O9PH
E158
c.2

UNIVERSITY OF ADELAIDE LIBRARY

THE UNIVERSITY OF ADELAIDE

The accompanying paper, which has
been submitted to
*The Journal of Mathematical Analysis
and Applications*,
is enclosed by request of an examiner.

If not delivered, please return to
THE UNIVERSITY OF ADELAIDE

O9PH
E158
c.2

UNIVERSITY OF ADELAIDE
6/3/87
LIBRARY

Characterization of Sub-gradients: 1

A.C. Eberhard
Department of Mathematics,
Advanced College
Royal Melbourne Institute of Technology,
G.P.O. Box 2476V,
Melbourne, Victoria 3001,
Australia.

1. INTRODUCTION.

Research into constrained non-linear optimization and Lagrangian theory has brought about the appearance of several sub-differentiability concepts. We concern ourselves with the following two types : the generalized gradient of Clarke [1] and the ϕ_2 - convexity sub-derivative of Dolecki and Kurcyusz [3] . Clarke's gradient is a generalization of the sub-derivative of a convex function, but *per se* has little to do with convexity . The ϕ_2 - sub-derivative and other related concepts generalize the idea of support planes of convex sets . In the context of "classical" convexity both of the corresponding convexity and sub-differentiability concepts are closely related . Developments in non-differentiable optimization have seen a separation of these concepts . This paper presents some results relating the corresponding generalizations of such concepts, for non-smooth functions .

"CHARACTERIZATION OF SUB-GRADIENTS:1"

A.C. Eberhard
Department of Mathematics,
Advanced College
Royal Melbourne Institute of Technology,
G.P.O. Box 2476v,
Melbourne, Victoria 3001,
Australia,

It is shown that the existence of the Clarke and ϕ_2 - sub-derivatives implies, under natural conditions, the existence of the Q^c - sub-gradient [3] within a given neighbourhood. Furthermore, the Clarke sub-gradient can be characterized as the convex closure of the derivatives of the Q^c - convexity sub-gradients.

These results are analogous to those of classical convexity. Any convex function can be obtained by taking the supremum of affine functions $\varphi(u) = \langle u,b\rangle + \beta$. In connection with the sub-derivative of a convex function at a point $\bar{u}$, the class of interest consists of all affine functions such that

$$f(\cdot) \geq \varphi(\cdot) \qquad (1) \quad \text{and}$$

$$f(u) = \varphi(\bar{u}) \qquad (2) \quad .$$

Combining (1) and (2) we arrive at the condition

$$f(u) - f(\bar{u}) \geq \varphi(u) - \varphi(\bar{u}) = \langle u - \bar{u},b\rangle.$$

Denote the class of functions satisfying (1) and (2) by $S_a(\bar{u})$. Then the sub-derivative of the convex function is given by

$$\partial f(\bar{u}) = \{b \in R^n: \quad f(u) - f(\bar{u}) \geq \langle u - \bar{u},b\rangle; \ \forall u \in u\}$$

$$= \{\nabla \varphi(\bar{u}): \quad \varphi(\cdot) \in S_a(\bar{u})\}.$$

With this in mind, it is natural to view the main result as stating the following .

Suppose a function $f(\cdot) : R^n \to R$ *is* ϕ_2-*sub-differentiable everywhere in a neighbourhood of* $\bar{u}$ *and also locally Lipschitz around* $\bar{u}$. *Then there exists a comstant* $\hat{c} > 0$ *and a compact set* $C \subseteq R^n$ *such that*

$$\partial f(\bar{u}) = \overline{\text{co}}\{\nabla \varphi(\bar{u}): \quad \varphi \in \hat{S}_2 \; f(\bar{u})\},$$

where

$$\hat{S}_2 \; f(\bar{u}) = \{\varphi(.)\in \hat{\phi}_2 : f(\bar{u}) - f(u) \leq \varphi(\bar{u}) - \varphi(u); \forall u \in R^n\}$$

and

$$\hat{\phi}_2 = \{\varphi(u) = a - \frac{c}{2}\|u - y\|^2; \; a \in R; \; 0 < c \leq \hat{c} \; ; \; y \in C\}.$$

2. PRELIMINARIES.

If U_1 and U_2 are sets, a mapping Γ of U_1 to the subsets of U_2 can be represented uniquely by its graph

$$G(\Gamma) = \{(u_1,u_2): \quad u_2 \in \Gamma(u_1)\} ,$$

a subset of $U_1 \times U_2$.

When U_1 and U_2 are topological spaces we will consider the concepts of lower semi-continuity (l.s.c.) and upper semi-continuity (u.s.c.) to be those generated by the lower and upper semi-finite topologies on 2^{u_2}.

A full treatment of these concepts is given in ([5] I, page 173) . See also [4] for a thorough account. We now state some properties.

<u>Properties 2.1</u> *Suppose* U_1 *and* U_2 *are topological spaces and* Γ_1 *and* Γ_2 *are multi-valued mappings from* U_1 to U_2.

(i) *If* $c\ell\, \Gamma_1(u_1) = c\ell\, \Gamma_2(u_1)$ *for all* $u_1 \in U_1$, *then we have* Γ_1 *is l.s.c. if and only if* Γ_2 *is l.s.c.*

(ii) *If* U_1 *is a topological linear space and* Γ_1 *is l.s.c. , then* Γ_2 *defined by* $\Gamma_2(u_1) = \text{co}\ \Gamma_1(u_1)$ *for all* $u_1 \in U_1$ *is l.s.c. (Here and subsequently co denotes the convex hull.)*

(iii) *If* U_2 is *regular and* Γ *is closed-valued (i.e.* $c\ell\, \Gamma(u_1) = \Gamma(u_1)$) *and u.s.c., then the graph* $G(\Gamma)$ *is closed.*

(iv) *Define*

$$\bar{m}(u_1) = \inf \{f(u_2): \ u_2 \in \Gamma(u_1)\} ,$$

where U_1 *and* U_2 *are metric spaces , f:* $U_1 \to U_2$ *is single-valued and* $\Gamma(u_1) \neq \phi$ *for all* $u_1 \in U_1$. *Then* $\bar{m}(u_1)$ *is u.s.c. at* $\bar{u}_1$ *as a single-valued function if* $\Gamma(\cdot)$ is l.s.c. *at* $\bar{u}_1$ *and* $f(\cdot)$ *is u.s.c. on* $\Gamma(\bar{u}_1)$ *as a single-valued function.*

For proofs see ([7] Proposition 2.3 and 2.6) for (i) and (ii) and ([6] chapter 5) for (iii) and (iv).

Within the literature there are various inequivalent concepts of continuity of multi-valued mappings which bear similar names. Most are equivalent in metric spaces. The major discrepancy occurs between u.s.c. and the requirement that $G(\Gamma)$ be closed. Without some extra condition equivalence fails to hold in general. We will call a multi-valued mapping Γ closed if it has a graph $G(\Gamma)$ which is a closed set in $U_1 \times U_2$ (endowed with the topology induced by the spaces U_1 and U_2) . The multi-valued mapping Γ will be called closed-valued if the set $\Gamma(u)$ is closed for each $u \in U_1$. A standard condition which forces equivalence is now given .

<u>Definition 2.2</u>

A mapping $\Gamma : U_1 \to 2^{U_2}$ *is said to be uniformly compact near* $\bar{u}_1$ *if and only if there is a neighbourhood* N *of* $\bar{u}_1$ *such that the closure of the set* $\cup \{ \Gamma(u_1) : u_1 \in N \}$ *is compact.*

<u>Proposition 2.1</u>

Let Γ *be uniformly compact near* $\bar{u}_1$ *. Then* Γ *is closed at* $\bar{u}_1$ *if and only if* $\Gamma(\bar{u}_1)$ *is a compact set and* Γ *is u.s.c. at* $\bar{u}_1$.

A proof of the above result may be found in [12] . It is shown in [6] that for a compact valued ,u.s.c. multi-valued mapping , on a compact set U , the set produced by taking the union of all the image sets , is itself a compact set . We shall be concerned with the multi-valued mapping

$$\Gamma(b) = \{c \geq \hat{c}: \quad c \geq h(b)\} ,$$

where $\hat{c} > 0$ and $h: R^m \rightarrow R_+$ is single-valued (positive) . Because of the simple structure of this multi-valued mapping there is a simple equivalence between closure and u.s.c.

<u>Theorem 2.1</u> *The following are equivalent for the mapping*

$$\Gamma(b) = \{c \geq 0: \quad c \geq h(b)\} , \text{ where } \quad h: \quad R^m \rightarrow R_+;$$

(i) $\Gamma(\cdot)$ *is closed at* $\bar{b}$,

(ii) $\Gamma(\cdot)$ *is u.s.c. at* $\bar{b}$, *and*

(iii) $h(\cdot)$ *is l.s.c. at* $\bar{b}$ *as a single-valued mapping.*

<u>Proof</u> Since (ii) ⇒ (i) is immediate, we need only show (i) ⇒ (iii) and (iii) ⇒ (ii).

Suppose $h(\bar{b}) = 0$. Since $h(b) \geq 0$ for all b we must have for any $\epsilon > 0$ that

$$h(b) \geq h(\bar{b}) - \epsilon = -\epsilon$$

for all b sufficiently close to $\bar{b}$, i.e., $h(\cdot)$ is l.s.c. at $\bar{b}$ without any further condition.

Now suppose $h(\bar{b}) > 0$, $h(\cdot)$ <u>not</u> l.s.c. at $\bar{b}$ and $\Gamma(\cdot)$ closed at $\bar{b}$. Then for any $\epsilon > 0$, there exists $b_n \in N(\bar{b},\frac{1}{n})$ such that $h(b_n) \leq h(\bar{b}) - \epsilon$ for all n sufficiently large. Since $h(\bar{b}) > 0$ there must exist an $\epsilon > 0$ such that $h(\bar{b}) - \epsilon > 0$ and a non-negative sequence $\{c_n\}$ such that

$$h(b_n) \leq c_n \leq h(\bar{b}) - \epsilon < h(\bar{b}) \quad \text{for all } n.$$

As $c_n \in [0, h(\bar{b})]$, there must exist a convergent subsequence and after relabelling we have

$$h(b_n) \leq c_n \to c < h(\bar{b}).$$

That is, there exists $c_n \in \Gamma(b_n)$ where $c_n \to c$ as $b_n \to \bar{b}$ and $c \notin \Gamma(\bar{b})$. This contradicts $\Gamma(\cdot)$ being closed at $\bar{b}$, establishing (i) $\Rightarrow$ (iii).

Now suppose $h(\cdot)$ is l.s.c. at $\bar{b}$. In order to show that (iii) $\Rightarrow$ (ii). we need to show that for any open set $A \subset R$ such that $\Gamma(\bar{b}) \subseteq A$, there exits a $\delta > 0$ such that

$$\Gamma(b) \subseteq A \quad \text{for every } b \in N(\bar{b},\delta).$$

If $\Gamma(\bar{b}) \subseteq A$ we must have $\bar{N}(\Gamma(\bar{b}), \epsilon) \subseteq A$ for some $\epsilon > 0$, whenever A is open. Since we have

$$\bar{N}(\Gamma(\bar{b}), \epsilon) = \{c \geq 0: \ c \geq h(\bar{b}) - \epsilon\},$$

we can use the l.s.c. of $h(\cdot)$ at $\bar{b}$ to deduce the existence of $\delta > 0$ such that, for all $b \in N(\bar{b},\delta)$, we have

$$h(b) \geq h(\bar{b}) - \epsilon.$$

This implies

$$\begin{aligned} \Gamma(b) &= \{c \geq 0: \; c \geq h(b)\} \\ &\subseteq \{c \geq 0: \; c \geq h(\bar{b}) - \epsilon\} \\ &\subseteq A. \end{aligned}$$

□

If (U_1,d_1) and (U_2,d_2) are metric spaces then $U_1 \times U_2$ has the metric

$$d((u_1,u_2), (\bar{u}_1,\bar{u}_2)) = \max \{d_1(u_1,\bar{u}_1), d_2(u_2,\bar{u}_2)\}.$$

As usual we define

$$d((u_1,u_2),A) = \inf \{d((u_1,u_2), (\bar{u}_1,\bar{u}_2)): \; (\bar{u}_1,\bar{u}_2) \in A\}$$

for $A \subset U_1 \times U_2$. The separation of two subsets $A,B \subseteq U_1 \times U_2$ is given by

$$d^*(B,A) = \sup \{d((u_1,u_2),A): \; (u_1,u_2) \in B\}.$$

We give a slightly reworded statement of part of the content of ([8], Theorem 1). In the following $K(U_2)$ denotes the closed subsets of U_2.

<u>Theorem 2.2</u> *Suppose* (U_1,d_1) *is a compact metric space and* (U_2,d_2) *is metric. If* $\Gamma: U_1 \to K(U_2)$ *is u.s.c., then we can approximate* Γ *by l.s.c. multi-valued mappings* $\Gamma_\epsilon: U_1 \to K(U_2)$ *such that*

$$\bigcap_{\epsilon>0} \Gamma_\epsilon(u_1) = \Gamma(u_1) \quad \textit{for all} \quad u_1 \in U_1 \quad \textit{and}$$

$$d^*(G(\Gamma_\epsilon), G(\Gamma)) \leq \epsilon \quad \textit{for all} \quad \epsilon > 0.$$

We have discussed these concepts in very general spaces and shall continue to use the corresponding notation. As is usual in the literature we shall however deal specifically with R^n (see [1] and [10]). As has been noted before much of the material extends to more generalized spaces. Generalizations of our results will , as a consequence , be self-evident.

3. SUB-DIFFERENTIABILITY

Ever since F.H. Clarke published his paper [1] on generalized gradients , much interest has surrounded the development of these theories. Locally Lipschitz functions play an important role as they imply the existence of this type of differentiability. We use the approach of [10] to define the sub-gradient of an arbitrary l.s.c. function . When the function is locally Lipschitz it will correspond to the sub-gradient of Clarke . We will consider this situation in section four .

Definition 3.1 *(i) For an arbitrary l.s.c. function* $f(\cdot)$ *we define the upper sub-derivative of* $f(\cdot)$ *at* $\bar{u}$ *with respect to* h as

$$f^{\uparrow}(\bar{u};h) = \limsup_{\substack{u \to_f \bar{u} \\ t \to 0_+}} \inf_{h' \to h} \frac{f(u + th') - f(u)}{t}$$

where $u \to_f \bar{u}$ *if and only if* $u \to \bar{u}$ *and* $f(u) \to f(\bar{u})$. *(Obviously this will be the same as* $u \to \bar{u}$ *when* $f(\cdot)$ *is a continuous function.)* .

(ii). For such a function we define the sub-gradients of $f(\cdot)$ *at* $\bar{u}$ *as the set*

$$\partial f(\bar{u}) = \{ z \in R^n : f^{\uparrow}(\bar{u};h) \geq \langle z,h \rangle \text{ for all } h \in R^n \} .$$

See ([10] page 31) for a discussion of the concept of the limit "lim sup inf" . We shall not use this concept directly in subsequent proofs . The set is always closed and convex . It follows that if $f(\cdot)$ is locally Lipschitz , the mapping $\partial f(u)$ is convex compact and non-empty and , as in the convex case , the mapping $u \to \partial f(u)$ is also an u.s.c. multi-valued mapping . Also $\partial f(u)$ is a singleton for all $u \in \Omega$ if and only if $f(\cdot)$ continuously differentiable on Ω. If $\partial f(u) = \{x\}$ then $\nabla f(u) = x$.

For a locally Lipschitz funtion a simpler definition exists . On can show for a such a function

$$f_c(u;h) = \limsup_{(y,t) \to (u,o^+)} \frac{f(y + th) - f(y)}{t}$$
$$= \max \{\langle x;h \rangle : x \in \partial f(u)\}$$

It follows that $f^{\uparrow}(u;h) = f_c(u;h)$, providing an alternative poceedure for defining $\partial f(\cdot)$ when $f(\cdot)$ is locally Lipschitz .

Definition 3.2 *We say that a l.s.c. function* $f(\cdot)$ *is differentially regular at* $\bar{u}$ *if*

$$f^{\uparrow}(\bar{u};h) = \liminf_{(h',t)\to(h,0_+)} \frac{f(\bar{u} + th') - f(\bar{u})}{t}$$

for all h .

Proposition 3.1 *Let* $f: R^n \to R$ *be a smooth function and let* $h: R^n \to R$ *be locally Lipschitz. Then the function*

$$F(x) = f(x) + h(x)$$

is locally Lipschitz and

$$\partial F(x) = \{\nabla f(x) + u; \quad u \in \partial h(x)\} \stackrel{\Delta}{=} \partial h(x) + \nabla f(x).$$

A proof is given in ([11], p 62) .

In the case when $f(\cdot)$: $U \to R$ is convex the sub-derivative $\partial f(\cdot)$ with respect to the affine mappings coincides with the Clarke sub-derivative at every point in int U, for $U \subseteq R^n$. For a convex function , the condition $0 \in \partial f(u)$ implies that $f(\cdot)$ achieves its global minimum at u. If $f(\cdot)$ is locally Lipschitz around u and achieves a local minimum at u , then $0 \in \partial f(u)$.

The other type of sub-differentiability we use is derived from generalizations of the concept of convexity. We may generalize convexity by simply allowing ϕ to be a family of arbitrary real functions which satisfy

$$\phi + c \triangleq \{\varphi + c: \ \varphi \in \phi\} = \phi.$$

In this situation f is ϕ - convex if

$$f(u) = \sup\{\varphi(u): \ \varphi \in \phi' \subseteq \phi\}$$

for some sub-collection ϕ' (if $\phi' = \emptyset$, then $f \equiv -\infty$).

<u>Definition 3.3</u> *For an arbitrary class* ϕ, *a* ϕ*-convex function* f *is said to be* ϕ *sub-differentiable at* $\bar{u} \in U$ *if there exists a* $\varphi \in \phi$ *such that*

$$f(\bar{u}) = \varphi(\bar{u}) \quad \textit{and}$$

$$f(u) \geq \varphi(u) \quad \textit{for all} \quad u \in U.$$

The set of all $\varphi(\cdot) + c$, *where* $c \in R$ *and* φ *is a subgradient of* f *at* $\bar{u}$ *is called the* ϕ *sub-differential of* f *at* $\bar{u}$ *and is denoted* $S\phi\ f(\bar{u})$. *Equivalently* $S\phi\ f(\bar{u})$ *consists of all* $\varphi \in \phi$ *such that*

$$f(u) - f(\bar{u}) \geq \varphi(u) - \varphi(\bar{u})$$

for all $u \in U$.

The class of convexity-generating functions we shall be concerned with is

$$\phi_2 = \{\varphi(u) = a - \frac{c}{2}\|u - y\|^2; \quad a \in R; \quad c \in R_+; \quad y \in U\}$$

and we shall denote the sub-differential of f at $\bar{u}$ by $S_2\ f(\bar{u})$. A function f is ϕ_2 bounded if there exists

$\psi(\cdot) \in \phi_2$ such that $f(u) \geq \psi(u)$ for all $u \in U$.

The other class of interest is

$$Q^c = \{\psi(u) = a - \tfrac{c}{2}\|u - y\|^2 : (y,a) \in S \ ; \ S \subseteq R^n \times R\} .$$

Suppose we have

$$f(u) = \sup \{ a - \tfrac{c}{2}\|u - y\|^2 : (y,a) \in S \ ; \ S \subseteq R^n \times R \}.$$

Since $\|u - y\|^2 = \|u\|^2 - 2\langle u,y\rangle + \|y\|^2$,

we have

$$f(u)+\tfrac{c}{2}\|u\|^2 = \sup\{\langle u,cy\rangle + \|y\|^2 + a:(y,a)\in S; S\subseteq R^n \times R\}.$$

a supremum of a class of affine mappings.

Thus $f(\cdot)$ is Q^c-convex if and only if $f(\cdot) + \frac{c}{2}\|\cdot\|^2$ is convex in the ordinary sense . In this situation we know that $f(\cdot)$ is Q^c-sub-differentiable at any point in int(dom f) ([2] Theorem 5.11) . The relationship between ϕ_2- convexity and ϕ_2- sub-differentiability is not quite as strong .

<u>Proposition 3.2</u> *Suppose* $f : R^n \to R$ *is lower semi-continuous and* ϕ_2*- bounded . Then*

(*i*) $f(\cdot)$ *is sub-differentiable with respect to the class* ϕ_2 *on a dense subset of its domain , and*

(*ii*) $f(\cdot)$ *is in fact* ϕ_2*-convex .*

The statement (i) is Theorem 6.2 of [3] with $\alpha = 2$ and $X = R^n$ while (ii) is Theorem 4.2 combined with Proposition 4.13 of [3] .

Any lower semi-continuous function that is ϕ_2 - sub-differentiable at any point will , as a consequence of ϕ_2 - boundedness , be ϕ_2 - convex. This allows us to write

$$f(u) = \sup \{ \varphi(u) : \varphi(\cdot) \in \hat{\phi}_2 \subseteq \phi_2 \}$$

for any $u \in dom(f)$. This does not imply that we may assume anything about the compactness of the set of parameters (c,y) generating the functions $\varphi(\cdot)$ that comprise the set $\hat{\phi}_2$. The resultant ϕ_2 - convex function may not be sub-differentially regular . The following almost trivial observation will give context to Proposition 3.4 .

<u>Proposition 3.3</u> *Suppose* $\varphi(\cdot)$ *is a function such that* $f(\bar{u}) = \varphi(\bar{u})$ *and* $f(u) \geq \varphi(u)$ *for all* u *in some neighbourhood of* $\bar{u}$ *. If* $\varphi(\cdot)$ *is differentiable at* u *, then* $z = \nabla \varphi(\bar{u})$ *is a lower semi-gradient at* $\bar{u}$ *, that is ,*

$$\liminf_{(h',t)\to(h,0_+)} \frac{f(u + th') - f(u)}{t} \geq \langle z,h \rangle$$

for all $h \in R^n$.

This result can be found in ([10] pages 28-29) . The following Proposition indicates when one can characterize $\partial f(\cdot)$ as exactly the set of lower semi-gradients .

<u>Proposition 3.4</u> *It is always true that*
$\partial f(\bar{u}) \supset \{ z : z$ is a lower semi-gradient of $f(\cdot)$ at $\bar{u}\}$
When $\partial f(\bar{u}) \neq \phi$ *, one has equality in the above if and only if* $f(\cdot)$ *is sub-differentially regular at* $\bar{u}$ *.*

For a proof see in ([10], page 37) .

If $f(\cdot)$ is ϕ_2- convex , then $\partial f(\bar{u}) \neq \phi$ at every point at which $f(\cdot)$ is ϕ_2- sub-differentiable . We are not assured of equality in the relation $\partial f(\bar{u}) \supseteq \overline{co}\{\nabla \varphi(\bar{u}) : \varphi$ is a ϕ_2 subderivative of f at $\bar{u}\}$. The function $f(\cdot)$ is l.s.c. and hence $\partial f(\cdot)$ is well-defined . By Propositions 2.2 and 2.4 , $\partial f(\cdot)$ must be non-empty on a dense subset of dom(f) .

This prompts one to ask whether it is possible to extend the sub-differentiation by taking limits , rather like Clarke originally did to define the sub-gradient . Unfortunately we can not use this approach to extend sub-differentiability to the whole of dom(f) without assuming either u.s.c. of the multi- function $\partial f(\cdot)$, or at least closure of its graph and the existence of bounded sequences . Uniform compactness would seem a natural assumption to augment closedness at some point . This in turn would imply "local" compactness of the parameter set (c,y) generating the functions in the class ϕ_2 . As we shall see this would allow us to extend ϕ_2-sub-differentiability to the whole of some neighbourhood as well .It would also imply that $\partial f(\cdot)$ is u.s.c. .

As is pointed out in ([10] pages 47-48) , directional Lipschitzness is closely related to the closure of the graph of $\partial f(\cdot)$. The reader is referred to ([10] pages 49-50) for two characterizations of $\partial f(\cdot)$ in terms of limits of lower semi-gradients . One class of lower semi-gradients is generated by a ϕ_2 - like class . We now consider what sort of compact set of parameters (c,y) will allow one to deduce Q^{c}-subdifferentiability from a dense ϕ_2 - like sub-differentiability .

<u>Theorem 3.1</u> *Suppose* $f(\cdot): U \to R$ *is continuous and* ϕ_2 *sub-differentiable on a dense subset of* U *with respect to the sub-class*

$$\hat{\phi}_2 = \{\varphi(u) = a - \frac{c}{2}\|u - y\|^2;\ a \in R;\ 0 \le c \le \hat{c};\ y \in C\},$$

where the compact set C *has the property that*

$$y^1 = \bar{u} - (c/\bar{c})(\bar{u} - y) \in C \quad \textit{for any} \quad \bar{u} \in U$$

$$\textit{whenever} \quad 0 \le c \le \bar{c} \le \hat{c} \quad \textit{and} \quad y \in C .$$

Then (i) $f(\cdot)$ *is sub-differentiable everywhere with respect to the class* Q^{c} *for some* $c > 0$.

(ii) *We may identify a sub-derivative* $\psi(u) = a - \frac{c}{2}\,\|u - y\|^2$ *with the pair* (c,y). *Then the multi-valued mappings*

$$S_2 f(\bar{u}) = \{(c,y) : (c,y) \textit{ is a } \hat{\phi}_2\textit{-sub-deriv. of } f \textit{ at } \bar{u}\}$$

and

$$S_c f(\bar{u}) = \{y : y \in C;\ (c,y) \textit{ is a } Q^c\textit{-sub-deriv. of } f \textit{ at } \bar{u}\}$$

are both non-empty and u.s.c. on U .

(iii) *The following holds*

$$\{\, c(y - \bar{u}) :\ y \in \overline{co}\, S_c\, f(\bar{u})\}$$
$$= \overline{co}\,\{\, c(y - \bar{u}) :\ (c,y) \in S_2 f(\bar{u})\ \} \neq \phi .$$

<u>Proof</u> We take a sequence $u_n \to \bar{u} \in U$ where, for each n, $f(\cdot)$ is $\hat{\phi}_2$ sub-differentiable at u_n. For each n there exists $0 \leq c_n \leq \hat{c}$ and $y_n \in C$ such that, for all $u \in U$, we have

$$f(u) - f(u_n) \geq \frac{c_n}{2}\,[\,\|u_n - y_n\|^2 - \|u - y_n\|^2].$$

There exist convergent subsequences of (c_n, y_n) tending to some (c,y) where $0 \leq c \leq \hat{c}$ and $y \in C$. When the appropriate limit is taken in the above inequality, the continuity of $f(\cdot)$ gives

$$f(u) - f(\bar{u}) \geq \frac{c}{2}\,[\,\|\bar{u} - y\|^2 - \|u - y\|^2]$$

for all $u \in U$.

This establishes that $S_2\, f(\bar{u}) \neq \phi$. Since $f(\cdot)$ is densely $\hat{\phi}_2$ sub-differentiable we have extended this sub-differentiability to the whole of dom f . This also establishes that the multi-valued mapping

$S_2 f(u)$ is closed at $\bar{u}$. Since $S_2\, f(\bar{u}) \subseteq [0,\hat{c}] \times C$, the images are compact and hence the multi-valued mapping is u.s.c. as well . No confusion can be created by identifying the functions $\varphi(\cdot)$ with the ordered pairs (c,y). Any limit of such functions will correspond to a limit in the topology of $R_+ \times R^n$. As a consequence the notions are interchangeable .

We now show that $\hat{\phi}_2$ sub-differentiability implies Q^c sub-differentiability. Take $(c,y) \in S_2 f(\bar{u})$, where $0 \le c < \bar{c} \le \hat{c}$ and hence

$$f(u) - f(\bar{u}) \ge \frac{c}{2}\,[\|\bar{u} - y\|^2 - \|u - y\|^2]$$

for all $u \in U$. First we show that $(\bar{c},y^1) \in S_2\, f(\bar{u})$, where

$$y^1 = \bar{u} - (c/\bar{c})(\bar{u} - y\ ).$$

Since we have $f(\cdot)$ $\hat{\phi}_2$ - subdifferentiable at any $\bar{u}$, and for any $\hat{\phi}_2$ sub-derivative (c,y) with $\hat{c} > c$ there must exist $y^1 \in C$ such that $(\hat{c},y^1) \in S_2\, f(\bar{u})$, we will as a result establish (i) .

By hypothesis any such y^1 belongs to C , and by using $cy = c\bar{u} - \bar{c}\bar{u} + \bar{c}y^1$ we have

$$f(u) - f(\bar{u})$$

$$\ge \frac{c}{2}\,[\|\bar{u}\|^2 - 2\langle\bar{u},y\rangle + \|y\|^2 - \|u\|^2 + 2\langle u,y\rangle - \|y\|^2]$$

$$= \frac{c}{2}\,[\|\bar{u}\|^2 - \|u\|^2] + \langle u - \bar{u},\ cy\rangle$$

$$= \frac{c}{2}\,[\|\bar{u}\|^2 - \|u\|^2] + \langle u - \bar{u},\ c\bar{u} - \bar{c}\bar{u} + \bar{c}y^1\rangle$$

$$= \langle u - \bar{u}, \bar{c}y^1\rangle + \frac{c}{2}[\|\bar{u}\|^2 - \|u\|^2] + (\bar{c} - c)[\|\bar{u}\|^2 - \langle u, \bar{u}\rangle].$$

We now show that

$$\frac{c}{2}[\|\bar{u}\|^2 - \|u\|^2] + (\bar{c} - c)[\|\bar{u}\|^2 - \langle u, \bar{u}\rangle] \geq \frac{\bar{c}}{2}[\|\bar{u}\|^2 - \|u\|^2].$$

On subtracting the right side of the inequality from the left we obtain, since $\bar{c} > c$, that

$$-\frac{(\bar{c} - c)}{2}\|\bar{u}\|^2 + \frac{(\bar{c} - c)}{2}\|u\|^2 + (\bar{c} - c)[\|\bar{u}\|^2 - \langle u,\bar{u}\rangle]$$

$$\geq -\frac{(\bar{c} - c)}{2}\|\bar{u}\|^2 + \frac{(\bar{c} - c)}{2}\|u\|^2 + (\bar{c} - c)[\|\bar{u}\|^2 - \|u\|\,\|\bar{u}\|]$$

$$= \frac{(\bar{c} - c)}{2}[\|u\|^2 - \|\bar{u}\|^2 + 2\|\bar{u}\|^2 - 2\|u\|\,\|\bar{u}\|]$$

$$= \frac{(\bar{c} - c)}{2}[\|\bar{u}\| - \|u\|]^2 \geq 0.$$

Hence

$$f(u) - f(\bar{u}) \geq \frac{\bar{c}}{2}[\|\bar{u}\|^2 - \|u\|^2] + \langle u - \bar{u}, \bar{c}y^1\rangle$$

$$= \frac{\bar{c}}{2}[\|\bar{u} - y^1\|^2 - \|u - y^1\|^2]$$

for all $u \in U$, establishing (i).

We derive the remaining part of (ii) as follows. Select $u_n \to \bar{u}$ and $y_n \in S_c f(u_n)$ such that $y_n \to y$. By taking limits in the inequality

$$f(u) - f(u_n) \geq \frac{c}{2}[\|u_n - y_n\|^2 - \|u - y_n\|^2],$$

we show $y \in S_c f(\bar{u})$. This establishes the closure of the graph and also proves that $S_c f(\bar{u})$ is a closed set. Since $S_c f(\bar{u})$ is contained in C it must also be compact. Hence co $S_c f(\bar{u})$ is a compact set and the

UNIVERSITY OF ADELAIDE LIBRARY

corresponding multi-valued mappings must be u.s.c. as well .

We now establish (iii) , that is

$$\Omega \triangleq \{\hat{c}(y - \bar{u}): \; y \in \text{co } S_{\hat{c}} f(\bar{u})\}$$
$$= \text{co } \{\nabla \varphi(\bar{u}): \; \varphi(\cdot) \in S_2 f(\bar{u})\}.$$

Since $y \in \text{co } S_{\hat{c}} f(\bar{u})$, there exists $\bar{y}, y^1 \in S_{\hat{c}} f(\bar{u})$ and $0 \leq \lambda \leq 1$ such that $y = \lambda\bar{y} + (1 - \lambda)y^1$ and

$$\hat{c}(y - \bar{u}) = \lambda \hat{c}(\bar{y} - \bar{u}) + (1 - \lambda) \hat{c}(y^1 - \bar{u}).$$

As $S_{\hat{c}} f(\bar{u}) \subseteq S_2(\bar{u})$ the inclusion of the set Ω is implied .

Suppose $(c,y), (\bar{c},\bar{y}) \in S_2 f(\bar{u})$. Then there exist sub-derivatives $\varphi_1(\cdot)$ and $\varphi_2(\cdot)$ corresponding to these vectors . If either of $\bar{c},c$ is less than $\hat{c}$, then there must exist $y', y'' \in C$ for which

$$\bar{c}(\bar{y} - \bar{u}) = \hat{c}(y'' - \bar{u}) \quad \text{and}$$
$$c(y - \bar{u}) = \hat{c}(y' - \bar{u}) \; .$$

The pairs $(\hat{c},y')$ and $(\hat{c},y'')$ correspond to sub-derivatives and we have

$$\lambda \nabla \varphi_1(\bar{u}) + (1 - \lambda) \nabla \varphi_2(\bar{u})$$
$$= \lambda c(y - \bar{u}) + (1 - \lambda) \bar{c}(\bar{y} - \bar{u})$$
$$= \lambda \hat{c}(y' - \bar{u}) + (1 - \lambda) \hat{c}(y'' - \bar{u})$$
$$= \hat{c}((\lambda y' + (1 - \lambda) y'') - \bar{u}) ,$$

establishing the other inclusion.

The closure of the set Ω is obviously $\{ \hat{c}(y - \bar{u}) : y \in \overline{co}\ S_{\hat{c}}\ f(\bar{u}) \}$, hence we have (iii) . □

4 . LOCALLY LIPSCHITZ FUNCTIONS

We now show the strength of assuming local ϕ_2 - sub-differentiability of a locally Lipschitz function . Under these conditions we have a locally Q^{c}- subdifferentiable function . That is , we can force such a function to become convex over some neibourhood , in the usual sense , by adding a fixed "penalty function" . We require the following results.

Lemma 4.1 *Let* $f(\bar{u}) = \max \{ \varphi_y(\bar{u}) : y \in M \}$, *where* M *is a compact space . Suppose each* $\varphi_y(\cdot)$ *is locally Lipschitz on* R^n, *the function* $y \to \varphi_y(u)$ *is upper semi-continuous , and the multi-function* $(y,u) \to \partial \varphi_y(u)$ *is upper semi-continuous and also locally bounded .*

For any point $\bar{u}$, *let*

$$M(\bar{u}) = \{ y \in M : \varphi_y(\bar{u}) = f(\bar{u}) \} .$$

Then $\partial f(\bar{u}) \subseteq \overline{co} \{ \partial \varphi_y(\bar{u}) : y \in M(\bar{u}) \}$.

If the functions $\varphi_y(\cdot)$ *are sub-differentially regular at* $\bar{u}$, *then so is* $f(\cdot)$ *and equality holds in the above relation .*

The precise statement above was taken from ([10] page 69) but it is a restatement of ([1] Theorem 2.1) . Of course when each $\psi_y(\cdot)$ is continuously differentiable they are differentially regular and the lemma reduces to Danskin's theorem .

<u>Theorem 4.1</u> *Suppose* $f(\cdot)\colon R^n \to R$ *is* ϕ_2 *sub-differentiable everywhere in a neighbourhood of* $\bar{u}$ *and also locally Lipschitz around* $\bar{u}$.

Then there exists a constant $c > 0$ *and a compact set* C , *such that* $f(\cdot)$ *is sub-differentiable with respect to the class*

$$Q^{C} = \{\psi(u) = a - \frac{c}{2}\|u - y\|^2; \quad a \in R, \quad y \in C\}$$

everywhere on a sufficiently small neighbourhood of $\bar{u}$, *and further*

$$\partial f(\bar{u}) = \{ c(y - \bar{u}): \quad y \in \overline{co}\, S_c f(\bar{u})\}$$
$$= \overline{co}\, \{ c(y - \bar{u}): (c,y) \in S_2 f(u)\} ,$$

our function $f(\cdot)$ *being differentially regular at* $\bar{u}$.

<u>Proof</u> Let $N(\bar{u},\delta)$ be a neighbourhood of $\bar{u}$ for which $\partial f(\cdot)$ exists as a compact convex set and $f(\cdot)$ is ϕ_2 sub-differentiable at every $u \in \bar{N}(\bar{u},\delta)$. Then for any $u' \in \bar{N}(\bar{u},\delta)$ there exists (c,y) such

that the function

$$u \to f(u) + \frac{c}{2}\,\|u - y\|^2$$

attains a global minimum at u'.

Hence

$$0 \in \partial(f(u) + \frac{c}{2}\,\|u - y\|^2)\big|_{u=u'}$$
$$= \partial\, f(u') + c(u' - y)\ ,$$

by Proposition 3.1. This implies that for some c we have

$$y \in \{\frac{x}{c} + u' :\quad x \in \partial\, f(u');\quad u' \in \bar{N}(\bar{u},\delta)\}.$$

For $\hat{c} > 0$, $\hat{\epsilon} > 0$ and δ sufficiently small , we define $C(\hat{c})$ to be the set $\{\frac{x}{c} + u' : x \in \bar{N}(\partial\, f(\bar{u}), \hat{\epsilon});\ u' \in \bar{N}(\bar{u}, \delta);\ x = 0;\ c \geq \hat{c}\}$. For $\hat{c} > 0$, the set $C(\hat{c})$ is compact. This follows from the compactness of $\bar{N}(\bar{u}, \delta)$, $\partial\, f(\bar{u})$ and the consequent compactness of $\bar{N}(\partial\, f(\bar{u}), \hat{\epsilon})$. Take a sequence $\{y_n\}$ in $C(\hat{c})$. Then there exists x_n, c_n and u'_n such that

$$y_n = x_n/c_n + u'_n\ ,\quad \text{where}$$
$$x_n \in \bar{N}(\partial\, f(\bar{u}), \hat{\epsilon}),\quad c_n \geq \hat{c}\ , \text{ and}$$
$$u'_n \in \bar{N}(\bar{u},\delta).$$

There must exist a convergent subsequence of (x_n, u'_n) tending to (x,u') with $x \in \bar{N}(\partial\, f(\bar{u}), \hat{\epsilon})$ and $u' \in \bar{N}(\bar{u}, \delta)$.

Two cases arise .

I. $c_n \to \infty$. In this event

$$y_n = x_n/c_n + u'_n \to u' \in \bar{N}(\bar{u}, \delta) \subseteq C(\hat{c}).$$

II. Suppose c_n remains bounded. In this case there exists a convergent sub-sequence of (c_n, x_n, u'_n) tending to (c, x, u') . With relabelling we have

$$y_n = x_n/c_n + u'_n \to x/c + u' \in C(\hat{c}),$$

where $$c_n \to c \geq \hat{c} .$$

In either case $C(\hat{c})$ is sequentially compact and hence compact.

For δ sufficiently small we have

$$\partial f(u) \subseteq N(\partial f(\bar{u}), \hat{\epsilon}) \subseteq \bar{N} (\partial f(\bar{u}), \hat{\epsilon})$$

for all $u \in \bar{N}(\bar{u},\delta)$. Hence if (c,y) determines a sub-derivative of $f(\cdot)$ at u , we must have $y \in C(c)$. Whenever $c < \hat{c}$ we may, as before, increase c to $\hat{c}$ and move y to $y^1 = u - (c/\hat{c})(u - y)$ to produce a new sub-derivative at u. Since $y = x/c + u$, we have

$$y^1 = u - (c/\hat{c})(u - \frac{x}{c} - u)$$
$$= u + x/\hat{c} \in \bar{C}(\hat{c}).$$

That is, for any $u \in \bar{N}(\bar{u},\delta)$ there exist $c \geq \hat{c}$ and $y \in C(\hat{c})$ such that (c,y) produces a sub-derivative of $f(\cdot)$ at u. This result is independent of how large we make $\hat{c} > 0$. As a consequence

$$S f(u) = \{(c,y): c \geq \hat{c},\ y \in C(\hat{c}) \text{ and } (c,y) \in S\phi_2 f(u)\},$$

where

$S\phi_2 f(u) = \{(c,y) : (c,y) \text{ is a } \phi_2\text{- sub-deriv. of } f \text{ at } u\}$

is a closed non-empty multi-valued mapping on $\bar{N}(\bar{u},\delta)$.

Define

$$H(u) = \{c: \exists y \text{ s.t. } (c,y) \in S f(u)\}$$

and let

$$h(u) = \inf \{c: c \in H(u)\}.$$

We note that $h(u) < \infty$ for all $u \in \bar{N}(\bar{u},\delta)$ and prove that the multi-valued mapping $H(u)$ must be closed at any $u \in \bar{N}(\bar{u},\delta)$. Suppose $u_n \in \bar{N}(\bar{u},\delta)$, $u_n \to u$ and $c_n \in H(u_n)$. We must show that $c \in H(u)$ whenever $c_n \to c$.

Since $c_n \in H(u_n)$, there must exist $y_n \in C(\hat{c})$ such that $(c_n,y_n) \in S\phi_2 f(u_n)$ with a convergent sub-sequence tending to $(c,y) \in S\phi_2 f(u)$, where $y \in C(\hat{c})$ and $c \geq \hat{c}$. That is, c belongs to $H(u)$ establishing closure . As a bonus this also establishes that $H(u)$ is a closed set for all $u \in \bar{N}(\bar{u},\delta)$.

If $\bar{c} \in H(u)$, then for any $c \geq \bar{c}$ we have $c \in H(u)$. Hence

$$H(u) = \{c \geq 0: c \geq h(u)\}$$

for all $u \in \bar{N}(\bar{u},\delta)$. By Theorem 2.1 , $H(\cdot)$ is u.s.c on $\bar{N}(\bar{u},\delta)$ and $h(\cdot)$ is l.s.c. on $\bar{N}(\bar{u},\delta)$.

Since $U_1 = \bar{N}(\bar{u},\delta)$ is a compact metric space and $H(\cdot)$: $U_1 \to K(R)$ is u.s.c. , we can invoke Theorem 2.2 to deduce the existence of a l.s.c. multi-valued mapping $H_\epsilon(\cdot)$ approximating $H(\cdot)$ in graph, i.e.,

$$d^*(G(H_\epsilon), G(H)) \leq \epsilon \quad \text{for all} \quad \epsilon > 0.$$

Thus for all $\epsilon > 0$ and $u \in \bar{N}(\bar{u},\delta)$, there exists $u^1 \in N(u,\epsilon)$ such that

$$H_\epsilon(u) \subseteq N(H(u^1), \epsilon).$$

We may take $H_\epsilon(u)$ to be a closed, convex set . For otherwise we could replace it by $\overline{co}\, H_\epsilon(u)$. Proposition 2.1 parts (i) and (ii) , ensure that $\overline{co}\, H_\epsilon(\cdot)$ is still l.s.c. on $\bar{N}(\bar{u},\delta)$. Since $H(\cdot)$ is closed and convex we must have for all $\epsilon > 0$ and $u \in \bar{N}(\bar{u},\delta)$ the existence of $u^1 \in N(u,\epsilon)$ such that

$$\overline{co}\, H_\epsilon(u) \subseteq N(H(u^1), \epsilon).$$

The mapping $\overline{co}\, H_\epsilon(\cdot)$ will still approximate $H(\cdot)$ in graph .

Let

$$h_\epsilon(u) = \inf \{c: \quad c \in \overline{co}\, H_\epsilon(u)\} ,$$

and note that

$$\overline{co}\, H_\epsilon(u) = \{c: \quad c \geq h_\epsilon(u)\}.$$

By Proposition 2.1 , part (iv) , $h_\epsilon(\cdot)$ is u.s.c. on $\bar{N}(\bar{u},\delta)$ and , since $H(u) \subseteq H_\epsilon(u)$ for all $u \in \bar{N}(\bar{u},\delta)$, we must have also

$$h_\epsilon(u) \leq h(u) \quad \text{for all} \quad u \in \bar{N}(\bar{u},\delta).$$

Putting this all together, we have for $u \in \bar{N}(\bar{u},\delta)$ the existence of $u^1 \in N(u,\epsilon)$ such that

$$\infty > h(u) \geq h_\epsilon(u) \geq h(u^1) - \epsilon \geq \hat{c} - \epsilon$$

for all $\epsilon > 0$. By letting

$$M = \sup \{h_\epsilon(u) : u \in \bar{N}(\bar{u},\delta)\},$$

we establish that for any $\epsilon > 0$ and $u \in N(\bar{u},\delta)$ there exists $u^1 \in N(u,\epsilon)$ such that $\infty > M + \epsilon \geq h(u^1)$. The constant M is finite since $h_\epsilon(\cdot)$ is u.s.c. and $\bar{N}(\bar{u},\delta)$ is compact.

We show that this in turn implies the existence of $(c,y) \in S\, f(u)$ where $M \geq c \geq \hat{c}$. The arbitrariness of $u \in \bar{N}(\bar{u},\delta)$ establishes a $\hat{\phi}_2$-type sub-differentiability on $\bar{N}(\bar{u}.\delta)$.

Let $\epsilon = 1/n$, where $n \in Z^+$. Take $u \in N(\bar{u},\epsilon)$, and for each n choose u^1_n as above. As $n \to \infty$, necessarily $u^1_n \to u$. If $c_n = h(u^1_n) \geq \hat{c}$ there must exist for each n some y_n such that $(c_n,y_n) \in S\, f(u^1_n)$. Since $M + 1/n \geq c_n \geq \hat{c}$ and $y_n \in C(\hat{c})$, there must exist a convergent sub-sequence converging to $(c,y) \in S\, f(u)$, by the closed mapping property of $S\, f(\cdot)$.

Take $C = C(\hat{c})$. Then we have established sub-differentiability with respect to

$$\hat{\phi}_2 = \{\varphi(u) = a - \tfrac{c}{2}\|u - y\|^2 : a \in R;\ 0 < c < M;\ y \in C\}.$$

The set C has the required properties and hence an application of Theorem 3.1 to the function $f : U \to R$, where $U = \bar{N}(\bar{u},\delta)$, establishes all except the equality of $\partial f(\cdot)$ with its sub-gradients .

We now complete our proof by noting that for $u \in \bar{N}(\bar{u},\delta) = U$,

$f(u) = \sup\{\varphi(u) : \varphi(\cdot)$ is a $\hat{\phi}_2$-sub-deriv. of f at $u \in U\}$.

The set $M = \cup \{ S_2 f(u) \; u \in U \}$ is compact since $S_2 f(\cdot)$ is u.s.c. and U is compact . Hence for $u \in U$

$$f(u) = \sup \{ \varphi(\cdot) = a - \frac{c}{2} \|u - y\|^2 : (c,y) \in M \}$$

and an application of Lemma 4.1 gives

$$\partial f(\bar{u}) = \overline{co} \{ \nabla \varphi(\bar{u}) = c(y - \bar{u}) : (c,y) \in S_2 f(\bar{u}) \},$$

where $S_2 f(\bar{u}) = \{(c,y) : (c,y) \in M \text{ and } \varphi(\bar{u}) = f(\bar{u}) \}$.

Using (iii) of Theorem 3.1 , we arrive at

$$\partial f(\bar{u}) = \{c(y - u) : y \in \overline{co}\; S_c f(u)\}$$

$$= \overline{co} \{c(y - u) : (c,y) \in S_c f(\bar{u})\},$$

which concludes the proof . □

REFERENCES

1. F.H. Clarke, Generalized Gradients and Applications, Trans. Amer. Math. Soc. 205 (1975), 247-262.

2. F.H. Clarke, A New Approach to Lagrange Multipliers, Math. of Operations Research 1, No. 2 (1976), 165-174.

3. S. Dolecki and S. Kurcyusz, On Fi-convexity in External Problems, SIAM J. Contr. Optimization 16 No. 2 (1978), 277-300.

4. S. Dolecki, Semi-continuity in Constrained Optimization Part I.1. Metric spaces, Control a Cyber 7 No. 2 (1978), 5-16.

5. K.Kuratowski, "Topology", Vols I and II, Academic Press, New York, 1966 and 1968.

6. C. Berge, "Topological Spaces", MacMillan, New York, 1963.

7. E. Micheal, Continuous Selection I, Ann. Math. 63, No. 2 (1956), 361-382.

8. A. Cellina, Approximation of Set Valued Functions and Fixed Point Theorems, Ann. d. Math. Pur. Applic. Ser.4, No. 82 (1969), 17-24.

9. R.T. Rockafellar, "Convex Analysis," Princeton University Press, Princeton, New Jersey, 1970.

10. R.T. Rockafellar, "The Theory of Subgradients and its Applications to Problems of Optimization, Convex and Non-convex Functions", Heldermann,Berlin, (1981).

11. R.S. Womersley, "Numerical Methods for Structured Problems in Non-Smooth Optimization", Ph. D. Thesis, University of Dundee, 1981.

12. W.W. Hogan, Point-to-Set Maps in Mathematical Programming,S.I.A.M. Reveiw 15, No. 3 (1973), 591-603.

Symbol	Name	first appearance
Φ	Greek u.c phi	p. 1
β	Greek l.c.beta	p. 3
$\langle\ ,\ \rangle$	angle brackets	p. 3
ψ	Greek l.c. psi	p. 3
$\forall$	universal quantifier	p. 3
∇	gradient / nabla	p. 3
$\in$	membership	p. 3
$\subseteq$	class containment	p. 4
∂	del	p. 4
Γ	u.s. Greek gamma	p. 4
$\times$	multiplication	p. 4
ϕ	empty set/l.c. Greek phi	p. 5
$\cup$	set union	p. 6
δ	l.c. Greek delta	p. 8
ϵ	l.c. Greek epsilon	p. 7
$\cap$	set intersection	p. 9
Ω	u.c. Greek omega	p.11
$\triangleq$	definition	p.12
λ	l.c. Greek lambda	p.21
$\exists$	existence quantifier	p.26