Departure processes from *MAP/PH/1* queues

David Anthony Green

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Signed Statement

This work contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text.

I consent to this copy of my thesis, when deposited in the University Library, being available for loan and photocopying.

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Dedication

This work is dedicated to Elizabeth Joseph, Margaret Edward and Carmel Anthony, three of my ten children who left this world before my wife and I could know them.

Requiem æternam dona eis Domine.
Et lux perpetua luceat eis.

Abstract

A *MAP/PH/1* queue is a queue having a Markov arrival process (*MAP*), and a single server with phase-type (*PH*-type) distributed service time. This thesis considers the departure process from these type of queues. We use matrix analytic methods, the Jordan canonical form of matrices, non-linear filtering and approximation techniques. The departure process of a queue is important in the analysis of networks of queues, as it may be the arrival process to another queue in the network. If a simple description were to exist for a departure process, the analysis of at least feed-forward networks of these queues would then be analytically tractable.

Chapter 1 is an introduction to some of the literature and ideas surrounding the departure process from *MAP/PH/1* queues.

Chapter 2 sets up the basic notation and establishes some results which are used throughout the thesis. It contains a preliminary consideration of *PH*-type distributions, *PH*-renewal processes, *MAP*s, *MAP/PH/1* queues, non-linear filtering and the Jordan canonical form.

Chapter 3 is an expansion of [7], where the question of whether a *MAP* description can exist for the departure process of a non-trivial *MAP/M/1* queue is considered. In a 1994 paper, Olivier and Walrand conjectured that the departure process of a *MAP/PH/1* queue is not a *MAP* unless the queue is a stationary *M/M/1* queue. This conjecture was prompted by their claim that the departure process of an *MMPP/M/1* queue is not *MAP* unless the queue is a stationary *M/M/1* queue. We show that their proof has an algebraic error, which leaves open the above question of whether the departure process of an *MMPP/PH/1* queue is a *MAP* or not.

In Chapter 4, the more fundamental problem of identifying stationary *M/M/1* queues in the class of *MAP/PH/1* queues is considered. It is essential to be able to determine from its generator when a stationary *MAP* is a Poisson process. This does not appear to have been discussed in the literature prior to the author's paper [5], where this deficiency was remedied using ideas from non-linear filtering theory, to give a characterisation as to when a stationary *MAP* is a Poisson process. Chapter 4 expands upon [5]. This investigation of higher order representations of the Poisson process is motivated by first considering when a higher order *PH*-type distribution is just negative exponential.

In Chapter 5, we consider the related question of minimal order representations for *PH*-type distributions, an issue which has attracted much interest in the literature. A discussion of other authors' ideas is given and these ideas are then inter-related to the work presented in Chapter 4 on the *PH*-type distributions.

The *MAP/M/1* queue is then considered in Chapter 6 from the perspective of whether having an exact level and phase independent stationary distribution of the geometric form

$$\boldsymbol{\Psi}_0 = [\boldsymbol{\pi}_0, \eta\boldsymbol{\pi}_0, \eta^2\boldsymbol{\pi}_0, \eta^3\boldsymbol{\pi}_0, \ldots],$$

where $0 < \eta < 1$ is a scalar, implies that the *MAP* is Poisson. The answer is in the affirmative for this question, but the converse is not strictly true. Apart from showing the ubiquitous asymptotic form of level and phase independence exhibited by all stable *MAP/M/1* queues, we prove that a very large class of stable queues, exhibits what we have termed *shift-one level and phase independence*. Stable *MAP/M/1* queues exhibiting shift-one level and phase independence, are characterised by a stationary distribution of the following form:

$$\boldsymbol{\Psi}_0 = [\boldsymbol{\pi}_0, \chi\boldsymbol{\xi}_0, \chi^2\boldsymbol{\xi}_0, \chi^3\boldsymbol{\xi}_0, \ldots],$$

where $0 < \chi < 1$ is a scalar and $\boldsymbol{\xi}_0$ is a positive row vector.

In Chapter 7, a family of approximations is proposed for the output process of a stationary *MAP/PH/1* queue. To check the viability of these approximations,

they are used as input to another single server queue. Performance measures for the second server are obtained analytically in both the tandem and approximation cases, thus eliminating the need for simulation to compare results. Comparison of these approximations is also made against other approximation methods in the literature.

In Chapter 8, we show that our approximations from Chapter 7 have the property of exactly matching the inter-departure time distribution. Our k^{th} approximation also accurately captures the first $k-1$ lag-correlation coefficients of the stationary departure process. The proofs of this direct association between lag-correlation coefficients and the level of complexity k are given.