A KEYNESIAN CRITIQUE OF RECENT FINANCE AND MACROECONOMIC APPLICATIONS OF RISK-SENSITIVE AND ROBUST CONTROL THEORY

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THESIS ABSTRACT

The objective of this thesis is to assess the strengths and weaknesses of recent economic applications of robust and risk-sensitive control theory from a Keynesian perspective. In particular, I review papers by Anderson, Hansen and Sargent (1999) and Hansen, Sargent and Tallarini (1999) that adopt this theoretical approach in an attempt to overcome certain limitations in the rational expectations literature. The first of these papers constructs a representative agent, permanent income model of optimal consumption-investment under habit persistence. The optimal sequence of consumption streams is then treated as an exogenous endowment within a Lucas-style asset-pricing model (Lucas, 1978).

For control purposes, the authors introduce risk-sensitive value functions based on Epstein and Zin's (1989) recursive utility framework. Theoretical analysis draws on the limiting relationships that hold between risk-sensitive control, *H*-infinity control and risk-neutral control. The solution for the risk-neutral control problem is determined using the Kalman-Bucy filter. The authors then apply the robust optimality conditions to calculate the range of parameter values that are consistent with observed data, while reflecting varying degrees of sensitivity to risk. In an asset-pricing context, the authors show that risk-sensitivity is manifested in the

stochastic discount factors, which can be decomposed into two multiplicative components: representing factor risk and uncertainty, respectively.

The second of the papers follows a similar pattern but is more general, accounting for more complex value functions and stochastic processes, in both discrete-time and continuous-time. In the main I focus, in my critique, on the limitations of this robust control framework in regard to its ability to capture what I see to be the essential aspects of a monetary production economy. These aspects include: the prevalence of nominal, non-indexed contracts; liquidity effects associated with the existence of transactions costs; the need to control and estimate time-varying systems to adequately account for financial instability; and the prospect that dynamic aspects of investment behaviour are governed by complex, non-linear relationships and concern non-ergodic stochastic processes. However, I also question the representative agent assumption on the basis that it precludes the possibility of insufficient effective demand or involuntary unemployment. For scholars working within the Keynesian tradition the existence of the latter phenomena is largely explained on the basis of liquidity preference effects associated with uncertainty. While a risk-sensitive control approach to asset demand can introduce liquidity preference effects, the usual accompanying assumptions of either a pure exchange setting or an exogenously determined accumulation process imply that these effects can exert no substantive influence over the macroeconomy.

In a control framework, I suggest that endogenously influenced fluctuations in the state of uncertainty perception and uncertainty aversion, which I interpret in the form of respective variations in the risk-sensitivity parameter and the norm bounds governing observation error, model uncertainty and external perturbation, are factors that would be difficult to accommodate within existing neoclassically inspired applications of risk-sensitive and robust control theory. I contend that the complexity introduced by the need to model all the relevant aspects of a monetary production economy could not be satisfactorily encompassed within such a framework because the neoclassical logic excludes interactions between environmental factors and preferences such as feedback from increasing financial instability onto the general level of investor uncertainty aversion. In contrast, I argue for a modeling strategy informed by the richer and broader notions of communicative or intersubjective rationality and bounded rationality that are to be found in the respective philosophical works of Jürgen Habermas and Herbert Simon.

DECLARATION

This thesis contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by any other person, except where due reference has been made in the text.

I give consent to this copy of my thesis, when deposited in the University Library, being available for loan and photocopying.

James Juniper

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INTRODUCTION: A WITTGENSTEINIAN PERSPECTIVE ON THE HAYEK-SRAFFA DEBATES

0.1. Introduction

In a recent address to the US Securities Industry Association, one that followed hard on the heels of the Asian and Russian financial turmoil, Alan Greenspan, Chairman of the Federal Reserve Board, made a series of observations about the rise in preference for liquidity on the part of investors. He viewed this rise as one reflected in a disturbing increase in collateral requirements imposed by lenders:

> The surge toward less risky assets reflected dramatic increases in uncertainty, but still a risk differentiation judgment among various assets. The surge toward liquidity protection, however, is a step beyond, since it implies that any commitment is perceived as so tentative that the ability to easily reverse the decision is accorded a high premium. Risk differentiation, despite its recent abruptness, is, of course, a straight forward feature of well-functioning capital markets. The enhanced demand for liquidity protection, however, reflected a markedly decreased willingness to deal with uncertainty--that is a tendency to disengage from risk-taking to a highly unusual degree.
>
> It is, of course, plausible that the current episode of investor fright will dissipate, and yield spreads and liquidity premiums will soon fall into more normal ranges. Indeed we are already seeing significant signs of some reversals. But that leaves unanswered the question of why such episodes erupted in the first place.
>
> It has become evident time and again that when events become too complex and move too rapidly as appears to be the case today, human beings become demonstrably less able to cope. The failure of the ability to comprehend external events almost invariably induces disengagement from an activity, whether it be fear of entering a dark room, or of market volatility. And disengagement from markets that are net long, the most general case, means bids are hit and prices fall (Greenspan, 1998).

At first glance, this fear of entering a dark room and its likely consequences for asset markets would seem to elude the sure grasp of theory. It is one thing to measure and account for the preferences of investors as manifested in the trade-off between risk and return, but it is another thing to account for the consequences of a refusal to contemplate the prospects for a trade-off between risk and return. Nevertheless, one fashionable entry-point for the modeling of uncertainty in financial markets, and the calculation of *uncertainty premia* rather than the more familiar category of *risk premia* that are embodied in the returns on financial assets, is through robust and risk-sensitive control theory.

In this thesis I review recent work by Lars Hansen, Thomas Sargent and their associates (Hansen, Sargent and Tallarini, 1999; and Andersen, Hansen and Sargent, 1999), which introduces uncertainty aversion into what would otherwise be a fairly conventional, recursive, representative agent, intertemporal general equilibrium model of permanent income and asset-pricing. The assets in question are ownership entitlements to the fruits of trees (i.e. claims over an exogenously determined endowment or dividend stream, measured in consumption goods). The endowment stream in question is determined elsewhere in the model through optimal consumption-investment decisions on the part of the representative agent under a habit-persistence mechanism with the underlying production technology (relating gross output to inputs of capital) determined in complete isolation.

Habit persistence is itself represented as *another* form of "production technology": one that relates the output of household services positively to current consumption and negatively to a weighted average of past consumption. Preferences are captured using a particular version of Epstein and Zin's (1989) family of risk-sensitive, recursive utility functions. Epstein and Zin establish that these recursive functions can encompass a variety of non-expected utility theories including those proposed by Kreps and Porteus (1978), Chew (1989), and Dekel (1986).

To accommodate uncertainty, Hansen and Sargent exploit the mathematical parallel existing between the risk-sensitive stochastic control theory, risk-neutral stochastic control theory, and the deterministic, differential game interpretation of robust control theory. In engineering applications of stochastic control theory, uncertainty is represented variously by a norm bound, a sum integral constraint or a relative entropy constraint (see the Technical Appendix). Through this stochastic uncertainty constraint, root mean squared error bounds are imposed over three types of uncertainty: observation error, additive or multiplicative model uncertainty, and external perturbation. In contrast, a relative entropy constraint imposes a bound over the divergence of candidate probability distributions from a reference distribution (the latter is usually of a predefined Gaussian form).

It would seem that this highly technical approach to representative agent modeling would have little in common with the largely discursive debates on monetary theory conducted earlier in the century. However, this notion would be incorrect. In this introductory chapter I review the Hayek-Sraffa debate because the issues raised by its protagonists provide a continuing refrain in economic discussions over subsequent years. In passing, I also review the capital debates that were instigated by Joan Robinson and formalized with the publication of Sraffa's *Production of Commodities by Means of Commodities*. In this review I argue for a Wittgensteinian interpretation of Sraffa's critique of neoclassical theory. In this Wittgensteinian light, I proceed to examine Duménil and Lévy's rejoinder to Frank Hahn's critique of the neo-Ricardian prices-of-production framework: an examination that puts the tâtonnment process under the critical spotlight and sets the scene for a somewhat selective review of adjustment processes in the rational expectations, dynamic disequilibrium and adaptive belief equilibrium literature. I briefly consider two approaches to questions of long-period disequilibrium, before arguing for a return to a Keynesian dynamic analysis of fluctuations in uncertainty, long-term expectations, and liquidity preference.

The debates occurring between European monetary theorists in the late 20s and early 30s raised fundamental issues in monetary theory, many of which are still largely unresolved. Most of these early works, including Myrdal's *Monetary Equilibrium (1939)*, Hayek's *Monetary Theory and the Trade Cycle* (1935a) and *Prices and Production* (1935b), and Keynes's *Treatise on Money* (1930) were essentially predicated on Wicksellian capital theory and the unquestioned notion of a natural rate of interest. The natural rate was viewed as being determined through interactions between the real forces of productivity and thrift and was therefore seen to be independent of any monetary influences [1].

[1] However, as Desai points out, while Hayek found Wicksell's concept of the natural rate useful he argued that "...the natural rate of interest could not be at the same time one that would equilibrate the supply of real savings and the demand for capital, and one that would prevail if the transactions were carried on *in natura* in a barter economy, i.e. the rate at which the price level would be stable" (p. 152).

Simultaneously, this period witnessed the seminal development and application of techniques of dynamic optimisation to economic growth based on the calculus of variations (e.g. as reflected in the famous Ramsey-Keynes golden rule). These early techniques continue to provide a framework for much of the new growth theory and also the representative agent models of macroeconomic dynamics (see Barro and Sala-i-Martin, 1995, on the former and Turnovsky, 1995, on the latter). Until the mid-fifties, for reasons of tractability, this technical literature did not advance beyond either the one-sector or the balanced growth model and largely ignored the role of money, thereby putting to one side the complexities associated with risk and uncertainty.

Keynes, in his transition from the *Treatise on Money* to *The General Theory* (1936), more or less abandoned many of the cornerstones of Wicksellian capital theory, rejecting in turn the notion of the natural rate of interest and the quantity theory of money (in either its Cambridge or Fisherian modes). And even the *Treatise* confined itself to explaining the disequilibrium process of short-run price, income and asset adjustment as the economy moved from one equilibrium position to another, under the assumption that the quantity theory of money still applied in the long run. Arguably, each of these conceptual legacies was responsible to a varying degree for hindering the development of monetary theory in the early part of this century.

One of the first critical volleys on Wicksellian capital theory was fired during the heated Hayek-Sraffa debate. This debate raised certain issues which were to remain the subject of contestation over the decades to follow. The following section of the paper examines these issues, dwelling in particular on Sraffa's comments about the relationship which obtains between asset-prices as the economy undergoes a transition from one monetary equilibrium to another.

0.2. The Hayek-Sraffa Debate

Early contributions to this critique of the Wicksellian heritage included those of Piero Sraffa and Richard Kahn. Sraffa's first theoretical volleys included his original work on increasing returns to scale and his severe critique of Hayek's writings on monetary theory and the trade cycle. The Hayek-Sraffa debate has recently been surveyed by Meghnad Desai (1982). This paper will not attempt to improve upon Desai's lucid coverage of this heated debate. Desai convincingly argues that Sraffa's critique of Hayek's work foreshadowed much later, and more extensive, critical inquiry into the foundations of neoclassical capital theory in *Production of Commodities by Means of Commodities* (1960). This comes through most clearly in Sraffa's attack on the natural rate of interest. Sraffa cites Hayek's use of this notion in the following passage:

> In a money economy, the actual or money rate of interest may differ from the equilibrium or natural rate, because the demand for and the supply of capital do not meet in their natural form but in the form of money, the quantity of which is available for capital purposes may be arbitrarily changed by the banks (Hayek, 1932, pp. 20-1).

To set the scene, Sraffa begins to set out his own contrasting perspective by making the distinction between a monetary and a non-monetary economy:

> If Dr. Hayek had adhered to his original intention, he would have seen at once that the difference between a monetary and non-monetary economy can only be found in those characteristics which are set forth at the beginning of every text-book on money.

> That is to say, that money is not only the medium of exchange, but also a store of value, and a standard in terms of which debts, and other legal obligations, habits, opinions, conventions, in short all kinds of relations between men, are more or less rigidly fixed (Sraffa, 1932, p. 43).

In reference to the natural rate, Sraffa argues that Hayek confusingly identifies 'actual' rates with 'money' rates and 'equilibrium' rates with 'natural' rates. However:

> If money did not exist, and loans were made in terms of all sorts of commodities, there would be a single rate which satisfies the conditions of equilibrium, but there might be at any one moment as many "natural" rates of interest as there are commodities, though they would not be "equilibrium" rates (Sraffa, p. 49).

At this point, to clarify the notion of an own rate of interest, Sraffa introduces the example of a forward market for cotton:

> In equilibrium the spot and forward price coincide, for cotton as for any other commodity; and all the 'natural' or commodity rates are equal to one another, and to the money rate. But if, for any reason, the supply and demand for a commodity are not in equilibrium (i.e. its market price exceeds or falls short of its cost of production), its spot and forward prices diverge, and the 'natural rate of interest on that commodity diverges from the 'natural' rates on other commodities. [...] It is only one step to pass from this to the case of a non-money economy, and to see that when equilibrium is disturbed, and during the time of the transition, the 'natural' rates of interest on loans in terms of the commodities the output of which is increasing must be higher, to various extents, than the 'natural' rates on commodities the output of which is falling; and that there may be as many 'natural' rates as there are commodities (Sraffa, 1932, p. 50).

It must be acknowledged that Hayek was, himself, moving away from the Wicksellian tradition in an attempt to lay the foundations for an intertemporal theory of monetary general equilibrium[2]. This was only to be fully accomplished in 1939, with the publication of Hicks' path-breaking work, *Value and Capital,* which precisely defined the concept of temporary equilibrium and began to analyse its methodological consequences[3]. Full consideration of this matter would have to be the subject of another thesis in which the complex issues raised by the capital debates would have to be reviewed (on these matters, see Rogers, 1989). Instead, I will confine my interests in this chapter specifically to conjectures about: 1) the difference between short-period and long-period monetary equilibrium, and 2) processes of adjustment between one equilibrium and another (see sections 5, 7 and 8 below).

0.2.1. Sraffa's Return to the Fray and the Capital Debates[4]

In his book, *Production of Commodities by Commodities*, Piero Sraffa clarified many of the notions that underpinned his earlier critique of Hayek's trade-cycle theory. In particular, he established that the choice of technique, made in response to a cheapening of the rental price of capital relative to the wage rate, would not necessarily lead to a rise in capital intensity[5]. Such a departure from the conventional assumption of a monotonic relationship between the value

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[2] This is an argument made by Roy McCloughry's note on Desai in the same volume from a perspective more sympathetic to Hayek's position (McCloughry, 1982:Chapter 10).

[3] While neo-Walrasian approaches evade some of the criticisms leveled at Wicksellian capital theory (see footnote 2), particularly those associated with the phenomenon of reswitching, they are exposed to criticism on the grounds of possible multiple equilibria and also capital deepening perversities (see Rogers, 1989 or Dougherty, 1980).

[4] As the capital debates have been widely discussed this overview will be brief.

[5] Capital deepening is the process in which capital intensity of techniques rise with a rise in the relative rental price of capital services. Reswitching occurs where a capital intensive technique may be favoured at both low and higher interest rates, but replaced by less capital intensive techniques at middle range levels of the interest rate. For an interesting historical account of the differences between Sraffa and Wicksell over the issue of capital reversal, see Ferguson and Hooks (1971).

marginal productivity of capital and the cost of investment funds is a damaging blow to those who would resort to the use of aggregate-production functions to represent the real side of a monetary production economy[6].

Pasinetti's *Lectures on the Theory of Production* distills Sraffa's insights through the abstract lens of linear algebra. Pasinetti commences with a simple *n* sectoral model, where each sector combines labour and circulating capital to produce a single good either for purposes of consumption or usage as capital. In this case the pricing equations for each sector can be represented in matrix notation as $\mathbf{p}^* = (1 + r)\mathbf{A}\mathbf{p}^* + w\mathbf{l}$. By recursively substituting for the price term appearing on the right hand side, this equation can be written in the form of an infinite series[7]:

$$\mathbf{p}^* = w\mathbf{l} + (1+r)w\mathbf{A}\mathbf{l} + (1+r)^2 w\mathbf{A}^2\mathbf{l} + \mathrm{K} + (1+r)^n w\mathbf{A}^n\mathbf{l} + \mathrm{K}$$

Here, each of the $\boldsymbol{A^j I}$ terms represents the quantity of labour expended in the production of the economy's commody output at step *j* in the iterative process of producing commodities by means of other commodities. In this (single product per sector) case, Sraffa shows that a unique set of non-negative multipliers exists that can be applied to the original pricing equation, to generate a hypothetical system whose product consists of the same composite commodity as its means of production (see Kurz, 1985, pp.17-8). This system is called the *standard system* and associated with it is the composite or *standard commodity*: an entirely artificial construct that has the characteristic that when the wage is measured using the standard commodity as a numeraire, the

[6] This is no less the case for followers of the New Growth Theory as well as those like William Brock, who build asset-pricing models on micro-foundations utilising multi-sectoral, but still *single-good* production-functions.

rate of profits can be expressed in the form $r = R(1 - w_s)$, where R is the maximum rate of profits and w_s is the standardized wage[8]. The determination of the standard commodity resolves an outstanding riddle, first posed by David Ricardo, as to the feasible construction of a standard whose value would be invariant to changes in the distribution of income between wage earners and profit-takers[9].

Following Sraffa, Pasinetti shows that this system possesses a finite maximum rate of profit $R > 0$ corresponding to the zero-wage. In addition, $\mathbf{p}^*/w$, the vector of prices in terms of the wage rate (i.e. in terms of *labour-commanded*) is positive and rises monotonically for $0 \leq r < R$, tending to infinity as r approaches its maximum level. At the maximum level of wages corresponding to the zero rate of profit, relative prices are proportional to their labour cost. For values of $r > 0$, prices deviate from their "embodied labour" values to a different extent within each industry, unless there are equivalent proportions of labour to means-of-production across all industries. Hence, the capital-to-labour and capital-to-output ratios cannot be ranked independently of the distribution of income.

When more than one technique is available for a given industry, Sraffa shows that a different economy-wide wage curve $w = w(r)$ will exist for each of the feasible techniques. For example, the following diagram depicts the case where two techniques are available, α and β, where w and

[7] The $\mathbf{p}^*$ notation is used here for prices of production to distinguish them from the undiscounted and discounted equilbrium prices to be introduced later.
[8] In a 1987 paper co-authored with Neri Salvadori, Kurz defends Sraffa's use of the standard commodity against criticism emanating from one of its main neoclassical detractors, Edwin Burmeister (1980, 1984). An earlier paper by Kurz and Hagemann (1976) defends the neo-Ricardian position against interpretations by John Hicks and certain neo-Austrian scholars.

p are expressed in terms of a commodity that is produced in both systems. Given the shape for each of the wage-curves, when

$0 < w < w_1$ then technique α will be chosen by a profit maximising firm. However, when $w_1 < w < w_2$, then technique β will be chosen. Finally, as wages rise again into the region $w_2 < w < W_\alpha$, then technique α will once again be preferred. At the switchpoints P and Q, both techniques will be equi-profitable. This *reswitching* phenomenon confirms that input proportions are not related monotonically to changes in relative factor prices. Note that W_α and W_β represent the net product per worker implying the existence of a higher capital stock at each switch point for technique α[10].

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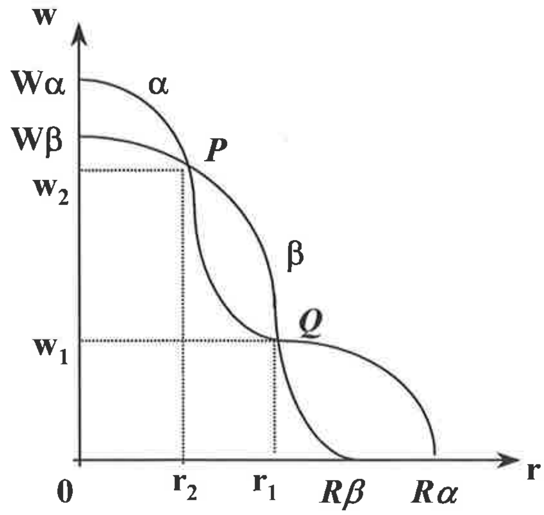
[9] Shefold (1980) discusses the conditions under which the construction of such a standard can be accomplished in the more complex case of joint-production. It should be noted that in the following section of the paper I question the traditional interpretation of Sraffa's resolution of the Ricardian invariant standard of value problem.

[10] In 1953, Champernowne proposed the construction of a *chain index* as one solution to the problem of measuring the capital employed. His proposal required the determination of a set of weights at each intersection between the factor price contours, with weights given by the relative slopes of each contour at the point of intersection. However, Champernowne's solution breaks down because reswitching will occur at each of the intersection points as the relative prices of the two types of capital vary in the move from one intersection point to another (see Nagatani, 1981, pp. 210-2).

Fig. 0: THE RESWITCHING PHENOMENON



In conclusion, the capital debates call for a rejection of the naïve, aggregative parables derived from the marginal productivity theory of income distribution[11]. This raises the question of whether neo-Walrasian general equilibrium theory remains immune to such criticism. However, as demonstrated by Christopher Dougherty, an advocate of the neo-Walrasian approach, the neoclassical alternative gives rise to multiple equilibria. The actual outcome will vary depending on the initial (intertemporal) resource endowments of economic agents[12]. This is because there

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[11] Kurz (1985) has noted that, "…much of the debate on capital theory fell short of the conceptual richness of Sraffa's analysis." In particular, many of the heuristic models ignored the complexities associated with joint-production, the rental of land, and the fact that optimal rates of depreciation will also depend on the distribution of income (for the joint-production case, see the collection of essays in Pasinetti (ed.), 1980). The issue of ground rent is given an excellent treatment in Kurz (1978). This latter dependency arises because the price of a fixed capital good, determined by discounting future streams of net receipts at the ruling rate of profits, may become negative:

> [m]ore particularly, it is possible that with a complex pattern of the time-profile of productivity, maintenance and repairs the same length of economic life of an instrument is optimal at disconnected intervals of values of the rate of profits, while different lengths are advantageous in between. Here we encounter a variant of the reswitching phenomenon, i.e. the return of the same truncation period. (Kurz, 1985, p. 21).

[12] Multiple equilibria arise because the offer curves (in interest rate space) of agents engaging in borrowing or lending are determined by the interaction of intertemporal exchange (borrowing and lending current entitlements to

are effectively too many degrees of freedom for adequate closure in the Neo-Walrasian model (Dougherty, 1980 & Rogers, 1989, Chapter 3).

0.2.2. A Wittgensteinian Reading of Sraffa

A recent "Wittgensteinian" interpretation of Sraffa's 1960 critique has convincingly argued against the view that Sraffa was a neo-Ricardian theorist of the long-period or prices of production as "centres of gravity" persuasion (Andrews 1996, pp. 763-77). Andrews suggests that Sraffa adopted the standard commodity as a prelude to a critique of neo-classical theory in the following sense: the standard commodity operates as a standard of value due to the invariance of its money price despite discrete jumps in the ratio of wages to the rate of profit on capital. However, it can only be defined under the assumption that the composition of output is held constant (the infamous "snapshot" assumption). Any variation in returns to scale would lead to instantaneous variations in the eigenvalues which determine the standard commodity's weighting of sectoral shares, thereby undermining the very notion of a standard which could be invariant to changes in both the composition of output *and* the ratio of wages to profits. Disequilibrium adjustments only add to the impossibilities associated with determining such an invariant standard of value.

To "put the cat amongst the pigeons", Andrews contends that this destruction of the possibility of an invariant standard of value not only operates as a critique of the neoclassical theory of income distribution, but also acts to undermine the foundation for any essentialist, long-period notion of

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consumption), production (the response of future output to current investment activity) and consumption (preferences over current and future levels of consumption). As a result, the offer curves can exhibit many turning prices-of-production as centres of gravity, around which actual prices would be constrained to fluctuate. Andrews suggests that this anti-essentialist Sraffian critique (with obvious philosophical affinities with the perspective of Wittgenstein in the *Philosophical Investigations*) is closer to a more dialectical Hegelian Marxist view of the economic process than a long-period neo-Ricardian view. On Andrews's reading of Sraffa, continual adjustments in the composition of demand, changes in the distribution of income between economic classes, and variations in both the rate of technological change and the degree to which economies of scale are exploited undermine the possibility of constructing or finding any stable standard of value—either natural or artificial[13]. In common with the arguments of certain fundamentalist Keynesians, Andrews suggests that, in the absence of such a standard, the *real* value of any type of capital or financial asset-price or rate of return cannot be defined or calculated, let alone ranked in an order which would be invariant with respect to changes either in income distribution or the composition of output. Of course, we have seen that an invariant ranking of techniques by capital intensity is necessary for the derivation of a surrogate production function or chain index of capital, if orthodox theorists are to evade the difficulties raised by reswitching or capital deepening (see Harcourt 1972, Chapters 1 and 4).

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points, which yield multiple equilibria at points of intersection with the offer curves of other agents.

[13] Of course, one way of dealing with the issues of long-period changes in the composition of demand and the sectoral structure of productivity growth differentials is to treat such changes as exogenous trends. Pasinetti (1981) effectively adopts this strategy in his book on structural change and economic dynamics. An approach of this kind makes a valuable contribution to the classically motivated literature on the long-period pricing of commodities. However, I would argue that although this sort of analysis provides useful insights into the structural aspects of the accumulation and reproduction process, it does not describe the "living reality" of the accumulation process in which actual variations in rates of technological change and the composition of demand are both highly erratic and also very difficult to forecast. Pasinetti's approach consists of a logical rather than a descriptive-historical analysis.

0.2.3. HAHN'S CRITIQUE OF THE NEO-RICARDIANS AND DUMÉNIL AND LÉVY'S REJOINDER

It is now appropriate to return to the issue of determining the relationship holding between own-rates and monetary rates of return. In a celebrated paper, Frank Hahn contended that the neo-Ricardian model was a special case of the neo-Walrasian model: one in which the vector of endowments was set to the value required to achieve a uniform rate of profit when expressed in terms of classical prices of production. In their rejoinder, Duménil and Lévy demonstrate that prices of production operates as centres of gravity for an intertemporal general equilibrium: i.e. the intertemporal equilibrium over an infinite horizon generally converges towards prices of production as its asymptotic limit. The Tâtonnment process then becomes both the major distinguishing feature of neoclassical analysis and its "Achilles Heel".

Duménil and Lévy (1985) follow Hahn in considering a simple infinite-horizon, general intertemporal equilibrium model with constant returns to scale, one non-reproducible resource and one capitalist consumer. The discounted price vector p_0 for the n goods at the beginning of the period negotiated at the origin of the period is $p_0 = (p_0^1, p_0^2, ..., p_0^n)$ and, similarly, for the n goods in each time period t is $p_t = (p_t^1, p_t^2, ..., p_t^n)$. A fixed coefficient production technology featuring a single technique for each good is denoted by the square matrix $\mathbf{A'}$ and the row vector of direct labour inputs is denoted by $\mathbf{L}$, with wages given by $\mathbf{d}$ the purchasing power of wage earners. The wage sector is absorbed within the sociotechnical matrix $\mathbf{A} = \mathbf{A'} + \mathbf{d} \otimes \mathbf{L}$, which now expresses the total amount of each good required for production, either directly as an input into production or indirectly as part of wage-earners consumption (p. 330). Enterprises maximize the sum of discounted profits for each period:

$$\Pi^i = \sum_{t=1}^{\infty} \Pi^i_t = \sum_{t=1}^{\infty} Y^i_t \pi^i_t$$

where

$$\pi^i_t = p^i_t - \left(\mathbf{p}_{t-1}\mathbf{A}'^{\mathbf{i}} + L^i w\right) = p^i_t - \mathbf{p}_{t-1}\mathbf{A}^{\mathbf{i}}.$$

With constant returns to scale, in equilibrium every unit profit must be less than or equal to zero. When unit profits are negative activity levels must be equal to zero. The authors only address the case where activity levels in each sector are strictly positive for

i = 1, 2,...n and for t = 1, 2,... so that the price vector **p** can be calculated iteratively from the zero profit conditions (p. 332):

$$\pi^i_t = p^i_t - \mathbf{p}_{t-1}\mathbf{A}^{\mathbf{i}} = 0; \quad i = 1,2,\ldots,n; \quad t = 1,2,\ldots$$

as a function of

$\mathbf{p}_0$ (i.e. $\mathbf{p}_t = \mathbf{p}_0 \mathbf{A}^t$)

Consumer behaviour is represented by an intertemporal utility function $U(c_0, c_1, c_{2,\ldots})$, with an initial endowment ω. In equilibrium consumers are at an optimum determined by:

$$\max_{c_0, c_1, c_2, \ldots} U(c_0, c_1, c_2, \ldots) \, s.t. \, \mathbf{p}_0 \omega = \sum_{t=0}^{\infty} \mathbf{p}_0 (\mathbf{A})^t \mathbf{c}_t$$

If U possesses appropriate properties, then this programme yields the solution: $\{\mathbf{c}_0(\mathbf{p}_0), \mathbf{c}_1(\mathbf{p}_1), \ldots\}$. Equilibrium is defined by the set of prices and quantities such that individual consumer optimization is compatible with the existence of supply larger or equal to demand for each commodity at each period (p. 332):

$$\begin{cases} \omega \geq \mathbf{AY}_1 + \mathbf{c}_0 \\ \mathbf{Y}_1 \geq \mathbf{AY}_2 + \mathbf{c}_1 \\ \vdots \\ \mathbf{Y}_t \geq \mathbf{AY}_{t+1} + \mathbf{c}_t \\ \vdots \end{cases}$$

Along with the zero profit conditions, these equations form an infinite system of equations with a double infinity of unknowns—the sequences p_t, $\mathbf{Y}_t$. The former sequence can be expressed in terms of prices $\mathbf{p}_0$ and the latter $\mathbf{Y}_t$ sequence can be eliminated by multiplying the first of the above inequalities by $\boldsymbol{I}$, the second by $\mathbf{A}$,..., the tth by $(\mathbf{A})^t$, etc. and summing together to yield the vector inequality (p.333):

$$\omega \geq \sum_{t=0}^{\infty} (\mathbf{A})^t \mathbf{c}_t(\mathbf{p}_0)$$

which, under certain regularity conditions over the utility function, can be shown to have a strictly positive solution in $\mathbf{p}_0$: $\mathbf{p}_0 = \mathbf{p}_0$ ($\mathbf{A}$, U, $\boldsymbol{\omega}$). Equality then prevails in the equilibrium market clearing constraints which can be used along with the above vector inequality to calculate $c_t(\mathbf{p}_0)$ and $\mathbf{Y}_t$.

Next, following the precedent set by Malinvaud (1972), Duménil and Lévy introduce the notions of the numeraire and undiscounted prices. A numeraire $\mathbf{N}$ is a bundle of n commodities in given proportions that can be chosen to normalize the general level of discounted prices, as in: $\mathbf{p}_0\mathbf{N} = 1$, independently of the period in which they occur. Alternatively, undiscounted prices $\overline{p}_t'$, can be normalized in every period such that: $\overline{\mathbf{p}}_o\mathbf{N} = \overline{\mathbf{p}}_1\mathbf{N} = \overline{\mathbf{p}}_2\mathbf{N} = \ldots\ \overline{\mathbf{p}}_t\mathbf{N} = 1$. Discounted prices can then be transformed into undiscounted prices through division by the price of the numeraire in discounted terms i.e.:

$$\overline{p}_t^i = \frac{1}{\mathbf{p}_t\mathbf{N}} p_t^i.$$

Deploying these definitions, Duménil and Lévy (p. 334) introduce ρ_t^i, the *own rate of interest* of good *i*: $p_t^i = (1+\rho_t^i)p_{t+1}^i$ and, in addition, the *rate of interest* ρ_i, applied to the numeraire:

$$1+\rho_t = \frac{\mathbf{p}_t\mathbf{N}}{\mathbf{p}_{t+1}\mathbf{N}}$$

Finally, they define the undiscounted price analogue to the own rate of interest, j_t^i which they call the *rate of variance*: $\overline{p}_{t+1}^i = (1+j_t^i)\overline{p}_t^i$.

This enables them to identify the relationship between changes in discounted prices from one period to the next:

$$p_{t+1}^i = p_t^i \frac{1+j_t^i}{1+\rho_t}.$$

Here, the rate of variance in the numerator of the second term accounts for the change in relative prices while the rate of interest in the denominator accounts for the discount effect. Finally, the *diachronic rate of profit* r_t^i, measured in undiscounted prices is given by:

$$1+r_t^i = \frac{\overline{p}_{t+1}^i}{\overline{\mathbf{p}}_t\mathbf{A}^i}.$$

Armed with these definitions, Duménil and Lévy show that, given the wage rate $\boldsymbol{\omega}^*$, there are three ways to locate prices of production **p*** and uniform *synchronic rate of profit* **r***, in the context of an intertemporal equilibrium. The first of these is identified by Hahn, the second by Malinvaud, and the third by the authors themselves. To establish the basis for Hahn's arguments, Duménil and Lévy introduce the familiar prices of production equation as expressed in:

$$\mathbf{p}^* = (1 + r^*)(\mathbf{p}^* \mathbf{A}' + \mathbf{L}\omega^*).$$

Using the fact that $\boldsymbol{\omega}^* = \mathbf{p}^*\mathbf{d}$, this equation can be written as:

$$\mathbf{p}^* \mathbf{A} = \frac{1}{1 + r^*}\mathbf{p}^*.$$

Thus, the vector of prices of production is the eigenvector of the matrix **A**, associated with the eigenvalue $1/(1 + r^*)$. Duménil and Lévy observe the existence of this eigenvector $\mathbf{p}^* = \mathbf{p}^*(\mathbf{A})$ and its semipositivity are guaranteed by the Perron-Frobenius theorem, which can also be used to establish that $1/(1 + r^*)$ is equal to $\lambda(\mathbf{A})$, the eigenvalue of **A** with the largest modulus (that must be less than one if the system is capable of reproduction). Significantly, the vector of prices of production is independent of endowments and the utility function. Duménil and Lévy first observe that there is no a-priori reason for the vector of discounted prices $\mathbf{p}_0$ to equal **p***. They interpret Hahn to be advancing the argument that a particular vector of endowments $\boldsymbol{\omega}^*$ must exist such that: $\omega^* = \sum_{t=0}^{\infty} (\mathbf{A})^t \mathbf{c}_t(\mathbf{p}^*)$.

However, because there is no way to guarantee that this particular vector of endowments will prevail, prices of production (and by association, the whole neo-Ricardian theoretical edifice) are completely irrelevant. To counter this claim, the authors show that an alternative approach can be taken. First they establish Malinvaud's claim that in intertemporal equilibrium the diachronic rate of profit is uniform and equal to the rate of interest, as shown in the following (p. 336):

$$1+r_t^i=\frac{\overline{p}_{t+1}^i}{\overline{\mathbf{p}}_t\mathbf{A}_i}=\frac{p_{t+1}^i\big/\mathbf{p}_{t+1}\mathbf{N}}{\mathbf{p}_t\mathbf{A}_i\big/\mathbf{p}_t\mathbf{N}}=\frac{p_{t+1}^i}{\mathbf{p}_t\mathbf{A}_i}\frac{\mathbf{p}_t\mathbf{N}}{\mathbf{p}_{t+1}\mathbf{N}}=1+\rho_t,$$

which follows from the definitions of the rate of profit, undiscounted prices, the equilibrium value of the unit profit, and the definition of the rate of interest. Both the diachronous rate of profit and rate of interest depend on the numeraire but if the latter changes, each variable must change in such a way that the above identity between the two variables is preserved. Note that synchronic rates of profit are not equalized under the manipulations that yield the above expression. However, Duménil and Lévy show that an equality can be established between the neoclassical rate of profit and classical superprofits $\overline{s\pi}_t^i$, that are derived by deducting normal profit $\rho_{t-1}\overline{\mathbf{p}}_{t-1}\mathbf{A}^i$ from total classical profit $\overline{p}_t^i-\overline{\mathbf{p}}_{t-1}\mathbf{A}^i$, as in:

$$\overline{s\pi}_t^i=\overline{p}_t^i-(1+\rho_{t-1})\overline{\mathbf{p}}_{t-1}\mathbf{A}^i=\frac{\pi_t^i}{\mathbf{p}_t\mathbf{N}}$$

where the last equality follows from the definitions of undiscounted prices and the rate of interest. Thus the well known *zero neoclassical rate of profit condition* is equivalent to the *zero classical super profits condition.*

Finally, Duménil and Lévy (p. 337) reproduce the findings of research by Dana and Lévy (1984) that, over an infinite horizon:

(1) the ratio of each discounted price of the intertemporal equilibrium to the corresponding price of production divided by $(1 + r^*)$ tends towards a constant C_1 as in:

$$\mathbf{p}_t(\mathbf{A}, U, \omega) \approx \frac{C_1}{(1+r^*)^t}\mathbf{p}^*(\mathbf{A}) \text{ if } t \to \infty$$

(2) undiscounted prices tend towards prices of production (multiplied by a constant C_2) as in:

$$\lim_{t\to\infty} \overline{p}_t(\mathbf{N}, \mathbf{A}, U, \omega) = C_2 \mathbf{p}^*(\mathbf{A})$$

(3) the rate of interest and all the rates of profit tend towards r^* as in:

$$\lim_{t\to\infty} \rho_t(\mathbf{N}, \mathbf{A}, U, \omega) = \lim_{t\to\infty} r_t^i = r^*(\mathbf{A})$$

Significantly, because the vector of prices of production is independent of endowments or utility, these factors exert less and less influence over the equilibrium outcomes for prices, profits and interest as the adjustment process continues. Duménil and Lévy state that the proof of these relationships is based on the equation for equilibrium prices,

$\mathbf{p}_t = \mathbf{p}_0\ \mathbf{A}^t$, is a direct consequence of a well-known theorem on non-negative matrices, and concerns the conditions of existence of the following limit:

$$\lim_{t\to\infty}\left(\frac{\mathbf{A}}{\lambda(\mathbf{A})}\right)^t$$

In the concluding sections of their paper Duménil and Lévy suggest that dynamic analysis in the Walrasian intertemporal model can take two forms: it can either address the tâtonnment process, or it can trace the outcome of the already accomplished tâtonnment as the economy advances from one period to the next. However, unlike their classical centre-of-gravity counterparts, neoclassical theories impose the implausible requirement that at each point in time, the diachronic rate of profit is equal to the rate of interest. No such equilibrium condition is required by the classical analysis, which only operates tendentially.

Nevertheless, Andrew's (1996) Wittgensteinian reading of Sraffa that I discussed in section four, serves to question Duménil and Lévy's neo-Ricardian analysis. Once the possibility of non-constant returns to scale and technological change is acknowledged, the vector of prices of production, itself, must vary over time in what has the potential to be a highly complex manner. Moreover, the **A** matrix that plays such an important role in the asymptotic analysis would vary with changes in the distribution of income, as reflected in the **d** vector, a fact that Ricardo tried to overcome through the construction of the standard commodity.

0.3. The Vexatious Issue of Dynamic Adjustment

Duménil and Lévy admit that classical dynamics is underdeveloped, an admission that may not go far enough. The following section of the paper looks at the issue of dynamics in the macroeconomic literature and finds it to be wanting.

0.3.1. SHORT-PERIOD ADJUSTMENT: DISEQUILIBRIUM MODELS OF MONETARY GROWTH

Applied economic modellers have been tempted to introduce ad-hoc processes of adjustment that represent the efforts of an economy to achieve dynamic equilibrium between supply and demand. Effectively, they introduce estimable lag relationships that enable prices to change in the direction required to gradually eliminate any excess supply or demand in the particular market that is being represented. These mechanisms are the empirical counterpart to the excess demand functions that motivate proofs of the existence (via fixed point theorems or the set-theoretic equivalents) and stability (via, say, the application of the relevant Liapunov stability theorem) of intertemporal general equilibrium. Enormous effort, for example, has gone into the theoretical elaboration and estimation of Phillips Curve relationships in labour.

Recently, Chiarella and Flaschel (1999) have argued for a revival of the dual Phillips Curve approach in macro-theory. On logical grounds, this approach transfers the conventional Phillips Curve (PC) mechanism from the labour market into the physical capital goods market. However, the ultimate objective of their research is to question the conventional rational expectations analysis of saddle-point dynamics. Under the rational expectations approach, adjustment processes are explicitly modelled (e.g. through the imposition of linear quadratic costs of adjustment and quadratic penalty terms reflecting the cost of deviations away from the decision-maker's desired position), and it is presumed that policy announcements or exogenous shocks to key parameters lead to movements in the saddle-path that leads to the steady-state. Jump variables then move instantaneously to place the representative economic agent at a point that will gradually take her onto the requisite saddle-path in accordance with the posited adjustment mechanisms.

Elsewhere, Chiarella (1990) has shown that for limit cycles—in the limiting case where adaptive expectations become equivalent to perfect foresight—an economic system will, instead, move in a complex saw-tooth trajectory, oscillating between instantaneous jumps and gradual adjustments. In this context, Chiarella and Flaschel (1999) initially observe that a minimal set of assumptions guarantee the existence of (overshooting) limit cycles in real wages and the rate of employment within the economy at large. First, the real wage PC that relates the rate of growth of real wages with the rate of employment, must determine a unique Non-Accelerating Inflation Rate of Utilization (NAIRU) of the labour force. Second, the PC exhibits negative values below (to the left of) this NAIRU, implying falling real wages. Third, the PC exhibits positive values above (to the right of) this NAIRU, implying rising real wages. Corresponding to this real wage PC and associated NAIRU is an accumulation curve (AC) associating positive variations in the rate of employment to the level of real wages—a Non-Accelerating-Growth Rate of Wages (NAGRW). Once again, to the left of the NAGRW level of real wages the change in the rate of employment is positive, while to the right of NAGRW the change in the rate of employment is negative. The authors apply a heuristic Liapunov function analysis to show the presence of a global sink and constancy of level curves—the conditions for a limit cycle.

To this basic Goodwin model, the authors follow Steve Keen (1999) in adding a simple debt-deflation process. Firms are assumed to employ both retained earnings and loans from asset-holding households to finance their investments. The excess of their nominal investment expenditures over pure profits are financed by these new loans. Investment activity, itself, is assumed to be (linearly) sensitive to increases in the pure rate of profit above a threshold value. Chiarella and Flaschel (1999) show that this debt financed investment makes the Goodwin growth cycle convergent for small shocks to the ratio of debt to capital but it is asymptotically globally unstable for sufficiently large shocks, from an economically feasible starting position.

Next, Chiarella and Flaschel (1999) introduce their goods market version of the PC. This extension also introduces a price level variable into the debt accumulation equation because capital must now be measured in nominal terms. Associated with this new PC is a second non-accelerating inflation rate of utilization curve, where the utilization rate now refers to the capital stock. A simple fixed proportions approach is taken to technology, at this stage, to link the rate of employment to potential output-to-capital ratio of the economy. Under varying parameter values, the model gives rise to either locally asymptotically stable dynamics, locally asymptotically unstable or even explosive dynamics (for real wages above their steady-state values and sufficiently high initial debt to capital ratios). They observe, with some irony, that sufficiently sluggish price level adjustments are favourable to local asymptotic stability, while sufficiently flexible price levels are destabilizing. The full-blown, closed-economy version of their model adds a Metzler type inventory cycle, and makes investment sensitive to not only the pure rate of profit but also the real rate of interest and the rate of capacity utilization. In this case, under varying parameter values the dynamics are very complex, potentially chaotic, and must be analyzed using techniques that have been drawn from the physical sciences, and which examine supercritical and subcritical regions of stability and instability for each of the three Hopf bifurcations arising within the model. Their important research has clearly identified the potential for dynamic adjustment procedures, especially those that are too responsive, to give rise to turbulence and chaotic dynamics.

0.3.2. SHORT-PERIOD ADJUSTMENT: ADAPTIVE BELIEF SYSTEMS AND CHAOTIC DYNAMICS

Another approach to the modeling of turbulent processes of market adjustment is captured in the literature on *adaptive belief systems* (Brock and Hommes, 1997, 1998; Brock 1999). The first of these papers, chronologically speaking, introduces an evolutionary dynamics into a simple cobweb model of supply and demand. In this linearized cobweb economy, sophisticated rational predictors such as those associated with rational expectations are presumed to be more costly to obtain than extrapolative and myopic predictors such as those associated with adaptive expectations. However, the relative accuracy of the more sophisticated predictors increases at positions further away from the rational expectations steady state. Nonlinearity is introduced into what would otherwise be a linear equilibrium model because those agents using the rational expectations predictors must know the relative proportions of each type of agent—those like themselves using rational, and those using adaptive predictors—in calculating the rational expectations trajectory. The second of the papers applies much the same approach to a model of asset-pricing, but allows for up to four different types of trader. The third paper extends this asset-pricing approach by introducing a performance measure of trading strategies based on *accumulated* net profits. In addition, it allows for a wider variety of heterogeneity in predictors.

While agents agree on the conditional volatility of one-period returns and the expected dividend process, heterogenous beliefs are represented by uncertainty over the conditional drift of fundamental prices (where the conditional asset-price adds to the rational expectations predictor a time stationary functional of past deviations from a commonly shared view of the fundamental).

The approach adopted in this last paper is justified on the basis of Bollerslev, Engle and Nelson's (1994) findings on continuous record asymptotics. Brock and Hommes demonstrate (p. 115) that as a result, the one-period realized excess return can be decomposed into two terms. One of these is a martingale difference term (as in conventional expositions of the efficient market hypothesis) and another is the term $x_{t+1} - Rx_t$, where R is the risk-free rate of interest and x_t is the deviation of the asset-price from its fundamental value. In a related study, Brock, Lakonishok, & LeBaron (1992) parameterize this martingale difference term using either a GARCH or E-GARCH model and use bootstrapping techniques to test for the presence of the extra term, whose existence is confirmed in their data. Brock and Hommes (1999, section 3) ignore the martingale difference term and analyze the deterministic skeleton of their asset-pricing model constructed from a market equilibrium condition (written in deviation from), an updating equation for the proportion of agents using a particular prediction strategy (based on a discrete-choice model that determines how fast agents switch between strategies based on a switching parameter β, and the given fitness measure), and a fitness or performance measure for agents of type h based on a weighted average of past realized profits for traders of type h. To derive an analytic solution they restrict trader types to the two familiar kinds of trader—adaptive trend-chasers and rational fundamentalists.

In the fourth section of their paper, Brock and Hommes (1999, pp. 124-9) allow for many different trading strategies or belief structures by introducing a Large Type Limit (LTL)—an ensemble limit analogous to the thermodynamic limits in statistical mechanics. In this way, Brock and Hommes build a theoretical bridge between their own work and the experimental research of LeBaron et al. (1999) and Arthur et al. (1997) on artificial stock markets. In the latter body of research heterogenous trading strategies are generated by a variety of genetic algorithms differing in their principles of coding and mutation, which compete with one another in accordance with their relative fitness. Using the deterministic skeleton under a further set of simplifying assumptions the authors derive a LTL for their asset-pricing equation in deviation form. This difference equation is an n +2 th dimensional dynamic system, where n is the number of lags in the time-stationary functional that each agent employs to determine their conditional expectation of the drift term. When there are no lags, this second-order system becomes unstable with increases in choice intensity β, decreases in risk aversion, a decrease in the conditional variance of asset returns, or an increase in the diversity of beliefs. For three lags, the authors demonstrate the existence of a variety of paths to chaos comprising both Hopf and pitchfork bifurcations. They examine a particular case where strange attractors arise with an increase in the diversity of bias in the linear forecasting rules of traders.

Thus, Rational Expectation Equilibria (REE) are built into Brock and Hommes's models as special linear sub-cases. Brock and Hommes suggest that departures from REE either reflect trader uncertainty about the behaviour of other traders (1999, p. 132) or imply that prices do not necessarily fully aggregate individual information. They caution against the notion that aggregation could "wash out" non-linearities across a sufficiently rich heterogeneity of agent types because heterogeneity is endogenously created in their models through evolutionary selection. From a statistical mechanics perspective, complementary expectations are not annihilated through the operation of the law of large numbers or central limit theorems because these theorems require weak dependence, whereas strong dependence is induced by the endogenous selection procedures.

0.3.3. Issues of Long-Period Disequilibrium

Panico (1988) has followed Sraffa in arguing that the rate of profit rather than the subsistence wage is better thought of as the exogenous variable in closing the neo-Ricardian multisectoral macroeconomic model. A further innovation is his introduction of a banking sector into a long-period, price-of-production framework. Panico models the banking sector's profit rate as determined by the average interest rate margin, net of exogenously-given illiquidity premia. In turn, banking sector profits "rule-the-roost" in so far as they set the standard for economy-wide rate of profit. Panico intentionally eschews any discussion of short-term market adjustment, beyond the presumption that a gradual equalization of sectoral rates of profit occurs, following the lead established by the banking sector at any given period[14].

From another perspective, Robert Brenner has returned to the 30s industrial economics literature in emphasizing the significance of the sunk costs in relation to investments in fixed capital. He

[14] In my opinion Panico goes a little too far in contending that Chapter 17 of The General Theory ought to be interpreted from a long-period perspective. Nevertheless, apart from this unfortunate hubris, his long-period analysis affords a logically coherent and useful contribution to the prices-of-production literature. However, given the dominance of equity markets in today's monetary economies, resulting in part from the privatization of social security and the rise of mutual funds, I would advocate further extensions to Panico's basic model to incorporate appropriate illiquidity discounts and required returns on equity as well as banking finance.

regards the latter as responsible for global onset of excess capacity in traded goods sectors, especially over the mid-70s and 80s. Flow-on effects to the wider economy arise due to technological interrelatedness and inter-sectoral dependency. Low profit rates on total capital employed, in turn, are viewed as responsible for low rates of accumulation and productivity growth over this period. This excess capacity operates as a structural constraint over effective demand policies. In fact, both Keynesian expansionary policies and Conservative "tight-money" policies hinder necessary structural adjustment that can only be achieved through an accelerated scrapping of excess capacity: the former because adjustment pressure is temporarily mitigated and the latter because adjustment is rendered too costly. Each of these classes of long-period disequilibrium have the potential to add depth to any classically motivated analysis of dynamic adjustment in the macroeconomy[15].

0.4. The "Tâtonnment" Process as the Key to Chaotic Dynamics?

In reviewing the preceding discussion, from the vertiginous heights of Andrew's Wittgensteinian reading of Sraffa, it should be obvious that, once technological change and non-constant returns to scale are recognised, it has to be acknowledged that even the long-period, asymptotic prices-of-production, featured in Duménil and Lévy's analysis, becomes a veritable "moving feast".

In the Post Keynesian monetary tradition the above factors are generally seen to be the ones most closely associated with fundamental uncertainty. Fundamental uncertainty is also viewed as

[15] In a series of papers, Duménil and Lévy (1985) and Duménil (1987, 1989) attempt to lay the foundations for a classical dynamics. This is a long overdue and worthy exercise but one that in my view should, nonetheless, not distract researchers from the equally invaluable task of attempting to theoretically reconcile the Keynesian analysis

responsible for fluctuations in *long-term* expectations, liquidity preference, and consequently in the level of financial and non-financial investment. Thus, Keynes would have regarded any prospective limit cycle relationships engendered by adjustments in either goods prices or nominal wages as being of *secondary* importance, governed as they are by *short-term* expectations. In Book *V* of *The General Theory*, downward flexibility in wages is regarded as a destabilizing influence, *primarily* because it leads to adverse shifts in the state of liquidity preference, debt-deflation induced movements in the marginal propensity to consume, and ambiguous shifts in the marginal efficiency of capital. Thus, emphasis is placed on the influence of price and wage volatility on the state of long-term expectations: a key variable that is seen to govern investment in financial and non-financial assets. Unfortunately, these aspects of uncertainty and asset demand are completely ignored within Chiarella and Flaschel's and Brock and Hommes' otherwise invaluable modelling work. I can only conclude this section of the introduction by asserting that much remains to be accomplished in restoring vigour to a quantitative Keynesian analysis of disequilibrium.

0.5. The Content and Structure of the Thesis

To set the scene for later analysis of risk-sensitivity and robustness, I take advantage of the ontological framework posited by critical realist philosophy to more rigorously ground my inquiry into the characteristics of a monetary production economy (sections 1.1-1.2). This

of animal spirits and liquidity preference with the classical "centers-of-gravity" slant on price formation and accumulation.

groundwork provides a continuing refrain throughout the thesis and helps me to elucidate certain crucial aspects of uncertainty in economic life. I then embark on a somewhat speculative review of how various authors have interpreted the views of Frank Knight and John Maynard Keynes on the role of uncertainty in economic behaviour (sections 1.3.1-1.3.4). I focus, in particular, on the question of whether market-based or learning-driven processes could, over time, lead to the resolution of uncertainty and its replacement by risk alone. Section 1.3.5 examines the axiomatic foundations of non-expected utility theory and uncertainty aversion, framed by the need for choice-theorists to respond to the paradoxes thrown up by Allais and Ellsberg. I then take up David Dequech's analysis of the difference between fundamental uncertainty and ambiguity (section 1.3.6) and outline his useful schematic analysis of the factors influencing uncertainty aversion and uncertainty perception, relating his analysis to the mathematical representation of uncertainty in robust and risk-sensitive control theory (section 1.3.7). The following section addresses a number of misconceptions about uncertainty aversion emanating from both the Post Keynesian and Austrian camps. Section 1.4 returns to the issue of how uncertainty aversion is accommodated by techniques grounded in risk-sensitive and robust control theory.

In section 2.1 of Chapter Two, I draw on Alessandro Vercelli's (1991) interpretation of *The General Theory*'s fix-price and Flex price models to examine various approaches that Post Keynesian monetary theorists have adopted to the phenomenon of liquidity preference. At the same time, I investigate the implications of imposing representative agent assumptions in applications of control theory that are informed by the *rational expectations* paradigm. My intention, here, is not only to champion the virtue of models that permit heterogeneity of preferences and endowments, but also to recognize the fact that decisions about consumption and savings, portfolio management and real investment are largely made by different groups of economic agents. Thus, the ultimate objects of my critique are modern versions of the Ramsey, one-sector growth model peopled by producer-consumer-saver-investors that may be either homogenous or heterogenous. In this type of single-good, Robinson Crusoe world, decisions to refrain from eating "corn" automatically result in the planting of more "seed-corn" ruling out, by assumption, the possibility of insufficient effective demand. I point to the additional implication that this structure therefore obliterates one of the essential channels of influence and mutual interaction between financial markets and markets for goods and services.

I then focus on liquidity preference as a critical influence over both the supply of and demand for financial assets and also real investment (sections 2.1.4 and 2.2.1). I argue that conventional models for the pricing of financial assets, the determination of optimal portfolios, and the planning of consumption and saving—including those of Tobin (1958, 1969) and, by implication, Lucas (1978), Merton (1973a) and Breeden (1979)—ignore many of the features of a real-world economy. In particular, I contend that they characterize the factors that determine the allocation of a certain proportion of one's portfolio to low interest money holdings in a narrow and reductionist form (section 2.3). In an effort to broaden current approaches to the demand for money and near money assets I would not only include uncertainty aversion, but also transactions costs, which reflect the option value of being able to respond flexibly to new information, and the ubiquitous presence of nominal, non-indexed contracts. For economic efficiency to obtain, these contracts would require that economic agents possess an unlikely ability to accurately forecast inflation (sections 2.3.5 and 2.3.4, respectively).

In section 2.3.4, I suggest that something akin to this notion of option values can be found in the work of Jones and Ostroy (1984). More recently, a paper by Jan Kregel (1998) has argued that Keynes' approach to investment in *The General Theory* is option-theoretic in nature. He suggests that for Keynes: "[...] the concept of user cost represents the influence of the future in determining the relative costs of holding or using a commodity in production or sale," (p. 123). In *The General Theory* the user cost of capital is defined as the difference between the current costs involved in the operation or use of assets relative to the maintenance costs of keeping them idle. The rate of discount "...that equates the expected stream of future returns to the cost of producing new capital goods, thus, has to compete with the rate of discount that equates the expected stream of future returns to the supply prices of existing capital goods as determined by user costs" (Kregel, 1998, pp 121-2). In opposition to critics who argue that market prices cannot be used for calculating user cost owing to the absence of a liquid market in second hand capital goods, Kregel suggests that, alternatively, the user cost could be evaluated using option prices.

In accordance with this option-based notion of user-cost, he goes on to argue that the user cost of expending money today can similarly be defined as:

> ...the present value of the potential future gain or loss that has been foregone or avoided by parting with money today. This future gain or loss will depend on the expected prices of investment goods, consumption goods, and financial assets at future dates. The user cost of money can be defined as the equivalent of a call option on a deposit at the current interest rate. Alternatively, holding money uninvested in a portfolio allows you to avoid the sale of an investment asset to meet an unanticipated need for liquid funds. This is equivalent to the value of a put on the investment position written at the expected future appreciation (or depreciation) of the

> asset. With respect to other financial assets, the user cost of money is the foregone gain (or loss) that could have been earned (or avoided) by waiting to purchase financial assets at lower (higher) prices and higher (lower) yields (Kregel, 1998, p. 123).

I would fully concur with this viewpoint, with the simple but all important qualification (one supported by Kregel's own lucid analysis of Keynes' views about expectations in the same paper), that a suitable framework would have to be constructed to adequately account for investor uncertainty in the valuation of options[16].

In Chapter Three I set the scene by reviewing Pliska's approach to discrete-time, multi-period finance theory and martingale methods (section 3.2). In particular, I describe his construction of the nominal and discounted gain and value process and the way that he incorporates dividend payments into the model. In addition, I review the key relationship between non-arbitrage and the derivation of risk-neutral probability measures (for the single period) and martingale measures (for multiple periods). This relationship enables Pliska to derive the fundamental pricing equation for securities. I then proceed to outline Pliska's dynamic programming and martingale-based solutions to the optimal portfolio and optimal consumption-investment planning problems. I briefly touch on ways for generalizing these problems to account for unconsumed terminal wealth, various constraints over trading, and incomplete markets. I then discuss Epstein and Zin's (1989) discrete-time recursive representation utility theory (section 3.4). Section 3.5 examines a series of applications of uncertainty aversion in both the finance and macroeconomics literature. In the last sub-section I depart from my discrete-time convention to review Hansen and Sargent's (2001) defence of robust control theory as a vehicle for dealing with both uncertainty aversion and the Ellsberg paradoxes, against cogent criticism emanating from Chen and Epstein (2000). This sub-section leads naturally into the following section's consideration of Epstein and Wang's (1994) approach to Knightian uncertainty, which focuses on the derivation of relevant Euler equations and asset-pricing equations. This provides a treatment of asset-pricing that differs notably from the robust-control theoretic version presented in the work of Hansen, Sargent and their associates.

In Chapter Four I establish a framework for thinking about the relationship between the real economy and monetary and financial markets (section 4.2). As a prelude to this discussion I briefly outline the distinction between production-based, consumption-based and complete general equilibrium models of financial markets (section 4.1). I raise three additional matters. First, I examine Vercelli's (1991) arguments in support of minimax control as a mechanism for accommodating what he chooses to call *k*-uncertainty (section 4.3). Specifically, I review his analysis of Heiner's (1983) notion of the gap that opens between an agent's *competence* to solve a problem and the *difficulties* involved in deciphering what is going on in a complex environment, and his integration of Heiner's notion of *atemporal* flexibility and Jones and Ostroy's (1984) notion of *intertemporal* flexibility within a two-stage decision-process. I caution that Jones and Ostroy operate with a conception of *increasing risk* rather than *k*-uncertainty.

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[16] Recursive, risk-sensitive control theory could provide a vehicle for the development of such a framework but at the moment its use has been confined to the valuation of options over processes characterized by stochastic volatility

Second, I make the claim that non-linearities and complex dynamics are inescapable aspects of economic behaviour, for which rigorous justification can readily be provided through the use of microfoundational arguments that are so dear to New Classical theorists (section 4.7). For this reason, I contend that proponents of economic applications of robust control should develop techniques for dealing with complex and non-linear, rather than linear systems (e.g. as in chaotic or non-linear control theory).

Third, I contend that variations in financial instability—reflecting movements along the investment continuum that ranges from hedge, through speculative to Ponzi financial positions—should be accommodated (in a control sense) through the use of adaptive techniques that trace the path-dependent trajectories of critical parameters (4.6).

Fourth, I review Vercelli's claim that the early rational expectations literature imposed a separation between cognitive activity or decision-making and predictive activity, to determine how this position is modified in the new robust control framework (section 4.8). I suggest that, despite the adoption of a more sophisticated view of the relationship between knowledge and uncertainty amongst economic agents and econometricians or calibrators (as modeled by changes in probability measure between reference and candidate distributions under associated stochastic uncertainty constraints), a more complex notion of separation is still at work in the new approach.

(e.g. see McEneaney, 1997).

In Chapter Five, I trace the recent history of economic applications of control theory, moving from linear quadratic Gaussian control, through sub-optimal, deterministic control and, finally, into risk-sensitive and robust control theory (section 5.1). This provides the context for a detailed review of the Hansen, Sargent and Tallarini (1999) and Andersen, Hansen and Sargent (1999) papers (section 5.2). Having established the basis for the authors' claim that, in addition to factor risk premia, asset-prices also incorporate uncertainty premia, I return to the themes identified in earlier chapters. I investigate the implications of moving beyond the representative agent by reviewing Thomas Palley's 1999 reconfiguration of quantity-constrained rationing models to incorporate debt-deflation effects (section 5.3.2). I then question Hansen and Sargent's reductionist approach to robust control theory (section 5.3.3), before reexamining the modeling implications of recognizing the existence of time-variation in parameters and dynamic and structural instability (sections 5.3.4 and 5.3.6, respectively).

Hansen and Sargent construct their full information version of the robust control problem to ensure both manageability and also containment within a more-or-less conventional rational expectation *cum* real business-cycle framework. My largely destructive critique of these attempts is motivated by my concern to show that, for the social sciences, Herbert Simon's (1982) notion of bounded rationality provides a more comprehensive and more realistic portrayal of the actual restraints over decision-making than one predicated on robust control under stochastic uncertainty constraints (section 5.3.7). A natural outcome of this critique is both a broader view of policy interventions and a richer appreciation of their significance for the satisfactory operation of private markets.

Chapter 6 sets out two case studies: the first examines debates around the estimation and use of monetary policy reaction functions, while the second examines the valuation of real options in incomplete markets. The first case study commences with a review of Rudebusch's (1998) critique of the use of VAR regressions for the analysis of monetary policy. I consider both the Rudebusch critique and Christopher Sims' response to it. Much of the discussion then gravitates around the issue of non-linearities in the macroeconomy. I consider evidence for non-linearities and recent research into tests for nonlinearity and techniques of estimation designed for non-linear systems. The first case study is rounded off with a somewhat tricky discussion about whether agents need to anticipate the level of robustness characterizing the preferences of other agents in their market dealings.

The second case study examines real options theory. The literature on real options theory is both diverse and rapidly expanding. By now standard techniques for the pricing of derivatives, given a stochastic process for the price of underlying assets, are increasingly applied to the case of *real investment* in productive assets, including property development (Trigeorgis, 1996). Typically, this latter class of investments is represented as a set of options over deferment of a project, termination and salvage, switching of inputs and/or outputs, spawning of related projects, and the expansion, contraction, and temporary shut-down of projects. Concurrently, finance theorists have drawn on the mathematical literature on risk-sensitive and robust control theory under norm bounds and relative entropy constraints as one vehicle for accommodating uncertainty and market incompleteness (Andersen, Hansen, and Sargent 1999, McEneaney 1997 & Tornell 2000).

This case study provides a heuristic understanding of the relationship between these two bodies of research by examining recent work that establishes range bounds over option prices in incomplete markets (e.g. that arising due to the stochastic volatility of stock prices or the existence of a stochastic interest rate) through the application of "good-deal" bounds over the Sharpe ratios and gain-loss ratios of basis assets (Bernardo and Ledoit, 2000; Cochrane and Saá-Requejo, 2000). By varying the stipulated bound, the valuer can move along a spectrum ranging from the set of non-arbitrage bounds through to the uniquely defined option price that is associated with the pricing kernel of a chosen asset-pricing model. The "good-deal" bound can thus be interpreted as a measure of investor uncertainty relative to a reference probability distribution for the equilibrium asset-pricing model. The paper identifies the precise relationship holding between the sup-norm bound on the pricing kernel, minimum cross entropy (Stutzer, 1995), and the stochastic uncertainty constraint that is adopted in certain robust control problems. In addition, it examines the derivation of martingale measures and the role of entropy techniques in Generalized Method of Moments estimation. As such, it sets out an agenda for future research into real options-based valuation of investment under uncertainty.

In the accompanying technical appendix, I review the relevant literature on robust and recursive control theory. The ultimate goal of my criticism has been Andersen, Hansen and Sargent's (1999) application of risk-sensitive control theory—with uncertainty appearing in the form of a relative entropy constraint—to both optimal consumption planning and asset-price determination. Andersen, Hansen and Sargent interpret this constraint as a bound over the size of the family of feasible probability distributions (i.e. those that cannot be rejected under known specification tests). However, their approach ignores observation error and model uncertainty, being essentially confined to a world of *full* rather than *partial* information.

The technical appendix provides some of the mathematical results that I have drawn upon in my discussion of the literature. To keep the mathematical requirements as straightforward as possible I confine my discussion to the *discrete-time* rather than the *continuous-time* setting. To set the context I first introduce the z-transform operator and examine its application to the discrete-time state-space model. In the following section of the appendix I discuss spectral analysis and norms. The next three sections introduce results for the robust (H_∞) control and filtering of linear state-space systems. This is followed by an overview of the non-linear risk-sensitive control and filtering problem. In this section I follow the information-state explication of Boel, James and Petersen (1997), which establishes the recursive relationship holding between risk-sensitive control, the Kalman-Bucy filter, and the H_∞ control problem. These authors also draw on the duality between free and relative entropy to establish error bounds over the minimum risk-sensitive estimator. In the next section I examine the relationship between stochastic uncertainty constraints (including the relative entropy constraint) and norm bounds imposed over model uncertainty, external perturbation, and observation error. I conclude by demonstrating that Andersen, Hansen and Sargent (1999) ignore observation error in their consideration of relative entropy constraints that govern their version of the risk-sensitive control problem.

CHAPTER ONE — UNCERTAINTY: FROM KEYNES TO CONTROL THEORY

1.1. Introduction

In this thesis I review recent work by Aaron Tornell (2000) and Lars Hansen, Thomas Sargent and their associates (Hansen, Sargent and Tallarini, 1999; and Andersen, Hansen and Sargent, 1999), which introduces uncertainty aversion into what would otherwise be a fairly conventional, recursive, representative agent, intertemporal general equilibrium model of permanent income and asset-pricing. The assets in question are ownership entitlements to the fruits of trees (i.e. claims over an exogenously determined endowment or dividend stream, measured in consumption goods). The endowment stream in question is determined elsewhere in the model through optimal consumption-investment decisions on the part of the representative agent for a given production technology (relating gross output to inputs of capital) under a habit-persistence mechanism. The latter mechanism can be thought of as another form of "production technology" that relates the output of household services positively to current consumption and negatively to a weighted average of past consumption. In the above-cited applications of risk-sensitive control theory to finance, agent preferences are represented by Epstein and Zin's (1989) family of risk-sensitive, recursive utility functions. Epstein and Zin establish that these recursive functions can represent a variety of non-expected utility theories including those introduced by Kreps and Porteus (1978), Chew (1989), and Dekel (1986).

To accommodate uncertainty, Hansen and Sargent exploit the mathematical parallel between risk-sensitive stochastic control theory, risk-neutral stochastic control theory (e.g. Linear Quadratic Guassian control and Kalman Filtering), and H^{∞} control—a deterministic, differential game version of robust control theory (for a specific example of the H^{∞} approach see Tornell, 2000).

In engineering applications of stochastic control theory, uncertainty is represented variously by a norm bound, a sum integral constraint or a relative entropy constraint. Typically, the stochastic uncertainty constraint imposes root mean squared error bounds over three types of uncertainty: observation error, additive or multiplicative model uncertainty, and external perturbation. An entropy constraint imposes a bound over the energy of candidate probability distributions relative to a reference distribution (usually, though not exclusively, of the Gaussian form). I argue, in later chapters, that Hansen and Sargent work with a somewhat idiosyncratic interpretation of this constraint, reducing it to a bound over the size of the family of feasible probability distributions (i.e. those that cannot be rejected under known specification tests). In other words, they assume complete information about the state of the economy. Much of the force of recent developments in H^{∞} control results from the ability of this body of techniques to introduce observation error and model uncertainty into the control and filtering framework. Tornell's own work explicitly attempts to go beyond these limitations of Hansen and Sargent.

In the first part of this Chapter I set the context for what is to follow by identifying the essential ontological properties of a modern economy. To this end, I outline the critical realist approach to the theory of knowledge and the precise way in which scientific development is understood to progress *differentially* within the human and natural sciences. My intention here is to clarify, on reasonably sound philosophical grounds, the need for theory to recognise those attributes that Keynes considered as being unique to a monetary production economy: that is, an economy in which money functions as a contractual unit of account, store of value, and medium of exchange and contractual settlement. These attributes include the pervasiveness of uncertainty as an influence over economic decision-making, the ubiquitousness of transaction costs that pertain to the transfer of titles to various financial and real assets, and the prevalence of nominal, non-indexed contracting.

This philosophical overview is followed by a specific inquiry into the nature of uncertainty and the manner in which various authors have interpreted the views of Frank Knight and John Maynard Keynes on the role uncertainty plays in economic behaviour. Initially, I focus on the distinction between *chance uncertainty* or risk and *fundamental uncertainty*: a distinction hinging on the question of whether uncertainty can be eliminated over time through market-based or learning-driven processes and replaced by an assured knowledge of relevant probability distributions. This is followed by an overview of non-expected utility theory in which I focus on the relationship between risk as uncertainty aversion and uncertainty aversion. Both the sub-additive and multiple-priors versions of the latter are considered in regard to their axiomatic foundations. In this section I also investigate properties of relative entropy that I associate with the S-shaped weighting functions that are to be found in rank-dependent utility theory and cumulative prospect theory.

In subsequent sections, I examine David Dequech's (2000) further distinction between *fundamental uncertainty* and *ambiguity*. However, my own views differ somewhat from those espoused in Dequech's work, specifically, in regard to whether the types of uncertainty featuring in financial applications of risk-sensitive stochastic control theory can *solely* be classified under the heading of ambiguity. I argue that in certain respects, these techniques possess attributes that are also associated with fundamental uncertainty. And even where they fall prey to the more skeptical aspects of the Keynesian critique of the "probability calculus", I argue that they nevertheless provide a formal mechanism for modeling the phenomenon of liquidity preference.

In my interrogation of this issue I have found Dequech's (1999b) schematic analysis of the factors influencing both uncertainty aversion and uncertainty perception to be of particular value. These matters, though subtle and seemingly pedantic, are far from trivial and, all too frequently, participants in the debate adopt conceptual grids that obscure otherwise fine distinctions, reducing shades of gray to overly harsh and simplistic tones of black and white.

1.2. The Ontological Context

1.2.1. Justification for a Critical Realist Starting Point

The discipline of economics originally evolved out of political economy that, in turn, grew out of a particular branch of applied ethical philosophy (Adam Smith, of course, wrote his first book on the Theory of Moral Sentiments, rather than political economy). Economists of a positivist orientation would like to firmly sever any lingering relationship with applied ethical philosophy, preferring instead to view economics as a positive science. In matters of methodology, this group of researchers typically ascribes to a hypothetico-deductive or instrumentalist approach to economic science.

One strand of philosophical thought that clearly stands opposed to the dominant logical positivist or empiricist school of philosophical thinking is *critical realism* (Bhaskar, 1978). Currently, this tradition has been adopted by a growing number of progressive economists; especially those associated with Cambridge University (UK). In this, and the following four sections of the chapter, I provide a sympathetic overview of critical realist philosophy. I go on to examine the implications of a critical realist approach for econometric practice, deferring to the work of the late Ari Spanos. However, while admiring the perspicacity and consistency of the critical realist project, I also argue, in passing, that many researchers in the humanities and social sciences hold to a philosophical perspective that is far more comprehensive than critical or transcendental realism.

However, for better or for worse, the critical realist framework serves as a vehicle for interrogating the ontological features of uncertainty in decision-making and control. To this end, it reinforces the argument that modern finance and monetary theory must attend to those aspects of a monetary production economy that are fundamental in an ontological sense of the term. In this regard, I follow those Post Keynesian researchers who emphasize the ubiquitous nature of nominal (non-indexed) contracts, the endogenous, credit-driven nature of the money supply, and those characteristics of money that make it into a safe haven for savings, when firms, financial institutions and households become more uncertain and apprehensive about their economic environment.

1.2.2. Critical realism[17]

Critical realism is founded on the fundamental distinction between realism and idealism: the former adhering to the notion that reality and real objects should be conceived as existing independent of, or irreducible to mind. As Northover establishes (1999, p. 36), the realist perspective can don a variety of garbs: in *perceptual realism,* the real exists independent of perception; in *predicative realism,* universals exist independently or as properties of material objects; in *scientific realism,* the objects of scientific enquiry exist and act either absolutely or, at least, relatively independent of such enquiry. Three different positions can be held in relation to theoretical statements or propositions: (i) *referential*—theories factually refer to elements in their extensions; (ii) *representational*—theories represent real entities by making assertions about their attributes and characteristics; (iii) *veristic*—linguistic statements about reality convey truth or falsehood by virtue of their correspondence with the referents that they represent. In conjunction, Northover argues that (i), (ii), and (iii) imply *semantic realism*. In *transcendental realism*–a form of semantic realism—the objects of science are viewed as structured and irreducible merely to events and their observable patterns. In this sense, transcendental realism is

[17] The following overview of critical realism is based on the work of Roy Bhaskar (1978, 1989) and principally draws upon a recent paper by Steven Pratten (1993) who applies critical realism to Marx's theories of the labour process.

opposed to *empirical realism*—the theory that reality consists of experience and phenomena that are the direct objects of experience alone.

1.2.3. DEDUCTION, INDUCTION AND RETRODUCTION

In driving home their opposition to positivism and empirical realism, critical realists distinguish between three species of inference: deduction, induction and retroduction. The deductivist account of scientific development is founded on Humean causality. Deductivists argue that only spatial and temporal continuity can be observed - that there is no knowledge of causal power. Accordingly, the idea of necessary connections between events must be projected onto our experiences of the external world. Hence, there is a necessary symmetry between explanation and prediction and truth is reduced solely to that which can be confirmed.

Countering this, is the notion that scientific statements get their meaning from complex networks of metaphors, models and theories. On this account of scientific description, the Humean criteria of causal laws can only be regarded as insufficient because their necessity can only be comprehended through models operating at the heart of theory. As such, empirical invariance must be supplemented by an examination of both the plausibility and the coherence of the underlying model. But this raises the ontological question of what it is exactly that is referred to by the models themselves (Pratten, 1993, pp. 405-7).

There can be no convincing and coherent account of science which implicitly accepts the empiricist ontology presupposed by the positivism under attack. Accordingly, the ontological approach of critical realism distinguishes between three separate domains:

1. the real, made up of entities and mechanisms;
2. the actual, made up of events and states of affairs;
3. the empirical, made up of experiences.

Given this structuring of the world into three realms, transcendental enquiry is predicated on "...an understanding of laws as transfactual generative mechanisms, dynamised causal powers, *tendencies*" (Northover, 1999, p. 39) that describe what happens when causal powers are first, activated, and then, dynamised. Events "can occur without being experienced, causal mechanisms can counteract one another and there can be real mechanisms in nature which never have effects though they would under certain circumstances" (Pratten, 1993, p. 406). This contingency holds because powers may be exercised without being manifest in states of the world - that is, they may operate as tendencies. Alternatively, one power may counteract, neutralise or temporarily overwhelm another. The ascription of a tendency can be interpreted as a statement of a law - whereby such laws are conceived as setting limits rather than prescribing uniquely fixed results (Foss, 1994, p. 40). To articulate the nature of these real, often tendential mechanisms, and the inter-relationships which obtain between each of the three ontological domains, researchers must distinguish between the *intransitive object* of scientific analysis that exists outside the scientific process and the *transitive* and changing *cognitive objects* that are produced within science.

Retroduction is a logic of discovery rather than truth. Where deduction proves that something *must* be and induction shows that something *actually* is operative, retroduction is a mode of inference that merely asserts its conclusions conjecturally. Empirical things are explained by postulating (and subsequently demonstrating) the existence of real, generative mechanisms (Marsden, 1993, p. 299-300). For a realist, retroduction takes the form:

1. a particular phenomenon P_1 is observed;
2. P_1 would be explained if H were to exist and act in the postulated way;
3. Hence, there is reason to think that H exists and acts in this way (Marsden, 1998 p. 300).

Marsden claims that Marx, in his method of political economy, adopts a critical realist ontology and frequently resorts to the inferential mode of retroduction. This, he argues, is most clearly to be seen in Marx's analysis of the internal connections between the relations of production, distribution and circulation and the internal relations between capitalist and worker (p. 304). For Marx, these connections and relations are both real and non-empirical and must be explained through a conceptual model that maps out the causal relations at work within civil society. However, Marsden notes that Marx takes pains to differentiate this retroductive analysis of social activity from a Hegelian dialectical analysis. Where the latter would interpret things as a product of conceptual determination alone, Marx distinguishes between the process by which the concrete comes into being and the process by which thought appropriates the concrete[18]. Reality is conceptually mediated, to be sure, but what we see and understand is also determined by the phenomena that social relations actually produce (p. 305). Marsden goes on to suggest that Marx firmly rejected the notion that the material and the ideal could be separated. He suggests that Marx's celebrated *inversion* of Hegel's dialectic does not amount to the replacement of the ideal with the material, but rather the replacement of the epistemological with the ontological and the transitive with the intransitive. Marsden claims that Marx has taken Hegel's dialectic among concepts and applied it to history as social reality in motion[19].

Marsden makes the important observation that it is the retroductive interrogation of internal structures which enables Marx to identify those characteristics of production that are common to all epochs and distinguish them from those that are unique to a particular mode of production. Marx faults orthodox economists for confusing the particular with the general and for mistakenly thinking that the preconditions of the modern, bourgeois mode of production are preconditions of production in all possible epochs: past, present and future (p. 302). Marsden sees in the Marxian *law of value* the internal dynamics of this structure of social relations, capital's way of acting, its causal powers (p. 309). However, capital creates 'man' as an abstraction (i.e. the value-form of the commodity and the value of labour-power, itself, is the expression of socially necessary,

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[18] Of course, Hegel makes a similar distinction. However, in appropriating the concrete Hegel argues that thought, in the form of *speculative reason* rather than the entirely negative and therefore circumscribed *understanding*, can overcome the primordial alienation of Logos into Nature. For Marx, on the other hand, the analogous alienation of social labour into capital and the commodity form cannot be arrested through thinking alone, but must be accomplished through revolutionary action.

[19] I would argue that Marx's inversion involves something quite different, but to best appreciate this point would require more space than I can afford to accommodate a deep interrogation of Hegel's own critique of both the Kantian philosophy of reflection and its romantic counterpart in the works of the absolute idealists, notably

abstract human labour) and thereby conceals the internal structure of production behind what is only a surface manifestation, one that takes the form of a series of free and mutually beneficial exchanges between abstract private individuals. This private and free individual becomes the apparent foundation of civil society, masking the underlying processes of exploitation, alienation and the extraction of surplus value. For my own purposes, retroduction operates within the Post Keynesian corpus to interrogate the underlying causal relations that determine the way that uncertainty influences economic decision-making on the part of firms, households and the institutions that provide credit finance.

1.2.4. Closure and Generalization Beyond the Experimental Situation

The logic of scientific discovery necessarily advances from the identification of invariances to the classification of the structures or mechanisms which generate them. First, an effect is identified and described. Second, a hypothetical mechanism is postulated which, if it existed, would explain the effect. Models are developed through a social practice of production by drawing upon antecedent knowledge and understanding of the experimental situation. Third, an attempt is made to demonstrate the existence and operation of the proposed mechanism through experimental activity, which either isolates the mechanism or eliminates alternative explanations of the underlying mechanism.

Most natural phenomena are the products of a plurality of deeper structures, and it is the task of scientific experiment to establish the conditions that isolate one particular mechanism allowing

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Schelling and Fichte (on this point see Habermas, 1987). Here, I merely wish to take on board a limited version of its causal affects to be realised. This process of isolation is called c*losure* and can take three forms:

Intrinsic: ensuring that under identically specified circumstances, individuals would always respond in the same predictable way.

Extrinsic: effectively isolating the mechanisms being investigated from the influence of those aspects of the environment not explicitly considered by the analysis.

Organisational: imposing some stipulation on individual behaviour so that the requisite sort of behaviour emerges at the level of the system as a whole (Foss, pp. 40-1; Pratten, fn. 7, p. 408; Bhaskar, 1978).

It is the nature of and the requirement for closure in scientific experiment, which most clearly demonstrates the inadequacies of the empiricist ontology.

1.2.5. Critical Realism and the Social

Critical realism views social phenomena as conditioned by, dependent upon and only manifest in natural phenomena, but causally and ontologically *irreducible* to them. Social forms and individuals have different properties:

> Society is at once the ever present condition and continually reproduced outcome of human agency: this is the duality of structure. And human agency is both typically work (generically conceived) i.e., normally conscious production, and reproduction of the conditions of production, including society: this is the duality of praxis (Bhaskar, 1986).

Critical Realism, essentially for pragmatic purposes.

In other words human agency, as the power to effect changes in the world, depends not only on our ability to understand, evaluate and to control our motivations and actions, but also on the existence of society and social relations which we reproduce and transform, often in unacknowledged and unintentional ways.

Furthermore, social structures are ontologically distinct from natural ones, first, because they "...exist only by virtue of the activities they govern, enable and constrain...do not exist independently of the agents' conceptions of what they are doing in their activity, ...(and) may be only relatively enduring so that the tendencies they ground may be liable to considerable space-time variance" (Pratten, 1993, p. 411). Furthermore, and most significantly, social phenomena only ever occur in open systems. At the most fundamental level "...the reality of choice presupposes that the agent could always have acted other than he or she did, and thus events could really have been different" (fn. 12, p. 411).

Because human action is intentional, reasons can be causal despite their status as mental concepts. However, "...while human action does not take place independently of agents' conceptualizations of what is being done, it frequently occurs independently of an adequate conceptualization. Agents' conceptions of 'external reality' are fallible, furthermore, the meaning of their actions may be opaque to themselves" (Pratten, 1993, p. 412). In their actions people often draw upon unconscious beliefs and motivations which rational argument cannot easily reveal. Much decision-making occurs at a tacit level (it cannot be codified or written down) because although we know what we do, we do not necessarily know how or why it is that what we do works in the way that it apparently does. In my analysis of liquidity preference, this conceptual fallibility is captured by the model uncertainty and observation error that operates as a constraint over the filtering and control efforts of economic agents.

1.2.6. Critical Realism and Econometric Practice

In the opening chapter of his book on the statistical foundations of econometrics, Ari Spanos (1986) takes aim at the conventional textbook approach to the issue of econometric modeling. Typically, this textbook approach views the progress of science as a linear feedback sequence: moving from estimation to improved economic theory, and back again to improved estimation, testing and inference. Similarly, assessments of forecasting accuracy, whereby competing models are compared and then further refined or rejected, are viewed as a direct expression of the same linear process.

Spanos notes that the practice of econometrics frequently strays from the high ideals enshrined by econometric theory into a questionable and arcane alchemy of data-mining and trial-and-error model building. For this reason Spanos places the Data Generating Process (DGP)—the underlying mechanism that gives rise to the *observable* data—at the heart of his modeling strategy. In this regard, his approach can be interpreted as one motivated by critical realist thinking. For Spanos, theory, as an idealised conceptual construct, must then relate specifically to this underlying mechanism with a view to providing an explanation of, and predictions relating to, the actual DGP. Given the DGP and actual observations of the data, the *theoretical* model is seen as a mathematical formulation of the theory, which can be specialised into a potentially estimable form. This *estimable* model is further refined into "...a probabalistic formulation purporting to provide a generalised description of the actual DGP with a view to analysing the estimable model in its context" (Spanos, 1986, p. 21): this is the *statistical* model. Finally, a reparameterisation and restriction of the statistical model in view of the estimable model can then be used for description, explanation and prediction. In relation to econometric practice, the importance of the DGP is that it influences decisions the modeler makes "...about characteristics of the random variables which gave rise to the observed data chosen such as normality, independence, stationarity, mixing before any estimation is even attempted" (Spanos, 1986, p. 22).

In this thesis I argue that filtering techniques must match the control or decision-making procedures that agents utilize. When recognition is granted to non-Gaussian forms of observation error, external perturbation and model uncertainty, then the Kalman filter in its rational expectations form must be rejected as an appropriate estimation mechanism. To adequately account for these factors, I suggest that Generalized Method of Moment techniques for the estimation of asset-pricing models should also be modified.

1.2.7. The Limitations of Critical Realism

Unfortunately any discussion about the limitations of critical realism as a philosophical system must be brief, because this thesis is not primarily a philosophical treatise. I contend that for critical realist thinking ontological considerations dominate those of metaphysics, epistemology, and ethics. Under the assumptions of a materialist realism metaphysical questions are confined to those concerning the articulation between the three levels of the real, the actualised and the experienced. Similarly, epistemological questions are confined to what the character of knowledge must be for scientific discovery to progress towards a deeper and deeper understanding of the relationship between the real, actual and experienced. As we have seen, this includes the interrogation of modes of reasoning (deduction, induction, retroduction) and experimental closure and then progress of scientific knowledge. Finally, ethical considerations are confined to an investigation into the implications for human conduct of the ontological freedoms that are constituted by the reality of social choice, intention and chance encounter, including the possibility that the collective consequences of actions may differ from those that were originally intended by those individuals or groups who were the originators of such actions.

Nevertheless, when confined solely to the specific set of issues relating to scientific knowledge and calculative action, critical realism can still function as both a useful and a powerful referential framework. This will become more apparent in the discussion that follows about the ontological nature of uncertainty. At various stages in the elaboration of my thesis I emphasize other important ontological properties of a monetary production economy that must be incorporated into theory. These include coordination failure, complexity and structural instability in economic dynamics, the presence of non-indexed, nominal contracts in labour and credit markets, and the transactions costs associated with the transfer of titles to illiquid assets.

1.3. Philosophical Approaches to Uncertainty

1.3.1. THE DISTINCTION BETWEEN RISK, CHAOTIC DYNAMICS AND UNCERTAINTY

In this section of the Chapter I clarify the distinction between economic uncertainty and risk and consider the implications that risk and uncertainty each have for decision-making and investor behaviour. To this end I review recent assessments of Keynes' views about the influence of probability on human conduct.

Bill Gerrard (1994) has argued that the traditional "Keynes-as-philosopher" research has largely neglected probability theory. However, he also criticizes those Keynesian fundamentalists who assert that Keynes rejected the "probability calculus" out of hand as an adequate vehicle for understanding economic behaviour under uncertainty. Gerrard complains that this fundamentalist viewpoint offers a negative critique only and does not put forward a constructive alternative in its place. He demonstrates that new Keynesian scholarship has remedied this defect. Gerrard shows how Keynes' early philosophical writings (1930, 1973b) can be gainfully employed to illuminate some of the essential arguments in his later economic works (1937), including *The General Theory* (1936). I provide an outline of Gerrard's own position to clarify the important distinction between risk, uncertainty associated with chaotic dynamic systems, and the fundamental uncertainty associated with liquidity preference and the irreversibility of investment decisions.

In a related vein, Bayesian theorists have questioned the relevance of Frank Knight's original distinction between risk and uncertainty by re-interpreting the latter as a form of subjective risk. These pernicious attempts to "tame the untameable" have been vigorously criticised by Joachem Runde, who has attempted to demonstrate that liquidity preference is incompatible with Savage's canonical version of the Bayesian model. I follow Runde in arguing that the Keynesian idea of fundamental uncertainty is incompatible with the notion that risk assessments based on subjective probabilities under incomplete knowledge could, over time, become more objective through the operation of some sort of evolutionary selection or learning process. However, I point to limitations in Runde's approach that result from a refusal to make the distinction between payoffs to an action and payoffs to the relevant value-function.

1.3.2. Frank Knight on Uncertainty and Risk

Frank Knight's perspective on uncertainty has exercised the greatest influence over two subsequent traditions, on the one hand:

- that embraced by followers of Ronald Coase (1937, 1960); including members of the property-rights school (Alchian and Demetz, 1972), those who promulgate principal-agent and transactions cost theory (Jensen and Meckling, 1976; and Williamson, 1975; respectively); and on the other hand,
- that propounded by Keynes (1936) and the Post Keynesian monetary theorists.

Frank Knight (1921) defines profit as the abnormal return over and above the normal factor rental income accruing to owners of capital, which reflects a reward for entrepreneurial action in the face of uncertainty[20]. One of his most significant achievements was the distinction between risk and uncertainty:

> The essential fact is that "risk" means in some cases a quantity susceptible of measurement, while at other times it is something distinctly not of this character; and there are far reaching and crucial differences in the bearings of the phenomenon depending on which of the two is really present and operating.[...] It will appear that a *measurable* uncertainty, or "risk" proper, as we shall use the term, is so far different from an *unmeasurable* one that it is not in effect an uncertainty at all (Knight, 1921, pp. 10-20).

He also introduces the notion of the confidence that agents feel about their estimates of risk and return:

> The businessman himself not merely forms the best estimate he can of the outcome of his actions, but he is likely also to estimate the probability that his estimate is correct. The "degree" of certainty or of confidence felt in the conclusion after it is reached cannot be ignored, for it is of the greatest practical significance. The action which follows upon opinion depends as much upon the amount of confidence in that opinion as it does upon the favourableness of the opinion itself. The ultimate logic, or psychology, of these deliberations is obscure, a part of the scientifically unfathomable mystery of life and mind (Knight, 1921, pp. 226-7).

[20] Of course, a Marxist would take a more sanguine view about these returns, acknowledging their existence, but attributing them to a shuffling of surplus value away from firms who were unwilling to relinquish outmoded techniques of production, to firms who were willing to embrace new techniques which reduced the quanta of either direct and indirect labour necessary for the production of a particular service or product line. Joseph Schumpeter acknowledged this fact in his own research.

After making the familiar distinction between *a-priori* and statistical or inductive types of probability judgements, Knight defines a third type of judgement which he calls *estimates:* these judgements arise in cases where objective probabilities cannot be determined:

> The liability of opinion or estimate to error must be radically distinguished from probability or chance of either type, for there is no possibility of forming *in any way* groups of instances of sufficient homogeneity to make possible a quantitative determination of true probability. Business decisions, for example, deal with situations which are far too unique, generally speaking, for any sort of statistical tabulation to have any value for guidance. The conception of an objectively measurable probability or chance is simply inapplicable (p. 231).

Langlois and Cosgel (1993) convincingly argue that Knight has been misunderstood by many commentators due to the tendency to interpret his work from the perspective of present-day theory—particularly asymmetric information and non-insurable risk. They contend that Knight's main concern was less with the difficulty in assigning probabilities to outcomes and more with the impossibility of classifying the relevant states of nature (Langlois and Cosgel, 1993, p. 459)[21].

Knight addressed a variety of techniques which could be adopted to confront both risk and uncertainty. For one thing, he favoured diversification through consolidation and aggregation of a range of activities. However, when it came to uncertainty as such, he specifically focused on what would later come to be known as the principal and agent relationship. The capacity of agency contracts to deal with uncertainty is most evident in the autocratic nature of the labour contract, which gives the employer a vertiginous freedom to dispose of the employees' energies and skills as warranted by any unpredictable changes in the contingencies of the commercial environment. In Knight's view, other contracts such as those obtaining between owners and managers, managers and supervisors and so on down through the decision-making hierarchy, often afford no lesser a freedom to deal with unpredictability. Langlois and Cosgel summarise Knight's theory of organisation in the following way:

> Because of the non-mechanical nature of economic life, novel possibilities are always emerging, and these cannot easily be categorised in an intersubjective way as repeatable instances. To deal with this "uncertainty" one must rely on judgment. Such judgment will be one of the skills in which people specialize, yielding the usual Smithian economies. Moreover, some will specialize in the judgment of other people's judgment. As the literature since Coase [1937] suggests, however, a theory of specialization is not by itself a theory of organization, since, in the absence of transaction costs, there is no reason why the division of labor could not be undertaken through markets rather than within a firm. Knight's answer is that the function of judgment is ultimately non-contractible (Langlois and Cosgel, p. 462; who cite Knight, 1921, p. 311).

1.3.3. KEYNES ON UNCERTAINTY - LAYING THE FOUNDATIONS

In the *Treatise on Probability*, Keynes (1973b, Chapter 16) argues that the imperative to act rationally provides a justification for the determination of rational degrees of belief[22]. An individual's preference for a more probable belief over alternative beliefs is always made with reference to action or conduct, implying that it is practical, not speculative reason that operates in

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[21] A similar Knightian notion of uncertainty appears to underpin Fama's recent intepretation of the "unnamed" state variables problem arising in Breeden's (1979) intertemporal capital asset-pricing model (Fama, 1998).

[22] For reasons of brevity I will not enter into all the subtleties and nuances of the arguments that Keynes developed in the *Treatise*, particularly in the celebrated Chapter 26. Other writers have provided admirable overviews to which the reader is referred (Gerrard, 1994; Davidson, 1994a; Runde, 1994; Lawson, 1993; Butos and Koppl, 1997; Rosser 2001).

the application of probability to human affairs. Nevertheless, in many cases, because the goodness or worth of a particular action may not be numerically measurable or additive, the determination of the mathematical expectation of an outcome or event may not be possible or meaningful[23].

In the *Treatise* Keynes argues that three conditions must hold jointly for application of the conventional Benthamite form of the probability calculus. These are measurability, the principle of indifference, and atomic uniformity. Another two supplementary conditions relate to the problem of induction, and consequently, the level of confidence we may have in our calculations of probability. For many outcomes we have "…no scientific basis on which to form any calculable probability whatever. We simply do not know" (Keynes, 1937, pp. 213-4). This is the problem of *measurability*. In the absence of measurability, Keynes intimated that we might not even be able to *rank* probabilities on an ordinal scale of greater or smaller likelihood. Although this is plausible for events such as the prospect of a European war, he suggested that it was also conceivable for the rate of interest twenty years hence, or the present value of current additions to capital.

The *principle of indifference* posits that "*equal* probabilities must be assigned to each of several arguments, if there is an absence of positive ground for assigning *unequal* ones" (Keynes, 1973b, p. 45). Keynes argues that this principle can only be applied when evaluating *indivisible* alternatives (i.e. those which are not a disjunction of two or more mutually exclusive

[23] In his 1911 paper on "The principal of averages and the law of errors which lead to them", Keynes applied what has come to be known as the maximum likelihood method to argue that the mathematical expectation was not

probabilities, and for one of these alternatives, therefore capable of being further split up into pairs of possible alternatives. In *The General Theory* he contends that the principle of indifference is not generally applicable to social and economic phenomena, for which the calculation of actuarial expectations on the "assumption of arithmetically equal probabilities based on a state of ignorance leads to absurdities" (Keynes, 1973a, p. 152)[24].

Atomic uniformity requires that causes work additively rather than organically so that the net outcome of any complex of causes acting in combination can be determined as the vector sum of each of their separate, independent and invariable effects (Keynes, 1973b, pp. 276-7).

The condition of *limited independent variety* applies when a finite number of ultimate axioms and laws of necessary connection govern the structural composition of the system under analysis such that the range of future realisations of the system can be identified and their relative likelihoods determined. In work subsequent to *The General Theory*, Keynes questions whether this condition could hold for future outcomes in unstable and volatile markets (Keynes, 1937, p. 214)[25].

Finally, the *weight of argument* refers to a measure of the sum of favourable and unfavourable evidence that must be distinguished from probability, which instead measures the difference between these two classes of evidence.

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necessarily the most probable forecast. Rather, the latter would depend on the precise form of error associated with a particular probability distribution (cited in Gerrard, 1994, p. 189).

[24] This reference of Keynes could be viewed as presaging the sub-additive probabilities approach to uncertainty aversion that is championed by Dow and Werlang (1992, 1994). See section 6.2 in Karni and Schmeidler, 1991, for a contextualised review.

Even when probabilities and worth are measurable, if only in an ordinal sense, any assessment must include both the weight of argument and the associated risk. The weight comprises both the amount of relevant evidence and the completeness of evidence. In other words, evidence determines the extent of our relative knowledge about the situation compared to the relative degree of our ignorance. Additional evidence may only confirm the extent of our relative ignorance about relevant processes and relationships rather than increase our level of confidence[26]. Furthermore, Keynes contends that any assessment of evidence must account for risk in the sense of the mathematical expectation of the loss attached to a particular action.

1.3.4. KEYNES' VIEWS ON UNCERTAINTY IN *THE GENERAL THEORY*[27]

Gerrard (1994) demonstrates that Keynes' later economic writing displays a strong sense of continuity in the presence of change - although practical empiricism replaces speculative reason, expectations still provide the link between earlier philosophical analysis and later economic research. In a manner analogous to the distinction made in the *Treatise* between mathematical expectation and weight, in *The General Theory*, expectations depend not only on most probable forecast but also on the state of confidence.

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[25] Incidentally, Butos and Koppl note the affinity between this condition and Kurt Gödel's incompleteness theorem, but acknowledge the later (1931) publication of Gödel's work.

[26] Keynes criticised Moore's position expounded in the *Principia Ethica* that growing ignorance was a justification for discounting future consequences relative to the present. Keynes argued that Moore was too attached to the relative frequencies view of probability, thereby ignoring the effect of uncertainty about the future on the weight of evidence (Gerrard. 1994, p. 184).

[27] I return to this issue once more in Chapter 1, where I re-examine the relationship between expectations and long-term investments.

Keynes consistently distinguishes between short-run and long-run expectations in *The General Theory*[28]. The former are assumed to apply to pricing and output decisions (e.g. to the expected amount of proceeds which determine the position of the aggregate supply and aggregate demand curves), while the latter are assumed to be relevant for decisions about real and financial investment. Keynes contends that the vagueness of probability distributions prevents the exact calculation of mathematical expectations and that awareness of vagueness changes the behaviour of agents. Accordingly, for Keynes short-run expectations are subject to constant, gradual revisions and he presumes that recently realised results exercise the most influence over expectation formation, simply due to the imperative to act. Keynes asserts that agents would generally expect the future to be like the recent past unless strong reasons exist for thinking otherwise (Keynes, 1936, p. 51). Gerrard (1994) contends that for Keynes, fluctuations in, or disappointments over short-term expectations are only a source of frictional unemployment and that relative frequency approach to the calculation of expectations is, thus, often a reasonable presumption if sufficient stability in market conditions obtains.

Keynes argues that long-term expectations are influenced by two components: the most probable forecast and the state of confidence (Keynes, 1936, p. 148). He considers the latter component to depend on both the size of any prospective change and uncertainty about the precise form that the change would take. In other words, confidence is related to the concept of *weight* rather than to *probability*. In a manner similar to the determination of short-term expectation, Keynes believes that long-term expectations are also affected disproportionately by the recent past due to

[28] This distinction is established across two chapters: Chapter 5 (Keynes, 1936, pp. 46-51) and Chapter 12 (pp.147-164).

the conventional practice of projecting the existing situation into the future. Nevertheless, he asserts that the investment decision at any particular moment in time is a unique choice for which no relevant frequency distribution exists, not least because market conditions change systematically over time. However, Keynes concedes that in stable conditions, comparability of probabilities might be reasonable for hypotheses of a similar type related to a similar choice system[29].

Gerrard summarises Keynes' views on uncertainty by distinguishing between *uncertainty*, *fundamental uncertainty* and *chance uncertainty* or *risk*. He imposes a two dimensional grid based on the completeness or incompleteness of knowledge and the structural determinacy or indeterminacy of the choice situation. Perfect certainty, for example, implies complete knowledge of a structurally determinate choice situation. Uncertainty implies either limited knowledge of a structurally determinate situation or more or less knowledge of a structurally indeterminate situation. Chance uncertainty implies complete knowledge of long-run relative frequencies and is equivalent to what conventional economists define as risk.

According to Gerrard, Keynes argues that the propensity to act on expectations depends on credence. For instance, certainty implies complete credence and probabilities equal to either zero

[29] In this context Gerrard has convincingly argued that the Keynesian notion of "animal spirits", defined as the spontaneous urge to action rather than inaction, is most appropriately viewed as being determined by the state of confidence rather than by some sort of non-rational motivation. In contrast, Davidson (1995) prefers to hold the view that "animal spirits" reflect a disposition on the part of an investor to act in the absence of any secure basis for determining the likely outcome of their actions. I examine these and other questions in further detail below, drawing on David Dequech's interesting schemata.

or one; while risk implies complete credence (otherwise known as absolute degrees of belief) and probabilities (referred to as relative degrees of belief) which are long-run relative frequencies. In conditions of ambiguity there is less than complete credence, and correspondingly, degrees of belief must reside somewhere between zero and one.

Here, Paul Davidson's use of the distinction between ergodic and non-ergodic reality may be useful (Davidson, 1994a)[30]. In ergodic environments agents either know the future in the sense of actuarial certainty equivalents; or alternatively, their knowledge is incomplete in the short-run due to bounded rationality. But in the latter case, it is presumed that subjective probabilities would *ultimately* converge to objective probabilities in the long run either due to the operation of certain learning processes or some form of Darwinian selection. However, in a non-ergodic environment, knowledge is *intrinsically* incomplete[31]. Davidson argues that non-ergodic reality is a feature associated with all long-term decisions about investment and wealth creation which:

[30] Paul Davidson argues that if a stochastic process is ergodic, then for an infinite realisation, time and space averages will coincide - here, space averages are calculated from cross-sectional data at a fixed point in time, while time averages are calculated from time-series data at a fixed realisation. For finite realisations, the space and time averages will gradually converge. The mean ergodic axiom is that space or time averages calculated from past data provide reliable estimates of the space averages that will exist at a future date. For a stationary process, the estimates of time averages do not vary with the historical calendar period under consideration. Non-stationarity is a sufficient but not a necessary condition for nonergodicity (see Davidson, 1994a p. 90, and also Vercelli, 1991, Chapter 5).
[31] This argument raises a series of issues, which concern the relationship between cognition and reality. From a critical realist perspective the cognitive capabilities of agents come to operate as a part of the very mechanisms and structures which the human sciences are seeking to explain. Therefore, in addition to changes in sentiments or the breakdown of conventions and other institutional arrangements, fundamental uncertainty about events, in and of itself, can give rise to structural breaks and unpredictable changes in the relationships determining these same events. As such, variations in subjective cognition and attitudes can give rise to variations in objective processes and conditions, but causality also flows in the opposite direction (this argument is also made in Chapter 4 in relation to equilibrium outcomes in financial markets under uncertainty aversion). Keynes suggested that for both reasons, economic processes were too changeable for aleatory notions of probability (based on the determination of relative frequencies) to apply with any degree of confidence to the world of human conduct. In this thesis it will henceforth be assumed that uncertainty reflects structural changes both to subjective judgements and also to objective processes.

> implies the possibility that there is a permanent and positive role for government in designing policies and institutions to provide results preferable to those that would be generated by competitive markets in a nonergodic environment (Davidson, 1994a).

To clarify these arguments, the following diagram merges Gerrard's exposition with that of Davidson:

Fig. 1: KNIGHTIAN/KEYNESIAN UNCERTAINTY: A SYNTHESIS

	INCOMPLETE KNOWLEDGE	**COMPLETE KNOWLEDGE**
STRUCTURALLY DETERMINANT	***UNCERTAINTY (eg. chaos)***	***PERFECT CERTAINTY***
STRUCTURALLY INDETERMINANT	***FUNDAMENTAL UNCERTAINTY (Type 2 or 3)***	***RISK (Type 1)***

IMMUTABLE (ERGODIC) REALITY:
Type 1 - Agents know the future in the sense of actuarial certainty equivalents
Type 2 - Agents' knowledge of ergodic reality is incomplete in short-run due to bounded rationality
Subjective probabilities converge to objective probabilities through learning or Darwinian selection by market of more rational techniques => become Type 1
TRANSMUTABLE (NON-ERGODIC) REALITY:
Type 3 - Agents believe sufficient information does not exist to predict future by means of frequency distributions
Relevant to decisions involving investment, accumulationof wealth, and finance

The top right-hand quadrant represents an environment of perfect foresight - the most straightforward and simple assumption that can be made in economic analysis. Most orthodox theories presume an environment of risk, represented by the bottom right-hand quadrant, in which economic agents can determine the optimal trade-off between variance and return for any transaction or group of transactions. In this quadrant, agents have complete knowledge of the future in the sense that outcomes can be described in terms of actuarially certain equivalents.

From the diagram, it is obvious that the formation of expectations over chaotic processes occupies the top left quadrant of the table: incomplete knowledge of a structurally determinant system. In the case of chaos, knowledge is incomplete due to the infinite sensitivity of the system to initial conditions. [32]

Grandmont has argued that the existence of non-linear chaotic macroeconomic systems justifies a permanent and positive role for Government (Grandmont, 1985, 1987). In this context, the role of policy is to nudge the economy onto more preferable, but still attainable, trajectories or to manipulate key parameters to prevent any transition from systems with well-behaved and stable dynamics to ones subject to complex, chaotic dynamics[33]. Nevertheless, non-linear dynamics

[32] In my introductory comments in section 0.3.2 I endorsed the Sante Fe Institute notion that, within asset markets, complexity is typically endogenously generated through multiple and heterogeneous-agent interactions. Thus, attempts to price derivatives or optimally manage portfolios in such an environment using risk-sensitive or robust control techniques merely introduces an additional dimension of heterogeneity reflecting individual differences in uncertainty aversion and uncertainty perception. The same thing could be said of techniques in mathematical finance that are predicated on the fractal nature of the underlying security price (Elliott and van der Hoek, 2000; Aase et al., 1999, Duncan et al., 2000, Hu and Øskendal, 1999). Essentially, I would argue that these pricing exercises can be thought of as a sophisticated form of contrarian strategy situated at the more costly end of a range of other less costly strategies (e.g. such as those using lattice models incorporating implied volatility corrections). In sections 4.8 and and 5.3.5 of the thesis I re-examine these forms of uncertainty, which are more generally associated with complex rather than merely chaotic systems.

[33] More formally, intergenerational fiscal and monetary transfers flatten the intertemporal offer curve which obtains in the space spanned by real balances today and real balances tomorrow. These interventions have the potential to convert a system subject to chaotic cycles into a well-behaved one which monotonically converges onto a golden-rule steady state without cycles (see Rosser, 1990). Of course, "...this requires not only that the government knows what it is doing, but that economic agents *believe* that the government knows what it is doing, a tall order indeed."

represents one source of volatility whereas uncertainty aversion represents another. It is this latter source that I intend to emphasise.

The bottom left-hand quadrant represents an environment in which agents have incomplete knowledge of a structurally indeterminant system. Many orthodox models allow for the existence of uncertainty in this sense but are predicated on the notion that knowledge which is initially incomplete can gradually become more complete. It is this deep-rooted article of faith of much neoclassical analysis which I ultimately wish to interrogate. However, before engaging in this interrogation I want to review recent developments in utility theory that in various ways go beyond the expected utility framework of Von Neumann and Morgenstern.

1.3.5. Beyond Expected Utility Theory

A growing body of literature covering decision-making, game-theory and finance draws upon generalizations of expected utility theory. Included here are behavioural theories that generalize the axiomatic framework developed by Von Neumann and Morgenstern. Some authors have extended the axiom of continuity (as in the work of Bewley, 1986) and transitivity, others have grounded their research in the notion that agents impose a lexicographic ordering over a hierarchy of feasible probability distributions, while yet another group of researchers have focused on the independence axiom (as in the work of Chew, 1989; Dekel, 1986; and Tversky and Wakker, 1995).

(Rosser, p. 279; Also see Rosser (2000). In section 4.8 I discuss the contribution that variations in robustness can make to the complexity of dynamic outcomes.

The independence axiom, first posited by Paul Samuelson, is related to the linearity of probabilities. It requires that if $P^* = (p^*_1, \ldots, p^*_n)$ and $P = (p_1, \ldots, p_n)$ are two lotteries over a common outcome set $(x_1, \ldots, x_n)$ and, given that the α:(1 - α) probability mixture of P^* and P is the lottery $\alpha P^* + (1 - \alpha)P$, then linearity in probabilities implies that

$$\sum U(x_i)(\alpha p^*_i + (1-\alpha)p_i) = \alpha \cdot \sum U(x_i)p^*_i + (1-\alpha) \cdot \sum U(x_i)p_i \ .$$

Hence, expected utility maximisers will exhibit the following property: If lottery P^* is preferred to the lottery P, then the mixture $\alpha P^* + (1 - \alpha)P^{**}$ will be preferred to the mixture $\alpha P + (1 - \alpha)P^{**}$ for all $\alpha > 0$ and P^{**}.

To illustrate the violations of the independence axiom that were first observed in the Allais paradox, let $[x, \alpha; 0, (1 - \alpha)]$ denote a lottery that assigns the probability of α to the prize \$$x$ and the probability of $(1 - \alpha)$ to the prize \$0. Machina introduces two pairs of lotteries:

a_1 = [1,000,000, 1.00; 0, 0.00] versus a_2 = [5,000,000, 0.10; 1,000,000, 0.89; 0, 0.01]

and

a_3 = [5,000,000, 0.10; 0, 0.90] versus a_4 = [1,000,000, 0.11; 0, 0.89].

Now consider the set of lotteries over fixed outcome levels $x_1 < x_2 < x_3$, which can be represented by the set of all probability triples of the form $P = (p_1, p_2, p_3)$, where p_i = prob (x_i). Since $\sum p_i = 1$ and $p_2 = 1 - p_1 - p_3$ these lotteries can be represented by points on the unit triangle in the (p_1, p_3) plane. Upward movements in this diagram would imply that p_3 is increasing at the expense of p_2, while rightward movements would imply that p_1 is increasing to the detriment of p_2.

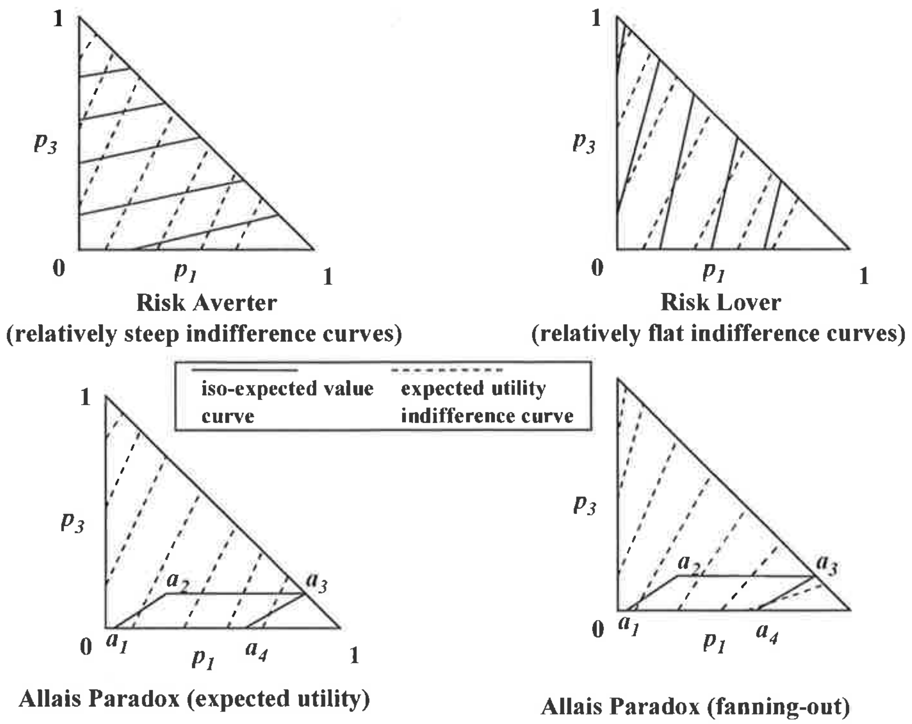
Iso-indifference and *iso-expected* utility curves can be plotted in the diagram to represent preferences. Iso-indifference curves are given by:

$$\bar{u} = \sum_{i=1}^{3} U(x_i)p_i = U(x_1)p_1 + U(x_2)(1 - p_1 - p_3) + U(x_3)p_3 = \text{constant}$$

Thus, the slope of the iso-indifference curve would be given by $[U(x_2) - U(x_1)]/[U(x_3) - U(x_2)]$, while iso-expected value curves are given by $\bar{x} = \sum_{i=1}^{3} x_i p_i = x_1 p_1 + x_2(1 - p_1 - p_3) + x_3 p_3 = \text{constant}$. In this case, the slope of the curve would be given by $[x_2 - x_1]/[x_3 - x_2]$.

Let $\{x_1, x_2, x_3\}$ ={\$0, \$1,000,000, \$5,000,000} so that these four gambles are seen as a parallelogram in the $(p_1, p_3,\}$ unit triangle. Under the expected utility hypothesis, indifference curves are assumed to have a common slope so that a risk averse individual might well prefer a_1 and a_4, while a risk lover might prefer a_2 and a_3. However, Allais and others have found that the modal, if not majority, preference is for the pair a_1 and a_3, implying that preferences *fan-out*, as in the fourth triangle diagram below:

Fig. 2: THE FANNING OF PREFERENCES



The independence axiom imposes the *common ratio effect* (CRE). Under the CRE, investors are asked to choose from amongst the following two pairs of prospects:

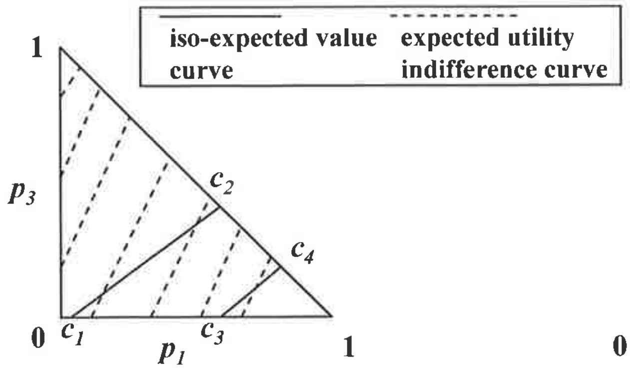
$a_1 = [X, p; 0, (1 - p)]$ versus $a_2 = [Y, q; 0, (1 - q)]$

and

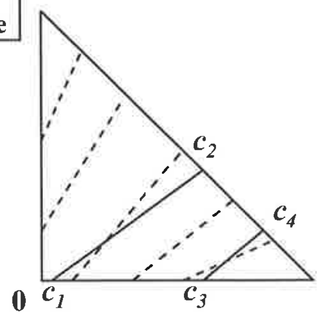
$a_3 = [X, rp; 0, (1 - rp)]$ versus $a_4 = [Y, rq; 0, (1 - rq)]$,

where $p > q$, $0 < X < Y$ and $r \in (0,1)$. Setting $\{x_1, x_2, x_3\} = \{0, X, Y\}$, the above prospects can be plotted in the standard unit triangle diagram:

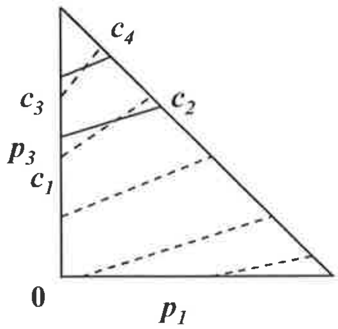
Fig. 3: THE COMMON RATIO EFFECT



The Common Ratio Effect (expected utility)



The CRE (fanning-out)



The CRE with negative payoffs (fanning-out)

Once again, in choices between each of these pairs of lotteries, observed preferences for c_1 in the first pair and c_4 in the second, or c_2 and c_3, respectively, violate the CRE.

In Savage's (1954) decision-making framework, the "sure-thing" principle plays the same role as the independence axiom (Gilboa and Schmeidler, 1989). Within this tradition, uncertainty aversion is accommodated by weakening the sure-thing principle, and is formalized mathematically through the application of Choquet expected utility theory (Choquet, 1955): a generalization of probability theory grounded in the notion of capacities. In this literature uncertainty is approached in two main forms; one of these involves *sub-additive probabilities*, while the other involves minimax optimization within a *multiple-priors* setting[34]. This research has largely motivated by the Ellsberg paradoxes (Ellsberg, 1961) that are associated with lotteries over unknown probabilities. Consider bets over draws from an urn containing 30 red balls and 60 others that are black and yellow. Fishburn (1993), in his summary of Ellsberg's findings, indicates that most people are observed to "...prefer to bet on *red* rather than *black*, and to bet on *black or yellow* rather than *red or yellow*". In Savage's model, the first preference suggests that the probability measure over *red* is preferred to the probability measure over *black*, whereas the second preference suggests the opposite.

SUB-ADDITIVE PROBABILITIES

Schmeidler (1982, 1984, 1989) introduces a weakening of the independence axiom (comonotonic independence) that allows him to construct a version of expected utility theory with non-additive probabilities. Two acts a, $b \in A$, are said to be *comonotonic* if for no states s, t $\in S$ (the state space), is it true that $a(s)\ \phi\ a(t)$ and $b(s)\ \phi\ b(t)$. A preference relation $\underline{\phi}$ on $A = \{a\text{:}\ s \rightarrow C\}$, with C a convex set, satisfies for all $a,b,c \in A$, a and c and b and c pairwise comonotonic and for all $\alpha \in]0, 1[$: $a\ \phi\ b \Rightarrow \alpha a + (1 - \alpha)c\ \phi\ \alpha b + (1 - \alpha)c$. The condition of comonotonic independence is situated somewhere between the independence axiom and Gilboa

[34] Significantly, Gilboa and Schmeidler (1989) and Chateauneuf (1991) prove that when an arbitrary closed and convex set of possible priors P is given, and a non-additive probability measure υ (convex) or v (concave) is defined on Ω such that all additive probability measures in P majorize υ or minorize v, the non-additive expected utility theory *coincides* with the maximin or maximax decision rule, respectively. The non-additive expected utility with respect to a convex (concave) capacity and the maximin (maximax) expected utility give the same solution if P is considered the core of υ (v), or a proper subset of υ (v), since by definition the core of υ (respectively v) consists of all finitely additive probability measures that majorize υ (minimize v) event-wise (Basili, 2000, p. 6).

and Schmeidler's (1989) somewhat weaker "smoothing" condition of certainty independence (to be discussed in section 2.3 below), which instead requires that $c \in C$ be a constant act[35].

Non-additive probability is by definition a set function π: **S** $\rightarrow [0, 1]$ such that $\pi(\varnothing) = 0$, $\pi(S) = 1$, and $E \subset F \Rightarrow \pi(E) \leq \pi(F)$. The following definition, due to Savage (1954), of the integral of a real value bounded function on S:

$$\int_S u(s)d\pi(s) = \int_0^M \pi(\{s \in S | u(s) \geq \alpha\})d\alpha + \int_{-M}^0 [\pi(\{s \in S | u(s) \geq \alpha\}) - 1]d\alpha,$$

still holds even when π is a non-additive probability. If $(E_i)_{i=1}^n$ is a partition of S and u: $S \rightarrow \Re$ such that $u(E_i) = \alpha_i$ for all $I = 1,\ldots,n$ and $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n$ and $\alpha_{n+1} = 0$ then:

$$\int_S u(s)d\pi(s) = \sum_{i=1}^n \left[(\alpha_i - \alpha_{i+1}) \times \pi\left(\sum_{j=1}^i E_j\right)\right]$$

Suppose a preference relation, $\succsim$, on A_f, the set of acts that obtain finitely many values, is given with the set of consequences $C = \Delta(X)$, the set of all simple probability measures on the algebra of all subsets of X (itself, the non-empty set of prizes or outcomes). Then, the following two conditions are equivalent (Karni and Schmeidler, 1991, 1802-1807):

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[35] Chateauneuf, Kast and Lapied (1996, p. 326) explain that two comonotone assets do not entail any hedging effect. Let X and Y two bounded measurable functions on the set of states S, and a σ-algebra **S**, of events in S, with **S** containing all singletons; then, they are comonotone iff their covariance is positive for any additive probability distribution on (S, **S**). Moreover, let P be such a probability distribution on (S, **S**) , let T and Z be two assets, and let F_Z^{-1} denote the generalized inverse cumulative distribution function on Z. The Bickel-Lehmann dispersion order then implies that T is more dispersed that Z if:
$\forall u, v;\ \ 0 < u < v < 1,\ \ F_T^{-1}(v) - F_T^{-1}(u) \geq F_Z^{-1}(v) - F_Z^{-1}(u)$.

(i) the preference relation $\succsim$ satisfies the Archimedean, comonotonic independence, monotonicity, and non-degenerency axioms.

(ii) There exists a unique non-additive probability π on subsets of S and a_unique up to a positive linear transformation utility u: $X \rightarrow \Re$ such that $a \rightarrow \int_S \left(\sum_{x \in X} a(s)(x)u(x)\right) d\pi(s)$ represents $\succsim$ on A_f.

Moreover, Schmeidler (1984) establishes that conditions (i) and (ii) are equivalent if A_f is replaced by $A(\succsim)$, the set of all bounded acts in A (an act $a \in A$ is said to be bounded if for some $x, y \in C$, it is the case that $x \succsim a(s) \succsim y$ for all $s \in S$). In this case, the preference relation satisfies uncertainty aversion (uncertainty aversion obtains if for $a,b \in A$, such that $a \sim b$ and for $\alpha \in [0, 1]$, it is the case that $\alpha a + (1 - \alpha)b \succsim b$); the non-additive probability π is convex (i.e. $\pi(E) + \pi(F) \leq \pi(E \cap F) + \pi(E \cup F)$); and $\int f d\pi = \min\left\{\int f dp \middle| p \in \text{core}(\pi)\right\}$, where core($\pi$) is the set of additive probability measures, p such that $p(E) \geq \pi(E)$ for all $E \subset S$.[36]

The theory of expected utility with rank-dependent probabilities may readily be deduced from the theory of expected utility with non-additive probabilities. Given the non-additive probability π on S, if there exists an additive measure P on subsets of S and an increasing function f: [0, 1] $\rightarrow$ [0, 1] onto, such that $\pi(E) = f(p(E))$ for all $E \subset S$, then any act $a \in A_f$ can be represented as a

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They then use the following result: if X and Y are comonotone, then $F_{X+Y}^{-1} = F_X^{-1} + F_Y^{-1}$. Hence, the comonotonicity of X and Y implies that $X + Y$ is more dispersed than either X or Y, individually.

[36] See the discussion on page 88 for a simple diagrammatic explanation of how this preference relation can be interpreted to imply uncertainty aversion in the form of a desire for smoothing of the utility distribution. Intuitively, uncertainty aversion can be thought of as a desire to smooth consumption both over states of nature and inter-temporally through the early resolution of uncertainty. Epstein (1999, p. 588) establishes that Choquet expected utility is a special case of a multiple-priors preference order.

lottery $p(a) \in \Delta(X)$ (not one to one). Moreover, if $X \subset \Re$, then $\int_S u(a(s))d\pi = \int u(x)df(F(x))$, where F is the distribution of $p(a)$. Karni and Schmeidler, (1991, p. 1810) observes that it has not yet been demonstrated that this deduction would continue to hold for the case of f non-decreasing, raising questions about the validity of applying non-expected utility theory to cases of economic decision-making under uncertainty[37].

MULTIPLE-PRIORS

Gilboa and Schmeidler (1989) construct their model of decision-making under a multiple-priors form of *uncertainty aversion* as follows (Karni and Schmeidler, 1991, pp. 1806-7). First, they introduce notation defining the set of consequences C, that coincides with the set of outcomes, X, the set of acts, A, consisting of all the functions from the set of states S to X; the set of acts A_f, that obtain finitely many values; the set of all bounded acts in A, $A(\phi\ )$; and finally the primitive of the model, a preference relation $\underline{\phi}$ defined over the space of acts A. Now let $a, b \in A,\ \ a \sim b$ and $\alpha \in (0, 1)$ imply $\alpha a + (1 - \alpha)b\ \underline{\phi}\ b$. This condition implies that substituting objective mixtures (in $\Delta(X)$; which denotes the set of all simple probability measures on the algebra of all subsets of

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[37] Sarin and Wakker (1992) have proposed a straightforward axiomatic foundation for expected utility with non-additive probabilities that has a Choquet integral representation. They introduce a seemingly intuitive *cumulative dominance* postulate that adapts the stochastic dominance principle to decision-making under uncertainty. However, Nehring (1994) has revealed a certain kind of arbitrariness at play in Sarin and Wakker's formulation of the cumulative dominance postulate, which either undermines its intuitive force or introduces an unintended restriction over preferences. When this arbitrariness is removed, at the same time eliminating the restriction over preferences, the resulting characterization Choquet expected utility preferences can no longer resolve the Ellsberg paradoxes that it was originally intended to explain. As a alternative to Sarin and Wakker's questionable axiomatic approach based on *cumulative dominance* Nehring favours an approach drawing on what he calls the *Indirect Stochastic Dominance* principle (Nehring, 1999). Instead, in section 2.4 I examine Tversky and Wakker's generalization of Cumulative Prospect Theory to accommodate uncertainty averison.

X, itself, the non-empty set of prizes or outcomes) for subjective mixtures can only increase the decision-maker's welfare. They also introduce the condition of monotonicity and certainty independence. Monotonicity requires that for all $a, b \in A.$, if for all s in S, $a(s)\ \underline{\phi}\ b(s)$, then $a\ \underline{\phi}\ b$. Certainty independence—a weakening of the comonotonic independence condition—stipulates that for all $a, b, c \in A.$, where c is a constant act defined by $c(s) = y \in \Delta(X)$ for all $s \in S$ and $a \in (0, 1)$, if $a\ \phi\ b$ then $a\ a + (1 - a)c\ \phi\ a\ b + (1 - a)c$. They also impose the usual Archimedean axiom that is employed within the conventional Savage framework. Suppose that a preference relation $\underline{\phi}$ on $A_f \subset \{a\colon S \rightarrow \Delta(X)\}$ is given, then the following two conditions are equivalent:

$\underline{\phi}$ satisfies certainty independence, monotonicity, uncertainty aversion, and the Archimedean axiom;

There exists a utility u: X $\rightarrow \Re$, unique up to a positive linear transformation and a convex compact subset, say K, of additive probability measures on subsets of S (compact in the weak star topology) such that $a \rightarrow \min\left\{\int_S u(a(s))dp(s) \middle| p \in K\right\}$ represents $\underline{\phi}$ on A_f;

The set K is unique iff $\underline{\phi}$ is non-degenerate (non-degeneracy implies that it is false that x $\underline{\phi}$ y for all ordered x, y $\in$ X); and,

The equivalence between (i), (ii), and (iii) holds if A_f is replaced by $A(\phi)$.

It should be noted that the above integral representation is defined, in accordance with Savage (1954), as the integral of a real-valued bounded function u on S with respect to the finitely additive probability measure p defined on S, the set of all subsets of S, as in:

$$\int_S u(s)dp(s) = \int_0^M p(\{s \in S | u(s) \geq \alpha\})d\alpha + \int_{-M}^0 [p(\{s \in S | u(s) \geq \alpha\}) - 1]d\alpha \ .$$

In Savage's framework, the integrals on the right-hand side of this expression are Riemann integrals and M is a bound on the absolute value of u. Existence of the integral is guaranteed by the monotonicity of the integrand (Karni and Schmeidler, 1991, pp. 1792-3). Under uncertainty aversion the relevant integrals are Choquet integrals defined over *capacities* rather than probability distributions.[38]

Cagliarini and Heath (2000) provide an elegant diagrammatic explanation of uncertainty aversion for the case of a binary lottery defined over two states of nature. Assets yield utility U_1 in state 1 and utility U_2 in state 2, where π is the probability of state 1 occurring. Combinations of U_1 and U_2 that yield the same utility must lie on an indifference curve with a constant slope of $\pi/(1- \pi)$. The set of constant acts or gambles, represented by C, is associated with outcomes positioned along the 45-degree line, which obviously yield constant utility irrespective of the ensuing state of nature.

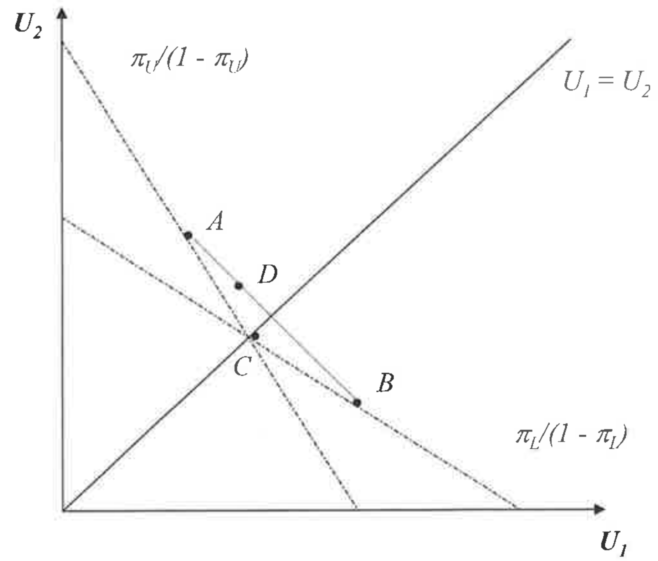
[38] Unlike the Lebesgue integral, it can be seen that the Choquet integral is defined in relation to the cumulative distribution rather than the probability density function (see Epstein, 1999, p. 588, for a more extended discussion). This form of integration establishes a clear link with the treatment of rank-dependent expected utility, which I intend to articulate in a heuristic fashion in the following pages of this chapter. For a more formal but less intuitive discussion, see Epstein (1999).

To allow this framework to encompass the multiple-priors situation, assume that a range of feasible probability distributions now applies that is bounded with an upper bound π_U and a lower bound π_L. For outcomes A and B, the usual properties of constant independence are assumed to apply in respect to mixtures with constant acts, namely:

$$A \phi B;\ \ \alpha A+(1-\alpha)C \ \phi\ \alpha B+(1-\alpha)C$$
$$A \sim B;\ \ \alpha A+(1-\alpha)B \ \phi\ \alpha B+(1-\alpha)B = B$$
$$A \phi B \ \text{ iff } \min_{\pi} E_{\pi}U(A) > \min_{\pi} E_{\pi}U(B)$$

Cagliarini and Heath interpret these conditions as implying that the smoothing or "averaging" of utility distributions would make the decision-maker better off. This sort of case can be illustrated in a straightforward fashion. In the following diagram, the consumer would be indifferent to any smoothed combination of outcomes along the line segments AC or CB, however, any combination of outcomes along the line segment AB would be preferred to the constant act C

Fig.4: CAGLIARINI AND HEATH (2000) ON UNCERTAINTY AVERSION



Tversky and Wakker's Treatment of Cumulative Prospect Theory under Risk and Uncertainty

One way of gaining an intuitive understanding of uncertainty aversion is to examine a version of non-expected utility that can easily be extended to accommodate uncertainty. This is the approach adopted by Tversky and Wakker (1995) who treat cumulative prospect theory (CPT). Other axiomatic approaches are no doubt feasible, but the one considered here is convenient in so far as it highlights the attributes of CPT under both uncertainty and risk. This distinction (one that amounts to a generalisation of the probability space) has been clearly identified in a recent paper by Tversky and Wakker (1995).

As we have seen in section 2.1, Machina shows how various forms of non-expected utility theory can be differentiated in terms of their respective value and weighting functions (1987, pp. 132-3). In expected utility theory, the weighting function is simply the relevant probability associated with each state of nature, while the value function is the utility associated with the payoff under each state. Under CPT, however, the weighting function takes on a much more complex form. For a prospect ($x_1, p_1; \ldots; x_n, p_n$), yielding outcome x_j with probability p_j, and in which outcomes are ranked in order of magnitude $x_1 \leq \Lambda \leq x_k \leq 0 \leq x_{k+1} \leq \Lambda\ x_n$, the value of the prospect is given by the following function (Tversky and Wakker, 1995, p. 1259):

$$\sum_{j=1}^{k} \pi_j^- v(x_j) + \sum_{j=k+1}^{n} \pi_j^+ v(x_j)$$

where the decision weights are defined by:

$$\pi_j^- = w^-\left(p_1 + \Lambda\ p_j\right) - w^-\left(p_1 + \Lambda\ p_{j-1}\right) \text{ and}$$
$$\pi_j^+ = w^+\left(p_j + \Lambda\ p_n\right) - w^+\left(p_{j+1} + \Lambda\ p_n\right)$$

Here, the w^+ and w^- are the weighting functions for gains and losses, respectively, and both the weighting functions and the value function $v(x_j)$ are assumed to be continuous and strictly increasing[39]. Tversky and Wakker (1995, p. 1259) show that CPT generalizes the concept of rank-dependent utility, first introduced by Quiggin (1982) and Yaari (1987). Rank-dependent utility only coincides with CPT only for *non-negative* outcomes[40]. In an earlier paper, Wakker and Tversky (1993) provide axiomatic foundations for cumulative prospect theory (CPT).

Tversky and Wakker (1995) examine the properties of the weighting function, $w(.)$ used in CPT and show that this particular function satisfies what they call *bounded subadditivity* (*SA*), as defined and illustrated below (Tversky and Wakker, p. 1260):

Bounded sub-additivity obtains if there exist boundary constants $\varepsilon \geq 0, \varepsilon' \geq 0$ such that:

$$w(q) \geq w(p+q) - w(p) \text{ whenever } p + q \leq 1 - \varepsilon, \text{and}$$
$$1 - w(1-q) \geq w(p+q) - w(p) \text{whenever } p \geq \varepsilon'$$

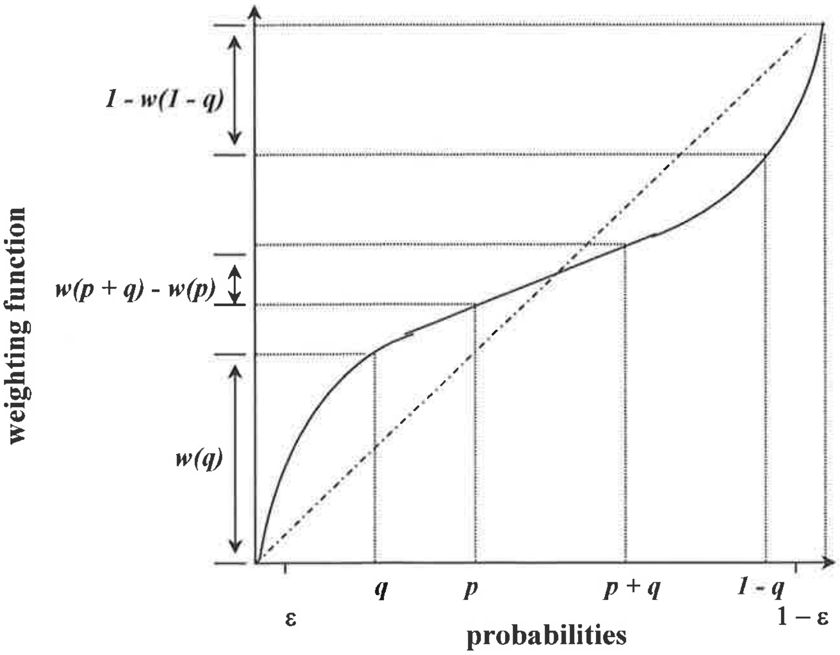
The first of these conditions—upper sub-additivity—implies that a shift in probability has more impact when it makes an event certain than when it makes an event more probable. The second

---

[39] In what follows, the + and – superscripts will be deleted for cases where only non-negative prospects are examined. The weighting function is a strictly increasing function from [0, 1] to [0, 1] with $w(0) = 0$ and $w(1) = 1$. Tversky and Wakker (fn. 4, p. 1259) also adopt the usual convention that, for $j = 0$, $p_1+\ldots+p_j = 0$, and for for $j = n$, $p_{j+1}+\ldots+p_n = 0$.

of these conditions—lower sub-additivity—implies that a shift is probability has more impact when it makes an event possible than when it merely increases the probability of an event.

Fig. 5: TVERSKY & WAKKER (1995) ON SUB-ADDITIVITY



The constants $\varepsilon, \varepsilon'$ are boundary constants, which are imposed to ensure that an interval that includes 0 or 1 is always compared with an interval that does not. The CPT weighting function, one of the variety identified by Machina (1987, pp132-6) as departing from linearity in probabilities, can be interpreted as implying that, on one hand, individuals are risk-seeking for gains and risk averse for losses of *low* probability, while on the other hand, they are risk averse for gains and risk seeking for losses of *high* probability. These characteristics are revealed in the shape of the weighting function that is concave in the lower portion of the curve and convex in

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[40] Rank dependent utility corresponds to the special case where the weighting function for losses under CPT is the *dual* of the weighting function for gains i.e., $w^{-}(p) = 1 - w^{+}(1-p)$ (Tversky and Wakker, p 1259).

the upper portion. The diagram clearly shows that it is this resulting *S*-shape that is responsible for bounded sub-additivity[41].

For applications of *CPT* under *uncertainty*, a prospect is defined in conventional terms as a function from the state space S to $\Re$ taking finitely many values. An uncertain prospect is described as $(x_1, A_1;\ldots;x_n,A_n)$, where the value of a prospect $(x_1, A_1;\ldots;x_n,A_n)$, with $(A_1,\ldots,A_n)$ being a partition of S and x_j being the outcome associated with the states in A_j, and in which $x_1 \leq \Lambda \leq x_k \leq 0 \leq x_{k+1} \leq \Lambda\ x_n$, is represented by decision weights modified in the following manner (Tversky and Wakker, 1995, p. 1264)[42]:

$$\sum_{j=1}^{k} \pi_j^- v(x_j) + \sum_{j=k+1}^{n} \pi_j^+ v(x_j),$$

where the decision weights are defined by:

$$\pi_j^- = W^-(A_1 \cup \Lambda \cup A_j) - W^-(A_1 \cup \Lambda \cup A_{j-1}) \text{ and}$$
$$\pi_j^+ = W^+(A_j \cup \Lambda \cup A_n) - W^+(A_{j+1} \cup \Lambda \cup A_n)$$

Under uncertainty a further assumption, which Tversky and Wakker call *solvability*, must be made about the nature of the weighting function. This is merely Gilboa's (1987) convex-range assumption that for all events $A \subset C$ and $W(A) \leq p \leq W(C)$ there exists B such that $W(B) = p$ and $A \subset B \subset C$.

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[41] In empirical applications the weighting function can be estimated by fitting the parametric form $w(p) = \delta p^\gamma / (\delta p^\gamma + (1-p)^\gamma)$.

Tversky and Wakker (1995, p. 1264) propose the following definition of sub-additivity under uncertainty:

> If there exist lower and upper boundary events $E \geq 0, E' \geq 0$, such that $W(B) \geq W(A \cup B) - W(A)$ whenever $W(A \cup B) \leq W(S - E)$ and,
> $1 - W(S - B) \geq W(A \cup B) - W(A)$ whenever $W(A) \geq W(E')$.

The first of the above conditions imposes *lower* sub-additivity, while the second condition imposes *upper* sub-additivity. Tversky and Wakker interpret this notion of sub-additivity under uncertainty, in the following intuitive fashion:

> ...[u]nder *SA,* an event B has a greater impact when it turns impossibility into possibility or possibility into certainty, than when it merely makes a possibility more likely (1995), p. 1264).

Sub-additivity is characterized by two preference conditions. As a preliminary, in the absence of probabilities Tversky and Wakker observe that inequalities such as $W(A) \geq W(B)$ must be defined in terms of preferences so that $A \ \underline{\phi}\ B$ only if there is a gain Z such that $(Z, A; 0, A^c) \ \underline{\phi}\ (Z, B; 0, B^c)$ (a dual definition applies to losses). The required conditions can now be written as:

$$z \sim (0, S - (A \cup B); z, B; Z, A) \Rightarrow (z, S - B; Z, B) \underline{\phi} (Z, A \cup B; 0, (A \cup B)^c),$$

whenever $0 < z < Z$ and $A \cup B \ \underline{\pi}\ S - E$, and

$$(z, S - B; 0, (S - B)^c) \sim (Z, A; 0, A^c) \Rightarrow z \underline{\phi} (0, S - (A \cup B); z, B; Z, A)$$

whenever $0 < z < Z$ and $A \ \underline{\phi}\ E'$, with E and E' being boundary events.

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[42] Tversky and Wakker (fn. 5, p. 1259) assume that, for $j = 0$, $A_1 \cup \ldots \cup A_j = \varnothing$, and for $j = n$, $A_{j+1} \cup \ldots \cup A_n = \varnothing$. Moreover, $W(\varnothing) = 0$, $W(S) = 1$, and $W(A) \geq W(B)$ whenever $A \supset B$.

To interpret these conditions note that lower sub-additivity implies the first condition, a fact which can readily be confirmed by noting that the CPT difference between the left prospects is $W(B)(v(Z)-v(z))$ while the difference between the right prospects is $(W(A \cup B) - W(A))(v(Z)-v(z))$, where both $v(z)$ and $(v(Z) - v(z))$ are > 0. Moreover, upper sub-additivity implies the second condition because the CPT difference between the left prospects is $(1 - W(S - B))v(z)$ while the difference between the right prospects it is $(W(A \cup B) - W(A))v(z)$.

Tversky and Wakker (1995, p. 1269) show how the notion of sub-additivity can be used to compare two different weighting functions. For such a comparison, one of the functions must be a strictly increasing transform of the other and must meet certain conditions relating to event unions and boundary events. As one would expect, these conditions over events carry over to similar conditions on the lower and upper sub-additivity of preferences. Tversky and Wakker (p. 1267) also introduce the notion of a sub-additive transformation to establish a definition of *more sub-additive than*. A transformation ϕ: [0, 1] → [0, 1] is *SA* if it has the same mathematical properties as a *SA* weighting function. In this case, one weighting function is then more *SA* than another if the first can be obtained from the second by an *SA* transformation. They examine the conditions that must obtain over both weighting functions and preferences to compare the sub-additivity of two or more weighting functions.

RISK-SENSITIVE CONTROL AND NON-EXPECTED UTILITY THEORY

In the mathematical finance literature, techniques of risk-sensitive stochastic control are often used to accommodate uncertainty aversion (Fleming, 1995; Hansen, Sargent and Tallarini, 1999;

Tornell, 2000; Epstein and Zin, 1989). In this technically demanding literature the curvature of the weighting function that Tversky and Sarin have emphasized in their version of non-expected utility theory is effectively reproduced through the use of a value function that has exponential form. It can be shown that this exponential value function is closely related to the relative entropy measure: a formal link that comes to the fore in applications of risk-sensitive value-functions to asset-pricing[43].

Space constraints prevent detailed discussion of this complex issue, but a simple diagrammatic exposition drawn from related work on entropy-based measures of income inequality (Conceiçào and Ferreira, 2000) might help to identify what is at stake here. The following table shows a simple binary lottery with respective probabilities—both actual (or a-posteriori) and expected (or a-priori)—for each state of nature s_1 and s_2:

A-POSTERIORI DISTRIBUTION		**A-PRIORI DISTRIBUTION**	
$\boldsymbol{P_1}$	0.82	$\boldsymbol{Q_1}$	0.36
$\boldsymbol{P_2}$	0.18	$\boldsymbol{Q_2}$	0.64

The following graph depicts three measures of the discrepancy between the actual or a-posteriori, and the expected or a-priori probabilities as the a-posteriori probability of the first state of nature (p_1) varies above and below its given value of 0.82. The first of these, M_1, represented by the dashed line, is simply the absolute value of the difference between the actual (p_1) and expected (q_1) probability of the first state of nature:

[43] See Kapur, and Kesavan (1994, Chapter 4). In risk-sensitive control applications entropy measures the tradoff

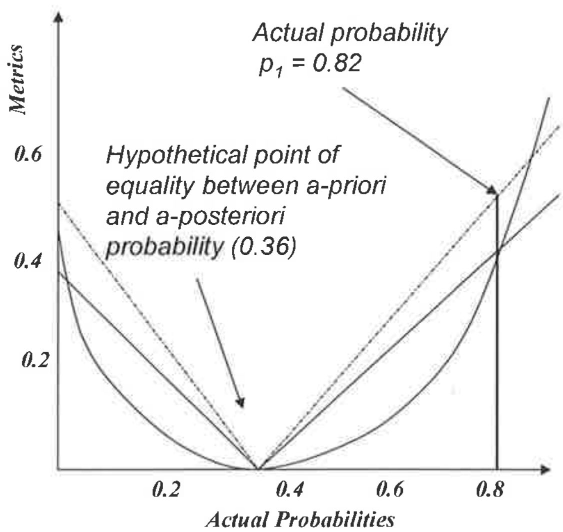
$M_1 = |p_1 - q_1| = |0.82 - 0.36| = 0.46$.

The second, M_2 represented by the continuous linear V-shaped line, is the absolute value of the weighted sum of the differences between each respective pair of probabilities, where the weights are the inverse of the expected probabilities:

$M_2 = |(1/q_1)(q_1 - p_1) + (1/q_2)(q_2 - p_2)| = 0.58$.

The third, *RE*, is the relative entropy measure represented by the continuous curve, which is the weighted sum of each of the logarithms of the ratios of a posteriori to a priori probabilities, where the weights are the respective a posteriori probabilities.

Fig. 6: RELATIVE ENTROPY AND CHANGES IN PROBABILITY MEASURE



$RE = p_1[\ln p_1 - \ln q_1] + p_2[\ln p_2 - \ln q_2] = 0.46$

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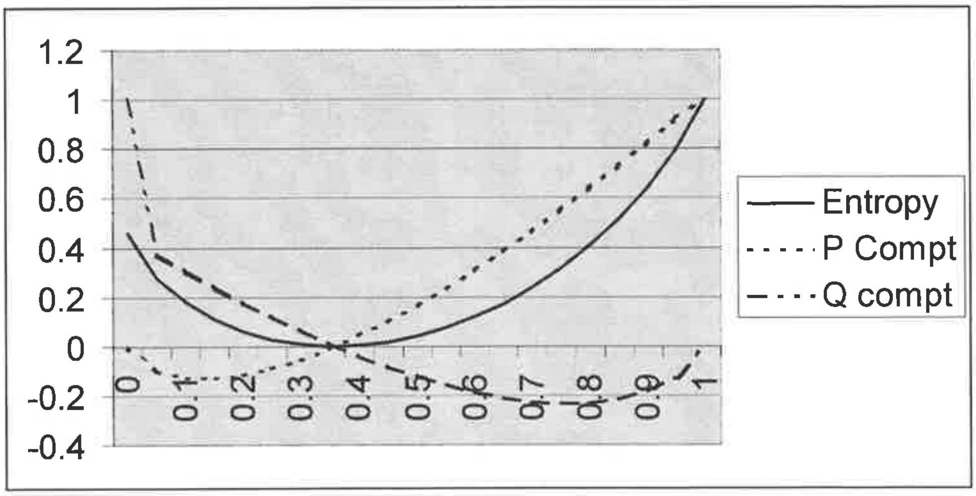
between linear quadratic gaussian and robust or H-infinity forms of mini-max control that are each individually incorporated into the risk-sensitive value-function as separately defined limit cases (Hansen, et al. 2001).

A dark vertical line has been drawn at the selected value of p_1 (the actual probability of the first state) which equals 0.82. When the actual probability happens to equal its conjectured value of 0.36 then all the measures of discrepancy become equal to zero. Conceiçào and Ferreira (p. 12) observe that to the right of the actual value of 0.82, the entropy measure increases at a greater rate than that of the weighted linear measure M_2. Similarly, to the left of this value, the entropy measure decreases at a more rapid rate than does M_2. However, at points close to the hypothetical value of 0.36, the entropy measure is far less sensitive to discrepancies between actual and expected or hypothesized probabilities. Additional insight can be gained from examining a graph that decomposes the entropy measure into the contributions made by each pair of probabilities (p_1, q_1) and (p_2, q_2). In the following diagram it can be seen that each pair's contribution changes sign as the actual share moves from positions below the hypothetical share to positions above the hypothetical share. However, the positive contributions are always higher than the negative contributions making the overall measure of entropy positive. The shape of each of the negative contributions (as they each possess a minimum point, both experience a change in sign of the first derivative) is responsible for the previously discussed finding that the entropy measure approaches zero more rapidly at first and then more slowly than for the M_2 measure as the actual and expected probabilities coincide (Conceiçào and Ferreira, pp.15-6)[44].

[44] Conceiçào and Ferreira note that the entropy measure of income inequality is the only class of measures that can be decomposed into between-region and within-region measures and where each region's inequality measure can itself be further decomposed into between-sub-region and within-sub-region measures and so on. When entropy measures are instead applied to the estimation of probability distributions, this characteristic of decomposability carries with it certain desirable properties of statistical inference that relate to the decomposability of the measure

Fig. 7: CONTRIBUTIONS TO THE ENTROPY MEASURE



EPSTEIN'S DEFINITION OF UNCERTAINTY AVERSION AND EVENT-WISE DIFFERENTIABILITY

Larry Epstein (1999) has developed a rigorous definition of *uncertainty aversion* that is sufficiently general to cover both the case of Choquet expected utility (CEU) theory (Schmeidler, 1989), and the multiple-priors model (Gilboa and Schmeidler, 1989). By analogy, he draws a direct link between a particularly general (and subjective) approach to risk-aversion and another, equally general (and subjective) approach to uncertainty aversion. Epstein establishes a direct correspondence between the benchmarking of risk-aversion against *risk-neutrality* and the benchmarking of uncertainty aversion against *uncertainty-neutrality*. Risk aversion is formally related to risk-neutrality through an implication to the effect that if one preference order establishes a weak (strict) ordering $\underline{\phi}$ (ϕ) over the set of outcomes x that are weakly (strictly)

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with respect to subsets of the filtration and associated sigma field of the given stochastic process (see Shore and Johnson, 1980).

preferred to a particular event e, then that initial preference order, say $\succsim_1$, implies that for a more risk-averse preference order, say $\succsim_2$, the same set of outcomes would also be ranked as weakly (strictly) preferred to the particular event e. In formal notation, for every act e and outcome x, $\succsim_2$ is more risk-averse than $\succsim_1$ if (p. 583):

$$x \succsim_1 (\succ_1) e \Rightarrow x \succsim_2 (\succ_2) e$$

By analogy, uncertainty aversion is defined by the implication holding between one preference relation and another, but in this case the weak ordering is established between two sets of acts: unambiguous acts and ambiguous acts. Associated with the set of unambiguous acts is a "probabilistically sophisticated" weak ordering $\succsim_{ps}$ that assigns subjective probabilities to each of the events encompassed within each unambiguous act so that any act is thereby transformed into a lottery or pure risk. Now, risk-neutrality is defined in reference to the expected value function that weights each act by the probability measure applying to it in a particular state of nature. Once again, by analogy, uncertainty neutrality is defined in relation to the probabilistically sophisticated ordering $\succsim_{ps}$, as follows: $\succsim$ is *uncertainty averse* if there exists an order $\succsim_{ps}$ such that: $\succsim$ is more uncertainty averse than $\succsim_{ps}$. That is, for any unambiguous act h and any ambiguous act e the following condition must hold: (p. 585)

$$h \succsim_{ps} (\succ_{ps}) e \Rightarrow h \succsim (\succ) e .$$

In the context of simple binary lotteries defined over the set of unambiguous acts A, events E, and outcomes x_1 and x_2 possessing the properties:

$$[x_1, A; x_2, A^c] \succsim_{ps} [x_1, E; x_2, E^c] \Rightarrow [x_1, A; x_2, A^c] \succsim [x_1, E; x_2, E^c]$$

Epstein shown that his definition of uncertainty aversion carries the following implication (p. 591):

$$\text{For } v(E)=U([x_1,E;x_2,E^c]), \qquad mA \geq (>) mE \Rightarrow vA \geq (>) vE .$$

Here, m is the supporting probability measure in the class of probability measures P defined over the state space S and its associated sigma algebra Σ and v is the convex capacity defined over both the families of ambiguous and unambiguous acts. Given a class **A** of unambiguous acts, all measures in P would be identical when restricted to this class (i.e., $mA = m'A$ for all m and m' in P and A in **A**). Epstein suggests that a multiple-priors preference order $\underline{\phi}$ defined over P could then be represented by a utility function U^{mp} of the form:

$$U^{mp}(e)=\min_{m\in P}\int_S u(e)dm .$$

Here u is a Von Neumann-Morgenstern index (vNM) u: $\mathbf{X} \rightarrow \Re^1$.

For the Choquet Expected utility (CEU) representation, Epstein defines P in terms of the core of the convex capacity v, as follows (p. 588):

$$P=\text{core}(v)=\{m : m(\cdot)\geq v(\cdot) \text{ on } \Sigma\}.$$

For outcomes ranked as $x_1 \ \phi \ x_2 \ \phi \ \Lambda \ \phi \ x_n$ and the act e such that $e(x_i) = E_i$, $i = 1,\ldots,n$: the CEU utility function U^{ceu} is given by:

$$U^{ceu}(e)=\sum_{i=1}^{n-1}[u(x_i)-u(x_{i+1})]v\left(\mathrm{Y}_1^i E_j\right)+u(x_n).$$

Epstein proves that the supporting utility function U^{ps} has the form:

$$U^{ps}(e) = W(\Psi_{m,e}) \text{ where } W(\Psi) = \int_{\mathbf{A}} u(x) d(g \circ \Psi)(x),$$

which is a member of the *rank-dependent* class. Here, g: [0, 1]→[0, 1] is a bijection such that:

$$m \in \text{core}(g^{-1}(v)) \text{ and } m(\cdot) = g^{-1}(v(\cdot)) \text{ on } \mathbf{A}.$$

.Another strength of Epstein's paper is that for CEU preferences, the definition of uncertainty aversion is adapted to the more relevant single-stage Savage domain of acts rather than the two-stage Anscombe-Aumann domain of acts based, initially, on subjective preferences over horse-lottery acts and, subsequently, on objective preferences over roulette-wheel acts.

Epstein also examines the conditions for *event-wise differentiability* of utility that assist in the determination of the "local probabilistic beliefs" implicit in an arbitrary preference order. These beliefs represent the agent's (possibly non-unique) "mean" or "ambiguity-free" likelihood assessments of acts implicit in utility from the perspective of a given act. However, Epstein confirms that event-wise differentiability ensures *uniqueness* of these likelihood assessments. Significantly, he notes that this uniqueness cannot be established under either Frechet or Gateux differentiability[45]. In the same sense that differentiability simplifies the task of testing for concavity and hence risk aversion of the vNM index, Epstein argues that event-wise

[45] Machina (1982) demonstrates that much of expected utility analysis is robust to modifications of the indepependence axiom required to explain behaviour associated with the Allais paradoxes if the preference function is suitably smooth. That is, locally, the preference function must have a local linear approximation, in which case it is *locally* (in the space of distribution functions) an expected utility functional. Gateux differentiability extends the notion of a directional derivative, and unlike the Frechet differentiability condition employed by Machina, its definition does not require a norm on the domain of the functional (for a summary exposition see Karni and Schmeidler, 1991, section 3.5, pp. 1781-6).

differentiability "…provides a practicable characterization of uncertainty aversion.." (p. 593)[46]. For further details on this technically demanding but insightful analysis I direct the reader to section 4 of Epstein's paper (pp. 593-9).

1.3.11. Ambiguity versus Fundamental Uncertainty

In a recent paper, Dequech's explicit intention is to identify aspects of "…near mainstream economics that, superficially at least, resemble PKE" (Dequech, 2000, p. 2). These aspects concern uncertainty, liquidity preference and asset choice. In this regard, Dequech examines research that I have already examined above, which generalizes on expected utility (EU) theory—typically, by drawing upon Knightian notions of *uncertainty* or *ambiguity*.

Initially, Dequech defines a notion of *strong uncertainty* that stands opposed to orthodox notions of *weak uncertainty* or *risk* that are "…characterized by the presence of a unique, additive and fully reliable probability distribution" (p. 4). However, he subsequently establishes a further dichotomy between what he terms *fundamental uncertainty* and *ambiguity*. In this distinction he follows Camerer and Weber (1992, p. 330), who identify ambiguity with "…uncertainty about probability, created by missing information that is relevant and could be known". In keeping with this notion Dequech describes the urn problem, delineated by Ellsberg, as a clear example of ambiguity because information about the contents of the urns exists and could potentially be revealed to the decision-maker. He contends that both the multiple-priors and subadditive

[46] For convenience, Epstein adopts a dual approach commencing with the inverse correspondence representation of an act e^{-1}, where $e^{-1}(x)$ denotes the event E on which the act assumes the outcome x (i.e. under this mathematically

probability approaches adopted by those seeking to generalize EU theory conform to this notion of ambiguity. In contrast, the latter case of fundamental uncertainty is characterized by

> ...the possibility of creativity and structural change and therefore significant indeterminacy of the future. The future cannot be anticipated by a fully reliable probabilistic estimate because *the future is yet to be created*. Surprises may occur, both as intended and as unintended consequences of human action. The very decisions that would require a fully reliable probabilistic guide may change the socio-economic future in an unpredictable way, and this possibility of change prevents such a fully reliable guide from existing (p. 8).

For Dequech, the key defining quality of situations of fundamental uncertainty is that some of the information relevant for decision-making purposes "*...cannot be known, not even in principle*" at the time of making the decision. In such cases, even though the agent might construct a subjective probability distribution for use in making decisions, he or she should openly acknowledge the unknowability of all possible events. Under this notion of fundamental uncertainty he subsumes both Paul Davidson's ontologically grounded concept of non-ergodicity in stochastic processes and also the philosophical approach adopted by those such as O'Donnell (1991), Carabelli and De Vecchi (2001) and Lawson (1999a,b), who follow the position articulated by Keynes in his *Treatise on Probability*.

In constructing a world wherein states of nature are defined as independent of acts, Dequech argues that proponents of EU theory preclude any possible consideration of fundamental uncertainty because people's acts, whether intended or unintended, cannot create new states of

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equivalent *dual* perspective acts are viewed as assigning a common set of outcomes to different event rather than the *primal* perspective that views distinct acts as assigning a different set of outcomes to common events).

the world. Although EU theory under ambiguity can certainly accommodate cases such as those constructed by Ellsberg, they cannot deal with situations of fundamental uncertainty. Dequech is careful to stress that situations of fundamental uncertainty do not necessarily imply complete ignorance. As such, "the ordinal degree of uncertainty regarding the result of a decision may vary over time" (p. 9). Moreover, while informational asymmetry may apply to existing information it obviously cannot apply to information that can never come into existence.

Returning to views expressed by Keynes in the *Treatise on Probability*, Deqeuch suggests that its conceptual framework can distinguish between both weak and strong uncertainty. Weak uncertainty would refer to the presence of numerical less-than-unity probabilities and maximum weight. Both cases of strong uncertainty would be characterized by low weight reflecting the lack of reliability of knowledge and incompleteness of evidence. However, the case of ambiguity would refer to the presence of interval probabilities and a predetermined list of possible events, whereas the case of fundamental uncertainty is open-ended in regard to the range of possible events. As such, "...[w]e cannot completely establish how complete our information about the future is." (Dequech, 2000, p. 12).

It is important to emphasize that certain characteristics of what Dequech terms fundamental uncertainty can be accommodated through the use of techniques of robust and risk-sensitive control under stochastic uncertainty constraints, including cases where the relevant stochastic processes are non-ergodic. For example, when the matrix of transition probabilities in a hidden-Markov model is irreducible, after departing from a given state of nature or regime, and once sufficient time has passed, there is no guarantee that a particular trajectory will ever return to that initial state of nature or regime[47].

Moreover, the imposition of a relative entropy constraint in risk-sensitive control theory implies that the actual conditional joint distribution function could depart fundamentally from the chosen reference model (say one that is Gaussian or Hidden-Markov): and in ways that to a large extent remain unknown, if not unknowable (at the very least, for the duration of the modeling horizon). All that is required is that the spectral energies of these processes, in so far as they deviate from the reference model, fall within the pre-specified relative entropy bounds.

Implicitly, the presumption is that these pre-imposed bounds do not diminish over time, either through the operation of some learning mechanism, the acquisition of new information, or some surreptitious, evolutionary process of market selection. Thus, at least in a narrowly conceived sense, the entropy constraint can be envisaged as representing our relative ignorance about the relevant states of nature that pertain to the control problem under consideration.[48]

In a broader sense, it can still be argued that the application of stochastic uncertainty constraints governed by model uncertainty, observation error and external perturbation could *potentially* be alleviated if more complete knowledge was available. It is thus conceivable that each of these

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[47] Nevertheless, strictly speaking, this form of non-ergodicity relies on a *contraction* in the dimension of the regime switching space, whereas, the surprises associated with Dequech's notion of fundamental uncertainty are equally likely to be associated with an unpredictable expansion in the number of attainable regimes.

[48] In a personal communication with me, Barkley Rosser has pointed to an obvious correspondence between this robust approach and the literature on consistent expectations equilibria (Grandmont, 1998; Hommes and Sorger, 1998), where agents mimic outcomes for an incompletely known model by using simple autoregressive rules.

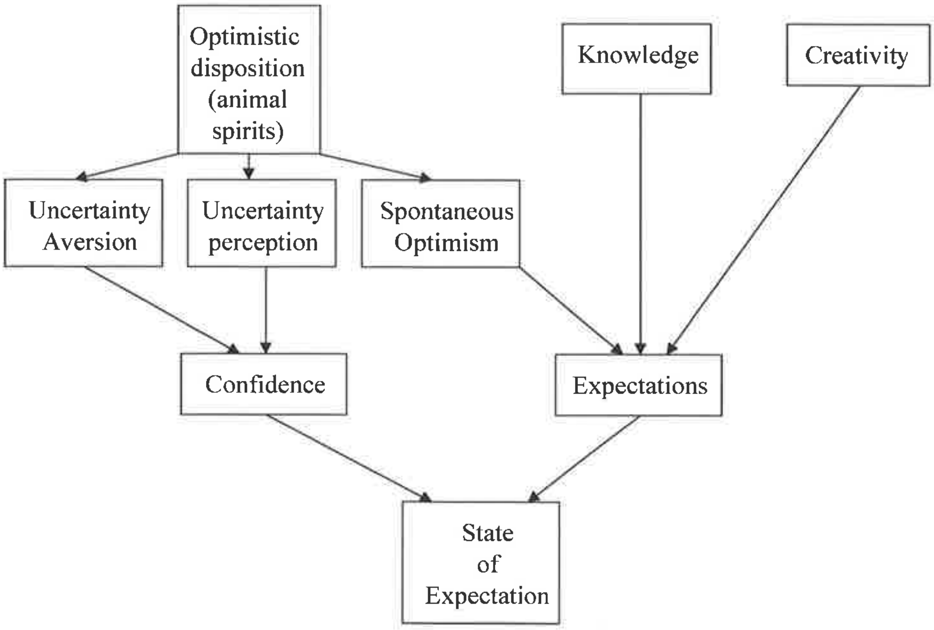
sources of error or perturbation could ultimately be removed or at least modeled in a completely deterministic form. Nevertheless, it is the contention of this thesis that, as they stand, these techniques of risk-sensitive control can still provide rigorous microeconomic foundations for modeling the uncertainty-related phenomena of liquidity preference and animal spirits.

1.3.12. David Dequech's Distinction between Uncertainty Aversion and Uncertainty Perception

Following on from the arguments presented in the first section of this Chapter, it should be clear that I, like David Dequech (p. 416), prefer to adopt an *ontological* approach to fundamental uncertainty: it is best viewed as an objective feature of economic life, not merely a subjective attitude reflecting one's level of confidence in the probability distributions that are applied in decision-making contexts (due solely to the incompleteness of knowledge).

Dequech views surprise as an intended or unintended consequence of creative human activity. This creative openness is a source of incomplete knowledge or ignorance. Although no absolute benchmark exists against which our ignorance can be measured, Dequech argues that we can nevertheless *grade* uncertainty (i.e. *ordinally*) in terms of whether we are uncertain to a greater or lesser degree (fn 4, p. 417). The diagram immediately below summarizes his framework (p. 418):

Fig. 8: DEQUECH (2000) ON UNCERTAINTY AND EXPECTATIONS



In this diagram, uncertainty is reflected in what he chooses to call the *state of expectation*. The *fundamental* determinants of state of expectation are seen to be knowledge, creativity and optimistic disposition to face uncertainty. *Immediate* determinants of state of expectation are seen to be confidence and expectations. While expectations are determined by knowledge, creativity and animal spirits in the form of spontaneous optimism, confidence is determined by both uncertainty aversion and uncertainty perception. Each of these in turn is seen as influenced by animal spirits: the latter defined as an optimistic disposition to face uncertainty (recognizable in Paul Davidson's use of the phrase: "damn the torpedoes and full speed ahead!" to capture attitudes towards corporate investment).

Dequech clarifies the notion of animal spirits by defining it to mean the (ordinally gradeable) spontaneous disposition to act in certain ways *combined* with an overall spontaneous optimism or pessimism about future outcomes. However, he favours a partially objectivist account of animal spirits in arguing that they are influenced by the institutional environment in which agents operate. He contends that uncertainty perception may be influenced by animal spirits to the extent that an optimistic disposition may encourage decision-makers to ignore certain evidence that uncovers uncertainty.

Dequech suggests that uncertainty aversion is directly and solely governed by animal spirits. In contrast, he argues that uncertainty perception is also influenced by knowledge—in the form of an actual awareness of the existence of uncertainty. Thus, uncertainty perception is a state that may be influenced, on one hand, by economic theory itself and, on the other hand, by an understanding on the part of agents about how social factors at their disposal might reduce uncertainty.

In his diagram, Dequech contends that knowledge also *directly* influences expectations, but he provides no further discussion of the matter. One could conjecture that Dequech sees knowledge as directly influencing decision-makers in their capacity to form their expectations—through the construction of appropriate models, their calibration, and intelligent application—in part, along the lines favored by rational expectations theorists. However, he views expectations as also being influenced by spontaneous optimism: a notion that he defines as "optimism *not based on any* *knowledge*". Creativity in expectations is conceived (p. 422) as the ability to imagine a future that is, at least in some respects, different from the present. Dequech emphasizes the innovative aspect of creativity as a perception of future technical and economic opportunities that might be exploited.

Dequech takes pains to distinguish confidence, as the combined outcome of uncertainty perception and uncertainty aversion, from 'weight', arguing that the narrower notion of weight relates to uncertainty perception alone. Thus weight and confidence certainly move in the same direction, but should not be collapsed into one concept.

Finally, Dequech establishes an inverse relationship between liquidity preference and the confidence a decision-maker has in his or her estimates of the returns from *holding* assets of varying liquidity, whereas he associates estimates of returns from *waiting to buy* liquid assets with the *speculative* demand for liquidity. The gradeability of liquidity preference can then be related in inverse proportion to the gradeable characteristics of factors that influence the state of confidence: including factors such as animal spirits and knowledge.

1.3.13. Barriers to Evolution from Uncertainty to Risk – Runde's Critique of the Bayesian Conquest

I have argued above, that a key aspect of risk is the notion that knowledge can gradually evolve from incompleteness to completeness or actuarial certainty. In many cases, this notion is grounded in a Bayesian framework that has agents updating their initial subjective probabilities in response to new information. Over time, these subjective probabilities asymptotically approach the true objective probabilities associated with the relevant events.

In a 1994 paper, Joachem Runde presents a rigorous critique of Bayesian attempts to tame uncertainty by placing them somewhere on a continuum with subjective probabilities or "hunches" at one end and objective probabilities at the other. For example, Ramsey (1931) represents the rational Bayesian agent's utility by a real-valued utility function (unique up to order and scale) defined over the set of possible consequences[49]. If the agent is indifferent between the certainty of *b* on one hand and a two way gamble between *a* if *h* obtains, and *d* if *not-h* obtains, then his or her degree of belief in *h* ($p(h)$) is equal to $[\mathrm{u}(b) - \mathrm{u}(d)]/[\mathrm{u}(a) - \mathrm{u}(d)]$. In other words, the probability of an event *h* is defined implicitly by the equivalence relation: $\mathrm{u}(b) = p(h)\mathrm{u}(a) + (1 - p(h))\mathrm{u}(d)$. However, the agent is presumed to be perfectly definite in his or her valuations of the consequences and their associated probabilities (i.e., the calculated mathematical expectations).[50]

Runde contends that, whereas Bayesians emphasise the agent's disposition to act, instead Keynes emphasises inaction or the potential reversibility of actions (i.e., asset markets provide a means by which investors are saved from having to make irrevocable investment decisions). Where a

[49] At this point Runde observes that he departs from Ramsey's approach in presuming, for convenience of exposition, that bets are made in terms of money rather than utility (Runde, 1994, p. 199). This is an important point that I shall shortly re-examine.
[50] It should be noted that Diaconis and Freeman (1986) have established that the claim that subjective probabilities converge on objective probabilities in Bayesian analysis is not true in an infinitely dimensional space.

bet promises a fixed net gain or loss when it is taken, assets can usually be liquidated when expectations are disappointed. Hence, their attractiveness may depend on the way they function as a mechanism for transferring purchasing power through time onto other assets that possess potentially more lucrative but yet to be determined returns.

In Savage's canonical version of the Bayesian model, illustrated below, agents are assumed to choose from a set of acts ($a_1, a_2,...a_n$) taking into account the consequences (c_{ij}) of each act for each possible state of the world ($s_1, s_2,...s_m$). Preference relations are then defined for any pair of acts defined in this way. Any assignment of consequences to states constitutes an act but these consequences may not include a reference to features of any particular act, state or choice problem.

Fig. 9: RUNDE'SCRITIQUE OF THE BAYESIAN APPROACH

Degree of belief in *h* determined by agent's indifference between:
•certainty of *b*
•two way gamble between *a* if *h*, *d* if not *h*

ie. $u(b) = p(h)u(a) + [1 - p(h)]u(d)$
hence: $p(h) = [u(b) - u(d)]/[u(a) - u(d)]$

Savage's canonical version of Bayesian model:

States / *Acts*		s_1	s_2	....	s_m
a_1		c_{11}	c_{12}	....	c_{1m}
a_2	*outcomes*	c_{21}	c_{22}	....	c_{2m}
....		....	....	....	....
a_n		c_{n1}	c_{n2}	....	c_{nm}

However, Runde argues that liquidity considerations lead to violations of this "rectangular field" assumption (Runde, 1994, p. 207). The return on a specific asset, for example, will depend on the particular characteristics of that asset. (e.g. liquidity of houses and tractors versus bonds or equities). As shown in the following diagram, Runde asserts that for each act liquidity will depend on counter expected events (i.e., on how the situation would look if other states had obtained and what might have been for each state if a different act had been chosen)[51].

> More formally, for any act a_i the utility of every c_{ik} will reflect a liquidity premium that is a function of the s_j's ($j \neq k$) and the a_j's ($i \neq j$)...Liquidity and Keynes's notion of liquidity premium make it impossible to separate consequences from acts, states and

[51] In an equally fundamental, Knightian fashion, Paul Davidson (1991) has questioned whether the full range of relevant states of nature can actually be determined by economic agents, let alone assigned an appropriate set of probabilities.

> particular choice problems in the way that the Bayesian model requires (Runde, 1994, p. 207-8).

It should be noted that liquidity preference implies incomplete markets for contingent commodities, due to the impossibility of insuring against the capital loss arising from a potential collapse in asset-prices, particularly for durable goods (Runde, 1994). Hence, it is impossible for optimising agents to determine the appropriate trade-off between risk and return. It is frequently argued that the government should act as a second-best insurer where equity or insurance markets operate inefficiently (e.g., the venture capital market) due to problems of moral hazard and adverse selection, because firms face collateral constraints on their borrowing, or may suffer from a lack of reputation. The uncertainty-based perspective provides a compelling alternative to the traditional neoclassical market-failure justification for government intervention in capital markets[52].

In conclusion, Runde argues that liquidity preference is less concerned with the choice between certain acts, each of which elicit varying outcomes depending on which randomly distributed states of nature come to pass, than with the choice of inaction over action, or the potential irreversibility of actions. Moreover, while the returns on assets depend on the specific characteristics of the act, he suggests that the magnitude of liquidity premia on various assets depends on uncertainty about the likely occurrence of counter-expected events:

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[52] Foss has convincingly argued that similar notions of uncertainty inform Oliver Williamson's transactions cost theory and serve to distinguish Williamson's work from that of the more formal principal-agent and property-rights theorists (Foss, 1994). In this regard he differs from the views espoused by Dunn (1999).

> In the Bayesian approach to choice under uncertainty an agent's degree of belief is a causal property of it, reflected in the extent that he or she is prepared to act on it. In *The General Theory*, in contrast, the emphasis is on uncertainty leading to investor inaction and on liquid assets making it possible to suspend judgement altogether, or at least to go for assets the consequences of which are not fixed and irrevocable. On the Keynesian view, then, the behaviour of agents under uncertainty reflects their inability to form subjective point probabilities rather than something in which such probabilities are implicit (Runde, 1994, p. 208)[53].

1.3.14. Barriers to Evolution from Uncertainty to Risk – The Neo-Austrian Revival

The notion that market-related mechanisms operate to bring rational, but subjectively based, assessments about economic outcomes into alignment with their real counterparts (i.e., the actual objective probability distributions relating time to states of nature and payoffs) is a commonly held doctrine of faith amongst neoclassical theorists. The works of the neo-Austrian School represent a subtle, but no less significant departure from this received wisdom. Butos and Koppl (1977), a representative pair of authors allied to this School, have drawn upon Friedrich von Hayek's "anti-Cartesian" theory of mind. They suggest that Keynesian rational epistemologies are both encompassed by, and determined as a special case of, Hayek's more general evolutionary notions of knowledge as a social practice.

[53] In a microeconomic study of industry types, Salais and Storper examine the modus operandi of firms producing with standardised technology for uncertain markets. They suggest that such firms are driven to contract out capacity to minimise the problems associated with the *irreversibility of investment decisions* and self-consciously point to analogies with the more general Keynesian notion of liquidity preference (Salais and Storper, 1992).

They follow Hayek in viewing social practice as determined by rule-governed procedures of classification, pre-cognitive or tacit patterns of action, and the unstable interplay between sensual perception and conceptual presupposition. Their Hayekian notion of knowledge could be interpreted in Kantian terms as a "synthetic a-priori" of forms constituting practical action, except for the fact that these forms are conceived to be neither *biologically given* nor *socially constructed.* Instead, they are seen to evolve culturally, through a Lamarckian process of learning and creativity, in the direction of ever greater complexity, efficaciousness, and goodness-of-fit.

For Hayek, the evolutionary principle of inheritance is the cooperative, cultural tradition which preserves successful rules of action, whereas the *prima-facie* principle of selection in a capitalist economy, arguably the most recent and historically contingent expression of a more generic process of "group selection", is the *market mechanism*. The latter is viewed as an immaculate mechanism operating without favour or discrimination to reward those who tacitly, if not unintentionally, apply increasingly appropriate, and hence increasingly successful, techniques and practices to the projection of prospective returns, the allocation of resources, and the creation and preservation of value. Nevertheless, Butos and Koppl stress the fact that this market-based evolutionary process can only operate successfully in achieving the coordination of individual plans, expectations, discretionary actions, and realisations if competitive forces are strong. Otherwise,

> ...when the context facing individuals is dominated by "Big Players"—in other words, participants whose discretionary actions

> have a disproportionate effect on the market—individual's expectations are more likely to generate perverse and incoherent outcomes [see Koppl and Yeager 1996] (Butos and Koppl, 1977).

On such a view, any Keynesian doubt that unbounded rationality will ever hold sway over the formation of long-term expectations must therefore be abandoned. Or at the very least, it must be confined to a small region of activity within less competitive markets, which through their very imperfections, preclude the evolution of more successful coordination mechanisms. In contrast, Butos and Koppl contend that the problem of bounded rationality can be overcome through the gradual accumulation, refinement and perfection of a tacit web of corporate and managerial practices which are, perhaps, intuitively grasped but never fully comprehended as to their logic or underlying rationale. Understandably, the Hayekian view of policy confines government to a merely shepherding role, which, in preserving the healthy and vigorous forces of market competition, allows these collective, largely unthought, and often unintended evolutionary mechanisms of inheritance, variety generation, and selection to obtain, unimpeded in their beneficial effects.

I would argue that there are fundamental flaws in such a conception. Without going in to excessive detail, it should first be noted that, even for the Hayekians, extra-market interventions are still warranted to ensure that the market process itself, and also its evolutionary mechanisms of reward and incentive, work smoothly. Processes of evolution are predicated on the existence of diverse mechanisms promoting variety, selection and inheritance. This range of mechanisms extends far beyond the market alone, conceived as a regulated site for the fair and equitable display, auction and exchange of title to goods, services or financial obligations. It would include institutions of government and commercial regulation, organizational practices, and research conducted with academic centers. Second, and more significantly, to the extent that Keynesian uncertainty is deemed to be objective, as well as subjective in nature, it must be acknowledged that no conceivable set of rule-governed actions (tacit or otherwise) can tame this ontological wilderness. I argue that practices and procedures must openly recognise our relative ignorance and insecurity, and operate on this basis to constrain speculation; promote long-term stability, trust and orderliness within market exchange; institute buffers against excessive volatility and hysteresis; and enable the development and implementation of sound monetary and fiscal policies.[54]

Only in this way can they adequately deal with the adverse consequences of fluctuations in uncertainty aversion, and resulting financial instability. Needless to say, the aims and objectives of such a diverse range of policies cannot be reduced, as Butos and Koppl would have it, to the

[54] In this regard I have adopted an *objectivist* and logical position on probability that can be contrasted to Hayek's *subjectivist* and pysychological interpretation of how beliefs are formed in the human mind. As Carabelli and De Vecchi (2000) demonstrate, while Hayek grounded his explanation of economic action on an evolutionary theory of mind, Keynes based his view of belief and the role of conventions on probable reason. For Hayek, human beings act rationally only by following accumulated and largely unconscious, impersonal, and habitual rules of conduct or convention. For Keynes, the validity of inductive argument depends on *reasonableness*: on the relation of a certain matter of fact to given evidence on the basis of which a probable judgement can be made. Economics is a moral science because it deals with introspection, motives, expectations and with values: it must consider the meaning that people impute to their actions. Although the sources of belief may well be subjective and psychological, the reasons for holding a belief belong to logic: logic investigates the rational principles of valid thought which form the basis of rational belief, action and choice. *Reasonable* expectations are grounded on logical probabilities, but *conventional* expectations are practical answers to the existence of total ignorance and uncertainty. Caprice, whim, and sentiment help to resolve situations of radical uncertainty, and while these sources of belief are no doubt irrational, Keynes still held to the view that the reasons for holding a belief are always rational. As such, Carabelli and De Vecchi (2000, p. 277) argue that the Keynesian notion of "animal spirits" should not be regarded as a manifestation of irrationality, as a purely psychological force replacing reasonableness as an explanator of investor behaviour.

mere promotion of competition. Rather, they would come under the umbrella of instruments that condition the actual workings of the market and of the evolutionary process itself. Hence, to the extent that they *condition* the evolutionary process, they cannot be *selected*, as to their *fitness* by the evolutionary process, itself!

1.3.15. The Limitations of Runde's Critique of the Savage axioms

Although I fully concur with the sentiments expressed in Runde's critique of the Savage axioms, I now believe that he does not go far enough in his justificatory arguments. Since the publication of Tobin's research on liquidity preference *as behaviour towards risk* (1958) and Jones and Ostroy's work on *liquidity preference as flexibility* (1984), it should be clear that orthodox monetary theory can capture certain aspects of liquidity preference in a manner similar to that espoused by Runde. On one hand, Tobin views liquidity preference as reflecting a risk-averse response to prospective capital losses *across* a diverse portfolio of asset-holdings. On the other hand, Jones and Ostroy represent liquidity preference as motivated by a requirement to minimize the losses associated with transactions costs that arise when financial investors have to readjust their portfolio holdings in the light of new information on the likely distribution of next period payoffs *over the set* of risky assets.

Runde cannot offer a deeper critique of these insightful but limited theories because he fails to make an important distinction between the *payoff* accruing to actions (e.g, portfolio holdings) in different states of nature, and the *value function* assigned to these payoffs in an optimal control context[55]. Tobin defines his value function in mean-variance utility terms, while Jones and Ostroy define it in terms of maximising expected portfolio returns *net* of transactions costs in a two-step dynamic programming problem. Those who, like Epstein and Zin (1989), recommend moving beyond expected utility theory, have adopted non-state separable value functions[56].

Alternatively, the characteristics of the optimization process itself can encompass many of the concerns raised in Runde's exposition. In later chapters, I follow Allesandro Vercelli in arguing for the need to extend both of the approaches developed by Tobin and Jones and Ostroy by incorporating uncertainty directly into the optimization procedure. Vercelli takes up the deliberations of Gädenfors and Sahlin (1982) when he suggests that liquidity preference can be modeled through a two-step minimax optimisation (Vercelli, 1991, appendix 5A, pp. 85-90). Building on Vercelli's descriptive conjectures, I describe how H^∞ control and risk-sensitive control under uncertainty constraints largely accommodates the concerns raised in Runde's critique. For example, in Tornell's (2000) H^∞ control approach to asset-pricing acts can no longer be separated from states of nature because the worst-case disturbance (a deterministic version of the states of nature) can no longer be identified separately from the agent's control law (which is assumed to be known by nature—one of the two agents in the differential game). Of course—and to this extent Runde is essentially correct—I have shown that in finance applications of risk-

[55] As noted in footnote 18, this problem could have been usefully discussed in relation to Runde's self-acknowledged departure from Ramsey's utility-theoretic approach to decision-making under risk.
[56] Analogously, attempts to explain the equity premium often resort to non-time-separable utility functions such as those reflecting habit persistence. In this case, one period's consumption affects another period's marginal utility so that the utility function can no longer be written as the sum of a series of current period utility functions. Cochrane notes that nonseparability can also be imposed across goods (2001 p. 440).

sensitive control that employ Epstein and Zin's (1989) aggregator functions, certain of Savage's axioms must be suitably modified.

1.3.16. Sources of Uncertainty in Social Life

I now wish to loosely employ the above-described Critical Realist ontology to identify the differing forms that can be adopted by uncertainty. In particular, I intend to distinguish between forms of uncertainty that reflect subjective aspects of preference, attitude and doubt and fundamentally objective forms of uncertainty that are grounded in the unpredictable nature of reality as it is actualized and experienced. To a large extent, Post Keynesians adhere to the latter, more skeptical perspective while orthodox theorists adhere to the former. For example, I have argued that Paul Davidson (1991, 1996) is a proponent of the view that underlying economic mechanisms generate price and quantity time-series that can be described as non-ergodic stochastic processes. The sources of this non-ergodicity range from human creativity and innovation, herd behaviour, and irrationality, through to disruptive forms of institutional transformation. Of course, particular institutions can also promote orderliness and stability, but this outcome must then, strictly, be recognized as a social product or artifact rather than an attribute of nature: a social attribute that can be both the object and the objective benchmark of policy intervention. Certain advocates of non-linear dynamic modeling also favour a non-ergodic viewpoint. Although chaotic systems can elicit outcomes that are ergodic in a statistical sense, other outcomes can be highly irregular and non-ergodic. I shall return to these important issues about non-linear economic dynamics in later chapters of the thesis. It should be noted, however, that certain recursive forms of control and filtering theory can actually accommodate non-ergodic processes.

No doubt, the rich discursive approach to uncertainty that David Dequech sets out is deliberately opposed to what are sometimes complex, but nevertheless, reductionist forms of mathematical formalism: including those that might draw on notions of risk-sensitive control. I have argued that the distinction he makes between *uncertainty perception* and *uncertainty aversion* can be given a control-theoretic interpretation in terms of the stochastic uncertainty constraint and the magnitude of the risk-sensitivity parameter, respectively. However, there is nothing in the formal toolbox that could separate out the individual effects of animal spirits or the effects of knowledge and creativity on expectations (except, perhaps, to the extent that an optimistic disposition encourages agents to ignore certain evidence, thereby imposing incorrect and underestimated uncertainty constraints). However, this difficulty should properly be regarded as a limitation of the robust control framework rather than of Dequech's more philosophically based arguments.

A similar failing is to be found in approaches to uncertainty that are grounded in generalizations of the Savage axioms—principally in an extension of the sure-thing principle—that collapse both uncertainty perception and uncertainty aversion, as defined above, into a single measure of uncertainty aversion (see sections 5.2 and 6.3 in Karni and Schmeidler, 1991)[57]. At least in a control framework, the influences of model uncertainty, external perturbation and observation error can be separately identified. The fundamental issue at stake, here, is whether models of robust or risk-sensitive control under stochastic uncertainty constraints have been confined to an interpretative field that is unnecessarily narrow, diminishing the potency of applications of such techniques in the social sciences.

1.4. Uncertainty in Robust and Risk-sensitive Control Theory

I have argued above, that attempts at estimating the DGP must grapple with the complex relationship holding between the real, the actual and the experienced, as underlying mechanisms are ultimately translated into observed events. Scientific understanding must dig beneath what is observed to uncover the underlying mechanisms: in all their overdetermined, contradictory, and tendential complexity. At the same time, this interrogation will reveal the precise manner in which the underlying mechanisms achieve *actualisation*.

From an ontological framework, the relationship between the *actual* and the *experienced* may be distorted by observation error. Alternatively, uncertainty may reside in the interactions between the *real* and what is *actualized* (i.e. the dynamic system under consideration may be exposed to external perturbations that exhibit unknown patterns and possess unknown energy). Finally, the limitations of scientific knowledge, itself, imply that our modeling of the dynamic system is necessarily simplified and incomplete (in a control theory framework, this *epistemic* uncertainty

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[57] As Epstein and Wang (1994, p. 387) concede, their extension of Schmeidler's (1989) multiple-priors framework collapses both the presence of uncertainty and the agent's attitude to uncertainty into a single mathematical entity—the probablity kernel correspondence.

is typically represented in the form of model uncertainty—an energetically-bounded additive or multiplicative gain factor driving the impulse response of the dynamic system).

The key question distinguishing the Post Keynesian position from that of other economists is whether these forms of incomplete knowledge can ever be overcome either through learning or through some sort of tacit, market-based, evolutionary process of selection that might well be largely unconscious (at least to the extent that economic agents acquire insights into the magnitude of energies and respective patterns of observation error, external disturbance and model uncertainty so that they can be accurately assessed, modeled and predicted). Runde argues that the Bayesian theorists favour the latter position, while Post Keynesians firmly reject it. I shall argue below that both Epstein and Wang (1995) and Hansen, Sargent and their associates have embraced a view of uncertainty with closer affinities to the Post Keynesian viewpoint. Uncertainty, in their control theoretic applications, is a *permanent* fixture that cannot be resolved—except, perhaps, through long-term improvements in diagnostic and information based statistical tests that might eventually enable the econometrician to reduce the family of feasible distributions to a singleton.

A second major issue is whether the market clearing assumption about the economic system can be justified. From a critical realist perspective, this relates to the need for scientific discourse to uncover the actual underlying mechanisms that are responsible for what can be observed. In an equilibrium asset-pricing context, I argue in favour of a quantity-constrained rationing approach that recognizes the reality of involuntary unemployment and opposes any artificial separation between financial and real economic variables, even over the long run. When the decisions of financial investors, firms and households are disentangled varying degrees of uncertainty and liquidity preference can arise as an influence over lending or portfolio choice, real investment activity and consumption behaviour. This disentanglement is accomplished in the following chapter.

CHAPTER TWO—THEORIZING THE EFFECTS OF UNCERTAINTY OVER REAL AND FINANCIAL INVESTMENT

2.0. Introduction

Currently, the most obvious and, often, the most adversarial division amongst theorists who study the influence of financial variables over the macroeconomy seems to be situated between those who favour a representative agent, intertemporal general equilibrium approach and those who embrace an endogenous, non-linear dynamic modeling approach that openly eschews representative agent microfoundations. The former is associated with members of the New Classical school like Hansen and Sargent (1980, 1981) and Lucas (1975, 1978, 1981), but also extends into much of what passes for modern finance theory. The latter is associated with advocates of the heterodox Harrod-Kaldor-Goodwin tradition (Kaldor, 1940; Goodwin, 1951, 1967; Minsky, 1985; Keen, 1995; Chiarella and Flaschel, 1999; and Chiarella, 1990). Another related branch is that represented by contributors to Post Keynesian monetary theory (notably, Paul Davidson, 1988, 1991, 1994a,b; L. Randall Wray, 1990, 1991, 1992; Jan Kregel, 1988, and Thomas Palley, 1994, 1995, 1999; Lavoie and Godley, 2000; and Cowen and Kroszner, 1994). In general, these authors favour graphical or more discursive and schematic analysis over sophisticated mathematical modeling (work by Lavoie and Godley and by Palley being exceptions to the rule). In a sense, this thesis attempts to bridge the gap between these three theoretical traditions in a manner that preserves what I consider to be the essential Keynesian insights into the workings of a monetary production economy. I shall argue that recent theoretical developments in asset-pricing theory by protagonists within the first of these traditions can be usefully employed by members of the second tradition to illuminate analysis by members of the third of these traditions.

Spurred on by the experience of financial volatility and crisis in regional financial markets (e.g. the Mexican Peso problems, the Russian debt default, and the Asian Meltdown), a number of recent developments in finance theory represent a notable departure from some of the more extreme, New Classical and Rational Expectations approaches that endorse the efficient markets hypothesis. Here, I would include the macroeconomic literature on self-fulfilling prophecies and sun-spot equilibria that features either some sort of indeterminacy (e.g. multiple equilibria in overlapping generations models or, alternatively, non-convexity in production such as Benhabib and Day, 1981; Azariadis and Smith, 1998; or coordination failure as in the models of Farmer and Guo, 1994; Salyer and Sheffrin, 1998; and Marshall, 1998). However, my complaint about this approach to financial modeling is that it largely ignores the phenomenon of liquidity preference and its link to uncertainty.

In addition, there the literature on Adaptive Rational Expectations Dynamics (ARED), where turbulence in asset markets is the product of rational choice: economic agents choose between using more costly rational predictors that perform well along trajectories far from the steady-state and less costly adaptive or myopic predictors that perform reasonably well close to the steady state (Brock and Hommes, 1997). In ARED models chaotic trajectories obtain, even for simple, linear cobweb models of equilibrium, because in calculating their expectations, rational agents must determine the relative proportions of those agents, like themselves, using rational predictors and those using adaptive predictors. It is this requirement that introduces nonlinearity into the dynamic relationships between key variables. However, despite the fact that this research affords insights into the endogenous characteristics of financial volatility, uncertainty plays no substantial role. The information gathering and decision making activities of economic agents are circumscribed by *cost* and not by *uncertainty*. As I argued in the introduction (section 0.3.2), the literature on ARED is closely related to research that models market outcomes as the result of interactions between multiple, heterogeneous agents (Lux, 1998; Le Baron et al., 1999, Arthur et al., 1994).

Yet another branch of finance theory, inspired by psychological insights into human behaviour, focuses on agent misperception, overconfidence and the tendency to associate current experiences of shocks with previous experiences so that current expectations are, if you like, erroneously coloured by the evocation of past emotional complexes (Hirschleifer, et. al. 1998; Barberis et. al. 1998; and Mullainathan, 1998). Similarly, the noise trading literature (Shleifer and Vishny, 1990; DeLong et al, 1990) and associated work on fads and fashions (reviewed in Shiller, 1989), also emphasizes irrational aspects of investor behaviour. In noise trading models, the investment horizons of rational investors are constrained so that they cannot trade in financial assets that they know to be mispriced, but will take too long to adjust to their appropriate levels. In my view, the advantage of risk-sensitive and robust control techniques is that they do not impose ad hoc and simplistic mechanisms for modeling misperceptions and do not have to import faddish or irrational forms of behaviour that cannot be explained within the asset-pricing model.

The primary focus of this thesis, however, is on economic applications of risk-sensitive and robust control-theory that allow for certain forms of uncertainty aversion. A notable example of this type of analysis is set out in a recent paper by Andersen, Hansen and Sargent (1999), one of three that I review in chapter 4. Using a representative agent, intertemporal pure-exchange equilibrium model, these authors demonstrate the theoretical existence of uncertainty premia in asset-prices. The equity premium puzzle is the context for their inquiries, a financial anomaly that can also be viewed as an expression of liquidity preference, once uncertainty has been incorporated into the analysis of asset-pricing.

In this chapter I remind readers about necessity to capture key ontological features of monetary economy. These include the presence of nominal contracting in labour and financial markets, the influence of uncertainty over long-term investment behaviour, transactions costs involved in extricating investments from illiquid positions. Having examined the issue of uncertainty in some depth in Chapter 1, I shall now examine the role played by uncertainty in relation to both financial and non-financial investment as represented in both *The General Theory* and, more generally, the Post Keynesian tradition of monetary thought. This sets the scene for an examination of orthodox approaches to liquidity preference. The fundamental problem that I identify is that Post Keynesian liquidity preference usually employ orthodox frameworks (e.g. Tobin's portfolio-based asset demand system). I argue that due to their very structure, these frameworks do not adequately account for the effects of uncertainty, which therefore appears solely as an extraneous influence.

In section 2.1.1, I turn to the core notion of the Point of effective Demand. This revolutionary notion is still not recognised in New Classical thought. The block recursive structure of most New Classical finance and Real Business Cycle macroeconomic models implies that movements in the real side of economy are completely governed by the condition of equilibrium in labour and product markets (albeit modified by efficiency wages, adjustment rigidities arising through optimal search behaviour etc.). Monetary factors are secondary playing no essential role. In section 2.1.5, I argue that one reasonable way to represent the Point of effective Demand is through a Quantity-Contrained Rationing (QCR) approach. As Victoria Chick and others have suggested, the graphical and discursive analysis of the point of effective demand has its closest algebraic analogue in the QCR framework: insufficiency of aggregate demand necessarily implies the imposition of binding QCs over labour market outcomes.

Section 2.1.2, examines *The General Theory's* Flex price Model. In the dynamic disequilibrium setting of Book V it is argued that, of its own accord, wage-price flexibility cannot restore the economy to equilibrium. Keynes contended that the real balance effect and Pigou effects associated with the increases in real wealth would probably countered by the adverse effects of downward price adjustment on liquidity preference, the Marginal Efficiency of Capital, and the marginal propensity to consume (the latter due to debt-deflation). Similar mechanisms have been incorporated into Chiarella and Flaschel's approach to dynamic disequilbirium macro-modelling.I shall consider this approach in more detail in Chapter 3.

In section 2.1.4, I examine how Post Keynesian's have theorized the influence of uncertainty over both corporate investment and the asset-demand for money within a portfolio-choice context. These two apparently separate decisions come together in approaches that acknowledge that, for most firms, investment choice is made over a liquidity continuum that stretches from money and near money at one end of the spectrum to highly illiquid investment in plant and equipment at the other.

In section 2.1.3, I reflect on the significance of nominal contracts in asset and labour markets. In particular, I emphasize the fact that debt-deflation reflects the ubiquitous nature of non-indexed nominal contracting in the borrowing decisions of both firms and households. In section 2.2.1, I follow Minsky in graphically "unpacking" the Marginal Efficiency of Capital Schedule to determine the precise influence of uncertainty and liquidity preference over corporate investment. In section 2.2.2, I also examine Minsky's analysis of debt-deflation in the capital goods sector, which is seen to operate in a manner that differs notably from debt-deflation effects in the consumer goods sectors. In passing, I examine Minsky's theory of the business cycle: one that is based on the distinction between hedge, speculative and Ponzi financing positions.

In section 2.3.1, I examine how Post Keynesian theorists have extended *The General Theory's* analysis of mnoey demand to incorporate wider portfolio choices amongst monies, fixed interest securities and equities. Although the money supply is assumed to be exogenous in *The General Theory*, most Post Keynesians follow the endogenous-money position espoused by Keynes in the *Treatise on Money* I briefly review these matters in section 2.3.2. In section 2.3.3, I examine the way that various Post Keynesian authors have attempted to model liquidity preference: specifically Paul Dalziel (1996) and Lavoie and Godley (2000). In general, these authors employ a Tobin-style approach to portfolio demand, yet I argue that this sort of framework cannot accommodate uncertainty perception/aversion as an influence over asset demand, equity prices, the q-ratio and the rate of accumulation.. To support this conjecture in section 2.3.3. I examine Tobin's approach to *Liquidity Preference as Behaviour Towards Risk.* This is followed in section 2.3.4. by a sympathetic review of Jones and Ostroy's notion of *Liquidity as Flexibility*: an attribute of the demand for money that comes in to play when investors wish to adjust the liquidity composition of their portfolios in response to new information. Flexibility is defined in relation to the transactions costs that are imposed when investors attempt to dispose of illiquid assets within their portfolios. Finally, in section 2.4.5. I summarize Magill and Quinzii's attempt to account for the effects of nominal, non-indexed financial contracts. Although their analysis explicitly recognise debt-deflation effects, I suggest that it does so *inappropriately* by employing a Clower constraint, alongside a naïve quantity-theory approach to the transactions demand for money.

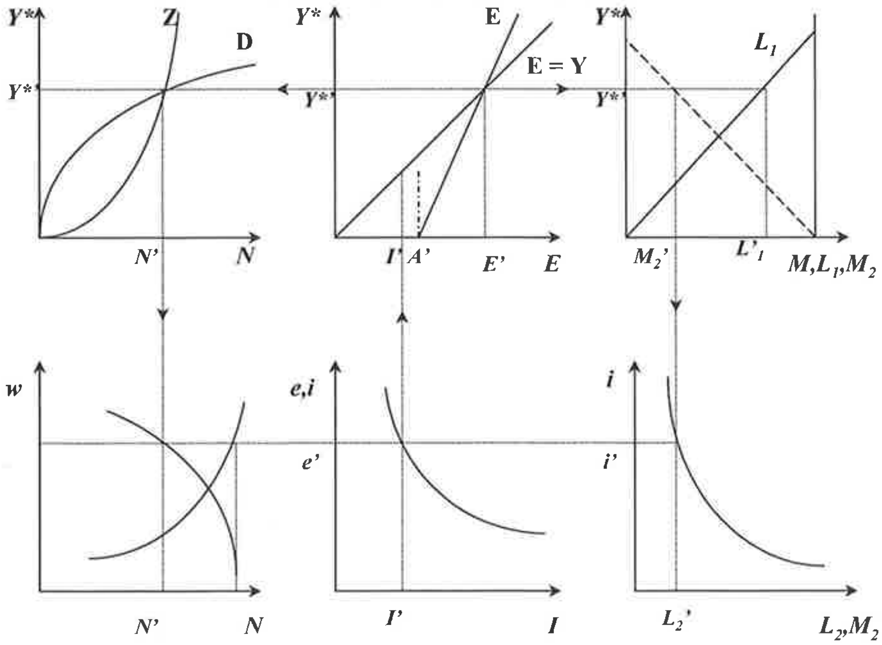
2.1. The General Theory's Fix-Price Model

2.1.1. THE CORE NOTION OF THE POINT OF EFFECTIVE DEMAND

For Post Keynesian theorists, the point of effective-demand is a core notion that distinguishes their work from models of a New Keynesian or New Classical persuasion. Although many valuable expositions of this notion are available in the literature, the need for some brevity has dictated that I draw upon Aleassandro Vercelli's lucid, graphical and block-diagram based analysis of *The General Theory*'s fix-price and Flex price models.

This has the advantage of combining inter-relationships between variables for all the key markets—capital, money, goods, and labour—into a single diagram. In particular, the linkage between the "IS-LM" block and other aspects of the Keynesian model, such as aggregate expenditure, aggregate supply and aggregate demand, stand out clearly.

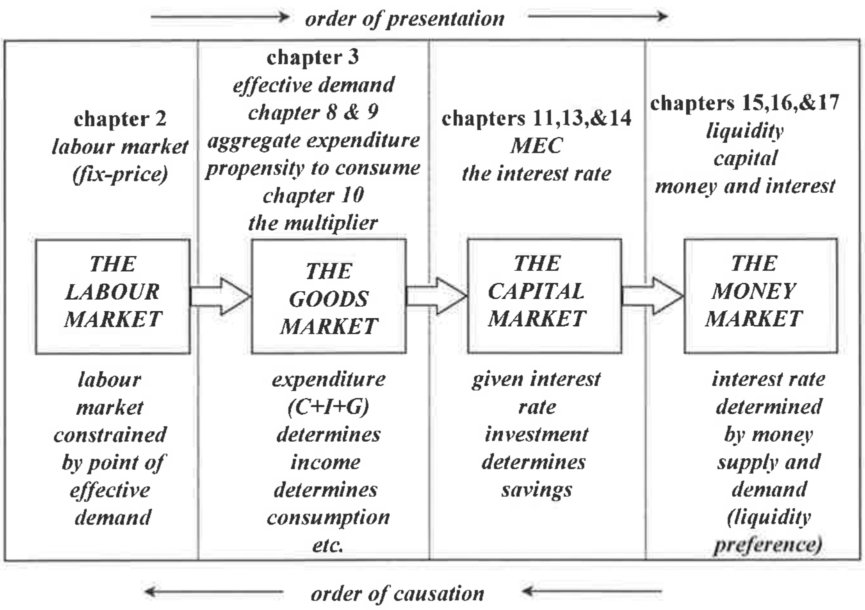
Fig. 10: VERCELLI (1991) ON THE GT's "HEURISTIC" MODEL



The variables appearing in this and the subsequent Flex price version of the model, which appears later, are as follows: *M* equals money supply (as determined by the policy stance of the monetary authorities), which can be further decomposed into M_1, the supply of money for transactions purposes and M_2, the supply of money to meet speculative and precautionary demand; matched by L_1, the income sensitive, and L_2, the interest sensitive components of the demand for liquidity; $1/k$ = the velocity of money; I = investment expenditure; Y^* = the point of effective demand (the point at which the aggregate demand curve, *D,* cuts the aggregate supply curve *Z*); *e* = marginal efficiency of capital, *s* = the marginal propensity to save; λ = the average productivity of labour; λ' = the marginal productivity of labour; *d'* = the marginal disutility of labour; N_d = the labour demand curve; N_s = the labour supply curve; and, *U* = unemployment.

As Vercelli observes, in *The General Theory*, the elaboration of what transpires within each market appears in a sequence that is opposite to the chain of logical causality. First, the labour market is analysed, followed by the goods market, capital market, then money market. The chain of causality, however, flows in the reverse direction as shown in the diagram below, that also lists relevant chapters from *The General Theory*.

Fig. 11: VERCELLI ON THE GT's ORDER OF CAUSATION/PRESENTATION



The lower right hand graph in the Figure 10 depicts the demand for speculative and precautionary balances as a function of the nominal interest rate, while the graph immediately above combines demand for speculative and precautionary balances with demand for transactions balances, as a function of nominal income. The upward sloping demand curve labelled L_l represents the demand for transactions balances, while the total stock of available money balances is represented by the vertical line. The downward sloping the dashed line, representing the amount of money balances available to meet speculative and precautionary demand, is found by subtracting transactions demand, at every level of income, from the given money supply. A certain point on this line is associated with any given level of income, and once translated onto the diagram immediately below, enables the appropriate level of interest rates to be determined, for that particular income level, that would achieve equilibrium in the money market. The bottom middle graph depicts the marginal efficiency of investment schedule that determines the level of investment that would occur at any given interest rate. This level of investment can be read off along the horizontal axis of the graph. To determine aggregate expenditure, this amount of investment, *I*, must be added to consumption and government expenditure. This is accomplished in the upper middle diagram, which features the "Keynesian cross". The slope of the aggregate expenditure curve in income and expenditure space would be determined by the marginal propensity to consumer and the marginal income tax rate. Where this curve cuts the 45° line, the multiplier effect would have fully worked itself out so that income would equal expenditure. This presumption that full equilibrium is rapidly attained through the workings of the multiplier represents one of the weak points in *The General Theory*'s analysis of short-run dynamics that Keynes took pains to qualify in his discursive analysis of dynamic adjustment.[58]

The four graphs discussed up to this point portray the set of relationships that are conventionally incorporated into the textbook IS-LM model. Two more graphs are needed to complete the model. In the upper left-hand graph, the point of intersection between the aggregate supply and demand curves in employment and nominal income space, now determines the point of effective demand associated with a given level of aggregate expenditure, under the assumption that firm *expectations* of proceeds ensuing from the sale of produced goods are correct (i.e. they match the *actual* level of aggregate expenditure). The amount of employment, N, offered by firms can then be read off the horizontal axis.

When transcribed into the lower graph, depicting labour supply and labour demand as a function of the real wage, the employment level determined by the point of effective demand then determines the real wage and level of involuntary unemployment. The so-called "labour demand" curve, therefore, is best understood as a pricing equation rather than as a summation of each firm's marginal value product curve (under the assumption of perfect competition in the goods and labour markets).

[58] As Paul Dalziel argues (1995a,b, 1996), the actions of the multiplier are best interpreted using a form of process analysis that, following an initial injection of investment spending, traces the repercussions on savings, money holdings, and equity holdings at each step or interation. As such, the equilibrium outcome is an artifact calculated as

Keynes was fully aware of the complications that would be introduced by contemplating forms of imperfect competition in product and labour markets (complications that would largely explain the Dunlop-Tarshis effect: the apparently perverse movements of prices and wages over the business cycle that were a feature of empirical studies) but, as I shall demonstrate, his central intention was to show that involuntary unemployment could arise even in a perfectly competitive economy due to the adverse consequences of uncertainty for liquidity preference and animal spirits.

The expectations of net proceeds from the sale of goods that are embodied in the aggregate demand curve are described by Keynes as being of a short-run nature to distinguish them from the set of long-run expectations that must be formed, by firms and investors, about future streams of returns from the activity of real investment and from holdings of financial assets[59].

2.1.2. *The General Theory*'s Flex price Model

In most textbook representations of the AS-AD model the key upper left-hand diagram that appears in Vercelli's portrayal of *The General Theory*'s model is replaced by one that erroneously depicts long-run equilibrium in the labour market, that obtains at the real wage that matches labour supply to labour demand. When "Keynesian" short-run effects are allowed to intrude into the process determining labour market equilibrium, they do so in the form of various

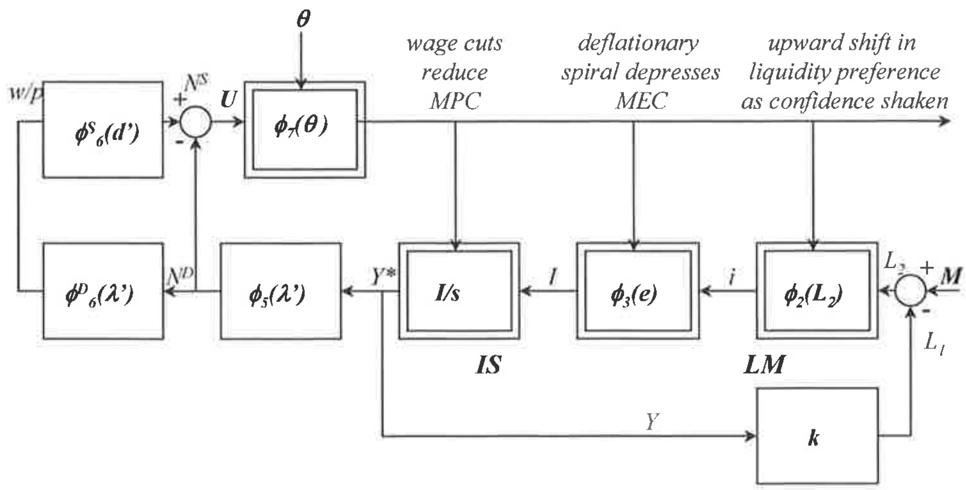
the limit of an infinite process, so that at any one time, the economy is responding to a series of lagged investments that will continue to ripple forwards through time.

[59] Of course, as will be explained below, long-run expectations of this nature are incorporated into the marginal efficiency of capital schedule and also the schedule of demand for speculative and precautionary balances.

nominal rigidities in the pricing of labour services (e.g. due to search costs, multi-period contracts, implicit contracts or efficiency wage payments etc.). However, in such cases, unemployment would be a voluntary rather than an involuntary phenomenon: a temporary aberration that would, in the fullness of time, be removed through processes of learning and renegotiation of contracts.

In orthodox theory, one of the crucial adjustment mechanisms that would supposedly preclude the possibility of any sustained involuntary unemployment is the Pigou Effect (and the closely related Real Balance Effect), whereby any fall in nominal wages that was induced by rising unemployment would flow through to prices, precipitating a general price deflation. In a deflationary environment the nominal value of both real and financial assets would rise, supposedly stimulating wealth-sensitive components of expenditure. I shall return to consider these deflationary effects in more detail. For the moment I intend to review the views that Keynes held about what would transpire in an economy where wages and prices adjust downwards in response to a downturn in economic activity. Once again, Vercelli's abbreviated block-diagram representation of the "Flex price" model that was first presented by Keynes in Book V of *The General Theory*, provides a useful benchmark. Note that the various ϕ (.) functions in the diagram below represent equations defining the respective functional relationships (with the independent variable appearing inside the brackets).

Fig. 12: VERCELLI'S BLOCK-DIAGRAM FOR THE FLEX-PRICE MODEL



This model is included here, to highlight Keynes's discussion about the consequences of permitting nominal wages to respond flexibly to variations in level of unemployment (this interaction has been incorporated in the functional relationship $\phi(\theta)$. Keynes suggested that variations in money wages would influence variables such as the state of liquidity preference, the marginal efficiency of capital, and the marginal propensity to save. He considered the impact of wage cuts over savings to be indirect, arguing that the level of savings would primarily be effected by the redistribution of income, induced by debt-deflation, from *borrowers* who generally evince a lower propensity to save to *rentiers* who exhibit a higher savings propensity.

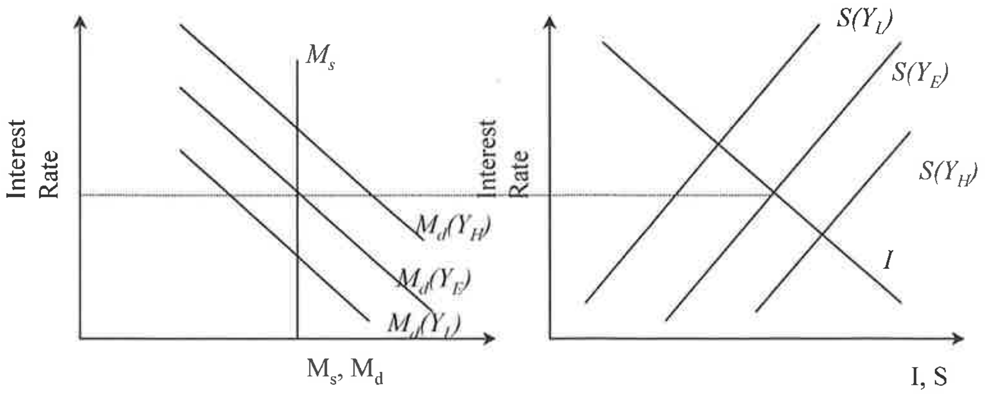
2.1.3. The Significance of Nominal Contracts in Asset and Labour Markets

As John C. Eckalbar has argued (1997), the notion that workers and employers bargain over nominal wages rather than real wages, and over nominal rather than real interest rates represents the essential difference between what goes on in a pure barter economy and what transpires in a monetary production economy. Moreover, this simple notion is also central to Keynes's arguments about the point of effective demand. Because wages are paid in kind in a barter economy, monetary factors could not interfere with the determination of a labour market equilibrium (Eckalbar, 1997, pp. 126-7). However, in a monetary production economy suffering from an insufficiency of effective demand, workers would be off their labour supply curves. Efforts on the part of individual firms to drive real wages lower by reducing nominal wages would not have their intended effect because prices would follow: the labour demand curve merely determines the aggregate level of product prices (Eckalbar, 1997, p. 129).

Eckalbar demonstrates that, in his critique of the classically inspired loanable funds theorem, Keynes applied a similar argument to the determination of equilibrium in the money market (Eckalbar, 1997, pp. 131-3). In a Robinson Crusoe economy, given the amount of time allocated to gathering food and seed, a decision to save merely diverts more produce from current consumption to investment giving rise to future consumption benefits. However, in a monetary production economy, although both planned savings and planned investment are obviously influenced by the level of interest rates, the interest rate does not serve to equate the supply of and demand for loanable funds. Equilibrium in the market for loanable funds is achieved through changes in income, with the level of interest rates being determined by the preference for liquidity (Eckalbar, 1997, p. 133).

This outcome is portrayed in the following diagram described in Eckalbar (1997, figure 4, p 132), which reveals an alternative way of visualising links within the IS-LM block of the fix-price model:

Fig. 13: ECKALBAR, 1997 ON THE GT'S IS-LM MODEL



The LM Curve *The IS Curve*

The graph of the IS curve actually appears in chapter 14 of *The General Theory* whereas Eckalbar has extracted the elements presented in the LM curve from what is implicit in Keynes's principally *discursive* critique of the classical theory of interest rate determination (Eckalbar, 1997, p. 132). Here, Y_H, Y_M, and Y_L stand for high, medium and low levels of income, respectively. It is evident from the diagram that, given the money stock M_S, only one level of income (Y_M) is compatible with equilibrium[60].

[60] It should be noted that nothing essential would be changed in Keynes's critique of the classical theory if it were presumed that the monetary authorities were targeting interest rates rather than the money supply. The money supply curve would then merely become a horizontal line at the chosen interest rate target.

2.1.4. Uncertainty as a Determinant of Volatility in Investment and the Demand for Money

In his celebrated 1978 *Mattioli Lectures*, Richard Kahn discusses various limitations of *The General Theory* (Kahn, 1984). In particular, he suggests that the liquidity preference and money demand schedules are presented in a manner which conveys the erroneous impression that they are stable, if not deterministic.

Moreover, in his fifth lecture Kahn observes that:

> (b)oth in the *Treatise* and in *The General Theory* the treatment of equities, as opposed to fixed interest securities, is limited in scope and hesitant as to the importance of the behaviour of equities as an influence on real investment (1984, p. 150)

Nevertheless, Kahn cites Keynes's well-known comment on the role played by equity prices which, during the 1928-29 boom, remained high in the US despite the Federal Reserve Bank's imposition of punitively high short-term interest rates. The high price-to-earnings ratios on common shares

> ...offered joint stock enterprises an exceptionally cheap method of financing themselves. Thus, whilst short-money rates were very high and bond rates somewhat high, it was cheaper than at any previous period to finance new investment by the issue of common stock. By the spring of 1929 this was becoming the predominant method of finance. Thus easy terms were maintained for certain types of investment, in spite of the appearance of very dear short-money (Keynes 1973a, vol. VI, pp. 174-5).

Kahn (1984) provides an overview of Keynes's views on the role of equity prices by compiling an impressive array of quotations from *The General Theory* and other works. Short-term movements in equity prices, owing to their very volatility and unpredictability, are often discounted by those engaged in real investment activity. Nevertheless, sustained movements cannot be so easily ignored. One of the well-known quotes, drawn from amongst those compiled by Kahn, is worth repeating in this context:

> ... the daily revaluations of the Stock Exchange, though they are primarily made to facilitate transfers of old investments between one individual and another, inevitably exert a decisive influence on the rate of current investment. For there is no sense in building up a new enterprise at a cost greater than that at which a similar existing enterprise may be purchased: whilst there is an inducement to spend on a new project what may seem like an extravagant sum, if it can be floated off on the Stock Exchange at an immediate profit (Keynes, 1973b, vol. VII, pp, 150-1).

Some observers have argued that fluctuations in equity prices now exert far less influence over investment decisions in the UK, because industrial and commercial firms, aided and abetted by low corporate tax burdens, have effectively become self-financing. Countering this institutional argument, Kahn reasons:

> (b)ut those who argue on this line overlook the wide polarisation between companies, at one extreme, which rely heavily on new issues of equities and those, at the other extreme, which accumulate financial assets and make successful take-over bids for other companies. The financial position would be much eased if the companies at the latter extreme bought the shares of companies at the former extreme. But - apart from takeover bids - they do not (Kahn, 1984, p. 166).

Due to the ever-present threat of take-over, or dislodgement of managers by dissident groups at the Board level, quite apart from the more routine impact of valuations placed by the Stock Exchange on the rate of increase of earnings per share:

> (t)he result is that, indirectly if not directly, pressure is exercised on them to take account of their shareholder's interests. The prospective yield per share, both on the real assets of the company (the internal rate of return) and on its shares, must not fall below the rate generally prevailing in the country. This factor acts as a restraint on the rate of growth of the company rather than Keynes' rate of interest. This is why the price of a company's shares matters to its management even if there is no question of making an issue of shares (Kahn, 1984, p. 167).

As Kahn (1984) argues, a comprehensive analysis of the role of equity can only be conducted within a portfolio framework.[61]. I shall return to this matter in a subsequent section, where I intend to outline a formal model of investment, which draws on the intepretative work of Hyman Minsky (1975) and Victoria Chick (1983) so that the separate effects of animal spirits and liquidity preference over investment can be traced and identified.

Minsky contends that:

> (t)o understand Keynes it is necessary to understand his sophisticated view about uncertainty, and the importance of uncertainty in his vision of the economic process. Keynes without uncertainty is something like *Hamlet* without the Prince (1975, p. 57).

Minsky focuses on the role that *conventions* play in when individuals make economic decisions in the face of uncertainty, distinguishing between the conventions adopted by banks, investors and consumers. He specifically argues that:

> ...in a capitalist economy the aspect which is least bound by technology or by fundamental psychological properties, which is most clearly a convention or even a fashion, subject to moods of

[61] Kahn's own efforts in this regard are outlined in his well-known 1954 paper on liquidity preference.

> optimism and pessimism and responsive to the visions of soothsayers, is the liability structure of both operating and financial organisations (1975, p. 128).

Because both fluctuations in liquidity preference and animal spirits are seen to be both volatile and difficult to predict, Keynes contends that these variables are the most important determinants of the trade cycle. However, in his discursive analysis of the trade cycle, in Chapter 22 of *The General Theory*, other lesser influences are considered, including inventory adjustments.

2.1.5. THE POINT OF EFFECTIVE DEMAND AND QUANTITY-CONSTRAINED RATIONING

One implication of moving away from a representative producer-consumer framework, to one in which different categories of agent make quite separate decisions about consumption and savings, financial intermediation and real investment, is that rationing could well obtain across interrelated markets. For example, in the market for real capital, the sensitivity of investment to uncertainty over prospective yields could be transmitted to the goods market via the multiplier. From here, the resultant quantity-constrained rationing effects could next be transmitted to labour markets. Unemployment and under-utilisation of capacity would be the ultimate outcome.

Moreover, dividend streams would not remain immune to the cyclical variations in profitability. The required return on equity would fluctuate in a counter-cyclical fashion to the extent that share prices responded myopically to cyclical variations in the dividend stream. Hence, capitalisation ratios on industrial investment would tend to vary inversely in response to counter-cyclical changes in the weighted-average cost of capital. Similarly, the average expected holding period return on industrial investment would tend to move pro-cyclically, to a greater extent than movements in the dividend stream, augmenting the effect of these movements. If the multiplier were also to vary in response to changing fortunes in equity markets (due to the heightened influence of accrued capital gains over the more speculative components of consumption), this would further augment the causal linkage between investment and aggregate expenditure.

Quantity constrained rationing models (QCRM's) were developed over the 1970's following the seminal work of Don Patinkin, Robert Clower and Axel Leijonhufvud (see Cuddington, Johanssen and Lofgren 1984, Chapter 2, for a review of this history). Essentially, these models are predicated on the notion that quantities adjust faster than prices and *tâtonnment* is assumed to occur over a sequence of Hicksian temporary fix-price equilibria. Quantity adjustments create a short-run equilibrium at fixed prices within each period, but prices can adjust between periods under the pressure imposed by effective excess demand in various markets.

The distinction between *effective* and *notional* demand is achieved through the incorporation of quantity constraints associated with the short side of transactions in other related markets. For example, the effective demand for labour on the part of firms may be constrained either by the amount of goods that they are able to sell to households or by the amount of labour that households wish to sell to them. In turn, the effective demand for goods on the part of households may be constrained by the amount of goods which firms wish to sell to them or the amount of labour which they can sell to firms. In each round of the *tâtonnment* process, prices are presumed to change in the direction indicated by the sign of effective excess demands. A long-run equilibrium is attained when there is no longer any tendency for prices to adjust over time so that the within-period equilibrium is sustained indefinitely.

Barro and Grossman's 1971 model can generate three different kinds of disequilibrium regime: (1) Keynesian unemployment with effective excess supply in both the labour market and market for goods; (2) classical unemployment with effective excess supply of labour; and (3) repressed inflation with effective excess demand in both markets.

Benassy (1975, 1976) extends Barro and Grossman's representative agent model to an economy consisting of H households and F firms. Rationing functions are applied to allocate volumes on the short side of the market to various agents on the long side of market. This makes the vector of equilibrium excess demands and equilibrium trades dependent on the particular rationing scheme assumed for the economy. Benassy also discusses the possibility that unscrupulous agents could manipulate these rationing schemes. Muellbauer and Portes (1978) further extend the scope of this class of models by incorporating chance uncertainty into decision-making.

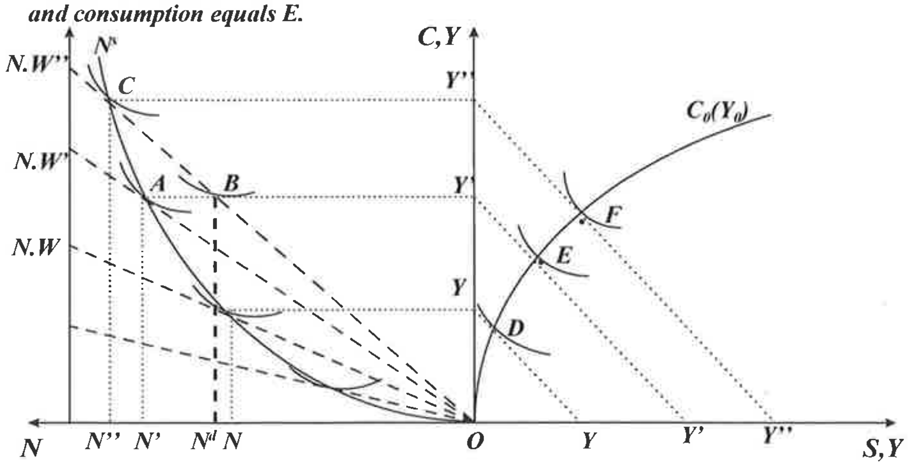
The initial development of quantity-constrained rationing models in macroeconomics was prompted by breakthroughs in the application of duality theory, which overcame certain complexities evidenced in the original post-war literature on the microeconomics of rationing. Nevertheless, despite an early and rapid growth phase quantity-constrained modeling approaches were abandoned with the subsequent advances that were made in real business cycle theory and

New Keynesian analysis of rigidities in goods, labour and credit markets. Despite the fact that the new models of price rigidity and financial rationing provide more respectable micro-foundations for quantity constrained trading, further development of this modeling framework has not occurred.

One reason for this demise could be the neglect of monetary factors over and above cash-in-advance approaches to transactions demand. In particular, researchers within the QCR tradition have largely ignored uncertainty, liquidity preference and volatility of demand for money and investment capital, two essential elements in *The General Theory*'s critique of economic orthodoxy. QCR models traditionally reflect interactions between rationing in labour markets and goods markets rather than interactions between the capital market and the goods market. In a Keynesian model, the impetus for rationing would have its source in the substantive decline in investment spending, *per se*, rather than in the interaction between quantity-constraints in both the goods and labour markets. Once rationing has come about, through adverse movements in long-run expectations that are formed under conditions of uncertainty, the effects of inertia and deepening pessimistic sentiment would explain the absence of any corrective mechanism. This QCR interpretation of involuntary unemployment is implicit in certain Post Keynesian readings of *The General Theory.* For example, Victoria Chick's interpretation of the Keynesian consumption function, depicted in the following diagram, is predicated on a quantity-constraint argument (Chick, 1983, figure 6.1, p 108):

<u>Fig. 14: CHICK (1983) ON CONSUMPTION AND EFFECTIVE DEMAND</u>

At income level N.W , N = unconstrained labour supply giving rise to income level Y and consumption level D. At income level N.W' , N' = unconstrained labour supply giving rise to income level Y' and consumption level E. At income level N.W'', N'' = unconstrained labour supply giving rise to income level Y'' and consumption F, but if labour supply is constrained to N^d, then only Y' income is generated from labour and consumption equals E.



As shown, Chick argues that consumption expenditure (at point *E*) can arise either from income derived from the unconstrained supply of labour at wage level *N.W*, or from the constrained supply of labour at wage *N'.W* (assuming that the point of effective demand obstructs further absorption of labour supply to the left of point N^d). Otherwise, workers would be willing to offer *N''* amount of labour to realise an income of *Y''* and expend *F* units of income on consumption. It is apparent from Chick's general argument, that she interprets the point of effective demand as being a product of constrained interactions between the consumption goods and investment goods sectors.

2.2. Unpacking the IS Schedule

2.2.1. Uncertainty and Real Investment

In his somewhat idiosyncratic homage to Keynes, Hyman Minsky puts forward a way of formally unpacking Keynes's arguments about the influence of uncertainty over investment:

> Since investment fluctuates, and since one of the basic ingredients in the analysis of investment - the supply schedule of investment goods - is a stable function, the observed fluctuations must be due to variations in (1) some combination of the prospective yields, as determined by both the production of income and views about the future; (2) the interest rate as determined in financial markets, or (3) the linkage between the capitalisation factor for prospective yields on real-capital assets and the interest rate on money loans [...] The linkage reflects the uncertainty felt by entrepreneurs, households and bankers. In fact, Keynes uses all three of these to explain the fluctuations of investment" (Minsky, 1975p. 95-96).

He suggests that the use by Keynes of the downward sloping marginal efficiency of capital schedule to discuss the influence of liquidity preference and interest rates on levels of investment, helped to obscure the sophistication of his analysis and encouraged later misinterpretations and distortions on the part of his neoclassical reviewers. Minsky favours an alternative representation of Keynes's views on investment, on the grounds that:

> (t)he capitalization of the prospective yields to generate a demand price for capital assets is a more natural way to approach the problems of fluctuating investment than the marginal-efficiency-of-capital schedule; a direct approach through the Q's (quasi-rents)[62]

[62] Following Keynes, Minsky defines quasi-rents as equal to the rentals arising from the difference between price and prime costs (material and labour). In chapter 17 of *The General Theory*, returns on each asset are defined to equal $q - c + l + a$, where a equals the expected capital appreciation, l is the liquidity premium on the asset, q is the own rate of return (Q in Minsky's notation) and c is the carrying cost. For money, $q - c$ equals zero, but the liquidity premium is the highest of all assets. For equities, q is the dividend, while a is the expected capital gain. For liquid goods, c is the cost of warehousing, insurance and the short-term borrowing rate, while a is the expected capital gain from resale. Finally, for capital goods q is the expected quasi-rents from sale of the product, while c is the interest

> and specific capitalisation factors is more precise than an approach by way of relative marginal efficiencies. First of all, the Q's are not submerged, as in the alternative approach; second, the capitalization factor, which can have a varying ratio to the market rate of interest on secure loans because of the different values placed upon liquidity, is explicitly considered. Furthermore, two market-determined prices are dimensionally equivalent to the capitalized value of the Q's: the market price for items in the stock of capital assets and the price of equities, of shares (1975, p. 100-101).

In the following diagram, I combine Chick's (1983) graphical analysis of the money market with Minsky's (1975) analysis of the capital goods market. Minsky's framework articulates the separate influence of changes in the capitalisation ratio on the demand price for capital assets, shifts in the supply curve for investment goods (P^I), and changes in borrowers' and lenders' risk ($P_{I/B}$ and $P_{I/L}$, respectively)[63] over desired levels of investment. However, whereas Minsky incorporates monetary influences over the capitalisation ratio implicitly into a $P^k = P^k(Q, M)$ curve relating the demand price of capital (P^k) to the supply of money balances (M) for a given amount of quasi-rents per period (Q), my synthetic model goes further by drawing on Chick's analysis of the money market.

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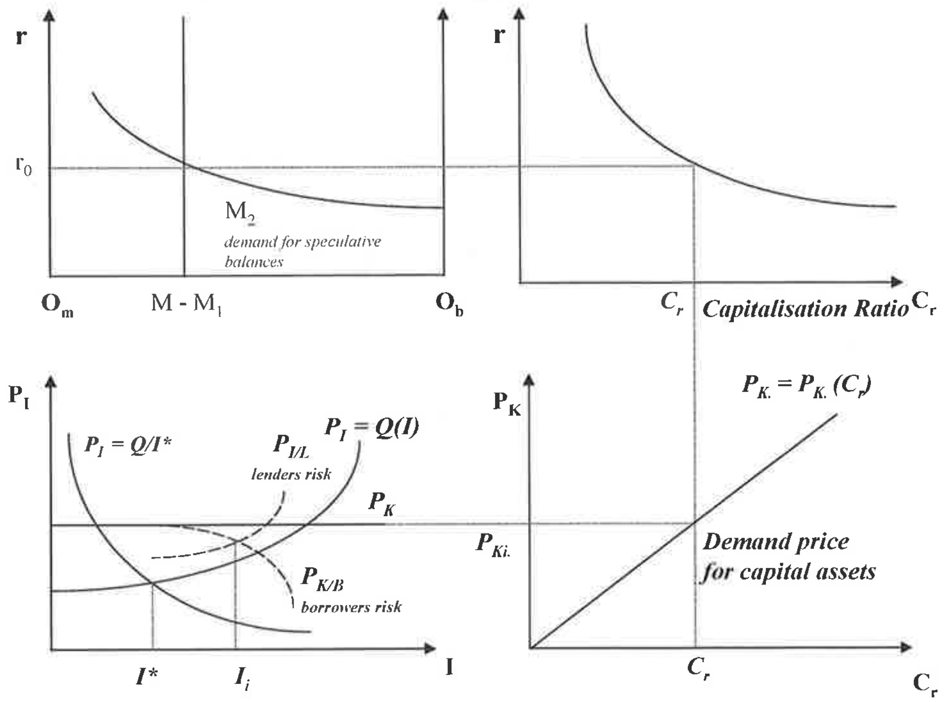
and dividends to be paid on external debt and equity finance. Production will only take place if the expected quasi-rents net of carrying costs are greater than the user cost (the latter is defined as the cost of putting assets to work productively over the period rather than deferring production).

[63] These are defined by Minsky in the following manner:

> Borrower's risk has two facets. First, in a world with uncertainty, where the fates of various capital assets and firms can differ, a risk averter will diversify. This means that beyond some point, which for an individual wealth owner or corporation depends on the size of his wealth, the capitalisation rate for any one type of capital asset to be used in a particular line of commerce declines as the amount owned increases. Second, since the borrower sees the cash flows due to debts (CC's) as certain and the prospective yields (Q's) as uncertain, increasing the ratio of investment that is debt-financed decreases the margin of security and thus lowers the capitalization rate the borrower applies to the Q's. (1975, p.109).
>
> Lender's risk...as it applies to a particular firm, takes the form of increased cash-flow requirements in debt contracts, as the ratio of debt to total assets increases. Lender's risk shows up in financial contracts in various forms: higher interest rates, shorter terms to maturity, a requirement to pledge specific assets as collateral, and restrictions on dividend payouts and further borrowing are some of them. Lender's risk rises with an increase in the ratio of debt to equity financing or the ratio of committed cash flows to total prospective cash flows. (1975, p. 110).

This unbundling of money-market relationships is analogous to Minsky's unbundling of the capital-market investment schedule in that it permits a more detailed analysis of the effect that changes in liquidity preference have over the speculative and precautionary demand for money balances.

Fig. 15: MINSKY (1975) ON THE DEMAND FOR INVESTMENT



The first quadrant, depicting equilibrium between the speculative demand for inactive money balances (M_2) and the residual supply of money ($M - M_1$) after accounting for the level of transactions demand (M_1) given income, determines the bond rate of interest (r). The resultant interest rate determines the capitalisation ratio (C_r) for expected returns on capital assets in the second quadrant. The third quadrant, shows how the selected capitalisation ratio translates expected quasi-rents into changes in the demand price of capital goods. Finally, the fourth quadrant brings investment supply and demand (augmented by lenders' and borrowers' risk, $P_{I/L}$ and $P_{K/B}$, respectively) together to determine the level of investment activity (I_i). The $P_I = Q/I^*$ curve then determines the proportion of investment (I^*) which is internally funded from cash inflows (Q) given the supply price (P_I).

I believe this integrated approach does more justice to the richness of Keynes's monetary analysis. As I have suggested, the representation of monetary relationships in the first quadrant provides the basis for a visual analysis of a variety of important financial influences, including those associated with: changes in velocity due to movements in the relative proportion of active and inactive balances; changes in the demand for money due to the operation of the finance motive (which would shift the $M - M_1$ curve inwards); and the application of monetary policy through either open market operations or pure expansions of the money supply (on this see Chick, 1983, Chapt. 11). In addition, declines in the marginal efficiency of capital can be represented by clock-wise rotations in the third quadrant's demand price of capital function. Increases in liquidity preference can be depicted by upward movements in the first quadrant's M_2 curve, downward shifts in the second quadrant's capitalisation curve, and inward movements in the fourth quadrant's curves which represent borrowers' and lenders' risk[64].

[64] It is important to realize that this investment model forms only a small component of a larger and more complete IS-LM and AS-AD model (as in Vercelli's work considered above) that identifies the links between investment activity, aggregate expenditure and income and induced changes in the demand for transactions balances.

2.2.2. Debt-deflation and the Supply and Demand Price of Capital

Minsky (1975) argues that as firms resort to higher levels of external finance (determined by I - I*), borrowers' and lender's risk will rise. In fact, over time an environment of stable growth will encourage all economic agents to adopt increasingly speculative financial positions:

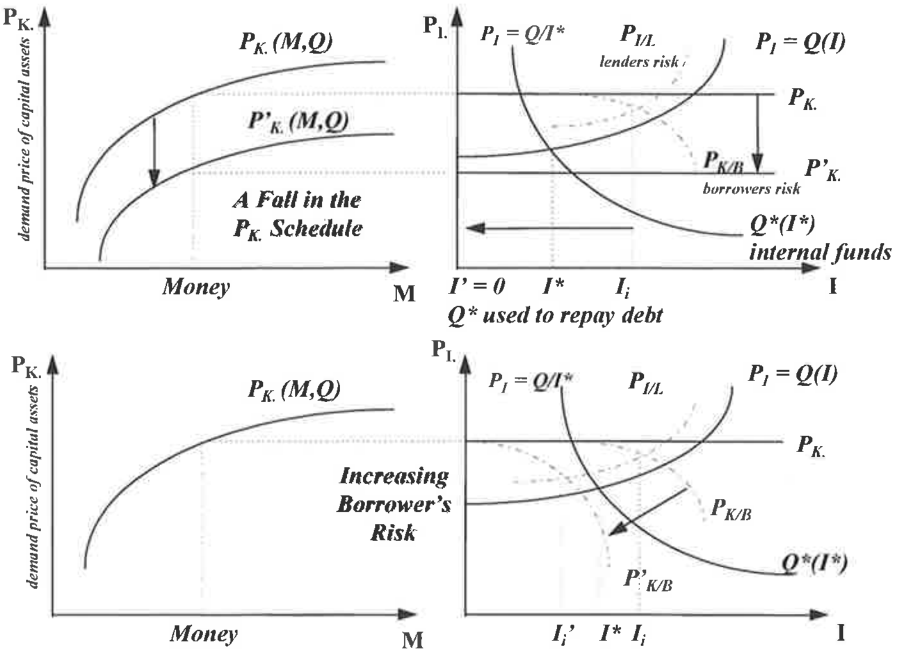
> Thus speculation has three aspects: (1) the owners of capital-assets speculate by debt-financing investment and positions in the stock of capital-assets; (2) banks and other financial institutions speculate on the asset mix they own and the liability mix they owe; (3) firms and households speculate on the financial assets they own and on how they finance their positions in these assets. [...] As a boom develops households, firms, and financial institutions are forced to undertake ever more adventuresome position-making activity. When the limit of their ability to borrow from one to repay another is reached, the option is to either sell out some position or to bring to a halt, or slow down, asset acquisition. For operating firms this involves a reduction in the leverage used in financing new investment (1975, p.123-4).

This attempt on the part of economic agents to reduce their exposure leads to the sale of assets to hasten the repayment of debts. The resultant fall in asset-prices can precipitate a full-blown episode of debt-deflation[65]. As shown in the following diagram, debt-deflation arises in two cases: first, when borrowers' risk increases to a point where investment drops below even the level which can be funded by internally generated funds; and second, the demand price of assets (P_k) falls below the supply price of assets (Pi). For simplicity, the debt-deflation process is

[65] Debt deflation was seen by Keynes as the ultimate consequence of downward wage flexibility as a 'solution' to the unemployment problem (see Palley, 1995 for a formal evaluation of this important aspect of Post Keynesian analysis). The fact that stability is a transient state, undermined by endogenous increases in the composition of speculative financial positions, is a core aspect of many dynamic Post Keynesian models of the business cycle. Because these compositional changes give rise to structural instabilities in the topological manifold of dynamic models (see Vercelli 1991, Appendices 4a and 4b) they also undermine the possibility of a definitive rational expectations equilibrium.

illustrated using Minsky's $P^k = P^k(Q, M)$ curve (which directly relates the level of real money balances to the demand price of capital).

Fig. 16: MINSKY (1975) ON THE TWO TYPES OF DEBT-DEFLATION



Minsky argues that:

> ...because a debt-deflation process has both an immediate and a lingering effect upon investment and desired debt positions, it will lead to a period of persistent unemployment. A relatively low income, high unemployment, stagnant recession of *uncertain* depth and duration will follow a debt-deflation process (1975, p. 126).

2.3. Extending the LM Schedule

2.3.1. Incorporating portfolio choice amongst monies, fixed interest securities and equities

In a formal sense, *The General Theory* collapses available financial assets into two classes, money and interest paying securities. As Kahn argues, a comprehensive analysis of monetary influences over the macroeconomy can only be sustained in the context of a portfolio-based model of asset supply and demand. Girol Karacaoglu has attempted a short-period analysis of this kind in a surprisingly neglected contribution to the *Journal of Post-Keynesian Economics*. The question to be answered is how changing capitalization factors can be embodied in portfolio models of asset demand.

In his paper, Karacaoglu (1984) utilises a static version of a Tobin-style portfolio model to examine the effects of public investment on the whole economy. His main intention is to investigate the consequences of abandoning the assumption that gross substitution obtains between liquid and illiquid assets in private portfolios. Following the arguments of Paul Davidson (1972), Karacaoglu regards resource-using reproducible durable goods as a poor substitute for money as a vehicle for transferring purchasing power into an uncertain future (Karacaoglu, 1984, pp. 1-2)[66].

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[66] Davidson's lifelong opposition to the axiom of gross substitution is based on the claim that it conceptually reduces the object of analysis to a "quasi-barter" economy.

Karacaoglu's linearized short-run version of Tobin's general equilibrium model of asset markets comprises a complete set of asset demand functions for money, bonds and physical capital (equation 7) and corresponding portfolio equilibrium conditions (equation 8), and an equation relating the real price of physical capital to the marginal efficiency of capital and the rate of return on capital (i.e. Tobin's q-ratio - equation 10). These are combined with a rudimentary IS model of the goods market, consisting of a simplified consumption function with a wealth effect (equation 1), an investment function (equation 2) and a goods market equilibrium condition (equation 5) setting income equal to consumption, investment and government spending (the latter, set exogenously in equation 3); and a government budget constraint (equation 9).

For tractability, several simplifications are introduced: first, to avoid explicit modeling of goods market supply conditions, unemployed resources are assumed to exist; second, goods prices are held constant; third, the initial equilibrium is characterised by a balanced budget; and fourth, it is assumed that investment does not lead to an increase in the capital stock within the period under analysis. Under these assumptions, the complete model can be represented as follows:

1. $C^* = c_0 + c_1(Y - T) + c_2 W$
2. $I^* = i_0 + i_1 q$
3. $G = \overline{G}$
4. $T = \overline{T}$
5. $Y = C^* + I^* + G^*$

6. $W = M + B + qK$

7. $$\begin{bmatrix} M^* \\ B^* \\ K^* \end{bmatrix} = \begin{bmatrix} m_0 \\ b_0 \\ k_0 \end{bmatrix} + \begin{bmatrix} m_1 & m_2 & m_3 \\ b_1 & b_2 & b_3 \\ k_1 & k_2 & k_3 \end{bmatrix} \begin{bmatrix} r_m \\ r_b \\ r_k \end{bmatrix} + \begin{bmatrix} m_4 \\ b_4 \\ k_4 \end{bmatrix} Y + \begin{bmatrix} m_5 \\ b_5 \\ k_5 \end{bmatrix} W$$

8. $$\begin{bmatrix} M^* \\ B^* \\ K^* \end{bmatrix} = \begin{bmatrix} M \\ B \\ qK \end{bmatrix}$$

9. $G^* - T = dM + dB$

10. $r_k q = R$

Where:

C^*, I^*, G^* = desired rates of private consumption, private investment, and government

expenditures, respectively

Y = rate of production
T = taxes
W = private wealth
M, M^* = actual and desired money stocks
B, B^* = actual and desired stock of interest bearing government bonds
K, K^* = outstanding stock of physical capital; actual and desired, respectively
r_m, r_b, r_k = rates of return on money, bonds and physical capital, respectively
dM, dB = differentials of M and B, respectively
R = expected marginal efficiency of capital
q = Tobin's Q-Ratio

From the balance sheet constraints, it follows that:

$m_j + b_j + k_j = 0,\ j = 0,1,...,4$
$m_5 + b_5 + k_5 = 1.$

Three symmetry constraints are also imposed:

$b_1 = m_2,\ k_1 = m_3,\ k_2 = b_3$

Under the balance sheet and symmetry constraints the asset demand equations can be reduced to two equations for money and bond market equilibrium. The remaining equations can be reduced through substitution to a single equation determining equilibrium in the goods market. Karacaoglu (1984) first considers a partial equilibrium version of the model in which income has not yet attained its final equilibrium value. This consists of the first two equations with the additional complication that the income variable is replaced by actual disequilibrium values for consumption, investment and government spending.

This substitution permits the finance demand for money to operate (conforming to the approach favoured by Davidson, 1994a, chapter 7). In a disequilibrium situation, an increase in desired expenditures will lead to an increased demand for loanable funds to meet contractual obligations at constant levels of income. Once a new equilibrium is attained, with desired expenditure equaling income, the finance demand for money will be absorbed into the transactions demand for money (see Tsiang, 1980). In the transition to this new equilibrium, the demand for money would be proportional to actual expenditure (C + I + G) rather than equilibrium production (Y).

Karacaoglu (1984) examines the response of Tobin's q-ratio to an increase in government spending for four different versions of this disequilibrium model: the general case incorporating gross substitution; Davidson's case wherein the axiom is violated; a money-capital version wherein money and bonds are perfect substitutes; and a Patinkin, or text-book IS-LM, case wherein bonds and capital are perfect substitutes. In this model, the absence of gross substitution is achieved by setting equal to zero the parameters representing cross elasticities of substitution in demand between physical capital and money in response to changes in their relative rates of return, (i.e. $m_3 = b_3 = 0$; $m_4 = -b_4$). Moreover, to prevent the possibility of conversion of physical capital to cash balances to meet increases in the transactions demand for money, the income elasticity of demand for capital is also set equal to zero. This implies that any increase in the demand for real cash balances can only be met through the conversion of bond holdings into cash[67]. The money/capital model assumes that m_2 approaches minus infinity, while the Patinkin model assumes b_3 approaches minus infinity.

Karacaoglu (1984) confirms that only for Davidson's case does increased government spending clearly boost investment and income, whereas crowding-out is implicitly assumed by both the money-capital and Patinkin versions of the model. Next, he considers the full equilibrium case by adding the goods market equilibrium equation to the two asset demand equations. Government spending in the Davidson version of the model raises both investment and income, while indeterminate results obtain in the general case and money-capital versions. In the Patinkin model, although income increases, government spending crowds out private investment activity. Similar results arise in the case of monetary policy achieved through open market operations. An increase in money supply has no effect on investment or output in Davidson's version of the model in marked contrast to the textbook IS-LM version.

Finally, Karacaoglu (1984) examines the outcome of an increase in liquidity preference (either for money or bond holdings) and shows that in the absence of gross substitution, both desired

[67] In Wray's analysis there is a liquidity continuum stretching from money through to durable capital goods. The outcomes of a rise in liquidity preference depend on the presence of a small elasticity of substitution between the most liquid and the least liquid assets. This requirement is a far less stringent condition than the one Karacaoglu imposes - that the elasticity of substitution be *zero,* in violation of the celebrated axiom of gross substitution - but is

private investment and the equilibrium rate of production would decline although outcomes are indeterminate for both the general case and the money-capital money. The Patinkin version of the IS-LM model yields the perverse result that investment would actually rise in the case of an increase in demand for bond holdings due to the reduction in interest rates.

Apparently, this simple model confirms Keynes' intuitions and insights about the adverse consequences of an increase in liquidity preference in response to rising uncertainty, the ineffectiveness of monetary policy (vis. Keynes' celebrated metaphor describing attempts to control the economy by monetary means as 'pushing on a piece of string'), and the overall correctness of Keynes' criticisms of Pigovian prescriptions to cure unemployment.

However, Karacaoglu commits three errors in his article. First and foremost, in applying parameter restrictions to the 'capital' rate of return coefficients in the money and bond demand equations, equity and physical capital are conflated. It is equity rather than physical capital which appears in the q-ratio expression in Tobin-style portfolio models (physical capital only operates implicitly in the marginal efficiency of capital variable which goes into the determination of Tobin's q-ratio). As Paul Davidson has defined it, the axiom of gross substitution would still hold because only a restricted set of assets - namely, money, bonds and equities rather than physical capital - is represented in the portfolio model's asset demand equations.

Second, he assumes that liquidity preference effects operate through changes in the intercept terms of the demand system. In contrast, I believe that liquidity preference can only be logically

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nonetheless sufficient to generate the desired Keynesian results on which Karacaoglu focuses, such as the absence of represented through the incorporation of a vector of liquidity premia into the vector of required rates of return (the r_m, r_b, and r_k terms)[68]. In addition, it could be argued that the wealth coefficients on more liquid assets (money and treasury bonds) should rise relative to those of less liquid assets (corporate bonds and equities). In Wray's (1990, 1992) analysis, these wealth effects operate through changes in the diversity of null prices and the subsequent changes in the offer of IOU's by borrowers and take-up by banks of these IOU's.

Third, and closely related to the above discussion of liquidity premia. Karacaoglu excludes by assumption the possibility of any capital gains component in the asset rates of return. In fact, asset stock/price and asset demand/flow-rate-of-return models can only be logically reconciled if it is recognised that the actual rates of return to equity and bonds must vary in the short-run to preserve equilibrium in response to changes in liquidity preference[69]. Initially, this can only occur through variations in the spot- or demand-prices of these assets, such that the expected capital gain or loss can compensate for any implicit change in liquidity premia. In conformity with the arguments in chapter 17 of *The General Theory*, a term for asset-price appreciation should appear in each of the asset demand equations[70]. Thereby, in a monetary equilibrium the money rate of interest would equal the own rate of return on each asset, net of carrying cost,

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public sector "*crowding out*" of private sector investment.

[68] In moving from an asset flow-demand *cum* rate of return model to an asset stock-demand *cum* asset-price model, such as Wray's version of a Boulding style identity that I consider in section 2.3.2 below, this procedure would be formally equivalent to Wray's adjustment of the desired asset ratios (the $R_{Ai's}$ and $R_{m's}$).

[69] In this regard, I am arguing against Wray's position that liquidity preference can only be analysed effectively by utilising stock-price models of asset markets (Wray, 1990). Ultimately, both stock-price and flow-rate-of-return approaches must potentially be open to reconciliation in this manner so as to be logically and semantically equivalent.

[70] In effect, this is the approach adopted by rational expectations theorists who utilise Tobin-style models to examine the effects of economic "news" on the demand for assets. For example, the portfolio-theoretic asset approach to exchange rate determination attempts to capture this influence by incorporating a term for the expected exchange-rate appreciation into the demand equation for foreign financial assets.

including a component for any expected capital appreciation or depreciation on the asset[71]. In Colin Rogers' analysis of *The General Theory*'s $[q - c + l + a]$ expression, the expected appreciation term a equals the difference between the money rate of return r and rate of return over cost r^* (i.e., $r^* = (Q_2 - Q_1)/Q_1$ *and* $r = (P_2Q_2 - P_1Q_1)/P_1Q_1$; where P_2, Q_2 are the expected prices and quantities in the subsequent period [equivalent to Davidson's P_t], while P_1,Q_1 are this period's prices and quantities [equivalent to Davidson's P_f]). Hence, $a = ((P_2 - P_1)/P_1)Q_2/Q_1$ (see Rogers 1989, chapter 9)[72] .

Note that while variations in the liquidity premium on money give rise to compensating changes in the spectrum of interest rates and expected capital gains on other assets, in the case of physical capital, quasi-rents can only rise gradually through some combination of disinvestment (net depreciation of the capital stock) or devalorisation of capital assets. Rogers' expression implies that if liquidity preference or the interest rate (r) rises, then P_1 (the short-period or spot-price of the asset) falls. As we have seen, in Paul Davidson's graphical analysis of the capital markets, this fall in the spot price occurs due to a leftward shift in the demand curve for capital.

As Carlo Panico (1993) has argued, a series of additional Post Keynesian (specifically Kaldorian) criticisms can be made of Tobin-style portfolio models. First, the adoption of an endogenous money approach, in which interest rates are targeted through open-market operations, would require modifications to those government sector equations which determine both the supply of

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[71] See footnote 8.

[72] In private discussions, Colin Rogers has emphasized the fact that, expressed in continuous time form, the somewhat confusing Q_2 / Q_1 term drops out of the formula for the expected appreciation.

money and the expected real rates of return to bonds and money[73]. Second, as I have also suggested above, in a more comprehensive model the firm sector equation determining the q-ratio (equation 10 in Karacaoglu, 1984) would normally incorporate a term for the expected capital gain to holders of equity. Following Kaldor, Panico contends that such an equation should be further modified to incorporate the effect of highly speculative and volatile expectations. I have suggested that this could be achieved through the incorporation of variable liquidity-premia into the rates of return. Moreover, a more detailed model would also include an equation for determining the dividend payout ratio as a function of its long-run equilibrium proportion, with net income or some other variable brought in to allow current economic conditions to exert some short-term influence over dividend payouts. Panico (1993) argues that variations in interest rates and dividends would also exert a short-term influence over investment and recommends that each of these should be incorporated into the equation determining investment as a function of the q ratio (equation 2, in Karacaoglu, 1984). Finally, while the real wage is determined in orthodox models of the firm sector by assuming that it is equal to the value marginal product of labour, in a Post Keynesian model the nominal wage would be set exogenously with prices determined by some sort of mark-up pricing behaviour.

Wray's (1991) criticism of Tobin's portfolio analysis is based on four classic papers, republished in the 1971 volume, *Essays in Economics*. Wray contends that, while Tobin allows a role for liquidity preference effects in his model, because wealth holders can adjust their asset-holdings

[73] Karacaoglu's minimalist model does not include equations to determine expected real rates of return, and equation 9 assumes that the government deficit is funded through equiproportionate expansions in both money and bond supply, the latter implicitly assumed to be under the complete control of the monetary authorities.

subject to the wealth constraint, there is no role for money demand. This is because it is assumed that 'deposits make loans' via the traditional deposit multiplier, rather than that 'loans make deposits'. In other words, Tobin takes flow variables as exogenously given so that the money supply cannot expand endogenously as spending increases. Therefore

> ...the demand for money is always a demand for hoards, and portfolio decisions are made independently of 'given' spending decisions (Tobin, 1971). [...] That is, money is obtained by selling assets, rather than by issuing new debt. Tobin is, therefore, concerned with portfolio allocation decisions, rather than with decisions made by economic agents to finance positions in assets (Wray, 1991, pp. 84-5).

In his *Metroeconomica* paper, Panico has also examined a long-run version of a Tobin-style financial macroeconomic model, comparing it unfavourably with its Kaldorian counterpart (Panico, 1993). He argues that the Kaldorian variant is more general in its incorporation of differential savings propensities for workers and rentiers. More fundamentally, reinforcing the point about capital perversities canvassed above, Panico endorses Kaldor's rejection of any presumed monotonic long-run relationship holding between the capital-share of income (K/Y) and the dividend return on equity (r_k)[74]. Accordingly, there is no inbuilt mechanism for ensuring long-run equilibrium at full-employment and the economy's steady-state. Or rather, the warranted growth rate can only be attained if the key fiscal and monetary policy variables are allowed to adjust endogenously to their appropriate levels - namely, those determining the level of the budget deficit and the extent to which it is financed through expansion of money or bond supply under the assumption that the nominal interest rate is set exogenously. Not only is this the case, but the distribution of income between labour and capital now becomes an exogenous variable capable of influencing the growth path (see the appendix to this chapter of the thesis for a detailed overview of Panico's analysis).

2.3.2. Endogeneity of the Money Supply

Both Hyman Minsky's and Victoria Chick's works follow *The General Theory* closely in adopting Keynes' assumption that the money supply is exogenously controlled by the monetary authorities. However, both argue in favour of the targeting interest rates rather than the money supply on theoretical grounds that the money supply endogenously expands or contracts to meet variations in demand. This occurs in two ways: first, through innovations which enable expansion of loans and deposits on a given reserve base and, second, because the central bank - operating as lender of last resort - will be obliged to supply required liquidity to prevent severe debt-deflation (for an overview of Minsky's monetary views see Wray, 1990, pp. 135-8). Wray defines the Post Keynesian endogenous money approach as follows:

> The endogenous money approach includes three essential propositions, first, loans make deposits, second, deposits make reserves, third, money demand induces money supply (Lavoie 1985). The first two propositions imply that banks do not passively await deposits so that they can issue loans. With a developed and integrated financial system, as loans are spent, the vast majority of expenditures return as deposits to the banking system (leakages into currency are minimal). Asset and liability management, the Fed funds market, international sources of liquidity, and central bank loans (either at the discount window or in lender of last resort operations) ensure that banks that need reserves are normally able to obtain them. [...] As Kaldor (1985) argues, money is supplied

[74] This brings us back to capital debates. Panico cites Pasinetti's rejection of the aggregate production function and its associated neoclassical fables of income distribution, all of which are undermined by capital-perversities that feature in re-switching and capital-deepening phenomena (Panico, 1993, fn. 18, p. 105).

> because someone wants it. Money supply and money demand are simply different sides of the balance sheet. From the firm's point of view, money demand is the willingness to go into debt, and money supply is the IOU it issues. Of course, the firm's IOU is not money unless someone is willing to accept it. From the bank's point of view, money demand is indicated by the willingness of the firm to issue an IOU, and money supply is determined by the willingness of the bank to hold that IOU and to issue its own liabilities to finance the purchase of the firm's IOU (1990, pp. 73-4).

This raises the issue of how to represent liquidity preference while recognising the endogenous nature of the money supply process. Within the Post Keynesian tradition, two contrasting stances have been taken on this issue - the horizontalist (or accommodationist) position associated with writers such as Basil Moore (1988) and Marc Lavoie (1985); and the structuralist position associated with writers such as L. Randall Wray (1990, 1991, 1992) and Thomas Palley (1995)[75]. Moore contends that Keynes' theory is based on a circular argument because the demand for money and the level of interest rates vary with income, so that each change of income would change interest rates and therefore affect investment, thereby leading to further changes in income and interest rates (Wray 1990, p.155)[76]. Moore favours replacing *The General Theory's* multiplier analysis and its "flawed" notion of liquidity preference with an endogenous money approach. Lavoie (1985) goes on to criticise Keynesian notions of the finance motive, suggesting

[75] As will become clear in the following discussion, the term "*Structuralist*" implies that there are structural barriers to the continuous, endogenous expansion of money supply at unchanged interest rates to meet growth in demand.
[76] On the face of it, this is a simple problem of feedback from expenditure to transactions demand (and the finance motive) which can readily be accommodated analytically through the use of feedback mechanisms, without resorting to the IS-LM model's simultaneous determination of both income and the interest rate (see Vercelli 1991, Chapter 11).

that a rise in the demand for finance cannot put pressure on interest rates in a world where the money supply responds endogenously to money demand[77].

In his comprehensive 1990 study of money and credit, Wray attacks the horizontalist conceptions of authors such as Marc Lavoie (1985), going to great lengths to show that liquidity preference theory is still applicable in a financial system with lender-of-last-resort facilities, sophisticated asset and liability management mechanisms, and a prevalence of underutilised lines of credit. He turns to Jan Kregel's writing for support, endorsing Kregel's argument that liquidity preference theory and the expenditure multiplier are two sides of the same coin (Kregel 1988):

> A decline in liquidity preference will lower the interest rate, which raises the demand price of capital assets and causes investment to rise until the marginal efficiencies of all assets fall to equality with the lower interest rate. This is equivalent to arguing that income rises through the multiplier until savings rises to equality with the new higher level of investment[78] (Wray 1990, p. 157).

Kregel (1988) argues that Keynes held to a monetary theory of the interest rate and a liquidity preference theory of asset-prices. In common with Minsky (1975), Kregel deems the demand price of capital goods to be a function of liquidity preference, the marginal efficiency of capital and the nominal interest rate, while the long-period supply price of capital goods is viewed as largely determined by expected prime costs. However, it is the marginal efficiency of money which sets the standard against which other rates of return must be compared. This standard is determined by the stock supply and demand for hoards. If the marginal efficiency of money falls,

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[77] In a recent paper, Paul Dalziel (1996) has utilised a process model of the multiplier, first developed by James Meade, to clarify a number of Post Keynesian controversies about liquidity preference, the multiplier, endogenous money and the finance motive.

investment increases because the demand price of capital rises above the supply price and there are more projects for which the marginal efficiency of capital exceeds the nominal interest rate. If, on the other hand, liquidity preference declines, then money demand falls and the marginal efficiency of money falls below the marginal efficiency of capital.

In an extension of Kregel's (1988) notion that liquidity preference and the multiplier process are two sides of the same coin, Wray (1990) emphasises the point that endogenous money must be included in any narrative about the determination of effective demand. In support of this argument, Wray (1991, 1992) applies the "Boulding's Balloons" identity to the analysis of liquidity preference. This accounting identity, which was first derived by Kenneth Boulding (1966), relates asset-prices linearly to the proportional ratios in which agents wish to hold assets and the predetermined supplies of each asset. Wray's extended matrix version of Boulding's identity is depicted below followed by the cross-equation restriction that desired asset ratios must sum to unity. The *i*th row of the identity is also written out to clarify the relationship, which obtains between desired asset ratios, asset stocks and prices.

$$\begin{bmatrix} P_{A_1} \\ P_{A_2} \\ \mathrm{M} \\ P_{A_3} \end{bmatrix} = \left|\frac{1}{Rm}\right| |M| \begin{bmatrix} R_{A_1} & 0 & \Lambda & 0 \\ 0 & R_{A_2} & \Lambda & 0 \\ \mathrm{M} & \mathrm{M} & \mathrm{M} & \mathrm{M} \\ 0 & 0 & \Lambda & R_{A_n} \end{bmatrix} \begin{bmatrix} \frac{1}{A_1} \\ \frac{1}{A_2} \\ \mathrm{M} \\ \frac{1}{A_n} \end{bmatrix}$$

[78] Wray has erroneously substituted the term "income" for "investment" at the end of this quoted sentence. However, the passage only makes sense in the latter case.

$$Rm + \sum_{i=1}^{n} R_{A_i} = 1$$

implying that $P_{A_i} = \left(\frac{1}{R_m}\right) M \left[R_{A_i} \left(\frac{1}{A_i}\right)\right]$

Here the R_{A_i}'s = desired asset ratios, R_m = liquidity preference (i.e. The desired asset ratio of money), the A_i's = quantities of each asset, and the P_{A_i}'s = respective asset-prices

In the short period, asset quantities are fixed, but in the long period, asset quantities may vary. The upper portion of the A vector is assumed to be more liquid, while the lower portion of the A vector is less liquid. Wray (1992) introduces the concept of the null price of asset - the price at which agents are indifferent between supplying or demanding the asset. This implies that, for a given asset quantity, exchange takes place only if there is a *diversity* of null prices, whereas the prices of exchanged assets depend on the *level* of null prices. If null prices diverge between banks and borrowers then banks become more willing to lend and firms become more willing to borrow. If null prices diverge between suppliers and demanders (e.g. of capital goods), the demand price of capital goods rises and firms' purchases of capital goods are funded by debt, which is taken up by banks as they issue money.

In these papers and his earlier book, Wray (1992) makes much of the distinction between changes in liquidity preference and changes in money demand - although the two phenomena are obviously related. A rise in liquidity preference (R_m) implies that desired asset ratios fall, especially for less liquid assets, so that the level and diversity of their prices will fall, leading to a decline in money demand. This is because borrowers become more reluctant to issue new debt and banks reduce the volume of their purchases of debt.

When many forms of money exist the multiplicative component in front of the matrix in the above expression for Boulding's identity can be expanded as shown in the first equation in the following diagram. The next three equations identify the cross-equation restrictions, which must hold in the extended version of the asset-pricing identity:

$$\left(\frac{1}{R_m}\right)M \Rightarrow \begin{vmatrix} \frac{1}{R_{m1}} & \frac{1}{R_{m2}} & \Lambda & \frac{1}{R_{mk}} \end{vmatrix} \begin{vmatrix} M_1 \\ M_2 \\ \mathrm{M} \\ M_k \end{vmatrix},$$

where

$$R_{mj} = Mj \frac{1}{\sum_{j=1}^{k} M_j + \sum_{i=1}^{n} A_i}$$

$$R_m = \sum_{j=1}^{k} R_{mj}$$

$$\sum_{j=1}^{k} R_{mj} + \sum_{i=1}^{n} R_{A_i} = 1$$

Here:

M_1 = high powered money	A_1 = government bonds
M_2 = demand deposits	A_2 = commercial paper
M_3 = other checkable liabilities	A_3 = highly rated corporate bonds
M_k = liabilities of non-bank fin'l instit's	A_n = physical capital

Wray argues that M_1 is dependent on the purchase of A_1 by the Central Bank; M_2-M_4 (bank money) is dependent on purchases of assets A_2-A_4 (liabilities issued by borrowers) by the commercial banks; and M_5-M_k (NBFI money) is dependent on purchase of assets A_2-A_k by Non Bank Financial Institutions. However, purchases of A_1-A_k by the general public do not lead to any change in money supply.

In this more complex identity, a rise in liquidity preference implies that the *left hand side* R_{mj}*'s* are greater than the *right hand side* $R_{mj,}$*'s*; the R_m*'s* are greater than the R_A*'s*; and the *left hand side* R_A*'s* are greater than the *right hand side* R_A*'s*. Hence liquidity preference has the greatest effect on the P_A*'s* in the lower portion of the *A* vector. On one hand, interest on bank liabilities (money) determines the banking sector's willingness to buy non-bank liabilities, while, on the other hand, changes in the upper portion of the *A* vector determine fluctuations in the term structure of interest rates.

Wray (1992) uses this identity to clarify Keynes' properties of money, namely: 1) a small elasticity of production, which implies that an increase in liquidity preference does not lead to diversion of labour into money production; instead, interest rates rise because a higher value for R_m implies that P_A falls and preferences for assets in the lower right hand side of the R_A matrix which require high labour inputs, also fall; 2) a small elasticity of substitution, which is expressed in the fact that a rise in liquidity preference implies a higher demand for assets in the upper portion of the *A* vector; hence P_A is lower for assets in the lower portion of the matrix, but substitution will not occur because these assets can't satisfy the desire for liquidity; and 3) a high and positive liquidity premium, which is reflected in the fact that the return on money does not fall quickly as *M* is increased, due primarily to its negligible carrying costs.

These properties of money explain how, for a given degree of liquidity preference, the nominal interest rate functions as the centre of gravity for returns on other less liquid assets through changes in their spot asset-prices (also see Chick, 1983 and Cowen and Kroszner, 1994). As we have seen, the Horizontalists argue that the price of high powered money is fixed by the central bank and banks use a stable mark-up to determine the spectrum of interest rates, giving rise to a horizontal demand curve at the long-run rate of interest. In contrast, Wray argues that:

> [e]xpansion of investment is subject only to a liquidity constraint: the banking system must allow its balance sheet to adjust by satisfying the additional demand for hoards at the prevailing interest rate. Normally, banks will advance credit to meet an increase in the demand for money used to finance investment expenditures with no pressure on interest rates. However, there will be upward pressure on interest rates if first, the liquidity preference of the public rises so that bank credit does not return to banks, and second, if banks will meet the extra money demand only at higher interest rates (1990, p. 159).

In the first case, the banking system is forced to remain illiquid so that households can hold liquid assets. In the second case:

> ...as a first approximation, short term interest rates on loans are a function of the costs incurred by commercial banks as they issue deposits, in competition with alternative liquid assets which might be held by the public, and make loans. The differential between the deposit and loan rates of interest must compensate banks for the perceived risk, and is set primarily through custom and rules of thumb. The long term bond rate of interest is then established at a level sufficient to induce the public to hold saving in the form of bonds, while the long term loan rate of interest is set high enough to provide a differential to compensate investment banks for perceived risk - again primarily determined by rules of thumb. A

> given level of investment (and saving) should place no pressure on interest rates as long as liquidity preference and rules of thumb don't change. A rise of investment spending will increase the size of balance sheets and may affect both short term and long term interest rates if perceived risk or liquidity preference rises, or if prudent leverage ratios are exceeded (Wray, 1990, p. 168).

In these matters Wray is in good company, as a number of Post Keynesians, including Wells (1983), Arestis and Howell (1996) and Sheila Dow (1996), have investigated how liquidity preference operates in a world in which, increasingly, the public can choose between a wide variety of highly liquid, short-term interest bearing assets. As Wells puts it:

> For some long period of time, wealth owners have had the option of holding liquidity in the form of savings deposits, NOW accounts, money market funds, certificates of deposit, and numerous other short-term debt instruments. These instruments are normally no riskier than money and are virtually as liquid as cash. But they do have the advantage of yielding their owners an interest income (Wells, 1983, p. 526).

Echoing Wray's position, Wells's own answer to this question is relatively straightforward, and still leaves the door open for liquidity preference effects to exert some influence over the final outcome[79]:

> With a financial system in which money no longer is held in hoards, expectations work to determine the spread between the short rate and the bond rate of interest. The expectation of an increased need for liquidity, for example, would prompt a sell-off of bonds and stocks in favour of short-term debt instruments. Long rates would rise and short rates would fall (Wells, p. 533).

[79] See Palley (1994) for a formal portfolio-theoretic representation of liquidity preference in this money supply context.

In his concluding comments on this matter, Wray argues that Basil Moore's Horizontalist case against liquidity preference rests on a false, but implicit, assumption that some institution is prepared to create liquidity on demand and buy sufficient quantities of assets to peg their prices in response to fluctuations in desired liquidity ratios (see Wray, 1991 pp. 81-2).

2.3.3. Tobin on Liquidity Preference as Behaviour Towards Risk

At first sight, James Tobin's 1958 application of the Capital Asset-pricing Model (CAPM) to the phenomenon of liquidity preference, seems to provide a rigorous justification for non-transactions forms of money demand. Keynes, himself, constructed the speculative demand curve for money on the basis that, under conditions of uncertainty, investors would hold heterogenous and changing beliefs about the future movement in interest rates. While one group of investors would hold to the belief that interest rates were likely to rise in the future, another group of investors would believe that interest rates were likely to fall. The relative magnitude and wealth of each of these classes of investor would then determine the equilibrium outcome in bond markets.

While Tobin (1958, 1983) is sympathetic to the monetary arguments of *The General Theory,* he accepts the specific criticisms that were made by both Leontief (1947, pp. 238-9) and Fellner (1946, p. 149), some time after the publication of Keynes's (1936) *magnum opus*. According to Tobin (1958, p. 248) these scholars questioned the validity of liquidity preference theory as an underpinning for the speculative demand for money. The basis for their critique was straightforward: a strong preference for liquidity could only be sustained if a large number of agents persisted in their belief that interest rates would soon rise, to the great detriment of those primarily holding bonds and other securities. If one accepted that expectations were formulated rationally in the light of past experience, then eventually, it was argued, these inelastic expectations would have had to be modified substantially in the face of continual falsification.

Tobin (1983) claims that it was this dissatisfaction with *The General Theory*'s arguments that motivated his efforts to ground liquidity preference in the rational efforts of agents to diversify their portfolios. In his 1958 work, a simple finance market is posited in which agents can only choose between holding either cash or consols. His derivation of a downward sloping liquidity preference curve requires two assumptions: agents' expectations of future interest rates are less than unit elastic with respect to variations in the current interest rate; and agents have diverse interest expectations. He modifies the capital asset-pricing model by taking into account the prospect of capital gains and losses on the holdings of consols (Tobin, 1958, p 247). In this context, Tobin interprets Keynes to be arguing the following: if certain agents expect these perpetuities to pay an interest rate, r_e, which is presumed to be independent of the current market rate, r; then they would expect a capital gain or loss of :

$$g = \frac{r}{r_e} - 1$$

on their holdings of consols (Tobin, 1958, p. 245). For these investors there would be a critical rate of interest, $r_c = r_e /(1 + r_e)$, above which they would hold only consols, but below which they would hold only cash (Tobin, 1958, p. 245-6). Tobin shows that a diversity of views about the appropriate level for the interest rate r_e would yield the familiar downward sloping liquidity preference schedule (Tobin, 1958, pp. 246-7).

In developing an alternative model, Tobin assumes that the expected capital gain or loss on the consols can be represented by a probability distribution with mean of zero and variance of σ_g. Agents hold a portfolio consisting of a proportion A_1 of cash and A_2 of consols. Hence, the return on the portfolio, R is given by (Tobin, 1958, pp. 249-51):

$$R = A_2(r + g) 0 \leq A_2 \leq 1$$

with expected return

$$E(R) = \mu_R = A_2 r$$

and variance

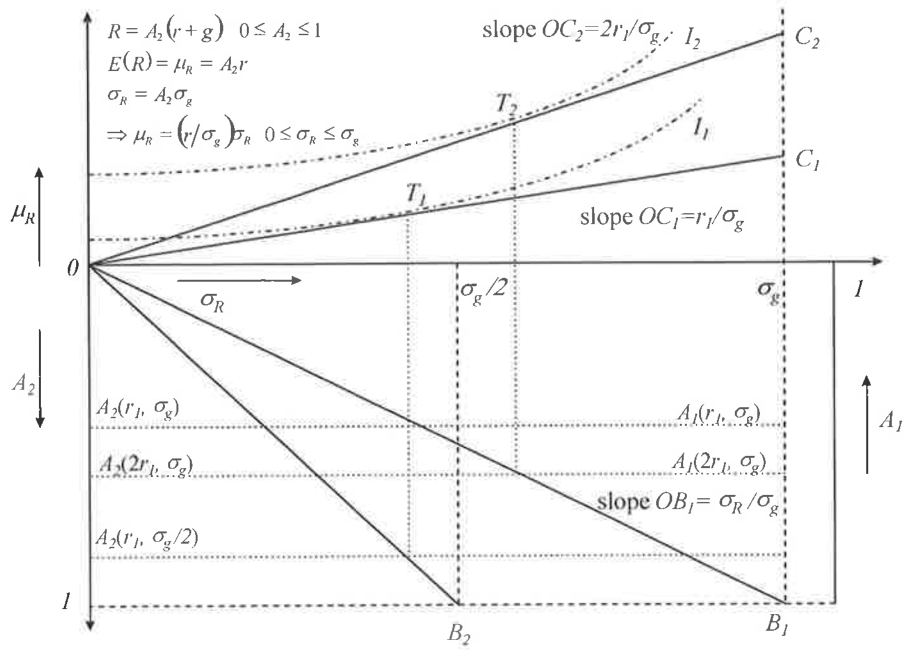
$$\sigma_R = A_2 \sigma_g$$

implying a trade-off between risk and return of

$$\mu_R = (r/\sigma_g)\sigma_R 0 \leq \sigma_R \leq \sigma_g$$

For an interest rate of r_1, this trade-off is represented by an opportunity locus OC_1, whereas for an interest rate of r_2 (where $r_2 = 2.\ r_1$), the locus is OC_2, as depicted in the upper half of the following diagram by the two lines drawn from the origin (Tobin, 1958, fig. 15.7, p. 259).

Fig. 17: TOBIN'S (1958) REPRESENTATION OF LIQUIDITY PREFERENCE



The OB lines, shown in the lower half, show portfolio risk as proportional to the share of total balances held in consols (A_2). Understandably, their slopes vary inversely with σ_g, the risk of capital gain or loss. A risk-averting mean-variance maximiser, with preferences shown by the indifference curves I_1, I_2 above, would choose the tangency points T_1 and T_2 when facing interest rates r_1 and r_2. It is easy to see that holdings of consols would increase with a doubling of interest rates from r_1 to r_2, and would increase even more if, instead, the variance of capital gain or loss were halved from σ_g to $\sigma_g/2$. Tobin (1958, pp. 261-5) establishes a separation theorem to confirm that similar results would hold when the model was extended to accommodate a wide variety of risky assets.

Tobin's reformulation of liquidity preference theory is both rigorous and skilful. In his response to criticism made after the publication of his paper, Tobin (1983) acknowledges the limitations of its asset-pricing framework. In particular, he concedes that mean-variance utility is a simplification of more general functional forms that might better take into account the possible influence of higher-order moments in the probability distribution governing prospective capital gains and losses. No doubt, his theoretical approach could also be extended by incorporating non-expected or recursive utility functions; thereby taking on some of the stochastic sophistication and complexity of modern-day continuous-time finance[80]. However, I believe that a more serious concern is that Tobin errs, first, by conflating the demand for money with the demand for a riskless asset (such as a zero-coupon bond, or more realistically, an artificially constructed zero-beta portfolio). Underlying this conflation is his contention that the only difference between the return on a riskless asset and the return on money is that the former offers a positive, but certain return, while the latter offers a zero return[81]. Therefore, in the standard two-period version of the CAPM, the riskless asset can act as a proxy for money and will be held

[80] Paul Mizen has constructed a model of money demand (1994) that incorporates Tobin's notion of liquidity preference as behaviour towards risk into what would otherwise be a conventional *buffer stock* model. His chosen quadratic objective function penalizes deviations from desired target holdings and captures the adjustment costs associated with moving towards the desired target. Standard Kalman filter techniques are used to estimate the model. Despite the sophistication of this model, uncertainty aversion is neglected.
[81] Of course, in an inflationary environment, investors may seek alternative havens for their savings if the liquidity premium on money cannot match the capital loss to inflation in the price of goods and services.

in positive amounts when fairly reasonable conditions are imposed over preferences and returns (Tobin, 1958, p. 245).

Second, Tobin's modeling of expectational dynamics is too simplistic. For example, the mere fact that interest rates remain above the level expected by a number of investors may cause them to reassess their expectations, leading to a narrowing in the diversity of null prices. Alternatively though, rising uncertainty may prevent any reassessment of expectations on the part of those whose views up until now have been repeatedly confounded by events. Again, the fact that an overvaluation of stock prices has not yet been corrected may, or may not, encourage more pessimistic investors to revise their opinions about the appropriate general level of stock prices. Indeed, as inflation in asset-prices continues beyond plausible levels, more investors may ultimately switch their allegiance away from the camp of those with more optimistic expectations into the camp of those with more pessimistic expectations. At such times, the speculative demand for money will rise and become a self-fulfilling prophecy as the majority of investors become increasingly pessimistic. Metaphorically, at least, this sort of dynamic is at play in the adaptive rational expectations dynamics models of Brock and Hommes (1997).

Tobin's model (1958) is based on the idea that money can still be held in positive amounts because it serves as a vehicle for portfolio diversification, even though the expected return on money balances (assumed to be equal in magnitude to the negative of the anticipated rate of inflation) is dominated by that of other assets. However, Nagatani notes that the CAPM model as a theory of demand for money has often been questioned because in practice, "…any number of near-monies are as safe and liquid as cash and yet have higher yields than cash" (Nagatani, 1981, p. 156)[82].

2.3.4. Transactions costs and Jones and Ostroy's notion of Liquidity as Flexibility

With some qualifications, I follow Vercelli (1991) in his contention that Jones and Ostroy's model of "liquidity as flexibility" comes fairly close to the position enunciated by Keynes (1936) in *The General Theory*. Keynes was less concerned with deriving the demand for money from notions of *time preference* (the latter typically defined as the desire on the part of an individual agent to reserve some portion of his or her income as a general instrument of command over future consumption), than he was with the *precise form*—liquid or illiquid—in which agents will choose to hold these reserves (Nagatani, 1981, pp. 100-2; Keynes, 1936, pp. 166-8; similarly, Paul Davidson, 1988, 1991, 1994a, 1994b, 1996).

Jones and Ostroy's work builds upon a neglected tradition in monetary theory, which they trace to seminal contributions by Marshak (1949), Hart (1942), Tintner (1942), and others (for an overview of their framework see the technical appendix). According to their reading, authors working within this tradition supposedly make an important distinction between risk and uncertainty: Jones and Ostroy (1984) argue that uncertainty is concerned with the prospects for learning from new information. With some justification, they relate this notion to the burgeoning

[82] Nagatani cites Tsiang's (1972) argument that, empirically, the slope of the efficient frontier in financial markets is typically much steeper than can be obtained from a reasonable class of utility functions. This observation can be linked to recent debates on the inability of the consumption-based capital asset-pricing model to explain either the equity-price premium and/or the low return on risk-free bonds. However, at this stage, a deeper analysis of this topic would distract us from our investigation of factors determining the preference for liquidity. In the next chapter I

literature on "postponement of choice" in which individual agents are seen to defer investment decisions in the face of concerns about the irreversibility of investment and the influence of asset liquidation costs over portfolio choice.

Jones and Ostroy (1984) consider a choice between shorter-term investments, which leave future options open, and longer-term investments which foreclose on those options. A basic consideration in the choice between the short-term and long-term investment, one that is predicated on the fact that they can be distinguished by differences in the probability distribution of their payoffs, is the recognition:

> …that beliefs about the risks governing these payoffs may change. Current doubts may be partially resolved in the near future. This prospect decreases the attractiveness of the longer term commitment, in that one is able to respond less fully to new information, and even if it does not directly affect the risks associated with shorter term choices, enhances their appeal (Jones and Ostroy, 1984, p. 13).

In interpreting their model, the authors observe that money is never held if its opportunity cost overshadows either the alternative's switching costs or the maximum second period yield at stake. Reducing the yield r^* on alternative assets enlarges the set of beliefs for which money is the optimal first period asset; raising switching costs c has a similar effect; increasing prior uncertainty about which asset offers the highest yield always expands the domain over which money is the preferred asset, increasing the information content of y; never causes a switch out

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return to examine this issue motivating the subsequent discussion of robust control through a consideration of Weil's (1989) analysis of the equity-premium puzzle.

of this domain, and finally, the greater is the information expected in the near term, the higher is the yield required for less liquid assets to be held.

Jones and Ostroy argue that their model was constructed to distinguish its motive for holding money as much as possible from the motives embodied in existing theories of money demand. They observe that risk was essential to the outcome:

> but not risk averse behaviour; differential asset liquidation costs were required, but not compulsory liquidations (to meet, for example, unforseen "cash requirements"); yields on alternative assets were uncertain, but money was dominated, in terms of both immediate (period one) and future (period two) yields, by all other assets—none yielded less than 0 in each period. Liquidity has value because it permits profitable exploitation of information not yet received (Jones and Ostroy, 1984, p. 24).

In reference to Tobin's (1958) approach to liquidity preference as *behaviour towards risk*, Jones and Ostroy simply note that "…[f]lexibility is not an issue since choice is confined to assets free of switching costs" (1984, p. 24). Their concluding comments are that:

> [w]e thus have a sixty year tradition of isolated recognition that flexibility choice is a component of a wide range of economic decisions. The difficulty of defining flexibility in such a way as to have universal application, and the difficulty of obtaining formal results without model-specific qualifications, may account for its limited role in conventional microeconomic theory. From a macroeconomic perspective, however, the tantalising prospect of portraying the connection between business cycles and public confidence as a relation between flexibility induced shifts in asset demands (away from capital investment and towards more liquid assets, especially money) and uncertainty is too compelling to be ignored (1984, pp. 26-7).

2.3.5. Magill and Quinzii on the Significance of Nominal, Non-Indexed Financial Contracts

Despite their concern with portfolio allocation and for the pricing of financial assets, modern general equilibrium asset-pricing models do not permit money to play a meaningful role. For example, Breeden's 1986 paper deals with a multiple good world that, essentially, comprises a variety of different types of 'corn': the corn withdrawn from consumption must automatically equate to the seed-corn available for investment. Similarly, the short and long term interest rates underlying the term structure, are specifically corn rather than money-rates of return. At certain corn-rates of interest, corn is made available to borrowers and lenders who can use it either as food-corn for eating here and now or as seed-corn for planting and later harvesting to meet future consumption needs. While the business cycle and the asset-pricing process alike might conceivably be governed by such productivity-factors such as weather patterns, crop disease, and even innovations in farming technology, it is never envisaged that they could be influenced by some transition between optimistic and pessimistic expectations: that is, by uncertainty-induced patterns of expansion and contraction in investment and aggregate demand which, in a real world of monetary production, would find clear expression in fields lying fallow, agricultural labourers left unemployed, and production and welfare violently truncated.

Similarly, Cox, Ingersoll and Ross (1985b) are willing to acknowledge the same limitations in regard to their model of financial market equilbrium:

> The model presented here deals with a real economy in which money would serve no purpose. To provide a valid role for money, we would have to introduce additional features which would lead

> far afield of our original intent However, for a world in which changes in the money supply have no real effects, we can introduce some aspects of money and inflation in an artificial way by imagining that one of the state variables represents a price level and that some contracts have payoffs whose real value depends on this price level. That is, they are specified in nominal terms. None of this requires any change in *The General Theory* (Cox, Ingersoll and Ross, 1985b, p. 104)[83].

In fact, money can only be introduced into the conventional stable of neoclassical financial models (including text-book IS-LM models and those building on work by Merton, 1973a; Breeden, 1979; Lucas, 1978; Rubinstein, 1976; or Cox, Ingersoll and Ross, 1985a), through fairly artificial means - specifically, either through incorporating money as a 'good' directly into the utility function or, following Clower's original lead (1967), through adhering to the quantity theory of money and imposing some sort of cash-in-advance constraint over transactions[84].

In the "money-in-the-utility-function" approach, money takes on the attributes of any other good serving as a source of utility, and in a general equilibrium context it can also function as the specific numeraire. Magill and Quinzii (1998) follow Hahn (1965) in questioning the validity of this approach, because models of this nature,

> ...in which preferences for real money balances satiate at a finite level, always have a non-monetary equilibrium in which the price of money is zero: seeking to avoid this difficulty, by assuming that

[83] After making one of the state variables a price level p_t, determining the payoff, $1/p_T$, at time T of a nominal unit discount bond, Cox, Ingersoll and Ross demonstrate that their version of the fundamental valuation equation (equation 6) and also their equation for the spot rate (equation 3) "...have exactly the same form when all variables are expressed in nominal terms as when all variables are expressed in real terms" (Cox, Ingersoll and Ross 1985b, see equation 57).

[84] See Sargent (1993) for examples of monetary equilibrium in either the currency-in-utility-function (Chapter 4) or the cash-in-advance (Chapter 5) class of model.

> agents' preferences for real money balances are always strictly monotonic, amounts to assuming that fiat money has intrinsic value (Magill and Quinzii, 1998, p. 442).

Cash-in-advance approaches effectively evade the problem of determining a non-transactions based demand for money. By assumption, agents must use money for transactions and no other motive is contemplated. Even models based on search theory view money as a temporary abode for purchasing power until agents can assess the relative merits of investing in non-monetary assets. Nevertheless, some valuable insights can be obtained even from otherwise simplistic transactions-based models when market failure is allowed to obtain and equilibrium is indeterminate. In a simple two-period general equilibrium model, Magill and Quinzii (1998, Chapter 7) establish that monetary policy can be non-neutral when non-indexed nominal contracts, incomplete markets, heterogeneity of agents, and unpredictable policy interventions are jointly present. They observe that their fairly elementary model has the potential for extension to incorporate nominal debt contracts and wage contracts. In particular, this would allow for a distinction to be clearly drawn between nominal and real rates of interest. In true Keynesian fashion, further modifications could then be made to permit the operation of Fisher effects.

2.4. Conclusion

In brief, this chapter has identified a number of features associated with both conventional and heterodox monetary theory, that are difficult to encompass within modern finance theory, irrespective of whether these financial models are or are not inserted into a complete general equilibrium framework. Included are problems of coordination across interrelated markets for money, capital, labour and consumption goods; the ubiquitous presence of nominal, non-indexed contracts that are, among other things, associated with debt-deflationary effect; and notions of liquidity preference that are predicated on the option value of intertemporal flexibility. These issues will be re-examined in more detail in the following two chapters.

To build a bridge between Post Keynesian monetary theory and modern finance theory, I have argued that we must examine alternative approaches to the modeling of investment, borrowing and portfolio choice that account for uncertainty or ambiguity. Although the multiple-priors and sub-additive probability perspectives into uncertainty in decision-making may not fully embody all the concerns raised by Keynes in the *Treatise on Probability*, I show that they at least generate liquidity premia, can explain animal spirits effects, and provide insight into a range of puzzling anomalies observed in financial markets, including the equity premium puzzle, return predictability, and excess volatility.

Unfortunately, the mathematical prerequisites for understanding risk-sensitive and robust control theory are demanding. The most sophisticated version appearing in the literature represents uncertainty through the imposition of a relative entropy constraint. This constraint takes the form of an energy bound over perturbations to the transition probabilities of a hidden-Markov reference model, and captures the observer's inability to distinguish between alternative specifications for the stochastic process that drives asset returns. In chapter four and in the accompanying technical appendix, the requisite lemmas and theoretical foundations are outlined, with references to where detailed proofs are to be found. There, I argue that a relative entropy constraint can be interpreted as a generalisation of the (Euclidean) norm bounds that are applied in standard applications of robust control, and govern the magnitude of external perturbations, observation error and model imperfection. In fact, I suggest that Hansen and Sargent (1995, 1999) focus on the first of these sources to the neglect of the latter two.

In the Andersen, Hansen and Sargent paper (1999), the uncertainty premia can be viewed as an expression of the investors' need to protect themselves against the worst-possible constellation of perturbations that could arise under the entropy constraint. Their most general results are attained for a continuous-time, infinite-horizon, dynamic programming version of the model, applied to a value function that belongs to a particular class of recursive utility functions (Duffie and Epstein, 1992). However, this continuous-time version is favored for its elegance and mathematical convenience rather than for its rigour. As they explain, analogous results can be obtained in a discrete-time setting, by utilising the equivalent discrete-time versions of recursive utility presented in the related paper by Epstein and Zin (1989).

One of the advantages of Epstein and Zin's (1989) recursive approach is that it sidesteps any possibility of dynamic inconsistency: a thorny problem that can arise in applications of game theory to equilibrium outcomes in financial markets. In addition, the family of recursive utility functions developed by Duffie and Epstein (1992) and Epstein and Zin, (1989) is sufficiently rich to accommodate various forms of non-expected utility theory including Kreps and Porteus's (1978) recursive version of dynamic choice theory, Chew's (1989, 1990) rank-dependent version of weighted utility theory, based on the "in-betweenness" principle, and Dekel's (1986) extension of weighted utility theory. As I explain in the following chapter, motivated by experimental evidence for systematic violations of the expected utility hypothesis, these latter two forms of utility theory are derived by weakening Von Neumann and Morgenstern's Independence axiom.

The use of recursive utility functions, under a relative-entropy constraint, represents one way of accommodating a "multiple-priors" approach to decision-making under uncertainty. Other congruent approaches include one based on a rejection of Savage's completeness axiom (as in Bewley, 1986) and Epstein and Wang's (1994) infinite horizon extension of Gilboa and Schmeidler's atemporal multiple-priors model. This extension is based on the construction of a probability kernel correspondence that, in a heuristic sense, constitutes the *set* of probability measures that represent beliefs about next period's state[85]. An alternative mechanism for accommodating uncertainty aversion would be to use sub-additive probabilities (see Dow and Werlang, 1992). Both the sub-additive and multiple-priors approaches can be grounded mathematically in the theory of capacities: a generalisation of probability theory based on the use of probability correspondences rather than probability singletons. Although Andersen et al (1999) do not discuss this issue explicitly, their control-based approach could be axiomatically grounded in such a manner.

In this thesis, I make the case that there is much to be learnt from the economic applications of risk-sensitive and robust control techniques, to the extent that they afford an opportunity to extend traditional approaches to decision-making under risk to incorporate uncertainty aversion. However, I also wish to argue that, in a macroeconomic context, the representative agent approach (i.e. combining consumption-savings-production-investment and borrowing/lending decisions within a single economic agent, or even a heterogeneous collection of such agents) suffers from profound limitations. This is because it cannot accommodate or allow for movements in the point of effective demand. The notion of "heterogeneity" at play, here, is considerably broader than the related notion propounded in the extensive literature on asset market equilbrium with heterogenous agents. The latter notion is confined to heterogeneity in respective information sets or opinions about non-fundamental states that are conjectured to exercise an influence over asset-prices despite the lack of any influence over fundamental determinants of consumption endowments and returns.

In a barter economy inhabited solely by a collection of individual producing-investing-borrowing-or-lending-and-consuming agents, decisions to defer the consumption of "corn" would automatically be reflected in decisions to either invest for oneself or to lend to other agents for their own consumption and investment. In aggregate, any saved corn would automatically be invested in the form of uneaten seed. Outside the secure confines of this

[85] In the next chapter it will be seen that this construction matches what Vercelli has called the "epistemic repertoire" of sufficiently reliable probabilities.

agricultural utopia, however, there could be no guarantee that food *uneaten*, would inevitably be transformed into *investment* capital.

Even in a pure-exchange asset-pricing context, it is questionable as to whether exogenously given movements in the point of effective demand, and the effect they would exercise over dividend income and prospective capital gains, could be adequately captured by a Gaussian, hidden-Markov, GARCH, fractional ARIMA or any other conventional time-series representation of asset returns, alone or in combination. This argument must be defended and that is the intention of what is to follow in Chapter Three.

CHAPTER THREE — ELEMENTS OF DISCRETE-TIME FINANCE THEORY

3.1. Introduction

In this chapter I provide an overview of modern and more conventional approaches to discrete-time finance theory. To set the scene, I first review Pliska's approach to discrete-time, multi-period finance theory and martingale methods. In particular, I describe his construction of the nominal and discounted gain and value process and how dividend payments are incorporated into the model. I also review the key relationship that holds between non-arbitrage and the derivation of risk-neutral probability measures (for the two period problem) and for martingale measures (over multiple periods). This key relationship enables Pliska to derive the fundamental pricing equation for securities. I then proceed to outline Pliska's dynamic programming and martingale-based solutions to the optimal portfolio and optimal consumption-investment planning problems. I briefly touch on ways of generalizing these problems to account for unconsumed terminal wealth, various constraints over trading, and incomplete markets. The papers by Hansen, Sargent and Tallarini (1999), Andersen, Hansen and Sargent (1999), and Tornell (2000) deliberately set aside such complications so that they can focus exclusively on the implications of probabilistic uncertainty rather than constraints over trading and the inability to completely diversify risk.

Modern approaches to finance theory based on the stochastic discount factors and martingales seem to be far removed from traditional perspectives on asset-pricing. Therefore, in section 3.3 I briefly examine the relationship between asset-pricing models with stochastic discount factors and more conventional mean-variance and factor-based approaches to asset-pricing.

In section 3.4. I examine Epstein and Zin's (1989) discrete-time recursive representation of utility theory in financial decision-making focusing, especially on their derivation of the relevant Euler equations and asset-pricing equations. Epstein and Zin (1989) derive various types of aggregator function for use in dynamic programming. As mentioned in the first chapter of the thesis, these aggregator functions enable the researcher to advance beyond the axioms of expected utility theory to encompass preference structures that account for the Allais paradox. Epstein and Zin's approach to asset-pricing differs slightly from those explicated in the three papers that I have critiqued in the thesis.

In the following section, 3.5. I examine four recent applications of uncertainty aversion to asset-pricing and savings and investment decisions. This section of the chapter leads directly to a review of Epstein and Wang's more complex and technical work on asset-pricing under 'Knightian' uncertainty.

3.2. The Optimal Portfolio and Consumption-Investment Problems in Discrete-time Finance Theory

3.2.1. A Model of a Multiperiod Securities Market

Because that I have chosen to approach stochastic and robust control theory from a discrete-time perspectives, Pliska's (1997) discrete-time representation of a multiperiod securities market offers a congruent and comprehensive foundation for financial modeling. Pliska (pp. 72-3) specifies the following elements as data for the basic multi-period model:

$T + 1$ trading dates: $t = 0,1,\ldots,T$.

A finite sample space Ω with $K < \infty$ elements : $\Omega = \{\omega_1,\omega_2,\ldots,\omega_K\}$.

A probability measure P on Ω with $P(\omega) > 0$ for all $\omega\in\Omega$.

A filtration $\mathbf{F} = \{F_t;\ t = 0,1,\ldots,T\}$ which is a submodel describing how information about security prices is revealed to investors.

A bank account process $B = \{B_t;\ t = 0,1,\ldots,T\}$, where B is a non-decreasing stochastic process with $B_0 = 1$ and $B_t(\omega) > 0$ for all t and ω, which represents the t value of a savings account when \$1 is deposited at time 0. Therefore the quantity $r_t = (B_t - B_{t-1})/\ B_t \geq 0$, $t = 1,\ldots,T$ can be thought of as an interest rate pertaining to the time interval $(t - 1, t)$.

N risky security processes $S_n = \{S_n(t)\text{: } t = 0,1,\ldots,T\}$ where S_n is a non-negative stochastic process for each $n = 1,2,\ldots,N$ representing the time t price of the risky security n. This stochastic process is a real-valued function $S_n(\omega, t)$ of both t and ω such that for each fixed $\omega\in\Omega$, the function $t \rightarrow S_n(\omega, t)$ is called the *sample path* and for each fixed t the function $\omega \rightarrow S_n(\omega, t)$ is a random variable.

The filtration **F** represents the evolution of the information process as a random sequence $\{A_t\}$ of subsets of Ω, where $A_0 = \Omega$, $= \{\omega\}$ for some $\omega\in \Omega$ and $A_0 \supseteq A_1 \supseteq,\ldots,\supseteq A_{T-1} \supseteq A_T$. The collection $\{A_t\}$ of all possible time t subsets forms a partition of Ω, denoted P_t, which has the property that each $A \in P_t$ is equal to the union of some elements in P_{t+1} for every $t < T$. The sequence of partitions $\{P_t\}$ is uniquely constructed from the collection of possible information sequences $\{A_t\}$. Conversely, given a sequence of partitions $\{P_t\}$, there is a unique, corresponding collection of possible information sequences $\{A_t\}$. This information structure can either be described as a tree using a network diagram or diagrammatically in the form of a finer and finer partitioning of a space containing each of the final period end states $\{\omega_1,\omega_2,\ldots,\omega_K\}$. Alternatively, it can be specified in terms of an algebra on Ω. A collection F of subsets of Ω is called an *algebra* on Ω if:

1) $\Omega \in \boldsymbol{F}$

2) $F \in \boldsymbol{F} \Rightarrow F^c = \Omega \backslash F \in \boldsymbol{F}$

3) F and $G \in \boldsymbol{F} \Rightarrow F \cup G \in \boldsymbol{F}$.

Corresponding to the algebra on Ω, denoted $\boldsymbol{F}_t$ is a partition of Ω, which is unique, that can be viewed as a collection $\{F_n\}$ of subsets F_n such that (p. 75):

a) each $F_n \in \boldsymbol{F}$
b) the subsets $\{F_n\}$ are disjoint
c) the union of the subsets $\{F_n\}$ equals Ω.

Thus, given a partition, a variety of elementary set operations (taking complements, forming unions and intersections etc.) can be performed to generate the largest possible number of subsets resulting in an algebra which is unique. A random variable X is set to be *measurable* with respect to the algebra $\boldsymbol{F}$ if the function $\omega \rightarrow W(\omega)$ is constant on any subset in the partition corresponding to $\boldsymbol{F}$. Equivalently, for any real number x, the subset $\{\omega \in \Omega\colon X(\omega) = x\}$ is an element of the algebra $\boldsymbol{F}$. A stochastic process $S_n = \{S_n(t)\colon t = 0,1,\ldots,T\}$ is said to be *adapted* to the filtration $\mathbf{F} = \{F_t;\ t = 0,1,\ldots,T\}$ if the random variable $S_n(t)$ is measurable with respect to $\boldsymbol{F}_t$ for every $t = 0,1,\ldots,T$. Pliska assumes (p. 77) that both the bank account B and the prices of each of the n risky securities are adapted stochastic processes. Given that investors know at time t that the true state ω is contained in the time t partition P_t, the time t price $S_n(t)$ of each security must be constant on this subset, so that investors can determine what the time t value of each security must be.

3.2.2. The Nominal and Discounted Value and Gain Processes

A *trading strategy* $H = (H_0, H_1,\ldots, H_N)$ is a vector of stochastic processes $H_n = \{H_n(t)\!: t = 1,2,\ldots,T\}$, $n = 0,1,\ldots,N$. $H_n(0)$ is not specified because $H_0(t)\ B_{t-1}$ equals the amount of money invested in the bank account at time $t-1$. If $H_n(t)$ is negative this either corresponds to the action of borrowing money from the bank (when $n = 0$) or selling short security n (when $n \geq 1$). In other words, a trading strategy specifies an investor's position in each security at each point in time and in every state of the world. A stochastic process H_n is said to be *predictable* with respect to the filtration $\mathbf{F}$ if each random variable $H_n(t)$ is measurable with respect to $\boldsymbol{F}_{t-1}$ for all $t = 1,2,\ldots,T$. Since $\boldsymbol{F}_{t-1} \subseteq \boldsymbol{F}_t$, this means that all predictable stochastic processes are *adapted*. Thus, it is posited that the investor establishes the trading position $H_n(t)$ taking into account information no later than that available within the time $t-1$ partition P_{t-1}.

The *value process* $V = \{\ V_t;\ t = 0,1,\ldots,T\ \}$ is a stochastic process defined by (Pliska, 1997, p. 81):

$$V_t = \begin{cases} H_0(1)B_0 + \sum_{n=1}^{N} H_n(1)S_n(0), & t = 0 \\ H_0(t)B_t + \sum_{n=1}^{N} H_n(t)S_n(t), & t \geq 1 \end{cases}$$

Hence, V_0 is the initial value of the portfolio and, for $t \geq 1$, V_t is the time-t value of the portfolio before any transactions are made at that same time. V_t is an adapted process because it is a function of stochastic processes that are adapted.

The *gains process* G_t, representing the cumulative gain or loss through time t on the given portfolio, is defined by the stochastic integral (being a weighted sum of the values of a stochastic process $H_n(t)$ where the weights are given by one-period changes in another stochastic process) of the trading strategy with respect to the price process:

1. $$G_t \equiv \sum_{u=1}^{t} H_0(u)\Delta B_u + \sum_{n=1}^{N}\sum_{u=1}^{t} H_n(u)\Delta S_n(u), \quad t \geq 1,$$

where $\Delta X_n = X_n(t)\text{-}X_n(t - 1)$ and $G = \{ G_t, t = 1,\ldots,T\}$ is, by definition, an adapted stochastic process.

A trading strategy is *self-financing* if (p. 82):

2. $$V_t = H_0(t+1)B_t + \sum_{n=1}^{N} H_n(t+1)S_n(t), \quad t = 1,\ldots,T-1$$

which, by simple manipulation, can be shown to imply that $V_t = V_0 + G_t$, $t = 1,2,\ldots,T$. Thus, a self-financing strategy implies that the time t values of the portfolio just before and just after any time t transactions are equal.

The *discounted price process* $S_n^* = \{S_n^*(t); t = 0,1,...,T\}$ is defined by (p. 83):

3. $S_n^*(t) \equiv S_n(t)/B_t, \quad t = 0,1,...,T; \; n = 1,2,...,N$

while the *discounted value* and *discounted gain processes* are defined by:

$$V_t^* \equiv \begin{cases} H_0(1) + \sum_{n=1}^{N} H_n(1) S_n^*(0), & t = 0 \\ H_0(t) + \sum_{n=1}^{N} H_n(t) S_n^*(t), & t = 1,...,T \end{cases},$$

hence, $V^*_t = V_t / B_t$, and

$$G_t^* \equiv \sum_{n=1}^{N} \sum_{u=1}^{t} H_n(u) \Delta S_n^*(u), \quad t = 1,...,T.$$

A trading strategy H is self-financing if and only if: $V^*_t = V^*_0 + G^*_t$.

The *return process* $R_n = \{ R_n(t); t = 0,1,\ldots,T\}$ corresponding to the price process S_n is defined by (p. 84):

$$\Delta R_n(t) \equiv \begin{cases} \Delta S_n(t)/S_n(t-1), & S_n(t-1) > 0 \\ 0, & S_n(t-1) = 0 \end{cases}$$

which is equivalent to $\Delta S_n(t) = S_n(t-1)\, \Delta R_n(t), \quad t = 1,\ldots,T$. It can be shown that:

$$\Delta S_n^*(t) = S_n^*(t-1)\left[\frac{\Delta R_n(t) - \Delta R_0(t)}{1 + \Delta R_0(t)}\right], \text{ where } \Delta R_0(t) = r.$$

3.2.3. The Dividend Process

To accommodate dividend-paying securities a *dividend process* D_n must be specified for each security: $D_n = \{D_n(t): t = 0,\ldots,T\}$ for $n = 1,\ldots,N$, where $D_n(0) = 0$ and $\Delta D_n(t)$ represents the dividend per security unit paid at time t (Pliska, 1997, pp. 87-8). Thus $D_n(t)$ represents the cumulative dividend payments associated with one unit of the security, while $S_n(t)$ now represents the *ex-dividend* price of the security (i.e. after any time t dividend payment). The corresponding one-period return now becomes:

$$\Delta R_n(t) = \frac{\Delta S_n(t) + \Delta D_n(t)}{S_n(t-1)} = \frac{\Delta S_n^*(t) + \Delta D_n(t)/B_t}{S_n^*(t-1)}, \quad t = 1,.., T, \; n = 1,..., N$$

3.2.4. Absence of Arbitrage and Martingale Measures

In a single-period securities market model the absence of arbitrage opportunities implies the existence of a risk-neutral probability measure. In a multiperiod setting the equivalent concept is that of a martingale. Given a filtered probability space $\boldsymbol{F_t}$ together with an adapted stochastic process $Z = \{Z_t;\, t = 0,1,\ldots,T\}$, the process Z is said to be a *martingale* if $E[Z_{t+s} \mid \boldsymbol{F_t}] = Z_t$, for all $s,t \geq 0$. In a multiperiod securities market an *arbitrage opportunity* is some trading strategy H such that (Pliska, 1997, p. 92):

a) $V_0 = 0$,

b) $V_T \geq 0$,

c) E $V_T > 0$, and

d) H is self-financing.

In view of the fact that $V^*_t = V_t / B_t$, the self-financing strategy H is an arbitrage opportunity if and only if:

a) $V^*_0 = 0$,

b) $V^*_T \geq 0$,

c) $EV^*_T > 0$.

Alternatively, from the definition of a discounted gain process, H is an arbitrage opportunity if and only if $G^*_T \geq 0$, and $E.G^*_T > 0$.

A *risk-neutral probability measure* or *martingale measure* is a probability measure Q such that:

1) $Q(\omega) > 0$ for all $\omega \in \Omega$, and

2) The discounted price process is a martingale under Q for every $n = 1,2,\ldots,N$.

In the single-period case a probability measure Q on Ω is said to be a risk-neutral probability measure if:

1) $Q(\omega) > 0$ for all $\omega \in \Omega$, and

2) $E_Q[\Delta S^*_n] = 0$, $n = 1,2,\ldots,N$, which is equivalent to the requirement that $E_Q[S^*_n(1)] = S^*_n(0)$, $n = 1,2,\ldots,N$.

That is, the expected time t discounted price of each security is equal to its initial price. Thus, a risk-neutral probability measure is merely a linear pricing measure π, satisfying: $S^*_n(0) = \sum_{\omega} \pi(\omega) S^*_n(1)(\omega)$, $n = 1,\ldots,N$, one that gives a strictly positive mass to every $\omega \in \Omega$[86].

[86] Pliska (1997, pp, 6-7) uses the linear programming duality theory to establish that a linear pricing measure exists if and only if there are no dominant trading strategies (i.e. strategies satisfying $V_0 = 0$ and $V_1(\omega) > 0$ or $V_0 < 0$ and

There are no arbitrage opportunities if and only if there exists such a risk-neutral probability measure Q.

In the multiperiod case, a risk-neutral probability measure must satisfy:

$$E_Q\left[S_n^*(t+s)\middle|F_t\right] = E_Q\left[B_t S_n(t+s)/B_{t+s}\middle|F_t\right] = S_n^*(t), \quad t, s \geq 0$$

Lemma (Pliska, 1997, p. 94,):

There are no arbitrage opportunities if and only if there exists a martingale measure Q.

The equivalent risk-neutral measure lemma for a single period can be established with relative ease using either the Minkowski-Farkas lemma or separating hyperplane arguments (see Pliska, 1997, pp. 13-4 for an example of the latter). Pliska (p. 95) observes that this lemma can be proved through a multiperiod generalization of the separating hyperplane argument, but he prefers a more intuitive proof - one using the single period risk-neutral measures - he demonstrates the multiperiod result by linking each of the appropriate one-period conditional probabilities, that are compatible with risk-neutrality, into a chain and then multiplying them together in accordance with the information structure of the multiperiod model. He confirms that if the multiperiod model does not have any arbitrage opportunities, then none of the underlying single-period models has any arbitrage opportunities in the single-period sense. Moreover, the following properties of the martingale measure must also obtain (pp. 96-7):

$$E_Q\left[\Delta S_n^*(t+1)\middle|F_t\right] = 0 = E_Q\left[\Delta R_n^*(t+1)\right], \text{ for } n = 1,\ldots, N \text{ and } t < T$$

$V_1(\omega) \geq 0$ for all $\omega \in \Omega$. If there are no dominant trading strategies, then the law of one holds. The converse, however, is not necessarily true.

For dividend paying securities: $\Delta R_n^*(t+1) = [\Delta S_n^*(t+1) + \Delta D_n(t+1)/B_{t+1}]/S_n^*(t)$. By substituting this equation into the second of the above equalities and using the law of interated expectations, Pliska (p. 98) arrives at the discrete-time version of the *fundamental pricing equation* for dividend-paying securities, namely:

$$S_n^*(t) = E_Q[\Delta D_n(t+1)/B_{t+1} + ... + \Delta D_n(t+s)/B_{t+s} + S_n^*(t+s)|F_t].$$

Thus, the time t discounted price of a risky security equals the conditional expected value of the discounted dividend payments up through time $t + s$ plus the time $t+ s$ discounted price, where the conditional expectation is taken with respect to the risk-neutral probability measure.

3.2.5. THE BASIC OPTIMAL PORTFOLIO PROBLEM

Pliska's approach to the basic optimal portfolio problem (1997, p. 149) is structured around the maximisation of end-period or time T wealth when there is no prior consumption. He specifies a *utility function u*: $\Re\times\Omega\to\Re$, where $u(w,\omega)$ represents the utility of wealth w at time T when $\omega\in\Omega$ is the state of the world. He assumes that $w \to u(w,\omega)$ is differentiable, concave, and strictly increasing for each $\omega\in\Omega$. He specifies an initial wealth holding v assuming that investors can choose any self-financing strategy consistent with this initial level of wealth. The investor is assumed to maximise the expected utility of terminal wealth given by:

$$Eu(V_T) = \sum P(\omega)u(V_T(\omega),\omega)$$

by solving the following portfolio problem:

$$\text{maximise}\, Eu(V_T)\, \text{subject to}\, V_0 = v,\ H \in \mathbf{H}$$

where **H** is the set of all self-financing strategies. Because $V_T = B_T V^*_T$, and $V^*_T = V^*_0 + G^*_T$ it follows that this is equivalent to the problem: maximise $Eu(B_T\{v + G^*_T\})$ subject to $(H_1,\ldots,H_N) \in \mathbf{H}_p$

where $\mathbf{H}p$ denotes the set of all predictable processes that take values in $\Re^n$. Given an optimal set of trading strategies $(\hat{H}_1,\ldots,\hat{H}_N)$, H_0 can then be chosen so that V_0 is equal to v and $(\hat{H}_0,\hat{H}_1,\ldots,\hat{H}_N)$ is self-financing.

This problem can be solved using standard calculus and optimisation theory or dynamic programming. However, for complex problems with many securities and states, Pliska (pp. 156-7) favors the more tractable martingale approach. This method is based on a two step procedure utilising the risk-neutral probability measure Q. Pliska (p. 152) shows that if (H,V) is a solution to the basic portfolio problem, then a risk-neutral probability measure exists defined by:

$Q(\omega) = \dfrac{P(\omega)B_T u'(V_T(\omega),\omega)}{E[B_T u'(V_T)]}$, $\omega \in \Omega$, where u' denotes the partial derivative of u with respect to the first argument. Pliska derives this expression from the first order necessary conditions corresponding to an arbitrary security n, time t and event A in partition P_{t-1}, corresponding to F_{t-1}. These conditions obtain for $H_n(t)1_A$, the investor's position in security n that is carried forward from time $t-1$, when event A occurs (note that $1_A(\omega)$ is the indicator function that takes the value of 1 if $\omega \in A$, but equals zero if $\omega \notin A$). The relevant first order condition is:

$$\sum_{\omega \in A} P(\omega)u'(B_T(\omega)\{v + G^*_T(\omega),\omega\})B_T(\omega)\Delta S^*_n(t,\omega) = 0$$

which is true for all $A \in P_{t-1}$, so that $E\left[u'\left(B_T\left\{v + G_T^*\right\}\right)B_T\Delta S_n^*(t)\middle|F_{t-1}\right] = 0$. Hence, $E_Q\left[\Delta S_n^*(t)\middle|F_{t-1}\right] = 0$, if Q is defined in accordance with the above expression and use is made of the fact that $V_T = B_T V^*_T = B_T \{v + G^*_T\}$

In the first step the set of *attainable wealths* $\mathbf{W}_v = \{w \in \Re^k$: $W = V_T$ for some self-financing H with $V_0 = v\}$, is determined. This represents the set of all time T contingent claims that can be generated by a self-financing strategy starting with initial wealth v. In the second step the following subproblem is solved:

maximise $Eu(W)$ subject to $W \in \mathbf{W}_v$

Having obtained the optimal solution, say $\hat{W}$, the trading strategy H that generates this optimal solution can be computed as a replicating contingent claim. The second step can be solved using Lagrange multiplier techniques. Pliska introduces the Lagrange multiplier λ into the objective function as follows:

maximise $Eu(W) - \lambda E_Q[W/B_T]$.

Next he substitutes for the state price vector or state price density:

$$L(\omega) \equiv \frac{Q(\omega)}{P(\omega)},$$

which can be derived by applying the discrete form of Girsanov's theorem. Thus, the objective function can be rewritten as:

$$E[u(W) - \lambda\, LW/B_T] = \sum_{\omega\in\Omega} P(\omega)[u(W(\omega)) - \lambda\, L(\omega)W(\omega)/B_T(\omega)].$$

The first order conditions, derived by maximising this expression, comprise a set of equations, one for each $\omega \in \Omega$, namely: $u'(W(\omega)) = \lambda L(\omega)/B_T(\omega), \quad \forall \omega \in \Omega$. Pliska (p. 157) now makes the assumption that the utility function u: $\Re \rightarrow \Re$ is solely a function of wealth and is independent of the state $\omega \in \Omega$. This assumption allows him to solve for W:

$$W(\omega) = I(\lambda L(\omega)/B_T(\omega)), \quad \forall \omega \in \Omega$$

where I is the inverse function, normally decreasing with range $(0, \infty)$, corresponding to u'. The correct value of λ can now be found by substituting this expression for W into the equation $v = E_Q[W/B_T] = E_Q[I(\lambda L/B_T)/B_T]$.

3.2.6. OPTIMAL CONSUMPTION AND INVESTMENT AND DYNAMIC PROGRAMMING

Following Pliska (p. 162), a consumption process $C = \{C_t\,; t = 0,\ldots,T\}$ is a non-negative, adapted stochastic process with C_t representing the amount of funds consumed by the investor at time t. A *consumption-investment plan* consists of a pair (C,H), where C is a consumption process and H is a trading strategy. The investor chooses a self-financing consumption-investment plan that maximises expected utility over the T periods. Let V_t represent the value of the investor's portfolio before *both* time t transactions and time t consumption and let $V_t = v$ denote initial wealth. Thus, (C,H) is self-financing if (p.163):

$$V_t = C_t + H_0(t+1)B_t + \sum_{n=1}^{N} H_n(t+1)S_n(t), \qquad t = 0,\ldots,T-1$$

Given initial wealth v, the self-financing consumption-investment plan (C,H) is said to be *admissible* if $C_T \geq V_T$. Since C is a non-negative process, this implies that $V_T \geq 0$. The investor's consumption-investment problem is (p. 163):

$$\text{maximise } E\left[\sum_{t=0}^{T}\alpha^t u(C_t)\right] \text{ subject to}: v = \text{initial wealth}, (C, H) \text{ is admissible.}$$

This problem can be expressed in a recursive form suitable for dynamic programming by starting at time $T-1$ with wealth w so that the single period problem becomes (p. 164):

$$\text{maximise } u(C_{T-1}) + E\left[\alpha u_T(W) \middle| F_{T-1}\right], \text{ subject to}$$

$$w = C_{T-1} + H_0(T)B_{T-1} + \sum_{n=1}^{N} H_n(T)S_n(T-1)$$

$$W = H_0(T)B_T + \sum_{n=1}^{N} H_n(T)S_n(T)$$

$$H_n(T) \in F_{T-1}, \text{ for } n = 1,\dots,N \text{ and } C_{T-1} \in F_{T-1}$$

The first of these constraints can be solved for $H_0(T)$ and substituted into the second to derive an expression for wealth W that can be substituted back into the objective function. Proceeding in this fashion, and generalising to the t-th time period, Pliska derives the following dynamic programming functional equation:

$$u_{t-1}(w) = \max\left\{u(C_{t-1}) + \alpha E\left(u_t\left(\left(w - C_{t-1}\right)B_t / B_{t-1} + B_t \sum_{n=1}^{N} H_n(t)\Delta S_n^*(t)\right)\middle| F_{t-1}\right)\right\}$$

where the maximum is taken over all $H_n(t) \in F_{t-1}$ for $n = 1,\dots,N$ and $C_{t-1} \in F_{t-1}$. The technical assumption that $u(w) = -\infty$ for all $w < 0$ ensures that the optimal solution to this problem satisfies $C_{T-1} \geq 0$ and $W \geq 0$. The value function $u_0(v)$ gives the optimal objective value for the original problem and the maximising values for C_{t-1} and $H_n(t)$ will determine the optimal consumption-investment plan. The value for H_0 can be determined from the self-financing equation.

3.2.7. OPTIMAL CONSUMPTION AND INVESTMENT AND MARTINGALE METHODS

The risk-neutral computational approach is applied in three steps. First, the set of attainable consumption processes is characterised. A consumption process C is called *attainable* if the exists a trading strategy H such that the pair (C,H) is an admissible consumption-investment plan satisfying the condition $C_T = V_T$ for an implicitly specified initial wealth v. From the definition of a self-financing consumption-investment plan (C,H) Pliska (pp. 168-9) establishes, by induction, that given an initial wealth $v \geq 0$, a consumption process C, and a self-financing strategy H, the following relationship holds:

$$V_t/B_t = v + G_t^* - \sum_{u=0}^{t-1} C_u/B_u, \quad t = 1,\dots,T$$

If $M_t = v + G^*_t$, then M is a martingale under the risk-neutral probability measure Q satisfying $M_0 = v$. Then $M_t = V_t/B_t + C_0/B_0 + \dots + C_{t-1}/B_{t-1}$ and $v = E_Q[V_t/B_t + C_0/B_0 + \dots + C_{t-1}/B_{t-1}]$, $t = 1,\dots,T$. If, in addition, $C_T = V_T$ then $v = E_Q[C_0/B_0 + \dots + C_T/B_T]$. Pliska demonstrates that this condition is not only a necessary condition for the consumption process to be attainable but also a sufficient condition[87]. Moreover, if V_0 is the value of the portfolio which replicates C then $V_0 \geq 0$. This result enables him to restate the optimisation problem as follows (p. 170):

[87] Sufficiency follows from the fact that $B_T\,[C_0/B_0 + \dots + C_T/B_T]$ is an attainable contingent claim representing a sequential combination of contingent claims, the t-th claim in the sequence being the receipt of C_t dollars at time t, which are then deposited and held in the bank account until time T. Thus, there are T self-financing strategies $H^1,\dots, H^T$ that replicate the T contingent claims $C_1,\dots, C_T$, respectively. Taking $H \equiv H^1,\dots, H^T$, it follows that H is a self-financing trading strategy such that (C,H) is admissible with $C_T = V_T$. Pliska (pp. 169-170) proves the non-negativity condition that follows by taking expectations of the replicating strategy under the martingale measure.

$$\text{maximise } E\left[\sum_{t=0}^{T}\alpha^t u(C_t)\right]$$
$$\text{subject to } E_Q\left[C_0/B_0+\ldots+C_T/B_T=v\right]$$

where C is an adapted process. Lagrange multiplier techniques can be used once the constraint

$$E_Q\left[\sum_{t=0}^{T}C_t/B_t\right]=E\left[L\sum_{t=0}^{T}C_t/B_t\right]=E\left[\sum_{t=0}^{T}E\left[C_tL/B_t\left|F_t\right.\right]\right]=E\left[\sum_{t=0}^{T}C_tN_t\right]$$

has been expressed in a congruent form through a change in measure, namely:

where $N_t = \mathrm{E}[L \mid F_t]B_t$ and $L = Q/P$. Introducing a Lagrange multiplier λ, Pliska can now express the consumption-investment problem in the following form (p. 171):

$$\text{maximise } E\left[\sum_{t=0}^{T}\alpha^t u(C_t)-\lambda\sum_{t=0}^{T}C_tN_t\right]$$

The conditions that $\lim_{C\to0^+}u'(C)=\infty,\ \lim_{C\to\infty}u'(C)=0$ suffice to ensure that the optimal C are strictly positive. The first order necessary condition is: $\alpha^t u'(C_t)=\lambda N_t,\quad \forall\omega\in\Omega,\ t=0,\ldots,T$, which can be inverted to solve for C_t. The correct value for the multiplier is found by solving for:

$$E\left[\sum_{t=0}^{T}N_tI(\lambda N_t/\alpha^t)\right]=v$$

where, as before, $I(.)$ is the inverse of the marginal utility function.

3.2.8. Accommodating Unconsumed Terminal Wealth

In later sections of the Chapter on *Optimal Consumption and Investment Problems*, Pliska extends his model of optimal consumption-investment to accommodate unconsumed terminal wealth. At time T only a portion C_T of wealth V_T is consumed, leaving V_T - C_T available for future investment (including the provision of bequests). This is achieved by splitting terminal wealth between consumption and terminal investment and making a distinction between the utility derived from each of these two sources. Utility of terminal wealth must now include an additional optimising decision over terminal wealth as represented by (pp. 174-5):

$$u_T(w) = \max_{0 \le c \le w}\{u_c(c) + u_p(w-c)\}$$

However, to apply the risk-neutral computational approach given initial wealth $v \ge 0$ and an admissible consumption-investment plan (C,H) the set of attainable consumption processes must be modified as follows:

$$\begin{aligned}&\text{maximise } E\left[\sum_{t=0}^{T}\alpha^t u_c(C_t) + \alpha^T u_p(V_T - C_T)\right]\\&\text{subject to } v = E_Q[C_0/B_0 + \ldots + C_{T-1}/B_{T-1} + V_T/B_T];\ V_T \ge C_T;\ V_T \in F_T,\ C \text{ is adapted.}\end{aligned}$$

Now, there are three necessary first order conditions that must be solved (p. 176):

$$\begin{aligned}&\alpha^t u_c'(C_t) = \lambda N_t \Rightarrow C_t = I_c(\lambda N_t/\alpha^t), \quad t = 0,\ldots,T,\\&\left.\begin{aligned}\alpha^T u_c'(C_T) &= \alpha^T u_p'(V_T - C_T)\\ \alpha^T u_p'(V_T - C_T) &= \lambda N_T\end{aligned}\right\} \Rightarrow \begin{aligned}&V_T = I_c(\lambda N_t/\alpha^T) + I_p(\lambda N_T/\alpha^T)N_T,\\&E\left[\sum_{t=0}^{T} I_c(\lambda N_t/\alpha^T) + I_p(\lambda N_T/\alpha^T)N_T\right] = v,\end{aligned}\end{aligned}$$

where, adopting an obvious notational convention, the subscripted I functions are the inverse functions of the respective subscripted marginal utility functions.

3.2.9. CONSTRAINTS AND INCOMPLETE MARKETS

Pliska demonstrates how the standard models for optimal portfolio investment and optimal consumption-investment can be modified to account for certain constraints over investment activity. Included, are short sales constraints, prohibitions on borrowing for the purchase of risky securities, and those constraints arising from the presence of incomplete markets.

To incorporate constraints, Pliska transforms the set of *trading strategies* so that they take the form $F = (F_1, \ldots, F_N)$ where each $F_n = \{F_n(t);\ t = 0,1,\ldots,T\}$ is a predictable stochastic process with $F_n(t) = H_n(t)S_n(t\text{-}1)/V_{t-1}(t)$ representing the fraction of time $t - 1$ wealth that is invested in security n and held until time t. The corresponding value function is now maximized subject to a set of *admissible* strategies **A:** $F(t) \in \mathbf{K}$ for $t = 1,\ldots,T$, , where the latter is defined in relation to the set of relevant constraints, **K** applying at each moment of time. For example, no short selling of the nth security requires that $F_n \geq 0$, whereas no borrowing from the bank account requires that $F_1 + \ldots + F_N \leq 1$. Under these conditions, the T value V_T of the portfolio can now be expressed as:

$$V_T = v\prod_{t=1}^{T}\left[1 + r_t + \sum_{n=1}^{N} F_n(t)\{\Delta R_n(t) - r_t\}\right], \quad v = V_0, \text{ subject to } F \in \mathbf{A}.$$

The dynamic programming and optimal portfolio problems must be modified accordingly. However, the martingale approach requires further modifications to account for the effect of the constraints over the securities market. Specifically, Pliska (pp. 181-2) introduces a convex support function $\delta(x): \Re^N \to \Re \cup \{+\infty\}$ of $\mathbf{K}$ defined by:

$$\delta(x) \equiv \sup_{F \in \mathbf{K}} (-Fx')$$

The effective domain of δ is the convex cone $\widetilde{\mathbf{K}} \equiv \{x \in \Re^N : \delta(x) < \infty\}$. Furthermore, Pliska assumes that $0 \in \mathbf{K}$ and that δ is continuous on $\widetilde{\mathbf{K}}$. He also introduces a predictable stochastic process $\kappa = \{\kappa(t);\ t = 0,1,\ldots,T\}$ which is required to satisfy $\kappa(t) \in \widetilde{\mathbf{K}}$ for all $t \geq 1$. $\kappa(t,\omega)$ is an N-vector whose nth component corresponds to the nth security. Pliska then defines an auxiliary market M_κ for each $\kappa \in \boldsymbol{N}$, where $\boldsymbol{N}$ is the set of all such κ processes by modifying the return processes for the bank account and risky securities as follows:

$$\begin{aligned}
&r_t \to r_t + \delta(\kappa(t)), \quad t \geq 0 \\
&\Delta R_n(t) \to \Delta R_n(t) + \delta(\kappa(t)) + \kappa_n(t), \quad n = 1,\ldots,N;\ t \geq 1.
\end{aligned}$$

For the market M_κ, and any trading strategy F, whether it is admissible or not, the time T value of the portfolio can now be expressed as (p. 181):

$$\begin{aligned}
V_T^\kappa &= v\prod_{t=1}^{T}\left[1 + r_t + \delta(\kappa_t) + \sum_{n=1}^{N} F_n(t)\{\Delta R_n(t) + \kappa_n(t) - r_t\}\right], \\
&= v\prod_{t=1}^{T}\left[1 + r_t + \sum_{n=1}^{N} F_n(t)\{\Delta R_n(t) - r_t\} + \delta(\kappa_t) + F(t)\kappa_n'(t)\right]
\end{aligned}$$

For each $\kappa \in \boldsymbol{N}$, an unconstrained optimal portfolio problem can now be solved after substituting V_T^κ for the normal terminal wealth variable V_T. Pliska shows that a *dual problem*:

$$\min_{\kappa \in \mathbf{N}} J_\kappa(v), \quad \text{where } J_\kappa(v) = \min Eu(V_T^\kappa) \text{ subject to } V_0 = v$$

must now be solved. If $\kappa*$ denotes the optimal solution, then the optimal trading strategy for the unconstrained problem in market $M_{\kappa*}$, is the candidate for the solution to the constrained problem in the original market M_0, providing the conditions $F \in \mathbf{A}$ and $\delta(\kappa^*(t)) + F(t)\kappa^*(t) = 0$, are both satisfied.

Incompleteness implies that the set of securities does not fully span the state space. In other words, for each single period, say from $\tau - 1$ to τ, within the time horizon $t = 0,1,\ldots,T$, the dimensions of the τ-indexed matrix made up of the column vectors of state-indexed security prices (including the bank-account $B_\tau(\omega)$) is of a rank that is smaller than the number of states, $\omega \in \Omega$. For the multiperiod setting, this implies that certain types of contingent claim may not be attainable: risk-neutral computational methods are still feasible, but the set of risk-neutral probability measures may no longer be unique. This problem of non-uniqueness, under martingale measures, implies the existence of a range of feasible expected values for securities and derivatives. Nevertheless, in many cases, upper and lower bounds on this feasible range can still be determined.

In Chapter Seven, Pliska examines extensions that must be introduced to accommodate models with infinite sample spaces Ω. He observes that the fundamental theorem of asset-pricing remains true in the case of infinite sample spaces provided the trading horizon remains finite. However, to establish this result the filtration representing the information submodel must be extended. An additional condition is required to convert the collection F of subsets of Ω, called an algebra, into a σ–algebra. This axiom, requiring that countable unions of subsets in F be included in F, is formally posited by (p. 244):

$$F_1, F_2, \ldots \in F \Rightarrow \bigcup_{n=1}^{\infty} F_n \in F$$

Hence, the filtration becomes $\mathbf{F} = \{F_t: t = 0,1,\ldots,T\}$ where $\{F_t\}$ is now an increasing sequence of σ–algebras, but it still remains the case that $F_t \subseteq F_{t-1}$. Moreover, a random variable is said to be *measurable* with respect to the σ–algebra F if, for every real number x, the subset $\{\omega \in \Omega: X(\omega) \leq x\}$ is an element of the σ–algebra F, indicated by $X \in F$. The stochastic process $X = \{X_t: t = 0,1,\ldots,T\}$ is said to be *adapted* if $X_t \in F_t$ for all t. The securities market model is essentially unchanged.

3.3. Traditional Portfolio Theory and the Modern Approach

Martingale techniques based on measure changes that have been inspired by Girsanov's theorem and Bayes law are situated at the core of current financial research. However, a casual observer might be forgiven for exhibiting some confusion about the precise relationship that is supposed to hold between the fundamental theorem of asset-pricing and the more conventional depictions of asset-pricing theory that appear in corporate finance texts, replete with betas and mean-variance frontiers; or for that matter, the multi-factor models that seem to dominate empirical and practitioner research. Fortunately, John Cochrane's (2000) recently archived textbook articulates these linkages very clearly.

Like Pliska (1997) Cochrane demonstrates that non-arbitrage implies the existence of stochastic discount factors (see Cochrane, 2000, Chapter 4). Similarly, he shows that these discount factors can also be derived within an infinite horizon optimal consumption framework in which agents determine portfolio composition to maximize the utility associated with endowments and returns on their holdings of financial assets (p. 33). However, he also clearly establishes the two-way linkages between the fundamental asset-pricing equation and the beta representation of the CAPM model, and the minimum variance return situated on the mean-variance frontier.

Cochrane's work begins with a simple derivation of the fundamental asset-pricing equation relating the appropriately discounted expected payoffs x_{t+1} to the asset's price p_t (p. 15):

$$p_t = E_t\left[\beta \frac{u'(c_{t+1})}{u'(c_t)} x_{t+1}\right] = E_t\left[m_{t+1} x_{t+1}\right]$$

The key variable in his analysis is the stochastic discount rate m_{t+1}. As we have seen, Pliska's analysis confirms the assumption that either the law of one price or non-arbitrage hold are sufficient to prove the existence of such a discount rate. In the first part of his text, in findings that are tied together in Chapter 6, Cochrane demonstrates that the conventional mean-variance frontier and also the beta representation of the CAPM can each be derived from the above asset-pricing equation. For example, when the payoff is a return R^i, the asset-pricing equation can be written in the form $1 = \mathrm{E}(m\, R^i)$. Use of the covariance decomposition:

$$\operatorname{cov}(m, x) = E(mx) - E(m)E(x),$$

allows Cochrane to write the asset-pricing equation in the form (p. 99):

$$1 = E(m)E(R^i) - \text{cov}(m, R^i) \Rightarrow E(R^i) = \frac{1}{E(m)} - \frac{\text{cov}(m, R^i)}{E(m)} = \alpha + \lambda_m \beta i_{i,m}$$

where $\alpha = 1/E(m)$, $\beta_{i,m} = \text{cov}(m, R^i)/\text{var}(m)$, $\lambda_m = -\text{var}(m)/E(m)$.

Alternatively, the projection of m on the space of payoffs **X** could be used instead of the original payoff. This projection of the stochastic discount rate can also be used to derive the asset return $R^* = x^*/\text{E}(x^{*2})$, which is orthogonal to the original return space and has the additional property that it is the minimum second moment return and is, thus, situated on the mean-variance frontier. Cochrane then defines another return R^{*e} which is defined by the projection of the unit vector on the space of excess returns **R**:

$$R^{e*} \equiv \text{proj}(1|\mathbf{R}) \text{ where } \mathbf{R} = \{x \in \mathbf{X} \text{ s.t. } p(x) = 0\}$$

He demonstrates that a particular return R^{mv} can only be situated on the mean-variance frontier if it can be represented as the sum $R^{mv} = R^* + w\, R^{*e}$ for some real number w. Any chosen return R^i can in turn be expressed in a form that decomposes the return into three orthogonal components $R^i = R^* + w^i\, R^{*e} + n^i$ where $\text{E}(n^i) = 0$ (Cochrane, 2000, p. 84).

Moreover, Cochrane (p. 118) shows that the derivations can proceed in the opposite direction from either the mean-variance frontier or the CAPM to the stochastic discount rate form of representation. For example, $\text{E}(R^i) = \text{E}(R^*) + w^i\, \text{E}(R^{*e})$. An expression for w^i can then be found by solving for w^i in the equation for the covariance between R^{mv} and R^i: $\text{cov}(R^i, R^{mv}) = \text{var}(R^*) - w\text{E}(R^*)\text{E}(R^{*e}) + w^i[w\ var(R^{*e}) - \text{E}(R^*)\text{E}(R^{*e})]$.

Cochrane relates Merton's (1973) Intertemporal Capital Asset-pricing Model (ICAPM) to the asset-pricing equation by expressing time t consumption as a function of a set of state variables z_t. This results in the following factor version of the stochastic discount rate m_{t+1}(p.156):

$$m_{t+1} = \frac{\beta u'[g(z_{t+1})]}{u'[g(z_t)]}$$

Associated with the direct utility function is an indirect utility function $V(W_{t+1}, z_{t+1})$ in wealth W_{t+1} and the state variables z_{t+1}. The actual transition to the ICAPM is accomplished by applying the envelope theorem, which equates marginal utility with the marginal indirect utility of wealth (p.157):

$$u'(c_t) = V_W(W_t, z_t) \text{so that } m_{t+1} = \frac{\beta V_W(W_{t+1}, z_{t+1})}{V_W(W_t, z_t)}.$$

In discrete time this relationship can be linearized using a Taylors series approximation or, if it can be assumed that returns are normally distributed, by invoking Stein's lemma to make the approximation exact (see Cochrane, pp. 154, 157). In continuous time linearity arises naturally through the application of Ito's lemma to the differential expression for the instantaneous change in the marginal utility of wealth (for details see Cochrane, 2000, pp. 36-7, and p.157). Cochrane's discrete-time approximation to the continuous-time expression for excess returns is (p. 157):

$$E(R) - R^f \approx rra \operatorname{cov}(R, \Delta W) + \lambda_z \operatorname{cov}(R, \Delta z),$$

where rra is the coefficient of relative risk aversion WV_{W}/V_{W}, and the market price of risk $\lambda_z = V_{Wz}/V_{W}$.

In chapter 19 of his text Cochrane (2000) describes the relationship between the stochastic discount rate approach to asset-pricing and term structure models of the interest rate. The lynch pin is the expectation hypothesis that associates the N period yield with the average of expected future one-period yields. Using the discount factor existence theorem, the researcher selects a statistical model for the positive discount factor and finds one period bond prices as the expectation of this discount factor. By solving the discount factor forward and taking the expectation of the resulting expression, the price of an N period zero-coupon bond can then be derived. Alternatively, a differential equation that prices must follow, can be derived from the basic the pricing condition $P_t^{(j)} = E_t\left(m_{t,t+j}\right)$, which can be solved backwards from $P_N^{(0)} = 1$. In practice, this is usually accomplished in continuous time using an Itô diffusion for the discount factor and another diffusion process for the state variable that accounts for the drift of the discount factor (i.e. the short-rate process—because the expectation of the discount factor must, by definition, equal the risk-free rate).

3.4. Epstein and Zin's Risk-sensitive Aggregator Functions

Both in terms of theoretical elaboration and application, risk-sensitive control theory and associated techniques have had a significant influence over approaches to the analysis of risk and uncertainty in finance theory. Seminal work by Epstein and Zin (1989) has paved the way for the use of risk-sensitive aggregator functions in financial economics so as to capture preference orderings that violate the axioms of Von Neumann and Morgenstern expected utility framework.

These more general utility-theoretic approaches to risk enable finance theorists to develop models with the capacity to distinguish between intertemporal elasticities of consumption and risk-aversion (under the expected utility approach, these two parameters are constrained to be inversely proportional to one another). As such, they can be employed to explain both the Allais and also the Ellsberg paradoxes: the former by allowing for risk-sensitivity, the latter by allowing for uncertainty aversion[88].

Epstein and Zin's approach is based on the properties of an aggregator function. In the deterministic, discrete-time case, aggregator functions are recursive operators defined over consumption streams that have been aggregated for the purposes of dynamic programming. The utility V associated with a consumption stream $(c_0, c_1, c_2,\ldots)$ can be represented as $V(c_0, c_1, c_2,\ldots) = W(c_0, V(c_1, c_2,\ldots))$ for some function W. Thus current consumption and future utility have been combined to determine current utility. However, Epstein and Zin's family of aggregator functions is stochastic, and as such, must be defined over two variables: one of these is the certainty equivalent m, of the future consumption stream, while the other f, is a measurable function, defined over the current commodity space. Although Epstein and Zin's (1989) approach is not one derived from first principles, they cite references that do provide an axiomatic foundation for their chosen aggregator function (Chew and Epstein, 1990; Skiadis, 1991). In addition, they confirm that the stochastic differential utility function they have developed is sufficiently flexible to capture standard, additive, expected utility, Machina's (1982)

[88] Duffie and Epstein (1992) have extended Epstein and Zin's (1989) earlier work on discrete-time recursive utility into its continuous-time counterpart: stochastic differential utility.

extension of expected utility theory, Kreps-Porteus utility (1978), and Dekel-Chew betweeness utility (Dekel, 1986 and Chew, 1989)[89].

Epstein and Zin (1989) derive their intertemporal asset-pricing model for the particular Dekel-Chew class of non-expected utility theories. They confirm that appropriate versions of both the temporal CAPM and intertemporal consumption-CAPM are nested as special cases within their more general model (1989, section 6, pp. 956-60). In this general version of the model, "…the systematic risk of an asset is determined by covariance with both the return to the market portfolio and consumption growth, while in each of the existing models only one of these factors plays a role (Epstein and Zin, 1989, p. 937).

I shall now provide a brief overview of their approach. After defining the intertemporal consumption space, Epstein and Zin examine the properties of a recursive utility function V satisfying the following equation on its domain (1989, pp. 944-5):

$$V(c_0, m) = W(c_0, \mu(V[m]))$$

where $W : \Re_+^2 \to \Re_+$ is an aggregator function that combines current consumption and future utility to determine current utility. Here, μ stands for the certainty equivalent (or generalised mean value) functional that defines a mapping consistent with first and second degree stochastic dominance, over the following domain:

[89] An accompanying appendix to their paper, written by Costis Skiadis, examines infinite horizon generalizations of their model. Further extensions enable the techniques to: be applied to cases where filtrations are generated by mixed Poisson-Brownian information; accommodate habit formation; and, model utility defined over a "kinked" certainty equivalent.

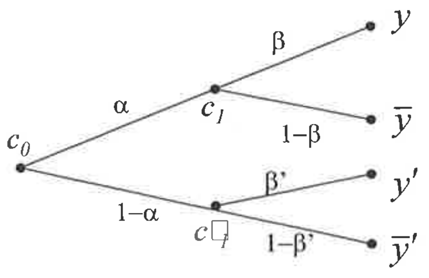
$\mu : dom\mu \subseteq \mathbf{M}(\Re_+) \rightarrow \Re_+$

for the metric space X governed by the Borel sigma algebra $\mathbf{B}(X)$ with $\mathbf{M}(X)$ representing the space of Borel probability measures on X endowed with the weak convergence topology. Also, $\mu(\delta_x) = x \quad \forall X \in \Re_+$ given δ_x, being a probability measure that assigns unit mass to the sequence $\{x\}$. Let D be the space of temporal lotteries whose members d can be described as an infinite probability tree (Epstein and Zin, 1989, p 940), in which each branch corresponds to a deterministic consumption stream $y \in \Re^{\infty}{}_{+}$. Epstein and Zin presume that each lottery d can be identified with a pair (c_0,m) where $c_0 \geq 0$ denotes the *non-stochastic* period 0 level of consumption and m is the probability measure over the set of $t = 1$ nodes in the tree. Because each such node can be identified with the whole probability tree emanating from it, Epstein and Zin (p. 940) argue that m can be envisaged as an element of the whole consumption space $\mathbf{M}(D)$. Next, they define a subspace $\hat{\mathbf{M}}(D(b))$, where b is a bound on the growth of consumption. They next show (p. 941) that consumption programs $D(b)$ defined over this subspace possess the requisite properties of compact support and homeomorphism to $\Re_+ \times \hat{\mathbf{M}}(D(b))$. In addition, they presume that the certainty equivalent functionals satisfy unusually weak properties of continuity and homogeneity (pp. 946-7).

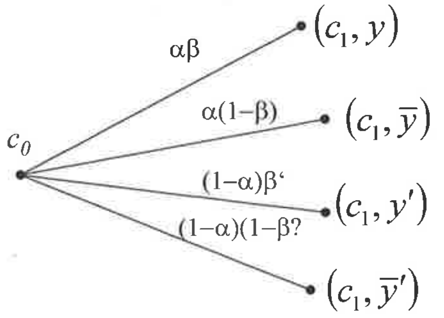
These properties enable them to prove the existence of an aggregator W and associated mean value functional μ such that there is a solution to the recursive utility function $V(c_0,m) = W(c_0, \mu(V[m]))$ under certain technical bounding conditions (Theorem 3.1, p. 946; and Appendix 3, pp. 961-5).

The homogeneity property merely requires that $\mu(p_{\lambda\tilde{x}}) = \lambda\mu(p_{\tilde{x}})\ \forall \lambda > 0$, where $p_{\lambda\tilde{x}}$ and $p_{\tilde{x}}$ are probability measures in the domain of corresponding to the random variables $\lambda\tilde{x}$ and $\tilde{x}$, respectively. This property plays an important role in the derivation of the Euler conditions for their asset-pricing model.

The significance of the defined subspace $D(b)$ and its derivative bounded subspace $D(b,l)$ is that it represents the way in which uncertainty about future consumption is resolved over time. Given a probability tree $d = (d_1,\ldots, d_t,\ldots)$ defined over $D(b)$, then $d_1 = (c_0, m_1)$ where $m_1 \in M(\Re_+^{\infty})$ and all uncertainty is resolved in period 1. Thus m_1 can be viewed as representing the *atemporal distribution* of uncertain future consumption (Epstein and Zin, 1989, p. 940). This notion is captured in the following diagram (figure 1, p. 942) which depicts two lotteries defined over the spaces D_2 and D_1, respectively, which induce the same probability measure on $\Re_+^{\infty}$ (as represented by the function $f(m)$) and differ only in the temporal resolution of uncertainty:



$d \in \mathbf{D}_2. \quad d = (c_0, m)$
$m = \alpha.\delta_{(c_1,d_1)} + (1-\alpha)\delta_{(c_1',d_1')}$
$d_1 = (c_1, [\beta.\delta_y + (1-\beta).\delta_{\bar{y}}]) \in \mathbf{D}_1$
$d_1' = (c_1', (\beta'.\delta_{y'} + (1-\beta').\delta_{\bar{y}'})) \in \mathbf{D}_1$
$y, \bar{y}, y', \bar{y}' \in \Re_+^\infty$



$(c_0, m) \in \mathbf{D}_1. \quad m_1 = f(m)$

$f(m)$ is the probability measure induced on $\Re_+^\infty$ by having all uncertainty resolve in the first stage?

Moreover, if V and V^* are comparable in the above sense, then they must rank non-stochastic consumption programs identically. In other words, $W = W^*$ or alternatively, $\rho = \rho^*$ and $\beta = \beta^*$. In addition, V^* is more risk averse than V if and only if $W = W^*$ and $\mu(\cdot) \leq \mu^*(\cdot)$ on the appropriate domain. Thus, Epstein and Zin observe (p. 950) that the certainty equivalent functional $\mu(\cdot)$ determines the degree of risk aversion for the corresponding intertemporal utility function. Secondly, they note that by definition, the mean value functionals $\mu^*(\cdot)$ exhibit risk aversion in the sense of second degree stochastic dominance so that $\mu^*(\cdot) \leq E(\cdot)$ where $E(\cdot)$ denotes the expected value operator. Thus, expected utility represents the least risk averse form of intertemporal utility function. For Kreps-Porteus functionals (considered below), comparative risk aversion is embodied in the α parameter with smaller values denoting greater risk aversion. Moreover, the well-known advantage of this class of utility function is that a separation between the intertemporal substitution parameter ρ and risk aversion parameter α has been realised. For the Chew-Dekel class of utility function, comparative risk aversion is captured in the ϕ parameter. Specifically:

$$\mu^*(\cdot) \leq \mu(\cdot) \Rightarrow \phi^{*''}(\cdot)/\phi^{*'}(\cdot) \leq \phi''(\cdot)/\phi'(\cdot)$$

implying, for Epstein and Zin's preferred empirical specification, that $\alpha \leq \alpha*$ and $a* \leq a$. The authors also examine the preference for early or late resolution (p. 952), confirming that the expected utility functional exhibits indifference towards timing, while for the Kreps-Porteus specification, early (late) resolution is preferred if $\alpha <$ (>) ρ. They were unable to find a similar characterization in terms of W or μ for the Chew-Dekel class of functional.

The CES specification is selected for the aggregator function W, as shown in the following expression (p. 946):

$$W(c,z) = \left[c^{\rho} + \beta z^{\rho}\right]^{\frac{1}{\rho}}, \qquad 0 \neq \rho < 1, \quad 0 < \beta < 1$$

The z variable appearing in the above aggregator function stands for the certainty equivalent term to be discussed in more detail below. For this specification, the elasticity of substitution $\sigma = (1 - \rho)^{-1}$ reflects substitution between current and future consumption. The certainty equivalent functional then determines the degree of risk aversion.

Epstein and Zin (1989, p. 949) confirm this key aspect of their approach by first defining a new variable $c = \lambda\,(m)$ for any (c_0, m) by the recursion: $V(c_0, m) = V(c_0, c, c, \ldots)$. The 'nearly' constant and deterministic path ($c_0, c, c, c, \ldots$) that is presumed indifferent to (c_0, m) then becomes the certainty equivalent for the latter. Risk aversion is then defined in relation to this certainty equivalent stream. Let V^* and V represent recursive utility functions possessing the distinct aggregators W^* and W, and consumption paths (c_0, $\lambda^*(m)$, $\lambda^*(m)$, $\lambda^*(m)\ldots$) and (c_0, $\lambda(m)$, $\lambda(m)$, $\lambda(m)\ldots$), respectively. Then V^* is more risk averse to V if and only if $\lambda^*(m) \leq \lambda(m)$ for all (c_0, m) in some common domain. An existence theorem for the utility function V, is proved using a weighted contraction mapping theorem specifically developed to deal with the case of unbounded aggregators (Appendix 3). Various classes of recursive utility function (expected utility, Kreps-Porteus, and Chew-Dekel) can be defined through differences in the $\mu(p)$s, their respective certainty equivalent functionals, as follows (pp. 947-8):

Expected Utility:

$$\mu(p) \equiv \left(\int x^\rho dp(x)\right)^{\frac{1}{\rho}} = \left(E_p \tilde{x}^\rho\right)^{\frac{1}{\rho}}, \quad p \in \mathbf{M}(\Re_+)$$

with the associated intertemporal utility function:

$$V(c_0, m) = \left[c_0^\rho + E_{m_1} \sum_1^\infty \beta^t \tilde{c}_t^\rho\right]^{\frac{1}{\rho}}.$$

Kreps-Porteus Utility:

$$\mu(p) \equiv \left(E_p \tilde{x}^\alpha\right)^{\frac{1}{\alpha}}, \quad p \in \mathbf{M}(\Re_+), \quad 0 \neq \alpha < 1$$

with the associated intertemporal utility function:

$$V(c_0, m) = \left[c_0^\rho + \beta\left(E_m V^\alpha(\cdot)\right)^{\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}}.$$

Chew-Dekel Utility, μ is defined on $\mathbf{M}(\Re)$ implicitly by the equation:

$$\int F(x, \mu(p))dp(x) = 0, \quad p \in \mathbf{M}(\Re_+)$$

where $F : \Re_+^2 \to \Re$ is continuous, increasing (decreasing) in its first (second) argument, $F(\cdot, z)$ is concave, and $F(x, x) \equiv 0$. Under the homogeneity condition F must be linearly homogenous so that the equation can be written in the form:

$$\int \phi(x/\mu(p))dp(x) = 0, \; p \in \mathbf{M}(\Re_{++}).$$

Epstein and Zin adopt the following specification for ϕ

$$\varphi(x) = (x^{\alpha} - 1)/\alpha + a(x - 1), \quad 0 \neq \alpha < 1, \quad a \geq 0.$$

If $a = 0$ the specification is equivalent to expected utility theory. Epstein and Zin (p. 948) observe that the Dekel-Chew class of mean-value functional has a major advantage over its Kreps-Porteus counterpart: namely, the latter specification effectively ranks *timeless* gambles (i.e. those where any uncertainty is resolved before further consumption takes place) by an *expected utility* ordering. However, most of the experimental evidence against expected utility is derived for timeless wealth gambles. However, the asset-price equation is only derived for Kreps-Porteus preferences.

I now consider in detail, Epsteins and Zin's (section 6, pp. 956-60) derivation of the Euler conditions for the most general case, that of Chew-Dekel preferences, because they establish results for other classes of preferences by specialization. This is because it provides an alternative derivation to that accomplished by Hansen, Sargent and Tallarini (1999), which, instead, utilised a sub-gradient inequality expression. Epstein and Zin (1989, p. 955) note that the recursive structure of utility functions implies the following Bellman equation (here, and in what follows, the curl notation indicates the presence of a random variable):

$$J(I_0, x_0) = \max_{c_0 \geq 0, w_0 \in S^K} \left[c_0^\rho + \beta\mu^\rho\left[p_{J(\tilde{I}_1,(x_0-c_0)w_0'\tilde{r}_0)}\middle|I_0\right]\right]^{\frac{1}{\rho}}.$$

Here, the argument of the mean value functional is the probability measure for $J(\tilde{I}_1, x_1)$ conditional on I_0[90]. The latter is formally defined as the information filtration composed of observed past returns on the K-vector of assets and other relevant exogenous variables $z_t \in \Re^z$ that, in combination, determine Ω and the Borel σ–algebra $\boldsymbol{B}(\Omega)$. The second argument in wealth x_1, has been replaced using the constraint defining the evolution of investor wealth (i.e. $x_1 = (x_0 - c_0)\, w'_0\, r_0$).

Given the assumed property of homotheticity of utility, J can be expressed in the form $J(I, x) = A(I)\, x$ so that the Bellman equation can be rewritten in the form:

$$A(I_0)x_0 = \max_{c_0 \geq 0, w_0 \in S^K} \left[c_0^\rho + \beta(x_0 - c_0)^\rho\, \mu^\rho\left[p_{A(\tilde{I}_1)w_0'\tilde{r}_0}\middle|I_0\right]\right]^{\frac{1}{\rho}}.$$

This implies the conventional portfolio separation property: decisions about optimal consumption-saving are entirely separated from decisions about optimal portfolio holdings. It can be seen that the latter are determined by the solution to:

[90] In the Hidden-Markov case, for example, the probability measure would refer to the set of transition probabilities determining switching from one state to another. However, Epstein and Zin's representation has greater generality.

$$\max_{w_0 \in S^K} \mu\left\lfloor p_{A(\tilde{I}_1)w_0'\tilde{r}_0} \middle| I_0 \right\rfloor.$$

Next, Epstein and Zin (p. 955) assume that optimal consumption c^*_0 can be written as $c^*_0 = a_0 x_0$. Substitution of this expression into the Bellman equation yields (after dividing through by x_0 and using the fact that $a_0 = (c_0/x_0)$):

$$A^\rho(I_0) = a_0^\rho + \beta(1-a_0)^\rho \mu^{*\rho}.$$

Moreover, the first order condition for consumption, derived from the separable Bellman equation, yields:

$$a_0^{\rho-1} = (1-a_0)^{\rho-1} \beta\mu^{*\rho}.$$

Epstein and Zin combine the last two equations to give $A(I_0) = a_0^{(\rho-1)/\rho} = (c_0^*/x_0)^{(\rho-1)/\rho}$.

Given recursive utility, and under the assumed property of stationarity, the following expression is implied:

$$A(\tilde{I}_1) = (\tilde{c}_1^*/\tilde{x}_1)^{(\rho-1)/\rho}.$$

Along with the wealth constraint this can be substituted in to the first order condition for consumption to yield:

$$\beta^{1/\rho} \mu\left\lfloor p_{(\tilde{c}_1/c_0)^{(\rho-1)/\rho} \tilde{M}_0^{1/\rho}} \middle| I_0 \right\rfloor = 1,$$

where $\widetilde{M}_0 = w_0'\widetilde{r}_0$, is the market portfolio return, and use has been made of the fact that:

$$\widetilde{a}_1\left(\frac{1-a_0}{a_0}\right) = \widetilde{a}_1\left(\frac{x_0 - c_0}{c_0}\right) = \left(\frac{\widetilde{c}_1}{c_0}\right)\left(\frac{x_0 - c_0}{x_1}\right) = \left(\frac{\widetilde{c}_1}{c_0}\right)\left(\widetilde{M}_0\right)^{-1}.$$

The latter also implies that: $A(\widetilde{I}_1)/\mu^* = [\widetilde{c}_1/c_0]^{(\rho-1)/\rho}\beta^{1/\rho}\widetilde{M}_0^{(1-\rho)/\rho}$, a result that proves useful below. Under Chew-Dekel preferences, the mean value functional μ is determined implicitly by $\int\phi(x/\mu(p))dp(x) = 0$, so that the optimal portfolio problem can be expressed in the form (p. 957):

$$\max_{w_0 \in S^K} E\left[\phi\left(A(\widetilde{I}_1)w_0'\widetilde{r}_0/\mu^*\right)I_0\right].$$

The first order conditions for this problem are:

$$E\left[\phi'\left(A(\widetilde{I}_1)\widetilde{M}_0/\mu^*\right)\cdot A(\widetilde{I}_1)\cdot(\widetilde{r}_{k0} - \widetilde{r}_{10})I_0\right] = 0, k = 2,\ldots, K.$$

Substituting for $A(\widetilde{I}_1)$ and $A(\widetilde{I}_1)/\mu^*$ gives:

$$E\left[\phi'\left(\left[\frac{\widetilde{c}_1}{c_0}\right]^{\frac{\rho-1}{\rho}}\widetilde{M}_0^{\frac{1}{\rho}}\beta^{\frac{1}{\rho}}\right)\cdot\left[\frac{\widetilde{c}_1}{\widetilde{M}_0}\right]^{\frac{\rho-1}{\rho}}\cdot(\widetilde{r}_{k0} - \widetilde{r}_{10})I_0\right] = 0, \quad k = 2,\ldots, K.$$

A second Euler condition is derived, in a similar fashion, from the expression derived from the first order conditions for consumption; namely:

$$E\left[\phi\left(\beta^{\frac{1}{\rho}}\left[\frac{\widetilde{c}_1}{c_0}\right]^{\frac{\rho-1}{\rho}}\widetilde{M}_0^{\frac{1}{\rho}}\right)\Big|I_0\right]=0.$$

The Euler conditions for the Kreps-Porteus specification are derived from this result using $\phi(x)=(x^{\alpha}-1)/\alpha,\ 0\neq\alpha<1$. When $\alpha=\rho$, the Euler conditions for the expected utility model are obtained. The Euler equation for the Kreps-Porteus specification is (Epstein and Zin, 1989, pp. 957-9):

$$E[\widetilde{r}_{k0}-\widetilde{r}_{10}/I_0]=\frac{\operatorname{cov}\left[\left[\frac{\widetilde{c}_1}{c_0}\right]^{\alpha(\rho-1)/\rho}\widetilde{M}_0^{(\alpha-\rho)/\rho},(\widetilde{r}_{k0}-\widetilde{r}_{10})/I_0\right]}{E\left[\left[\frac{\widetilde{c}_1}{c_0}\right]^{\alpha(\rho-1)/\rho}\widetilde{M}_0^{(\alpha-\rho)/\rho}/I_0\right]}$$

where $\widetilde{M}_0$ is the market portfolio return, I_0 is the informatio n set at time $t = 0$, r_{10} is the risk - free rate of interest and r_{k0} for $k = 2,...K$ is the return on the kth risky asset.

For this specification it is apparent that both consumption and the market return enter into the covariance that defines systematic risk. The conventional consumption-CAPM is obtained if $\alpha=\rho$, while the static CAPM obtains if $\alpha=0$ (see Epstein and Zin, 1989, p. 958-9 for a derivation of the Euler equations representing Dekel-Chew preferences). The recursive asset-price relation establishes that prices reflect the discounted expected value of future dividends, where the discount factors involve both consumption and market returns (pp. 959-60)[91].

[91] The recursion relation for the asset-price has the solution:

Notational Differences between Authors:

In reviewing the work of Hansen, Sargent, and Tallarini (1999), Epstein and Zin (1989) and Weil (1989), I have maintained the preferred notation of each author or group of authors so that readers can refer back to the original papers. However, minor notational differences arise, especially in descriptions of the gross return process. For comparative purposes, the following table identifies the key differences in notation for the relevant Euler equations (notation is identical for both consumption and the discount rate):

	Asset-price	**Dividend**	**Gross Return**
Hansen et al (1999)	q_t	Not specified explicitly	p_{t+1}
Weil (1989)	p_t	y_t where $\lambda_t = y_t / y_{t-1}$	$R_{t+1}= (p_{t+1}+y_{t+1})/ p_t$
Epstein & Zin	p_t	q_t	$r_{kt} = (P_{kt} + q_{kt})/ p_{kt-1}$

3.5 Applications of Uncertainty Aversion to Asset-pricing and Investment

Empirical studies of non-expected utility theory include Shumway (1997) and Benartzi and Thaler (1995). However, most economic applications of uncertainty aversion have concentrated on optimal stabilization policy conducted under various forms of observation error, external

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$$P_0 = E\left[\sum \beta^{(\alpha/\rho)t} \left[\frac{\bar{c}_t}{c_0} \right]^{\alpha\frac{(\rho-1)}{\rho}} \left(\widetilde{M}_0, \ldots, \widetilde{M}_{t-1} \right)^{\frac{(\alpha-\rho)}{\rho}} \widetilde{q}_t / I_0 \right],$$

if the right-hand side is finite. Thus, price equals the discounted expected value of future dividends, but the discount factors involve both consumption and market returns.

perturbation, and model uncertainty. However, four examples in the economics literature are of direct relevance to the question of investment and capital budgeting. Another is Lehnert and Passmore's (1999) study of investment and savings behaviour under uncertainty. In addition, there is Cherubini's (1997) fuzzy measure analysis of option pricing. Finally, there is Chateauneuf, Kast and Lapied's (1996) examination of a series of asset-pricing anomalies that are treated using the concavity property associated with the sub-additive form of uncertainty aversion.

3.5.1 Dow and Werlang's Analysis of Asset-pricing under Sub-additive Probabilities

Dow and Werlang (1992) employ the following definition of uncertainty aversion:

Definition: Let P be a probability and $A \in \Omega$ an event. The uncertainty aversion of P at A is then defined by:

$$c(P, A) = 1 - P(A) - P(A^C).$$

This number measures the amount of probability "lost" by the presence of uncertainty aversion in the form of the deviation of P form additivity at A. they also draw on the following lemma:

Lemma: $c(P, A) = 0$ for all events $A \subset \Omega$ if, and only if, P is additive.

Dow and Werlang (1992, p.202) provide an example of how this representation of uncertainty aversion could be applied in a decision-making context. They consider a random variable X with the following properties:

$\underline{X} = \inf_{\omega \in \Omega} X(\omega) \geq 0$and $\overline{X} = \sup_{\omega \in \Omega} X(\omega) \leq \infty.$

Now let P be a non-additive probability which is obtained by uniformly increasing uncertainty aversion from P: letting $P_c(\Omega)=1$, and $P_c(A)=(1-c)P(A)$ for $A \neq \Omega$. Then it can be shown that:

$c(P_c, A) = c \,\forall A \neq \Omega, \varnothing$, and that,

$E_{P_c}X = c\underline{X} + (1-c)E_P X$ and $-E_{P_c}(-X) = c\overline{X} + (1-c)E_P X$.

Thus:

$-E_P(-X) - E_P X = c(\overline{X} - \underline{X})$.

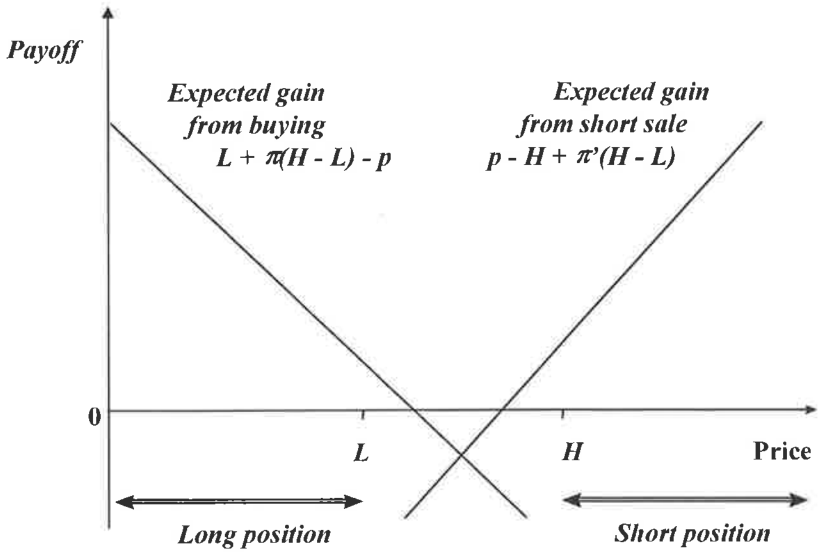
Thus, it is the case that: (1) the difference between $-E(-X)$ and $E(X)$ is increasing in uncertainty aversion c; and, (2) a risk-neutral agent whose behaviour is represented by this distribution will maximise a weighted average of the worst possible outcome and the expectation of the additive distribution, where the weight on the worst outcome is given by the coefficient of uncertainty aversion c.

Within this framework, the main result Dow and Werlang establish is that there will be a range of prices from *E(X)* to *-E(-X)*, at which the investor will take no position in the asset. They provide a simple example of this phenomenon by considering an asset which can only take one of two possible values, a high value *H*, and a low value *L*, with probabilities π and π', respectively. The expected return from buying one unit of the asset at a price *p* will be at worst (*L* - *p*) net of the price, but with probability π it will be *(H - p)*. Therefore, the expected payoff will be $L + \pi(H - L) - p$. The return from selling the asset will at worst be *(p - H),* but with probability π' it will be *(p - L)* yielding an expected payoff of $p - H + \pi'(H - L)$. Because $\pi + \pi' < 1$, $H - \pi'(H - L) > L + \pi(H - L)$ and at prices in between these two values, investors will not take a position in the asset, as shown in the following diagram reproduced from Dow and Werlang (p. 199).

More formally, let W be the investor's initial wealth, $u \geq 0$ the utility function (where it is assumed that u is C^2, $u' > 0, u'' \leq 0$), and X a random variable with nonadditive distribution P. Dow and Werlang establish the following lemma and related theorem:

Lemma: Suppose $EX < \infty$ and $-E(-X^2) < \infty$. For $\lambda \in \Re$ define $f(\lambda) = Eu(W + \lambda X)$.

Fig. 18: DOW & WERLANG'S (1992) MODEL OF ASSET DEMAND



Then, (i) f is right-differentiable at $\lambda = 0$; (ii) $f'_+(0) = u'(W)EX$.

Theorem: A risk-averse or risk-neutral investor with certain wealth W, who is faced with an asset which yields X per unit, whose price is $p > 0$ per unit, will buy the asset if $p < EX$ and only if $p \leq EX$. He will sell the asset if $p > -E(-X)$ and only if $p \geq -E(-X)$[92].

3.5.2 Lehnert and Passmore on "Pricing Systemic Crises"

Lehnert and Passmore (1999) develop an overlapping generations model of investment and savings behaviour under risk, which they then extend to incorporate uncertainty aversion[93]. To achieve this end, they draw upon aspects of Dow and Werlang's (1992, 1994) analysis of sub-additive version of uncertainty aversion. They assume that agents possess a subjective probability distribution over events of $P_\in = (1 - \in)Q$, where Q is an additive probability distribution over events in the σ-algebra **G** constructed from the set of states Ω. Thus $\in$ is the degree of uncertainty aversion. If investor utility in each state is $u(\omega)$ then expected utility is:

$$E_{P_\in}\{u(\omega)\} = \in \min_{\omega} u(\omega) + (1 - \in)E_Q\{u(\omega)\}$$

Thus, all of the missing probability is ascribed to the term on the left-hand side of the above equation that represents the worst-case scenario. The balance is ascribed to the expected value calculated under the additive distribution. It is assumed that the rate of return on investment in risky assets $\rho(\cdot)$ is a function of both the state of nature ω, and the uncertainty-contaminated level of aggregate investment $X^*_{\in t}.(r_t)$, which itself depends on the riskfree rate of interest. So that they can isolate the pure effects of uncertainty aversion, Lehnert and Passmore assume that savers are risk-neutral with respect to consumption while old and possess no assets other than a

[92] Proof of this theorem is by Jensen's inequality.

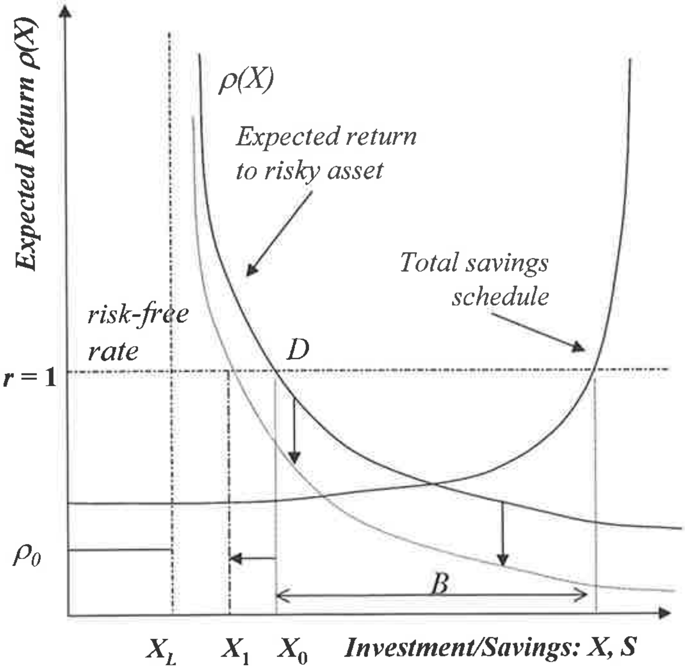
fully diversified share in a large number of risky projects. Under this assumption, the level of aggregate investment is defined implicitly by:

$$(1-\in_t)\int_{\Omega}\rho\left[\omega, X^*_{\in_t}(r_t)\right]d\pi^* + \in_t \rho_0 = r_t$$

The level of savings is likewise determined by the same variables, $S_t = s(\omega_t, r_t)$ Here, the ρ_0 term is the worst-case return on risky assets if investment happens to falls below a critical level X_L. The amount of the riskless asset (bonds) held by savers B, is then determined as a residual by the difference between savings and investment for a given riskfree rate of interest, $S_t - X^*(r_t)$. Lehnert and Passmore (proposition 1, pp. 26 and 43) show that a generation j born with an uncertainty parameter $\in_j$ will invest less than all generations i born with a smaller uncertainty parameter $\in_i < \in_j$. In addition, in times of high uncertainty, spreads between risky assets and riskless assets will widen. They further demonstrate (proposition 2, pp. 26, 44) that in each period, given an announced riskfree rate r_t, if $r_t > \rho_0$ there is some level of uncertainty $\in^* > 1$ such that, if $\in_j \geq \in^*$, no equilibrium with positive investment exists. The first of these outcomes is depicted in the following diagram (Lehnert and Passmore pp. 16, 25)[94].

[93] I shall not describe the actual model in any detail. I only intend to focus on aspects of uncertainty aversion and investment behaviour.
[94] Lehnert and Passmore go on to examine the effect of monetary policy interventions. Here, the effects are more complicated because changes to the riskfree rate also influence the level of savings.

Fig. 19: LEHNERT & PASSMORE'S (1999) MODEL OF INVESTMENT UNDER UNCERTAINTY AVERSION



Lehnert and Passmore's analysis is highly schematic, heuristic, but relatively straightforward manner, it does illustrate some of the likely effects of changes in uncertainty aversion. Rustem (1992), for example, has developed a robust algorithm for minimax control that could have practical application in a more micro-level, multiple-priors or scenario-based capital budgeting context.

3.5.3 CHERUBINI'S FUZZY MEASURE APPROACH TO OPTION PRICING

Cherubini (1997) proposes a parametric representation of uncertainty aversion by means of a special class of fuzzy measures known as g_λ-measures, where the λ parameter operates as an indicator of uncertainty. Given an additive probability distribution, a benchmark utility function, and a value λ in $(0, \infty)$, a sub-additive (super-additive if λ is negative) expected utility can be derived that represents uncertainty aversion. Cherubini maintains (p. 136) that fuzzy measures, which can be traced back to Dempster's (1967) work on probability intervals, are closely related to Gilboa and Schmeidler's (1989) maximin expected utility. Given a measurable space $\{\Omega, \mathbf{F}\}$, a set function μ is a fuzzy measure if:

(i) $\mu(\varnothing) = 0$

(ii) $A_i, A_j \in \mathbf{F}, A_i \subset A_j \Rightarrow \mu(A_i) \leq \mu(A_j)$

(iii) $\{A_n\} \in \mathbf{F}, A_1 \subset A_2 \subset \ldots$ and $\bigcup_{n=1}^{\infty} A_n \in \mathbf{F} \Rightarrow \lim \mu(A_n) = \mu(\bigcup_{n=1}^{\infty} A_n)$

(iv) $\{A_n\} \in \mathbf{F}, A_1 \supset A_2 \supset \ldots$ and $\bigcap_{n=1}^{\infty} A_n \in \mathbf{F} \Rightarrow \lim \mu(A_n) = \mu(\bigcap_{n=1}^{\infty} A_n)$

(v) if in addition, $\mu(\Omega) = 1$, then the fuzzy measure is called regular.

Additivity is replaced by three weaker assumptions: point (ii) represents a monotonicity assumption with respect to set inclusion, while points (iii) and (iv) represent upper and lower semi-continuity assumptions, respectively.

Cherubini (p. 139) restricts his attention to the g_λ sub-class of continuous fuzzy measures defined by:

$$\mu(A_i \cup A_j) = \mu(A_i) + \mu(A_j) + \lambda\mu(A_i)\mu(A_j),$$

for any pair $A_i, A_j \in \mathbf{F}, A_i \cap A_j = \varnothing$.

In addition, the classical additivity requirement:

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right)=\sum_{j=1}^{\infty}\mu(A_j),$$

is replaced for $\lambda \neq 0$, by the non-linear requirement:

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right)=\frac{1}{\lambda}\left\{\prod_{j=1}^{\infty}\left[1+\lambda\mu(A_j)\right]-1\right\}.$$

The motivation for this requirement will soon become apparent, at least from a heuristic perspective.

Cherubini (pp. 140-1) introduces an auxiliary concept—the quasi-measure μ— a set function, which is defined if there exists a continuous, strictly increasing function T: $[0, a] \rightarrow [0, \infty]$, $a \in (0, \infty]$, with $T(0) = 0$, $T^{-1}(\infty) = \varnothing$ if a is finite, and $T^{-1}(\infty) = \infty$ otherwise, such that $\mu' = T(\mu)$ is an additive measure. The function that transforms the quasi-measure into an additive measure is formally known as a proper T-function.of μ (and if the respective quasi-measure is regular it is known as a quasi-probability).

For μ, a given member of the g_λ-class, with $\lambda \neq 0$, the T-function is defined by $T_\lambda(y) = \ln(1+\lambda y)/k\lambda$, with $k > 0$. Thus, $\mu' = T(\mu)$ is an additive measure. Cherubini (p. 140) provides the following sketch proof:

$$\mu'\left(\bigcup_{j=1}^{\infty} A_j\right)=\frac{1}{k\lambda}\ln\left[1+\lambda\mu\left(\bigcup_{j=1}^{\infty} A_j\right)\right]=\sum_{j=1}^{\infty}\frac{\ln(1+\lambda\mu(A_j))}{k\lambda}=\sum_{j=1}^{\infty}\mu'(A_j).$$

A symmetrical result follows, such that for an additive measure μ', there exists a function $T_k^{-1}(y) = \left[\exp(k\lambda y)-1\right]/\lambda$, with $\lambda \neq 0$, whereby the measure $\mu = T^{-1}(\mu')$

is a member of the g_λ-class. The proof, by substitution of the relevant non-linear requirement and simple algebraic manipulation, is similar to that provided for the previous result.

Starting from a given additive distribution $H(\cdot)$, the g_λ-measure is defined by setting (p. 140):

$$g_\lambda([a,b)) = \frac{H(b) - H(a)}{1 + \lambda H(a)}.$$

Cherubini draws on a duality result to assert that for any subset A, a super-additive measure can be constructed such that:

$$g_\lambda(A) + g_{\lambda*}(A^c) = 1,$$

where $g_{\lambda*}$ is parameterized by $\lambda^* = -\dfrac{\lambda}{1+\lambda}$.

Non-additive expected utility is defined in an analogous fashion to standard expected utility U, where the latter:

$$U = \int u(x) dH = \int H(x : u(x) \geq \alpha) d\alpha,$$

is replaced, firstly, by the *lower Choquet integral* (using the decumulated distribution) as in:

$$U_* = \int g_\lambda(x : u(x) \geq \alpha) d\alpha = \int \left(\frac{1 - H(\alpha)}{1 + \lambda H(\alpha)} \right) d\alpha = \min\left\{ \int u(x) dP : P \in \Gamma(H, \lambda) \right\}$$

where the set of probability functions $\Gamma(\cdot)$, is defined by:

$$\Gamma(H, \lambda) = \left\{ P : g^H_{\lambda*}(A) \geq P(A) \geq g^H_\lambda(A), \forall A \in \mathbf{F}, \lambda^* = -\lambda/1 + \lambda \right\}.$$

Here, P represents the probability measures of the set, $\mathbf{F}$ is a sigma-field, and $g^H_\lambda(\cdot)$ represents the measure of the g_λ-class (core) constructed using the distribution $H(\cdot)$.

Secondly, the duality result yields the upper Choquet integral (using the cumulated distribution):

$$U^*(H, \lambda) = U_*(H, \lambda^*) = \int \left(\frac{1 - H(\alpha)}{1 + \lambda^* H(\alpha)} \right) d\alpha = \max\left\{ \int u(x) dP : P \in \Gamma(H, \lambda) \right\}.$$

Assuming a linear representation for utility, Cherubini (pp. 142-3) uses the above expressions to derive upper and lower bounds for the valuation of a corporate debt contract promising to pay min(β, v)—the minimum of the nominal value of debt β, or the value of the firm v. The latter is assumed to be distributed in the interval [0, 1], while β is measured in terms of the support of v, and the rate of interest is zero (a simplification Cherubini justifies on the basis that interest risk and credit risk are likely to be orthogonal). Again, for convenience, $H(\cdot)$, the additive probability distribution governing the value of the firm is assumed to be rectangular. This yields a particularly simple form for the calculated bounds. This valuation technique provides the basic ingredient for Cherubini's second example: levered option replication. Under standard replication arguments, the time t value of a call option $c(V, t; K)$ with strike price K, risk free rate r, expiration date T, and underlying value $V(t)$ at time t, is given by:

$$c(V,t;K) = V(t) - \exp[-r(T-t)]E[\min(V(T).K)|V(t)].$$

Here, the replicating portfolio is a leveraged position that is long on the underlying asset and short on a debt contract with a position on the value of the asset posted as collateral. Cherubini argues that unlike trading in the underlying, the debt position would be built in an unofficial market with considerable uncertainty attached to the credit standing of the two trading parties. As such, he contends that, for the debt position, a fuzzy measure approach would be pertinent.

3.5.4. Chateauneuf's Analysis of Option Pricing Irregularities

Chateauneuf, Kast and Lapied (1996) introduce a non-linear valuation formula based on Choquet integrals of random payoffs to determine the selling and buying prices of securities that are set by dealers. With this machinery, the authors investigate several pricing puzzles: the premium that is paid for a short position, violation of put-call parity and the fact that the components of a security —primes yielding the dividends plus the strike price at expiration) and scores (yielding the excess value to the strike price only)— can sell at a premium to the underlying security. Unlike the previous characterizations of portfolio choice under uncertainty aversion in Dow and Werlang (1992) and Epstein and Wang (1995), where convex capacities are representations of individual behaviour, Chateauneuf, Kast and Lapied (1996) derive convex capacities from prices and agents are assumed to be price-takers. The pricing relationships so derived are non-linear rather than linear due to the presence of market frictions.

Uncertainty is described in familiar terms in relation to a set of states S, and a σ-algebra $\mathbf{S}$, of events in S. Assets are defined by the random variable X: $S \rightarrow \Re$ of its payoffs $X(s)$ in state s—a bounded measurable function from (S, $\mathbf{S}$) to ($\Re$, $\mathbf{B}$), where $\mathbf{B}$ is a Lebesgue σ-algebra. The price at which a dealer sells an asset Y to agents is $q(Y)$, while the price at which the dealer buys the asset is $-q(Y)$. Chateauneuf, Kast and Lapied (1996, p. 325), introduce three axioms which are sufficient to derive their sublinear Choquet integral representation for buying and selling prices.

Axiom 1.1 is monotonicity of the price functional: *If an asset pays more in all states than an asset X, its price must be higher*: $Y \geq X \Rightarrow q(Y) \geq q(X)$. Axiom 1.2 is absence of transaction costs on riskless assets: *Let* $\mathbf{1}_s$ *be the asset which pays* 1 *in all states in S. Riskless assets are of the form* $\alpha\mathbf{1}_s$ *for all* α *in* $\Re$. *It is assumed that* $q(\alpha\mathbf{1}_s) = \alpha$ *for all* α *in* $\Re$. Axiom 1.3. defines the comonotonicity premium: $X, Y \in \mathbf{X}$: $q(X + Y) \leq q(X) + q(Y)$ equality holds if X and Y are comonotonic[95].

[95] The authors explain how comonotonicity rules out arbitrage opportunities in the sense that an investor is unable to form a portfolio comprising purchase amounts $\alpha_i \geq 0$; $1 \leq i \leq n$ of asset X_i, and sales amounts $\alpha_j \geq 0$ of asset X_j, $1 \leq j \leq m$ such that the portfolio would yield positive payoffs and incur a negative cost. By assumption, the portfolio would yield:

Chateauneuf, Kast and Lapied (1996, proposition 1.1., p. 326) establish that, together with additivity ($q(X + Y) \leq q(X) + q(Y)$), axioms 1.1 and 1.2 are sufficient to establish the *Linear Pricing* rule: *there exists a unique probability distribution* μ, *on* (S, **S**) *such that for any asset* X *in* **X**, $q(X) = \int_S X d\mu$. In the usual manner, a *capacity* on the measurable space (S, **S**) is then defined as a set function v: $S \rightarrow [0, 1]$ satisfying $v(S) = 1$, $v(\varnothing) = 0$, monotonicity with respect to inclusion (if $B \supseteq A$, then $v(B) \geq v(A)$, and concavity ($v(A \cup B) + v(A \cap B) \leq v(A) + v(B)$). Chateauneuf, Kast and Lapied (1996, theorem 1.1, p. 327) then draw on results by Schmeidler and others to establish that axioms 1.1, 1.2 and 1.3 are sufficient to establish *Choquet Sublinear Pricing*: *there exists a unique concave capacity* v *on* S *such that the value of an asset* $X \in$ **X** *is defined by*:

$q(X) = \max\left\{\int_S X d\mu : \mu \text{ is an additive probability s.t } \mu \leq v\right\}$. *Moreover, the price of* X *is the Choquet integral of its payoffs*:

$q(X) = \int_S X dv = \int_{\Re-} [v(X \geq t) - 1] dt + \int_{\Re+} v(X \geq t) dt$; *and q is sublinear.*

For the particular case where $X = \alpha_1 A^*_1 + \alpha_2 A^*_2 + \ldots + \alpha_n A^*_n$, where α_i are increasing real numbers and the A^*_i are characteristic functions of events A_i , forming a partition of S, then the Choquet integral takes the form:

$q(X) = \alpha_1 v(A_1 \cup A_2 \cup \ldots \cup A_n) + (\alpha_2 - \alpha_1) v(A_2 \cup A_3 \cup \ldots \cup A_n) + \ldots + (\alpha_n - \alpha_{n-1}) v(A_n)$.

$X = \sum_{i=1}^{n} \alpha_i X_i - \sum \alpha_j X_j \geq 0$.
However, formation cost would equal:
$\sum_{i=1}^{n} \alpha_i q(X_i) + \sum_{j=1}^{m} \alpha_j q(-X_j) = \sum_{i=1}^{n} q(\alpha_i X_i) + \sum_{j=1}^{m} q(-\alpha_j X_j) \geq q\left(\sum_{i=1}^{n} \alpha_i X_i - \sum_{j=1}^{m} \alpha_j X_j\right) = q(X) \geq 0$.

This latter form is the one used by the authors to analyze the various asset-pricing puzzles for a simple binomial case in which either the underlying stock price and/or dividend payouts jump by $\pm.h$ and $\pm\, d$, respectively. The property of concavity plays an essential role in the results thereby obtained.

3.5.5. Hansen et al. (2001) on Max-min Expected Utility Theory

This section reviews a paper by Hansen et al (2001) that was motivated, in part, by arguments presented in Chen and Epstein (2000). These latter authors raised doubts about the validity of Hansen and Sargent's applications of risk-sensitive control theory focusing, in particular, on the issue of time-inconsistency and an inability to accommodate the Ellsberg paradoxes. In response, Hansen et al. (2001) demonstrate, first, that their approach conforms to Gilboa and Schmeidler's (1989) multiple-priors version of uncertainty aversion and, second, that it fulfills all the necessary requirements for time-consistency. For this reason, it is worthwhile considering Hansen et al's counter arguments in some detail, despite the fact that I have to briefly renounce my discrete-time for a continuous-time treatment. This also enables me to raise some doubts about their justification for recursivity in preference orderings.

Hansen et al. (2001) commence their analysis of uncertainty aversion by setting out Gilboa and Schmeidler's (1989) representation of max-min expected utility theory with preference orderings defined over decisions c and states x:

$$\inf_{Q \in \mathbf{Q}} E_Q\left[\int_0^\infty \exp(-\delta t) U(c_t, x_t) dt\right].$$

Here, **Q** is a set of measures over c, x and the discount rate δ. The set **Q** can be specified in different ways as required by the particular problem. In Hansen et al. (2001) **Q** is specified as a statistical perturbation of a single approximating model that is *implicitly* parameterized by a single penalty variable θ. Let C denote the set of admissible control processes, $\{B_t: t \geq 0\}$ denote a d-dimensional, standard Brownian motion on the underlying probability space (Ω, **F**, P), $\{F_t: t \geq 0\}$ denote the completion of the filtration generated by this Brownian motion, and U, the instantaneous utility function. Arguing by analogy, Hansen et al. (2001, p. 5) relate this representation to what they call the *multiplier* and the *constraint* robust control problems. The *Multiplier* robust control problem is given by:

$$\sup_{c \in C} \inf_{Q} E_Q\left[\int_0^\infty \exp(-\delta t) U(c_t, x_t) dt\right] + \theta \mathbf{R}(Q)$$

subject to:

$$dx_t = \mu(c_t, x_t) dt + \sigma(c_t, x_t) dB_t$$

where $dB_t = d\hat{B}_t + h_t dt$ and m and s are restricted so that any progressively measurable control c in C implies a progressively measurable state vector process x. Let P denote the stochastic process for x_t, generated by the above stochastic differential equation (the approximating model).

The *Constraint* robust control problem, however, is given by:

$$\sup_{c \in C} \inf_{Q} E_Q\left[\int_0^\infty \exp(-\delta t) U(c_t, x_t) dt\right]$$

subject to:

$$dx_t = \mu(c_t, x_t)dt + \sigma(c_t, x_t)dB_t$$

and $\mathbf{R}(Q) \leq \eta$

Hansen et al. (2001, p. 6) relate this pair of problems to a *Lagrangian* problem:

$$\sup_{c \in C} \inf_{Q} \max_{\theta \geq 0} E_Q\left[\int_0^\infty \exp(-\delta t)U(c_t, x_t)dt\right] + \theta[\mathbf{R}(Q) - \eta]$$

$$= \sup_{c} \max_{\theta \geq 0} \inf_{Q} E_Q\left[\int_0^\infty \exp(-\delta t)U(c_t, x_t)dt\right] + \theta[\mathbf{R}(Q) - \eta]$$

with Q convex and $\mathbf{R}$ a convex function mapping onto the real line. The second equality above follows from the Lagrange multiplier theorem. With c given by the constraint robust control problem, θ must be chosen from the Lagrangian:

$$= \sup_{\theta \geq \Theta} \inf_{Q} \inf_{Q} E_Q\left[\int_0^\infty \exp(-\delta t)U(c_t, x_t)dt\right] + \theta[\mathbf{R}(Q) - \eta]$$

Having found the optimizing value θ^*, we are led to solve:

$$\sup_{c \in C} \inf_{Q} E_Q\left[\int_0^\infty \exp(-\delta t)U(c_t, x_t)dt\right] + \theta^* \mathbf{R}(Q)$$

which is the multiplier robust control problem with - $\theta^* \eta$ removed from the objective because θ^* is already given. Hansen et al. (2001, p. 7) then set out the following two claims:

Claim 4.3. Suppose that for $\eta = \eta^*$, c^*, and Q^* solve the constraint robust control problem. There exists a $\theta^* \in \Theta$ such that the multiplier and constraint robust control problem have the same solution.

Following Luenberger (1969), let $J(c, \eta)$ satisfy:

$$\inf_{Q} E_Q\left[\int_0^\infty \exp(-\delta t)U(c_t, x_t)dt\right]$$

subject to $\mathbf{R}(Q) \leq \eta$

and $J^*(\eta) = \sup_{c \in C} J(c, \eta)$. $J(c, \eta)$

is decreasing and convex in η. The same properties carry over to J^*. Given η^*, let θ^* be the absolute value of the slope of the line tangent to J^* at η^* (i.e. the negative of the slope of the sub-gradient of J^* at η^*). Thus, given θ^*, we can find a line with slope $-\theta^*$ that lies below J^* and touches J^* at one point. We let η^* denote this point of contact (existence may only be guaranteed if θ^* is in the interior of Θ).

Claim 4.4. Suppose J* is strictly decreasing, θ* is in the interior of Θ, and there exists a solution c* and Q* to the multiplier robust control problem. Then that c* also solves the constraint robust control problem for η = η* = **R**(Q).[96]

Hansen et al. (p. 8) next contend that the multiplier robust control problem has the same solution as a *recursive risk-sensitive control problem*, where $-\theta^1$ is the risk-sensitivity parameter, directing their readers to Anderson, Hansen and Sargent (2000) for confirmation of this result.

Next they define the exponential average of $E(g_t)$ by $E^*(g) = \delta \int_0^\infty \exp(-\delta t)E(g_t)dt$. A similar discounted probability measure $Q^* = Q \times M$, can be constructed for Q as in:

[96] On page 248 the equivalence between solutions for these two problems will be drawn upon to establish a local equivalence of preference odering for each problem.

$$E_Q^*(g) = \delta \int \exp(-\delta t) E(q_t g_t) dt$$

where the process { q_t} is the Radon-Nikodym derivative for Q^* with respect to P^*. The discrepancy between P and Q, given by the relative entropy measure can be manipulated as in (Hansen et al. p.5):

$$\begin{aligned}
\mathbf{R}(Q) &= \delta \int_0^\infty \exp(-\delta t) E_Q(\log q_t) dt \\
&= \delta \int_0^\infty \exp(-\delta t) E_Q\left(\int_0^t h_\tau \cdot B_\tau - \frac{|h_\tau|^2}{2} d\tau \right) dt \\
&= \delta \int_0^\infty E_Q\left(\frac{|h_\tau|^2}{2} \right) \int_\tau^\infty \exp(-\delta t) dt d\tau \\
&= \int_0^\infty \exp(-\delta t) E_Q\left(\frac{|h_\tau|^2}{2} \right) d\tau.
\end{aligned}$$

On substitution of this expression into the multiplier robust control problem we arrive at a stochastic version of a robust game given by (Hansen et al. 2001, claim 5.1, p. 8):

$$\sup_c \inf_Q \hat{E}_Q\left[\int_0^\infty \exp(-\delta t)\left[U(c_t, x_t) + \frac{\theta}{2}(h_t \cdot h_t) \right] dt \right]$$

subject to $dx_t = \mu(c_t, x_t) dt + \sigma(c_t, x_t)(h_t dt + d\hat{B}_t)$

where the operator $\hat{E}$ denotes integration with respect to the Brownian motion $\hat{B}$. Hansen et al. (2001) draw upon research by Fleming and Sougandis (1989), which establishes the requisite Bellman-Isaacs condition for the existence of a recursive solution to this robust game problem. This condition on the characteristics of the value function "…is needed to relate a solution to a date zero *commitment* game to a Markov perfect game in which the decision rules of both agents are functions of the state vector x_t.

Using the Lagrange Multiplier Theorem, Hansen et al. (2001, p. 10), then show that an implied preference ordering can be associated with each of the multiplier and constraint robust control problems. They introduce an endogenous state vector s_t, defined by:

$ds_t = \mu_s(s_t, c_t)dt$, where this differential equation can be solved uniquely for s_t, assumed to be progressively measurable, given s_0 and process $\{\, c_t\!: 0 \leq s < t\}$. This endogenous state vector is used to make preferences nonseparable over time. For this s_t, two preference orderings, $W(c;\eta)$ and $\hat{W}(c;\theta)$ associated, respectively, with each of the two robust control problems, are then defined in relation to the following additively separable representation of utility:

$$D(c) = \int_0^\infty \exp(-\delta t)u(s_t, c_t)dt$$

Namely:

$$W(c;\eta) = \inf_{\mathbf{R}(Q)\leq\eta} E_Q D(c)$$

such that for any progressively measurable c and c^*, $c^* \succeq_\eta c$ if $W(c^*;\eta) \geq W(c;\eta)$

and,

$$\hat{W}(c;\theta) = \inf_Q E_Q D(c) + \theta\mathbf{R}(Q)$$

such that for any progressively measurable c and c^*, $c^* \succeq_\theta c$ if $\hat{W}(c^*;\theta) \geq \hat{W}(c;\theta)$. Provided that $\theta \geq 0$, this multiplier preference ordering coincides with a recursive, risk-sensitive preferences ordering. Hansen et al. (2001, p. 11) observe that, although globally, the two preference orderings differ, at a given point c^* in the consumption set (i.e. at the solution to the optimal resource allocation problem), the indifference curves for each ordering are tangential, implying that each is supported by the same prices. This is established using the Lagrangian multiplier theorem. Let θ^* denote the maximizing value of θ in the following problem:

$$W(c^*;\eta^*) = \max_{\theta} \inf_{Q} E_Q D(c^*) + \theta[\mathbf{R}(Q) - \eta^*]$$

where θ^* is assumed to be strictly positive. Supposing $c * \hat{\phi}_{\theta}\, c$, then:

$$\hat{W}(c;\theta^*) - \theta^*\eta^* \leq W(c;\eta^*) \leq W(c^*;\eta^*) = \hat{W}(c^*;\theta^*) - \theta^*\eta^*$$

Thus, $c * \hat{\phi}_{\theta^*}\, c$

A final section 7 (pp. 11-6) establishes the recursivity and time consistency of these preference orderings.

3.6 Epstein and Wang's Multiple-priors Model of Knightian Uncertainty

This section of the chapter reviews Epstein and Wang's (1994) model of asset-pricing under a multiple-priors form of Knightian uncertainty. Significantly, their model is structured in recursive form to ensure that dynamic consistency obtains. Epstein and Wang (1994, p. 286) let the set of states comprising the environment be represented by Ω, a compact metric space with Borel sigma algebra B(Ω). The space of all Borel probability measures is also a compact metric space under the weak convergence topology, $M(\Omega)$. At time t, the representative decision-maker observes a particular realization $\omega_t \in \Omega$. Beliefs about the evolution of the process $\{\omega_t\}$ are presumed to accord with a time-homogenous Markov structure, but beliefs conditional on ω_t are too imprecise to be represented by a Markov *probability kernel* and its derivative singleton probability measure. Instead, beliefs are modeled by a *probability kernel correspondence*.

$\mathbf{P}$: $\Omega \to M(\Omega)$, assumed to be continuous, compact and convex-valued. Epstein and Wang suggest that for each $\omega_t \in \Omega$, $\mathbf{P}\{\omega_t\}$ can be thought of as *a set of probability measures representing beliefs about next period's state of nature* and, hence, the multi-valued nature of $\mathbf{P}$ is seen to represent uncertainty aversion. The recursive value function is defined over $\mathbf{P}$ as follows (p. 291):

$$V_t(c;\omega^t) = u(c_t(\omega^t)) + \beta \int V_{t+1}(c;\omega^t,\cdot)d\mathbf{P}(\omega_t,\cdot)$$

Epstein and Wang's departure from the traditional expected utility model is accomplished through a weakening of Savage's *Sure-Thing* principle. Savage's *single prior* representation of preferences and probability beliefs is replaced by a *multiple-priors* representation. Epstein and Wang contend that this recursive model represents a sensible intertemporal extension of the atemporal model originally conceived by Gilboa and Schmeidler (1989). For any bounded Borel-measurable f: $\Omega \to \Re$ and for any set $\mathbf{P} \subset M(\Omega)$ (p. 287):

$$\int_\Omega f dP \equiv \inf\left\{\int_\Omega f dm : m \in P\right\}$$

and thus,

$$P(A) \equiv \inf\{m(A) : m \in P\},\ A \in B(\Omega).$$

In particular, if $\mathbf{P} = \mathbf{P}(\omega)$ for some ω, then $\mathbf{P}(\omega, A) \equiv \inf\{m(A) : m \in \mathbf{P}(\omega)\}$ and for continuous f:

$$\int f(\cdot)d\mathbf{P}(\omega,\cdot) \equiv \int f d\mathbf{P}(\omega) \equiv \min\left\{\int f dm : m \in \mathbf{P}(\Omega)\right\}.$$

The last integral is a generalization of the Choquet integral, under which the map $A \to P(A)$ defines a capacity. As in Epstein and Zin (1989), consumption, utility and price processes are each assumed to lie in the complete normed space **D** (Epstein and Wang, p. 290). Under standard regularity conditions, Epstein and Wang demonstrate both existence of utility and dynamic consistency: the latter holding in a strong rather than a weak sense if **P** has full support (see p. 288 for a discussion of this assumption, which is made to avoid unnecessary notational clutter).

$\{V_t\}$ is *dynamically consistent* if for all $\omega_1 \in \Omega$, c' and c in **D** and $T \geq 1$, $V_1(c'; \omega_1, \cdot) > V_1(c; \omega_1, \cdot)$ if (p. 292):

i. $c'_t = c_t$ for $t = 1, \ldots, T-1$,
ii. $V_T(c'; \omega_1, \cdot) \neq V_T(c; \omega_1, \cdot)$, and
iii. $V_T(c'; \omega_1, \cdot) \geq V_T(c; \omega_1, \cdot)$ on Ω^{T-1}

$\{V_t\}$ is *weakly* dynamically consistent if properties (i) to (iii) only imply that $V_1(c'; \omega_1, \cdot) \geq V_1(c; \omega_1, \cdot)$. One of the limitations of this model is that risk aversion is not well defined unless probabilities exist that can be used to define actuarial fairness. Because their favoured model of utility for decision-making under uncertainty is non-differentiable in the Gâteaux sense, Epstein and Wang are obliged to examine utility supergradients[97]. Fortunately, since $V_t(.;\omega)$ is concave, it does possess one-sided Gâteaux derivatives, derived through the application of an appropriate "envelope theorem," and given by (p. 295):

[97] Epstein and Wang (1994, p. 295) explain that non-differentiability holds because utility is defined via a (non-differentiable) point-wise minimum of functions (unless the probability kernel correspondence collapses into a probability kernel).

$$\frac{d}{d\xi}V(e+\xi h;\omega)\Big|_{0^-} = u'(e^*(\omega))h_1 + \beta\min\left\{\int u'(e^*)h_2 dm\!: m \in Q(\omega)\right\},$$
$$\frac{d}{d\xi}V(e+\xi h;\omega)\Big|_{0^-} = u'(e^*(\omega))h_1 + \beta\max\left\{\int u'(e^*)h_2 dm\!: m \in Q(\omega)\right\}.$$

Here, $e \in \mathbf{D}$ is a strictly positive, Markov, time-homogenous consumption process. Epstein and Wang argue that it is sufficient to consider the effect on utility of perturbations in "today's" and "next period's" consumption only, represented by the change from e to $e + \xi h$, where $\xi \in \Re^1$ and $h = \{h_t\}_1^\infty$ is a continuous real-valued process, such that $h_t \equiv 0$ for all $t \neq 1,2$, $h_1 \in \Re$ and $h_2 \in C(\Omega)$. Moreover, the convex-valued and compact-valued correspondence Q: $\Omega \rightarrow \mathbf{M}(\Omega)$ is defined by:

$$Q(\omega) = \left\{m \in \mathbf{P}(\omega)\!: \int V^* dm = \int V^* d\mathbf{P}(\omega)\right\}$$

where $V^*(\omega) \equiv V(e,\omega)$, $\quad \omega \in \Omega$.

A *feasible* intertemporal consumption and portfolio plan for period t is defined in the conventional manner for n securities, presumed to be traded in a competitive market at prices $q_{i,t}$ and paying dividends $d_{i,t}$, $i = 1,\ldots,$n.. On the basis of a perturbation argument and the application of both Fan's theorem and the maximum theorem, the authors derive the following system of Euler inequalities which must be satisfied in equilibrium for all (t, ω^t) (see Epstein and Wang, pp. 297-9 for details):

$$\min_{m \in Q(\omega_t)} \max_i \left\{\left|\beta E_m\left[\frac{u'(e_{t+1})}{u'(e_t)}(q_{i,t+1} + d_{i,t+1})\right] - q_{i,t}\right|\right\} = 0$$

Existence and the characterisation of equilibria are established in Epstein and Wang's Theorem 2 (p. 300) under the assumption that **P** satisfies the strict Feller property (a general form of continuity appropriate to probability kernel correspondences such as the mapping Q). Satisfaction of the same assumption, in Theorem 3 (p. 301) enables the authors to establish that (a) the set of all equilibria, ε, is a closed and connected subset of $\mathbf{D}^n$; and (b) for each i, the equations

$$\overline{q}_{i,t} = \beta \max_{m \in Q(\omega_t)} E_m \left\{ \frac{u'(e_{t+1})}{u'(e_t)} \left(\overline{q}_{i,t+1} + d_{i,t+1} \right) \right\} \text{and}$$

$$\underline{q}_{i,t} = \beta \min_{m \in Q(\omega_t)} E_m \left\{ \frac{u'(e_{t+1})}{u'(e_t)} \left(\underline{q}_{i,t+1} + d_{i,t+1} \right) \right\}$$

have unique solutions in **D** denoted $\overline{q}_i$ and $\underline{q}_i$ respectively. These solutions satisfy the condition that for any $q \in \varepsilon$ and for any i and t (p. 301):

$$\underline{q}_{i,t} \leq q_{i,t} \leq \overline{q}_{i,t} \quad \text{on } \Omega^t$$

Moreover, given any i and t, and any $\varepsilon > 0$, there exist q^1 and q^2 in ε such that:

$$q^1_{i,t} \leq \underline{q}_{i,t} + \varepsilon \quad \text{and } q^2_{i,t} \geq \overline{q}_{i,t} - \varepsilon \quad \text{on } \Omega^t$$

Finally, the ith security price is indeterminate if and only if for some t:

$$\underline{q}_{i,t} \neq \overline{q}_{i,t}$$

in which case $\{q_t\colon q \in \varepsilon\}$ is an uncountably infinite set.

Thus, equation 3.5.6 provides bounds for the equilibrium price of the security, equation 3.5.7 establishes that these bounds are tight and equation 3.5.8 provides the necessary and sufficient conditions for price indeterminacy in the model. Epstein and Wang associate this feature with excess volatility, a phenomenon observed by New Keynesians scholars such as Shiller (1989, 1990, 1995), and Campbell and Shiller (1991), and explained by investor irrationality and faddishness. Instead, Epstein and Wang argue the following:

> Intuitively, we would expect a link between indeterminacy of asset-prices and intertemporal price volatility. This intuition can be confirmed in the special case of "i.i.d." beliefs," that is, where $P\{\omega_t\}$ is independent of ω, in which case the correspondence Q is also constant. Hence, for any security with time-homogenous dividend process, if the price of the security is determinate, then it must be constant (across time and states). Consequently, any fluctuation in price is a reflection of indeterminacy. More generally, the link between indeterminacy and volatility can be thought of in the usual way in terms of the existence of "sunspot equilibria." That is, if the selection $\{\pi_t\}$ from Q is made to depend on a "sunspot" or "extrinsic" variable, then the corresponding equilibrium price process will also depend on that variable (p. 302).

Finally, in their Theorem 4, Epstein and Wang characterize equilibria in a manner which establishes an "empirical content" for the restrictions imposed on the stochastic discount factors utilized in their chosen model of asset-pricing. This is achieved by establishing the existence of a *measurable selection* ξ such that, (pp. 302-3):

$$\xi_t : \Omega' \to \mathbf{M}(\Omega) \text{measurable}, \quad \xi_t(\omega',\cdot) \in Q(\omega_t)\,\forall\, \omega' \in \Omega'$$

such that q is in ξ if and only if q is in $\mathbf{D}^{\mathbf{n}}$ and for some $\{\xi_t\}$, q satisfies:

$$q_{i,t} = \beta E_{\xi_t(\omega',\cdot)}\left\{\frac{u'(e_{t+1})}{u'(e_t)}\left(q_{i,t+1} + d_{i,t+1}\right)\right\},\ \forall t, i\,.$$

Epstein and Wang denote by $z_{t+1}(\omega^t, \cdot):\Omega\rightarrow\Re_+$ the Radon-Nikodym derivative of $\xi_t(\omega^t, \cdot)$. They then adopt the assumption that the actual evolution of $\{\omega^t\}$ is described by a probability kernel π^*, but rather than presume that agents know this kernel exactly it is merely presumed that Q is absolutely continuous with respect to π^* .(i.e. **P** is absolutely continuous). Then the asset-price is given by:

$$q_{i,t} = \beta E_{\pi^*_t(\omega_t,\cdot)}\left\{\frac{u'(e_{t+1})}{u'(e_t)} z_{t+1}\left(q_{i,t+1} + d_{i,t+1}\right)\right\} \text{where}$$
$$\int_\Omega z_{t+1} d\pi^*(\omega',\cdot) \equiv 1, \text{and } d\xi_t(\omega',\cdot) \equiv z_{t+1}(\omega',\cdot) d\pi^*(\omega_t,\cdot)$$

Thus the empirical content of Epstein and Wang's asset-pricing model is represented by the above restrictions imposed on the stochastic discount factors. They note that the standard Lucas based rational expectations model imposes the far stronger restriction that $z_{t+1} \equiv 1$.

In their 1994 paper, Epstein and Wang sidestepped some of the more complex problems of measurability "by restricting attention to consumption processes that are continuous as functions of the state." However, in their subsequent paper published in the following year, they concede that there is no economic justification for restricting agents to continuous processes if the state space is not discrete (Epstein and Wang , 1995, p. 42). They show that more general specifications for the representation of beliefs give rise to economies in which equilibrium price processes are *discontinuous* as functions of the state, even though consumption and dividend processes are continuous. In other words, "booms" and "crashes" can occur "without apparent changes in the fundamentals[98]".

[98] They formally achieve this generalisation by abandoning the characterisation of Mackey lower semi-continuity for utility defined for consumption processes over the set of states Ω, which is a compact metric space with Borel σ-algebra Σ denoted by B(Ω, Σ). The failure of Mackey lower semi-continuity implies that supporting prices (as characterised by the utility supergradients) lie in a finitely additive signed measure space $ba(\Omega)$, rather than in the countably additive signed measure space $ca(\Omega)$, let alone M(Ω), the space of set functions on Σ that are *probability measures*. The major implication of this generalisation, based on Schmeidler's (1989) model of preferences, is that the existence of a risk-neutral probability measure is no longer assured. Formally, the probability kernel correspondence P is *capacity-based* if: (i) P(ω,.) is a convex capacity for each ω; and (ii) P(ω)= $\{m \in M(\Omega): m(A) \geq P(\omega, A)\ \forall A \in \Sigma\}$; which for any $f \in B$, yields the Choquet integration formula

$$\int_\Omega f d\mathbf{P}(\omega) = \int_0^\infty \mathbf{P}(\omega, \{f \geq t\}) dt,$$

and for such a capacity-based **P**, and for a given endowment process e^*, suppose that $\inf_\Omega V(e,\omega)$ is not attained and that for some ω $\sup_{\alpha > \inf V(e,\omega)} (\omega, \{V(e,\omega) \geq \alpha\}) < 1$, then every equilibrium for any economy (e^*, d) fails to admit a risk-neutral measure representation. However, the practical implications of this theorem are difficult to establish "first, because empirical discrimination between finite and countable additivity requires an infinite set of data, and, second, because any charge in $Q(\omega t)$ can be approximated arbitrarily closely $\sigma(ba$, B) by a probability measure in P(ω_t)" (Epstein and Wang, 1995, p. 58). Here, $\sigma(ba$, B) is the weak topology on the finite additive signed measure space for which all linear functionals in B are continuous.

CHAPTER FOUR — ESSENTIAL ASPECTS OF A MONETARY PRODUCTION ECONOMY

4.1. Introduction

In this Chapter I establish a framework for thinking about the relationship between the real economy and monetary and financial markets. As a prelude to this discussion I briefly outline the distinction between production-based, consumption-based, and complete general equilibrium models of financial markets.

I raise four additional matters. First, I examine Vercelli's (1991) arguments in support of minimax control as a mechanism for accommodating what he chooses to call *k*-uncertainty. Specifically, I review his analysis of Heiner's (1983) notion of the gap that opens between an agent's *competence* to solve a problem and the *difficulties* involved in deciphering what is going on in a complex environment and his integration of Heiner's notion of *atemporal* flexibility and Jones and Ostroy's (1984) notion of *intertemporal* flexibility within a two-stage decision-process. I caution that Jones and Ostroy operate with a conception of *increasing risk* rather than *k*-uncertainty.

Second, I make the claim that non-linearities and complex dynamics are inescapable aspects of economic behaviour, for which rigorous justification can readily be provided through the use of microfoundational arguments that are so dear to New Classical theorists. For this reason, I contend that exponents of robust control should develop and apply techniques for dealing with complex and non-linear rather than linear systems (e.g. as in chaotic or non-linear control theory).

Third, I contend that variations in financial instability—reflecting movements along the investment continuum that ranges from hedge, through speculative to Ponzi financial positions—should be accommodated (in a control sense) through the use of adaptive techniques that trace the path-dependent trajectories of critical parameters.

Fourth, I review Vercelli's claim that the early rational expectations literature imposed a separation between cognitive activity or decision-making and predictive activity, to determine how this position is modified in the new robust control framework. Despite the fact that a more sophisticated view of the relationship between knowledge and uncertainty has been adopted—for both economic agents and econometricians or calibrators - I argue that a new, but still *recognizable*, notion of separation can be identified.

4.2. Finance Theory and arbitrage-based, versus consumption-based or production-based modeling

At this juncture it is worthwhile to contemplate the position taken, in relation to the complete Keynesian fix-price tableaux, by various authors whose models have become definitive benchmarks in the canon of modern finance theory. To this end, the work of Merton (1973a), Lucas (1978), Breeden (1979) and Brock (1982) will briefly be examined.

Lucas (1978) derives a pure-exchange model which prices assets that represent claims to a future stream of stochastic dividends that take the form of endowments of a depreciable consumption good. By assumption, returns that are not consumed cannot be stored. Merton (1973a) and Breeden's (1979) models fall into the class of Consumption-Based Asset-pricing Models (CBAPM's). This class of model relates asset returns to marginal rates of substitution inferred from the relationship between the representative consumer's first order conditions for optimal intertemporal consumption demand and observed patterns in consumption data (or state variables thought to determine consumption). Cochrane (1991) has identified the strict analogy obtaining between this class of model and another class of Production-Based Asset-pricing Models (PBAPM's). This latter class of model relates asset returns to marginal rates of transformation inferred from the relationship between the representative producer's first order conditions for optimal intertemporal investment demand and observed patterns in investment data. The testable content of CBAPM's is a restriction on the joint stochastic process of consumption and returns and for PBAPM's, a restriction on the joint stochastic process of investment and returns.

Cochrane observes that if the return process is modeled and predictions are made about consumption we arrive at a theory of consumption behaviour (e.g. like the permanent income hypothesis). Alternatively, if we model the consumption process and make predictions about returns we arrive at a consumption-based theory of asset-pricing. Cochrane goes on to identify the respective analogies with production-based analysis. Modeling the return process enables predictions to be made about investment (e.g. like the q-theory of investment). Alternatively, if the investment process is modeled and predictions are made about returns we arrive at a production-based theory of asset-pricing (Cochrane 1991, pp. 209-10)[99].

Each class of model, however, is partial in nature. For example, many CBAPM's treat the consumption stream as an endowment which may be lent to others, but cannot be stored or used for investment purposes. Moreover, Cochrane indicates that for PBAPM's, the restrictions between asset returns and production variables will continue to hold no matter what consumers do, while for CBAPM's, the restrictions between asset returns and consumption variables will continue to hold for any technology. However, he suggests that there are good reasons for pursuing research specifically into the PBAPM because it "...ties asset returns directly to production variables such as output and investment, whose relatively large movements characterize economic fluctuations, rather than to the relatively smooth nondurable and services consumption series." (Cochrane 1991, p. 212). In addition, because firms are larger than consumers, difficulties associated with transactions and information costs or indivisibility of goods that plague CBAPM's may be more attenuated for PBAPM's.

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[99] Under the assumption of complete markets, Cochrane derives an expression relating asset-prices and implied contingent claim prices which "...describes a linear space in which all asset returns must lie to prevent arbitrage from portfolio formation" (p.216). Noting that it has the same form as another expression that was derived from the first order conditions for optimal investment, he interprets these non-arbitrage conditions to imply that the firm should adjust investment until no arbitrage possibilities remain by forming portfolios of asset returns and the investment return. In other words, the firm should adjust investment until the investment return equals the return derived from a mimicking portfolio that is obtained by trading a portfolio of assets whose payoffs across states of nature at date t+1, mimic exactly those of the investment return. Then:

> "[i]f the price of this portfolio is greater than 1, the managers should short the portfolio, invest one dollar of the proceeds, pay off the mimicking portfolio with the investment return, and make a sure profit (and vice versa)" (p. 211).

Whereas the production-based model derived by Cochrane is expressed as a relation between returns, much of the consumption and investment literature is based on formulations of the relationship between marginal costs and the present value of future streams of benefits. Again, Cochrane favours the former approach, first, because "...returns emphasize high frequency aspects of the data that the models may be better able to capture in the presence of slow moving and unobserved changes in technology" (p. 218), and second, that most present value models exclude time-varying risk premia for tractability, relating changes in consumption or investment solely to variations that are induced by changes in the interest rate[100].

Brock's (1982) model is the only one that can claim to be derived from a complete general equilibrium optimisation framework. Brock brings together an arbitrage pricing model of asset-prices that has been integrated together with a Lucas-type model of asset-pricing and a stochastic growth model based on a single, representative agent (consumer-producer-investor). Effectively, this model combines both the CBAPM and PBAPM into a unified framework.

However, Brock's model falls firmly within the real business cycle tradition. Asset-prices are determined through arbitrage pricing over a dividend stream, whose stochastic characteristics ultimately reflect exogenously given productivity shocks to a conventional neoclassical

[100] In asset-pricing models, risk premia are generally related to the second moments of the diffusion processes governing prices or returns. Because prices fluctuate with arrival of new information and information arrival rates are time-varying this implies that the variance of rates of return on stocks will be time-varying. See Bodie *et al.* 1996; Chapter's 12 and 29, for a concise overview of the extensive literature on time-varying volatility and its implications for the efficient market hypothesis. The literature on excess volatility of asset returns suggests that the arrival of news is not a sufficient explanator for observed volatility.
Cochrane goes on to argue that his PBAPM escapes certain anomalies which trouble other APM's. While CCAPM models have difficulty explaining the equity premium and risk free rate puzzles, present-value derived, Q-theoretic models of investment require that an unreasonable proportion of GNP is absorbed in capital stock adjustment costs. In comparison, his single-period estimates of adjustment costs are relatively modest.

production function. In fact, one of Brock's main concerns was to deliberately trace and quantify this specific causal linkage between the asset-pricing and stochastic growth components of his intertemporal model. There is no recognition of effective demand or of the possibility that negative financial outcomes can have adverse impacts on the real economy. The question that naturally arises is whether the new macroeconomics and finance theory grounded in a robust control framework, can succeed in this endeavour. Not wishing to pre-empt arguments that I raise in the next chapter, I merely wish to observe that, apart from the imposition of risk-sensitive value functions and stochastic uncertainty constraints (the latter of very limited character) Hansen and Sargent's robust control approach to asset-pricing adheres closely to the structure of Brock's model. A simple stochastic growth model is solved *independently* and then interfaced with a Lucas "tree" model to determine asset-prices (although only a single asset—a tree—is identified and priced, the model can readily be extended to account for a multiplicity of assets).

4.3. The Macroeconomic Context of Monetary and Finance Theory

I wish to argue that a number of inter-related features of real world financial activity and monetary behaviour should lead researchers to the point where they embark on a profound questioning of current control and filtering applications in the economic sphere. This is not a question of complexity for the sake of complexity. It is rather a question of how social agents respond to the real world presence of complexity in their key decisions about investment, borrowing and lending, and portfolio choice. Moreover, it is a question of recognizing the macroeconomic implications of the fact that decisions on the part of households about savings and consumption, financial institutions about lending and investment in financial assets, and firm decisions about capital budgeting are not coordinated by some invisible hand.

Chiarella and Flaschel (1999, 2000) argue that debates in macroeconomics have been overshadowed by arguments for the desirability of sound *microeconomic foundations*. To some extent, this thesis explores the microfoundational issues. However, Chiarella and Flaschel also look at the other side of the coin: identifying a similar need for sound *macroeconomic foundations*:

> First in order to know what is to be micro-founded and because in our view macroeconomics' most important subject is the investigation of interdependence, and not so much the detailed of all possible types of optimizing behaviour which often has not much in common with actually observed situations (Chiarella and Flaschel, 1999, p.6).

At the same time, they also question the notion that agents are endowed with either sufficient information about the model structure or sufficient computational ability "...to form expectations in a way that is currently referred to as 'rational' in a large body of the literature" (Chiarella and Flaschel, 1999, p. 4). Consequently, they reject what they term the "jump-variable" technique employed in solving dynamic economic models under rational expectations. Nevertheless, they acknowledge that their treatment of anticipated future events must be a topic for further research, especially "...to incorporate the effects of heterogeneity of expectations and of learning on the part of the various economic agents of our models" (Chiarella and Flaschel, 1999, p.4).

Elsewhere, they identify the literature on Adapative Rational Expectations Dynamics as one possible vehicle for handling these expectational concerns. I would argue that the use of risk-sensitive and robust control is an alternative route to accommodate forward looking expectations formation under uncertain conditions, which Chiarella and Flaschel could have considered. Nevertheless in section 4.5, I argue that to be truly useful this kind of research must be extricated from its confinement within the conventional representative-agent framework.

4.4. Vercelli on Minimax optimisation under uncertainty

In his book *Keynes After Lucas* (1991), Vercelli discusses Jones and Ostroy's notion of *liquidity as flexibility* in some detail, relating it to Heiner's arguments that rising uncertainty will lead to a narrowing rather than a widening of behavioural repertoires (Heiner, 1983). Heiner's justification for this pessimistic position is predicated on the notion that increasing uncertainty reduces the likelihood of a correct selection of probabilities and, thus, increases the likelihood of an erroneous selection. Because the reliability ratio (pertaining to the probability of acting correctly given the state of nature) would fall with increasing uncertainty this would lead to a shrinking in the repertoire of sufficiently reliable actions and the growing predominance of "rule-governed" rather than optimal behaviours. Vercelli's integrated approach combines what he calls Heiner's *atemporal* flexibility with Jones and Ostroy's (1984) *intertemporal* flexibility by embedding both within a two-stage decision process (Vercelli 1991, section 5.3. pp. 79-82; Appendix 5A, pp. 85-90).

In the first stage, the agent defines a repertoire of actions which are sufficiently reliable to be considered for a choice (i.e. formally, by restricting the set of epistemically possible probability distributions Φ to a subset of sufficiently reliable probability distributions Φ *, which Vercelli (1991, p. 86) calls the "epistemic repertoire"). Heiner's reliability condition applies to decisions at this first stage. Over time a rise in uncertainty will lead to a narrowing of the atemporal repertoire of actions—in this sense reducing flexibility. In the second stage, an agent will make a choice within the defined repertoire, but will take account of the need for intertemporal flexibility. Formally, the agent restricts the set of possible actions A to a subset A* of sufficiently reliable actions, which Vercelli calls the "pragmatic repertoire", before choosing the action that best suits his or her objectives (Vercelli, 1991, p. 87). At this stage, an increase in the degree of uncertainty will encourage the choice of a more intertemporally flexible action. Although agents must now work within a smaller stage one repertoire, they will make a choice from within this repertoire of actions (or rules) which leaves open a wider set of possible repertoires or options in future periods than was desired when conditions were more certain.

In his discussion about the requirements for traditional optimising behaviour, Heiner (1983) makes a distinction between the competency of the agent and the difficulty of the problem. When agents are unable to decipher all the complexities of the environment, a gap opens up between competence (C) and difficulty (D). Vercelli (1991, pp. 78-9) suggests that a systematic C-D gap is unavoidable when the following conditions break down:

1. the decision-maker is confronted by a stationary stochastic process,
2. the process has persisted long enough for the decision maker to fully adjust to it,
3. the process is ergodic in the sense that it will converge over time towards a stationary state.

Lucas (1986) has argued that the first two conditions are necessary for any possibility of regularity in economic behaviour to be detected and analysed. Seen in this light, the extreme Keynesian position on unmeasurable risk is therefore cast as an atheoretical limbo of skepsis and analytical impotence. In contrast, Heiner (1983) argues that in the absence of a positive C-D gap, the behaviour of a perfectly rational agent, when confronted by risk, would be extremely irregular and unpredictable, because optimally-grounded actions would change in response to every minute perturbation in the environment. Accordingly, Heiner suggests, paradoxically, that predictability and regularity would only emerge with some degree of uncertainty. His justification for this view is predicated on the notion that increasing uncertainty reduces the likelihood of a correct selection of probabilities and increases the likelihood of an erroneous selection. Because the reliability ratio would fall with increasing uncertainty, this would lead to a shrinking in the repertoire of sufficiently reliable actions and the growing predominance of "rule-governed" rather than optimal behaviours.

Vercelli (1991, pp. 80-1) contends that both Heiner's and Lucas's positions are special cases of a more general framework: unpredictable behaviour can occur under two additional conditions; first, when there is a positive C-D gap despite the presence of a stationary process; and second, when there is a zero C-D gap in the presence of a non-stationary process. However, this somewhat trivial point is less significant than his efforts to integrate the apparently contradictory arguments of Heiner and Jones and Ostroy to which I now turn.

In his own terms, Vercelli (1991) summarises the arguments of Jones and Ostroy as a defence of the notion that an increase in the degree of k-uncertainty will encourage agents to make decisions which promote structural flexibility. This is achieved in the sense that a larger set of actions will remain feasible in following decision periods. He contrasts this intertemporal notion of flexibility with Heiner's view that a rise in k-uncertainty will lead to a narrowing of actions and behavioural repertoires.

Vercelli observes that the process of making the rigidity or flexibility of behaviour endogenous to the decision-making process provides a rigorous foundation for a distinction between *historical* time and *logical* time in economic theory (Vercelli, 1991, p. 89). The fact that a process of endogenous change determines the set of viable options and repertoires open to economic agents introduces an inescapable element of path-dependency or historicity into the analysis.

Vercelli (1991, p. 87) contends that the "optimal choice" of an action from amongst the pragmatic repertoire of sufficiently reliable actions operates in accordance with what he calls the Maximin Criterion for Expected Utility (MMEU). Expected utility is calculated in the conventional way for each alternative a_i in the pragmatic repertoire *and* for each probability distribution in the epistemic repertoire—then, the alternative with the greatest minimal utility expected utility with respect to each probability distribution within the epistemic repertoire is selected.

In the limit the decision situation is characterised by no information at all, and all probability distributions over all possible states have equal epistemic reliability. Vercelli cites Gädenfors and Sahlin (1982) who argue that "...in such a case, the minimal expected utility of an alternative is obtained from the distribution which assigns the probability 1 to the worst outcome of the alternatives" (Gädenfors and Sahlin, 1982, p. 373). At the other extreme, when only one probability distribution is included in the epistemic repertoire, decision-making can be represented by the conventional Bayesian approach to decision-making under risk.

However, Vercelli fails to acknowledge that Jones and Ostroy's model is based on *increasing risk* rather than *Keynesian uncertainty.* Following Marshak and Miyasawa (1968), Jones and Ostroy define an information structure (Π, q) over two possible probability vectors. The first of these vectors, Δ_S, is defined on S the set of states:

1. $\Delta_S \equiv \left\{\pi = \left(\pi_s\right): \pi_s \geq 0, \sum \pi_s = 1\right\}$

with elements $\pi \in \Delta_S$ representing a belief about the possibilities in S. The second of these, Δ_Y, is defined on Y, an index set of messages or observations:

2. $\Delta_Y \equiv \left\{q = (q_y) : q_y \geq 0, \sum q_y = 1\right\}$

with elements y. The set of possible conditional distributions of s given y is $\Delta^Y{}_S \equiv (\Delta_S)^Y$ with elements $\pi(y) \in \Delta_S$ denoting a belief (about the possibilities in S) conditional on observing the message y. Π is a matrix whose columns are $\pi(y)$, $y \in Y$, hence, $(\Pi, q) \in \Delta^Y{}_S \times \Delta_Y$ with a mean $\bar{\pi} \equiv \sum q_y \pi(y)$, which represents the prior belief for (Π, q) (i.e. before any message is received). A partial ordering of belief structures, denoted by $(\Pi, q) \succeq (\Pi', q')$ is defined by the condition that:

$$\sum q_y \Phi(\pi(y)) \geq \sum q'_y \Phi(\pi'(y))$$

for all convex functions Φ: $\Delta_S \rightarrow \Re$. Letting $\mathbf{B}$ be a finite set of actions and $u(b, s)$ be a payoff function defined on $\mathbf{B} \times \mathbf{S}$, the above expression is equivalent to the following definition of (Π, q) as more valuable than (Π', q') if:

$$\sum_y q_y \max_{b \in \mathbf{B}} \sum_s \pi_s(y) u(b, s) \geq \sum_y q'_y \max_{b \in \mathbf{B}} \sum_s \pi'_s(y) u(b, s)$$

for all bounded $u(b, s)$.

The authors distinguish this notion of increasing uncertainty from the more conventional notion of increasing risk as espoused by Rothschild and Stiglitz (1970), which assumes that beliefs are determined in relation to the states associated with the realisation of a random variable , $x(s)$. In this case, π is more risky than π' if for all convex functions $\psi: \Re \rightarrow \Re$,

3. $\sum \pi_s \psi(x(s)) \geq \sum \pi'_s \psi(x(s))$

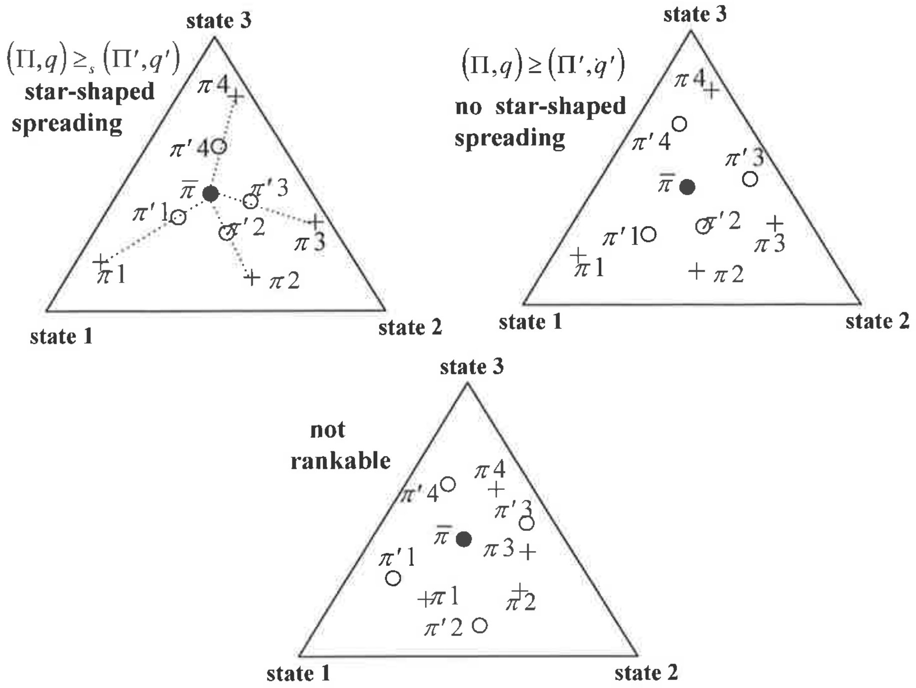
Yet another interpretation of the information structure (Π, q), proffered by Jones and Ostroy, is that it represents the outcome of an "experiment" where y will be observed with probability q_y, after which one's beliefs about S will be $\pi(y)$. In this case, (Π, q) is more informative than (Π', q') if there exists a non-negative $n \times n$ matrix M with columns summing to 1, where n is the number of elements in Y, such that:

4. $\Pi' = \Pi M$ and $q = Mq'$

In practice, Jones and Ostroy adopt a more restricted version of the preference ordering $\underline{\phi}$ indicated by (Π, q) $\underline{\phi}_S$ (Π', q'), which obtains if $\Pi q = \Pi' q'$, $q = q'$ and there exists $0 \leq \lambda_y \leq 1$ for each $y \in Y$ such that $\pi'(y) = \lambda_y \pi'(y) + (1 - \lambda_y)\, \overline{\pi}$.

This ordering they term a "star-shaped spreading" of (Π', q'), as indicated in the following diagram:

Fig. 20: JONES & OSTROY (1984) ON SPREADING OF PROBABILITIES



Only the first of these belief structures (plotted in the unit simplex) depicts star-shaped spreading; the second depicts the case where $(\Pi,q)\succeq(\Pi',q')$ without star-shaped spreading; and the third an information structure which cannot be ranked by either of the two orderings.

In a very Keynesian manner, the authors note that from an economic perspective there are two distinct sources of increased variability of beliefs. First, "…there may be an improvement in the information content of available observations" (e.g., through more accurate forecasts or better experiments. Second, "…there can be a change in the confidence with which prior beliefs are held" in relation to both the amount and relevance of any new, relative to existing information.

In the context of a sequential decision problem Jones and Ostroy define the flexibility of an initial position $a \in A$, relative to the opportunity to choose a second position $b \in B$, in period 2, after which time, in period 3, the state of nature s is revealed. The consequence for the individual agent is described by a payoff function $f: A \times B \times S \rightarrow \Re$, which can be decomposed into three components:

$$f(a,b,s) = r(a,s) + u(b,s) - c(a,b,s)$$

where $r(a,s)$ is the direct return on the selected first period position, $u(b,s)$ is the return on the second period action, and $c(a,b,s)$ is the cost of switching from a to b. Flexibility is defined in relation to the characteristics of the switching cost function $G: A \times S \times \Re \rightarrow 2^B$—a mapping used to define a partial ordering $\underline{\phi}_F$ on A—which is defined by:

$$G(a,s,\alpha) \equiv \{b : c(a,b,s) \leq \alpha\} \text{where}$$
$$G(a,s,\alpha) = \varnothing \, \forall \alpha < 0, \, (a,s) \in A \times S$$
$$\text{and } \exists g : A \rightarrow B \text{ such that } g(a) \in G(a,s,0) \; \forall (a,s) \in A \times S$$

The first of the above conditions imposes non-negative switching costs, while the second establishes the existence of a set of zero switching cost alternatives. Consequently, position a is more flexible than position a' when, for all $\alpha \geq 0$ and $s \in S$,

$$G(a,s,\alpha) \supset G(a',s,\alpha) \setminus g(a')$$

This expression is interpreted to mean that $a \, \underline{\phi}_F \, a'$ if the set of positions attainable from a always contains the set attainable from a', excluding its zero cost option $g(a')$.

With these theoretical prerequisites and assumptions, Jones and Ostroy construct a three period, two-choice sequential decision problem, which gives rise to two related partial orderings: one ranking information structures in accordance with the variability of beliefs, and the other ranking initial positions according to flexibility. For this problem, an optimal strategy is defined by the choice of an initial position, a, and a set of second period positions, $\{b_y\}$, conditional on the observation $y \in Y$ received, which maximises the expected total payoff. This is expressed recursively in accordance with the maximum principle of dynamic programming

Considerable work must be done to develop models of *liquidity as flexibility* under uncertainty rather than risk. One outcome might be a stronger justification for Keynes's (1936) arguments about the speculative demand for money. Jones and Ostroy's analysis is based on transactions-costs that are incurred by exchanging illiquid assets. By definition, money has zero transactions-costs: this fact is the source of the demand for money in the face of uncertainty.

One paper that has incorporated uncertainty of the Choquet form into a Jones and Ostroy framework is that of Klaus Nehring (1999). Nehring builds on the approach, first formalized by Kreps (1979), which characterizes a class of preferences that rank opportunity sets in terms of their Expected Indirect Utility (EIU). Nehring argues that preferences over opportunity sets may exhibit a *preference for flexibility* due to implicit uncertainty about future preferences that may be influenced by unforeseen contingencies. He makes use of the conjugate Möbius inversion, a technique that yields an explicit characterization of the class of EIU-maximizing utility functions over opportunity sets. The preference for flexibility is formally represented by a new axiom of Indirect Stochastic Dominance (ISD) that expresses a preference for "more opportunities in expectation". He also shows that EIU maximisation can be viewed as Choquet integration with respect to a " plausibility function". This technique has also been exploited by other researchers: notably, Gilboa and Schmeidler (1995).

In his discussion of the earlier work of Jones and Ostroy, Nehring notes that these authors relate the value of flexibility to the amount of information received. In the context of a similar two-stage decision, Nehring identifies three forms of uncertainty arising under his EIU approach: that relating to the opportunity set which results from a particular current choice, that associated with new information on the relative value of alternative final choices, and that relating to the final choice itself. His paper focuses on the first of these forms. In addition, he compares his approach to that of Sarin and Wakker (1992) who apply a Cumulative Dominance axiom that performs a similar role to that of ISD in providing and axiomatic basis for Choquet expected utility theory in the Savage framework. However:

> [c]umulative dominance "calibrates" explicitly ambiguous acts in terms of equivalent unambiguous ones and is responsible for the rank-dependent character of the integration, while neutral to the nature of the capacity. By contrast, ISD pertains to the implicit uncertainty of future choice and preference, and in effect singles out a particular type of capacity, namely, a plausibility-function..." (p.113)

This chapter is predicated on the notion that a better mechanism for generalization of Tobin's analysis can be found in formal representations of uncertainty aversion. In this case, uncertainty aversion could coexist with indeterminacy of equilibrium, ultimately associated with movements in the point of effective demand. These circumstances could very well ensure that pessimistic expectations about future levels of activity were rational and well-founded, despite the passage of lengthy time periods over which it might seem that such pessimism had been misplaced.

4.5. The point of effective demand, nominal (non-indexed) contracts and quantity-constrained rationing

One of the most questionable of underpinnings for many of the recent applications of LQG and robust, risk-sensitive control, is undoubtedly the representative agent assumption. Typically, in general equilibrium models of asset-pricing the world is populated by a set of clones: each one replicating a single agent who combines in one entity the decisions of producer-consumer-borrower-lender. In such a world, what is debt for one agent is an asset for another, and what is food put aside for later consumption for one agent becomes 'borrowed' seed-corn for immediate planting for another. In this sort of world, all contracts are written in real terms (for all intents and purposes, equivalent in outcomes to a nominal world that has been governed through the issue of fully indexed contracts). Therefore, it is extremely difficult to represent problems of coordination failure that would otherwise manifest themselves in the form of quantity-constraints (i.e. constraints affecting inter-related expenditure and income generation in closely linked markets such as those for the exchange of capital and those for the exchange of consumption, and also those for labour hire).

Alan Kirman is one author who has also questioned the representative individual presumption, noting, with some irony, that macroeconomic models that purport to represent real outcomes associated with the coordinating mechanism of the market often have almost no activity which needs such coordination (Kirman, 1992, p. 117). This is because modelers typically assume that, within the relevant sector, a single "representative" decision-making agent, whose choices coincide with the aggregate choices of the heterogenous individuals, can represent the choices of these diverse agents. Kirman's basic point is that such analytically convenient reductions often lead to conclusions that are both misleading and often erroneous. He contends, first, that there is no formal justification for such representative assumptions; second, that the reaction of the representative agent to parametric variations may not reflect likely outcomes aggregated across heterogenous individuals; third, that the actual choices of the representative may be diametrically opposed to those of each of the individuals they purportedly reflect; and fourth, that the resulting complex dynamic adjustments may be inappropriately obscured (p. 118). As examples of the second of these contentions he cites previous studies of how individuals are differentially affected by the imposition of subsidies to production and margin requirements on share trading so that the representative agent constructed before the change would be different from the respective agent constructed after the change (p. 123). Kirman also considers a simple graphical model of a two-agent equilibrium to confirm the third of his contentions (pp. 124-5).

Kirman argues that economists have in part been motivated by the supposed virtue of constructing models on the basis of robust microeconomic foundations. Moreover, in the absence of clear results on the appropriate linkage between individual and aggregate choice it has often been convenient to assume that aggregate behaviour could effectively be described by the choices of a representative individual. In many cases this assumption is rationalized and explicitly enforced through the imposition of implausible constraints over the preference structure of individuals so that collective behaviour conforms to that of a single individual[101]. Another justification is based on the view that individual acts of arbitrage reflect specific adjustments towards an equilibrium outcome that is conveniently described by the behaviour of the representative individual. Kirman emphasizes that in such cases the "representative" individual assumption is being used to provide properties of stability and uniqueness of equilibrium which may not be adequately guaranteed by the underlying model (p. 120). As such, the prospect of indeterminacy could effectively invalidate any standard forms of comparative static analysis. Kirman unfortunately reminds us of the fact that the literature on general equilibrium has shown that the properties of choice that actually guarantee uniqueness and stability do not carry over from individuals to the heterogenous collective[102]. In particular, the weak axiom of revealed preference may not be satisfied at an aggregate level, even when individuals are constrained to have linear Engel curves[103]. Moreover, Kirman argues that attempts to fall back on the "local uniqueness" properties of equilibria are neither typical, nor any more likely to meet with success.

[101] Kirman cites a number of well-known studies of the specific conditions over preferences that lead to exact aggregation, the most recent and general being that of Lewbel (1989).
[102] At this point Kirman notes that Debreu's (1974) theorem guarantees that only three properties of excess demand functions carry over from individual agents to the aggregate excess demand curve: continuity, the satisfaction of Walras' Law, and homogeneity of degree zero.
[103] Here, Kirman (p. 122) cites the findings of Mantel (1976).

Kirman considers a range of examples that further confirm that the representative agent assumption is far from innocent, including studies of the consumption-based asset-pricing model and the aggregate consumption-income relationship (pp. 126-7). In each case, he cautions that what might appear to be a very complex process of decision-making and optimal adjustment on the part of the representative agent over long time-horizons, may often reflect aggregation over heterogenous groups of agents, each of whom is making simple, but still rational, decisions over much shorter time-horizons.

However, he also considers a range of studies, which appear to confirm the fact that allowing for heterogenity and for the increasing dispersion of characteristics across agents can, in many cases, contribute to the overall smoothness, gross substitutability, and stability of outcomes in the aggregate economy. Paradoxically, these results show that structure can be regained at the aggregate level despite what goes on at the level of the individual agent.

In part to resolve these apparently contradictory findings, Kirman wraps up his critique of representative agent modeling by pointing to recent developments in non-cooperative and evolutionary game theory in which localized phenomena (including communication and coalition building activities) can propagate through the economy. He notes that models of this kind can generate potentially chaotic endogenous cycles and fluctuations (pp. 132-3).

I have argued that a fundamental notion implied by the Post Keynesian narrative is that effective demand can fall below what is required for full employment and the full utilization of capacity.

Shortfalls in investment, in particular, are seen as likely to arise endogenously and, subsequently, spill-over into consumer markets and, thence, into labour markets. Significantly, Keynes argued in Book V of *The General Theory* that flexibility in wages and prices, alone, would be unlikely to overcome these effects of coordination failure, given the volatility of both investment and liquidity preference. However, it should be emphasized that considerations of this nature to a large extent transcend the lesser concerns identified by Kirman. His main area of focus is on problems associated with aggregation across individual consumers or portfolio investors. The concern of Post Keynesians is with lack of coordination across different groups of economic agents: consumers/savers, producers/investors and financial intermediaries. The extent of liquidity preference (or uncertainty aversion and uncertainty perception) can differ amongst each of these differing groups of decision-makers (for example, the corporate sector may wish to invest but be unable to attract external finance due to the liquidity preference of their creditors and banks may wish to lend, but consumers or firms may be unwilling to purchase goods on credit). In each case, even though Walras' law continues to hold, a different kind of coordination failure will ensue, but one with similar consequences for overall employment and national levels of activity.

One recent model that, despite its simplicity, incorporates debt-deflation effects and quantity-constraints is presented in Palley (1999), which builds on earlier theoretical analysis (Palley, 1998). To demonstrate the significance of these inclusions, even for a simple, schematic model, I shall provide a brief overview of his findings. My intention is to show that the introduction of these mechanisms, corresponding in part to those described by Keynes in Book V of *The*

General Theory, suffice to raise strong doubts about the key role ascribed to price flexibility by New Classical theorists.

Palley (1998) establishes that Walras's law *continues to hold* when an economy is subject to quantity-constrained rationing, despite the existence of involuntary unemployment in labour markets. He demonstrates that the existence of involuntary unemployment (i.e. excess supply of labour) would simply correspond to a situation of excess demand for money income. As such:

> …the goods, bond, and portfolio money markets all clear, but there is involuntary unemployment that is matched by an excess demand for wage income. Market demand and supply schedules are effective demands and supplies, and actual outcomes confirm the expected constraints on which agents have formed their effective demands and supplies. As a result, agents have no incentive to change the effective demands and supplies that gave rise to an equilibrium (Palley, 1998, p. 338).

Significantly, the fact that expectations are fulfilled implies that a rational-expectations equilibrium could obtain and persist despite the lack of market clearing. The key question to be asked is whether downward price flexibility could restore full employment. Palley provides an answer to this question in a more recent paper (1999) by introducing the Fisher debt-deflation effect into a quantity-constrained rationing model of the kind pioneered by Barro and Grossman (1971) and Malinvaud (1977). The variables defined in Palley's model are:

y = output
y^d = aggregate effective demand
$f(\cdot)$ = aggregate production function
$n^s(\cdot)$ = labour supply function
w = real wage

$n^d(\cdot)$ = labour demand function
n = employment
i = nominal interest rate
P = the price level
L = the level of nominal inside debt
W = the nominal wage
g = real government spending
M = the nominal money supply

The model consists of five equations. The first equation determines the level of output, as constrained by the minimum of the level of aggregate demand, the availability of labour supply, and the willingness of firms to supply labour:

$$y = \min\left[y^d, f\left(n^s(w)\right), f\left(n^d(w)\right)\right]$$

The second equation determines the *notional* demand for labour, which is obtained from the inverse of the marginal product of labour function:

$$n^d = f'^{-1}(w)$$

The third equation determines the level of employment as conditional on the level of output produced, which is obtained from the inverse of the production function:

$$n = f^{-1}(y)$$

The fourth equation determines the level of aggregate demand as a function that depends positively on the level of employment, negatively on the price level in accordance with the Pigou effect, positively on the level of real wages, reflecting the Kaleckian income distribution effect (workers are assumed to have a higher marginal propensity to consume than non-workers), negatively on the debt-service burden of borrowers, and positively on government spending[104]:

$$y^d = y(n, P, i, w, iL/Wn, g)$$

Palley makes the assumption that workers are debtors and accordingly scales the debt-burden in relation to nominal wages. A decline in nominal wages would raise the debt service burden as a proportion of worker household income. The final equation determines the nominal interest rate as a function of the price level, in accordance with the Keynes effect, employment levels and the nominal money supply:

$$i = i(P, n, M)$$

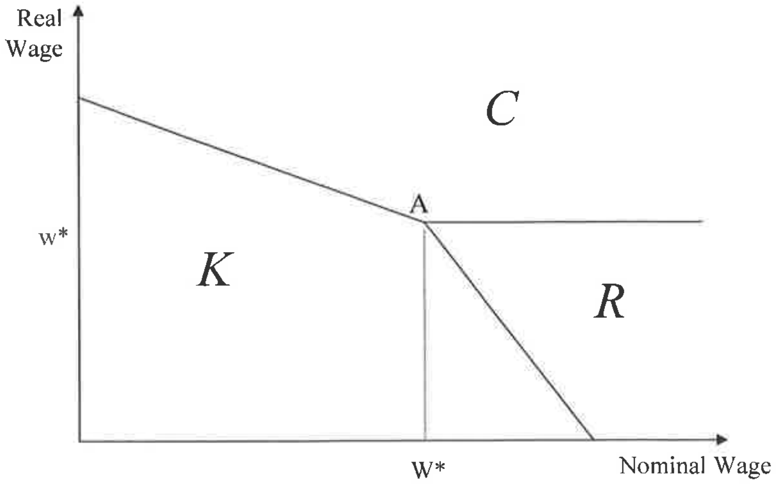
Four regimes arise in the model, characterized by the nature of the constraints, as shown below:

Regime	**Constraints**
Unconstrained Walrasian	$y = y^d = f(n^s(w)) = f(n^d(w))$
Keynesian Regime	$y = y^d \leq f(n^s(w)), f(n^d(w))$
Classical Regime	$y = f(n^d(w)) \leq y^d, f(n^d(w))$
Repressed Inflation	$y = f(n^s(w)) \leq y^d, f(n^d(w))$

Palley maps each of these regimes on a diagram in real and nominal wage space for a general case (and for a number of nested special cases that will not be considered further below):

[104] Palley notes that the Kaleckian effect could be negative if higher real wages also lower the profit rate (Palley, 1999, fn. 1, p. 788).

Fig. 21: PALLEY (1999) ON RATIONING AND DEBT-DEFLATION



Instrumental to these depicted outcomes is the assumption that increasing employment has a net positive effect on aggregate demand, with the direct effect of wages on income, and therefore expenditure overcoming the negative effects of employment over interest rates, which would otherwise exert a negative influence over the interest-sensitive components of expenditure and also the debt-servicing burden (Palley, 1999, p. 789).

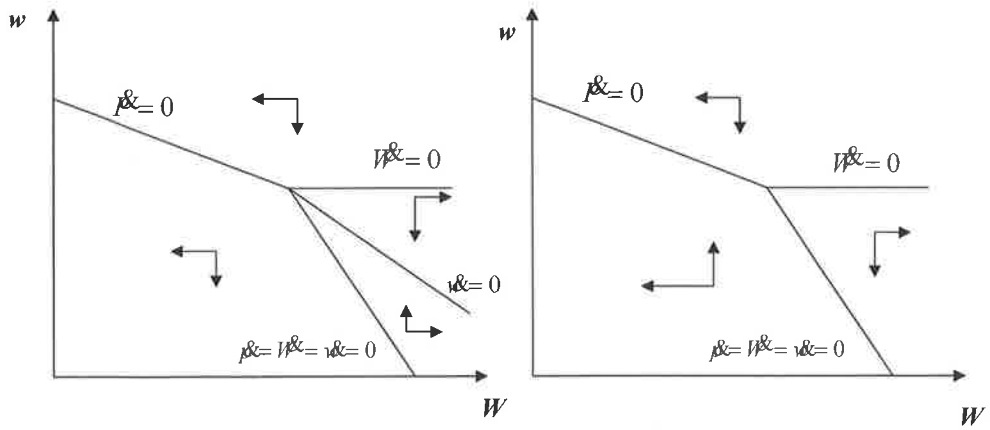
To investigate problems of dynamic adjustment, Palley includes two further equations for price and nominal wage adjustment (p. 795):

$$dP/P = P\left(y^d - \min\left[f\left(n^d(w)\right), f\left(n^s(w)\right)\right]\right), \quad P' > 0$$
$$dW/W = W\left(\min\left[f^{-1}\left(y^d\right), n^d(w)\right] - n^s(w)\right), \quad W' > 0$$

The rationale for each of these equations is that, on one hand, goods prices are presumed to be driven by the gap between aggregate demand and the minimum of firms' desired supply or ability to supply. Nominal wages, on the other hand, are presumed to be driven by the gap between effective labour demand (the minimum level of employment as determined by goods demand or firms' desired employment at the going real wage) and labour supply.

For the general case, as shown below, these equations give rise to two possible adjustment outcomes, depending on whether nominal wages adjust faster or slower than prices:

Fig. 22: PALLEY (1999) ON QUANTITY-CONSTRAINED ADJUSTMENT



Nominal wages adjust faster than prices

Nominal wages adjust slower than prices

Inspection of the phase diagrams shows in each case, downwardly flexible nominal wages generate instability, driving the economy away from Walrasian full employment equilibrium. In the first diagram's Keynesian region, falling nominal and real wages give rise to adverse Kaleckian and debt-deflation effects, which aggravate the shortage of aggregate demand. In the repressed inflation region, rising nominal wages give rise to a positive Fisher-effect, which aggravates excess demand. In the second diagram real wage changes are destabilizing in both the Keynesian and repressed inflation regimes. Palley observes that while real wage rigidity would solve the instability problem, activist monetary and fiscal policies would be required to overcome unemployment. These would operate by shifting the K-C border and point of Walrasian equilibrium downwards and to the left to a point where the economy could find itself in the classical regime where falling real wages might restore full employment (p. 800).

Chiarella and Flaschel's (1999) have constructed applied dynamic disequilibrium models around Goodwin-style predator-prey relationships. Nevertheless, their basic Keynes-Metzler-Goodwin (KMG) model incorporates disequilbrium debt-deflation effects similar to those described by Palley that build on the Minsky-inspired research of Steve Keen (1995). Moreover, when discussing the features of a six-dimensional version of the model, the authors observe that the resulting low dimensional IS-LM-Phillips Curve dynamics are explosively unstable when rapid adjustment of wages is allowed to occur in response to changes in inflationary expectations. However, this instability is significantly dampened when a kink is introduced into the Phillips Curve to reflect the fact that, in true Keynesian style, workers resist reductions in their money wages (Chiarella and Flaschel, pp. 21-2). Once again, the existence of non-indexed, nominal contracts, such as can be found in both credit and labour markets, must be seen to exercise a profound influence over macroeconomic outcomes, and should not be dismissed in models that purport to represent realistic outcomes in a monetary production economy.

Ultimately, my objective in this thesis has been to outline a future research agenda for studying general equilibrium interactions that occur between the financial and real parts of a monetary production economy. I would emphasize the fact that, in my view, quantity-constrained rationing mechanisms ought to be an important aspect of any such research. For example, if Hansen, Sargent and Tallarini's (1999) model of permanent income were replaced by one permitting quantity constrained rationing, then the fluctuations in investment, credit creation, and consumption that would be induced by variations in uncertainty aversion would provide an additional source of dynamic adjustment and complexity. In particular, the introduction of expectations about future prices and nominal wages under uncertainty would provide a further incentive for agents to adjust their debt holdings. Appropriately, Palley observes that an adequate analysis of these adjustments would obviously require formal modeling of the credit-creation and borrowing process (p. 801).

4.6. The Capital Debates and Indeterminacy in capital markets

The usual assumption that the economy produces a single good (corn or potatoes) which can be divided with equal ease into food for consumption, capital (seed) for investment purposes, or monetary assets for measuring and allocating obligations amongst would-be borrowers and lenders is far less benign than it looks. The composite commodity theorem would have us view this assumption as a first order approximation and a truism for a world where all prices move more-or-less together. However, once a distinction is made between goods of a capital and those of a current nature (leaving aside the vexatious question of where to place durable consumption goods), then the relative prices of these goods may well deviate markedly, both between and within each category, as aggregate wage and aggregate profit shares vary relative to one another. As revealed in the *Capital Debates*, this particular index number problem is difficult to resolve in such a way as to preserve the aggregative parables of neoclassical capital theory (see Harcourt, 1972; Kurz, 1985; Kurz and Salvadori, 1987). The question of whether we can transcend these problems by embracing more sophisticated forms of Walrasian general equilibrium is not at issue here. But the prospects for, and the implications of indeterminacy are very much an issue. New Keynesians have uncovered indeterminacies of their own in overlapping generations models, under the rhubric of sunspot equilibria (e.g. see Azariadis, 1981; Azariadis and Smith, 1998; and Farmer and Guo, 1994). However, the Neo-Ricardian position would be one favouring a class-based approach to financial markets: one that focuses on the way that variations in interest rates and the rate of profit change the distribution of income between rentiers, industrialists and workers and the long-run rate of accumulation (e.g. see Panico, 1998, pp. 93-4 and 186-7). Thus, the state of the class-struggle, or perhaps the current balance of forces between the financial and non-financial fractions of capital, would be viewed as an essential political force driving the system's dynamics over the long haul.

4.7. Time-variation in parameters

From a Post Keynesian perspective there are many reasons to suspect that real-world economies, to an extraordinary degree, are best described as time-varying parameter systems. For those of a Marxist or Neo-Ricardian persuasion, time-variance could be attributed to changes in the relative strengths of capital and labour, or perhaps in the balance of power between different fractions of capital, say financial and industrial. For those of a Minskyian persuasion, time-variance could be attributed to changes in the level of financial instability as borrowers, lenders and investors take on balance-sheet positions that feature increasingly speculative and time-deferred returns. For those with Neo-Keynesian leanings, time-variation could be attributed to variations in the magnitude of quantity-constraints and movements from one type of quantity-constrained regime to another or, for those drawing on complexity and non-linear dynamics, to endogenous variations in key model parameters (e.g. in animal spirits variables that govern investment).

I have argued above, that endogenous changes in uncertainty aversion or in uncertainty perception associated with increases in observation error, external perturbation, and model uncertainty could be incorporated into a robust and risk-sensitive control framework. However, these sources of variation require endogenous changes to the objective function or stochastic uncertainty constraint, respectively, rather than to the dynamic system and is therefore much more difficult to accommodate within the confines of existing theory and practice.

In the tradition of Kaldor, Goodwin and Minsky, a number of Keynesian models of macroeconomic and financial disequilibrium have recently been developed (Andersen, 1998, 1999a, b; Taylor and O'Connell, 1985, Foley, 1987, and Franke and Semmler, 1989, Flaschel, Franke and Semmler, 1996). Common to many of these models is the notion that key parameters determining the sensitivity of investment and/or money holdings to interest rates and debt levels is time-varying (albeit, often in a mean-reverting form). One of my intentions is to examine the possibility of reconciling modeling approaches of this nature with those associated with the modeling of decision-making under uncertainty aversion in a robust control context. For this reason, I defer further detailed consideration of this issue to later sections of the thesis.

4.8. Non-linearities, limit-cycles and chaos and Rational Expectations

Despite the fact that it represents a particularly extreme position, Vercelli (1991, pp. 127-8) contends that it is still worthwhile to examine the scientific framework developed by Lucas in his work over the 1970's. In my view it provides a useful benchmark for reviewing the more recent developments in robust control theory.

Vercelli contends that Lucas openly regarded Keynesian theory as a temporary, pathological departure from the main currents in economic thought: a departure that was occasioned by the failure of contemporary theory to explain or predict the severity of the economic crisis over the 1930's (Vercelli, 1991, p. 128). In Vercelli's eyes, Lucas found the short-lived nature of the Keynesian revolution to be a completely understandable phenomenon: one entirely justified by the fact that *The General Theory* lacked a sound general equilibrium foundation (Vercelli, 1991, p. 129). In particular, it could not provide policy makers with a theoretical basis for determining how the behaviour of economic agents might respond to anticipated or announced changes in the government's policy regime. This simple notion comprises what is now recognised as the Lucas critique of policy invariance.

Vercelli (1991, pp. 132-3) contends, for example, that in his work on business cycles—designed to usurp the position held by neoclassical synthesis Keynesianism—Lucas adheres to a fairly *sanitized* notion of economic cycles. Essentially, they are viewed as systematic phenomena characterized by superimposed oscillations of the principal economic time series. Moreover, Lucas regarded the two fundamental principles of rational self-interest and market clearing presumed by these models as completely justified, on the grounds that the incentive effects and informational efficiencies that were imposed through the operation of competitive markets would prevent non-market clearing outcomes. He reasoned that, in conjunction, these principles ought to be sufficient to explain observed market outcomes, without the need to resort to any notions of disequilibrium, money illusion or any other type of irrationality on the part of economic agents.

The fact that Lucas (1981), in his early work, concentrated exclusively on unanticipated monetary shocks as the source of cyclical oscillation, made him an easy target for Keynesian counter-attacks. But Vercelli (1991, pp. 136-9 and pp. 148-9) rightfully observes that the Lucas critique of model-based policy evaluation represents a very general and quite a significant insight. At its most abstract, Lucas's argument is that the invariance of behavioural parameters reflecting the decision-making rules of economic agents cannot be guaranteed *a-priori*, whereas parameters related to technology and preferences could be expected to remain stable in the face of changes to the policy regime. Nevertheless, Vercelli is clear in his belief that this prejudice is one that must ultimately be confirmed by observation, rather than on the grounds of logic alone (Vercelli, 1991, p. 139). For one thing, he suggests that:

> ...different economic policy rules can significantly influence investment activity in both its quantitative and qualitative aspects, and that this cannot but affect technological progress. It seems to me quite legitimate, therefore, to conclude that the technological structure cannot be considered invariant with respect to different alternative rules of economic policy (1991, p.139).

Moreover, while Lucas (1981)—the New Classical purist—admonishes the neoclassical synthesis Keynesians for surreptitiously introducing "free parameters" into their models, especially those linking excess supply or demand to changes in variables such as prices and factor rentals, Vercelli notes that on occasions, Lucas is forced to admit similar intrusions himself (Vercelli, 1991, p. 140). One example of this is the notion of insufficient "*credibility*" which Lucas employs in explaining the fact that monetary policies often have very real effects before they are transmitted into purely nominal outcomes[105]. However, a more profound critique of the Lucasian paradigm is exhibited in Vercelli's book, which deserves extensive consideration because it relates directly to issues examined in Chapter 4 of this thesis.

In his critique of Lucas, Vercelli (1991, pp. 100-4) examines the implications of dynamic and/or structural instability and indeterminacy for models predicated on the notion of rational expectations-based equilibrium. He takes care to distinguish between two separate forms of stability—*dynamic* and *structural* (Vercelli, 1991, appendix 3A, pp. 39-40; sections 3.3, pp. 34-5;

[105] Of course, a new classical theorist could respond by arguing that credibility effects can at least be modelled within a rigorous, microeconomic theoretical, if not empirical, framework by adopting Bayesian intertemporal game theory—as in the literature on dynamic inconsistency.

3.4, pp. 35-7; and 4.2 pp. 44-5). He defines the former concept in terms of the adjustment of a dynamic system towards the steady-state. The latter concept, he defines in relation to the functional structure of the dynamic system: specifically: for a system of autonomous differential equations (Vercelli, 1991, appendix 3A, pp. 39-40):

$$\dot{x} = f(x), x(0) = x_0, f : R^n \to R^n \text{ is } C^1$$

1. A set in $\Re n$ is a bounded set strictly stable of a trajectory $y(t)$ if the set is bounded and a $T \geq 0$ exists such that $y(t)$ belongs to the set for $t \geq T$.
2. The above differential equations have a strictly stable set if a strictly stable set exists for all solutions $x(t)$ of the above system[106].

To clarify what he means by the *functional structure* of the dynamic system, Vercelli argues that a dynamic system can be written in terms of a block diagram with a *connective structure* of oriented arrows connecting the functional components (blocks) of the system (Vercelli, 1991 pp. 54-6). Each block contains a relevant set of *functional coefficients* which specifies the relations holding between endogenous and exogenous variables. Vercelli defines as the functional structure of the system the matrix of functional coefficients written in the order specified by the connective structure. Such a system, he contends, undergoes a *prima facie* structural change when its functional structure and/or its connective structure is modified. A *genuine* structural

[106] Vercelli also considers a stochastic analogue for the notion of dynamic stability under which a stationary stochastic process (defined in accordance with either strictly or wide-sense stationarity) can be shown to converge to the original probability distribution after perturbation in its *systematic* part.

change occurs when a *prima facie* structural change has the effect of modifying the behaviour of the system in terms of both its equilibrium and disequilibrium properties.

A system is structurally unstable when a small perturbation is sufficient to induce a *discontinuous* structural change. Here, discontinuity is defined in regard to whether, either, (a) the number and type of equilibria is changed, or (b) the disequilibrium properties are altered in sign, and not just in their speed of convergence or divergence. Vercelli argues that this *strong* definition of structural stability can be further refined by distinguishing between *small* and *arbitrarily small* perturbations, and by applying a somewhat weaker notion of structural change by defining discontinuity in relation to a certain pre-assigned standard[107].

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[107] In his Appendix 4.B3 (pp 62-4), Vercelli examines more formal topological definitions of structural stability. The evolution of the dynamic system, he suggests, can be represented by a transformation $T{:}X\rightarrow X$, where X is the state space of the system and Tx gives the state at time $t + 1$ of a system which at time t is in state x. He analyses three main cases:

1. the case of differentiable dynamics—X is a differentiable manifold and T is a diffeomorphism
2. the case of topological dynamics—X is a topological space and T is a homeomorphism
3. the case of ergodic theory—X is a measure space and T is a measure preserving transformation

Structural stability holds if "...for any small change induced by a perturbation in the vector field, the system thus obtained is equivalent to the initial system" (Vercelli, 1991, pp. 62-3). However, the first and second of the above cases can be distinguished by the manner in which equivalence is defined. For case one, equivalence holds between two diffeomorphic systems when they have similar smooth manifolds. Case two—topological equivalence—is less stringent in requiring the existence of a homeomorphism of the phase space of one system onto the phase space of another, which translates the phase flow of the former onto the phase flow of the latter. A further relaxation of topological equivalence is that of topological orbital equivalence which obtains if there exists a homeomorphism of the phase space of one system onto another converting the oriented phase curves of the first into the oriented phase curves of the second. Vercelli describes on-going efforts by mathematicians to make the definition of structural stability even less demanding by building a bridge between two specific branches of topological analysis. One branch focuses on the classification of phase-portraits. The other emphasises the generic properties of vector fields, and in this context, discusses the phenomenological perspective of Thom (Vercelli, 1991, Appendix 4B.4). However, this complex discussion contributes little to Vercelli's later analysis of rational expectations and can therefore be passed over.

Vercelli examines Muth's original definition of a rational expectations equilibrium as that which is attained when the subjective probability distribution of outcomes held by economic agents tends, for the same information set, to be congruent with the objective or theoretically predicted probability distribution of outcomes (Vercelli, 1991, pp. 97-8). He notes that this equilibrium notion can be interpreted in both a *strong* and a *weak* sense, where each is defined in relation to the interactions that are presumed to take place between knowledge and decision making. In other words, changes in knowledge or understanding can lead to modifications in conduct. These changes can, in turn, generate new information leading to subsequent improvements in knowledge. The *strong* sense of equilibrium imposes the condition that these interactions have achieved a permanent state of rest. The *weak* sense acknowledges that cognitive and decision-making processes co-evolve in some sort of a cumulative process (Vercelli, 1991, p. 98)[108].

Vercelli contends that the early work of rational expectations theorists imposed strong regularity assumptions on the workings of the economy, through the supposition that the systematic component of the dynamic process was entirely predictable and could be represented in statistical terms by a stationary and ergodic stochastic process. Moreover, he examines an additional series of ad-hoc assumptions that he identifies as typical; namely:

a unique equilibrium exists

this unique equilibrium is a saddle-point

[108] As I suggest below, Thomas Sargent's recent work on search and learning processes and bounded rationality is firmly grounded in the latter weak sense of equilibrium.

the stable variety of the saddle-point is of unitary dimension, or in any case not greater than the number of control variables

there are reasonable auxiliary assumptions that justify the restriction of admissible values to the stable variety

there are 'providential' variables subject to discontinuous leaps (jump variables) which guarantee the immediate and perfect compensation of stochastic shocks (Vercelli, 1991, pp. 100-1).

Vercelli considers each of the above-discussed stability assumptions in turn (pp. 101-4). Citing the literature on overlapping generations models, he dismisses the first assumption on the basis that many reasonable representative agent models give rise to indeterminacy and a continuum of stationary equilibria. He describes the second assumption in the chain as "perplexing", because models characterized by saddle-points are structurally unstable. This attribute of instability is evaded through the imposition of the third assumption, which Vercelli suggests is clearly arbitrary. The fourth assumption is completely independent of the rational expectations hypothesis, seeking justification from a variety of extraneous directions, including:

- the constraining force of transversality conditions (which, of course, only hold sway over an infinite horizon);
- the dynamic inefficiency of non-convergent paths (which can only be a plausible supposition for certain types of optimising model wherein representative agents have *finite* rather than *infinite* lives); or
- the fact that economic policy conveniently selects initial conditions which position the economy on the stable path.

The final assumption is required to accommodate the likelihood that either agents make slight errors of judgement, or stochastic shocks could occur that jolt the economy off the stable path. The appropriate collection of jump variables must then compensate, discontinuously if necessary, for these otherwise destabilizing influences. Vercelli contends, however, that this sort of jump mechanism would have to become fully integrated into the model to be completely credible.

Chiarella and Flaschel (1999, 2000) have shown that chaotic dynamics can arise in simple macroeconomic models of sufficient dimensionality, even when the model equations for time rates of change and growth rates of the endogenous variables take a linear form. The core model they construct and analyze—identified as one belonging to the Keynes-Goodwin-Metzler family—possesses six dimensions. This reflects dynamic variations in prices and wages (determined by interaction between the labour market and goods market Phillips Curves), in the capital to labour ratio (governed by the IS- LM block), and in expected sales and desired stock-to-capital ratios (determined by the inventory adjustment block). However, because the price and wage Phillips Curve equations are expressed in growth terms, and because the employment rate is a bilinear function of the labour-to-output ratio and expected output-to-capital ratio, non-linearities are introduced into the model. The analysis of the resulting trajectories shows them to be associated with super-critical, sub-critical, and degenerate Hopf bifurcations[109]. As parameter

[109] The Hopf bifurcation involves a non-hyperbolic stationary point with complex (linearized) eigenvalues. Thus, in contrast to the saddle-node, pitchfork, and transcritical bifurcations, the solution trajectories of the Hopf bifurcation are defined over a two-dimensional rather than a one-dimensional center manifold. The bifurcating solutions are therefore periodic rather than stationary (see Glendinning, 1994; sections 8.8 and 9.3).

values are varied, otherwise stable limit cycle trajectories become degenerate and unstable, exhibiting sometimes chaotic dynamics.

This sort of dynamic behaviour is even more likely to arise when further dimensions are recognised. For example, Chiarella and Flaschel's most sophisticated model (1999 pp. 51-61) also incorporates open-economy effects associated with exchange-rate variations as well as the usual Fisher-style debt-deflation effects. The authors observe that the introduction of conventional lagged responses for savings, investment and expectational variables merely introduces delayed adjustment effects into the model that lead to no substantive changes in the structural characteristics of the model solutions.[110]

Although rational expectations modeling is sometimes applied to non-linear systems, it is usually taken for granted that in such cases, the usual optimization techniques can be carried over to a linearized version of the system at points close to the steady state. It is vital to recognize that, when chaotic dynamics arise, this linearization approach is certain to fail because *the conditions necessary for Poincare's linearization theorem to hold do not obtain.* Instead, techniques of adapative non-linear and chaotic control must be utilized.[111]

Barnett and Serletis (2000) provide a cogent review of methods of testing for the presence of nonlinearity or chaos in both macroeconomic and financial time series. They observe that under

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[110] See Rosser, 2000, Chapter 7 for further elaboration of models of this nature.

the efficient market hypothesis the stochastic process for asset-prices should follow a martingale (i.e. changes in price, adjusted for dividends, are unpredictable conditional on the relevant information set). They also note that Fama (1970) originally defined three forms of market efficiency with respective to three particular versions of the information set: the strong form (based on all information, both public and private), which implies a semi-strong form (based on all public information), which in turn implies a weak form (based solely on past returns and prices). They further point out that the martingale difference model is less restrictive than the conventional random walk model because it requires only independence of the conditional expectation of price changes from the available information, whereas the random walk model requires independence involving all the higher conditional moments of the probability distribution of price changes.

Barnett and Serletis (2000) briefly review the burgeoning literature on testing for unit roots under time stationarity and difference stationarity, noting Perron's (1989) recommendation that one-off changes in either the intercept and/or the slope of the trend function should be included to accommodate recognizable structural breaks associated with "big shocks" like the oil crises or the "great crash". Perron's justification is conventional: "big shocks" do not represent a realization of the underlying data generating mechanism for the time series under investigation and such breaks should be allowed for under both the null and any alternative hypotheses that are to be tested.

[111] In adaptive non-linear control, the conventional linear transfer functions are preplaced by a cascade of non-linear Liapunov-based blocks that replicate, to the desired order of approximation, a Lokta-Volterra expansion of the non-linear system. This approach captures the complex poly-spectral characteristics of the non-linear system.

However, Barnett and Serletis (2000) urge researchers to include tests for fractional integration when testing for integration because estimated autoregressive-moving average regressions of long-memory processes can exhibit spuriously high persistence close to a unit root. Conditioning on the alternative of fractional integration rather than stationarity alone would support any rejection of an autoregressive unit root that might otherwise result from the presence of fractional integration in the time series.

The primary purpose of the Barnett and Serletis paper is to discuss various tests for nonlinearity and chaos. They observe (pp. 710, 712, 721) that the earlier generation of nonlinear tests based on the correlation integral and Lyapunov exponents of an unknown dynamic system are sensitive to dynamic noise, and the problem grows as the dimension of the chaos increases (as are the more recent tests of the fractal dimension). Barnett and Serletis review the recent battery of improved diagnostic tests which include the BDS test (Brock et al. 1996), the Hinich bispectrum test (Hinich, 1982), the NEGM test (Nychka et al. 1992), the White test (White, 1989), and the Kaplan test (Kaplan, 1994). The null hypothesis for the BDS test is whiteness (independent and identically distributed observations) so that it only provides evidence for the existence of non-white linear or non-white nonlinear dependence; whereas for the Hinich test the null is lack of third order nonlinear dependence; for the NEGM test it is chaos; for the White test it is linearity in the mean; and for the Kaplan test it is linearity in the process.

Barnett and Serletis also gather together the evidence on nonlinearity and chaos in both macroeconomic and finance time series (summarized in Table 1, p. 716). They note the trade-off between the need for large amounts of data and the need to assume the absence of structural breaks in the data generating process over the data period. Under these conditions it is no surprise to find that researchers have reached different conclusions when using high-frequency data over short periods. In their discussion of the on-going controversy Barnett and Serletis note that even when the economic system is linear it can be shocked by nonlinear or chaotic impulses from the surrounding physical environment (i.e. related to climate or weather patterns). Accordingly, they suggest that findings of chaos in non-parametric tests should provoke little reaction while findings to the contrary should be greeted with skepticism. Instead, the opposite response is more frequently observed. They acknowledge that, in part, these responses may reflect concern about the apparent lack of robustness in tests for chaos and nonlinear dynamics across variations in sample size, test method and data aggregation method.

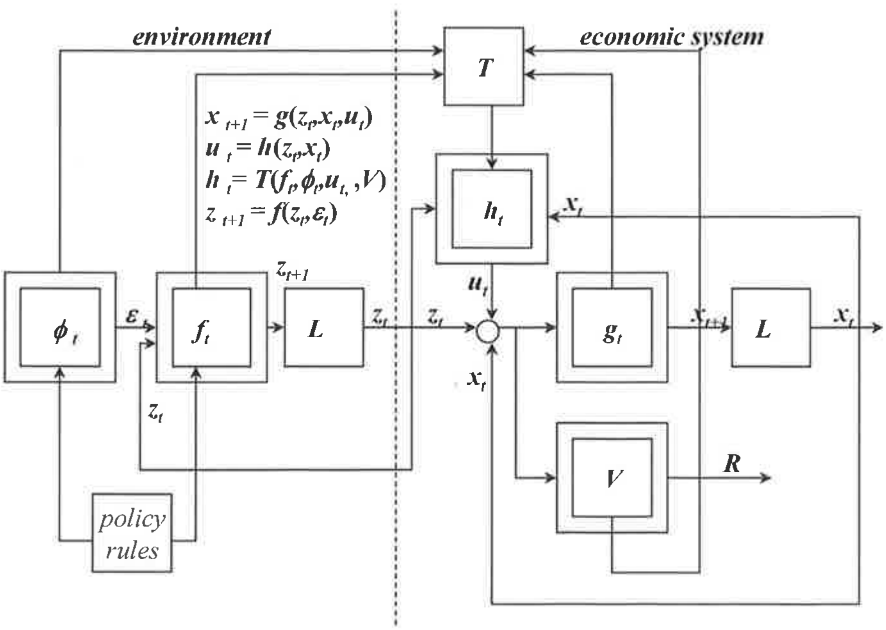
Some evidence of this diversity is captured in the reported success rate for each of the above-mentioned tests in a single-blind controlled competition, which is discussed in Barnett et al. (1997). Five generating models were provided to various researchers, both in small and large samples: a deterministic chaotic recursion, a GARCH process, a nonlinear moving average process, an ARCH process, and an ARMA process. Only the Kaplan and NEGM tests were free of failure for both sample sizes. Barnett et al. (1997) suggest a ranked approach, first using the Hinich and White tests for 3rd order nonlinear dependence or linearity in the mean, and if the null is rejected, the application of the NEGM test to determine whether chaos is present. Alternatively, if the Hinich and White tests each lead to acceptance of the null, the BDS or Kaplan test can be applied to determine if the process exhibits full linearity.

4.9. The Separation between Decision-making and Prediction

Vercelli identifies a fundamental set of core notions within the Lucasian rational expectations framework that enables stability issues to be displaced. Specifically, he identifies an epistemological *separation* that opens up between the stochastic evolution of the economic environment (including, here, the contribution made by economic policy interventions to the innovation process), processes of rational decision-making, and predictive rules and procedures (section 9.2, pp. 143-8).

This comes to the fore most visibly in Vercelli's block-flow diagrammatic presentation of Lucas's heuristic model, reproduced directly below (fig. 6, p. 147):

Fig. 23: VERCELLI'S (1991) READING OF LUCAS



In this diagram, the decisions made by the representative agent are conditioned by a state variable $x_t \in S_2$ which is a function of, $z_t \in S_1$, the state of the environment at time t and, $u_t \in U$, the action chosen at time t

$$x_{t+1} = g(z_t, x_t, u_t)$$

where g: $S_1 \times S_2 \times U \rightarrow S_1$ is an expression of the technology of the dynamic system.

The state of the environment is assumed to evolve through time in accordance with the function:

$$z_{t+1} = f(z_t, \varepsilon_t),$$

where the innovations $\varepsilon \in E$ are independent extractions of a given probability distribution ϕ, and f: $S_1 \times E \rightarrow S_1$ describes the law of evolution of the environment.

The agent is assumed to choose an action u_t, given the state of the environment and the current value for the state variable x_t in accordance with the following decision rule (pp. 144-5):

$$u_t = h(z_t, x_t), \text{ where}$$
$$h_t = T(f_t, \phi_t, g_t, V)$$

and V is the agent's objective function V: $S_1 \times S_2 \times U \rightarrow R$.

On the basis of the fundamental premise of rationality, the agent is presumed to select a decision rule h to maximise the objective function V. It is also presumed that the objective function is already known to the theoretician. As such, the function h that would be chosen in any environment can therefore be calculated. One possible source of difficulty arises in distinguishing that very large group of variables, say z_{2t}, which may well influence the law of evolution of the environment even though they do not *directly* affect technology or preferences, from those, say z_{1t}, that most certainly do affect them (i.e. like productivity shocks). Vercelli observes that the f function is exogenous, thus, while economic reasoning can identify the z_{1t} vector it is of no avail in identifying elements of the z_{2t} vector. At this point, however, rational expectations theorists resort to Granger causality testing[112].

Vercelli notes that in practical applications, the heuristic model is often further simplified to evade the possibility of complexity and instability and thereby guarantee the existence of a unique rational expectations equilibrium. Specifically, it is assumed that h and g are linear and that V is quadratic, imposing *certainty equivalence*, which allows the optimization problem to be decomposed into two simple, sequential components, as in (p. 145):

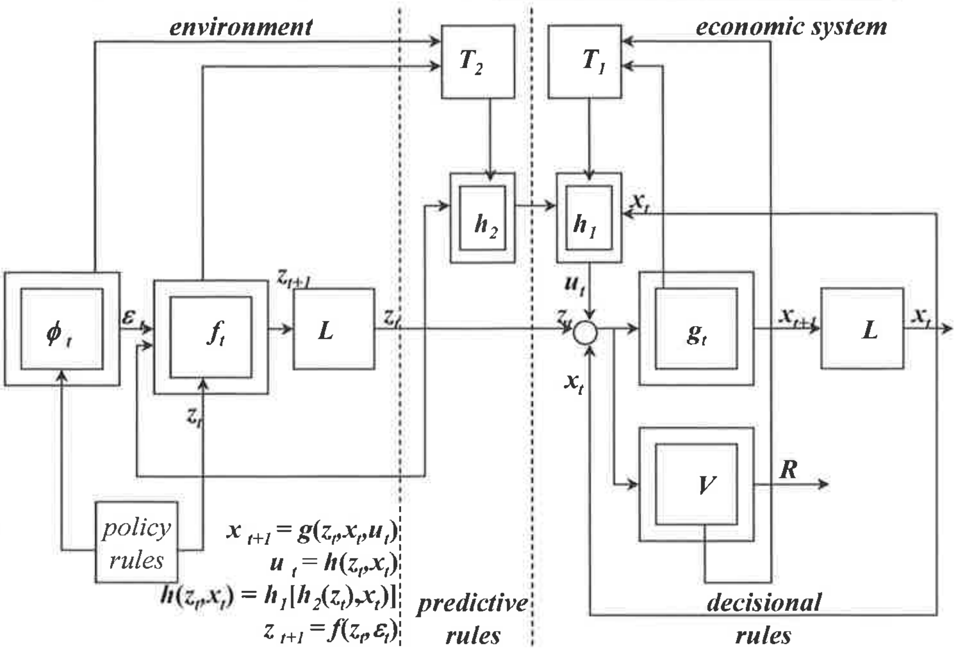
$$h(z_t, x_t) = h_1[h_2(z_t), x_t]$$

Here, $h_2(z_t)$ represents the optimal prediction of the law of evolution given the current state z_t such that h_2: $S_1 \rightarrow S_1$.

[112] Another interesting aspect of Vercelli's critique concerns the notion of probabilistic causality. He argues that Keynes shared with Suppes, a comprehensive philosophical notion of probable cause. While this notion appears to share similarities with Granger causality, a conceptual framework that rational expectations theorists draw upon for the testing of exogeneity, optimal forecasting and policy evaluation; in practice there is one essential difference. For the former writers the role of prior theoretical knowledge is central for distinguishing *prima facie* causality from spurious causality. For the latter, although new information could reveal spuriousness, there is no explicit recognition of the theory-laden character of empirical analysis and econometric procedure (see Vercelli, 1991, Chapter 7). Unsurprisingly, one of the examples that Vercelli propounds to illustrate spurious causality is the notion that an increase in real wages is responsible for an observed rise in unemployment, when the real culprit is an adverse shift in the point of effective demand.

It can be seen that this decomposition serves to isolates predictive procedures entirely from the rules of decision-making[113]. The effects of this additional presumption on the heuristic model are

Fig. 24: VERCELLI (1991) ON THE MODIFIED LUCASIAN MODEL



portrayed below (fig. 7, p. 147):

Vercelli observes that, arising from these convenient simplifications,

> [t]he cognitive process is separated from the decisional one, which allows one to circumscribe the problem of structural instability solely to the formation of expectations. The internal functional structure of the system becomes completely independent of the

[113] As Vercelli notes, an additional assumption, that h_2 is a linear Minimum Least Squares (MLS) predictor, is also common, though by no means essential to this issue of separability.

> environment and of the uncertainty, which characterises it[114] (1991, p. 146).

Ultimately, Vercelli contends that either indeterminacy and instability must be acknowledged as inescapable features of the real world, or that a model must be adopted which possesses the attributes of stability, while at the same time accommodating some margin of indeterminacy. In both instances, regularity is only acknowledged to be qualitative and transitory. To a large extent, the move to embrace robust and risk-sensitive control follows the latter of Vercelli's two trajectories.

Nevertheless I have suggested that in models of consumption, savings, portfolio decisions and investment decisions, as research into complexity and non-linear dynamics has shown, fairly plausible, microfoundational characterizations of the decision-making process are likely to yield

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[114] Esther-Mirjam Sent (1997) has also recognised a similar separation between cognitive and decision-making activity in her study of the work of Thomas Sargent and Lars Hansen. Like his colleague Lucas, Sargent has been motivated by the objective of ensuring conceptual integrity between theory and method. Sent notes a number of key influences over Sargent's methodology, including Sims development of time-series econometrics "without theory", especially ARMA (p,d,q) and techniques for dealing with vector stochastic processes; the need for achieving forecasting performance on a par with the large structural, neoclassical-synthesis models; the problem of observational equivalence; and the Lucas critique of policy invariance, that has already been discussed above. Rather than provide a complete review of her critique, because many of these issues will be revisited in later sections of the thesis, we merely quote her own summary of the role played by *convenience* in Sargent's early writings:

> From covariance stationary, linearly indeterministic processes to Wold's decomposition theorem to moving average representations to autoregressive representations to Wiener-Kolmogorov linear least-squares prediction to feedback-feedforward symmetry to Kalman filtering to innovations representations—convenience was the mother of all rationality in Hansen and Sargent's contributions. Faced with disarray in economic dynamics, Hansen and Sargent sought to find unity in the use of engineering mathematics, and were left with a whole slew of very restrictive assumptions and with an interpretation of rationality motivated by convenience. This convenience was instrumental and the very opposite of atemporal law-governed rationality (Sent, 1997, p 33)[114].

non-linear relationships. As a result, complex dynamic outcomes would be fairly common: at the very least, taking the form of limit cycles, and, with more likelihood, the form of full-blown chaotic dynamics. In such cases, a resort to control and filtering techniques that have been designed for linear dynamic systems would be sure to fail. The slightest hint of observation error, model uncertainty or non-Gaussian perturbation would result in devastating control failure.

Burmeister (1980) provides an illuminating commentary on this very problem in his discussion of saddle-point instability in asset-pricing models:

> Because the demand for assets depends upon relative price changes, a price *increase* for a particular asset makes it relatively more attractive due to capital gains, thereby increasing demand for that asset and causing its price to rise even further (Burmeister, 1980, p. 96).

Burmeister examines four possible mechanisms that could ensure that the economy starts at a point on the convergent manifold. Divergent trajectories would eventually lead either to zero or to finite prices for aggregate output. He rejects one possible mechanism—the existence of future markets for commodities—as being dubious relevance, especially if the time horizon for large movements in the price level were of long duration. A second mechanism, a transversality condition requiring that the limiting price level is finite as t goes to infinity, could be imposed on the basis of intertemporal utility maximisation. However, Burmeister also rejects this possible resolution of the convergence problem because it would depend on planning over an infinite horizon. The fourth mechanism, predicated on the assumption that agents believe that the path of expected prices is convergent, is also rejected because "...expected and actual price paths coincide with perfect foresight, and the assumption of "convergent expectations" is equivalent to assuming the answer to the question at hand!" (1980, p. 95). The final mechanism, an optimal stabilisation policy rule that favours a monetary expansion when inflation is too high and a contraction when per capita capital stock is too high, is rejected on the grounds that:

> [s]uch counterintuitive stabilisation rules should not be taken as serious policy advice; our perfectly competitive model with full employment is not appropriate for addressing traditional macroeconomic policy questions (Burmeister, 1980 p. 95).

In a later chapter on stochastic models and rational expectations Burmeister quotes extensively from Samuelson's discussion of the tulip-mania phenomenon in a 1957 work, before citing a later paper that demonstrates a continuing skepticism on Samuelson's part, about the ability of existing economic theory to deal adequately with speculative bubbles:

> Even though every tulip mania and stock-market bubble have come to an end in history, economists have no good theory to explain why they last as long as they do and not twice or half as long (Samuelson, 1967, p. 230).

In his more recent analysis of the same issue Scarth (1996) considers another three mechanisms raised in the literature (and understandably overlooked by Vercelli). The first is Taylor's (1977) suggestion that agents would choose (and therefore, analysts should select) the solution that involves the smallest variance. Scarth responds:

> [b]ut for which endogenous variable and over what time horizon should this rule be applied? (1996, p. 117).

Second is McCallum's (1983) proposal to ignore feasible rational-expectations solutions if the reduced-form coefficients are not continuous functions of the structural parameters. And third, there is Scarth's own attempt to extend the correspondence principal to second-order discrete-time systems by requiring the admissible characteristic root to be positive given that two-period saw-tooth cycles are not observed[115].

To see what Scarth means by saw-tooth dynamics in second-order dynamic systems, we can turn to Chapter 7 of Carl Chiarella's (1990) work on non-linear economic dynamics. This section is entitled, *Perfect Foresight Models and the Dynamic Instability Problem from a Higher Dimension.* Interestingly, Chiarella takes Burmeister's model of monetary dynamics as his starting point. Burmeister constructs this simple model to investigate both the source of, and resolution of dynamic instability. It is derived by taking the natural logarithm of the following demand and supply functions for real cash balances:

$$\frac{M^d}{P} = Be^{-\alpha\left(\dot{P}/P\right)}, \quad M = e^m$$

where B and α are exogenously given positive constants. Using small typescript for the log of the respective untransformed variable, Burmeister derives the following equation:

$$m^d - p = b - \alpha\dot{p}.$$

[115] Ultimately, Scarth hopes to leave such unpleasant arbitrariness behind as modellers build more robust and comprehensive microfoundations, which supposedly would enable more structural restrictions to be imposed on the underlying macro model. Of course, a growing number of New Keynesian and New Classical models actively exploit non-uniqueness of equilibrium in their explanations of macroeconomic processes.

In equilibrium $m^d = m$. With no loss of generality Burmeister sets $m = 0$, yielding the following equation:

$$\dot{p} = \frac{b}{a} + \frac{p}{a}.$$

This equation has a unique rest point at $p^* = -b$, but since $\frac{d\dot{p}}{dp} = \frac{1}{\alpha} > 0$, this rest point is unstable and will diverge unless $p^* = p_0$ (i.e. the initial price is set equal to the dynamic equilibrium price).

While Burmeister (1980, pp. 268-9) presumes that the asset demand is a decreasing linear function of expected inflation π (with elasticity α), Chiarella, instead, favors a logistic function[116]. He demonstrates that this particular choice gives rise to a marginal money demand function $f'(\pi)$, with a relatively flat base over much of its range but, unlike Burmeister's, it curls out like the lip of a bowl at either end.

Chiarella's equations for money demand function, the formation of adaptive expectations, and money-market adjustment are given by:

$$m^d = \log M^d = p + f(\pi) = \log P + \log(T + \alpha(\pi))$$

$$\dot{\pi} = \frac{1}{\tau}(\dot{p} - \pi)$$

and

$$\dot{p} = \beta(m - p - f(\pi))$$

[116] Chiarella (1990) contends that the shape of the function can be justified by appealing to a utility maximising framework. Furthermore, he argues that aggregation over many individuals will also explain the asymptotic

In obvious notation, β is the coefficient of adjustment, m is the log of the money supply, p is the log of the actual price level P, $\alpha(\pi)$ is asset demand, and T is the transactions demand for money balances. Chiarella shows that the dynamics of the model are now governed by the following second-order differential equation:

$$\frac{\tau}{\beta}\ddot{\pi} + \left(\frac{1}{\beta} + \tau + f'(\pi)\right)\dot{\pi} + \pi = \mu$$

where μ is the constant growth rate of the money supply. For $\tau > 0$, expectations are adaptive, while $\tau = 0$, yields the perfect foresight solution.

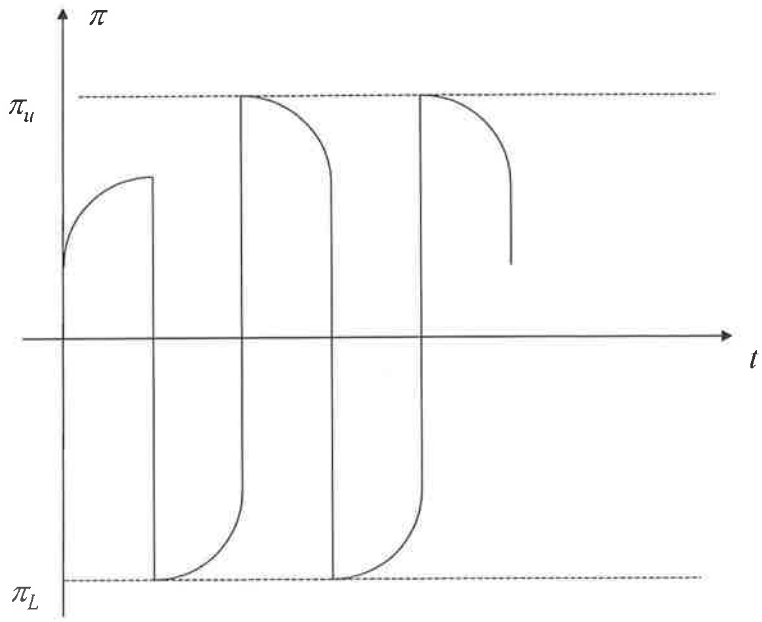
For a linearized version of this second-order model under adaptive expectations, Chiarella demonstrates that the locally unstable case with $-f'(\mu) > 1/\beta$ (the case typically associated with saddle-point instability in Burmeister's original model), now gives rise to a limit cycle in accordance with the conditions of the Hopf bifurcation theorem. He investigates the properties of the limit cycle using the method of averaging. To a first-order approximation, the amplitude of the limit cycle is shown to depend on the speed of adjustment parameters, $1/\tau$ and β, but is "barely affected by moderate changes in the rate of monetary expansion μ" (Chiarella, 1990, Chapter 7). Presumably, the significance of this observation is that some kind of optimal policy intervention on the part of the monetary authorities—one that would otherwise be expected by

adjustment of the function towards its upper and lower limits of 1 and 0, representing the respective polar cases in which either all or no wealth is allocated to the monetary asset.

the non-government, rational, representative agents and, therefore, factored into their economic decisions—would not effectively moderate the limit cycle.

In an accompanying appendix 7.2, Chiarella (1990) applies two theorems from the theory of non-linear differential equations to confirm that the properties identified through linear approximation are, in fact, globally valid. In appendix 7.3, he examines what happens to the limit cycle under perfect foresight (i.e., as $\tau \to 0^+$). This technical analysis is supplemented by a heuristic analysis of the phase plane for a (slightly modified) first-order version of the second-order differential equation system. Without going into all the technical details, Chiarella establishes that the time-path of expected inflation follows a regular periodic motion comprised of a sequence of two types of adjustment—instantaneous and slow-moving—to yield a saw-toothed pattern as shown in the diagram below.

Fig. 25: CHIARELLA (1990) ON PERFECT FORESIGHT AND NON-LINEARITY



Instantaneous jumps occur onto either the upper and lower bounds of the expected inflation variable, followed by gradual movements away from these bounds until the next flip-flop jump occurs.

In other words, the perfect foresight solution also exhibits limit cycle behaviour. Chiarella argues, moreover, that if the manifold over which the perfect foresight trajectory moves was to be embedded within a higher dimension, the resulting division of the phase space into fast and slow regions would naturally allow for jump-wise changes in economic variables. There is no need to resort to arbitrary assumptions, as is common in the early rational expectation literature, about the occurrence of the requisite jump-wise changes[117].

In his concluding comments to the Chapter, he contends that similar non-linearities may play an important role in a wide range of dynamic models displaying saddle-point instability including models of futures markets and foreign exchange and portfolio models in finance theory. Of course, many Post Keynesian models are predicated upon similar forms of nonlinearity in real investment (notably, Kaldor, 1940), in representations of the class struggle between capital and labour (Goodwin's famous 1951 predator-prey model), in adjustments to profit margins over the business cycle (Shaikh, 1989).

[117] Chiarella (1990) goes on to examine a discrete version of his non-linear monetary dynamics model that yields the map $\pi_{t+1} = F(\pi_t)$. All trajectories exhibit bounded oscillatory motion, but Chiarella suggests that, for certain parameter sets, chaotic rather than periodic behaviour may result. However, the discussion at this point is largely conjectural and the author stops short of full computer simulation of the discrete system.

Chiarella's (1990) analysis of monetary dynamics under adaptive expectations and perfect foresight raises doubts about the core presumptions of rational expectations dynamics. However, more sophisticated models of financial and productive investment must obviously account for optimisation over multiple-period expected returns. This requires some sort of (stochastic) optimal control approach. The resultant dynamics may also differ markedly from the saw-tooth pattern: one rejected by Scarth solely for reasons of empirical implausibility. I shall return to this fundamental issue in subsequent chapters of the thesis.

4.10. Conclusion

In the next chapter, after a review of Tornell's (2000) and Hansen and Sargent's (1995, 1999) embrace of risk-sensitive and robust control theory, I shall reconsider each of the issues examined above. Needless to say, in their entirety they raise doubts about the ability of this new theoretical framework, in its current form, to adequately capture the driving forces within a monetary production economy. There, it will be argued that although the resulting asset-pricing models can price "trees" in terms of their expected "harvest" of the numeraire consumption good, in this stylized pure-exchange world without money, the very factor that would actually serve to explain the mysterious source of the demand for liquidity—uncertainty—can sadly explain nothing of the sort. Absent, are the core ontological features of a monetary production economy that have been discussed in this chapter: nominal, non-indexed contracts, the possibility of coordination failure, and transactions costs—the true hallmarks of a real world monetary economy. The likely existence of time-varying observation error, model uncertainty, time-varying system parameters, and complex non-linear dynamics only serves to undermine the purported robustness of these existing applications of robust control. Nevertheless, somewhat paradoxically I maintain that the techniques themselves, once extricated from their New Classical integument, can serve as a source of new inspiration for Post Keynesian research into liquidity preference and decision-making under uncertainty.

CHAPTER FIVE — RISK-SENSITIVE CONTROL THEORY AND LIQUIDITY PREFERENCE

5.0. Introduction

In this chapter of the thesis I examine economic applications of control theory, tracing their development from straightforward linear quadratic Gaussian control (in section 5.1.1), through sub-optimal forms of deterministic control (in section 5.1.2) into risk-sensitive and robust control theory (section 5.1.3). My intention is to demonstrate that this move towards more sophisticated techniques has been motivated by the need to operate within increasingly general stochastic environments. In section 5.1.3, I provide an intuitive explanation of what is achieved when quadratic objective functions are replaced with their risk-sensitive counterparts. In addition, I review recent applications of risk-sensitive control to finance theory. Other macroeconomic applications are discussed in footnotes. In section 5.1.4, I provide a robust control interpretation of the Least Mean Squares algorithm that is utilized in recursive control and in the estimation of neural networks.

This overview of applications provides the context for a more detailed study of the papers by Hansen, Sargent and Tallarini (1999), Andersen, Hansen and Sargent (1999) and Tornell (2000). In the introduction to this section I look at how these New Classical authors have interpreted the move into risk-sensitive control theory. I show that these new techniques have been adopted to compensate for the inadequacies of earlier methods based on LQG control, but in a manner that preserves much of the New Classical heritage.

To provide a context for my study of what are essentially revamped asset-pricing models, I review Weil's (1989) paper on the equity premium puzzle. This celebrated study shows that the pricing of assets using discount factors derived from very general, Kreps-Porteus utility functions, under reasonable assumptions about the value of the key parameters for intertemporal elasticities of substitution and relative risk aversion, still cannot resolve the equity premium puzzle.

Hansen, Sargent and Tallarini (1999) apply risk-sensitive control to a Lucas-style equilibrium asset-pricing model, where the dividend stream is constructed from an entirely separate, habit-persistence model of consumption and growth. Uncertainty about the stochastic process is captured by a norm bound over the external perturbation term that appears in the state variable equation. Adopting a multiple-priors interpretation, Hansen, Sargent and Tallarini argue that the *non-uniqueness* that is implied by this stochastic constraint embodies a form of Knightian uncertainty. Ambiguity of beliefs is not fully specified in probabilistic terms but is instead described by a feasible set of specification errors, with a range defined by the imposed norm bound. Using the equivalence between risk-sensitive control, robust control and the Kalman filter—under limiting conditions over the risk-sensitivity parameters σ— the authors are able to apply standard, Kalman-filter based estimation techniques for the linear quadratic Gaussian case.

From this estimate they determine the magnitude of the respective uncertainty premia that could be embodied in asset-prices, by considering variations in the magnitude of both the σ parameter and another parameter that reflects the size of the exogenous shocks to preferences in their habit-persistence based model of consumption. Andersen, Hansen and Sargent (1999) generalize these results to the continuous-time case, in the process allowing for more general forms of uncertainty (e.g. perturbations to the transition probabilities of a hidden-Markov model and generalized diffusion processes). Tornell's work (2000), described in section 5.2.3, achieves similar results within a framework that is at once far simpler, in terms of the objective function—namely, quadratic utility—but also more complex, because it is accomplished in a partial information setting (the state variable is an incompletely observed autoregressive process). He shows that a concern for robustness explains various anomalies, including excess volatility of asset-prices, predictability of returns (in regressions against the dividend-price ratio), and the equity-premium and low risk-free rate puzzles. His H^{∞} forecasts of asset-prices, based on calibrated simulations for the dividend sequence, also map the actual US data more closely over the 1871-1997 period.

I return to the themes identified in earlier chapters. I investigate the implications of moving beyond the representative agent by reviewing Thomas Palley's 1999 reconfiguration of quantity-constrained rationing models to incorporate debt-deflation effects. I then question Hansen and Sargent's reductionist approach to robust control theory, before reexamining the implications of recognising time-variation in parameters and dynamic and structural instability.

Hansen and Sargent construct their full information version of the robust control problem to ensure both manageability and also containment within a more-or-less conventional rational expectations *cum* real business-cycle framework. My largely destructive critique of these attempts is motivated by my concern to show that, for the social sciences, Herbert Simon's (1982) notion of bounded rationality provides a more comprehensive and more realistic portrayal of the actual restraints over decision-making than one predicated on robust control under stochastic uncertainty constraints. A natural outcome of this critique is both a broader view of policy interventions and a richer appreciation of their significance for the satisfactory operation of private markets.

5.1. An Overview of Developments in Stochastic Control Theory

5.1.1. LQG Control Applications

Linear Quadratic Gaussian (LQG) techniques have a long history in economics and finance theory and practice. Sargent demonstrates that LQG techniques can be applied to a wide variety of economic decisions, including consumption with a random return (Sargent, 1987, pp. 31-3), dynamic portfolio theory (pp. 33-5), inter-related factor demand (1987, pp. 51-2), and optimal investment with time-to-build lags (pp. 49-51) or quadratic adjustment costs (pp. 53-4). Researchers have used Kalman filter techniques to estimate time-varying parameter models that include the Capital Asset-pricing Model (Wells, 1996, section 4.2), and buffer-stock models of money demand (Mizen, 1994; Cuthbertson and Taylor, 1989). For example, Wells (1996, chapter 5) reviews some 25 studies of time-varying parameter models in the finance literature and discusses a series of econometric techniques for discriminating between random walk, random coefficient, mean reversion, mean reversion to a random mean, and moving mean models.

Tucci (1997) provides a recent example of adaptive learning where the Kalman filter is applied recursively to generate the optimal control for a model subject to time variation in parameters. The coefficient matrices on the vector of intercept terms, control variables and state variables are assumed to each be a function of a single parameter β that, itself, varies in accordance with a mean-reverting model. Tucci acknowledges the Lucas critique implication that agent responses to anticipated changes in government policy should be explicitly modeled but argues that:

> [i]f the policy is new and not announced by the policy-maker, the economic agents may learn using a bayesian rule and the application of the Kalman filter, the basis for most of the algorithms estimating time-varying parameters, is appropriate. Only when the policy is new and announced the use of parameters following a 'return to normality' model may be a poor, and misleading, approximation of the rational expectations hypothesis (e.g. Lucas 1976, pp. 39-40).

He uses Monte Carlo simulation to compare the relative performance of adaptive control with passive learning (which takes into account the uncertainty of the parameters but ignores the fact that they will be remeasured in the future) and certainty equivalent (under which the controls are estimated assuming that the parameters are known with certainty but that they are reestimated any time a new piece of information becomes available). His results confirm that adaptive control performs well, in terms of the lowest variance of outcomes, relative to the certainty equivalent method. Moreover, it does reasonably well even in cases where it does not perform better than either the certainty equivalent or the passive learning methods, while the reverse is not true (Tucci, 1997, p. 46).

More recently, Real Business Cycle practitioners have employed LQG methods to solve complex stochastic growth models (McGratten, 1994; Taylor and Uhlig, 1990; Roche, 1998). The resulting optimal trajectories are presumed to reflect Gaussian disturbances to the underlying parameters representing both intertemporal substitution between current and future streams of consumption and aggregate productivity.

5.1.2. Sub-optimal deterministic control

In many economic applications, the limitations of LQG control are obvious. In particular, many optimal control problems give rise to saddle-point solutions. This characterization of the phase-space carries the implication that any form of non-Gaussian perturbation or model error will force the controlled system off the unique "razor-edge" saddle-path trajectory into the region of instability, ultimately leading to a violation of requisite transversality conditions. In response to this inadequacy a number of theorists have championed a sub-optimal, but more robust alternative. For example, Infante and Stein (1972) favour the use of *sub-optimal*, recursive, feedback-control techniques that are robust in situations with a "…high probability of errors in measurement, formulation and implementation, as well as the likelihood of outside disturbances," (p. 47). They contend that their proposed alternative is implementable, has a solution that is closely related to market processes, and can both track and ultimately reach the desired steady-state, even when the latter is not known with any certainty and may be changing over time. Through the utilization of a second-order Taylor's series approximation to the model's first-order conditions around the steady state, the authors derive a recursive expression for the sub-optimal feedback control law[118].

Stein has applied a similar technique in recent econometric studies of the long-run *natural real exchange rate* (NATREX). He estimates values for this key economic variable using a recursive sub-optimal filter in an open-economy version of the neoclassical growth model. Imbalances between savings and investment give rise to complementary capital flows that lead to long term departures from purchasing power parity[119].

Another example of sub-optimal deterministic control is provided by Nagatani (1981, pp. 30-2) who examines a sub-optimal (robust) deterministic control problem that is based on the early work of Goldman (1968). Goldman's model is a derivative of the basic Ramsey-Keynes growth model with a representative, utility maximising agent. The sub-optimal control is calculated through the determination of a continuously revised sequence of plans. Stationarity is built into the solution by through an assumed equality that is presumed to hold between initial (current)

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[118] Of course, a limitation of this approach is that a unique approximating (sub-optimal) algorithm would have to be constructed for each particular model.

[119] Although in principle favouring a sub-optimal approach to exchange-rate determination, I find Stein's work on exchange rate determination unconvincing because it relies on the neoclassical notion that the real rate of interest, embodying the real forces of productivity and thrift, serves as a long-run mechanism of equilibration in the macroeconomy. In my opinion, this sort of position has been severely compromised by the capital debates (see Harcourt, 1972).

stocks and terminal stocks over the planning horizon T. Nagatani observes that welfare losses will be incurred in following a sub-optimal path, but of course, these may well be minor in comparison with the outcomes that might arise through unsuccessfully following an infeasible optimal path (i.e. one that could only be followed accurately in a perfect information state)[120].

5.1.3. Robust and Risk-Sensitive Control

Risk-sensitive control obtains when the conventional linear quadratic objective function is replaced by one that is non-linear. For example, Caravani (1987) considers two risk-sensitive (H_2 Norm) functions:

$$f(x) = x + \mu|x|$$
$$f(x) = \frac{1}{2}x[1 + \exp(\mu x)]$$

that he incorporates into a criterion function that preserves some of the mathematical convenience of the linear quadratic case:

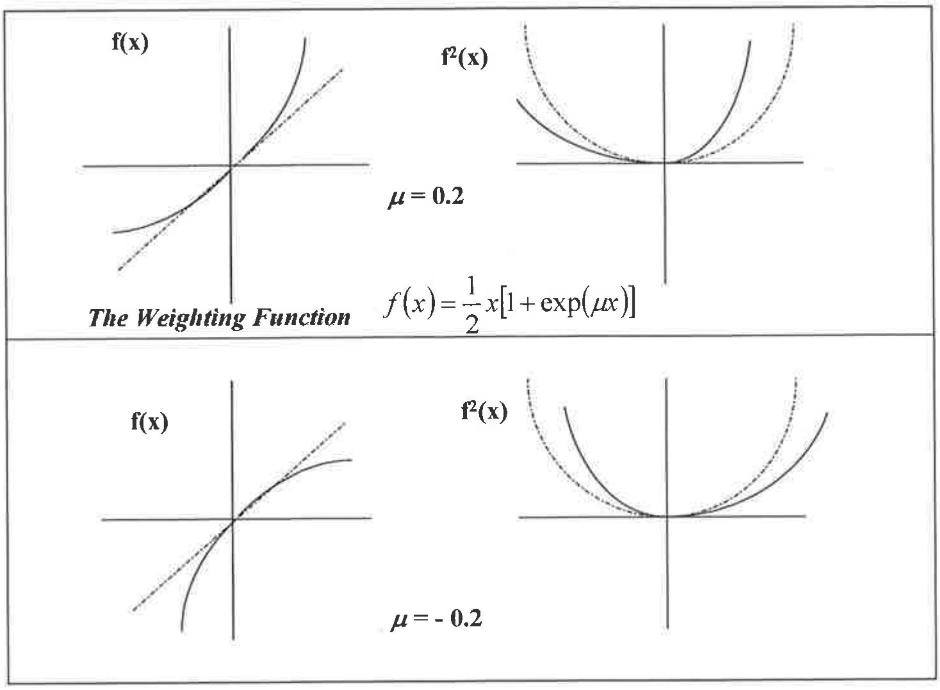
$$\min \sum_{t=0}^{T} \{f'(x(t))Qf(x(t)) + u'(t)Ru(t)\}$$

Immediately below, the exponential function (for a scalar state variable) is graphed for two different values of the μ parameter (thew solid line) and compared with its linear quadratic

[120] Nagatani also notes in passing, that a recursive planning approach lends itself to political manipulation (i.e., the economy may be subject to a cycle generated by the political process). He concedes that this may further increase the resultant welfare losses.

counterpart (the dashed line) to show how this particular function can assign asymmetrical weight to positive and negative values of the state variable:

Fig. 26: CARAVANI'S (1987) RISK-SENSITIVE WEIGHTING FUNCTION



Applications of risk-sensitive and robust control and filtering principles to finance theory are less common than applications to optimal stabilization policy. Nevertheless, notable exceptions include Lefebvre and Montulet's (1994) utilization of risk-sensitive, calculus-of-variations techniques to investigate a firm's optimal choice of the mix between liquid and illiquid assets, Fleming's (1993) risk-sensitive approach to portfolio management, and McEneaney's (1997) work on robust pricing of financial options under stochastic volatility. When continuous trading is impossible (e.g. during stockmarket crashes that are frequently modeled as Poisson jump processes), or when interest rates and stock volatility are stochastic the law of one price breaks down. A replicating portfolio of securities cannot be constructed to perfectly hedge against the corresponding shocks (see Cochrane, 2000, Chapter 18). McEneaney shows how robust control techniques can be employed when the stock volatility is stochastic, using the well known Black and Scholes formulas (see Cochrane, 2000, chapter 18) to derive upper bounds on the relevant option price[121].

One obvious reason for the recent proliferation of these techniques is that typical time-series for the return sequences of most financial assets exhibit significant kurtosis and skewness. Risk-neutral control techniques based on Gaussian processes only attend to the mean and variance of the relevant series rather than to higher-order moments and moments about the mean. It is easy to confirm that exponential objective functions are sensitive to all relevant moments within the joint-probability distribution. For example, Caravani (1987, p. 456) demonstrates that in the scalar case:

$$Ef'(x)Qf(x) = g(x_0) + \frac{\partial g}{\partial x}\Big|_{x_0} E(x - x_0) + \frac{1}{2!}\frac{\partial^2 g}{\partial x^2}\Big|_{x_0} E(x - x_0)^2 \ldots ,$$

Moreover, the shape of the $f(x)$ and $f^2(x)$ functions illustrated above shows that the application of risk-sensitive control techniques can yield a maximal asymptotic ratio of "up-side chance" to "down-side risk".

This asymmetry between up-side and down-side risk is a notable feature of Bielecki and Pliska's (1999) derivation of a continuous-time portfolio optimisation model under risk-sensitive control. The authors' chosen objective function is (Bielecki and Pliska, 1999, p. 339):

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[121] I discuss McEneaney's findings in more detail in the second of two case-studies appearing at the end of this chapter. There, I also provide an intuitive explanation of what occurs by drawing on the research of Cochrane and Saá-Requejo (2000) and Bernardo and Ledoit (2000).

$$J_\theta = \liminf_{t\to\infty}\left(\frac{-2}{\theta}\right)t^{-1}\ln Ee^{-(\theta/2)\ln V(t)}, \quad \theta > -2, \theta \neq 0$$

By taking a second-order Taylor's series expansion about $\theta = 0$ the authors confirm that

> [...] J_θ can be interpreted as the long-run expected growth rate minus a penalty term, with an error that is proportional to θ^2. Furthermore, the penalty term is proportional to the *asymptotic variance* ... (p. 339).

Maximizing J_θ protects an investor interest in maximizing the expected growth of their capital against large deviations of the actually realized rate from their expectations, where θ plays the role of the risk-aversion parameter and $R(t)$ = ln $V(t)$ is the cumulative reward.

Non-financial applications include Hughes-Hallet and Rees (1983) for macroeconomic planning, reviewed by Brandsma (1986)[122]; Sengupta's (1999) incorporation of risk-sensitive intertemporal

[122] Brandsma focuses on two characteristics of the resulting outcome. The first involves the application of more (less) weight or relative penalty on high (low) risk variables that exhibit large (small) variance. The second aspect involves a further scaling of each target vector in proportion to the relative priority each element possessed in the original quadratic objective function. Brandsma comments on the fact that
[...] mean-variance decisions that optimise the expectation of a second-order approximation to a Von Neumann-Morgenstern utility function fit into the expected utility analysis of Machina (1982). But the difficulty remains that the underlying utility function and probability distributions are unknown. From a practical point of view it is therefore attractive to be able to treat measures of uncertainty as an amendment to the preferences in their original quadratic form—which was itself an approximation to a more general objective—without having recourse to make too many extra assumptions (Brandsma, 1986, p 304).
Using a standard planning model for the Netherlands economy, supplemented by Monte Carlo simulation, Brandsma confirms that risk-sensitive control implies smaller policy adjustments, less wage moderation, and a more lenient budgetary policies than for its certainty equivalent counterpart. This results in more stable economic growth with lower inflation and a smaller government deficit, but after an initial favourable effect unemployment is observed to rise above the certainty equivalent level (Brandsma, p 305).

planning into data envelope analysis[123]; Caravani's (1995) application of H_∞ control to stabilisation policy for a simple lagged multiplier-accelerator model of income growth and investment; and Kenneth Kasa's (1999) examination of a scalar robust control problem with a simple quadratic objective function capturing the trade-off between variations in the state variable and variations in the control[124]. Kasa presumes that agents are uncertain about both the stochastic properties of the disturbance and the underlying model, where the latter form of uncertainty is parameterised by a bound on the H_∞ norm of the loss function[125].

5.1.4. Adaptive Filtering and Robustness

In an adaptive filtering setting, the fundamental recursive filters are those associated with the least mean squares, minimum least squares and recursive least squares algorithms. Haykin (1996) outlines the derivation of, and justification for, these various filtering techniques. The robustness properties of the LMS algorithm are now well understood by control theorists (Haykin, 1996, section 9.10) and this, no doubt, serves to explain the remarkable growth in LMS applications

[123] Sengupta's approach distinguishes between current and capital inputs and results in a dynamic cost frontier that permits a separate analysis of short-run and steady-state solutions. He then extends the approach to accommodate stochastic variation in input prices.

[124] Further simplifying assumptions are no discounting, the absence of cross-product terms representing interaction between the state variable x and control variable u, and that initial conditions are known to be zero. Kasa correctly observes that a relaxation of either the first or last of these assumptions would not change the analysis markedly. Nevetheless, in the interactive case, the standard change of variable technique to eliminate cross-product terms in the loss function would imply that a norm bound must enter into the coefficients of the transformed state equation with some influence over the outcome.

[125] Kasa's unique contribution is to show that for a given feedback policy, there is a strictly convex function relating values of the H_∞ norm to values of the parameter summarising the relative cost of the state versus control variability:

> Thus, if one wants to produce a less activist policy, one can either make changes in the control variable more costly or increase the degree of model uncertainty. (Kasa, 1999, p. 175)

He establishes this result through the application of the implicit function theorem to the expression he derives for the robust feedback control law (Kasa, 1999, p. 179). However, Kasa observes that the analysis is highly specific to the scalar value, state-feedback case. For the general multiple-input-multiple-output case, the trace or largest eigenvalue of the feedback matrix would have to be utilised. Output feedback would further complicate the analysis.

across a wide range of fields that draw upon techniques of recursive control and filtering. Haykin demonstrates (pp. 430-32) that, under the LMS algorithm, the sum of squared estimates is always bounded by the sum of two terms: the first is a scaled version of the Euclidean distance between the initial weight or coefficent vector and the true weight vector that is being recursively estimated, while the second is the sum of the squared noise over the interval. Via the singular value decomposition method, Haykin also shows that the LMS algorithm is closely related to a particular category of Linear Minimum Least Squares problems: those that are underdetermined, of rank unity (pp. 517-32). In addition, he identifies the relationships holding between the Kalman filter and the recursive least squares algorithm (p. 572).

From another perspective, Hassibi et al. (1995) confirm these findings by setting out the limiting assumptions that enable the Least Mean Square and Normalized Least Mean Square recursive algorithms to be interpreted as special cases of the *H*-infinity a-priori and a-posteriori filters, respectively. In addition, they show that the limiting relationships between *H*-infinity and risk-sensitive filtering also carry over to these algorithms. In chapter 16 of his text, Haykin takes up the issue of tracking time-varying systems. His analysis draws on a simple signal-processing model whose tap-weight or coefficient vector is presumed to follow a first-order Markov process, but could easily be extended to cover the tracking of more complex, n^{th} order Markov models. In subsequent discussions about the need to adopt methods for modeling time-varying systems (see section 4.3.4), this is the sort of approach I have in mind.

5.2. Robust and Risk-sensitive Control as the latest New Classical Research Strategy

The robust and risk-sensitive control approach adopted in the Hansen, Sargent and Tallarini (1999) and Andersen, Hansen and Sargent (1999) papers is, in each case, motivated by concerns about the inadequacies of rational expectations modeling. For example, in dynamic models based on the linear quadratic regulator it is presumed that agents know the stochastic process but are not concerned about specification error. Under the certainty equivalence principal the optimal control is determined as if the agent were operating under deterministic conditions of perfect certainty. In contrast, the econometrician generally has to apply a battery of diagnostic tests that compare and evaluate alternative specifications. In particular, information criteria are employed to distinguish between two sets of specifications, those that can be rejected by the data, and those that cannot. Andersen et al. suggest that their new robust control methodology enables them to overcome this asymmetry that has plagued earlier efforts at rational expectation modeling by allowing decision-making agents to hold doubts about plausible and implausible specifications which are similar to those entertained by the econometrician.

Hansen and Sargent's embrace of risk-sensitive control and filtering theory can be interpreted in various ways. On one level it represents an understandable evolution in technique. Engineers have developed robust control to overcome the inadequacies of linear quadratic control and Kalman filtering that are embodied in the certainty equivalence principal. As I have shown, on another level it represents an attempt at resolving the troublesome notion that, under rational expectations, agents were assumed to possess more accurate information about relevant distributions that the econometrician. However, another interpretation is that it can be viewed as a systematic effort to incorporate uncertainty into economic thought, in a manner that resolves certain paradoxes in the theoretical and empirical analysis of decision-making (e.g. the Ellsberg paradoxes); and provides an explanation for certain well-known anomalies that have appeared in the literature on asset-pricing (e.g. the equity-premium puzzle).

In this regard, a brief review of Philippe Weil's (1989) work will be of great service. Like Hansen, Sargent, and their colleagues, Weil employs recursive utility functions in a hidden-Markov, dynamic programming framework. Moreover, he adopts a particular version of the Kreps-Porteus recursive utility specification to examine whether this method of escaping the constraints of the Von Neumann-Morgenstern expected utility approach could better explain observed outcomes in financial markets. As I shall demonstrate, Weil answers in the negative. Seen in this light, therefore, Hansen and Sargent's work on robustness can be viewed as providing a plausible and rigorous explanation for the equity premium puzzle.

Weil assumes that the growth rate of dividends $\lambda_{t+1} \equiv y_{t+1}/y_t$ is Markovian over a finite state space, with transition probabilities given by (Weil, 1989, p. 403):

$$\phi_{ij} = \Pr\{\lambda_{t+1} = \lambda_j | \lambda_t = \lambda_i\} \qquad i, j = 1,2,\ldots, I < \infty,\ \lambda_j > 0,\ \sum_{j=1}^{I} \phi_{ij} = 1,\ \forall i$$

The representative agent is assumed to solve the following functional equation (p. 405):

$$V(w_t,\lambda_t)=\max_{c_t} U[c_t, E_t V(w_{t+1},\lambda_{t+1})], \quad s.t.\ w_{t+1}=R_{t+1}(w_t-c_t).$$

Here, $w_t \equiv [p_t + y_t]\, x_t$ is beginning-of-period wealth when the state of nature at time t yields dividend growth of λ_t, p_t is the price of a tree at time t, x_t is the number of (shares of) trees held by agents at the beginning of period t, c_t is the consumption of the representative agent at time t, and $R_{t+1} \equiv [p_{t+1} + y_{t+1}]/\, p_t$ denotes the one-period rate of return on a tree. The above constraint is a simple transformation of the one-period budget constraint facing representative consumers: $c_t + p_t x_{t+1} = [p_t + y_t]\, x_t$. Weil adopts the following parametrization of Kreps-Porteus preferences (p. 404):

$$U[c,V]=\frac{\left\{(1-\beta)c^{1-\rho}+\beta[1+(1-\beta)(1-\gamma)V]^{\frac{1-\rho}{1-\gamma}}\right\}^{\frac{1-\gamma}{1-\rho}}-1}{(1-\beta)(1-\gamma)}.$$

Here, $\rho > 0$ is the inverse of the (constant) elasticity of intertemporal substitution, $\gamma > 0$ is the Arrow-Pratt (constant) coefficient of relative risk aversion for static gambles, and $\beta \in (0,1)$ is the subjective discount factor under certainty. The Von Neumann-Morgenstern (VNM) time-additive expected utility specification emerges if $\gamma = \rho$.

Weil solves the agent's problem for the following Euler equation (p. 406, and his Appendix *A.1*):

$$E_t\left\{\left[\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\rho}\right]^{\frac{1-\gamma}{1-\rho}}[R_{t+1}]^{\frac{1-\gamma}{1-\rho}-1}R_{kt+1}\right\}=1,$$

where R_{kt+1} is the return on any asset k willingly held by the representative consumer.

Weil adopts the standard assumption that there are an equal number of trees and agents so that, in equilbrium, each agent holds one tree ($x_t = 1$ for all t). By Walras law this implies that $c_t = y_t$ for all t. Weil searches for a stationary equilibrium such that $p_t = w_i\, y_t$ if the level of output at t is y_t and i is the state of nature at t. He shows that the w_i's ($I = 1,\ldots,I$) that fully characterize equilibrium are the non-negative solutions to the following system of I non-linear equations:

$$w_i = \beta\left\{\sum_{j=1}^{I}\phi_{ij}\lambda_j^{1-\gamma}\left(w_j+1\right)^{\frac{1-\gamma}{1-\rho}}\right\}^{\frac{1-\rho}{1-\gamma}}, \quad i = 1,\ldots,I.$$

This system and the Euler equation (which, for the tree economy, implies that $R_{kt} = R_t$) enables Weil to determine the risk-free rate of return and the equity premium. This he does for Mehra and Prescott's two-regime Markov characterization of the aggregate consumption process for the US economy over the period 1889-1978.

The respective dividend growth rates and transition probabilities, and the long-run (ergodic) probability of being in the good state (Φ) are given below together with the appropriate formulae for calculating the long-run risk-free rate (RF) and equity premium (Π) from their respective regime-specific values:

$$\lambda_1 = 1.054, \quad \lambda_2 = 0.984$$
$$\phi_{11} = \phi_{22} = 0.43, \quad \phi_{12} = \phi_{21} = 0.57$$
$$\Phi = \frac{1-\phi_{11}}{2-\phi_{11}-\phi_{22}} = \frac{1}{2}$$
$$RF = \Phi RF^1 + (1-\Phi)RF^2$$
$$\Pi = \Phi\Pi^1 + (1-\Phi)\Pi^2$$

Table 1, drawn from Weil's paper (1989, p. 413), serves to illuminate the equity-premium puzzle. The table is constructed for a β value of = 0.95 for a range of elasticities of intertemporal substitution ($1/\rho$) and constant relative risk-aversion (γ) values (the bold number in each cell is the net risk-free rate, while the value above it in each cell is the net risk premium):

EIS	CRRA (γ)						
	0	0.5	1	5	10	20	45
∞	0.00 **5.25**	0.05 **5.24**	0.10 **5.21**	0.48 **5.01**	0.94 **4.78**	1.77 **4.40**	3.01 **4.09**
2	0.01 **6.02**	0.06 **6.16**	0.11 **6.12**	0.51 **5.79**	1.01 **5.40**	1.89 **4.73**	3.14 **3.93**
1	0.01 **7.14**	0.07 **7.08**	0.12 **7.03**	0.55 **6.56**	1.08 **6.02**	2.00 **5.06**	3.27 **3.76**
0.2	0.10 **15.02**	0.18 **14.81**	0.26 **14.61**	0.88 **13.02**	1.64 **11.11**	2.91 **7.75**	4.34 **2.45**
0.1	0.24 **25.73**	0.35 **25.32**	0.45 **24.96**	1.31 **21.68**	2.33 **17.87**	4.04 **11.23**	5.72 **0.85**
0.05	0.56 **50.51**	0.72 **49.55**	0.87 **48.61**	2.12 **41.26**	3.66 **32.80**	6.25 **18.65**	8.66 **-2.23**
1/45	1.13 **138.91**	1.36 **135.56**	1.60 **132.25**	3.58 **107.44**	6.22 **80.69**	11.22 **40.39**	17.11 **-9.22**

The dilemma facing the Walrasian equilibrium theorist arises when trying to explain both the average level of the risk-free rate (0.75%) and the premium between average rates of return on equity (6.95%) and on riskless securities during the period 1889-1978, under plausible ranges of values for γ (1 to 5) and the EIS (0.2 to 1). The main diagonal reveals that this dilemma is particularly acute under VNM preferences ($\gamma = \rho$) because a low risk-free rate requires an extremely large EIS, which results in a low risk premium, while a large risk premium can only be explained by a large γ coefficient, which results in an extremely high risk –free rate. However, Weil's main intention was to show that the dilemma cannot be overcome simply by adopting his specification of Kreps-Porteus preferences. Under the latter, a reasonable combination for the risk-free rate and risk premium (say, 5.72% and 0.85%, respectively) could only be attained with a γ value of around 45 and an EIS of 0.1 (i.e. the γ coefficient would be implausibly high).

Of course, a Post Keynesian theorist would view the equity premium puzzle as a theoretical artifact. The size of the discrepancy between returns to fixed interest securities and stocks would be determined by the state of liquidity preference. An advocate of risk-sensitive control theory under stochastic uncertainty constraints would agree wholeheartedly with this contention. Presumably, she would also point out that her techniques are the only ones that can articulate the precise formal relationship holding between uncertainty aversion, model uncertainty, observation error, and external perturbation.

5.2.1. Hansen, Sargent and Tallarini's New Research Agenda

Hansen et al's representation of the risk-sensitive control problem combines the following state transition equation (p. 875):

$$x_{t+1} = Ax_t + Bi_t + Cw_{t+1}$$

where i_t is a control vector, x_t is a state vector and w_{t+1} is an i.i.d. Gaussian random vector with $E\, w_{t+1} = 0$, $E\, w_{t+1}\, w'_{t+1} = I$. The recursive objective function is:

$$U_t = u(i,x) + \beta \mathbf{R}_t(U_{t+1})$$

where $u(i,x) = -i'Qi - x'Rx$, and $\mathbf{R}_t(U_{t+1}) \equiv \frac{2}{\sigma}\log E\left[\exp\left(\frac{\sigma U_{t+1}}{2}\right)\middle| J_t\right]$.

Here, Q is positive definite and R is positive semidefinite and $\mathbf{R}_t$ is a risk-sensitive recursive utility specification closely related to those presented in Epstein and Zin (1989). Hansen et al. (1999) draw on the results presented in Hansen and Sargent's (1995) discounted, infinite horizon, recursive extension of Jacobsen's original work (1973). Jacobsen was the first to arrive at a solution for a full-information, discrete-time, linear exponential quadratic Gaussian version of the risk-sensitive control problem. Using these results, Hansen et al. (1999) express the optimal or efficient value function in the form:

$$U_t^e = W(x_t) = x_t'\Omega x_t + \rho$$

Furthermore, they show that[126]:

[126] For this problem, Hansen and Sargent (1995) establish that the solution for the recursion $U_t = u(i,x) + \beta \mathbf{W}_t(U_{t+1})$ is linear and time-invariant. In operator form this solution is expressed as (p. 970):

$i = -\mathbf{F} \circ \mathbf{D}(V)x$

where,

$\mathbf{F}(V) = \beta[Q + \beta B'VB]^{-1} B'VA$, and

$\mathbf{D}(V) \equiv V - \sigma VC(I + \sigma C'VC)^{-1} C'V$

Moreover, the minimized value of the criterion is :

$$\mathbf{R}_t\left(U^e_{t+1}\right) = x_t'\breve{\Omega}x_t + \breve{\rho}$$

where,

$$\breve{\rho} = \rho - (1/\sigma)\log[\det(I - \sigma C'\Omega C)],$$
$$\breve{\Omega} = A'^*\left[\Omega + \sigma\Omega C(I - \sigma C'\Omega C)^{-1}C'\Omega\right]A^*$$

Next, Hansen et al. (1999) introduce v_t, a distorted law of motion that perturbs the mean of the innovation w_{t+1} as in:

$$x_{t+1} = Ax_t + Bi_t + C(w_{t+1} + v_t)$$

It is assumed that the distortion is governed by the following "continuation pessimism bound" (Hansen et al. 1999, equations 8a, 8b, p. 877):

$$\hat{E}_t\sum_{j=0}^{\infty}\beta^j v_{t+j}\cdot v_{t+j} \leq \eta$$
$$\eta_{t+1} = \beta^{-1}(\eta_t - v_t\cdot v_t),$$

$$x'\mathbf{T}\circ\mathbf{D}(V)x + U(V,d)$$

where,

$$\mathbf{T}(V) = R + A'\left(\beta V - \beta^2 VB(Q + \beta B'VB)^{-1}B'V\right)A$$

When $\sigma = 0$, then $\mathbf{T}\circ\mathbf{D}$ is just $\mathbf{T}$ because $\mathbf{D}(W) = W$. Obviously, $\mathbf{T}$ is the operator associated with the matrix Ricatti equation for the ordinary (risk-neutral) discounted version of the optimal linear regulator problem. This operator notation is reproduced here because it will help to clarify later analysis of the more complex Andersen, Hansen and Sargent (1999) paper.

With a Gaussian information structure, the aggregator U maps a transformed quadratic cost measure for next period's costs (i.e. utility) into a transformed quadratic cost measure today. Because the infinite horizon cost is computed by iterating on U as in $U_{t,T} = f\left(i_t, x_t, U_{t+1,T}\middle|J_t\right)$, the resulting initial cost measure, when finite, is quadratic in the initial state vector, say $x'_0V^*x_0$, plus a constant term. A feasible control process $I = \{i_t: t = 0,1,\ldots\}$ is a stochastic process of controls adapted to the sequence of sigma algebra's $\{J_t: t = 0,1,\ldots\}$, for which a corresponding, adapted state vector process $X = \{x_t: t = 0,1,\ldots\}$ can be recursively defined using the dynamic equation for the state variable, taking x_0 as given. Given I and X, the time-zero infinite horizon cost U_0 can then be computed by evaluating the almost sure limit of $\{U_{0,T}: T = 1,2\ldots\}$. Hansen and Sargent establish the finitude of U_t by using convexity property of the exponential function and certain proven properties of the aggregator function; namely: that it is monotone increasing and convex in U (for details see Hansen and Sargent, 1995, lemma 3.1, 3.2, theorem 4.1, and theorem 5.1).

where η_0 is given and the conditional expectation is taken with respect to the distorted law of motion. Under this distortion the original risk-sensitive control problem is then set up as a recursive version of a zero-sum, two-player, Lagrangian multiplier game. The Lagrange multiplier enforces the continuation pessimism bound. The authors cite the results of a mimeo on discounted robust control and filtering (Hansen and Sargent, 1999), that demonstrates the equivalence, under the Markov perfect equilibrium concept, of outcomes for this recursive form and a version of the game where players can precommit at time zero. Exploiting the equivalence between robust control and a specific form of the risk-sensitive control problem, the solution of the robust version of the game is found by allowing the first player to determine a control or state feedback rule $\{ i_t\}$ that maximises a modified form of the original value function subject to the distorted law of motion. The second player chooses a feedback rule for v_t to minimise the value function, effectively choosing a worst-case feedback rule for the distortion v_t. The risk sensitivity parameter σ determines the constant multiplier $-\sigma^{-1}$ on the bound η_t.

Hansen et al. (p. 878) note the close relation between their approach to robust, risk-sensitive control and Gilboa and Schmeidler's (1989) multiple-prior generalisation of expected utility theory based on a maxmin criterion. Following Epstein and Wang (1994) they intepret the non-uniqueness of the stochastic constraints that govern the distorted law of motion as "depicting a form of Knightian uncertainty: an ambiguity of beliefs not fully specified in probabilistic terms but described by the set of specification errors $\{v_t\}$ defined by restriction (8)" (i.e. equation 38 above). Using the risk-sensitive control result to compute values for $\{v_t\}$, Hansen et al. are able to measure the degree of uncertainty aversion associated, on the one hand, with alternative values of σ, and on the other hand, with uncertainty-contaminated measures of risk aversion extracted from asset-prices.

Following Whittle (1990), Hansen, Sargent and Tallarini (1999, pp. 878-9) make use of the equivalence between the rational expectations, risk-sensitive solution and a robust, distorted expectations, risk-neutral solution to the recursive linear quadratic optimisation problem[127]. This equivalence, which they call the *modified certainty equivalence principle*, yields the same linear, time-invariant decision rules $i_t = -Fx_t$, for the control i_t. Under the former version of the problem, the value function can be written as: $U_t^e = -x_t' R^* x_t + \beta \mathbf{R}_t(U_{t+1}^e)$, where

$$\mathbf{R}_t(U_{t+1}) \equiv \frac{2}{\sigma} \log E\left[\exp\left(\frac{\sigma U_{t-1}}{2}\right) \middle| J_t\right] \quad R^* = R - F'QF$$

and the true law of motion is $x_{t+1} = A^* x_t + Cw_{t+1}$, where $A^* = A - BF$.

Under the latter version, the value function can be written as: $\widetilde{W}(x_t) = \widetilde{U}_t = -x_t' R^* x_t + \beta \widetilde{E} \widetilde{U}_{t+1} - (\beta/\sigma)\hat{v}_t' \hat{v}_t$, where the expectation is taken with respect to the distorted law of motion: $x_{t+1} = \hat{A} x_t + Cw_{t+1}$, with $\widetilde{U}_t = x_t' \Omega x_t + \widetilde{\rho}$, $\hat{A} = \left[I + \sigma C(I - \sigma C'\Omega C)^{-1} C'\Omega\right] A^*$, and $\hat{v}_t = \sigma(I - \sigma C'\Omega C)^{-1} C'\Omega A^* x_t$. The Ω matrix is the same under each representation of the control problem.

Hansen et al. apply this certainty equivalence principle to a permanent income model featuring habit persistence, and estimate the model from data on US consumption and investment for the period1970.I-1996.III. The model variables include c_t = real scalar consumption, s_t = scalar household services, h_t = a geometric weighted average of past and current consumption, δ_h = the weighting term, $\{b_t\}$ = an exogenous preference shock process, i_t = time t gross investment, k_t = the capital stock, δ_k = the depreciation factor for capital, γ is the (constant) marginal product of capital, $R \equiv (\delta_k + \gamma)$ = the physical (gross) return on capital, $\{z_t\}$ = the driving process for preference and productivity shocks, and $J_t = \{w_t, w_{t-1}, \ldots, w_1, z_0\}$ is the information filtration.

There are five equations in the model. Preferences are defined by the specific recursion (p. 879):

$$U_t = -(s_t - b_t)^2 + \beta \mathbf{R}_t(U_{t+1})$$

Household services are produced by scalar consumption in accordance with the following household technology (p. 880):

$$s_t = (1+\lambda)c_t - \lambda h_{t-1}, \text{ where}$$
$$h_t = \delta_h h_{t-1} + (1-\delta_h)c_t.$$

Under this conventional model of rational addiction or habit persistence, for $\lambda > 0$, consumption services depend positively on current consumption, but negatively on a weighted average of past consumption[128]. Investment and consumption goods output is determined by a linear production

[127] In section A.7 of my Appendix I establish the analogous result for the more complex, *partially observed* case.
[128] A criticism of this approach is that it adopts a model that was initially designed to explain addiction to various substances, where excessive usage diminishes the marginal utility of consumption associated with previous levels of consumption resulting in higher and higher levels of intake, and inappropriately applies it to models of aggregate consumption and asset-pricing. Such applications are motivated by the fact that mild forms of habit persistence can in part explain certain anomalies in finance theory, such as the equity price premium. This line of criticism will be noted, but pursued no further here.

technology: $c_t + i_t = \gamma k_{t-1} + d_t$, where the capital stock evolves according to: $k_t = \delta_k k_{t-1} + i_t$. Together, these equations imply:

$$c_t + k_t = (\delta_k + \gamma)k_{t-1} + d_t.$$

Preference and endowment shocks are presumed to be governed by:

$b_t = U_b z_t$, $d_t = U_d z_t$, and $z_{t+1} = A_{22} z_t + C_2 w_{t+1}$, where w_{t-1} is independent of $J_t = \{w_t, w_{t-1}, \ldots, w_1, z_0\}$[129]. The planner's problem is to choose a process $\{c_t, k_t\}$ given k_0, with components in $L^2{}_0$, to maximise U_0 subject to the equations for household services production and output production. The simple form of this problem enables the authors to apply straightforward Lagrangian multiplier analysis to derive the solution for a benchmark risk-neutral case ($\sigma = 0$)[130].

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[129] Expressed in terms of the standard control theory notation, the state variable vector:

$x_t = \begin{bmatrix} h_{t-1} \\ k_{t-1} \\ z_t \end{bmatrix}$, $i_t = s_t - b_t$, and evolution of the state variable is determined by solving the household service equation for c_t, as a function of s_t - b_t, h_{t-1}, and z_t, then substituting into the stack of equations defining h_{t-1}, output $c_t + i_t$ and the evolution of z_t.

[130] The solution equations include the equation defining the evolution of consumption and capital $c_t + k_t$, and the following equations for consumption and habit persistence:

$$c_t = \frac{1}{1+\lambda}(b_t - \mu_{st}) + \frac{\lambda}{1+\lambda} h_{t-1},$$
$$h_t = \frac{\delta_h + \lambda}{1+\lambda} h_{t-1} + \frac{1-\delta_h}{1+\lambda}(b_t - \mu_{st}),$$
$$\mu_{st} = (1 - R^{-1})\sum_{j=0}^{\infty} R^{-j} E_t b_{t+j} + \psi_0 \sum_{j=0}^{\infty} R^{-j} E_t d_{t+j} + \psi_1 h_{t+1} + \psi_2 k_{t+1}$$

Here, μ_{st} is the Lagrangian multiplier on the household technology constraint and ψ_0, ψ_1, ψ_2 are constants.

Another important aid is the fact that risk-neutral and risk-sensitive solutions are observationally equivalent for this model (pp. 882-3)[131]. Estimation procedures are also rendered easier because standard techniques for the estimation of linear quadratic risk-neutral rational expectations models can be applied to the benchmark case (i.e. the Kalman filter and Gaussian likelihood maximisation). Significantly, under more sophisticated, non-linear, recursive representations of preferences, these simplifying principals would no longer apply (see the discussion about non-quadratic objective functions on p. 902). The solution for the planner's problem reveals that s_t and c_t depend on the difference between b_t and a weighted average of current and future values of b_t. Thus, preferences exert no level effect, although the process for b_t, will affect equilibrium asset-prices.

Observational equivalence also means that Hansen et al. can examine various combinations of the paired parameters (σ,μ_b) that give rise to the same measure of market risk in an asset-pricing context. Here, μ_b is an exogenously set parameter that determines the preference shock b_t under the assumption that the evolution of shocks is best represented by a constant term for b_t and the following set of second-order AR regression equations for d_t:

$$d_t = \mu_d + d_t^* + \hat{d}_t, \text{where } c_{d*}w_t^{d*} = (1-\phi_1 L)(1-\phi_2 L)d_t^* \text{ and } c_{\hat{d}}w_t^{\hat{d}} = (1-\alpha_1 L)(1-\alpha_2 L)\hat{d}_t$$

Thus, the forcing process is determined by a set of seven parameters $(\alpha_1, \alpha_2, c_{\hat{d}}, c_{d*}, \phi_1, \phi_2, \mu_d)$, with μ_b set equal to 32. It should be remembered that although μ_b alters marginal utilities it has

[131] Hansen et al. (1999, fn. 13, p. 883) observe that, for this model, the risk free rate is pinned done solely by technology ($R \equiv (\delta_k + \gamma) = \beta^{1}$). For more complex models, featuring adjustment costs to capital, the risk free rate would also depend on σ.

no effect over planned $\{c_t, k_t,\}$. The value for δ_k is set at 0.975, and the restriction $\beta R = 1$ is first tested and then imposed.

One of the key objectives of the Hansen et al. paper is to study the effect of robust decision-making on security market prices. To this end, the authors treat the consumption process as though it were an exogenous endowment process, as in the Lucas(1978) asset-pricing model (Hansen et al. 1999, p. 889). The state for the model is specified as $x_t = [h_{t-1}, k_{t-1}, z'_t,]'$, so that equilibrium processes for consumption and services and the endowment shocks can be represented by $c^e{}_t = S_c x_t$, $s^e{}_t = S_s x_t$,. $d_t = S_d x_t$, and $b_t = S_b x_t$, respectively. Under robust decision-making, the equilibrium law of motion can be represented by:

$$x_{t+1} = A^0 x_t + C w_{t+1}$$

and the value function by: $U_t^e = x_t' \Omega x_t + \rho$ where $\Omega = -(S_s - S_b)'(S_s - S_b)/2 + \beta\hat{\Omega}$, and $\rho = \beta\hat{\rho}$ with A^* evaluated at A^0.

The key instrument of analysis is a sub-gradient inequality, which can be interpreted as a conditional expectation, similar to those applying in finance theory under non-arbitrage (p. 889 and Appendix A):

$$\mathbf{R}_t(U_{t+1}) - \mathbf{R}_t(U_{t+1}^e) \leq G_t(U_{t+1}) - G_t(U_{t+1}^e)$$

where $G_t(U_{t+1}) \equiv \dfrac{E[V_{t+1}U_{t+1}|J_t]}{E[V_{t+1}|J_t]}$ and $V_{t+1} \equiv \exp\left[\dfrac{\sigma U_{t+1}^e}{2}\right]$.

Here, s_t is any service process measurable with respect to J_t, with associated utility index U_t [132]. This inequality can be combined with the conventional gradient inequality for quadratic functions to arrive at the expression:

$$U_t - U_t^e \leq \left(s_t - s_t^e\right)\mathbf{M}_\mathbf{t}^\mathbf{s} + \beta G_t\left(U_{t+1} - U_{t+1}^e\right),\ \text{where}\ \mathbf{M}_\mathbf{s}^\mathbf{t} \equiv 2\left(b_t - s_t^e\right)$$

Here, $\mathbf{M}_\mathbf{t}^\mathbf{s}$ is the equilibrium time t price of services, while βG_t values the time t + 1 state-contingent utility of services. Thus, the equation implies that any pair (s_t, U_{t+1}) preferred to ($s^e{}_t$, $U^e{}_{t+1}$) costs more at time t. Hansen et al argue that G_t can be computed as the conditional expectation of the state in the transformed state equation system:

$$x_{t+1} = \widetilde{A}x_t + \widetilde{C}w_{t+1},$$

where the modified C variable satisfies: $\widetilde{C}\widetilde{C}' = C\left(I - \sigma C'\Omega C\right)^{-1}C'$, and $\widetilde{A} = \left(I + \sigma C\left[I - \sigma C'\Omega C\right]^{-1}C'\Omega\right)A^*$ [133]. It can be seen that both the conditional mean and the

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[132] The inequality is established using the convexity property of the exponential function, the conditional version of the Hölder inequality applied to a convex combination of U_t and $U^e{}_t$, with weights (1 - δ) and δ, and the limit rule for logarithms: $\log\lambda \approx \lim_{\delta\to 0}\left[\frac{\lambda^\delta - 1}{\delta}\right]$.

[133] Hansen et al (1999, p. 890) mention the fact that, given the modified C and A matrices, asset-prices can be computed using algorithms described in Hansen and Sargent (1996, 1999). In appendix B of their paper they confirm that for $U = x'\Omega x + \rho$, where $x = \mu + Cw$ and w is distributed normally with mean zero and covariance matrix I, the G_t operator is the conditional expectation with respect to a new probability measure. Moreover, this measure is constructed using the Radon-Nikodym derivative V/EV where:
$V = \exp(\sigma U/2) \propto \exp(\sigma w'C'\Omega Cw/2) + \sigma w'C'\Omega\mu$. Thus, for any bounded Borel measurable function ϕ mapping $\Re^n \to \Re$:

conditional variance associated with the operator G_t is greater than that for $C'C$ because $\sigma < 0$, Ω is negative semidefinite, and I is replaced by the larger matrix $(I - \sigma C'\Omega C)^{-1}$. In a footnote (fn. 24, p. 890), the authors note that Epstein and Wang's non-unique "beliefs" (multiple-priors) approach to uncertainty aversion permits a distinction between games with zero-time commitment and those with sequential choice that cannot be made using the risk-sensitive, robust control approach to Knightian uncertainty (for an overview, see Appendix section A.8). In contrast, under the latter approach, "beliefs" are unique and prices are completely determinant.

Hansen et al. show that multiple period contingent claims can be evaluated by constructing a family of operators $G_{t,\tau} = G_t, G_{t+1}, \ldots, G_{t+\tau-1}$, where $G_{t,0}$ is the identity map. Now, $G_{t,\tau}$ is a conditional expectation, under the transformed probability measure, of a sequence of random variables that are measurable with respect to $J_{t+\tau}$.

Under the permanent income model, consumption goods are a bundle of claims to future consumption services. Therefore, the equilibrium prices of a stream of services can be used to price titles to consumption goods. This is accomplished by iterating on the expression for

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$E[V\phi/EV] \propto \int \phi(w)\exp(\sigma w'C'\Omega Cw/2 + \sigma w'C'\Omega\mu)\exp(-w'w/2)dw$.

The following identity:

$\sigma w'C'\Omega Cw/2 + \sigma w'C'\Omega\mu - w'w/2 = -w'(I - \sigma C'\Omega C)w/2 + w'(I - \sigma C'\Omega C)(I - \sigma C'\Omega C)^{-1}\sigma C'\Omega\mu$

reveals that the operator $E[V\phi/EV]$ can be evaluated by integrating ϕ with respect to a normal density with mean vector $\hat{\mu} \equiv (I - \sigma C'\Omega C)^{-1}\sigma C'\Omega\mu$ and covariance matrix $\hat{\Sigma} \equiv (I - \sigma C'\Omega C)^{-1}$.

deviations from optimal utility previously derived from the sub-gradient inequality to yield an infinite horizon version of the inequality (p. 891):

$$U_t - U_t^e \leq \sum_{\tau=0}^{\infty} \beta^\tau G_{t,\tau}\left(\mathbf{M}_{t+\tau}^s s_{t+\tau}\right) - \sum_{\tau=0}^{\infty} \beta^\tau G_{t,\tau}\left(\mathbf{M}_{t+\tau}^s s_{t+\tau}^e\right).$$

This particular version of the inequality can be interpreted to mean that any *stream* of services $\{ s_t \}$, strictly preferred to $\{ s^e_t \}$, must cost more. In each case, the respective summation term can be viewed as the price of an asset offering a claim to a specific stream of services.

To arrive at an observable consumption goods counterpart to this expression, the authors replace $\mathbf{M}_{t+\tau}^s$ with $\mathbf{M}_{t+\tau}^c = M_c x_t$. The FOC's from the solution to the permanent income problem are used to arrive at an expression for the indirect marginal utility of consumption by pricing M_c, the implicit service flow associated with a unit of consumption [134]. Hansen et al. (p. 891) then utilize this derived indirect marginal utility for pricing one-period securities, using the conventional formula[135]:

$$q_t = G_t\left\{\left[\beta \mathbf{M}_{t+1}^c / \mathbf{M}_t^c\right] p_{t+1}\right\}$$

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[134]Specifically: $M_c \equiv 2\left[(1-\lambda) + (1-\delta_h)\sum_{\tau=0}^{\infty} \beta^\tau (\delta_h)^\tau (-\lambda)(\hat{A})^\tau\right](S_b - S_s)$

[135] For comparison, Epstein and Zin's (1989) and Epstein and Wang's (1994) derivations of the Euler and asset-pricing equations are included in Appendix section A.8 of the thesis.

where q_t is the time t price of the one period security and p_{t+1} is the time t + 1 total payoff (dividend plus capital gain). This expression can also be written in the equivalent form of a conditional expectation under either the distorted expectations operator $\check{E}$, or the original expectations operator E, as follows:

$$q_t = \check{E}_t\left\{\left[\beta \mathbf{M}^c_{t+1}/\mathbf{M}^c_t\right]p_{t+1}\right\} = E\left[m_{t+1,t}\, p_{t+1} \middle| J_t\right]$$

The authors note that, in this case, the $m_{t+1,t}$ variable can be interpreted as a one-period stochastic discount factor (or alternatively, as an intertemporal marginal rate of substitution that has been scaled by a random variable with conditional expectation of one).

This expression for q_t can be transformed using the *covariance decomposition* to yield (p. 892):

$$q_t = E_t(p_{t+1})E_t(m_{t+1}, t) + \mathrm{cov}_t(m_{t+1,t}, p_{t+1})$$

The Cauchy-Schwartz inequality can now be applied to the covariance term to determine a price bound:

$$q_t \geq E_t(p_{t+1})E_t(m_{t+1,t}) - \mathrm{std}_t(m_{t+1,t})\mathrm{std}_t(p_{t+1})$$

From this equation, Hansen et al. then calculate an expression for the price of risk relative to expected return (i.e., the market price of risk) along the efficient frontier: $\mathrm{std}_t(m_{t+1,t})/E(m_{t+1,t})$.

Although the risk-sensitivity parameter σ and preference curvature parameter μ_b are not identifiable from quantity data, Hansen et al. use the above expression to ascertain the effect these parameters exert over the market price of risk. The Kalman filter is used to compute the expectation $E(x_t \mid y_t, y_{t-1}, \ldots, y_t)$ for $y_t = \{c_t, i_t\}'$ and $x_t = \left[h_{t-1}\; k_{t-1}\; 1\, \hat{d}_t\; \hat{d}_{t-1}\; d_t^*\; d_{t-1}^*\right]$, providing a sequence of fitted states for estimating the median market price of risk over the sample period for alternative (σ, μ_b) pairs (tables 3a, 3b; p. 893).

To calculate the risk-sensitive and robust components of these estimates, Hansen et al. show that in each case the stochastic discount rate can be decomposed into two multiplicative parts (pp. 894-5, and their Appendix C):

	Stochastic discount factor	**Uncertainty component**
Risk Sensitive	$m_{t+1,t} = m^f_{t+1,t} m^r_{t+1,t}$	$m^r_{t+1,t} \equiv \dfrac{\exp\left(\sigma U^e_{t+1}/2\right)}{E\left[\exp\left(\sigma U^e_{t+1}/2\right) J_t\right]}$
Robust	$m_{t+1,t} = m^f_{t+1,t} m^u_{t+1,t}$	$m^u_{t+1,t} = \dfrac{\exp\left[-\left(w_{t+1} - \hat{v}_t\right)'\left(w_{t+1} - \hat{v}_t\right)/2\right]}{\exp\left(-w'_{t+1} w_{t+1}/2\right)}$

Here, $m^f_{t+1,t} \equiv \beta \dfrac{\mathbf{M}^c_{t+1}}{\mathbf{M}^c_t}$. In each of the two cases, the market price of risk approximately equals the standard deviation of the respective uncertainty component. For the robust case, the authors show that the market price of risk is approximately equal to the time t specification error $|\hat{v}_t|$.

The solution for v within the Markov perfect equilibrium $\hat{v}_t = \sigma(I - \sigma C'\Omega C)^{-1} C'\Omega A^* x_t$ makes the worst-case specification error a linear function of the Markov state. Thus, the calculated value for v can be decomposed into two components, $\hat{v}_{\hat{d},t}$ and $\hat{v}_{d*,t}$, associated, respectively, with the two (subscripted) endowment shocks. Hansen et al. have utilised the estimated values of these two shocks over the sample period to report the sizes of median, maximum and minimum specification error components under different values for the (σ, μ_b) pairs (see their tables 4a, 4b, p. 897). Understandably, the most important determinant of the overall error is the persistent component of income d^*_t.

In the final section of their paper (section 7, pp. 896-901) Hansen, Sargent and Tallarini advance beyond the calculation of one-period tradeoffs between the mean and standard deviation of asset returns to investigate local and global *intertemproal* tradeoffs associated with different values of σ and μ_b. The local tradeoff is achieved by introducing a small scalar perturbation ε to the state evolution equation, starting at time t – 1 and continuing into the future, and then obtaining the derivative calculated from the ratio of the change from the pre- and post-perturbation value functions to the perturbation. To this end, the authors exploit the usual subgradient inequality. Increasing σ in absolute value has a stronger effect on the mean-risk tradeoff than on the market price of risk, while increases in μ_b have a slightly larger impact. The global measure is calculated by setting ε = -1, which sets to zero the shock variance for the endowment process and holds the permanent income decision rule for the two competing specifications of the endowment process.

These measures can be interpreted as an estimate of the welfare cost associated with fluctuations in the endowment process.

5.2.2. Andersen, Hansen and Sargent's Paper

Andersen, Hansen and Sargent (1999) go beyond Hansen, Sargent and Tallarini (1999) by replacing their continuation bounded, linear quadratic framework with one based on non-quadratic current value functions and a more general reference set of stochastic processes, including Markov and Markov jump-diffusion specifications. In addition, the approach to bounded perturbations is extended using relative entropy constraints to encompass distortions to the transition probabilities of an *n*th-order continuous-time hidden-Markov model[136]. All that is required to ensure that the log likelihood ratio is well defined, is that the transition probability distribution for the candidate model be absolutely continuous with respect to the transition probabilities for the reference model. Nevertheless, from Girsanov's theorem, it can be seen that the absolute continuity requirement is very demanding. Andersen et al. observe that instantaneous covariance matrices can be inferred with arbitrary accuracy from high frequency data. Therefore, models that are statistically close will possess identical diffusion matrices (p. 25). For this reason, they confine their investigation of misspecification to the drift component of candidate models.

[136] More general material on robust control under relative entropy constraints (i.e. for the partially observed case) is provided below in Appendix section A.9 of the thesis.

Andersen et al. (1999, p. 8) examines robustness in a discrete-time setting using operator notation. They assume that a Markov process governs the evolution of the state vector. They model this process by specifying the conventional one-period conditional expectation operator:

$$\mathbf{T}(\phi)(y) = E[\phi(x_1)|x_0 = y],$$

defined over the space of bounded continuous functions. Following Epstein and Zin (1989), they begin with the value function $W(x)$ that solves the following recursive functional equation:

$$W(x) = U(x) + \exp(-\delta)\mathbf{T}W(x)$$

over an infinite horizon. Here, $U(x)$ is the current-period reward function and δ is the subjective rate of discount.

As in the Hansen et al. paper. Andersen, Hansen and Sargent then generalize this formulation to incorporate risk sensitivity by replacing the operator **T** with the following specification of **R**:

$$\mathbf{R}(W) = -\theta \log\left(\mathbf{T}\left[\exp\left(\frac{-W}{\theta}\right)\right]\right)$$

The recursive utility formulation based on this operator is motivated by a demonstration that **R** bounds the conditional tail probabilities of W. They establish (pp. 9-10) that:

$$\Pr\{W(z) \leq -r | y\} \leq \exp\left\{-\frac{1}{\theta}\mathbf{R}(W)(y)\right\}\exp\left\{-\frac{r}{\theta}\right\}.$$

The tail probability on the left is bounded by an exponential in r that declines at the rate $-1/\theta$, with **R** influencing the constant term in the bound. Decreasing θ increases the exponential rate at which the bound sends the tail probabilities to zero[137].

They define the class of candidate models in relation to the reference Markov process by introducing the following distorted expectation operator (p. 12):

$\mathbf{T}^{w}(\phi)(y) = \dfrac{\mathbf{T}(w\phi)(y)}{\mathbf{T}(w)(y)} = \dfrac{E[w\phi(x_1)|x_0 = y]}{E[w(x_1)|x_0 = y]}$, defined for strictly positive functions w. Under this construction $\dfrac{w(z)}{\mathbf{T}(w)(y)}$, with z indexing tomorrow's state and y indexing today's state, is the altered transition law with respect to the reference law. The discrepancy between candidate and reference models is measured using relative entropy (the expected value of the log-likelihood

[137] The tail probability bound comes from the theory of large deviations and is based on the inequality $\mathbf{1}_{\{W: W \leq -r\}} \leq \exp\left[\frac{-(W+r)}{\theta}\right]$, where **1** is an index function. Taking the log of the conditional expectation of this inequality gives:
$\log\Pr\{W(z) \leq -r | y\} \leq \log\left\{E\left(\exp\left[-\frac{W(z)}{\theta}\right]\middle| y\right)\exp\left(\frac{-r}{\theta}\right)\right\} = -\frac{1}{\theta}\mathbf{R}(W)(y) - \frac{r}{\theta}$. Taking the exponential elicits the desired tail probability bound.

ratio, which is equal to the log of the Radon-Nikodym derivative defined above). For a candidate model indexed by w, the relative entropy is (p. 13):

$$I(w)(y) \equiv E^{w}\left[\log \frac{w(z)}{\mathbf{T}(w)(y)} \middle| y\right] = \mathbf{T}^{w}(\log w)(y) - \log[\mathbf{T}(w)(y)] \geq 0$$

Substituting for $\mathbf{T}^{\mathbf{w}}$ gives:

$$I(w) = \frac{\mathbf{T}(w\log(w))(y)}{\mathbf{T}(w)(y)} - \log(\mathbf{T}(w)(y)) = E\left[\frac{w(z)}{E[w(z)|y]}\log\left(\frac{w(z)}{E[w(z)|y]}\right)\middle| y\right].$$

Andersen et al. (p. 14) consider the following problem defined for a continuation function W and given parameter $\theta > 0$:

$$\inf_{w>0} J(w), \text{ where } J(w) \equiv \theta I(w) + \mathbf{T}^{w}(W)$$

Assuming that $\mathbf{T}$ can be evaluated at $\exp(-W/\theta)$ in their proposition 1, the authors prove that this problem has the solution:

$$w^* = \exp\left(\frac{-W}{\theta}\right),$$

which attains the minimized value $J(w^*)$ = $\mathbf{R}(W)$ where $\mathbf{R}(W) = -\theta \log\left(\mathbf{T}\left[\exp\left(\frac{-W}{\theta}\right)\right]\right)$.

Although the solution w^* is only unique up to scalar multiple, the minimised value $J(w^*)$ and the associated probability law are unique.

The authors verify that w^* is a solution by writing (p. 15):

$$I(w) = I^*\left(w/w^*\right) + \frac{\mathbf{T}\left(w \log w^*\right)}{\mathbf{T}(w)} - \log \mathbf{T}\left(w^*\right),$$

where,

$$I^*(w) = \frac{\mathbf{T}^*(w \log w)}{\mathbf{T}^*(w)} - \log \mathbf{T}^*(w) \quad \text{and} \quad \mathbf{T}^*\phi \equiv \frac{\mathbf{T}\left(w^*\phi\right)}{\mathbf{T}\left(w^*\right)}.$$

They observe that I^* itself, can be interpreted as a measure of relative entropy and hence, $I^*(w/w^*) \geq 0$. Thus, the criterion J satisfies the inequality:

$$J(w) = \theta\left[I^*(w/w^*) + \frac{\mathbf{T}(w \log w^*)}{\mathbf{T}(w)} - \log \mathbf{T}(w^*) \right] + \mathbf{T}^{\mathrm{w}}(W)$$
$$\geq \theta\left[\frac{\mathbf{T}(w \log w^*)}{\mathbf{T}(w)} - \log \mathbf{T}(w^*) \right] + \mathbf{T}^{\mathrm{w}}(W) = -\theta \log \mathbf{T}\left[\mathbf{exp}\left(-\frac{\mathbf{W}}{\theta} \right) \right] = J(w^*)$$

From the definition of $J(w)$ and the fact that $J(w^*) = \mathbf{R}(W)$, corollary 1 immediately follows:

The conditional expectation of the value function W evaluated under the transition $\mathbf{T}^{\mathrm{w}}$ *satisfies the bound:*

$$\mathbf{T}^{\mathrm{w}}(W) \geq \mathbf{R}(W) - \theta I(w)$$

This inequality identifies θ as a type of utility-related price of robustness When w is a constant function, $I(w) = 0$, and $\mathbf{R}$ underestimates the expected value of W. Otherwise, the risk-sensitive expectation operator $\mathbf{R}$, puts a bound θ on the rate at which the expected value of W can fall with increasing discrepancy, as measured by the relative entropy $I(w)$.

Andersen et al. (1999, p. 16) outline a procedure for generalizing the above class of perturbations. They note that, in principle, the Radon-Nikodym derivative h_{t+1} can be made a function of the entire past history of the Markov state for the reference process:

$$h_{t+1} = \frac{w(z_{t+1}, z_t, z_{t-1}, \ldots)}{E\left[w(z_{t+1}, z_t, z_{t-1}, \ldots) \middle| z_t, z_{t-1}, \ldots \right]}.$$

The time t conditional entropy and control objective can be similarly modified to accommodate non-Markov and nth-order Markov candidate models.

With this theoretical background, the analysis proceeds in continuous time for reasons of simplicity. Continuous-time Markov and diffusion processes are specified using infinitesimal generators that are obtained as a limit of discrete-time generators. Perturbations are defined in terms of distorted generators of the continuous-time processes associated with w. Similarly, continuous-time entropy measures are formulated as the limiting case of their associated discrete-time counterparts. Unsurprisingly, the resulting robust value function recursion turns out to be one of the stochastic differential utility recursions studied in Duffie and Epstein (1992) whose discrete-time counterparts were examined in Epsten and Zin (1989).

Equipped with these technical instruments, Andersen et al. follow in the footsteps of Hansen, Sargent and Tallarini (1999). They consider a social planner's optimal allocation of consumption and capital accumulation. Equilibrium prices are calculated under both a preference for robustness and under risk-neutrality: the former, by solving a recursive continuous-time version of the familiar two-player Markov game. Equilibrium asset-prices are then computed by treating the optimal robust and risk-neutral allocations as though they had emerged from a pure endowment economy. Alterations in these asset-prices, arising in response to the activation of a preference for robustness, represent the pricing premia due to uncertainty. Once again, stochastic discount factors are decomposed into a pair of multiplicative components, as in Hansen et al.

(1999), with one component representing factor risk prices and the other representing uncertainty prices.

Although more general than the material covering the discrete time case, essentially little of real significance is added beyond what has already been discussed in the Hansen et al. (1999) paper. The main addition is the construction of a "Chernoff entropy metric" for measuring model misspecification that includes relative entropy as a limiting case, and which Andersen et al. (1999) show to be closely related to the market price of risk. This metric enables them to bound type 1 and type 2 errors for discriminating between candidate and reference Markov processes.

5.2.3. Aaron Tornell's H^{∞} Asset-pricing Model

Aaron Tornell (2000) adopts an H^{∞} control framework to interpret asset-pricing anomalies such as excess volatility in response to news. His model is worth detailed consideration because it demonstrates the depth of the insights that can be gained from an H^{∞} control interpretation of ambiguity in decision-making. Moreover, despite its simplicity, the specific asset-pricing application that has been chosen clearly demonstrates how the state equation/observation equation framework can be put to good use in modeling portfolio decisions.

Pointedly, Tornell rejects the view that asset-pricing anomalies are a result of misperception or irrationality (p. 1). While continuing to accept the assumption that agents, knowing the model that generates payoffs, must filter the persistent and transitory components of a sequence of observation in order to estimate the unobservable state of the economy, his point of departure is that agents are not perfectly sure about this model. For one thing, the disturbance process may be misspecified and, for another thing, the payoff model, itself, may be misformulated. In Hansen, Sargent and Tallarini (1999), risk-sensitive approach shocks are assumed to be normally distributed. In contrast, uncertainty is not parameterized in Tornell's analysis. His H^{∞} approach models uncertainty in the form of unknown disturbance sequences with a bounded l_2-norm. Thus, while the rational expectations solution to the control problem is designed to achieve the best performance conditional on the absence of misspecification in the relevant probability distribution, the H^{∞} solution to the control problem is designed to perform well under any norm-bounded misspecification. Moreover, while under rational expectations forecasts (that are based on a recursive version of the Kalman filter) can be formed independently from agent's choices, under H^{∞} control forecasts and robust portfolio choices are jointly determined. Critically, Tornell no longer assumes that the state is perfectly observed. Observation error is both an intrinsic and also an important part of the story he wants to tell about asset-pricing.

Tornell constructs an overlapping generations model of a single-good, two-asset exchange economy without a government sector. Each financial asset pays a one-period return: one of these returns is risk-free while the other return is risky. For the risky asset, the observed dividend process (y_t) is assumed to combine a persistent component (x_t) and a transitory component (v_t) so that $y_t = x_t + \sigma_v v_t$. However, the persistent component is unobservable and autocorrelated: $x_t = a\ x_{t-1} + \sigma_w w_{t-1}$. Agents have time-additive quadratic utility:

$u(c_t)+\beta u(c_{t+1})$ where $u(c_t)=-[m-c_t]^2$, with m a constant term.

At the beginning of time t, the younger generation is endowed with one unit of the risky asset. At time t, taking prices as given, the agent chooses consumption c_t, and the amount of risky and safe assets (q_t^r, q_t^s) she wishes to hold. The relevant budget constraint becomes:

$$c_t + p_t^r q_t^r + p_t^s r_t^s \leq p_t^r + y_t$$

At $t+1$, when the agent is old, she consumes all her wealth, given by:

$$c_{t+1}^o = y_{t+1} q_t^r + q_t^s .$$

The dividend process is expressed in deviation form with trend term d_j:

$$y_j = x_j + d_j + \sigma_{vj} v_j, \qquad j \geq 1$$

$$x_{j+1} = a_j x_j + \sigma_{wj} w_j, \qquad x_0 = 0$$

Tornell assumes that while the parameters $\{a_j\}_{j=1}^t, \{\sigma_{wj}\}_{j=0}^t, \{\sigma_{vj}\}_{j=1}^{t+1}$, and $\{d_j\}_{j=1}^{t+1}$ are known, neither the disturbances $\{v_j\}_{j=1}^{t+1}$ and $\{w_j\}_{j=0}^t$ nor the initial value of the unobservable state x_1 are known (to achieve the latter condition, x_0 is set equal to 0 so that x_1 equals $\sigma_{w0}\ w_0$).

For the rational expectations economy, under which the disturbance terms are presumed to be i.i.d. $N(0,1)$ random variables[138], given asset-prices (p_t^r, p_t^s) the young agent chooses (c_t, q_t^r, q_t^s) to maximize:

$$E\left[-(m-c_t)^2 - \beta(m-c_{t+1}^o)^2 \middle| I_t\right]$$

subject to the above budget constraints and dividend process where the information set equals $\{y_1, ..., y_t, p_t^r, p_t^s\}$.

For the H^∞ economy, given asset-prices and the dividend history $\{y_t\}_{j=1}^t$, the agent chooses (c_t, q_t^r, q_t^s) to ensure that:

$$\frac{[m-c_t]^2 + \beta[m-c_{t+1}^o]^2}{\sum_{j=0}^{t+1}[w_j^2 + v_j^2]} \leq \gamma_{t+1}^2$$

for all non-zero disturbances $\{v_j, w_j\}_{j=0}^{t+1} \in l_{2[0,t+1]}$ that are consistent with the dividend history $\{y_j - d_j\}_{j=1}^t \neq 0$ and satisfy the above budget constraints. Here, the γ-term reflects the degree of disturbance attenuation so that a lower γ implies a greater degree of robustness.

[138] Of course, with a quadratic utility function, it can be shown that the respective value function is also quadratic in consumption. Moreover, when disturbances are i.i.d. normally distributed it can be shown that conditional expectations for the consumption CAPM collapse into unconditional expectations yielding a conventional beta-CAPM model with constant intercept terms and betas (see Cochrane, 2000, section 9.1.3).

After the budget constraints have been substituted into the quadratic value function Tornell derives a new version of the value function consisting of two quadratic terms in current and next period asset quantities and observed dividends and an additional term in the sum of squares of the observed and unobserved disturbances. It transpires that the multiplier on this last term is the next period's robustness parameter γ_{t+1}.

The H^∞ *certainty equivalence principle* breaks the original problem into three sub-problems. First, for a given value of the state $x_{t+1} = x$, portfolio and consumption strategies are derived through *backward* dynamic programming. In this intermediate stage a sequence of disturbances is selected from the set that is compatible with observed dividends that is the worst possible given the objective function for the dynamic game. Second, the agent solves a *forward* dynamic programming problem to extract the persistent component from past dividend's observations y_t, conditional on $x_{t+1} = x$. At this stage it transpires that the sequence of unknown disturbances in the state equation has no bearing on the chosen optimum. However, the optimal amounts of each asset are chosen to maximize the objective function, given that nature has already selected the worst possible sequence of observation errors. Third, H^∞ estimates of x_{t+1} and y_{t+1}, and equilibrium asset-prices are derived using the value functions associated with forward and backward dynamic programming solutions to the previous two sub-problems.

The formulas for the optimal forecast of dividends, the unobservable state and prices of the risky and risk free assets derived from the three-stage optimization are set out in Tornell's proposition 2.2 (pp. 13-15). Let $\underline{\gamma}_{t+1} = \sqrt{\beta\left[Z_{t+1} + \sigma_{vt+1}^2\right]}$, where Z_{t+1}, the variance of the unobservable state, is the tth step in the recursion:

$$Z_{j+1} = \frac{a_j^2}{Z_j^{-1} + \sigma_{vj}^{-2}} + \sigma_{wj}^2, \quad Z_1 = \sigma_{w0}^2.$$

As is generally the case in robust control problems, too much robustness makes it impossible to find a feasible mini-max solution. Then, if $\gamma_{t+1} \le \underline{\gamma}_{t+1}$, there exists no equilibrium; while if $\gamma_{t+1} > \underline{\gamma}_{t+1}$, there exists an equilibrium in which the H^∞ one-period-ahead forecast of dividends is:

$$\mathbf{F}_t(y_{t+1}) = x_{t+1}^* + d_{t+1} + \frac{x_{t+1}^* + d_{t+1} - m}{\gamma_{t+1}^2 \beta^{-1} \sigma_{vt+1}^{-2} - 1}.$$

The appearance of the lower bound on γ_{t+1} reflects the fact that the value of the game between nature and the agent is bounded only if $\gamma_{t+1} > \underline{\gamma}_{t+1}$. Moreover, the estimate of the unobservable state is:

$$x_{t+1}^* = \frac{\hat{x}_{t+1} + Z_{t+1}\left[d_{t+1} - m\right]\left[\gamma_{t+1}^2 \beta^{-1} - \sigma_{vt+1}^2\right]^{-1}}{1 - Z_{t+1}\left[\gamma_{t+1}^2 \beta^{-1} - \sigma_{vt+1}^2\right]^{-1}}$$

where Z_{t+1} is the tth step in the above recursion and $\hat{x}_{t+1}$, the mean of the unobservable state, is the tth step in the recursion:

$$\hat{x}_{j+1} = a_{j+1}\hat{x}_j + h_j\left[y_j - d_j - \hat{x}_j\right], \quad h_j = \frac{a_j Z_j}{Z_j + \sigma_{vj}^2}, \hat{x}_1 = 0.$$

Finally, the equilibrium prices for the risky and safe assets are:

$$p_t^{r*}(y^t) = \beta \frac{\mathbf{F}_t(y_{t+1}) - m}{y_t - m} . \mathbf{F}_t(y_{t+1}), \quad p_t^{s*}(y^t) = \beta \frac{\mathbf{F}_t(y_{t+1}) - m}{y_t - m}$$

Tornell observes that the estimate of the unobservable state x^*_{t+1} is a linear function of $\hat{x}_{t+1}$, which is obtained recursively by adding to $\hat{x}_j$ a term proportional to the innovation. Similarly, the forecast of dividends $\mathbf{F}(y_{t+1})$ is obtained by adding to x^*_{t+1} the trend coefficient d_{t+1} and the forecast of the disturbance v_{t+1}.

In the limit when $\gamma_{t+1} \to \infty$ the H^∞ forecasting formulas coincide with their rational expectations counterparts. In particular, the recursive estimator for the state is replaced by the conventional Kalman filter.

In the third part of his paper Tornell assumes that the disturbance sequences are i.i.d. $N(0,1)$ and that the parameters of the data generating process are constant. He sets these constants equal to their maximum likelihood estimates for US stockmarket data over the period 1871-1997. Forecasts and prices are then derived using both the rational expectations and H^∞ formulas. The m parameter in the utility function was selected so that the average risk-free rate matched the 4-6 month commercial paper rate for the chosen period.

Regressions of excess returns on observed dividend price ratios revealed no significant predictability when the sequences derived from rational expectations formulas were employed whereas estimates from the H^{∞} formulas, with low values for the γ parameter, exhibited significant predictability in conformity with empirical evidence (Table 1, p. 30). Moreover Tornell shows that, for low γ, the H^{∞} prices also tend to violate the variance bounds, whereas the rational expectation prices do not. The variance bounds are calculated by comparing the variance of price estimates with the variance of price sequences that would have prevailed if the discount factor was constant and agents had perfect knowledge of future dividends. In turn, the latter sequences are calculated by substituting y_{t+1} in place of $\mathbf{F}(y_{t+1})$ in the price formulas (Table 2, p. 32). Tornell calculates 100 dividend sequences and tabulates the number of times the variance of the forecast price sequences exceeds that of the perfect foresight price sequences. In addition, Tornell presents data on the magnitude of the equity premium and the risk-free rate for various values of the γ parameter. As expected, the equity premium increases as γ falls. For $\gamma = 0.5$ the premium attains a value of 5.9% while maintaining a low risk-free rate of 1.49% (Table 3, p. 34). Finally, Tornell examines how well the rational expectations and H^{∞} formulas track actual US stockprices over the period 1871-1996. The H^{∞} price sequences track the actual S&P500 index much better than their rational expectations counterparts. Tornell attributes the ability of H^{∞} prices to exhibit similar anomalies to the actual US data to the fact that H^{∞} forecasts are more sensitive to dividend news (and as we have seen, this sensitivity is inversely related to the size of the robustness parameter). He concludes:

> The point we want to make is that, in a simple asset-pricing model, excess sensitivity to news can result from either: (a) *misperception*

> of the duration of shocks in a behavioural setup, or (b) from a desire for *robustness* in an H_∞ setup (Tornell, 2000, p. 38).

A key difference is that in an H^∞ approach, agents effectively use the same 'nominal' model as RE agents, whereas in a behavioural setup agents employ a different model. Tornell reviews some of the approaches taken in this behavioural literature (pp. 38-9), observing that in some models agents are *overconfident* about the precision of their private signals (i.e. they perceive the noise-to-signal ratio to be lower than it actually is), or else, the noise-to-signal ratio is reduced through the *association* of current events with memories evoked of similar past events, while in other models agents overreact to several signals pointing in the same direction because they project trends whereas, in fact, the earnings series is presumed to follow a random walk. We could add to this review the literature on *rational belief equilibria* (Kurz and Motolese, 1999), in which agents estimate regime switching models with fewer dimensions than exist in actuality. Nevertheless, the unconditional moments of the estimates conform closely to those of the real world. In this case, a particular form of bounded rationality is responsible for agent misperception.

In defence of the robust or risk-sensitive control approach I would point out that an enormous amount of intellectual power and technical virtuosity is applied to analyzing outcomes in financial markets so that certain investors can profit from the misperceptions of others. The robust approach has the advantage that it does not have to rely on what is really a set of elaborate metaphors for agent misperception. Any concern for robustness is firmly grounded in *substantive* and *persistent* ontological features of the control and filtering environment.

Tornell's work shows that we do not have to resort to the most sophisticated forms of risk-sensitive control under stochastic uncertainty constraints to arrive at results that are similar to those of Hansen and Sargent. Robust control techniques using the most basic form of utility function suffice to explain predictability of returns, excess volatility and the equity premium. However, the theoretical advantage of the risk-sensitive approach is that non-state separable utility functions reconcile the new control techniques with current research into the axiomatic foundations of non-expected utility theory. Towards the end of his paper Tornell notes that:

> "…there are several asset-pricing anomalies that we have not observed. Our purpose has been simply to illustrate how the robust approach can help to explain some anomalies. In this respect, a promising direction for future research is to consider environments in which the H_∞ approach generates richer price dynamics." (pp. 39-40).

He notes that one opportunity for further research involves the construction of models in which the robustness parameter γ exhibits endogenous fluctuations over the business cycle. This has been one of my key arguments. As models like this are developed, we start to move away from the neoclassical world into a very Keynesian world.

5.3. Questioning the Foundations

In Chapter Four, I identified what I saw as the key ontological features of a monetary production economy that had been ignored in much of what passes for finance theory. I shall now reexamine each of these features to provide a critique of the work by Andersen, Hansen and Sargent (1999), Hansen, Sargent and Tallarini (1999), and Tornell (2000). However, I want to stress that this critique should not be regarded as negative in its entirety. What I am trying to establish is the fact that existing techniques of risk-sensitive and robust control, once they have been extricated from their orthodox integument, can potentially be utilized to reinvigorate Keynesian modelling of uncertainty in macroeconomics and finance.

5.3.1. The Implications of Moving Beyond the Representative Agent Growth Model

Modern finance theory is firmly based on Walrasian General Equilibrium foundations. As De Vroey observes (1999):

> [i]n the Marshallian perspective, involuntary unemployment should be characterized as an *equilibrium* result, whereas in the Walrasian perspective it can only be characterized as a *disequilibrium* state. Thus, in reference to the Marshallian approach, a theory of involuntary unemployment aims at turning the Marshallian standard market-clearing disequilibrium result upside down and replacing it with models of non-market clearing equilibrium. The 'Keynesian-Walrasian' involuntary unemployment program is quite distinct as it consists of substituting a point-in-time market rationing result for a point-in-time market clearing result, thus replacing the temporary equilibrium outcome by its converse—a 'temporary disequilibrium' (p. 180)[139].

[139] A classical intepretation of liquidity preference is provided in the 1988 book by Carlo Panico. There, liquidity premia are treated as long-period phenomena driving a wedge between interest rates on short-term bonds, long-term bonds and bank lending rates. Long-period equilibrium in capital markets is defined by the state of backwardation, under which the spot price of capital goods would exceed the forward price by an amount sufficient to cover the difference in net yield that would be recouped through immediate delivery rather than actual delivery at the end of a set, and well understood, period of time.

I have argued in the previous chapter that one vehicle for introducing quantity-constrained rationing into a Walrasian model is to recognise coordination failure across inter-related markets—specifically, across markets for money, capital, goods and labour. This is a simple but vital point that I have made repeatedly throughout this thesis. The object of my critique has been the representative "consumer-saver-corporate investor-asset-manager" model of a moneyless (single-commodity) corn-world (with its pure-exchange counterpart in a Lucas tree financial market).

In the last chapter I reviewed the work of Thomas Palley (1996, 1999), which establishes anew the mechanics of quantity-constrained rationing and debt-deflation effects in macroeconomics. My main purpose, there, was to show that conventional faith in wage and price flexibility as a solution to unemployment was misplaced. This simple and obvious truth is belied by daily call for labour market deregulation as a cure-all for our economic ills.

However, my intention was also to call for a return to what many economists regard as a defunct research agenda. My point is that robust and risk-sensitive control techniques can be gainfully utilized to model endogenous variations in constraints as uncertainty impacts differentially on the decisions of investors, banks, firms and households. Each of these decisions has already been modeled in isolation by rational expectations theorists armed with LQG control and Kalman filter techniques. If these endeavours were superceded by ones based firmly on methods of robust and risk-sensitive control, but one that that incorporated quantity constrained rationing, a very different picture of the macroeconomy would come to the fore. This is by no means an easy task, but in my view it is a worthwhile one.

At this point, I want to return to the specific matter of uncertainty aversion so that I can question Hansen, Sargent and Tallarini (1999) and Andersen, Hansen and Sargent's (1999) interpretation of the continuation norm bound and relative entropy constraint representations of this phenomenon. I also intend to raise some fundamental questions about the difference between engineering and social science applications of risk-sensitive control.

5.3.2. Relative Entropy Constraints and Norm Bounds

In the section of the appendix covering risk-sensitive control under relative entropy constraints, I discuss the relationship between stochastic uncertainty constraints based on relative entropy and those based on sum quadratic norm bound constraints (section A.9). Tornell is alone in recognizing the limitations of the full information framework adopted in the Hansen, Sargent and Tallarini (1999) and Andersen, Hansen and Sargent (1999) papers. This concern is more than a theoretical curiosum because it goes to the heart of what these new techniques are supposed to achieve—namely, a resolution of the inadequacies besetting earlier, rational expectations approaches in economics and finance. The key point I have argued is that the choice of a full information setting with continuation norm bounds is deliberate: it essentially enables the New Classicals to bypass the implications of model uncertainty and observation error for rational decision making. In Chapter Three I suggested that one of the fundamental sources of model uncertainty is the use of linear approximations for what are highly non-linear processes. Of course, in the context of pure exchange models, the assumption that dividend sequences are linear stochastic processes might seem reasonable. But, I have also argued against the use of pure exchange models in macroeconomics and in favour of quantity constrained rationing across related markets. While the actions of an individual investor have no macroeconomic consequences, the same cannot be said for a representative agent standing in for the decisions of an entire financial community.

5.3.3. Time Varying Parameters: A Discussion

In a previous chapter, I discussed the importance of time-varying parameters in models that exhibit financial instability. Another source of time-invariance is processes of learning. Andersen, Hansen and Sargent (1999) comment on the fact that large deviation theory can be used in both robust decision theory and learning theory. In the latter case, "...expressions from large deviation theory provide bounds on rates of learning". Nevertheless, in their application of robust control theory, learning processes are excluded by assumption. The fact that learning processes are an undeniable feature of real-world decision-making—one that has been incorporated in various ways into econometric studies—conveys the obvious implication that robust control theory should be applied to time-varying systems (e.g. see Savkin and Petersen, 1995), that can represent these learning effects as integral aspects of the dynamic system.

5.3.4. Post Keynesian Models of Animal Spirits and Liquidity Preference

In the previous chapter, I foreshadowed further consideration of dynamic models that use time-varying parameters to represent variations in liquidity preference and animal spirits. I intend to focus informally on two of these models, in Andresen (1999b) and Flaschel et al. (1996), respectively.

A central idea underpinning Andresen's work (1998, 1999a, b) is the notion that changes in liquidity preference can be modeled as a variation in the magnitude of a key parameter within a transfer function. This parameter can be conceived as governing the rate of outflow of money stocks from a "reservoir". The parameter value represents the sensitivity of outflow to the height of stored "fluid" in the reservoir. The basic model can readily be extended to represent heterogenous agents who not only hold a certain proportion of money on their own account, but also channel the remainder into two streams, one of which flows to agents residing outside their own sector, while the other flows to agents residing within the sector. Now, the sectoral build up of money stocks is influenced by two parameters, one determining personal stock holding and the other determining the division between inter- and intra-sectoral flows[140].

On this basis, Andresen (1999b) has developed a four-component dynamic model to replace the conventional IS-LM model. The first of the model's components or sub-systems is the rentier capitalist sector, the second is the accumulation sector, the third represents what Andresen calls the loss mechanism, and the fourth is the real economy, which engages in consumption and investment activity. The accumulation sector determines the amounts of interest and principle

[140] In this regard, Andresen claims to be following a precedent set by A.W. Phillips (1954). In his block diagram model a lag block appears, that is traditionally interpreted to represent delays in the adjustment of demand to production. In contrast, Andresen interprets the lag as representing his money stock based model of liquidity preference.

that have to be paid on outstanding debt. These receipts flow on to the rentier sector from where they are split into two streams. The first of these streams represents investment on the part of rentier agents amongst one another, while the second represents rentier savings channeled out to the real sector of the economy for the credit financing of both investment and consumption expenditures.

Andresen posits that rising indebtedness amongst agents in the real economy is equivalent to what Hyman Minsky (1985) has termed financial fragility. The adverse effects of increasing fragility are embodied in the loss mechanism in the form of a negative feedback from debt levels to the asset accumulation process. Andresen interprets this negative feedback as an expression of a decline in the total value of assets due to increasing loss and insolvency as real economic activity is curtailed by rising debt servicing obligations.

The loss rate also affects liquidity preference within the rentier sector by increasing the time-delay functions that determine the total outflow of savings to the real economy. A rise in the preference for liquidity is matched by a reduced willingness of rentiers to spend money outside the rentier sector. This influence is compounded by allowing the coefficient determining the split of flows between those remaining within the rentier sector and those flowing outside, to depend inversely on the relative share of total money stock absorbed by the real sector relative to the rentier sector.

Andresen's modeling attempts to account for endogenous variations in liquidity preference that arise due to increasing financial fragility. It is in many respects a courageous and invaluable effort. However, in dealing solely with monetary flows (the mirror image of real savings and investment decisions), Andersen has excluded the real process of accumulation. As such, his analysis of liquidity preference is not only detached from any modeling of investment activity, but also from the pricing of goods and financial assets. Thus, he cannot establish explicit links between liquidity preference, asset-pricing and accumulation: the very links that are articulated in seminal form in the research of Dalziel (1995), Randall Wray (1991), and Lavoie and Godley (2000). It remains to be seen whether these issues can be resolved in Andresen's future research.

In Chapter 12 of their 1996 book, Flaschel, Franke and Semmler make a notable contribution to efforts to integrate monetary and financial variables in a macroeconomic setting. Their model, building on earlier research by Taylor and O'Connell (1985), permits endogenous variations in "animal spirits" to operate as a direct influence over investment. Flaschel, Franke and Semmler refer to *The General Theory*'s Chapter 16 argument that, due to unsteady beliefs about an uncertain future, the volatility of investment is an expression of fluctuations in the market's evaluation of the marginal efficiency of capital. These expectations about future yields on investment are highly sensitive to variations in the general state of confidence. In response, they introduce a state of confidence variable into their model as a direct influence over both real investment and financial conditions.

Net profits, in the model, are defined as the remainder of gross profits after the deduction of interest payments, while gross profits are presumed to be a function of both the debt-to-capital stock ratio and the state of confidence. This adjusted gross profit variable is then capitalised by dividing it by the interest rate, to determine the ratio of the demand price or market value of capital over the replacement cost of capital. The value of equity, in turn, is determined by the replacement cost of capital net of any outstanding debt obligations. Under the assumption that workers do not save, Flaschel, Franke and Semmler presume that the division of rentier wealth between equities, interest-free deposits, and interest bearing deposits is governed by a function whose arguments are the adjusted gross rate of profit variable and the interest rate. Given variations in the rate of interest, this division of wealth influences short-run equilibrium in the stock market (yielding an equity-based analog to the familiar LM curve).

On the real side of the economy investment is made a continuous function of the state of confidence. Government spending is assumed to be funded through the issue of base money which is set as a given proportion of the replacement cost of capital. Short-run goods market equilibrium is achieved by equating gross profits with government expenditure, rentier consumption and net investment (giving rise to the IS curve). Short-run equilibrium outcomes in both goods and stock markets then determine both the rate of interest and debt-to-capital ratio.

Long run dynamics in the model are determined by the adjustment of the debt to capital ratio and endogenous movements in the state of confidence. This dynamic adjustment occurs under the assumption that firms finance investment through retained earnings, new equity raising and borrowing from the commercial banking sector. The authors presume that new shares are issued in direct proportion to the level of investment and the retention ratio is also fixed, thereby determining growth in loans as a residual. Confidence, they assume, is adversely affected by the level of the debt to equity ratio, beneficially effected by the gap between net profits and the interest rate, and also beneficially by changes in its own level. The latter allows optimistic and pessimistic expectations feed on themselves. Flaschel, Franke and Semmler interpret this feedback, from the level of confidence onto changes in the level, to be an expression of "noise-trading" activity on the part of *self-referential* or myopic trend chasers. Conversely, they attribute the influence of the debt-to-equity ratio to the activity of *hetero-referential* or fundamentals traders: those who try to stay ahead of the crowd by anticipating future turnarounds in average sentiment. They suggest that another fundamental variable—the differential between net profits and the interest rate—represents a risk-premium on equity finance relative to bonds.

Under reasonable assumptions about the relative magnitude of various parameters Flaschel, Franke and Semmler show that the long run equilibrium, which determines steady-state values for both confidence levels and the debt-to-capital ratio, is locally and asymptotically stable. However, they demonstrate that saddle-point instability results if the forces determining conventional views about the state of confidence are sufficiently responsive to movements in the level of confidence so that their influence dominates the other more fundamental variables. They suggest that a plausible outcome is one in which repelling forces keep the economy from converging towards its steady state while contracting forces that are dominant further away from equilibrium keep fluctuations within a bounded interval. In such cases, a limit cycle can arise. Moreover, corridor instability may be exhibited wherein small perturbations may be enough to flip the economy from a narrowly arcing limit cycle to one with a much wider locus reflecting greater volatility of income and employment.

The issue I would now like to raise is whether it is possible to establish any link between Flaschel, Franke and Semmler's (1997) models of animal spirits and investment, Andresen's (1998, 1999a,b) models of liquidity preference and Hansen, Sargent and Tallarini's (1999) or Tornell's (2000) models of uncertainty aversion. The modeling principles of each group of authors appear to be violently opposed. Andresen, in particular, refuses to truck with any notion of equilibrium, quantity-constrained or otherwise. Nevertheless, it is possible to view uncertainty premia embodied in the prices of financial assets (i.e. ownership titles to Lucas trees) as one side of the coin (i.e., in a stock-price context) and volatility in the flow demand for liquid and illiquid assets as the other side of the coin (i.e. in a flow-return context). This link has already been identified in relation to general equilibrium asset-pricing models in terms of the relationship between stochastic discount factors and asset-prices. One way to model (real) investment under *fundamental* uncertainty is to exploit the analogy between financial and real options (Dixit and Pindyck, 1994).

In the second case study, appearing in Chapter 6, I take up this suggestion to focus on the valuation of real options under incomplete markets. I endeavor to build a bridge between applications of robust control theory to options pricing under conditions of stochastic volatility and research on the use of minimum cross-entropy methods for pricing options under inequality constraints over the relevant Radon-Nikodym derivative. I discuss the relationship between this research and related analysis of good-deal bounds and the imposition of inequality constraints over the gain-loss ratio. In passing, I also discuss new entropy-based techniques for Generalized Method of Moments estimation.

This detailed discussion of real options theory provides the linkage I have been seeking between the risk-sensitive control literature and investment theory. I would envisage the incorporation of such modeling within a complete macroeconomic model, in turn incorporating an asset-pricing block, so that it would be possible to contemplate a two-way interaction between the respective financial and non-financial components of the overall model. On one hand, one could investigate the way that fluctuations in uncertainty perception and aversion could directly influence investment and, via the multiplier process, the point of effective demand and, ultimately, the economy-wide dividend process that determines asset-prices. On the other hand, for a given dividend structure one can trace the influence of uncertainty aversion and uncertainty perception over asset-prices and thence, via Tobin's q-ratio, over investment. In the first case, emphasis is placed on the decisions of investors, while in the second case, the focal point is the decisions of portfolio investors.

What remains to be explained is how time-varying volatility can arise in a robust or risk-sensitive control framework. This is where the real difficulty arises. In my discussion of uncertainty in the first chapter of the thesis, I suggested that uncertainty perception and uncertainty aversion could be related, respectively, to the magnitude of the stochastic uncertainty constraint and the risk-sensitivity parameter. Time-varying liquidity preference of the form modeled in Trond Andresen's or Flaschel, Franke and Semmler's work could only be represented either by way of a time-varying uncertainty constraint and/or by a time-varying risk-sensitivity parameter within the value function, itself. At this stage, risk-sensitive control theory—even that which can be applied to systems with time-varying system parameters—has not evolved to accommodate time-variation of this particular character[141].

5.3.5. Dynamic and Structural Instability

As I have argued above, Vercelli, in his critique of the early writings of Robert Lucas, raised concerns about the extent to which rational expectations theory is willing to countenance structural and dynamic instability. In particular, issues of stability are confined entirely to the process of expectations formation. Vercelli pointed to the ontological separation between environmental innovations, predictive procedures and decision-making rules in the Lucasian paradigm as the key to justifying, however inadequately, such a process of confinement.

One important issue is the extent to which Hansen and Sargent's move into risk-sensitive control theory represents a departure from this presumption of separability. On one level, our answer must be in the affirmative. From the perspective of robust control theory, the certainty equivalence principle must be transformed to allow for the fact that, in determining a robust

[141] In section 4.6 of the thesis I noted Vercelli's observation that investment could, in a Schumpeterian fashion, influence the supposedly "exogenous" rate of change in productivity. In the more Keynesian scenario that I discuss here, financial instability and unemployment-induced volatility in nominal wages could influence the supposedly "exogenous" set of preference "shocks". It should be noted that here, we are discussing uncertainty aversion rather than the more conventional set of parameters representing intertemporal substitution in consumption or relative risk aversion.

control, the control law (as well as bounds over observation error, external perturbation and incomplete knowledge about model dynamics) exerts an identifiable influence over the filtering algorithms for the robust estimator. In the words of Green and Limebeer:

> For this reason, the filter cannot be designed independently of the control objective, as the filter required depends on the control law[142] (Green and Limebeer, 1995, p. 291).

Nevertheless, the solution to the control problem is one based upon full information: the controller is assumed to possess *complete* knowledge about the characteristics of the malicious (deterministic) perturbation. A separation principle of one form or another still operates.

In the previous chapter I suggest that, for a number of reasons, non-linear systems exhibiting complex dynamic trajectories are probably commonplace in real-world economies. For the would-be control theorist, one feasible solution to the problems raised by this admission would be to adopt techniques of non-linear or chaotic control to model robust and risk-sensitive stabilisation policy, consumption-investment planning, or portfolio management. However, on closer inspection, the apparent uniformity of approach conceals some important differences. Certainly, optimal stabilisation policy can readily be applied to non-linear and complex models

[142] In addition, unlike the situation with LQG control, *existence* questions must be addressed under H_∞ control because the generalised regulator problem only has a solution if:

1. the Ricatti equation associated with the full-information control problem has a solution;
2. the Ricatti equation associated with H^∞ estimation of the optimal filter has a solution (on the same interval); and,
3. a coupling condition is satisfied (i.e. the second Ricatti equation depends on the solution to the first). See Shahian and Hassul (1996, pp. 392-393 and 440) for a practical overview of robust control implementation.

of the macro-economy[143]. However, bringing the latter two types of problems under a robust control umbrella, in ways that allow for non-linear responses to expected returns (net of transactions costs and inflation), requires more than a simple modification of intertemporal portfolio constraints. For one thing, the utility function itself might have to be noticeably modified in ways that may render infeasible existing techniques of risk-sensitive control. Here, I am thinking of certain modifications required to allow for non-linear elasticities of money demand in response to asset-price inflation (e.g. the logistic form suggested by Chiarella, 1990, in his Chapter 7). However, other modifications include endogenous changes to the robustness (or uncertainty aversion) parameter, as suggested in Tornell's paper (2000). These difficult issues cannot adequately be addressed here, but must remain a topic for future research. I reconsider them briefly in one of the two case studies—the first, entitled *Monetary policy reaction-functions*—which appear in Chapter 6.

5.3.6. Extending the Notion of Rationality

In this section, I explore notions of rationality that are broader and more comprehensive than those associated with applications of robust and risk-sensitive control theory. I adopt as my starting point, the notion of rationality set out in early rational expectations theory. Vercelli (1991, p. 95), in a further development of his critique of Robert Lucas, takes up Herbert Simon's (1982) distinction between *adaptive* and *creative* rationality. The former category would apply in the case where an agent cannot modify his or her *option set* by affecting the structure of the system or its environment, either directly or indirectly; in the contrary case, the latter category

[143] For example, Leo Kaas (1998) applies techniques of chaotic control that have been developed by mathematical physicists to achieve macroeconomic stabilisation in a quantity-constrained rationing model.

would apply. *Substantive* rationality, however, refers to the *equilibrium* properties of the system, while *procedural* rationality refers to the attributes of the dynamic process seen in all of its dimensions, including *disequilibrium* (Vercelli, 1991, pp. 92-3). For Vercelli, the notion of procedural rationality would take on special import in cases where the equilibria identified under substantive rationality were neither unique nor stable (Vercelli, 1991, p. 93). These differing notions of rationality are gathered together in the following diagram (Vercelli, 1991, Table 2, p.

FIG. 27: VERCELLI'S TAXONOMY OF TYPES OF RATIONALITY

	Adaptive Rationality (option-taker)	**Creative Rationality (option-maker)**
Equilibrium	*Substantive*	*Utopic*
Dynamic	*Procedural*	*Designing*

97):

Viewed from the perspective of equilibrium (i.e. substantive rather than procedural), creative rationality is identified as *utopic,* suggesting a transformation of the existing economic structure into an ideal one "...corresponding to some sort of intrinsic and invariant optimum" (presumably, here Vercelli has in mind something like a stable, full-employment equilibrium). From the perspective of global dynamics, creative rationality is identified as *designing,* implying a search for procedures which, taking existing circumstances, means and ends as given, can modify the economic system in a desired direction[144]. Vercelli draws on this tableaux to make the simple point that, amongst rational expectations theorists, policy debates are confined to interventions governed solely by substantive rationality. Another set of issues concerns the extent to which rational interventions are conceived to be collective or individual, and consciously intended or unconscious (in a very weak sense perhaps, because the required competencies are tacit rather than codifiable).

In the last of his twelve lectures on *The Philosphical Discourse of Modernity* Jürgen Habermas (1987) takes aim at one of the most eloquent of the philosophical readings of systems theory: to be found in the works of Niklas Luhmann (1984). Luhmann offers a general theory of society that is, itself, grounded in a critique of the Enlightenment notion of knowledge. He aims to replace the duality between the knowing subject and the knowable object, instituted by modernist thought, with the metabiological duality holding between the system and its environment. A similar notion is at work in ideological interpretations of "the invisible hand" that figure

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[144] A further conceptual demarcation can be made following Max Weber's distinction between formal and substantive rationality. From the perspective of a corporation, a set of managerial techniques may conform to the criterion of *formal* rationality insofar as they are based on optimising algorithms and rigorous accounting procedures. However, they may be *substantively* irrational in a goal-oriented sense (e.g., in terms of the long-term survival of the corporation). A finance theory analogy springs to mind. A finance practitioner may price a derivative security under the principle of non-arbitrage in a manner which may fail to account for the fact that the price of the underlying security may be subject to severe jumps over the horizon of the option. As a result, the derivatives may be traded at unrealistic prices, threatening the very survival of a financial or industrial corporation.

An example of formal rationality, but substantive irrationality, achieving *utopic* forms of expression could arise when so-called experts force dangerous institutional changes in accordance with a quite mythical notion of how the financial system (or world trading environment) operates. An example of *designing* form of irrationality could be the development of more sophisticated techniques for estimating conditional volatility when fluctuations are driven more by uncertainty aversion rather than by predictable temporal patterns in the release of new information.

prominently in the works of neo-Hayekian scholars (Butos and Koppl, 1977). It is important to realize that the notion of rationality at play in the rational expectations literature does not have to be conscious or codifiable rationality. From Hayek, to Milton Freidman and beyond, we have been told that certain things happen the way they do for complex reasons that may escape the full comprehension of individuals. However, economic outcomes and decisions can be described as though they were the products of fully conscious rational calculation.

Habermas (1987, p. 369) argues that for Luhmann, the Enlightenment's knowing subject is constituted by the "I" of the apperceptive "I think" that accompanies all representation. In Kant's work, for example, the identity of self-consciousness is established through the transcendental workings of the *a-priori synthetic* However, Habermas observes that a similar notion of the reflexive system as something that, first, has to relate to itself before it can relate to anything else in its environment, can also be found in Luhmann's preferred philosophy (Habermas, 1987, p. 369). Nevertheless, for systems theory reflexivity does not have to attain to self-consciousness. The meaning processing systems that Luhmann bestows upon philosophy, as a replacement for the knowing subjects of modernist discourse, are conceived on the basis of a very narrow and constrained interpretation of *meaning-as-code*. Luhmann turns to Husserlian Phenomenology to arrive at his idea of meaning as equivalent to the phenomenological intention: a pre-linguistic intuition of such notions as connectivity, limit, boundary, pulse and duration. Because language is seen to be derivative of pre-linguistic meaning connections, Habermas observes that "communication carried on by linguistic means cannot be explained in terms of specifically linguistic conditions of possibility" (Habermas, 1987, p. 380). There can be no basis for understanding linguistic structures that ground identity of meaning, consensus or dissensus, and the validity or invalidity of propositions.

Luhmann's meaning-processing systems can encode and decode, act on coded instructions, and apply algorithms to resolve obstacles, all without the need for any form of self-consciousness. Presumably, the effectiveness of some of those algorithms would have to be based on a pre-linguistic recognition of system boundaries that define inputs, outputs, and feedback and feedforward relationships. Nevertheless, Habermas suggests that there is nothing to be found in Luhmann's thought that corresponds to what we understand by socio-cultural life (Habermas, 1987, p. 369).

Habermas argues that Luhmann's replacement of metaphysics by metabiology has five immediate consequences (Habermas, 1987, p. 370). First, because the system constitutes the environment as a universal horizon of meaning for itself with no possible distinction between the empirical and the transcendental, there can be no pre-harmonisation of the existing plurality of system-relative environments.

Second, Luhmann intends to overcome the limitations of subjective idealism, but without forming any structure of self-consciousness outside the knowing subject, he cannot distinguish between consciousness as a psychic phenomenon and consciousness as a social phenomenon. All systems are environments for each other. Society is merely an aggregate of the totality of meaning processing systems (Habermas, 1987, p370-1).

Third, Luhmann accomplishes a naturalisation of spirit that has analogies to the Marxist replacement of self-consciousness with *praxis*. For Marx, through the expenditure of social labour the species transforms external nature and regenerates labour power in the both the production and the use of commodities. Similarly, for Luhmann all knowledge is mediated by the mastery of environmental complexity on the part of meaning producing systems that achieve *autopoesis* through this very mastery. However, systems theory differs from Marxism in relinquishing enlightenment notions of truth, validity, and intentional emancipation from internal and external constraints (Habermas, 1987, pp. 371-2).

Fourth, although Luhmann makes a distinction between the social system and the scientific system (naturally classifying systems theory, itself, as a sub-system of the latter), Habermas contends that there is nothing in his thought to usurp metaphysics as the comprehension of rationally ordered entities, epistemology as a comprehension of the relationship between representable objects and knowing subjects or linguistic theory as the semantics of assertoric sentences and states of affairs. The self-enhancement and self-maintenance of the system replace reason as proposition, being, and thought. Notions of Being, Thinking and Truth are proscribed and identified as the meagre remnants of a redundant classical humanism. Habermas therefore claims that, in rejecting the power of self-maintenance as the latent essence of subject-centered reason, Luhmann takes a functionalist turn that is similar in nature to Foucault's replacing of the notions of truth and validity with the mere effect of "holding something as true." (Habermas, 1987, pp. 372-3) In Luhmann's metabiology, reason simply becomes a superstructure of life.

Fifth, the possibility of a centering comprehension of the whole in self-knowledge disappears to be replaced by an a-centric *autopoesis* of individual elements. Thus, the social system has no possibility of "shaping a rational identity": there can be no linguistic inter-subjectivity or communicative action on the basis of shared meanings and therefore, no possible "opposition to adverse environmental interdependencies". Equally, there can be no possible recognition of crisis or the "*reification*" of subjectivity. Unable to explain communicative participation in a common life-world, as something grounded in specifically linguistic conditions of possibility, the social can only be envisaged as a mere intermeshing of individual perspectives (Habermas, 1987, pp. 373-4).

In this communicative critique of systems theory, Habermas is accomplishing something that is similar to Esther-Mirjam Sent's critique of Thomas Sargent's *post-Sante Fe* approach to bounded rationality and artificial intelligence (Sent, 1997). Sent describes Herbert Simon's notion of the human being as a symbolic processor: (a) putting symbols in; (b) pulling symbols out; (c) storing symbols and relational structures of symbols; (d) constructing, modifying and erasing such symbol structures; (e) comparing two symbol structures; and (f) following one course of action or another, depending on the outcome of such a comparison (Sent, 1997, p. 334). In this comparison and choice, agents face both internal, cognitive constraints and also external, social constraints that promote their use of heuristics, rules of thumb, and the attainment of satisfactory rather than fully optimal outcomes. For Simon, not all alternative strategies can be contemplated, not all consequences of each strategy can be determined, and not all sets of consequences can be evaluated (Sent, 1997, p. 333; citing Simon, 1975, p. 67). Cognitive limits are not only limits over information but also limits on the adequacy of scientific theories (Sent, 1997, p. 333)[145].

Sent contrasts this rich, linguistic notion of intelligence with Sargent's notion of agents as adaptive computing systems. For the *post-Sante Fe* Sargent, the adoption of an adaptive computing systems framework—including neural networks, connectionist systems and classifier systems—has enabled him to overcome a fundamental weakness in earlier rational expectations research. This weakness is traced to the notion that econometricians and calibrators, in

[145] In certain Post Keynesian circles, it is fashionable to claim that Simonian notions of bounded rationality such as those to be found in both Transactions Cost Theory and Macroeconomic analysis under assumptions of incomplete information, are related to what Dequech has defined as *ambiguity* and should therefore be clearly differentiated from Keynesian notions of *fundamental uncertainty* (e.g. Dunn, 1999). For instance, Dunn associates bounded rationality with constraints that arise when "…agents are unable to provide for an exhaustive list of (future) states of the world and possible courses of action that relate to the problem at hand because of limitations in their computational (and linguistic) ability *even* if the relevant information exists (p. 203). In contrast, Dunn relates non-ergodicity to the situation where future knowledge would be impossible to foresee even if agents *could* make full use of present knowledge (pp. 204-205). This is because the future is path-dependent and unique and thus can no longer be reduced to a statistical reflection of the past. Dunn (p. 206) contends that Williamson, in rejecting the assumption of non-ergodicity while retaining the notion of bounded rationality, still clings to the notion of a world in which the future is *potentially* knowable. Even though agents cannot form expectations that are efficient, unbiased and free of persistent error *in the short run*, over a longer time period, learning can facilitate more efficient market-based contracting as the subjective assessments of agents converge on the objective ergodic environment. Dunn argues that Williamson embraces a weak-form notion of selection that is defined in relative rather than absolute terms, so that he can continue to regard market forces as optimal in a broad sense (p. 210). However, in a non-ergodic world of unrepeatable, non-routine decisions, Dunn rightfully contends that markets are incomplete even *in the long run*, and institutions to manage this incompleteness must also persist over the longer term.

However, I would argue that in Williamson's work, market incompleteness is seen to be an expression of the joint workings of both uncertainty and bounded rationality. Although Williamson, following the precedent established by Ronald Coase, cites Frank Knight as the source of the distinction between risk and uncertainty, for both of these authors one of the essential aspects of this Knightian uncertainty is that associated with the *uniqueness* and *unpredictability* of the business environment. Without a doubt, economic processes are non-ergodic and path-dependent due to the ever-present influence of innovations in technology and changes in institutional organization. The open-ended creative power of social action is the ultimate source of uniqueness and unpredictability in the circumstances of business. However, in my view neither Frank Knight nor Herbert Simon would disagree with these arguments. In my view, bounded rationality subsumes within itself both ambiguity *and* fundamental uncertainty. It is an umbrella concept that pertains to the *whole* complex of constraints over human cognition. And in my view, Simon's own work is dedicated to interrogating the implications of *both* forms of uncertainty for human decision-making.

attempting to model the economy, must engage in complex specification searches and diagnostic testing, whereas agents within the economy are presumed to know the stochastic process with no real concern about specification error (Andersen et al. 1999, p. 1). Nevertheless, Sent observes (Sent, 1997, p. 330) that the new-born Sargent is still:

> [r]eluctant to give up ideas like representative agents or completed arbitrage and to renounce Lucas-Chicago general equilibrium analysis, Sargent did not go all the way with Sante Fe. Rather than using classifier systems to think about populations, he saw them as models of the neurons of an individual's brain (see Sargent, 1993, p. 76). Rather than relinquishing the notion of an equilibrium, he focused on convergence to equilibrium (see Sargent, 1993, p. 153; Marimon, McGrattan and Sargent, 1990, p. 372) .

In contrast, Sent argues that Herbert Simon "analysed the architecture of mind at the symbolic level without a theory of how these symbolic processes were implemented by neuronal structures" (Sent, 1997, p. 334; citing Simon, 1993, p. 664). His theories operated at the level of *semantics* rather than *neuronal processes*. In Hansen and Sargent's applications of risk-sensitive control, human consciousness, intention and communicative meaning are, in a very real sense, extraneous complications. A well-programmed machine could conceivably accomplish all the requisite calculations. Certainly, Hansen and Sargent would probably accept the fact that agents could, over time, improve on their system models, and even learn how to incorporate more sophisticated risk-sensitive value functions to measure tracking performance. Nevertheless, there is no room in their framework for any form of creative rationality. Agents cannot communicate with one another to establish new forms of corporate governance or more robust types of market institution that could actually alleviate perturbation, tame observation error, and make the modeling of markets a more routine and predictable activity. Much of the Keynesian inheritance concerns these matters of institutional design and communicative practice, often on a global scale. The motivation for these policies is to *mitigate* the adverse effects of uncertainty rather than to determine how best to make decisions *despite* the uncertainty. There is no place for these policies in Hansen and Sargent's system theoretic world. Some of these issues are discussed in more detail in the first of two case studies which appear in the final chapter of this thesis.

5.4. Conclusion

In this chapter I have provided a detailed critical review of Hansen, Sargent and Tallarini (1999), Andersen, Hansen and Sargent (1999) and Tornell's (2000) applications of risk-sensitive and robust control theory to decisions about consumption and saving and asset-pricing. This review was embedded within a thumb-nail survey of developments in sub-optimal, LQG, and adaptive control and filtering techniques. Although I focused on economic and financial applications, in part my motivation in providing this survey was to draw out the historical forces at work within the engineering and mathematical control literature that reflected efforts to attain higher levels of statistical robustness and general applicability. I was more than happy to uncover a certain *structural inevitability* behind the observation that many economists have been quick to embrace these techniques within their own discipline. However, I was also motivated by the desire to identify those very factors which serve to distinguish the social sciences and economics from the natural sciences and engineering. These factors came to the fore in the third section of the

Chapter, achieving their most philosophical expression in section 5.3.6. Despite their obvious abstraction, the philosophical issues that I discuss at that point have served as a virtual anchor for my critique throughout my writing of this thesis. Hopefully, their resonance will be detected most clearly in the two case studies that appear in the next and last chapter of the thesis.

CHAPTER SIX — TWO CASE STUDIES AND CONCLUDING COMMENTS

6.0. Introduction

This Chapter considers two case-studies that elaborate on certain aspects of what has been discussed in previous chapters. The first of these focuses on the estimation of monetary policy reaction functions. It considers current debates over the use of vector autoregressions for policy evaluation and touches on the actual characteristics of policy interventions on the part of the US Federal Reserve Board over the 1990s. The second case study examines issues concerning the valuation of real options in incomplete markets. It investigates the use of minimum cross-entropy techniques for estimating probability distributions under moment constraints as a conceptual bridge between risk-sensitive asset-pricing theory and theories of aggregate investment. The Chapter concludes with a section that draws upon the issues identified within each of the case studies to provide an integrative review of the arguments that have been made in the thesis, as a whole.

6.1. Case Study 1: Monetary Policy Reaction Functions

In his paper Rudebusch (1998) argues that the interest rate equation in VAR models can only be interpreted as representing an implicit monetary policy reaction function. Monetary policy reaction functions are key equation blocks both within large-scale real business cycle models and also within Keynes-Klein style macroeconometric models, and their various derivatives. In macroeconomics, the rational expectations revolution spurred on efforts to construct and estimate plausible reaction functions within models that would thereby become immunized against the Lucas critique: researchers had to account for the fact that private agents must anticipate likely policy responses of government in determining their own optimal actions.

Initially, rational expectations theorists viewed policy interventions as having a largely detrimental effects: in attempting to fine-tune a largely self-correcting economy, government agencies were seen to be introducing additional sources of volatility to the economic system in the form of unanticipated policy shocks. Thus, the efforts of private agents to extract signal from noise would be hindered by this government-initiated augmentation of existing volatility. With the New Keynesian emphasis on adjustment costs, price rigidities, indeterminacy, self-fulfilling prophecies and coordination failure, these policy interventions came to be regarded somewhat more benignly. However, with further developments in risk-sensitive and robust control theory, we can now acknowledge that certain types of intervention by relevant authorities can reap additional benefits for the macroeconomy, at least to the extent that they serve to reduce uncertainty perception and uncertainty aversion.

The Elements of Rudebusch's Critique of VAR

Rudebusch points out a range of inadequacies in the VAR modelling approach including: structural instability (due to time-varying parameters associated with changes to Fed Board membership and objectives); likely non-linearities in policy reaction functions; the use of an information set for estimation that is too narrow (ignoring asset-prices, Current Account Deficit, exchange rates etc) and also errs in conflating preliminary and revised data sources; and the finding of long and implausible lags amongst VAR coefficients (in part suggesting spurious correlation). In particular, he compares the literature on modelling reaction functions, in isolation from other components of a macroeconomic model and compares VAR residuals with unanticipated shocks implied by relevant futures markets.

6.1.2 Christopher Sims' Response

Christopher Sims' response to the Rudebusch critique is to concede many of the points raised, but, at the same time, to complain about the absence of an alternative modeling approach. He argues, "[t]he best evidence is that nonlinearity and time variation are of modest quantitative significance" (Sims, 1998, p. 939). Moreover, although additional variables can be added to regression equations for monetary policy reaction functions, they "...have not proved to be of major importance." He questions whether the finding of significant coefficients on variables with long lags implies, as Rudebusch suggests, that the Fed is reacting to old information. Moreover, he accepts the need to introduce variables with appropriate time lags that match the actual observation delay. Sims therefore ends up recommending cautious and minor extensions to the VAR framework to include both a broader range of exogenous monetary variables, and to various kinds of non-stochastic break or shift in regime.

6.1.3. Who is Closer to the Truth?

In contrast to both the early rational expectations literature and the more idiosyncratic Simsian position, I have argued in this thesis that once you move beyond a representative agent, "Robinson Crusoe", corn-model framework, in which all prospective returns appear in the form of own/corn-rates-of-return, and all the corn not eaten is either planted in the ground as seed-corn or lent to other agents for eating or planting, then each agent (household, firm, financial institution and government) has to make their own decisions while attempting to account for the likely cognitive and behavioural responses of other agents. Effectively, each agent would be trying to conduct robust control and filtering for an economic system replete with non-linearities and various kinds of regime-switching. For one thing, coordination failure implies the existence of quantity-constrained rationing across related markets. For another, in a world where non-indexed nominal contracts are ubiquitous, unanticipated price shocks give rise to debt-deflation effects. As such, in my view the modest extensions to VAR estimation that Sims endorses fall far short of what may actually be required.

I have observed that limit cycles are very easy to generate in many low-dimensional macroeconomic models and could just as easily give rise to chaotic bifurcations in models of larger dimension. Complexity would also be likely if key elasticities (e.g. inflation-elasticities in asset demand equations, or elasticities of the price margin to profitability) were non-linear. I have also emphasized the fact that *uncertainty* premia in asset markets (as distinct from risk-premia) would exhibit endogenous fluctuations that would undoubtedly augment volatility over the business cycle. Finally, I have argued that allowing for the presence of Minskyian forms of financial instability in physical and financial asset markets (as banks, households and firms adopt more precarious financial positions along the hedge-speculative-Ponzi continuum) would require the use of models which incorporate time-varying parameters.

6.1.4. Evidence for Nonlinearity?

Without interrogating the sources of nonlinearity, recent empirically grounded research into the effects of monetary policy (Potter, 2000; Jones and Nesmith, 1999) has questioned the validity of conventional Wold decomposition techniques for cases where non-linear dynamic modeling may be more applicable. Although any covariance stationary stochastic process can be transformed into a moving average using the Wold decomposition, this research recognizes that the resulting residuals may not coincide with those estimated through the application of a more appropriate non-linear systems approach. As I have suggested, localized movements around attractors cannot be accurately represented using linearizations based on first-order Taylors series expansions because the relevant eigenvalues would equal either zero or minus one: implying the possible existence of period doubling, Hopf, pitchfork, saddle-node, or transcritical bifurcations.

Using a truncated third-order Volterra series expansion, Simon Potter (2000) shows that the use of the Wold decomposition (i.e. as one expression of an ARMA process) for estimating residuals imposes dramatic symmetry and non-skewness constraints on the coefficients of the Volterra series. In a monetary policy evaluation context, he examines generalized (non-linear) impulse response functions generated from estimates of both a SETAR and regime-switching model, to confirm levels of persistence that well and truly exceed what would be expected from a conventional ARMA process.

In a similar vein, Jones and Nesmith (1999) from the Federal Reserve Board, show how to use the Hinich (1982) bispectrum test to determine whether estimated cointegration relationships associated with monetary interventions are non-Gaussian or non-linear. If nonlinearity is confirmed they advise that researchers can adopt two different approaches. On one hand, they can choose to estimate one of the three classes of stationary non-linear error correction models identified by Granger (1991). On the other hand, they can model the cointegrated relationship using univariate non-linear models such as the bi-linear, threshold autoregression, non-linear moving average, etc.[146].

6.1.5. The Estimation and Control of Non-linear Systems

Of necessity, signal processing theorists have long been engaged in the estimation of highly non-linear systems. For example, James and Yuliar (1995) employ the information state approach to derive the risk-sensitive or robust control and filter to solve nonlinear, partially observed games with quadratic running costs, rather than rely upon the certainty equivalence principal. Their approach draws on a special case of a result first derived by James, Baras and Elliott (1994), but it possesses the further advantage that the relevant information state is actually finite-dimensional. The actual solution to both the primal (control) and the dual (filter) is presented in terms of a Hamilton-Jacobi-Isaacs equation that is, typically, highly nonlinear and whose solution can only be arrived at using numerical methods. Viscosity techniques, though, can be applied to prove the convergence of the approximating solutions that have been found. Moreover,

[146] See Tong (1990), chapters 4 and 5..

for the conventional smooth, quadratic case, James and Yuliar establish the existence of a closed form solution.

The information state approach can also be utilized for estimating bilinear models. However, despite that fact that models of a bilinear-Markov form can capture a variety of asymmetric, conditionally heteroskedastistic, and time-irreversible stochastic processes, they cannot represent time-series generated by stochastic differential equation systems whose deterministic "skeletons" give rise to limit-cycles, nor can they mirror the behaviour of highly erratic stochastic processes such as the Poisson jump[147].

In part to overcome these obvious limitations, a variety of existing techniques are currently being refined for the estimation of complex non-linear models, which feature non-Gaussian error terms. Einecke and White (1999) combine the Extended Kalman Filter (to estimate terms in a second-order Taylor's series expansion) with robust control techniques (to accommodate norm-bounded,

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[147] The sub-diagonal Bilinear form:

$$X_t - \sum_{j=1}^{l} a_j X_{t-j} = \sum_{j=0}^{r} c_j \varepsilon_{t-j} + \sum_{i=1}^{q} \sum_{j=o}^{p} b_{ij} X_{t-i-j} \varepsilon_{t-i}$$

where $c_0 = 1$, and ε_t is i.i.d. $(0, \sigma^2)$

is treated extensively in Tong (1990, pp. 132-135), who shows that despite the appearance of the product term on the right-hand-side of the expression, like the ARIMA process, it too can be expressed in a particular version of the usual state-equation and observation-equation system. Significantly, Jones and Nesmith (1999) endorse the bilinear form as a useful approach for capturing the nonlinear and asymmetric characteristics revealed by empirical tests of time-series estimates of the US term structure. Tong notes that the square of a basic ARCH process also conforms to a simple case of the general bilinear form.

Tong constructs a new state variable ξ_t, from a complex arrangements of lagged linear terms in X_t, ε_t and another composite variable γ_t, itself, comprised of a constant term and additional lagged linear terms in X_t. The systems can now be written in terms of the new state variable as:

$$\xi_t = \mathbf{A}\xi_{t-1} + \mathbf{B}\xi_{t-1}\varepsilon_t + \mathbf{c}\varepsilon_t + \mathbf{d}\left(\varepsilon_t^2 - \sigma^2\right)$$

$$\mathbf{X} = \mathbf{H}\xi_{t-1} + \varepsilon_t$$

higher order terms in the expansion). Bayesian particle filtering, or sequential Monte Carlo Methods are also wide-spread. Alternatively, there are Adaptive control techniques that use non-linear transfer functions to represent Volterra series expansions of the underlying state equation system. The transfer function coefficients are then estimated recursively using partial Liapunov functions[148]. One of the most user-friendly approaches is outlined in Ozaki et al. (2000). These authors show how a relatively straightforward local linearization filter can be derived based on maximum likelihood methods, and applied in the estimation of a variety of chaotic models.

In summary, when non-linearities are present, both conventional LQG rational expectations filtering and control approaches and VAR estimation techniques would give completely erroneous predictions. And any impulse reaction functions derived using these methods, which purport to represent unanticipated policy shocks would, in all likelihood, be highly inaccurate. The increasing prevalence of sophisticated techniques for both confirming the existence of, and estimating non-linear relationships means that the practitioner should no longer presume that linear models will suffice for policy evaluation or prediction.

6.1.6. Characterizing and Modeling FRB Interventions

On a more anecdotal level, it is worth drawing the attention of the reader to the range of research projects conducted by the Federal Reserve over the late 1990s. Policy has been informed by a variety of sophisticated studies into such things as the extent of "irrational exuberance" and

Tong examines sufficient conditions for ergodicity in this sub-diagonal system and analyzes a predictor-space representation of the model that can be estimated using numerical methods. See Todis et al. (1994) for a recursive information-state approach to the estimation of the bilinear model.

"uncertainty aversion" in asset markets (Lehnert and Passmore, 1999; Tevlin and Whelan, 2000; Kiley, 2000), the extent of productivity growth, depreciation and obsolescence in the computer hardware sector and feedback onto productivity growth in other computer-using industries in the US (Oliner and Sichel, 2000), and concerns about Minskyian forms of financial instability within the international economy (Meyer, 1999). To the extent that recent Federal Reserve Board interventions have been influenced by the outcomes of such research, the penchant of macroeconomists for estimating of monetary policy reaction functions based on Taylor's rules must be viewed as one that is extremely crude and reductionist[149].

A growing body of literature also attests to the adoption by the FRB of robust control techniques for the determination of optimal policy interventions. A recent Australian contribution, which also provides an extensive overview of the growing international literature, is the paper by Cagliarini and Heath (2000). These authors search for a characterization of Knightian uncertainty that might explain the sluggishness of monetary policy interventions in the face of model, parameter, and data uncertainty. They favour Bewley (1986) preferences over Gilboa and Schmeidler's (1989) axiomatic characterization of preference because they give rise to more plausible forms of inertia rather than excessive responsiveness for observed interest rate adjustments. However, my arguments above support a completely different interpretation. I have

[148] Incidentally, mathematicians have long recognized that the maximum entropy functional can be interpreted as a form of Liapunov function.

[149] Of course, one common response has been to declare that Taylor's rule estimates merely served the purpose of affording a simple benchmark against which actual policy interventions could be assessed and calibrated. Undoubtedly, here the proof of the pudding must lie in the actual tasting. In what sense are Taylor's rules being utilized as a benchmark within macroeconometric models? Usually, they are meant to represent actual policy interventions, albeit in a simplified form amenable to estimation, not just schemas for benchmarking and classifying possible interventions.

suggested that monetary policy interventions ought to be viewed as taking place in the context of a dynamic game, but one that differs markedly from the conventional approach first mooted by Kydland and Prescott (1977). I have argued that, to a considerable extent, monetary authorities intervene to counter episodes of "irrational exuberance" and excessive uncertainty aversion on the part of investors in financial markets. In other words, Cagliarini and Heath (2000) are applying techniques of robust control under Knightian uncertainty to the wrong category of agents. The interventions of *public* agents are designed to mitigate the adverse consequences of risk-sensitive control on the part of *private* agents!

One approach that can accommodate Knightian uncertainty on the part of private agents has been developed by Hansen and Sargent (2000). With remarkable clarity, these authors demonstrate how robust control theory can be employed to overcome notable weaknesses in the earlier rational expectations literature while essentially preserving the core structure and intent of the latter. In section 2 of their paper, Hansen and Sargent offer the, by now, standard primer on robust control theory, introducing concepts of model ambiguity, robust decision rule, conditional relative entropy, and their preferred choice of distortion mechanism for perturbing the Markov transition kernel of the pre-given approximating model. In section 3, they introduce a robust version of the forward-looking Ramsey problem incorporating the usual specification error representation. In this endeavor they follow the precedent set by Currie and Levine (1987) and Pearlman (1992), who have studied a non-robust version of a problem in which the government and private agents share a common model of the economy, but the government must find a sequence of rules that expresses the time t control as a function of the history of the relevant state variable vector y_t, up to that point in time. As the decisions of private agents are forward-looking and, as such, are influenced by their anticipation of future policy interventions, governments must take these influences into account in planning their decisions.

Hansen and Sargent show how this seemingly complex problem can easily be incorporated into the much simpler framework of a single-agent linear regulator problem. Initially, the problem is set up as a robust version of the standard Lagrangian problem. They derive first order conditions for the control variable vector (combining both private agent and government decisions), the state variable vector x_t, and the vector of specification error shocks w_t. These Euler conditions can be solved for the optimal (linear) regulator using either the Schur decomposition of the resulting symplectic matrix pencil, or the more familiar Ricatti equation approach. This yields a pair of feedback rules for both the government and the private agents. In each case, the solution represents a planned strategy for dealing with the worst-possible vector of conceivable shocks that could be imposed on the given model. The elements of the state variable vector that expresses the agents' optimal control x_t, are all jump variables determined by the model at time t.

In addition, the stabilizing solution to the Ricatti equation yields a matrix P that gives a recursive representation for $\mu_t = Py_t$, for the shadow price μ_t of the state (i.e. μ_t is the Lagrangian multiplier on the constraint that represents the stochastic equation of motion for the evolving system and its shadow price reflects the marginal effect that an "easing" of this constraint has on the quadratic objective function). Hansen and Sargent demonstrate how the original problem can be transformed through partitioning the multiplier into two parts: one μ_{zt}, for the government and the other μ_{xt}, for the private agents, where each of the multiplier components is matched by the relevant component z_t and x_t, respectively, in a congruent decomposition of the state variable vector, y_t. Replacing the jump variables x_t that appear in the original robust version of the optimal linear regulator problem by the multiplier variables μ_{xt}, Hansen and Sargent (2000; section 5.1) argue that these multiplier variables now play the role of *implementability constraints* on the government's choice of sequences for the relevant component of the control variable vector u_t. To accomplish their aim, Hansen and Sargent (2000) partition the P matrix conformably, so that the agents' choice of the jump variable vector x_t, can be expressed as a function of the government's state variable vector z_t, and the implementation multiplier μ_{xt}.

In section 7 of the paper, Hansen and Sargent (2000) provide a telling example of their approach in setting out a robust adjustment to Woodford's (1998) "new synthesis" macroeconomic model. These "new synthesis" models add a simple Phillips Curve mechanism onto what is essentially a real business cycle model to account for the stickiness of nominal prices and to thereby accommodate more familiar kinds of monetary influence over the absolute price level. In Woodford's model, the "IS curve" is derived from the private agent's Euler equation for consumption that has been modified using the production function. In true Wicksellian form, this approach allows the marginal productivity condition to play the key role of determining the natural rate of interest. The latter is modeled as a stationary univariate exogenous process enabling Woodford to linearize the model around a non-stochastic steady state.

Hansen and Sargent's main innovation (p. 21) is to introduce a vector of specification errors into the state variable process that represents the arrival of information in the economy. They adopt Woodford's first-order autoregressive specification for the natural rate so that the growth rate of potential output takes the form of a random walk and aggregate demand can deviate from potential output by a drift term plus a stable first-order autoregressive process. To accommodate a concern for robustness, Hansen and Sargent introduce a vector of specification errors into the equation set for the information-state. The chosen objective function, which is intended as an approximation of the welfare of the representative household is a log-linear quadratic function in inflation, the deviation of the output gap from its pre-specified target, and the deviation of the nominal interest rate from its target.

What we finally end up with is a robust Ramsey rule for a partially observed system. Potential GDP is a latent variable that must be estimated along with the parameters for the autoregressive process that are seen to drive discrepancies between aggregate demand and potential output. Under the certainty equivalence principle, Hansen and Sargent (2000, section 8, pp. 22-24) show how an innovations representation of the standard Kalman filter can be utilized to estimate both the hidden variable and other parameters.

6.1.7. Concluding Comments on the First Case Study

In conclusion, the observation that policy has exhibited sensitivity towards uncertainty aversion amongst private investors highlights many of the core issues that I have emphasized repeatedly in this thesis.

When viewed from the critical perspective adopted elsewhere in this thesis, it would appear that Hansen and Sargent (2000) have utilized a highly specialized set of robust control techniques to firm up and consolidate the rational expectations approach to policy analysis.[150] For obvious reasons, linearization techniques are applied without questioning their validity. Not only would non-linear forms of robust control theory be much harder to apply, but also they would lead to a deeper concern about the appropriateness of using "jump variable" techniques for dealing with dynamic issues associated with inflation, capital gains, and debt-deflation.

Once again, the capital debates raise doubts about the soundness of Woodford's reliance on essentially Wicksellian mechanisms as drivers of the accumulation process. Finally, Hansen and Sargent account for forward-looking choices on the part of private agents. Despite its recognition of both government and private sector agents, Woodford's (1998) framework completely ignores the matter of coordination failure. The prospect of coordination failure would suggest the need for a further partitioning of control variables and implementability constraints. Investors would have to make decisions on the basis of their anticipations not only about the likely behaviour of government, but also consumer/savers, and financial intermediaries; providers of external finance would have to make decisions on the basis of anticipations about government, investor and consumer behaviour, and so on. Both quantity constraints and implementability constraints would then have to come into play.

[150] Of course, Hansen and Sargen are not policy makers although Sargent has advised Central Bankers, including the Fed.

In a game-theoretic context, the use of conventional LQG control to express strategic decision-making on the part of each of the representative players ignores the fact that, in a more realistic setting, each player would also have to presume a certain exogenous level of uncertainty aversion on behalf of their opponents. But at this point, a fundamental asymmetry comes into effect: policy interventions on the part of government are often aimed directly at *mitigating* levels of uncertainty aversion in the community at large. All that a private agent can and hopes to achieve is to *protect themselves* against the likely adverse effects of an increase in the uncertainty aversion of other players. However, any anticipation of the degree of uncertainty aversion or robustness of other agents' filtering and control preferences can not be formally represented in probabilistic terms. There is nothing of a probabilistic nature that can be attributed to robustness, itself. Rather, in determining both the appropriate estimator and the appropriate control robustness characterizes an individual's sensitivity to the norm bounds imposed over model uncertainty, external perturbation, and observation error.

Once again, I wish to stress the fact that in robust control the optimal filter cannot be determined in isolation from the relevant control law. Thus, an economic agent with forward-looking expectations about the behaviour of other agents must somehow take into account the influence of those agents' (as well as her own) preferences for robustness! Because the certainty equivalence principle obtains in the LQG case, and also for robust versions of the linear quadratic regulator, an individual agent does not have to second-guess the preferences of other agents: the same set of risk-neutral or distorted probabilities, respectively, will be applied by every potential investor. In non-linear versions of robust control, for which government and private-sector agents possess different levels of uncertainty aversion and uncertainty perception, this will no longer be applicable. Not only would the worst-case perturbation now be associated with the structural properties (i.e. maximum eigen-value) of the whole (matrix) transfer-function relating the perturbation w to the performance index variable z but, as I argued above, a separate set of implementability multipliers would have to be calculated for each class of private sector agents—investors, consumer/savers, and financial intermediaries[151]. These are all issues about which engineers do not have to trouble themselves. Economists are not so lucky.

6.2. Case Study 2: Incomplete Markets and the Theory of Real Options

In this case study, I selectively review the literature on real options theory and investment in both complete and incomplete markets. I then attempt to provide a heuristic bridge between this field of research and the literature on risk-sensitive and robust filtering and control techniques. To this end I examine related research on the use of good-deal bounds over Sharpe ratios and gain-loss ratios, and minimum cross entropy methods for pricing options in incomplete markets. In regard to the latter, I examine new entropy-based Generalized Method of Moment techniques that appear to have improved small sample properties. Finally, I review Gzyl's discrete-range "maxentropic" derivation of martingales that can be used in option pricing models that are based on the binomial lattice, Markov-chain and finite-difference methodologies. At the same time, I draw on Shore and Johnson's (1980) research based on desirable properties of statistical inference to which entropy measures seem to conform. I observe that Shore and Johnson's

[151] On properties of the transfer function matrix, see the discussion in section A.6 of the technical appendix.

analysis relates both to equality and inequality constraints: the latter providing a direct link to approaches based on good-deal bounds and gain-loss ratios.

To the best of my knowledge, no other authors have identified the relations holding between real options theory and this latter strand of inquiry. Nor have the links between this literature and finance-based applications of risk-sensitive stochastic control theory been clearly articulated. Although this is principally a review paper, I hope to identify practical opportunities for future research and quantitative analysis. My ultimate intention, however, is to identify elements which are common to both the risk-sensitive control and option pricing literature to better explain the observed volatility of investment over the investment cycle. Alan Greenspan's comments about episodes of "irrational exhuberance" and "uncertainty aversion" in equity markets were equally applicable to the "boom" and "crash" characteristics of the global property cycle over the late 80s (and more recently amongst the Asian economies). It is these aspects of the investment cycle are what I am trying to grasp, albeit, in largely theoretical terms.

6.2.1. Real Options, Financial Options and the Influence of Uncertainty over Investment

One way to model (real) investment under increasing *risk* is to exploit the analogy between financial and real options (Dixit and Pindyck, 1994). The following simple correspondences hold between the elements that determine the value of financial (call) options and those that determine the value of real investment projects considered as options over the present value of expected cash flows over the project horizon (Trigeorgis, 1996):

Call	Real Option
current value of stock	gross present value of expected cash flows
exercise price	investment cost
time to expiration	window of opportunity
stock value uncertainty	project value uncertainty
riskless interest rate	riskless interest rate

These real options theories of investment, initially developed from the perpetual options models of Paul Samuelson and Robert Merton, have now secured a prominent position in both the theoretical and empirical literature on corporate investment decisions (for theoretical examples see Dixit and Pindyck 1994, Abel *et al.* 1996, and Guiso and Parigi 1996; and for influential empirical studies see Price 1996, Leahy and Whited, 1996, and Hurn and Wright, 1994). To a large extent this is because the managerial flexibility to revise current or adapt later decisions cannot be captured by conventional discounted cash-flow techniques. Although some aspects of choice and flexibility can be accommodated by combining net present value techniques with decision-tree analysis, Trigeorgis (1996, Chapter 5) shows theoretically that, to properly account for the sequential exercise of real options, discount rates would have to be continually modified as each decision is taken.

Dixit and Pindyck (1994, Chapter 11) develop a simple model of irreversible investment in which a representative firm faces uncertain demand given by $P = YD(Q)$, where Q equals output, P equals price, and Y is a shift variable following geometric Brownian motion

$$dY = \alpha Y dt + \sigma Y dz$$

The solution is characterized by the fact that a critical threshold value for Y must be exceeded for investment to take place[152]. The threshold is determined as $Y(K)=\frac{\beta_1}{\beta_1-1}\frac{\delta\kappa}{H'(K)}$, where κ is the unit price of capital. The term $\frac{\delta\kappa}{\mathrm{H'(K)}}$ reflects the cost of installing an extra unit of capital relative to the expected increase in the present value of the firm. The ratio $\frac{\beta_1}{\beta_1-1}$ gives the multiple by which the expected present value must exceed the marginal capital cost. In the Cobb-Douglas case, $H(K) = K^{\theta}$ and the long-run growth rate of capital can be shown to equal:

$\frac{1}{2}\sigma^2\frac{\left(\frac{2\alpha}{\sigma^2}-1\right)}{1-\theta}=\frac{\alpha-\frac{1}{2}\sigma^2}{1-\theta}$, so that *increasing* risk *reduces* capital accumulation.

Real options theory can readily be extended beyond the simple option to defer an investment project, to incorporate option values associated with a contraction in the scale of an existing project, an expansion at a particular stage in the life of a project as new information becomes avaiable, the termination and salvage of a project, default on future installments, and switching

[152] In their model (Dixit and Pyndyck 1994; also see Price 1996, for an empirical application), the firm faces uncertain demand given by $P = YD(Q)$ where Q = output, P = price and Y is a shift variable with geometric Brownian motion $dY = \alpha Ydt + \sigma Ydz$. Given a production function $G(K)$, the firm's profit can be expressed as: $\pi = \mathrm{yD}[\mathrm{G(K)}]G(K) = YH(K)$ where $H''(K) < 0$. Setting up the problem as a dynamic program, the authors derive the following expression for the value function: $W(K,Y) = YH(K)dt + e^{-\rho dt}E(W(K,Y+dY))$, which satisfies the second-order differential equation $\frac{1}{2}\sigma^2Y^2W_{YY}(K,Y)+\alpha YW_Y(K,Y)-\rho W(K,Y)+YH(K)=0$. The solution to this equation is given by:
$W(K,Y) = B_1(K)Y^{\beta_1} + YH(K)/\delta$ where $\delta = \rho - \alpha$ and β_1 is the positive root of

$\phi = \frac{1}{2}\sigma^2\beta(\beta-1)+\alpha\beta-\rho=0$

Here, $YH(K)/\delta$ is the expected value of profits the firm would receive if it maintained a constant value of K, while $B_1(K)Y^{\beta 1}$ is the current value of its future option to expand capacity. The constant of integration B_1 can be solved using the 'value-matching' and 'smooth-pasting' conditions.

between various resource inputs or between project outputs[153]. For example, investments in advanced manufacturing technology are frequently a source of switching option values (Lei et. al., 1996).

Downing and Wallace (2000) consider an extension of the Dixit and Pindyck model (1994) to account for stochastic volatility. A calibrated version of this extended model is used heuristically to identify influences over residential investment in the US. The authors assume that housing prices are a function of a vector of attributes, each generating a flow of services to the homeowner. Rental rates are set in a competitive market subject to demand shocks that evolve stochastically as geometric Brownian motions. The user cost of capital is the risk-free spot rate of interest adjusted for depreciation and costs of repair. In addition, the instantaneous risk-free rate evolves in accordance with a stochastic volatility model. Parameter values for this interest rate model are calibrated in accordance with a range of estimates from related empirical studies. Theoretical predictions from the calibrated real options model for an ascribed range of attribute values—changes in the spread between the rental rate and spot rate of interest, and in the volatility of the spread—are then compared with estimates taken from a mixed logit regression model of housing investment decisions to determine the validity of the theoretically inspired

[153] On a tangential note, real options theories have also been deployed to attack the current obsession with shareholder value-added benchmarks and incentive schemes. These fashionable metrics are largely based on discounted residual income measures of project value: an approach that *completely* ignores the often sizable option multiples that are embodied in project worth.

findings. The latter regression is estimated using panel data from the American Housing survey over the period 1985-1997 (Downing and Wallace, 2000; section 3)[154].

In a simple two period setting, Abel has extended the options pricing approach to investment to accommodate varying degrees of investment reversibility. He shows that the naïve net present value rule can only be applied after the cost of purchasing an additional unit of capital has been adjusted to take into account: the negative cost of extinguishing the marginal call option to purchase that same unit in the following period; and, the positive cost of acquiring a marginal put option to sell that additional unit in the following period (Abel et al., 1996). Abel goes on to demonstrate the relationship holding between the option pricing approach, the user cost of capital, and marginal q.

Nevertheless, over and above the issue of stochastic variations in the risk-free rate, there are additional limitations in the Dixit and Pindyck framework. These include the fact that it accounts neither for the effects of transaction costs, nor for the effects of stochastic volatility in the underlying asset. These inadequacies are the major focus of this chapter. However, the discussion inevitably leads to the consideration of notions of uncertainty that are broader than those examined in the existing literature on options pricing. For example, within the specialized literature on asset-pricing the implications of Knightian or Keynesian uncertainty is currently a rapidly expanding area of active research models. One inroad into these issues is via applications

[154] The variables in the regression model include volatility and spread variables, a user cost of capital that includes maintenance costs, federal and state marginal tax rates and property taxes, per capita income to account for business cycle effects, a variable for the average age of the house, and dummy variables to account for length of tenure and time-on-market.

of risk-sensitive and robust stochastic control theory, a topic that I shall examine in the next section of the chapter.

6.2.2. OPTION PRICING AND GENERAL EQUILIBRIUM ASSET-PRICING UNDER COMPLETE MARKETS

An excellent resource on options pricing is Cochrane's (2000) book on asset-pricing. Cochrane begins with a simple derivation of the fundamental asset-pricing equation relating the appropriately discounted expected payoff x_{t+1} to the asset's price p_t (p. 15):

$$p_t = E_t\left[\beta \frac{u'(c_{t+1})}{u'(c_t)} x_{t+1}\right] = E_t[m_{t+1}x_{t+1}]$$

The key variable in his analysis is the stochastic discount rate m_{t+1}. In accordance with convention, Cochrane demonstrates that either the law of one price or non-arbitrage is sufficient to prove the existence of such a discount rate (see Cochrane, 2000, Chapter 4). In findings that are tied together in Chapter 6, Cochrane demonstrates that both the minimum variance return situated on the conventional mean-variance frontier and also the beta representation of the CAPM can each be formally linked to, and derived from, the asset-pricing equation, appearing directly above.

When it comes to the pricing of options the traditional approach used in deriving the Black-Scholes (1972) formula is one based on the construction of a portfolio of stocks and bonds that replicates the instantaneous payoff of the option. Under the law of one price, the price of the option and the price of the replicating portfolio must be equal. After some algebraic manipulation using Itô's lemma, this equivalence gives rise to a partial differential equation that can be solved for the option price. An alternative approach, increasingly in vogue and one that Cochrane prefers, draws upon the stochastic discount factor formulation.

At each date the option is priced using the discount factor m, that prices both the stock and the bond. For a call option, Cochrane (2000, sections 17.2.1-17.2.2) shows that one can either solve the discount factor forward and then find the call option value by using $C = E(mx^C)$, or characterize the price path for the option using Itô's lemma and solve it backwards from expiration. This sort of analysis is now standard fare in up-to-date finance texts and will not, therefore, be examined in further detail. However, I shall review the pricing of options in incomplete markets—a field that is very much the focus of current research—in more detail.

6.2.3. Option Pricing and General Equilibrium Asset-pricing under Incomplete Markets

During extreme events such as stockmarket crashes, when continuous trading is impossible (events that are frequently modeled as Poisson jump processes in finance theory) or when interest rates and stock volatility are stochastic, then the law of one price breaks down. In these conditions a replicating portfolio of securities—one that provides a perfect hedge against the corresponding shocks—cannot be constructed (see Cochrane, 2000, Chapter 18).

This is one area of finance where techniques of robust control have been applied with some measure of success. As discussed in the previous section, McEneaney (1996) has used a robust control approach to price options where volatility is stochastic and bounded. It transpires that the option price is one generated by the familiar Black-Scholes formula with a constant volatility equal to the upper bound over the volatility[155]. His paper is technically demanding, employing viscosity solutions for the resulting Isaacs equations, and less mathematical readers would probably appreciate a more heuristic or intuitive description of what is involved in pricing options when markets are incomplete.

Cochrane sets out his own "good deal bounds" solution to the problem of imperfect replication in Chapter 18 of his text following Cochrane and Saá-Requejo (2000). The good deal bounds are derived by "systematically searching over all possible assignments of the 'market price of risk'" of the residual, constraining the total market price of risk to a reasonable value, and imposing no arbitrage opportunities, to find upper and lower bounds on the option price" (p. 301). The residual to which Cochrane refers is the error term in a projection of the option payoff onto the space of portfolio payoffs that can be constructed from the basis assets (a stock and a bond in the conventional Black and Scholes framework).

Essentially, the option is priced using a dynamic recursive programming approach maximizing the expected discounted payoff to the option under a series of one-period constraints (for

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[155] McEneaney (p.15) assumes that the asset-price process is a continuous semi-martingale driven by a Brownian motion process as given by the stochastic differential equation:

$$dP_t = bP_t dt + \sigma(P_t Y_t)P_t dB_t$$

while the wealth of the contingent claims writer is given by:

$$dX_t = [rX_t + (b-r)\lambda_t]dt + \sigma(P_t, Y_t)\lambda_t dB_t$$

where the volatility of the asset-price process is dependent on another, independent, stochastic process:

$$dY_t = f_1(Y_t) + f_2(Y_t)dB_t^{(2)}.$$

example, for a European call the objective function becomes: $\max_{\{m\}} E(mx^c)$, $x^c = \max(S_T - K, 0)$. Here, S_T is the price of the underlying stock while K is the strike price. One constraint imposes non-arbitrage (i.e. the stochastic discount rate must be non-negative), another characterizes the asset-pricing model that has been used to price the underlying stocks and bonds (p = E[mx]), while the last imposes a bound over the Sharpe ratio of the imperfectly replicating portfolio[156]. By restricting the range of discount factors to those falling within a sphere around the origin determined by a volatility constraint that takes the form $E(m^2) \leq A^2$, a range of option prices is generated that are narrower than the conventional arbitrage bounds[157]. The positivity constraint obviously rules out negative prices that are not ruled out merely through the imposition of the volatility bound. Cochrane demonstrates how to solve the multiple-period problem using Kuhn-Tucker techniques to arrive at the relevant upper and lower bounds over the option price.

However, following the work of Bernardo and Ledoit (2000) Cochrane observes (p. 318) that an alternative approach could be taken that, instead, would entail the imposition of bounds over the

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Here, $B^{(2)}$ is another Brownian motion on the given filtration (Ω,**F**,P) independent of B, r is the riskfree rate of return, λ is the amount of wealth invested in the underlying asset at any given time, $\sigma(p,y) \in [0, \sigma_2]$ where $\sigma_2 < \infty$ is the relevant upper bound, f_1 is Lipschitz and $f_2 \in C^4_b$. The differentiability assumptions on f_2 can be relaxed.

[156] The Sharpe ratio is the absolute value of the excess return of an asset divided by the return variance. Cochrane's approach draws on the duality between discount factor volatility and the Sharpe ratios established by Hansen and Jagganathan (1991). Using the covariance decomposition, the following identity can be derived from the asset-pricing relationship for excess returns R^e:

$$0 = E(mR^e) = E(m)E(R^e) + \rho_{m,R^e}\sigma(m)\sigma(R^e).$$

Because $|\rho| \leq 1$ and $E(m) = 1/R^f$ when a riskfree rate R^f exists, the following relation must hold between the Sharpe ratio and discount factor volatility:

$$\frac{\sigma(m)}{E(m)} \geq \frac{|E(R^e)|}{\sigma(R^e)}.$$

From a geometrical analysis of this relationship Hansen and Jagganathan derive the precise duality relationship between discount factor volatility and Sharpe ratios that Cochrane compares to Bernardo and Ledoit's gain-loss ratio, as discussed immediately below.

[157] The non-arbitrage bounds for the call option are $C \geq 0$ and $C \geq K/R^f$.

ratio of the gain ($\left[R^e\right]^+ = \max\left(R^e,0\right)$) and the loss ($\left[R^e\right]^- = -\min\left(-R^e,0\right)$) of the excess return[158]. Cochrane emphasizes the exact analogy holding between the key duality relationship that Bernardo and Ledoit derive, which compares the gain-loss ratio to the ratio of the supremum and infinum of the stochastic discount rates[159]:

$$\max_{\{R^e \in \mathbf{R}^e\}} \frac{E\left[R^e\right]^+}{E\left[R^e\right]^-} = \min_{\{m:0=E(mR^e)\}} \frac{\sup(m)}{\inf(m)}$$

and the well known Hansen and Jagannathan duality relationship between discount factor volatility and the Sharpe ratio:

$$\max_{\{R^e \in \mathbf{R}^e\}} \frac{\left|E\left(R^e\right)\right|}{\sigma\left(R^e\right)} = \min_{\{m:0=E(mR^e)\}} \frac{\sigma(m)}{E(m)}$$

He notes that this analogy "...hints at an interesting restatement of asset-pricing theory in L^1 with sup norm rather than L^2 with second moment norm." (p. 318).

The gain-loss ratio summarizes the attractiveness of a zero-cost investment for the benchmark investor. When it equals unity the investment is fairly priced for the investor, but if it exceeds unity the benchmark investor would receive more gain than necessary for him or her to increase

[158] It should be noted that the asymmetry between up-side and downside risk is a notable feature of Bielecki and Pliska's (1999) continuous-time, risk-sensitive portfolio model.

holdings in that asset. If markets were complete, the set of pricing kernels $E(mR^e)$ that correctly price all portfolio payoffs would have a unique element; otherwise it would have many elements. Using an asterisk to designate equilibrium outcomes using the benchmark pricing kernel, Bernardo and Ledoit's duality result can be framed in terms of deviations from the benchmarking pricing kernel (Bernardo and Ledoit, 2000, p. 151):

$$\max_{\{R^e \in \mathbf{R}^e\}} \frac{E*[R^e]^+}{E*[R^e]^-} = \min_{\{m: 0=E(mR^e)\}} \frac{\sup_{j=1,\dots,S}\left(\frac{m_j}{m_j*}\right)}{\inf_{j=1,\dots,S}\left(\frac{m_j}{m*_j}\right)},$$

where $j = 1,\dots,S$ is the relevant state space and the expectation $E*$ is taken under risk-adjusted probabilities:

$$p_j* = p_j u'(c_j*)/E[u'(\mathbf{c}*)]; \quad j = 1,\dots,S \ .$$

Thus the gain is the expectation of the excess payoff computed over those states in which the excess payoff is positive[160]. Bernardo and Ledoit see the main advantage of their gain-loss

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[159] In the draft of Cochrane's text that I have cited, the expectations operators appear to have been erroneously left off in the ratio appearing on the left hand side of the depicted equation.

[160] In a continuous–time setting Bernardo and Ledoit show that it can be calculated by decomposing the expression for the gain into three or fewer terms, each representing a linear function of the stock price over an interval (if the strike price K falls within an interval that interval must be broken into two subintervals at K). Each term can be computed given the interval bounds S_1 and S_2, using the formula:

$$E*[(\alpha+\beta S)\mathbf{1}_{[S_1 \le S \le S_2]}] = \alpha[\Phi(d_1-\sigma\sqrt{t})-\Phi(d_2-\sigma\sqrt{t})]+\beta Se^{rt}[\Phi(d_1)-\Phi(d_2)],$$

where $d_i = \frac{\log(S/S_i e^{-rt})}{\sigma\sqrt{t}}+\frac{1}{2}\sigma\sqrt{t}, \quad i=1,2$ and α and β are the coefficients of the linear function. This gives the gain (and by symmetry, the loss) for any portfolio weights w_S and w_C, given initial prices S and C. The authors next show that by setting each of the weights to: $w_S = wE*[\mathbf{C}-e^{rt}C]$; $w_C = 1-wE*[\mathbf{S}-e^{rt}S]$, where $\mathbf{C} = (\mathbf{S}-K)^+$ the free parameter can now be varied and the portfolio value $E*[R^e]$ will stay constant. The value of w can then be chosen to minimize the first absolute moment (L_1 norm) of the excess payoff—a straightforward univariate convex optimization problem. The final stage requires the imposition of bounds on the maximum gain-loss ratio. Bounds on the option price can be derived by inverting the maximum gain-loss function with respect to its argument in C.

duality result being the fact that it characterizes the set of arbitrage and approximate arbitrage opportunities. The existence of a bound on the gain-loss ratio is equivalent to imposing the restriction that $\alpha \geq m/m^* \geq \beta$ so as to sharpen the non-arbitrage restriction that $\infty \geq m/m^* > 0$ (i.e. the maximum gain-loss ratio $\overline{L}$ is just the ratio β/α). If the benchmark model is reasonable, then high gain-loss investments are inconsistent with well-functioning capital markets. If $\overline{L}$ increases (decreases) this reflects less (more) confidence in the ability of the benchmark model to price non-basis assets. In the limit, as $\overline{L}$ approaches infinity, the bounds over the price of the chosen non-basis asset will approach the non-arbitrage bounds, and as $\overline{L}$ approaches 1, the bounds for an option would approach the Black-Scholes price.

Bernardo and Ledoit (p. 168) observe that several other duality results in the literature could, in principle, be used to derive asset-price bounds. For example, the Hansen and Jagannathan bounds can be generalized to account for restrictions on the kth moment of the pricing kernel. However, Bernardo and Ledoit also cite research by Stutzer (1995), who demonstrates that a restriction over the maximum expected utility attainable by an investor whose preferences conform to *constant absolute risk aversion* is equivalent to a restriction on the entropy of the pricing kernel $E[m\log(m)]$. Stutzer's duality result enables Bernardo and Ledoit to link their gain-loss approach to the related literature on entropy measures and Bayesian estimation. As we have seen, entropy also features in risk-sensitive control theory, where it appears as in the form of a relative entropy constraint.

6.2.4. STUTZER'S ENTROPY-BASED ANALYSIS OF ASSET-PRICING MODELS

Stutzer (1995) introduces the unconditional moment conditions that are defined by an asset-pricing model over gross real returns R^i for each of the I assets, under the state probability measure μ (p. 369):

$$E[R^i m] = \int R^i m d\mu = 1, \quad i = 1,\ldots,N$$

He then introduces the risk-free asset with unit real payoff $X_t^0 = 1$, gross return (i.e. gross real interest rate) r, and, therefore, time-varying price $1/r$. In the usual way, he next obtains the unconditional moment condition for this asset, namely:

$E[m] = E[1/r] = c$, where c is the fixed mean.

He then proceeds to derive the familiar Hansen and Jagganathan (1991) affine benchmark—an affine combination of excess returns that delivers the minimum variance amongst those stochastic discount factors (SDF) satisfying the above two moment conditions (Stutzer, 1995, p. 371):

$$m^a = (R - E[R])' w^a + c$$

Where w^a is a vector of coefficients. On substitution into the first moment condition, the resulting expression can be solved for the unique vector w^a (when invertibility conditions are satisfied). This vector also determines the particular affine combination, m^a amongst all possible combinations $m(w)$ given by the preceding expression, that is closest to any SDF m satisfying the moment conditions, in the sense of mean squared distance. In addition, it is the vector of weights determining the mean-variance efficient portfolio.

Stutzer arrives at another version of the moment conditions by dividing the first by the second introducing a measure change ($dv = m/E[m]d\mu$) that enables him to establish the following equivalence between expectation operators (p. 374):

$$E\left[R^i \frac{m}{E[m]}\right] = \int R^i \frac{m}{E[m]} d\mu = \frac{1}{c} = \int R^i dv = E_v\left[R^i\right].$$

Stutzer derives a variational characterization of the set of *risk-neutral measures*, v^l, which satisfy the latter expectational relation and have a *state price probability density* (SPD), $dv/d\mu$. He achieves this result by minimizing the relative entropy or *Kullback-Leibler Information Criterion* (KLIC) $I(v,\mu)$ given by (p. 375):

$$v^l = \arg\min_{v} I(v, \mu) \equiv \int \log(dv/d\mu)dv$$

over the set of SPDs satisfying the unconditional moment condition above. Under appropriate regularity conditions, Stutzer demonstrates that, associated with this convex problem is the Gibbs density[161]:

[161] The Gibbs SPD is derived from the first order conditions for the following problem:

$w^l = \arg\min_{w_1, w_2, \ldots, w_N} M(w) \equiv E\left[\exp\left[\sum_{i=1}^N w_i^l \left(R^i - 1/c\right)\right]\right]$,

dividing through by $\exp\left(-\sum_{i=1}^N w_i^l c\right) E\left[\exp\left(\sum_{i=1}^N w_i^l R^i\right)\right]$,

$$\frac{dv^l}{d\mu} = \frac{\exp\left(\sum_{i=1}^{N} w_i^l R^i\right)}{E\left[\exp\left(\sum_{i=1}^{N} w_i^l R^i\right)\right]}$$

Stutzer provides four interpretations of the Gibbs SPD benchmark vector. First, he shows that the change of measure relative to the candidate SDF m^c, with mean c as defined by the moment conditions also satisfies the following information bound inequality or minimum distance criterion (p. 376):

$$I\left(v^l, \mu\right) \leq E\left[\frac{m^c}{E\left[m^c\right]} \log\left(\frac{m^c}{E\left[m^c\right]}\right)\right] = I\left(v^c, \mu\right)$$

This inequality plays the same role as Hansen and Jagganathan's variance bound inequality. Second, he provides a quasi-maximum likelihood interpretation of the benchmark (pp. 377-8). Stutzer's quasi-maximum likelihood interpretation of the affine benchmark portfolio and associated Gibbs density is based on the *directed orthogonality property* (Stutzer, p. 377) satisfied by the KLIC criterion[162]. Stutzer's version of the principle of directed orthogonality is given below:

$$v^l \equiv \arg\min_{v} I(v, \mu) \left| \int R_i \frac{m}{E[m]} d\mu = \frac{1}{c} = \arg\min_{v(w)} I(v, v(w)) \right. .$$

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and solving for 1/c, confirming the fact that the Gibbs SPD satisfies the requisite moment conditions. The information bound can, itself, be calculated from the same problem as $I(v^l, m) = - M(w^l)$.

162 The directed orthogonality principle is similar to the orthogonality of best approximation in Hilbert space for generalized inverse solutions to underdetermined systems of linear equations, where the latter linear equations are the constraints, the distance measure employed is the squared Euclidean and the prior estimate is the zero vector. Jones (1989) shows that directed orthogonality is a natural extension of this problem to the task is one of specifying a positive measurable function on a domain of positive measure subject to a series of linearly independent, locally bounded integral constraints. In this more general case, the prior P is an estimate of the density arrived at without knowledge of the linear functional values (constraints) and for admissible functions Q, the solution is chosen by minimizing the directed distance between members in the class Q and the prior P.

A series of straightforward manipulations of $I(v, v(w))$ yields the following:

$$I(v,v(w)) = \int \ln \frac{dv}{dv(w)} dv = \int \ln \frac{dv/d\mu}{dv(w)/d\mu} \equiv E_\mu\left[\frac{dv}{d\mu}\ln\frac{dv}{d\mu}\right] - E_v\left[\ln\frac{dv(w)}{d\mu}\right].$$

Because the first term in the above identity is independent of w, selecting a vector v that maximizes the second term minimizes $I(v, v(w))$. Thus, the Gibbs benchmark SPD v^l (based on the benchmark portfolio weights w^l) that minimizes the constrained KLIC criterion $I(v, \mu)$ (i.e. subject to the relevant moments conditions) also minimizes the unconstrained KLIC criterion $I(v, v(w))$[163].

Third, Stutzer shows that the Gibbs benchmark weights w^l determine the composition of optimal portfolio for an investor with constant absolute risk aversion with a constant of proportionality equal to $-1/a$ where a is the coefficient of absolute risk aversion[164].

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[163] Csiszár (1985) clarifies the point that Akaike's Bayesian approach differs markedly from applications of the extended maximum entropy principle to the derivation of risk-neutral measures (note that in Cziszár's notation Q stands for the prior distribution and $D(\cdot \mid \cdot)$ represents the distance or maximum entropy function):

> Akaike considers statistical estimation problems, adopting the view that an unknown distribution (rather than a parameter) is to be estimated, and he uses $D(P \mid Q)$ as a loss function measuring the loss when the unknown true distribution P is estimated by Q. The maximum entropy principle, as understood in this paper, relates to problems of a different kind, not within the scope of standard statistical decision theory, namely to updating priors to conform to evidence typically consisting in moment constraints. Although in both cases $D(P \mid Q)$ appears as a measure of "distance" which should be minimized, a formal difference is that in Akaike's model minimization is performed with respect to the second variable while the maximum entropy principle calls for minimization with respect to the first one. (Csiszár, 1985, p. 98)

[164] For a CARA investor with initial wealth W_0, who invests $\sum_{i=1}^{N} w_i$ in risky assets and the remainder in the riskless asset, terminal wealth would be:

$$W = W_0\left(\frac{1}{c}\right) + \sum_{i=1}^{N} w_i\left[R_i - \frac{1}{c}\right]$$

The associated optimal utility can therefore be determined from:

Fourth, Stutzer offers a Bayesian interpretation of the Gibbs benchmark SPD based on axiomatic arguments that the KLIC-based density minimizes the information gained by a change of measure satisfying the moment conditions, without incorporating any extraneous information. In other words, when μ is uniformly distributed (a common representation for the Bayesian prior distribution under uncertainty) the solution v^l to the directed orthogonality problem is a Bayesian posterior update of the prior risk-neutral distribution in the light of sample information consistent with the relevant moment conditions.

Shore and Johnson (1980), in work which is cited in Stutzer's paper, provide an alternative axiomatic justification for both maximum entropy and minimum cross-entropy based on a set of four reasonable principles of statistical inference: thus departing from earlier justifications that are instead based on the information-theoretic properties of entropy measures[165]. Without going too far into the technical detail of their analysis, these axioms include: *uniqueness* (i.e. for a specified prior and for new information restricted to a set that includes at least one density with

$$V \equiv \max_{w_1, w_2, \ldots, w_N} E[U(W)] = -e^{-\frac{aW_0}{c}} E\left[e^{\sum_{i=1}^{N} aw_i \left(R_i - \frac{1}{c}\right)} \right] = U\left(W_0 \frac{1}{c}\right) E\left[e^{\sum_{i=1}^{N} -aw_i \left(R_i - \frac{1}{c}\right)} \right].$$

Now $U(W) < 0$ but marginal utility is positive. Therefore, the maximum of V is equivalent to minimization of the second term. When compared with the Gibbs benchmark SPD it can be seen that $w^* = -w^l/a$. Stutzer also derives a utility-based characterization of the information bound:

$$\left| \log\left(\frac{V}{U(W_0/c)} \right) \right| = I(v^l, \mu).$$

165 Henri Theil's (1974) approach to *Rational Random Behaviour* exploits the information-based interpretation of Jaynes' entropy measure. Here, entropy functions as a measure of likelihood, which allows Bayesian decision-makers to arrive at a posterior distribution by multiplying their prior distribution by the entropy measure. Using a calculus of variations approach, Theil shows that the appropriate measure is constructed by taking the exponent of the (negative of the) ratio of the loss function (i.e. the loss arising from a decision based on the incorrect control variable) over the marginal cost of information (given by Jaynes entropy). For a lucid overview see Theil, 1978, pp. 255-61.

finite distance measure, the resulting posterior should be unique); *invariance* (a change in coordinate system that can be represented by a transformation with an invertible Jacobian should not matter to the result); *system independence* (it should not matter whether one accounts for independent information about independent densities separately in terms of different prior densities, obtaining separate posterior densities, or together in terms of a joint density for the prior, because there should be no interaction between the two systems); and *subset independence* (it should not matter whether one treats an independent subset of system states in terms of a separate conditional density or in terms of the full system density in obtaining the posterior density).

6.2.5. Kitamura and Stutzer's Entropy-based GMM Framework

Kitamura and Stutzer (1997) have developed an alternative to the Optimal Minimum Distance (OMD) estimator first proposed by Hansen and Singleton (1982) for Generalized Method of Moments (GMM) estimation. The former estimator, based on minimisation of the Kullback-Leibler Information Criterion, is asymptotically as efficient as the latter OMD, has the same data requirements and computational feasibility, but is more efficient in small samples. The OMD is biased in small samples because sampling errors in the second moment are correlated with sampling errors in the estimate of the covariance matrix of the sample moments. Kitamura and Stutzer use their estimator to construct a χ^2-specification test of the moment conditions as well as Wald, Lagrange multiplier and likelihood ration tests of parametric restrictions, analogous to those commonly used in applications of OMD.

Kitamura and Stutzer closely follow the presentation in Hansen (1982), commencing with a stochastic vector process x_t, $t = 1,2,\ldots$, a parameter vector β from a set Θ of possible parameter vectors, and an r-component vector of observable, real-valued functions $f(x,\beta) = (f_1,\ldots,f_r)'$. The authors denote the observed time-series by $f(x_1,\beta),\ldots, f(x_T,\beta)$. Theory is represented by the prediction $E^{\mu}[f(\mathbf{x},\beta^*)] \equiv \int f(\mathbf{x},\beta^*)d\mu(\mathbf{x}) = \mathbf{0}$, where β is a parameter vector from Θ, E^{μ} is the expectation with respect to probability measure μ, and $\mathbf{0}$ denotes an r-component vector of zeroes. Empirical content is given to the theoretical representation by assuming that: $\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T} f(x_t,\beta^*) \equiv \mathbf{0}$, for most realizations of the process. Hansen's GMM estimator of β^* satisfying theoretical priors is achieved by finding $\hat{\beta}$ which makes the observed vector of sample means $\hat{f}_t(\beta) \equiv (1/T)\sum_{t=1}^{T} f(x_t,\beta)$ close to $\mathbf{0}$. Specifically:

$$\hat{\beta} = \arg\min_{\beta\in\Theta} \tilde{f}_T(\beta)' W \tilde{f}_T(\beta)$$

As Hamilton shows (1994, Chapter 14), the weighting matrix W is calculated from the inverse of the asymptotic variance $\boldsymbol{S}$, where the latter given by:

$$\mathbf{S}_T^* \equiv (1/T)\sum_{t=1}^{T}[f(x_t,\beta_0)][f(x_t,\beta_0)]' \xrightarrow{P} \mathbf{S}$$

An iterative procedure is required using an arbitrary weighting matrix such as $\mathbf{W}_T = \mathbf{I}_r$, to estimate β_0, which is then used in the above expression to produce a new estimate of $\mathbf{W}_T = [\mathbf{S}^{(0)}{}_T\,]^{-1}$, for use in deriving a new estimate of β. The iterative procedure is repeated until a convergence criterion is met. The Newey-West adjusted-estimate of **S** can be used if there is serial correlation in the $[f(x_t, \beta)]_{t=-\infty}^{\infty}$ process.

Hamilton (1994) demonstrates that the GMM technique is sufficiently general to embrace a range of other econometric models as special cases, including: Ordinary Least Squares, instrumental variable estimation, estimators for systems of non-linear simultaneous equations, and dynamic rational expectations models. For example, OLS estimation implies the following set of orthogonality conditions:

$$E[\mathbf{x}_t(y_t - \mathbf{x}_t'\beta_0)] = 0 .$$

The terms inside the brackets conform to what is required of the set of $f(x,\beta)$ functions. For instrumental variables estimation the relevant r-vector of orthogonality conditions becomes:

$E[\mathbf{x}_t(y_t - \mathbf{z}_t'\beta_0)] = 0$, where $\mathbf{z}_t$ is a vector of explanatory variables, and $\mathbf{x}_t$ is a vector of predetermined explanatory variables that are correlated with **z** but uncorrelated with u_t, the residual vector from the regression model $y_t = \mathbf{z}_t'\beta + u_t$. For non-linear systems of simultaneous equations of the form:

$\mathbf{y}_t = \mathbf{g}(\mathbf{x}_t, \beta) + \mathbf{u}_t$, where $\mathbf{z}_t$ is a $(k \times 1)$ vector of explanatory variables, and β is an $(a \times 1)$ vector of unknown parameters, the orthogonality conditions can be expressed in the required form:

$$\mathbf{f}(\mathbf{x}_t,\beta)=\begin{bmatrix}[y_{1t}-f_1(\mathbf{x}_t)]\mathbf{z}_{1t}\\ [y_{2t}-f_2(\mathbf{x}_t)]\mathbf{z}_{2t}\\ \vdots \\ [y_{nt}-f_n(\mathbf{x}_t)]\mathbf{z}_{nt}\end{bmatrix}.$$

Here $\mathbf{z}_{it}$ is a vector of instruments that are uncorrelated with the ith element of $\mathbf{u}_t$. Finally, for estimating dynamic rational expectations models, the relevant set of Euler equations becomes the set of real-valued functions that is to be estimated through GMM techniques (see Hamilton, 1994, pp. 416-24 for details).

Kitamura and Stutzer's proposed replacement for the Hansen estimator is one based on a non-linear projection problem:

$$\min_{P\in\mathbf{P}(\beta)} D(P:\mu)\equiv \min_P \int \log(dP/d\mu)dP$$
subject to $E^P[f(\mathbf{x},\beta)]=\mathbf{0}$

where $D(P:\mu)\equiv \int \log(dP/d\mu)dP$ is the Kullback-Leibler Information Criterion distance from P to μ. The optimal estimator β is found by making $D(P{:}\mu)$ as close to zero as possible. Kitamura and Stutzer (1997, pp. 864-5) show that this is achieved by finding the saddlepoint of the following function:

$$\mathbf{M}(\beta,\gamma)\equiv E^{\mu}\left[e^{\gamma f(\mathbf{x},\beta)}\right]$$

where $\gamma(\beta)=\arg\min_{\gamma}\mathbf{M}(\beta,\gamma)$, and $\beta^*=\arg\max_{\beta}\mathbf{M}(\beta,\gamma(\beta))$

As we have seen above, this function arises naturally from the expression for the *Gibbs canonical density* (p. 864):

$$\frac{dP(\beta)}{d\mu}=\frac{\exp\left[\gamma(\beta)' f(\mathbf{x},\beta)\right]}{E^{\mu}\left[\mathbf{exp}\left[\gamma(\beta)' f(\mathbf{x},\beta)\right]\right]}$$

Under a series of standard assumptions (p. 866), the authors show that an asymptotically efficient estimator can be calculated by replacing the observation $f(\mathbf{x}_t,\beta)$ with:

$$\hat{f}(t,\beta)\equiv\sum_{k=-K}^{K}\frac{1}{2K+1}f(x_{t-k},\beta)$$

where $K^2/T\to 0$ and $K\to\infty$ as $T\to\infty$[166]. The estimator is then determined by:

$$\left(\hat{\beta}_T,\hat{\gamma}_t\right)=\arg\max_{\beta}\min_{\gamma}\left[\hat{Q}_T(\beta,\gamma)=\frac{1}{T}\sum_{t=1}^{T}e^{\gamma'}\hat{f}(t,\beta)\right]$$

In a further discussion of other practical asset-pricing problems that can be addressed using their approach Bernardo and Ledoit have observed that:

> Real options are difficult to value using arbitrage methods since the stochastic component of the options return often cannot be replicated because the underlying asset does not exist, does not trade, trades in an illiquid market, or is not spanned by a portfolio of traded assets. If one can construct an imperfect hedging strategy by using some combination of existing assets, then our gain-loss

[166] These assumptions primarily relate to characteristics of the $\mathbf{x}_t$ process that guarantee asymptotic normality (i.e. strong mixing, stationary and ergodic), but they also include moment existence conditions, differentiability of $f(x,\beta)$ at the optimum, non-singularity of the denominator in the Gibbs canonical density, and continuity and uniqueness of the β parameter satisfying the constraints (1).

> restriction yields bounds consistent with the inability to construct extremely attractive portfolios using these basis assets (p. 168).

This simple observation underlies what I am attempting to achieve in this paper. I have argued that the gain-loss and entropy approaches to options pricing mirror the findings of researchers who have used techniques of risk-sensitive and robust control to price assets. However, for deriving effective bounds on option prices in incomplete markets, the former body of literature affords more straightforward methods—that could, for example, be incorporated into spread-sheet or lattice-based models—with results which are, in essence, similar to those that can only be attained with much greater effort and more demanding levels of technical virtuosity through the application of via robust control techniques. This characteristic of entropy-based methods will be demonstrated in the next section of the paper.

6.2.6. Minimum Cross-Entropy and Martingale Measures: the Discrete-Range Case

The relationship between minimum entropy measures and martingales can be seen most clearly in the discrete-time and discrete-range case. Unfortunately, a different notation is required for the discrete range case because expectations and constraints have to be defined over a finite partition of the probability space. Consider a filtration $\mathbf{F}_N = \mathbf{F} = \sigma(P_N) = \{\omega_1, \omega_2, \ldots, \omega_N\}$ representing the evolution of the information process as a random sequence $\{A_l\}$ of subsets of the finite sample space Ω. Moreover, the P_t are sequences of partitions of Ω such that for each block A in P_t there exist blocks $A_1, \ldots, A_k$ in P_{t+1} such that $A = \cup A_j$. Also, for any $\omega \in \Omega$ there exists a sequence of blocks $B_N\{\omega\} = \{\omega\} \subset B_{N-1}\{\omega\} \subset \ldots. B_0\{\omega\} = \Omega$, each contained in P_{t+1}. Any probability Q on (Ω,

F) is determined by a sequence of conditional probabilities $Q[\cdot \mid A_t]$ for A_t in P_t and $t = 0,1,\ldots,N$ as follows:

$$Q\{\omega\} = Q_{B_0\{\omega\}}(B_1\{\omega\})Q_{B_1\{\omega\}}(B_2\{\omega\})\Lambda\ Q_{B_{N-1}\{\omega\}}(B_N\{\omega\})$$

When defined over such a filtration, we know that a risk-neutral probability measure Q turns S^*_t into an (F_t, Q) martingale that satisfies the following equation:

$$E_Q\left[S_n^*(t+s)\middle|F_t\right] = E_Q\left[J_t\, S_n(t+s)/J_{t+1}\middle|F_t\right] = S_n^*(t); \quad t,s \geq 0$$

Here $J=\{J_t : t = 0,1,\ldots,T\}$ is the bank account process with $J_0 = 1$ and the risk-free interest rate $r_t = (J_t - J_{t-1} / J_t) \geq 0$; while $S_n = \{S_n(t)$: $t = 0,1,\ldots,T\}$ is the non-negative stochastic process representing the time t price of the nth risky security and $S^*_n(t) \equiv S_n(t)/ J_t$, $t = 0,1,\ldots,T;$ $n = 1,2,\ldots,N$ is the discounted security price. Since the S^*_t vector is F_{t-1}–measurable its components are constant on the blocks of P_{t+1}. Let $S^*_{t+1} = \mathbf{y}_j$ on A_j. Assume that each B_j is partitioned into blocks $B_{kj} \in P_t$, for $j = 1,2,\ldots,M(j)$; $M(j) \geq 1$ and $\Sigma M(j) = N(t)$. Keeping t fixed, let the values of S^*_t on the block B_{ij} be denoted by $\mathbf{y}_{ij} \in \Re d$ with $\mathbf{s}_j$ denoting the value of S^*_{t-1} on the block B_j. In this context, Henryk Gzyl (2000) develops a risk-neutral measure through the application of the maximum entropy principle. First, he introduces the usual discrete-range non-arbitrage conditions:

$$\sum_{j=1}^{k} \mathbf{y}_j q_j = \mathbf{s}, \quad \sum_{j=1}^{k} q_j = 1, \text{ with the } F_{t+1} \text{ measurable prices } S^*_{t+1} = \mathbf{y}_j \text{ on the jth partition } A_j \cdot$$

The convex class P(p,**s**), assumed to be non-empty, is then defined in relation to these conditions as follows:

$$P(p,\mathbf{s}) = \{q(j) = a(j)p(j) | a(j) > 0;\ j = 1,\dots,k; \sum q(j) = 1;\ \sum \mathbf{y}_j q(j) = \mathbf{s}\}$$

Second, a concave function $S_p(q)$ is defined over this convex space as follows:

$$S_p(q) = -\sum q(j) \ln a(j).$$

This function is arrived at through the substitution of $q(j)$ and $p(\lambda,j)$ for the functions $f(j)$ and $g(j)$ in the discrete-range version of the Kullback-Leibler formula $K_p(f,g) = \sum f(j)\ln(f(j)/g(j))p(j)$.

Third, Gzyl defines the following exponential family (obviously related to the Gibbs SPD), parameterized by $\lambda \in \Re^d$:

$$p(\lambda, j) = \frac{e^{-\langle \lambda, y(j)\rangle} p(j)}{E_p\left[e^{-\langle \lambda, \mathbf{Y}\rangle}\right]} = \frac{1}{Z(\lambda)} e^{-\langle \lambda, y(j)\rangle} p(j)$$

where the $\mathbf{y}_j$ are values of the random variable $\mathbf{Y}$, such that P($\mathbf{Y} = \mathbf{y}_j$) = p(j). It must be he case that:

$S_p(q) \le \Sigma(\lambda) := \ln Z(\lambda) + \langle \lambda, \mathbf{s}\rangle$. As $Z(\lambda)$ is convex on $\Re^d$, and as $\Sigma(\lambda) \to \infty$ when $\|\lambda\| \to \infty$, then $\Sigma(\lambda)$ achieves its unique minimum at a certain point λ^* in $\Re^d$. Moreover, at this point it can be confirmed that $p(\lambda^*) \equiv p^*$ is in P(p,s), and at this point of minimum entropy $S_p(p^*) = \Sigma(\lambda)$ (Gyzl, 2000, p. 6; Csiszár, 1985, pp. 86-7)[167].

Pulling together the all the previous strands of analysis, it must be the case that the requisite risk-neutral measure Q on (Ω, F) can now be constructed from its sequence of conditional expectations with respect to the filtration $\{F_t; t = 1,\ldots,N\}$ in accordance with:

$$Q_{ij} = Q_{B_j}(B_{ij}) = \frac{e^{-\langle \lambda_t(j), y_{ij} \rangle} \mathbf{1}}{Z(t, \lambda_t(j))} p_{ij}$$

Because $Z(t, \lambda_t(j)) = \sum_{B_{ij} \subset B_j} e^{-\langle \lambda_t(j), \mathrm{y}_{ij} \rangle} p_{ij}$, then $p_{ij} = P(B_{ij})/P(B_j)$ can be replaced with $P(B_{ij})$. Moreover, $\lambda_t(j)$ will be constant on each B_j on F_{t-1} as will $Z(t, \lambda_t(j))$, so that these variables are both F_{t-1}-measurable. Thus:

$$\rho_t := \frac{e^{-\langle \lambda_t, S_t^* - S_{t-1}^* \rangle}}{\sum_k e^{-\langle \lambda_r(j), (\mathbf{y}_{ij} - \mathbf{s}_j) \rangle} p_{kj}} = \frac{e^{-\langle \lambda_t, \Delta S_t^* \rangle}}{Z_\Delta(t, \lambda_t)\big|_{B_j}} = \frac{e^{-\langle \lambda_t, S_t^* \rangle}}{Z(t, \lambda_t)}$$

since S^*_{t-1} is constant on the blocks B_j of P_{t-1}. Moreover:

$$\Sigma(t, \lambda_t) = \ln Z(t, \lambda_t) + \langle \lambda_t, S_{t-1}^* \rangle = \ln Z_\Delta(t, \lambda_t)$$

Now consider an F_{t-1}-measurable function H. The risk-neutral expectation of H can be calculated in accordance with the following:

[167] Incidentally, Imre Csiszár (1985) calls the updated probability density derived by minimizing the relative entropy or directed divergence (as reflected in the Kullback-Leibler number for all admissible densities) the "*I*-projection" and provides a Bayesian justification for its use that is similar to that offered by Stutzer.

$$E_Q[H|F_{t-1}]I_{B(j)} = E_Q[H|B_j] = \Sigma_k H(k)q_{kj} = \sum_{B_{ij}\subset B_j} H(k)\frac{e^{-\langle \lambda_t(k), y_{kj}\rangle}}{Z(t,\lambda_t,(j))}P(B_{kj}) = \int_{B_j}\rho_t H dP$$

Thus,

$$E_Q[H|F_{t-1}] = \frac{E_p[\rho H|F_{t-1}]}{E_p[\rho|F_{t-1}]} = E_p\left[\frac{\Psi_t}{\Psi_{t-1}}H|F_{t-1}\right], \text{ where } \Psi_t = E_p[\rho|F_{t-1}] \text{ and } \rho = \prod_{t=1}^{N}(\Psi_t/\Psi_{t-1}) = \Psi_N/\Psi_0$$

Here, $\Psi_0=1$, so that $\rho=\Psi_N$.

Accordingly, $\rho_t = \Psi_t/\Psi_{t-1} = \dfrac{e^{-\langle \lambda_t, \Delta S_t^*\rangle}}{Z_\Delta(t,\lambda_t)}$, and the function $\Sigma_\Delta(t,\lambda_t)$: $\Re^d\times\Omega\rightarrow\Re$ defined above has a minimum at $\lambda_t(\omega)\in\Re^d$, for each $\omega\in\Omega$, $\lambda_t:\Omega\rightarrow\Re^d$ is F_{t-1}-measurable, and (Gzyl, 2000, theorem 3.1, p. 10):

$$\rho = \exp\left\{-\sum_{t=1}^{N}\left[\langle \lambda_t, \Delta S_t^*\rangle + \ln Z_\Delta(t,\lambda_t)\right]\right\}$$

defines a density for Q with respect to P such that S^*_t is an (F_t, Q)-martingale.

Gzyl (pp. 11-13) uses this result to derive risk-neutral measures for the jump probabilities in a binomial, trinomial and a generic Markov chain model. However, at this juncture two specific aspects of Shore and Johnson's research (1980) are particularly noteworthy. First, for the discrete-range case, they establish that maximum entropy is a special case of minimum cross-entropy when the linear constraints are known to be binding, but no prior is available to the researcher (Shore and Johnson, 1980, p. 33). Second, using the same axiomatic properties, they demonstrate that minimum cross-entropy is still the appropriate distance measure to adopt, even for cases where the linear constraints that are to be imposed take the form of *inequalities* rather than *equations* (Shore and Johnson, 1980, equation 4, p. 43). This latter result provides a direct link to efforts by researchers such as Cochrane (2000) and Bernardo and Ledoit (2000), who endeavour to extend option pricing to the case of incomplete markets by imposing range bounds over the Sharpe ratio or gain-loss ratio. It also confirms that Stutzer's minimum cross entropy approach to the diagnosis of asset-pricing models can, likewise, by generalized to the case of incomplete markets. Rather than imposing a bound over the gain-loss ratio of the form $\alpha \geq m/m^*$ $\geq \beta$, as do Bernardo and Ledoit, the researcher can impose a pair of inequality bounds directly over the non-arbitrage conditions[168].

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[168] Kapur and Kesavan (1994, pp. 344-345) consider the original Markowitz portfolio choice problem, wherein each of the given expected returns on each security can be viewed as a moment condition, supplemented by non-negativity conditions on the amounts of each security in the portfolio and the requirement that portfolio shares must sum to unity. They demonstrate that if the Havrada-Charvat measure of minimum cross entropy is employed as an alternative to variance in this problem, then minimizing this measure is equivalent to maximizing the expected utility of a person with a CARA utility function of the form, $u(x) = x^{1-\alpha}$. This result establishes a clear entropy-based link between Stutzer's stochastic discount rate approach to estimating the SDF and the earlier literature focusing on the Markowitzian minimum-variance frontier.

Later in the same chapter (p. 351), they observe that a particular case may arise when the researcher may wish to impose inequality constraints over the actual probabilities that a state may arise. If, for the discrete-range case, one wishes to minimize cross entropy subject to the constraints:

$$\sum_{i=1}^{n} p_i = 1; \quad \sum_{i=1}^{n} p_i g_{ri} = a_r, \quad r = 1,2,\ldots,m; \quad a_i \leq p_i \leq b_i,$$

then Kapur and Kesavan recommend the adoption of the following generalized measure of minimum cross entropy:

$$\sum_{i=1}^{n} (p_i - a_i) \ln \frac{p_i - a_i}{q_i - a_i} + \sum_{i=1}^{n} (b_i - p_i) \ln \frac{b_i - p_i}{b_i - q_i}.$$

This measure is more convenient than the KLIC since the probabilities obtained automatically satisfy the inequality constraints unlike the former. To apply this technique to Stutzer's problem let $q_i = \mu_i$ and $p_i = v_i$. The moment constraints can then be written as:

$$\sum_{i=1}^{n} p_i g_{ri} = a_r \Rightarrow \sum_{i=1}^{m} R_i \frac{m}{E(m)} \mu_i = \frac{1}{c} = E_v[R_i].$$

6.2.7. HOWE AND RUSTEM'S MINIMAX HEDGING ALGORITHM

I shall now review Howe and Rustem's minimax approach to portfolio hedging in the presence of transaction costs to establish the setting for a brief sojourn into the field of minimum martingale measures. Howe and Rustem (1997, pp. 1073-1075) discuss various hedging strategies that the writer of a call option can adopt to cover potential downside risk. The standard approach is delta hedging whereby the writer holds a number of units of the underlying stock so that any decrease in the value of the stocks is offset by an increase in the value of the option, and vice versa. The amount of underlying stock is given by the delta—the instantaneous derivative of the option price with respect to the actual stock price—that will change with time. When transactions costs are significant, the writer cannot engage in continuous rebalancing and must, instead, fall back on discrete delta hedging, rebalancing at discrete intervals of time. This results in the accumulation of hedging errors *HE,* over each of the intervals from t to t+1 as in:

$$HE = N(B_t - B_{t+1}) + n_t(S_{t+1} - S_t)$$

where the contract is initiated at time 0, $B = B(S, t)$ is the call price, S is the stock price at time t, n is the number of shares to hold, and N is the contracted number of shares of stock. The larger the hedging error the larger the cost of rebalancing. Howe and Rustem (p. 1075) observe that minimax control can be applied in the case of discrete delta hedging with the objective of finding the hedge ratio that minimizes the worst case hedging error. They follow Leland's approach to the problem (1985), that introduces a modification σ' to the volatility σ appearing in the original Black and Scholes formula as in the expression:

The inequality constraints thus impose bounds over the discrete-range version of the Radon-Nikodym derivative $dv/d\mu = m/E[m]$.

$$\sigma' = \sqrt{\sigma^2\left[1+\sqrt{\frac{2}{\pi}}\frac{K}{\sigma\sqrt{\Delta t}}\right]}$$

where K is the roundtrip transaction cost arising from a buy and a sell of the same security, in either order. A formal representation of the problem is given by:

$$\min_{x\in R^n}\max_{y\in Y} f(x, y)$$

where y is a convex and compact infinite set $Y \subset \Re^m$, $f: \Re^n * Y \Rightarrow \Re^1$ such that f and $\nabla_y f(x, y)$ are continuous on $\Re^n * Y$. Constrained nonlinear programming cannot be used here, because it is a semi-finite optimization problem with an infinite number of constraints, each corresponding to the infinite number of elements in the set Y. In its place, Howe and Rustem present a quasi-Newton algorithm based on a quadratic approximation to the original problem. They consider a one-period and two-period application of the algorithm to both individual options and portfolios of options. In these applications, upper and lower bounds are imposed on the underlying asset-price for each period and the worst-case hedging cost is determined within these bounds.

6.2.8. Minimum Martingale Measures

A recent development in the option pricing literature involves the application of minimum martingale measures (MMM) to the pricing of options in markets that feature volatility and transaction clustering. Typically, options are priced over a binomial lattice characterized by a mixed point process with constant jump sizes a, occurring over random or irregular rather than fixed time arrival intervals, and with the jump probabilities of up-moves and down-moves given by a logistic transformation of an autoregressive process (see Prigent, Renault and Scaillet, 1999a).

These Marked Point Process (MPP) models allow for the possibility of either herding behaviour or mean reversion in the stock prices and capture the volatility smiles and smirks that are observed in actively traded markets. They are directly analogous to the GARCH and log-GARCH models but are applied to durations. Previous research by the same group of authors (1999b) led them to prefer the log form of the autoregressive conditional duration (ACD) model of Engle and Russell (1998) over the unlogged form (to guarantee positivity of the durations), over a latent geometric brownian motion process, observable only when its logarithm crossed boundaries spaced by the pregiven jump size a (which was difficult to express in kernel form), and over a Poisson version (Bossaerts et al. 1996) of the MPP (because exponentially distributed inter-trade durations were not supported by the data for small values of a). The relevant durations are a product of two terms: a residual and the conditional expectation of the duration. The distribution of the residual, which can be determined from the data, can take either the exponential or Weibull form.

The conditional distribution of marks was represented by the logistic linear model of Cox (1981) extended to incorporate lagged values of the conditional probabilities: a binomial (ACB) version of the autoregressive conditional multinomial (ACM) model of Russell and Engle (1998). This econometric model possesses the obvious advantage that it can be estimated from actual high-frequency transaction data.

The presence of jumps in the model implies that the market is incomplete so that a method must be chosen to select one measure from amongst the class of equivalent martingale measures. Prigent, Renault and Scaillet, (1999a) reject the optimal variance measure that minimizes total risk from time t to T, under historical probabilities, because its existence cannot be guaranteed and, in general, it does not possess an analytical form. Instead, they choose Schweizer's (1991) approach based on minimizing local risk over successive small periods between time t and T. This measure is characterized by the fact that it sets to zero all risk premia on sources of risk orthogonal to the martingale part of the underlying's price process. An explicit form for the Radon-Nikodym derivative can always be constructed for this particular measure, which possesses good convergence properties (Musiela and Rutkowski, 1997, pp. 99-108 and pp. 252-64). Moreover, jump boundedness ensures that the MMM is always positive so that the value of the trading strategy is an actual non-arbitrage price. Notably, when the mean-variance tradeoff (i.e. market price of risk) is deterministic the MMM is the closest of all EMMs to the original probability measure when measured by the relative entropy or directed divergence criterion (Föllmer and Schweizer, 1991)[169].

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[169] Recently, Elliott and Madan (1998) have questioned the applicability of minimal martingale methods in a discrete-time continuous-state setting. They show that "...the requirements for minimality are typically inconsistent with those of being a probability law, in that minimality often leads to signed measures and a loss of non-negativity" (Elliott and Madan, 1998, p. 128). To overcome these limitations Elliott and Madan develop an extended Girsanov principle for discrete-time continuous-state processes showing that this always defines an equivalent martingale measure. However, the resulting measure is only weak form efficient in that arbitrage opportunities that are excluded solely depend on past price information. Under the extended Girsanov principle, discounted asset-prices take on the probability law of their martingale components prior to the change in measure (theorem 3.1, p. 137). Moreover, the extended Girsanov principle is supported by weak form efficient hedging strategies that minimize the variance of the risk-adjusted costs of hedging, where risk-adjusted prices deflate asset-prices by the asset's excess return (theorems 4.1, 4.2, p. 140).

However, this literature is less relevant to my application of option pricing theory in this chapter, which has examined real options pricing under uncertainty aversion. For this purpose an approach based solely on minimum cross entropy methods is completely adequate.

6.2.9. Concluding Comments on the Second Case Study

In this case study I have focused on entropy as the unifying vehicle for investigating aspects of real options theory under incomplete markets. Stutzer's (1995) research provided a range of alternative interpretations of the benchmark, state price probability density in asset-pricing models. I then examined Kitamura and Stutzer's application of entropy-based techniques to GMM estimation of asset-pricing models. Finally, I reviewed Gzyl's (2000) use of entropy-based methods to derive martingale measures, which he had applied to binomial, trinomial and Markov-chain models to price both real and financial options. I drew on material that facilitated entropy-based estimation of option values and Martingale measures for cases where inequality constraints must be imposed over the requisite measure changes. These results provided a clear link between the entropy techniques I have discussed and recent investigations into "good-deal" bounds and norm-bounds over gain-loss ratios for pricing options in incomplete markets. In future research I intend to utilize these entropy-based techniques to value various kinds of real and financial options.

Andersen, Hansen and Sargent (1999) draw on Dupuis and Ellis's (1997) characterization of the duality between free energy and relative entropy to construct error bounds for risk-sensitive filters. As in the technical appendix I show, Boel, James and Petersen (1997) use the same characterization to calculate error bounds for the more general case in which the actual state is observed with error. Boel, James and Petersen (1997) assume that the true probability model is fixed, but unknown, and that the estimation procedure makes use of a fixed nominal model. They demonstrate that the resulting error bound for the risk-sensitive filter is the sum of two terms, one of which coincides with an upper bound on the error one would obtain if one knew exactly the underlying probability model, while the other is a measure of the distance between the true and design probability models. In the particular option pricing case under investigation here, the divergence represented by each of these two terms has been replaced by another pair of terms: one representing the measure change required to convert the stochastic return process into a martingale and the other representing the inequality bound over the resulting Radon-Nikodym derivative (i.e. $m/E(m)$, the stochastic discount rate). In both cases this divergence is measured by the relative entropy metric, but in the former case relative entropy is implicitly embodied in the exponential characterisation of the risk-sensitive filter, while in the latter case it is embodied in the set of inequality constraints over the relevant moments that are reflected in a set of induced bounds over the Gibbs benchmark state price probability density (i.e. via the KLIC or generalized entropy measure). If Kapur and Kesavan's (1994) generalized entropy measure is employed as a substitute for the KLIC, then ordinary Lagrangian multiplier techniques can be used to solve the minimum cross entropy problem rather than the more demanding Kuhn-Tucker techniques because they possess the advantage of automatically satisfying the inequality constraints.

In the option pricing literature it is typically assumed that the individual investor can exert no influence over the stochastic properties of the underlying asset. In such cases it is difficult to see why a change in the degree of uncertainty aversion would influence the dimensions of inequality constraints obtaining over the generalized minimum cross entropy measure. However, if the option valuation process were embedded in a more comprehensive model, with a monetary asset made available as a potential hedge against uncertainty over prospective returns, then one could readily associate rising uncertainty aversion with a rising preference for liquidity as real investments are postponed or abandoned[170].

Further research is progressing in the field of options pricing in incomplete markets that will have direct relevance to real options theory. I briefly reviewed Howe and Rustem's (1997) study of optimal hedging strategies when investors face significant transaction costs. McEneaney (1997) has foreshadowed similar research that he intends to conduct utilizing techniques of robust control. An investor can readily be conceived of as managing a portfolio of both real and financial options, and the transactions costs associated with hedging real options exposure are often sizeable. Accordingly, this field of research into minimax hedging would potentially have significant practical benefits.

With the increasing availability of high frequency data, evidence is accumulating that financial time-series exhibit the fractal characteristics that are associated with non-linear chaotic dynamics

[170] As discussed in Chapter One, Klaus Nehring (1999) has developed an axiomatic basis for choice that exhibits a "preference for flexibility" under uncertainty, in the sense that the agent wants to keep her options open so that she can respond to *anticipated* but *unforseen* contingencies. In future, this sort of framework could potentially be applied to portfolio choice and real options theory, in incomplete markets, where the money asset operates as a hedge against such forms of uncertainty.

(Dacorogna et al. 1993). In the first case study section I discussed a range of approaches to the estimation of non-linear systems. Techniques of this nature have obvious applications in finance for cases where the underlying dynamic system is known to be non-linear.

Tornell (2000) envisages further developments in finance-related applications of robust control that allow for time-variation in the robustness parameter over the business cycle. These variations would reflect changing investor sentiment or uncertainty aversion as the economic environment improves or deteriorates. Treating uncertainty aversion as an endogenous parameter seems to violate the neoclassical penchant to treat changes in preferences and technology as exogenous to the system. However, it mirrors related research into *adaptive belief systems* (Brock and Hommes, 1997), which presumes that investors are able to switch predictors in response to endogenous changes in their relative cost and predictive performance.

The inverse relationship between the robustness parameter γ and the risk-sensitive parameter θ, a parameter which also appears within the related expression for the entropy integral, confirms the effect that uncertainty aversion has on outcomes irrespective of whether the relevant decisions have been modeled using stochastic risk-sensitive control, maximum entropy or deterministic robust control techniques[171]. I have established that such endogenous fluctuations in the robustness parameter are directly related to the entropy-based alternative to the variance bounds, good-deal bounds, or the gain-loss ratio. A change in uncertainty aversion alone, quite apart from any variation in the stochastic characteristics of the underlying assets, would result in a widening

[171] See the relevant section of the technical appendix for details on this matter.

or narrowing of these bounds and ratios. This is because an increase in uncertainty aversion would have a direct impact on the drift and volatility of the *underlying* price!

Further progress in this direction affords the exciting prospect of combining neoclassical finance theory with Keynesian research into uncertainty, liquidity preference and animal spirits as influences over investment behaviour. As uncertainty increases, and prices move further along the continuum away from a unique reference value more towards the non-arbitrage bounds over the value of the real options, the spread between the "bid" and "ask" prices that reflect "market incompleteness" would widen. As a result, investment activity would inevitably decline. This is the ultimate insight that I have been striving for in this case study.

6.3. Conclusion

With their more recent venturing into risk-sensitive and robust control, Sargent, Hansen and their colleagues have come close to operationalizing certain aspects of Herbert Simon's notion of bounded rationality. However, recognising norm-bounded perturbations is not quite the same thing as recognising the implications of inadequacies in one's scientific theory. The notion of an entanglement between observation noise, system perturbation and ignorance (an entanglement that Hansen and Sargent appear to be reluctant to fully acknowledge) is still far more *circumscribed* than Simon's concept of an inability to identify all strategies, and fully evaluate their outcomes. And although robust control as a representation of Knightian uncertainty comes a little bit closer to satisficing rather than optimizing behavior, it still represents a very narrow and formal interpretation of this rich and complex concept. For Simon, satisficing represents choices made on the basis of barely acceptable and minimal criteria from amongst an incompletely identified and assessed set of alternatives. For Sargent, however, it represents risk-sensitive optimization over a pre-specified class of feasible (i.e. twisted) diffusion or Markov processes.

It is important to realize that the philosophical ideas about bounded rationality and creative intervention that I outlined in Chapter 5 are not merely to be found in the works of radical academics or obscure German philosophers. In their introductory discussion of neural networks Wang et al. (1995) characterize the Intelligent Algorithms (IA) that provide the foundations for Intelligent Control (IC) as having the ability to accomplish the following tasks: sensing the surrounding event and the world; influencing their interaction with the environment; and modeling their cause and effect relationships. However, they stress the oft-ignored reality that it is not the *technique* used that is intelligent, but rather the *application* of the technique.

Intelligent application of these algorithmic techniques requires a capacity to plan and reason about these actions. It also requires a capacity to learn from interactions with the environment, generalize information to similar situations, and abstract common concepts autonomously. For example, designers have the freedom to determine the network architecture and interconnections, the network's functional approximation and representation ability, and the learning laws by which the network adapts itself to external influences or experiences. In economic systems, policy makers can intervene to influence both norm bounds and also levels of robustness in the preference-structure of individual agents.

Those who propound rational expectations or model-consistent approaches in economics often confuse technique with application. The LQG or H-infinity solution is meant to *transparently* represent actual behavior on the part of the representative agent. Intelligent application is implicitly presumed and never discussed other than, perhaps, in the ill-defined evolutionary sense that success in the market-place will ultimately benefit the sensible and wean out the stupid. Instead, a more realistic concern with intelligent application would favour a more sophisticated and cautious behavioral approach to the modeling of equilibrium and disequilibrium within commodity and financial markets: one that recognizes not only varying degrees of ignorance about system dynamics, the differentials in observation error and differing levels of systemic exposure to external disturbances, but also the localized and heterogeneous character of information and the difficulties involved in coordination across related markets. Habermas, for one, would presumably choose to emphasize collective forms of coummunicative action and understanding that have the potential to transform our institutions and to forever change the environment within which agents make economic decisions.

Herbert Simon's focus on bounded rationality was a natural response to what he saw as the unquestioning allegiance on the part of his colleagues at Carnegie-Mellon, to the canons of methodological individualism, underpinned by the resort to an engineering-based optimal control theory. In this thesis I have argued that, for modern finance theory to adequately grasp the true nature of asset-pricing, it would have to account for the following vital features of a monetary production economy:

1. The ubiquitous use of nominal, non-indexed contracts, predicated on accurate forecasts of inflation;
2. The significance of transactions costs for liquidity preference as a form of investment in flexibility;
3. The fact that stochastic discount rates can be decomposed into factor risk premia and uncertainty premia (that are closely associated with the existence of nominal contracts and the imposition of transactions costs on trading illiquid assets);
4. The existence of differentials in the psychological preference for liquidity on the part of banks, households and firms;
5. Economy-wide switching between Classical, Repressed Inflation and Keynesian macroeconomic regimes, in part driven by fluctuations in uncertainty premia, so that gross rates of return on financial assets cannot be treated as exogenously determined, as they might be in pure exchange models of asset-price determination;
6. The fact that a range of social and epistemic factors would influence uncertainty premia through their direct effect on uncertainty aversion and uncertainty perception (including policy interventions of various kinds). This implies that parameters of risk-sensitivity and relative entropy bounds are best viewed as a "moving feast": a fact that is impossible to accommodate within current forms of robust control theory;
7. Variations in financial instability, reflecting movements along the investment continuum that ranges from hedge, through speculative to Ponzi financial positions, should be dealt with (in

a control sense) through the use of adaptive techniques that trace the trajectories of critical time-varying parameters;

8. Non-linearities and complex dynamics are inescapable aspects of economic behaviour, whose existence can readily be justified on a micro-foundational basis. For this reason, exponents of robust control should adopt techniques for dealing with non-linear rather than linear systems (as in chaotic control theory).

Needless to say, any modeling research program that attempted to meet all of these specified and justifiable criteria would have to move within a theoretical world of enormous complexity. For that very reason, the endeavours of Hansen, Sargent and their associates should be seen for what they really are: a set of elaborate, sophisticated and intellectual challenging metaphors (dare I say "fairy-tales") for dealing with a very limited range of bounds over human decision-making. And what should therefore be highlighted, is the ubiquitousness of Simonian "bounded rationality" in the broadest sense of the term: one that cannot be reduced to a concern for the modeling of "large deviation bounded" learning processes. In this world, satisficing behaviour based on consensual decision-making and "rules of thumb" is unavoidable. The role of the social scientists should be to better understand this reality, and not to fabricate another one, that is both easily digestible and comprehensible, but all the same utopian.

Similarly, despite a need to rely upon quantitative techniques for policy analysis, forecasting and prediction, the practitioner should never be blind to the fact that policy and strategy operates within a much richer universe of discourse and practice than the one conceived by either old or new forms of New Classical Theory. In progressive management circles, it is universally acknowledged that successful enterprises are those that can go beyond mere "optimization within given constraints" to focus on transforming the constraints and option sets that confront them. Intelligent application can overtake the mere acquisition and replication of various technical capabilities. The creative rearrangement of existing techniques into new combinations and sequences can be of far greater importance for innovation than the mere application of known techniques in conventional and familiar settings.

Remaining within a corporate setting for the moment, I wish to emphasize the policy-relevance of this distinction. It is one that recognizes the straightforward notion that social and cultural factors can become an important influence over economic activity because they can influence the adopted attitudes and the actual behaviour of agents who contract with one another. For example, cultural factors can confer obvious economic advantage through the promotion of long-term, trustworthy relationships and the discouragement of anti-social predation and rapacity. Notably, relationships of trust that promote collaboration between user and producers of technology, between the science sector and industry, and within supply chains that encompass more than one down-stream purchaser of materials, components or know-how can foster innovation and engender competitive success.

In the context of macroeconomic policy, this managerial notion of "changing the constraints" rather than optimising within them finds its direct counterpart in Alessandro Vercelli's distinction between adaptive and creative rationality: the former applying to decisions made within the existing environment and structure of relationships and the latter to decisions that directly or indirectly affect this environment and this structure. While institutional reforms can directly effect changes of this nature, a government-induced transformation of the commercial or financial culture can indirectly effect like changes. In game theory, credibility and time-consistency are important attributes of successful policy within the given environment, but they can also be important attributes of policies that aim to reform the economic environment.

Robust control and uncertainty aversion are useful concepts because they have the potential to question the more extreme presumptions of old-school rational expectations theorists. As I argued in the Introduction to this thesis, I see them as complementary to other strands of thought that have introduced notions of dynamic instability, indeterminacy and heterogeneity of beliefs into economic theory. It would be unfortunate if these efforts were seized upon once again, as mere vehicles for grounding conservative political interventions on supposedly more secure foundations. In this regard, a critique of anti-humanist system-theory marks the boundary and the "zone of contestation" for those who believe that sensible, transparent, policy interventions have their role to play in reducing unemployment, financial instability and economic hardship.

TECHNICAL APPENDIX — ELEMENTS OF DISCRETE-TIME FINANCE THEORY, HIDDEN-MARKOV MODELING, AND RISK-SENSITIVE AND ROBUST CONTROL THEORY

A.1. Introduction

In this technical appendix, I review the relevant literature on recursive control theory. The boundary of my task has been defined by Andersen, Hansen and Sargent's (1999) presentation of a sophisticated and mathematically challenging version of risk-sensitive control theory, where uncertainty appears in the form of a relative entropy constraint. The authors interpret this form of uncertainty as a norm-bounded perturbation of the matrix of transition probabilities in a hidden-Markov or regime switching model. They apply this version of risk-sensitive control to both optimal consumption and (financial) investment planning and asset-price determination. However, their analysis is set within a complete information context. Therefore, I examine a still more general, partial information representation of the control and filtering problem.

To keep the mathematical requirements as straightforward as possible I confine my discussion to the discrete-time rather than the continuous-time framework. To set the context I first introduce the z-transform operator and examine its application to the discrete-time state-space model. In the following section of the appendix I discuss spectral analysis and norms. The next three sections introduce results for the robust (H_∞) control and filtering of linear state-space systems. This is followed by an overview of the non-linear risk-sensitive control and filtering problem. In this section I follow the information-state explication of Boel, James and Petersen (1997), which establishes the relationship between risk-sensitive control, the Kalman-Bucy filter, and the H_∞ control problem. These authors also draw on the duality between free and relative entropy to establish error bounds over the minimum risk-sensitive estimator. In the next section I examine the relationship between stochastic uncertainty constraints (including the relative entropy constraint) and norm bounds imposed over model uncertainty, external perturbation, and observation error. I conclude by demonstrating that Andersen, Hansen and Sargent (1999) ignore observation error in their consideration of relative entropy constraints that govern their version of the risk-sensitive control problem..

A.2. THE DISCRETE-TIME STATE SPACE MODEL AND THE Z-TRANSFORM OPERATOR

Consider a discrete time-series denoted by $\{u(n)\}$ representing the sample sequence $u(n)$, $u(n - 1)$, $u(n-2)$,...,. The z-transform of the sequence is defined as (Haykin, 1996, p. 80):

$$U(z) = z[u(n)] = \sum_{n=-\infty}^{\infty} u(n) z^{-n}$$

where z is a complex variable and the z-transform is an operator expressed as an infinite power series in z. For the z-transform to be meaningful the Laurent power series defined in the preceding equation must be absolutely summable (i.e. $U(z)$ must be absolutely convergent). The set of values for which the z-transform is uniformly convergent is called the *region of convergence*. The inverse of the z-transform is given by the *z-transform inversion integral formula*:

$$u(n) = \frac{1}{2\pi j} \oint_C U(z) z^n \frac{dz}{z}$$

where C is the closed contour that encloses the origin and is itself contained in the annular domain $R_1 < |z| < R_2$ defining the region of convergence. The z-transform is a linear transform that satisfies the following properties: first, the principle of superposition. Given two sequences $u_1(n)$ and $u_2(n)$ whose z-transforms are given by $U_1(z)$ and $U_2(z)$, respectively, the associated z-transform pair is expressed as:

$$au_1(n) + bu_2(n) \Leftrightarrow aU_1(z) + bU_2(z)$$

where the $\Leftrightarrow$ symbol is intended to denote a *one-to-one correspondence* between each variable. The second is the time-shifting property: let $U(z)$ denote the z-transform of the sequence $u(n)$. The z-transform of $u(n - n_0)$ is then described by the relation:

$$u(n - n_0) \Leftrightarrow z^{-n_0} U(z)$$

where n_0 is an integer. For the special case of $n_0 = 1$ the time-shift has the effect of multiplying the z-transform by the factor z^{-1} and is often designated as the unit-delay element (Haykin, 1996, p. 81). The third property is the convolution theorem. For two sequences $u_1(n)$ and $u_2(n)$ whose z-transforms are given by $U_1(z)$ and $U_2(z)$, respectively, the convolution theorem stipulates that:

$$\sum u_1(i) u_2(n - i) \Leftrightarrow U_1(z) U_2(z)$$

over the region of convergence that encompasses the intersection of the respective regions of convergence for each of the two convoluted sequences.

The z-transform can be employed to define the *impulse response function* of a linear time-invariant filter. Here, the linearity property implies that a filter subject to two excitation inputs v_1 (n) and v_2 (n), where u_1 (n) and u_2 (n) are the filter responses, would elicit a combined response to the composite excitation $av_1(n) + bv_2(n)$ that would take the form $au_1(n) + bu_2(n)$. The response $u(n)$ to excitation $v(n)$ is given by the convolution sum (Haykin, p. 81):

$$u(n) = \sum_{i=-\infty}^{\infty} h(i)v(n-i)$$

Applying the z-transform to both sides of this expression and invoking the convolution theorem yields the following expression:

$$U(z) = H(z)V(z)$$

The z-transform $H(z)$ of the impulse response function is called the *transfer function*. For an important subclass of linear time-invariant filters, the input sequence v(n) and output sequence u(n) are related by a difference equation of order N as follows:

$$\sum_{j=0}^{N} a_j u(n-j) = \sum_{j=0}^{N} b_j v(n-j)$$

In this case the transfer function $H(z)$ is given by:

$$H(z) = \frac{U(z)}{V(z)} = \frac{\sum_{j=0}^{N} a_j z^{-j}}{\sum_{j=0}^{N} b_j z^{-j}} = \frac{a_0}{b_0} \frac{\prod_{k=1}^{N}\left(1 - c_k z^{-1}\right)}{\prod_{k=1}^{N}\left(1 - d_k z^{-1}\right)}$$

where the last equality gives the factor form of the transfer function. In this expression the numerator has a zero at $z = 0$ and a pole at $z = c_k$ while the denominator possesses a pole at $z = d_k$ and a zero at $z = 0$.

Now consider the following state space model where x is the state variable in the dynamic system (subject to perturbations u) and y is the observed variable (i.e. the state variable is not observed directly but is subject to observation error u):

$$
\begin{aligned}
x_{k+1} &= Ax_k + Bu_k \\
y_k &= Cx_k + Du_k
\end{aligned}
$$

and z-transform the equations to arrive at the representation (for convenience, assuming zero initial conditions):

$$
\begin{aligned}
zX(z) &= AX(z) + BU(z) \\
Y(z) &= CX(z) + DU(z)
\end{aligned}
$$

The first of these equations can readily be solved for $X(z)$:

$$(zI - A)X(z) = BU(z) \Rightarrow X(z) = (zI - A)^{-1} BU(z).$$

This equation can then be substituted into the observation equation to yield an equation for $Y(z)$:

$$Y(z) = C(zI - A)^{-1} BU(z) + DU(z)$$

Hence, the transfer function relating the input $U(z)$ to the observed output $Y(z)$ is given by:

$$G(z) = C\Phi(z)B + D \text{ where } \Phi(z) = (zI - A)^{-1}$$

Typically, a discrete signal $f^*(t)$ is constructed as the product of an impulse train $\delta(t-kT)$ and a continuous signal $f(t)$ as in (Shahian and Hassul, 1993, p. 256):

$$\begin{aligned} f^*(t) &= f(kT) \text{for } k = \ldots -3,-2,-1,0,1,2,3\ldots \\ &= f(t)\sum_{k=-\infty}^{\infty}\delta(t-kT). \end{aligned}$$

Because the impulse train is a periodic function it can be represented with a Fourier series:

$$\sum_{k=\infty}^{\infty}\delta(t-kT) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{j(2\pi n/T)t}$$

Thus the Fourier transform of the sampled signal becomes:

$$\begin{aligned} F^*(\omega) &= \int_{-\infty}^{\infty}\left(f(t)\frac{1}{T}\sum_{n=-\infty}^{\infty}e^{j(2\pi n/T)t}\right)e^{-j\omega t}dt = \frac{1}{T}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}f(t)e^{-j(\omega-2\pi n/T)t}dt \\ &= \frac{1}{T}\sum_{n=-\infty}^{\infty}F(\omega - 2\pi n/T), \end{aligned}$$

where the last equality follows from the fact that the original continuous signal has the Fourier transform:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt\,.$$

Thus the spectrum of the sampled signal is a periodic replication (in ω) of the continuous spectrum. This factum is responsible for the aliasing problem associated with attempts to recover the continuous signal by filtering the sampled signal with a low pass filter of appropriate bandwidth (see Shahian and Hassul, section 9.3.2). The Laplace transform of the sampled signal is (pp. 260-1):

$$F^*(s)=\int_0^\infty f^*(t)e^{-st}dt=\int_0^\infty\left(\sum_{k=0}^\infty f(kT)\delta(t-kT)\right)e^{-st}dt$$
$$=\sum_{k=0}^\infty\int_0^\infty f(kT)e^{-st}\delta(t-kT)dt=\sum_{k=0}^\infty f(kT)e^{-skT}.$$

This transform is not a rational function of s but it can be expressed in rational form by defining a new variable z as follows:

$$z=e^{sT}\Rightarrow F(z)=\sum_{k=0}^\infty f(kT)z^{-k}.$$

A.3. Spectral Representations and Norms

This first section of the appendix introduces the 2-norm and H_∞-norms that are routinely applied in a control theory setting. For any series of numbers $\{x_t\}$ the Fourier transform is defined by:

$$x(\omega)=\sum_{t=-\infty}^\infty e^{-i\omega t}x_t\,.$$

This operation transforms a series that is a function of time into a complex-valued function of ω.

Given $x(\omega)$, we can recover x_t by the inverse Fourier transform:

$$x_t=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{+i\omega t}x(\omega)d\omega=\frac{1}{\pi}\int_0^\pi|x(\omega)|\cos(\omega t+\phi(\omega))d\omega\,.$$

The above expression follows from the identity:

$e^{i\theta} = \cos\theta + i\sin\theta \Rightarrow \cos\theta = \left(e^{i\theta} + e^{-i\theta}\right)/2$.

The Euclidean or 2-norm of a matrix is defined by (Shahian and Hassul, 1995, pp. 442-3):

$$\|\mathbf{A}\|_2 = \max_{x\neq 0}\frac{\|\mathbf{A}x\|_2}{\|x\|_2} = \max_{\|x\|_2=1}\|\mathbf{A}x\|_2 = \bar{\sigma}(\mathbf{A})$$

where the barred sigma notation stands for the largest singular value of the **A** matrix in the associated singular value decomposition (SVD). The SVD decomposes a rectangular matrix **A** with rank ρ into the product:

$$\mathbf{A} = \mathbf{U}\hat{\Sigma}\mathbf{V}^*, \quad \text{where } \mathbf{U}^*\mathbf{U} = \mathbf{I}_m, \quad \mathbf{V}^*\mathbf{V} = \mathbf{I}_n,$$

$$\text{and } \hat{\Sigma}\begin{cases}[\Sigma \mid \mathbf{0}] \text{ if } n > m \\ \begin{bmatrix}\Sigma \\ \hline \mathbf{0}\end{bmatrix} \text{ if } n < m\end{cases} \quad where\,\Sigma = \begin{bmatrix}\sigma_1 & & \\ & \sigma_2 & \\ & \text{O} & \\ & & \sigma_\rho\end{bmatrix}, \; \rho = \min\{m, n\}$$

Here, the eigenvalues are subscripted in descending order of magnitude. Therefore, the largest singular value $\bar{\sigma}$ is then defined as the highest eigenvalue, σ_1.

For signals or time functions $x(t)$, the 2-norm defined by:

$$\|x(t)\|_2^2 = \int_0^\infty x(t)^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}|X(j\omega)|^2\,d\omega$$

can be interpreted as the gain of the system. When the above norm is finite, the function is said to belong to the L_2 Hilbert space (i.e. it is square integrable). If a matrix is conceived as a system with x as its input and Ax as its output, then the 2-norm represents the maximum gain.

The H_2- Norm for the *single-variable* transfer function G is defined by

$$\|G\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}|G(j\omega)|^2 d\omega.$$

The power spectral density of the system output is given by:

$$S_y(\omega) = |G(j\omega)|^2 S_x(\omega).$$

Hence, the root mean squared value of the output is given by:

$$y_{rms} = \left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}|G(j\omega)^2 S_x(\omega)d\omega|\right\}^{\frac{1}{2}}$$

Shahian and Hassul observe that for white noise inputs, $S_x(\omega) = 1$ for all frequencies (p.446). Therefore, the H_2-Norm can be interpreted as the RMS value of the output when the system is driven by white noise input. Similarly, they argue that for a *single-variable system* the H_∞-norm defined by:

$$\|G\|_\infty = \sup_{x\neq 0}\frac{\|Gx\|_2}{\|x\|_2} = \sup_{\omega}|G(j\omega)|$$

must satisfy the following inequality bound:

$$\|G\|_\infty \geq \frac{\left\{\int_{-\infty}^{\infty} S_y(\omega)d\omega\right\}^{\frac{1}{2}}}{\left\{\int_{-\infty}^{\infty} S_x(\omega)d\omega\right\}^{\frac{1}{2}}} = \frac{y_{rms}}{x_{rms}}$$

In other words, the H_∞-Norm is bounded from below by the *rms gain* of the system. In the *multivariable* case the H_2- and H_∞-Norms are defined by:

$$\|G\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} trG^*(j\omega)G(j\omega)d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_i \sigma_i^2[G(j\omega)]d\omega$$

and,

$\|G\|_\infty = \sup_\omega \bar{\sigma}[G(j\omega)]$, respectively.

In summary, the H_∞-norm minimizes the worst-case *rms* value of the regulated variables when the disturbances have unknown spectra, whereas the 2-norm minimizes the *rms* values of the regulated variables when the disturbances are unit intensity white-noise processes.

A.4. A HEURISTIC OVERVIEW OF H$_2$ AND H$_\infty$ CONTROL

Consider the following schematic representation of a deterministic, finite-horizon version of the continuous-time control problem with the following state equation and observation equation (for convenience, time subscripts have been suppressed):

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

where (A,B) are stabilizable, $R > 0$, $Q = C'qCq$, (Cq, A) are detectable, and the quadratic objective function is:

$$J = \frac{1}{2}\int_0^T (x'Qx + u'Ru)dt$$

The appropriate Hamiltonian is:

$$H(x, \lambda, t) = \frac{1}{2}(x'Qx + u'Ru) + \lambda'(Ax + Bu)$$

yielding the following first-order conditions:

$$\dot{x} = \frac{\partial H}{\partial \lambda} = Ax + Bu; \quad x(0) = x_0$$
$$-\dot{\lambda} = \frac{\partial H}{\partial x} = Qx + A'\lambda; \quad \lambda(T) = 0$$
$$\frac{\partial H}{\partial u} = 0 \Rightarrow u^* = -R^{-1}B'\lambda$$

After substitution of the last of the above three first-order conditions into that for the state variable x, the resulting two equations can be expressed in matrix notation as:

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B' \\ -Q & -A' \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} \overset{\Delta}{=} H \begin{bmatrix} x \\ \lambda \end{bmatrix}$$

This matrix expression represents a two-point boundary value problem that can be solved using conventional PDE techniques. However, due to the linear nature of the problem, it can be solved more easily using Ricatti equation techniques. Let $\lambda = Px$, then differentiate both sides of this expression with respect to time and substitute the relevant first order conditions to yield:

$$\frac{d\lambda}{dt} = \frac{dP}{dt}x + P\frac{dx}{dt} = \frac{dP}{dt}x + PAx - PBR^{-1}B'Px = -Qx - A'Px$$
$$\Rightarrow -\frac{dP}{dt} = A'P + PA + Q - PBR^{-1}B'P; \quad P(T) = 0 \; (\lambda(T) = 0)$$

The control variable u possesses a linear solution in the form of:

$u(t) = -K(t)x(t)$, where $K(t) = R^{-1}B'P(t)$.

This Ricatti solution for the Hamiltonian problem is typically presented in the following abbreviated form:

$H = \begin{bmatrix} A & -R \\ -Q & -A' \end{bmatrix}$, given that (A, -Rx) is stabilizable. This notation implies that the algebraic Ricatti Equation has the form: $A'X+XA-XRX+Q = 0$ with solution $X = \text{Ric}(H)$.

The linear quadratic Gaussian problem has the following representation:

$$\dot{x} = Ax + Bu + \Gamma w$$
$$y = Cx + v$$

where (C,A) are detectable, R_0 is positive definite, $Q_0 = H_0 H'_0$ and (A, H_0) is stabilizable.

$$E[w(t)] = 0, \; E[v(t)] = 0$$
$$E\left[w(t)w(t+\tau)'\right] = Q_0\delta(t-\tau); E\left[w(t)v(t+\tau)'\right] = 0;$$
$$E[v(t)v(t+\tau)] = R_0\delta(t-\tau).$$

For this stochastic problem, the objective function is written in expectational form:

$J_0 = E\left[\tilde{x}(t)'\tilde{x}(t)\right]$ where $\tilde{x}(t) = x(t) - \hat{x}(t)$.

In this case, the certainty equivalence principle can be invoked to establish the optimal control under complete information (perfect certainty). Under duality, the optimal filter for $\hat{x}(t)$ is derived from the above primal representation using the following equivalences:

$$A \to A',\ B \to C',\ Q \to Q_0,\ R \to R_0,$$
$$N \to N_0,\ K' \to L',\ P \to \Sigma.$$

Accordingly,

$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$

where $L = \Sigma C' R_0^{-1}$,

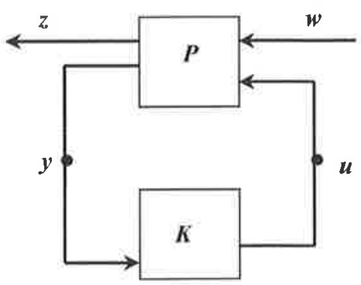
Σ is derived from the Ricatti equation $A\Sigma + \Sigma A' + \Gamma Q_0 \Gamma' - \Sigma C' R_0^{-1} C \Sigma = 0$

and where $tr\Sigma = trE\left[\tilde{x}(t)\tilde{x}(t)'\right] = E\left[\tilde{x}(t)'\tilde{x}(t)\right].$

The most general form of the stochastic LQG control problem is usually expressed in the two port formulation starting from the following set of equations:

$$\dot{x} = Ax + B_1 w + B_2 u$$
$$z = C_1 x + D_{11} w + D_{12} u$$
$$y = C_2 x + D_{21} w + D_{22} u$$

As depicted in the following block-flow diagram:



the two port formulation can be derived from the previous set of equations and expressed as a four block system by first solving for the x variable using the Laplace transform (in much the same way that x was eliminated for the discrete-case using the z-tranform) and by then assuming that the control variable can be written as a linear function of the observed variable y, as expressed in the following set of equations::

$$z = P_{zw}w + P_{zu}u$$
$$y = P_{yw}w + P_{yu}u$$
$$u = Ky$$

First, the equation for u is substituted into both the state and observation equations—the latter equation yielding the following expression for y:

$$y = (I - P_{yu}K)^{-1} P_{yw}w$$

so that u becomes:

$$u = K(I - P_{yu}K)^{-1} P_{yw}w$$

and thus, in succession, z becomes:

$$z = \left[P_{zw} + P_{zu}K(I - P_{yu}K)^{-1} P_{yw}\right]w = T_{zw}w\ .$$

The linear fractional transform for the transfer function T_{zw} is conventionally expressed in dense-packed matrix form:

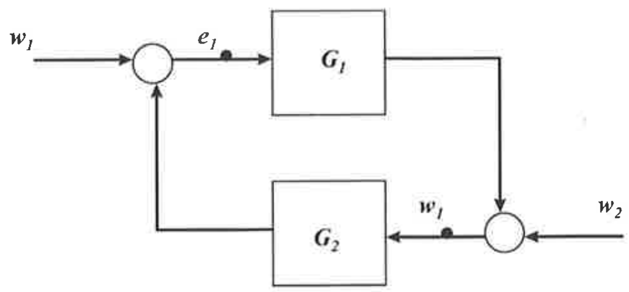
$$P(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]$$

where the dense-packed matrix notation for the matrix G(s) has the following interpretation:

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = C(sI - A)^{-1} B + D$$

Here, s is the familiar Laplace transform variable. The solution of optimal control and filtering problems is based on the properties of the T_{zw} transfer function in accordance with the small gain theorem. Suppose the systems $\mathbf{G}_1 : L_{2e}\ \alpha\ L_{2e}$ and $\mathbf{G}_2 : L_{2e}\ \alpha\ L_{2e}$ shown in the diagram below have finite incremental gains such that $\gamma(\mathbf{G}_1)\gamma(\mathbf{G}_2) < 1$, then:

i. For all ω_1, $\omega_2 \in L_{2e}$ there exist unique solutions e_1, $e_2 \in L_{2e}$.
ii. For all ω_1, $\omega_2 \in L_{2e}[0, \infty)$, there exist unique solutions e_1, $e_2 \in L_{2e}[0, \infty)$. That is, the closed loop is internally stable.



The small gain theorem suggests that the closed feedback loop system for a stable plant $G(s)$ and compensator $K(s)$ will remain stable if

$|G(s)K(s)| < 1$
which from the inequality $|G(s)K(s)| \le |G(s)||K(s)|$
implies that closed loop stability can be guaranteed if
$|G(s)||K(s)| < 1$

A.5. Doyle, Glover, Khargonekar and Francis (1989): The LQG Problem

More formally, we can follow Doyle, Glover, Khargonekar and Francis (1989), who initially assume that:

1. (A, B_1) is stabilizable and (C_1, A) is detectable,
2. (A, B_2) is stabilizable and (C_2, A) is detectable,
3. $D_{12}''[C_1 D_{12}] = [0 \quad I]$
4. $\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} D_{21}' = \begin{bmatrix} 0 \\ I \end{bmatrix}$

Under these assumptions they confirm (Doyle et al. 1989, Theorem 1, p. 835) that, for the general LQG control problem, the unique admissible controller K that minimizes $\|T_{zw}\|_2$ is:

$$K_{opt}(s) = \left[\begin{array}{c|c} A + B_2F_2 + L_2C_2 & -L_2 \\ \hline F_2 & 0 \end{array}\right]$$

where

$F_2 := -B_2'X_2, \quad L_2 := -Y_2C_2'$

and

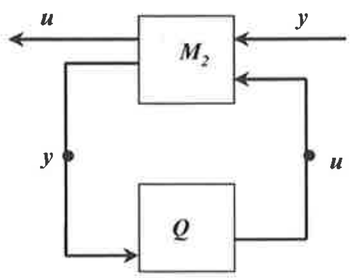
$$X_2 := Ric(H_2); \; H_2 := \left[\begin{array}{c|c} A & -B_2B_2' \\ \hline -C_1'C_1 & -A' \end{array}\right]$$

$$Y_2 := Ric(J_2); \; J_2 := \left[\begin{array}{c|c} A' & -C_2'C_2 \\ \hline -B_1B_1' & -A \end{array}\right]$$

Moreover, min $\|T_{zw}\|_2^2 = \|G_cB_1\|_2^2 + \|F_2G_f\|_2^2 = \|G_cL_2\|_2^2 + \|C_1G_f\|_2^2$, where:

$$G_c(s) = \left[\begin{array}{c|c} A + B_2F_2 & I \\ \hline C_1 + D_{12}F_2 & 0 \end{array}\right]; \quad G_f(s) = \left[\begin{array}{c|c} A + L_2C_2 & B_1 + L_2D_{21} \\ \hline I & 0 \end{array}\right].$$

The family of all admissible controllers such that $\|T_{zw}\|_2 < \gamma$, equals the set of all transfer matrices from y to u in (Doyle et al. 1989, Theorem 2, p. 835):



Where $M_2(s) = \left[\begin{array}{c|cc} A + B_2F_2 + L_2C_2 & -L_2 & B_2 \\ \hline F_2 & 0 & I \\ -C_2 & I & 0 \end{array}\right]$

and $Q \in \Re H_2$, $\|Q\|_2^2 < \gamma^2 - \left(\|G_cB_1\|_2^2 + \|F_2G_f\|_2^2\right)$ If the free parameter Q is set equal to 0, K_{opt} is recovered.

A.6. Doyle, Glover, Khargonekar and Francis (1989): The H_∞ Problem

The suboptimal H_∞ problem involves finding the family of all admissible controllers such that $\|T_{zw}\|_\infty < \gamma$. An admissible controller exists if the following assumptions hold:

5. $H_\infty \in \mathrm{dom(Ric)}$ and $X_\infty := \mathrm{Ric}(H_\infty) \geq 0$,
6. $J_\infty \in \mathrm{dom(Ric)}$ and $Y_\infty := \mathrm{Ric}\,(J_\infty) \geq 0$,
7. $\rho(X_\infty, Y_\infty) < \gamma^2$.

Where:

$$H_\infty = \begin{bmatrix} A & \gamma^{-2}B_1B_1' - B_2B_2' \\ -C_1'C_1 & -A' \end{bmatrix}; \quad J_\infty = \begin{bmatrix} A' & \gamma^{-2}C_1'C_1 - C_2'C_2 \\ -B_1B_1 & -A \end{bmatrix}.$$

When these conditions hold, one such controller is (Doyle et al. 1989, Theorem 3, p. 835):

$$K_{sub}(s) := \left[\begin{array}{c|c} A + \gamma^{-2}B_1B_1'X_\infty + B_2F_\infty + Z_\infty L_\infty C_2 & -Z_\infty L_\infty \\ \hline F_\infty & 0 \end{array}\right]$$

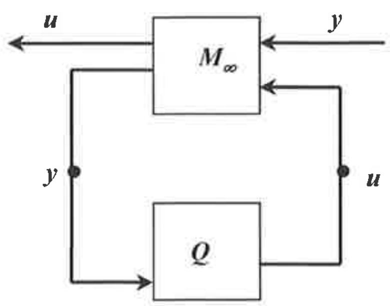
where $F_\infty := -B_2'X_\infty$; $L_\infty := -Y_\infty C_2'$; $Z_\infty := \left(I - \gamma^{-2}Y_\infty X_\infty\right)^{-1}$.

Moreover, the set of all admissible controllers such that $\|T_{zw}\|_\infty < \gamma$, equals the set of all transfer matrices from y to u in (Doyle et al. 1989, Theorem 4, p. 835):

where

$$M_\infty(s) = \left[\begin{array}{c|cc} A + \gamma^{-2}B_1B_1'X_\infty + B_2F_\infty + Z_\infty L_\infty C_2 & -Z_\infty L_\infty & Z_\infty B_2 \\ \hline F_\infty & 0 & I \\ -C_2 & I & 0 \end{array}\right],$$

and $Q \in \Re H_\infty$, $\|Q\|_\infty < \gamma$. The central controller K_{sub} is recovered when $Q = 0$.



A.7. LQG, H-INFINITY CONTROL, AND MAXIMUM ENTROPY

For the more technically minded, the following section of the appendix summarizes the somewhat remarkable relationship between stochastic LQG control problems, deterministic H_∞-problems and maximum entropy. Glover and Doyle (1988, p. 170) consider a discrete-time, risk-sensitive, LQG stochastic control problem for the state equation:

$$x_t = Ax_{t-1} + B_1 w_t + B_2 u_{t-1}$$

$$z_t = C_1 x_{t-1} + D_{11} w_t + D_{12} u_{t-1}$$

$$y_t = C_2 x_{t-1} + D_{21} u_{t-1}$$

where the process/observation noise w_t is white and Gaussian with unit variance. The relevant quadratic cost function is:

$$G = x_T^* \Pi x_T + \sum_{t=0}^{T-1} z_t^* z_t$$

The risk-sensitive optimal controller minimizes:

$$\gamma_T(\theta) = -\frac{2}{\theta T} \log E[\exp(-\theta G/2)]$$

For a stabilizing linear time invariant (LTI) controller with transfer function $K(s)$ connected from y to u, and cost function $\gamma_T(\theta)$, the output z_t will be a stationary Gaussian process with spectrum:

$$f(\lambda) = (1/2\pi)\mathbf{H}(e^{j\lambda})\mathbf{H}(e^{j\lambda})^*$$

$\mathbf{H}$ is the closed loop transfer function $\mathcal{F}_l(P,K)$ from w to z. Accordingly, $\mathbf{T}_{zw} = \mathcal{F}_l(P,K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21} = P_{zw} + P_{zu}K\left[I - P_{yu}K\right]^{-1}P_{yw}$ (in accordance with the dense-packed matrix notation introduced in the preceding section of the appendix) and:

$$P(s) = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & \mathbf{0} & D_{22} \end{array}\right] = \begin{pmatrix} D_{11} & D_{12} \\ \mathbf{0} & D_{22} \end{pmatrix} + \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}(sI - A)^{-1}\begin{bmatrix} B_1 & B_2 \end{bmatrix}$$

Glover and Doyle introduce two lemmas which allow them to establish the following equivalence (see lemmas 3.1, 3.2, p. 171):

$$\lim_{T\to\infty} \gamma_T(\theta) = \begin{cases} \dfrac{1}{2\pi\theta}\displaystyle\int_{-\pi}^{\pi} \log\left(\det\left[\mathbf{I} + \theta\mathbf{H}\mathbf{H}^*\right]\right)d\lambda & \text{if } \theta\|\mathbf{H}\|_\infty^2 > -1 \\ \infty & \text{if } \theta\|\mathbf{H}\|_\infty^2 < -1 \end{cases}$$

In this expression the absence of the cost term $x_T^*\Gamma x_T$ can be ignored since K is stabilizing and $T \to\infty$. Thus, any LTI optimal controller must be such that:

$$\|T_{zw}\|_\infty = \|\mathcal{F}_l(P,K)\|_\infty \le (-\theta)^{-1/2}, \qquad \text{for } \theta < 0.$$

The integral in the preceding limit expression is the frequency domain version of the entropy integral, which can be minimized over all LTI controllers meeting the H_∞-norm bound. As θ is made more negative the H_∞ control robustness parameter becomes larger. Beyond a certain critical point, all controllers would give infinite cost.

A.8. Boel, James and Petersen's Minimum Risk-Sensitive Estimator

Like Andersen, Hansen and Sargent (1999), Boel, James and Petersen (1997) draw on Dupuis and Ellis's (1997) characterization of the duality between free energy and relative entropy to construct error bounds for risk-sensitive filters. They assume that the true probability model is fixed but unknown, and that the estimation procedure makes use of a fixed nominal model. They show that the resulting error bound for the risk-sensitive filter is the sum of two terms, the first of which coincides with an upper bound on the error one would obtain if one knew exactly the underlying probability model, while the second term is a measure of the distance between the true and design probability models.

For the measurable space (Ω,F) and random variables X and Y, where the former represents the state, the latter represents the observation variable, let $\phi = \phi(X)$ be a (real valued) function of X to be estimated by the random variable $\hat{\phi} \in \mathbf{Y} = \sigma\{Y\}$ (i.e. the estimator of ϕ is Y measurable). The true distribution $P_{\alpha 0}$ is assumed to belong to the family of probability measures $\{P_{\alpha 0}\}_{\alpha \in A}$ indexed by the arbitrary index set A. A candidate or reference probability distribution $P_{\alpha d}$ defined by the design parameter α_d is utilized to construct the estimator $\hat{\phi}$. Let $\rho_\alpha \in C(\Re)$ be a strictly convex function satisfying $\nabla^2 \rho_\alpha(z) \geq c_\alpha > 0$, which is bounded from below and attains its global minimum at 0, and set the cost function:

$f(\hat{\phi}) = E_{\alpha l}\left[\rho_\alpha(\phi - \hat{\phi})\right]$, for any **Y**-measurable $\hat{\phi}$ (where the expectation is taken with respect to the design distribution $P_{\alpha d}$).

Defining a *minimum cost estimator* (MCE) by:

$$\hat{\phi}^* = \arg\min_{\hat{\phi} \in \mathbf{Y}} f(\hat{\phi}),$$

Boel, James and Petersen (1997, proposition 2.1, p. 3) establish that if $f(\hat{\phi}) < \infty$ for some $\hat{\phi} \in \mathbf{Y}$, then the minimum defined by $\hat{\phi}^*$ exists and is unique. Defining the conditional distribution π_α of X given Y under P_α as $\pi_\alpha(A) = \mathbf{P}(X \in A \mid Y)$, then the MCE can be written as (Theorem 2.2., p. 3):

$$\hat{\phi}^* = \arg\min_{\hat{\phi} \in \Re} \int \rho_\alpha\left(\phi(x) - \hat{\phi}\right)\pi_{\alpha l}(dx)$$

When $\rho_\alpha(e) = |e|^2$, the MCE is equivalent to the *minimum mean square estimator* (MMSE) or Kalman filter for the design probability $P_{\alpha d}$. Here, the variable e stands for the estimation error given by the difference between the function $\phi(X)$, governed by the true probability distribution $P_{\alpha 0}$, and the Y-measurable estimator of this function $\hat{\phi}$, calculated under the reference probability distribution $P_{\alpha d}$. While the Kalman filter is a quadratic function, the risk-sensitive filter is an exponential function in the estimator error: it is this exponential term in the objective function that is responsible for the robustness properties of the resulting filter.

Consider the following discrete time non-linear system:

$$x_{k+1} = b_\alpha(x_k) + g_\alpha(x_k)w_k$$
$$y_{k+1} = h_\alpha(x_k) + v_k$$

where k = 0,1,2,... is the time index, $x_k \in \Re^n$ is the signal state at time k, $y_k \in \Re^p$ is the observation, and w_k, v_k are i.i.d. random processes with densities $p_{\alpha,w}(w)$, $p_{\alpha,v}(v)$ respectively. It is assumed that $y_0 = 0$ and x_0 has distribution $P_{\alpha 0}$ independently of w_k, v_k and that α_d is the design value used for estimation. Under the presumption that the estimator $\hat{\phi}_k$ for ϕ_k can be computed in a recursive fashion, the previously defined cost functions and cost estimators are now subscripted accordingly with the time index k and are presumed to be adapted to the filtration $\mathbf{Y_k} = \sigma\{y_i, 0 \le i \le k\}$. Elliott, Aggoun and Moore (1995) and James, Baras and Elliott (1994) have established the existence of linear recursions for the unnormalized conditional measure σ_k, from which the conditional distribution π_k of x_k given $\mathbf{Y}_k$ under $P_{\alpha d}$, can be calculated via $\pi_k = \sigma_k / \int \sigma_k(dx)$. The requisite recursions are given by (Boel, James and Petersen, 1997, p.6; and Elliott, Aggoun and Moore, 1995, sect. 11.2):

$$\sigma_{k+1} = F(\sigma_k, y_{k+1})$$
$$\sigma_0 = P_{\alpha 0}$$

where $F(\sigma, y)$ is the operator defined by:

$$F(\sigma, y)[dx] = \int_{\Re^n} (2\pi)^{-\frac{n}{2}} \psi(dz, \xi) p_{\alpha d, v}(y - h_{\alpha d}(\xi)) \sigma(\xi) d\xi \cdot$$

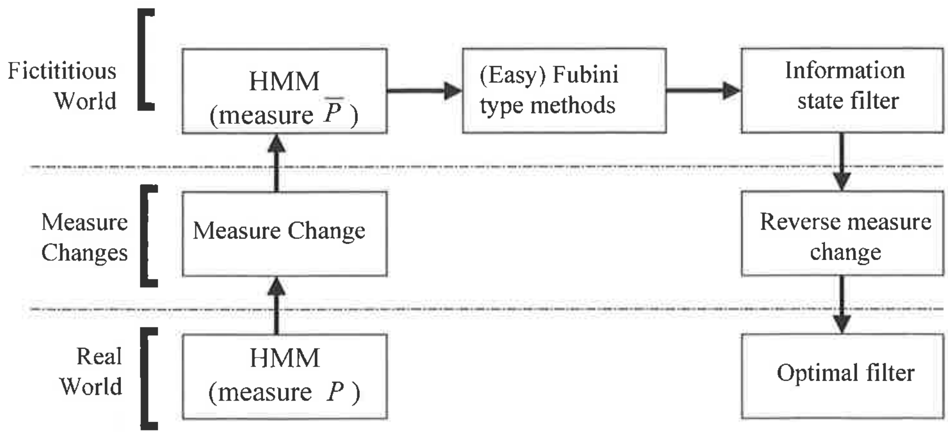
Also, $p_{\alpha d,v}(\cdot)$ is the density of the observation noise v_k under the design model $P_{\alpha d}$ and

$\psi(dz; \xi) = P_{\alpha d}(b_{\alpha d}(\xi) + g_{\alpha d}(\xi) w \in dz)$. The ξ variable (the integrand) in this expression stands for the state variable X under the reference probability density for the observation noise v_k, as given by $P_{\alpha d,\, v}$, and the reference probability density for the external perturbation w_k, as given by $P_{\alpha d,\, w}(\cdot)$. As explained below, because the observation process is $N(0, 1)$ i.i.d., under the requisite measure change, the density function for the normal distribution appears in the expression for the recursion operator $F(\sigma, y)$. Given this recursion, the minimum cost estimator (MCE) ϕ_k is then defined by (Boel, James and Petersen, 1997, p. 7):

$$\hat{\phi}_k = \Phi^*(\sigma_k) = \arg\min_{\hat{\phi} \in \Re} \int \rho_\alpha(\phi(x) - \hat{\phi}) \pi_k(dx) = \arg\min_{\hat{\phi} \in \Re} \int \rho_\alpha(\phi(x) - \hat{\phi}) \sigma_k(dx) \quad P_{\alpha d} - a.s.$$

Elliott, Aggoun and Moore's (1995) derivation of these recursions is based on a discrete-time version of Girsanov's theorem. A particular probability measure $\overline{P}$ is constructed under which all the components of the observation process are $N(0,1)$ i.i.d. variables. The use of such measure changes enables the practitioner to convert a complex non-linear recursive estimation problem into a linear one (within a 'fictitious' world) whose resolution is more amenable to straightforward Fubini-style manipulations (involving the simple interchange of expectations and summation signs). The information state approach adopted by Elliott, Aghoun and Moore (1995) for solving discrete-time non-linear control and filtering problems enables the control theorist to derive recursion equations for the unnormalized distributions of the form, $\sigma^{\mu}_{k+1} = \left(Y_0,\ldots,Y_{k+1};\mu_k;\sigma^{\mu}_k\right)$, which describe the observable dynamics of a separable problem (i.e. separable in the optimal control and the filter). Note that for hidden Markov problems, the information state σ^{μ}_k, is a positive measure on $S = \{e_1,\ldots, e_N\}$. Inverse measure changes then convert mathematical results back into the real world that operates under probability measure P. This approach is depicted schematically in the following diagram (Elliott et al. 1995, p. 9):

Fig. 28: ELLIOTT, AGGOUN & MOORE (1995) ON THE USE OF MEASURE CHANGES IN OPTIMAL FILTERING



Given an expected cost function in the state variable X and the control variable μ, the recursive structure of the information state means that straightforward dynamic programming rules can be obtained. Once again, under the requisite measure changes, the resulting simplifications allow for interchange of expectations and minimisation operations, but this time with respect to the programming algorithm. For the particular case where the system is linear with Gaussian noise and $\rho_\alpha(e)=\frac{1}{2}|e|^2$ Boel, James and Petersen (1997) show that the MCE reduces to the Kalman filter version of the MMSE. Another specialization, the hidden-Markov process, obtains when X_k is a finite state vector process taking values in $\{E_i, i = 1,2,\ldots,N\}$, where E_i is a vector with all the elements 0 except for a 1 in the i-th position and the coefficient on the lagged state variable is a transition matrix for the Markov process. As will be seen below, however, it should be noted that on occasions when a risk-sensitive cost function is employed the requisite recursions also come to depend on the cost function through the presence of an exponential term in the estimation error. In the differential game interpretation of risk-sensitive control problems, the presence of this cost function term is the source of the robustness property of both the control law and the filter: nature is malicious and chooses a perturbation process to maximize this cost term. The (sub-) optimal control or filter then minimizes the cost function, given this worst-case perturbation.

For the hidden-Markov model, given the sampled observations, unnormalized recursive estimates and smoothers can be derived for the number of jumps of the Markov chain from one state to another, the occupation time in any one state, and the variance and drift of the observation process (see Elliott, 1994). The estimation approach is implemented using the EM Algorithm. Following Elliot et al. (1995, pp. 35-6), let $\{P_\theta, \theta \in \Theta\}$ be a family of probability measures on a measurable space $(\Omega, \mathbf{F})$ all absolutely continuous with respect to a fixed probability measure P_0 and let $\mathbf{Y} \subset \mathbf{F}$. The likelihood function for computing an estimate of the parameter θ based on the information available in $\mathbf{Y}$ is:

$$L(\theta) = E_0\left[\frac{dP_\theta}{dP_0} | \mathbf{Y}\right]$$

where the derivative inside the brackets is the Radon-Nikodym derivative of the new measure P_θ with respect to the fixed measure. The Maximum Likelihood Estimate (MLE) is defined by:

$$\hat{\theta} \in \arg\max_{\theta \in \Theta} L(\theta)$$

The justification for this approach is that, based on the information available in $\mathbf{Y}$, the most likely value of the parameter θ is the one that maximises the conditional expectation of the density. The EM algorithm provides an iterative approximation method for calculating the MLE that might otherwise be difficult to calculate directly. The following steps are taken:

Step 1. Set $k = 0$ and choose $\hat{\theta}_0$.
Step 2. [E-step] Set $\theta^* = \hat{\theta}_k$ and compute $Q(\cdot, \theta^*)$, where:

$$Q(\theta, \theta^*) = E_0\left[\log \frac{dP_\theta}{dP_{\theta^*}} | \mathbf{Y}\right]$$

Step 3. [M-step] Find $\hat{\theta}_{k+1} = \arg\max_{\theta\in\Theta} Q(\theta,\theta^*)$

Step 4. Replace k with k + 1 and repeat beginning with step 2, until a stopping criterion is reached. The sequence generated gives non-decreasing values of the likelihood function. Moreover, it follows from Jensen's inequality that:

$$\log L(\hat{\theta}_{k+1}) - \log L(\hat{\theta}_k) \geq Q(\hat{\theta}_{k+1}, \hat{\theta}_k)$$

with equality iff $\hat{\theta}_{k+q} = \hat{\theta}_k$

For the strictly convex and continuous functions ρ_1 and ρ_2, satisfying $\nabla^2 \rho_i(z) \geq c_i > 0 \ (i = 1,2)$, bounded from below and attaining global minima at 0, Boel, James and Petersen (p. 8) define a risk-sensitive cost function, with risk parameters μ_1 > 0, μ_2 > 0, for the estimate sequence $\hat{\phi}_{0k} = \hat{\phi}_0, \ldots, \hat{\phi}_k$ (each $\hat{\phi}_l \in \mathbf{Y}_l$):

$$f_{rs,k}(\hat{\phi}_{0k}) = E_{ad}\left[\exp\left(\mu_1 \sum_{l=0}^{k-1} \rho_1(\phi(x_1) - \hat{\phi}_l) + \mu_2 \rho_2(\varphi(x_k) - \hat{\phi}_k)\right)\right].$$

The summation term within the brackets represents the accumulated error cost up to time k – 1, which is followed by the error term at time k. For computational simplicity, the authors assume that optimal decisions have been made for all times 0,…,k – 1 so that $\hat{\phi}_0^*, \ldots, \hat{\phi}_{k-1}^*$ are known and $\hat{\phi}_k^*$ remains to be determined. Following the construction of James, Baras and Elliott (1994) they define the minimum risk-sensitive estimator (MRSE) in terms of the information state σ_k^μ as follows (Boel, James and Petersen, 1997, pp. 9-10):

$$\sigma^{\mu}_{k+1} = F^{\mu}\left(\sigma^{\mu}_{k}, \hat{\phi}^{*}_{rs,k}, y_{k+1}\right), \; \sigma^{\mu}_{0} = P_{\alpha 0}$$
$$\hat{\phi}^{*}_{rs,k} = \Phi^{*}_{rs}\left(\sigma^{\mu}_{k}\right) = \arg\min_{\hat{\phi}\in\Re} \int \exp\left(\mu_2 \rho_2 \left(\phi(x) - \hat{\phi}\right)\right)\sigma(dx)$$

where $F^{\mu}\left(\sigma, \hat{\phi}, y\right)[dx] = \int_{\Re^n} (2\pi)^{-\frac{n}{2}} \psi(dx;\xi) p_{\alpha d, v}\left(y - h_{\alpha d}(\xi)\right) \exp\left(\mu_1 \rho_1 \left(\phi(\xi) - \hat{\phi}\right)\right)\sigma(\xi) d\xi$

Error bounds for the MRSE are based on the duality between free energy and relative entropy:

$$\log \int e^{\psi} dP = \sup_{Q} \left\{ \int \psi dQ - D(Q:P) : Q << P, \psi \in L_1(Q) \right\}$$

where $L_1(Q)$ is the space of **F**-measurable variables ψ with $\int |\psi| dQ$ finite, and:

$$D(Q:P) = \begin{cases} \int \log \dfrac{dQ}{dP} dQ & \text{if } Q << P \text{ and } \log \dfrac{dQ}{dP} \in L_1(Q) \\ +\infty & \text{otherwise} \end{cases}$$

is the relative entropy of the probability measure P, and $e^{\psi} \in L_1(P)$. From the definitions it can be seen that $\hat{\phi}^{*}_{rs}$ is also a *minimax* estimator given by the following (Boel, James and Petersen, p. 5):

$$\hat{\phi}^*_{rs} = \arg\min_{\phi \in \mathbf{Y}} \sup_{Q} \left\{ E_Q \left[\rho\left(\phi - \hat{\phi}^*_{rs}\right) \right] - D(Q : P_{ad}) : Q << P_{ad} \right\}$$

Boel, James and Petersen (1997, theorem 4.2, p. 10) show that the MRSE filter, so defined, has the following error bound:

$$E_{\alpha 0} \left[\mu_1 \sum_{l=0}^{k-1} \rho_1 \left(\phi(x_l) - \hat{\phi}^*_{rs,l} \right) + \mu_2 \rho_2 \left(\phi(x_k) - \hat{\phi}^*_{rs,k} \right) \right] \leq \log f_{rs,k} \left(\hat{\phi}^*_{rs,k} \right) + D_k \left(P_{\alpha 0} : P_{ad} \right)$$

where $D_k(P_{\alpha 0}: P_{\alpha d})$ is the relative entropy on the time interval $0,\ldots,k$.

Although the information state σ^μ_k is not a probability measure, it is a finite measure. Thus, the duality formula can be applied to the normalized measure $\sigma^\mu_{\alpha d,k} / \left|\sigma^\mu_{\alpha d,k}\right|$, where $|\sigma| = \int \sigma(dx)$ enabling Boel, James and Petersen (1997, Remark 4.3, p. 10) to establish the following *conditional* error bound for the MRSE:

$$C_k E_{\alpha 0} \left[\mu_1 \sum_{l=0}^{k-1} \rho_1 \left(\phi(x_l) - \hat{\phi}^*_{rs,l} \right) + \mu_2 \rho_2 \left(\phi(x_k) - \hat{\phi}^*_{rs,k} \right) \right] \leq \log \int \exp\left(\mu_1 \sum_{l=0}^{k-1} \rho_1 \left(\phi(x_l) - \hat{\phi}_l \right) + \mu_2 \rho_2 \left(\phi(x_k) - \hat{\phi}_k \right) \right) d\sigma^\mu_{\alpha d,k}$$
$$- \log \left| \sigma^\mu_{\alpha d,k} \right| + R_k \left(\frac{\sigma^\mu_{\alpha 0,k}}{\left|\sigma^\mu_{\alpha 0,k}\right|} \middle| \frac{\sigma^\mu_{\alpha d,k}}{\left|\sigma^\mu_{\alpha d,k}\right|} \right), \qquad \text{where } C_k = \left| \frac{\sigma_{\alpha 0,k}}{\sigma^\mu_{\alpha 0,k}} \right|$$

Boel, James and Petersen specialize this result for the case where the system is linear and the cost function has $\mu = (\mu_1, \mu_2)$, $\phi(x) = x$, $\rho_1(e) = \rho_1(e) = |e|^2$ and noise is Gaussian. In this case, for all k, σ_k takes the form of an unnormalized Gaussian distribution provided $\mu_1 > 0$ is sufficiently small (for details see pp. 10-11). Significantly, when $\mu_1 = 0$ (i.e. the accumulated error term is ignored and only the current time error is penalized) and $\mu_2 > 0$, the resulting estimator for the linear Gaussian system reduces to the standard MMSE Kalman filter.

Consider the deterministic signal model:

$$x_{k+1} = b_\alpha(x_k) + g_\alpha(x_k)w_k$$
$$y_{k+1} = h_\alpha(x_k) + v_k$$

where w_k, v_k are x_0 are unknown deterministic disturbances, x_k is the signal state and y_k is the observation sequence. In this context the worst-case (H^∞) cost function is given by (Boel, James and Petersen, p. 14):

$$f_{H^\infty,k}(\hat{\phi}_k) = \sup_{w_{0,k}, v_{0,k}, x_0} \left\{ \sum_{l=0}^{k-1} \left[\mu_1 \rho_1 \left(\phi(x_l) - \hat{\phi}_l^*\right) - \frac{1}{2}\left(|w_l|^2 + |v_l|^2\right)\right] + \mu_2 \rho_2 \left(\phi(x_k) - \hat{\phi}_k\right) - \frac{1}{2}|x_0|^2 \right\}$$

where ρ_1, ρ_2, and the $\hat{\phi}_0^*, \ldots, \hat{\phi}_{k-1}^*$ are as previously defined and $\hat{\phi}_k^*$ remains to be determined. The *minimum H^∞ estimator* (MHIE) is defined by:

$$\hat{\phi}_{H^\infty,k}^* \in \arg\min_{\hat{\phi}_k \in \mathbf{Y}_k} f_{H^\infty,k}(\hat{\phi}_k)$$

If the set of $\hat{\phi} \in \mathbf{Y}$ for which $f_{H\infty,k}(\hat{\phi}) < \infty$ is convex and the function $f_{H\infty,k}(\hat{\phi})$, then if $f_{H\infty,k}(\hat{\phi}) < \infty$ for some $\hat{\phi} \in \mathbf{Y}_k$, then Boel, James and Petersen establish that the minimum defined by

$\hat{\phi}^{*}_{H^\infty,k}$ * exists and is unique (proposition 5.1, p. 14). Under these conditions the MHIE enjoys the following error bound (p. 15):

$$\mu_1 \sum_{l=0}^{k-1} \rho_1\left(\phi(x_1) - \hat{\phi}^{*}_{H^\infty,l}\right) + \mu_2 \rho_2\left(\phi(x_k) - \hat{\phi}^{*}_{H^\infty,k}\right) \leq f_{H^\infty,k}\left(\hat{\phi}^{*}_{H^\infty,k}\right) + \sum_{l=0}^{k-1} \frac{1}{2}\left(|w_l|^2 + |v_l|^2\right) + \frac{1}{2}|x_0|^2$$

At this stage, the authors point out the similarity between this expression and its counterpart for the risk-sensitive estimator, noting that for the former the relative entropy term has been replaced by the sum-of-squares terms on the right-hand side. In the next section of this appendix, I identify the precise relationship that holds between stochastic uncertainty constraints and norm bounds over the various sources of system error.

The authors establish that the risk-sensitive cost function and H^∞ cost functions are related by small noise asymptotics. If for the stochastic signal model the noises have variance ε (in place of 1), the initial state x_0 is $N(0, \varepsilon)$, and the risk-sensitive cost function is rescaled by replacing μ_1 and μ_2 by μ_1/ε and μ_2/ε then:

$$\lim_{\varepsilon \to 0} \varepsilon \log f_{rs,k}\left(\hat{\phi}_k\right) = f_{H^\infty,k}\left(\hat{\phi}_k\right)$$

The MHIE filter can then be computed recursively by (Boel, James and Petersen, 1997, remark 5.4 and theorem 5.5, p. 15):

$$
\begin{aligned}
&p_{k+1} = F\left(p_k, \hat{\phi}^*_{H^\infty,k}, y_{k+1}\right), \; p_0(x) = -\frac{1}{2}|x|^2 \\
&\hat{\phi}^*_{H^\infty,k} = \Phi^*_{H^\infty}(p_k),
\end{aligned}
$$

where $F(p,\hat{\phi},y)(x) = \sup_{w,x'}\left\{\mu_1\rho_1\left(\hat{\phi}-\phi(x')\right) - \frac{1}{2}|w|^2 - \frac{1}{2}|y-h(x')|^2 + p(x') : x = b(x') + g(x')w\right\}$

and $\Phi^*_{H^\infty}(p) \in \arg\min_{\hat{\phi}} \sup_x \left\{\mu_2\rho_2\left(\hat{\phi}-\phi(x)\right) + p(x)\right\}$

A.9. Stochastic Uncertainty Constraints and Norm Bounds

In this section of the appendix, I discuss the relationship between stochastic uncertainty constraints based on relative entropy and those based on sum quadratic norm bound constraints. In Ugrinovski and Petersen (1999) it is argued that relative entropy constraints are a generalisation of sum quadratic constraints and similarly, therefore, can be given a norm bound interpretation. I now wish to justify this postulate by examining the work of Savkin and Petersen (1998), who demonstrate the equivalence between a stochastic uncertainty constraint and a set of norm bounds imposed on observation error, model uncertainty and external perturbation. This relationship is fundamental to my efforts to distinguish between what I have termed uncertainty aversion and uncertainty perception in the first chapter of the thesis.

Savkin and Petersen (1998) derive a robust state estimator for a time-varying, uncertain, discrete-time system represented by the following:

$$
\begin{aligned}
&x(t+1) = A(t)x(t) + B(t)w(t) \\
&z(t) = K(t)x(t) \\
&y(t) = C(t)x(t) + v(t)
\end{aligned}
$$

where $w(t)$ and $v(t)$ are the uncertainty inputs, $z(t)$ is the uncertainty output, $y(t)$ is the observation variable, and $x(t+1)$ is the state variable. The authors consider a sum quadratic uncertainty constraint of the form:

$$(x(0)-x_0)' X_0 (x(0)-x_0) + \sum_{t=0}^{T-1} w'(t)Q(t)w(t) + v'(t+1)R(t+1)v(t+1) \leq d + \sum_{t=0}^{T-1} \|z(t)\|^2$$

where $X_0 = X'_0 > 0$ is a given matrix, $x_0 \in \Re^n$ is a given vector, $d > 0$ is a given constant, $Q(t)$, $R(t)$ are given positive definite symmetric matrices defined for t = 0,1,...,T, and $\| \ \|$ denotes the Euclidean norm. Savkin and Petersen contend that the system and uncertainty constraint represented by these two sets of equations allow for uncertainty that satisfies a standard norm bound constraint as depicted in the following state equations:

$$\begin{aligned} &x(t+1) = [A(t) + B(t)\Delta(t)K(t)]x(t) + B_2(t)n_1(t); \\ &y(t) = C(t)x(t) + n_2(t); \quad \left\|\Delta(t)' Q(t)^{1/2}\right\| \leq 1 \end{aligned}$$

and the following inequality defined over initial conditions and noise sequences:

$$\begin{aligned} &(x(0)-x_0)' X_0 (x(0)-x_0) + x(0)' K(0)' Q(0) K(0) x(0) + \sum_{t=0}^{T-1} n_1(t)' Q(t) n_1(t) \\ &+ \sum_{t=0}^{T-1} n_2(t)' R(t) n_2(t) \leq d. \end{aligned}$$

Here, $\Delta(t)$ is the uncertainty matrix, $n_1(t)$ and $n_2(t)$ are noise sequences, $B(t) = [B_1(t) \quad B_2(t)]$ and $\| \ \|$ denotes the standard induced matrix norm.

This can be easily verified by letting $w(t)=\begin{bmatrix}\Delta(t)K(t)x(t)\\ n_1(t)\end{bmatrix}$, for t = 0,1,...,T and $v(t)=n_2(t)$ for t = 1,2,...,T, where $\left\|\Delta(t)Q(t)^{1/2}\right\|\leq 1$ for all t. As a consequence of these substitutions, the sum quadratic uncertainty constraint will be satisfied.

A.10. New Classical Interpretations of the Norm Bounds

In the Hansen, Sargent and Tallarini (1999) and Andersen, Hansen and Sargent (1999) papers, uncertainty is introduced into the risk-sensitive control problem in a narrowly circumscribed fashion. In the former paper, both observation error $v(t)$ (in terms of Savkin and Petersen's notation introduced above) and model uncertainty (i.e. the $\Delta(t)K(t)$ term incorporating the uncertainty matrix) are ignored, and only external perturbation $n_1(t)$ is modeled explicitly. In both papers, perturbation is acknowledged as a source of distortion, either to the conditional probability measure (in the former paper), or to the transition probabilities in a Markov chain and the drift term in a diffusion process (in the latter paper). This is because the control and filtering problems are set up and solved under the assumption of full information, following a precedent set by Jacobsen (1973) and Hansen and Sargent (1995). While this full information assumption has no effect on the determination of the robust or risk-sensitive control law, it does influence estimates of the robust filter. Hansen, Sargent and Tallarini (1999) appear to acknowledge the matter of model uncertainty in a footnote:

> [i]t can be argued that risk sensitivity is simply repairing a defect in quadratic preferences, a criticism to which we are certainly vulnerable in this paper. The usual measure of relative risk

> aversion in the absence of habit persistence is $-cU''(c)/U'(c)$. In the case of our quadratic preferences, this is given by $c/(b - c)$, which requires that the bliss point process be twice the consumption level to attain a risk aversion coefficient of one (fn 27, p. 894).

However, considerable caution must be exercised at this point. The authors are actually expressing concerns about the adequacy of the chosen *objective function*. Here, the "model" is to be understood in the widest possible sense, as referring to the complete set of system of equations that includes those for the state variable, the observation equations, *and* the objective function itself. In a control context, the engineer is concerned with optimal tracking, disturbance attenuation, noise rejection and overall system stability. The role of the objective function is to penalize (possibly in a risk-sensitive manner) deviations of system outputs from desired targets. In economic theory, the way agents are presumed to assess these deviations of outputs from targets is, itself, a matter for theoretical and empirical investigation and it relates directly to the axioms of choice-theory posited by the researcher. This matter, therefore, lies at the heart of current debates on the nature of human values, behaviour and action. Are quadratic preferences appropriate? Are they approximations to more complex forms of decision-making? Are additively separable intertemporal preferences a reasonable representation of investor behaviour? These are issues that the engineer can largely ignore.

Another acknowledgement of the possibility of model limitations, comes in the conclusion of the Hansen, Sargent and Tallarini paper (p. 902), where the authors discuss the inadequacies of the representative agent framework:

> [m]aybe we take the representative agent paradigm too seriously. We use the representative agent as a convenient starting point to understand the workings of risk sensitivity and robustness in decentralized economies. In other words, we know how heterogeneity of preferences and incomplete risk sharing affect investment behaviour and the market price of risk. In our model (and Epstein and Wang's (1994)), agents agree on the amount and location of the Knightian uncertainty. Thus, models like ours can contribute an additional dimension upon which homogeneity alters equilibrium quantities and prices.

Indeed! The issue of agency is also something that engineers do not have to interrogate. In an engineering context there is usually a single agent attempting to control events: whether the system is a remote lunar module, a nuclear reactor or a chemical reaction. For the economist, the agents are, instead, members of a complex society: a society that is marked by deep divisions of culture, class, gender and wealth, yet, one that must somehow be formally modeled. Hansen, Sargent and Tallarini's discussion of heterogeneity relates to abstract differences in agent preferences and endowments. This notion of heterogeneity falls far short of the recognition that there is an important distinction between those agents who consume and save, those who invest and offer employment, and those who manage portfolios of financial assets and liabilities (separation theorems and mutual fund theorems not withstanding). However, embracing this more fundamental kind of heterogenity would require a rejection of the foundational role played by the neoclassical Ramsey-style growth model in driving the growth and endowment process.

In the context of Hansen, Sargent and Tallarini's permanent income model the possibility of model uncertainty could, at the very least, be related to the overly simplistic features of the habit persistence model and the linear production technology that is assumed to govern output.

Moreover, the concern for robustness could also be extended to cover the investment decision and not just the consumption/savings decision. Recognition of the fact that model uncertainty and observation error also contribute to the magnitude of the stochastic uncertainty constraint has an additional, somewhat disturbing implication. In the context of a social, rather than a purely natural or mechanistic decision-making problem, the extent of model uncertainty would depend on uncertainty perception, in David Dequech's sense of the term, which combines awareness of the actual existence of uncertainty with an optimistic disposition to ignore certain unpalatable yet relevant strands of evidence. Thus, to the extent that social actors lose faith in their ability to adequately model what transpires in real-world (i.e. time-varying, complex, and structurally unstable) economic systems, their assessment of model uncertainty would inevitably rise.

In the context of the social sciences, observation error is also something that may be exceedingly difficult to evaluate. Observations are not made in a laboratory setting with scientific instruments of known calibration error or noise input. With more knowledge, agents may come to realize the inaccuracy and internal bias in survey data that was previously considered to be robust. This realization would inevitably contribute to a heightened perception of uncertainty.

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