xi|\gamma_\varepsilon(|x-\xi|)\,dH_m .$$

Now using the Cauchy inequality

$$\left| \sum_{i,j=1}^{n} \tilde{g}^{ij} \frac{x_i-\xi_i}{|x-\xi|} \frac{\partial \tilde{h}}{\partial x_j} \right| \leq \left\{ \sum_{i,j=1}^{n} \tilde{g}^{ij} \frac{x_i-\xi_i}{|x-\xi|} \frac{x_j-\xi_j}{|x-\xi|} \right\}^{\frac{1}{2}} \left\{ \sum_{i,j=1}^{n} \tilde{g}^{ij} \frac{\partial \tilde{h}}{\partial x_i} \cdot \frac{\partial \tilde{h}}{\partial x_j} \right\}^{\frac{1}{2}},$$

together with the identities $\sum_{i,j=1}^{n} \tilde{g}^{ij} \frac{\partial \tilde{h}}{\partial x_i} \frac{\partial \tilde{h}}{\partial x_j} = |\delta h|^2$ (by (4)) and $\sum_{i=1}^{n} \tilde{g}^{ii} = m$, together with

$$0 \leq \tilde{g}^{ij} \frac{x_i-\xi_i}{|x-\xi|} \frac{x_i-\xi_j}{|x-\xi|} \leq 1,$$

it follows from (7) that

$$m\int_M h\,\gamma_\varepsilon(|x-\xi|)\,dH_m - \int_M h\,\gamma'_\varepsilon(|x-\xi|)\,|x-\xi|\,dH_m \leq$$

$$\int_M |x-\xi|\,\gamma_\varepsilon(|x-\xi|)\,|\delta h|\,dH_m .$$

Using the definition of γ_ε and noting that

$m\gamma_\varepsilon(t) - \gamma'_\varepsilon(t)t = 0$ for $t \geq \varepsilon$ and $= m\varepsilon^m$ for $t < \varepsilon$,

it follows that

$$m\frac{\int_{S_\varepsilon(\xi)} h\,dH_m}{\varepsilon^n} \leq \int_M |\delta h|\,|x-\xi|^{1-m}dH_m ,$$

and letting $\varepsilon \to 0$ this gives

$$h(\xi) \leq \frac{1}{m\omega_m}\int_M |\delta h|\ |x-\xi|^{1-m}dH_m \tag{8}$$

for all $\xi \in M$, where ω_m is the volume of the unit sphere in R^m.

Multiplying by $h^{\kappa-1}(\xi)$, $\kappa = \frac{m}{m-1}$, integrating over M, and changing the order of integration on the right, (8) gives

$$\int_M h^\kappa(\xi)\,dH_m \leq \frac{1}{m\omega_m}\int_M |\delta h|\left\{\int_M |x-\xi|^{1-m}h^{\kappa-1}(\xi)\,dH_m(\xi)\right\}dH_m(x). \tag{9}$$

Now for $\tau \geq 0$ let $M_\tau = \{x \in M; h(x) > \tau\}$, and replace h in (9) by the Lipschitz function $\eta_\varepsilon(h)$, where $\eta_\varepsilon(t) = 0$, $t \leq \tau$, $\eta_\varepsilon(t) = 1$, $t \geq \tau + \varepsilon$ and $\eta_\varepsilon(t)$ increases linearly for t between $t = \tau$ and $t = \tau + \varepsilon$. Then using $\delta\eta_\varepsilon(h) = \eta_\varepsilon'(h)$ and letting $\varepsilon \to 0$ it follows from (9) that

$$(10) \quad \int_{M_\tau} dH_m \leq \frac{1}{m\omega_m} \sup_{x \in S_{\rho_0}(x_0)} \int_{M_\tau} |x-\xi|^{1-m} dH_m(\xi) \; .$$

$$\cdot \left\{ -\frac{d}{d\tau} \int_{M_\tau} |\delta h| \, dH_m \right\}$$

at all points where the derivative exists(which is for almost all τ because $\int_{M_\tau} \chi \, dH_m$ is a decreasing function if $\chi \geq 0$).

Now let

$$\delta = \left\{ \rho_0^m \left(\int_{M_\tau} dH_m \right) \Big/ \left(\int_{S_{2\rho_0}(x_0)} dH_m \right) \right\}^{\frac{1}{m}},$$

and note that

$$\int_{M_\tau} |x-\xi|^{1-m} dH_m = \int_{M_\tau - K_\delta(x)} |x-\xi|^{1-m} dH_m + \int_{K_\delta(x) \cap M_\tau} |x-\xi|^{1-m} dH_m \; .$$

Then since $|x-\xi| \geq \delta$ on $M_\tau - K_\delta(x)$, it follows that

$$(11) \quad \int_{M_\tau} |x-\xi|^{1-m} dH_m \leq \delta^{1-m} \int_{M_\tau} dH_m + \int_{S_\delta(x)} |x-\xi|^{1-m} dH_m \; .$$

Now for $\rho \leq \rho_0$ we have $K_\rho(x) \subset K_{2\rho_0}(x_0)$ for all $x \in K_{\rho_0}(x_0)$, and hence using Theorem 1 with $\chi \equiv 1$ we have

$$\frac{\int_{S_\rho(x)} dH_m}{\rho^m} \leq \frac{\int_{S_{2\rho_0}(x_0)} dH_m}{(2\rho_0)^m},$$

and it follows from this (see Lemma (4.3) p.59 of [7]) that

$$\int_{S_\rho(x)} |x-\xi|^{1-m} dH_m(\xi) \leq c\rho \frac{\int_{S_{2\rho_0}(x_0)} dH_m}{(2\rho_0)^m},$$

where c depends on m,n.

In particular, since $\delta \leq \rho_0$, we have

$$(12) \quad \int_{S_\delta(x)} |x-\xi|^{1-m} dH_m(\xi) \leq c\delta \int_{S_{2\rho_0}(x_0)} dH_m \cdot (2\rho_0)^{-m}.$$

Then combining (11) and (12) and using the definition of δ it follows that

$$\sup_{x \in S_{\rho_0}(x_0)} \int_{M_T} |x-\xi|^{1-m} dH_m \leq c' \left\{\int_{M_T} dH_m\right\}^{\frac{1}{m}} \left\{\rho_0^{-m} \int_{S_{2\rho_0}(x_0)} dH_m\right\}^{1-\frac{1}{m}},$$

where c' depends on m and n. Using this inequality in (10), it follows that

$$\left\{\int_{M_\tau} dH_m\right\}^{1-\frac{1}{m}} \leqslant c''\left\{\rho_0^{-m}\int_{S_{2\rho_0}(x_0)} dH_m\right\}^{1-\frac{1}{m}}\left\{-\frac{d}{d\tau}\int_{M_\tau}|\delta h|\,dH_m\right\},$$

where c'' depends on m and n.

Now this is a version of the isoperimetric inequality, and the remainder of the proof follows [3] p.258 exactly. (Note particularly that it is now not necessary to consider the quantity corresponding to $H_{n-1}(\partial A(t))$ on p.258 of [3]).

Finally we mention that in the case of non-parametric minimal hyper-surfaces the inequality of Theorem 2 can be somewhat refined.

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