Aspects of Quantum Game Theory

by

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Contents

Abstract

Quantum game theory is an exciting new topic that combines the physical behaviour of information in quantum mechanical systems with game theory, the mathematical description of conflict and competition situations, to shed new light on the fields of quantum control and quantum information. This thesis presents quantizations of some classic game-theoretic problems, new results in existing quantization schemes for two player, two strategy non-zero sum games, and in quantum versions of Parrondo's games, where the combination of two losing games can result in a winning game. In addition, quantum cellular automata and quantum walks are discussed, with a history-dependent quantum walk being presented.

Statement of Originality

This work contains no material that has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text.

I give consent to this copy of the thesis, when deposited in the University Library, being available for loan, photocopying, and dissemination through the digital thesis collection.

Adrian Flitney	4th January, 2005
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Acknowledgments

A work of this magnitude could not possibly have been undertaken without scorning the advice of too many people to mention here. George W. Bush and Rupert Murdoch are just two of the people whose advice was not even sort, nor did they provide any funding. However, generous funding was provided by the GTECH Corporation with help from the SA Lotteries Commission. Travel funding was provided by The University of Adelaide postgraduate travel award, the D. R. Stranks postgraduate travel scholarship and the Department of Electrical and Electronic Engineering at The University of Adelaide. The support, direction and encouragement of my supervisor, A/Prof. Derek Abbott is gratefully acknowledged. I would also like to thank various colleagues and collaborators for useful discussions and help with various parts of this work: Prof. Jens Eisert of Potsdam University, Prof. Neil F. Johnson of Oxford University, Wanli Li of Princeton University, Prof. David Meyer of the University of California San Diego (UCSD), Joseph Ng of the University of Queensland, Dr. Arun Pati of the University of Bangor, and a number of others with whom I spoke at conferences and during interstate or overseas visits. I would like to thank my fellow students at the Centre for Biomedical Engineering and those I spent time with at the Physics Departments at Melbourne, Oxford and Potsdam Universities. Finally, I would like to thank my family and friends, without whom life would be impossible.

—Adrian Flitney

"Returning home I read a book on Physics. I don't understand it very well . . . Why isn't nature clearer and more directly comprehensible?"

—Shin'ichirō Tomonaga, Nobel prize winner in Physics, 1965

Thesis Conventions

Typesetting. This thesis is typeset using $\LaTeX$2e software. Plots were generated by *Mathematica* 4.1. CorelDRAW 7.467 was used to generate some of the schematic diagrams, while the remainder were generated with standard $\LaTeX$ picture commands.

Spelling. Australian English spelling has been adopted throughout, as defined by the Macquarie English Dictionary (A. Delbridge (ed.) Macquarie Library, North Ryde, NSW, Australia, 2001). Where more than one spelling variant is permitted such as biassing or biasing and infra-red or infrared the option with the fewest characters has been chosen.

Mathematics. The International Standards Organization has established the recognized conventions for typesetting mathematics. The most important points are given below.

1. Equations are treated as part of the text and include the appropriate punctuation.
2. Simple variables are represented by italic letters, e.g., x, y or z.
3. Vectors are written in bold face italic, e.g., $\boldsymbol{B}$ or $\boldsymbol{\pi}$.
4. Superscripts or subscripts that are descriptions and not variables are in upright font, e.g., k_{A} where A stands for Alice as opposed to k_i where $i = 1, \ldots, n$.

Referencing. The Harvard style is used for referencing and citation.

Publications

Flitney-A. P and Abbott-D (2005). Quantum games with decoherence, *J. Phys. A*, **38**, 449–59.

Flitney-A. P and Abbott-D (2004c). A semi-quantum version of the game of Life, *in* A. S. Nowak and K. Szajowski (eds.), *Advances in Dynamic Games: Applications to Economics, Finance, Optimization and Stochastic Control (Proc. 9th Int. Symp. on Dynamic Games and Applications, Adelaide, Australia, Dec. 2000)*, Birkhäuser, Boston, pp. 667–79.

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Corrigenda

Typographic errors in names: "Laundauer" should be "Landauer" (page 2), "Morgenstein" should be "Morgenstern" (pages 2 and 177), "Wohl" should be "Wolf" (pages 2 and 170), and "Wilkins" should be "Wilkens" (pages 3, 17, 50, 55 and 170).

Page 15: Replace the sentence above Eq. (2.7) with:
"A particular choice of maximally entangling operator $\hat{J}$, for a general N player two strategy game, may be written (Benjamin and Hayden 2001b) as"

Page 37: In the paragraph below Eq. (4.7) replace the phrase "... invert a general $|\psi\rangle$" with "... perform a universal not on a general $|\psi\rangle$" and drop the word "unitarily" from the end of the following sentence (that begins with "A general complementing operation ...").

Page 40: In the caption to Figure 4.6 the last sentence should end with " ... with the subscript A referring to Alice and B to Bob."

Page 66: In the first paragraph, change the sentence beginning "These techniques work by encoding ..." to "The former technique works by encoding ..."

Page 67: Equation (6.4) is more clearly written as

$$\rho \rightarrow \sum_{j_1,\ldots,j_N=0}^{2} \mathcal{E}_{j_1} \otimes \ldots \otimes \mathcal{E}_{j_N}\, \rho\, (\mathcal{E}_{j_1} \otimes \ldots \otimes \mathcal{E}_{j_N})^\dagger,$$

Page 69: In the fourth line of Eq. (6.6), $\hat{M}_k$ should be $\hat{U}_k$ in both instances.

Page 138: The command `Unprotect[Play]` should be inserted before `Play[...]`.

Page 142: In the code for `ResultsRandom` the `NextStep[c, p]` command should be `NextStep[c, p, p]`.

Page 144: In the usage commands, `inithist` has been omitted from the descriptions of `Results`, `Results2` and `ResultsRandom`. For these commands the usage should read `Results::usage = "Results[cap, inithist, p, ... ]"` etc.

Page 150: In the second line of the code for `Results[ cap_List, p_List, n_Integer]` the braces inside the `Table` command have been omitted. This line should read `Module[ { results = Table[0, {i,n}], nc = cap, j },`

Page 157: The command `PlotPeakPosn` is redundant and should be deleted. It relies on another command that has been removed from the package.

Page 159: Braces are missing in the `MakeEmpty` and `SetQubit` commands. These command should read:

```
MakeEmpty[n_Integer] := Table[ 0. I + 0., {i,1,2^n} ]
SetQubit[ Q_List, n_Integer, val_ ] :=
   Module[ {newQ = Q}, ... ]
```

Page 160: In the code for `ApplyRule[ Q_List, t_, a_, b_, p_, sh_, m_Integer ]`, the variable `newqca` should be `newQ`.

Sections A.2–4: The code works best if the `BeginPackage[ ... ], Begin["Private`"]` and the corresponding `End[]` and `EndPackage[]` commands are omitted.

Known bugs in the code: (a) In Section A.2.2, if `n` is not a multiple of `(na + nb)` then executing `Results[cap, hist, p, p1, p2, p3, p4, na, nb, n]` will produce a list of results with trailing zeros. That is, only `Floor[n/(na + nb)] * (na + nb)` actual data points are produced. (b) In Section A.4.1, the configuration must have an even number of qubits. That is, `n` should be even when setting up the configuration with `MakeEmpty[n]`.

Chapter 1

Motivation and Layout of the Thesis

THIS chapter provides a brief background of, and motivation for, the study of quantum games. It gives a guide to the contents of the thesis and a list of the important original contributions.

> *"Landauer based his research on a simple rule: information is physical. That is, information is registered by physical systems such as strands of DNA, neurons and transistors; in turn the ways in which systems such as cells, brains and computers can process information is governed by the laws of physics. Landauer's work showed that the apparently simple and unproblematic statement of the physical nature of information had profound consequences."*

—Seth Lloyd on Rolf Laundauer (Lloyd 1999)

1.1 Background and motivation

Game theory is the mathematical language of competitive scenarios where the outcome is contingent upon interacting strategies of two or more agents with conflicting, or at best self-interested, motivations. Originally developed for use in economics by von Neumann and Morgenstein (1944), with important contributions by Nash (1950), it has now found a wide variety of uses in the social sciences, biology, computer science, international relations and, more recently, physics (Abbott et al. 2002).

Computers that exploit the inherent features of quantum mechanics, such as superposition and entanglement, are known as quantum computers [see, for example, Eisert and Wohl (2004)]. The rise of interest in quantum computing has brought with it increasing attention to the field of quantum information, the study of information processing tasks using quantum systems (Nielsen and Chuang 2000). At the intersection of game theory and quantum information is the new field of quantum game theory, created in 1999 when two groups independently had the idea of applying the rules of quantum mechanics to game theory (Eisert et al. 1999, Meyer 1999). Replacing the classical probabilities of game theory by quantum amplitudes creates the possibility of new effects resulting from superposition or entanglement. To date quantum game theory has concentrated on observing these new effects amongst the traditional settings of game theory, but ultimately quantum game-theoretic techniques could be used in quantum communication (Brandt 1998) or quantum computing (Lee and Johnson 2002a) protocols. For example, quantum communication in competition with an eavesdropper (Gisin and Huttner 1997), optimal cloning (Werner 1998), or quantum gambling (Goldenberg et al. 1999) can be considered as games. There has been a proposal to use quantum game theory in a quantum teleportation protocol (Pirandola 2004) and in quantum state estimation and quantum cloning (Lee and Johnson 2003b). A review of suggested applications of quantum games is given by Piotrowski and Sładowski (2004a). In the meantime quantum games have stimulated popular discussion (Peterson 1999, Collins 2000, Klarreich 2001b, Cho 2002, Lee and Johnson 2002b, Siegfried 2003a, Siegfried 2003b) and their study adds to our understanding of quantum information theory.

This thesis is concerned with developing the topic of quantum game theory. Only theoretical aspects are considered in this thesis—the details of the physical systems are omitted. The thesis is written for a readership with familiarity with basic quantum mechanics. Some necessary background on game theory is given in Chapter 2.

1.2 Layout of thesis and original contributions

The original work in this thesis has been the subject of peer reviewed publications, the full list of which is given in the Publication list. The topics covered by the thesis and the important original contributions are indicated below.

Chapter 2 is a review chapter giving some background in classical game theory and an introduction to its quantum analogue. A description of the popular Eisert model of two player, two strategy quantum games is presented (Eisert et al. 1999, Eisert and Wilkins 2000) as well as a review of other models.

Chapter 3 presents an original quantization (Flitney and Abbott 2002c) of the game show situation known as the Monty Hall problem (vos Savant 1990). The classical problem has generated much interest because of its counterintuitive optimal play. This is one of the few games with three classical strategies that has been quantized. Although two other quantum protocols for the game exist (Li et al. 2001, D'Ariano et al. 2002) all three models are quite distinct.

Chapter 4 presents an original quantum protocol for multiplayer duels, concentrating on the three person case (Flitney and Abbott 2003b, Flitney and Abbott 2004b). This is an example of a three person, three strategy, multi-stage game of which there are no other examples in the quantum game literature.

Chapter 5 considers the advantage a quantum player can achieve over one restricted to classical strategies, as a function of the degree of entanglement (Flitney and Abbott 2003a). The work of Eisert et al. (1999) and Du et al. (2003c) on quantum Prisoners' Dilemma is extended to a number of different two person non-zero sum games.

Chapter 6 examines how decoherence can be incorporated in quantum games of the Eisert scheme (Flitney and Abbott 2004a, Flitney and Abbott 2005). In the existing literature there is a single publication on decoherence in quantum games and this only considers Prisoners' Dilemma (Chen et al. 2003b). This thesis gives a model for decoherence in a more general quantum game setting and considers a number of two player, two strategy games. The advantage a quantum player has over a classical player is used as a measure of the "quantum-ness" of the games.

Chapter 7 gives an introduction to classical and quantum Parrondo's games (Flitney and Abbott 2003c). Parrondo's games occur when a mixture of two losing games can result in a winning game (Harmer and Abbott 1999a). A quantum analogy to a capital-dependent Parrondo's game exists (Meyer and Blumer 2002a). Here, new results are presented for a history-dependent quantum Parrondo game (Flitney et al. 2002) as well as further results for the earlier model (Flitney and Abbott 2002b). The model of the history-dependent quantum Parrondo game was first formulated in 2000 by Ng and Abbott (2004) but the calculations developed in this thesis are original.

Chapter 8 discusses quantum walks, the quantum analogue of classical random walks. A new multi-coin model of a quantum walk with history dependence is presented and its features are discussed (Flitney et al. 2004). Introduction of a bias through the history dependence distinguishes our model from existing work on multi-coin quantum walks (Brun et al. 2003b). The new model can produce another example of a quantum Parrondo's game and thus this chapter is an extension of the work detailed in Chapter 7. The work was carried out in collaboration with Prof. Neil F. Johnson of the Physics Department, Oxford University.

Chapter 9 gives a brief introduction to quantum cellular automata. A new semi-quantum version of the John Conway's famous two-dimensional cellular automata Life (Gardner 1970) is presented and some novel structures in the new model are discussed (Flitney and Abbott 2004c).

Chapter 10 gives a comprehensive summary of the thesis and possible future directions.

These contributions further the body of knowledge of quantum game theory. The layout of the material in the thesis is shown in Figure 1.1.

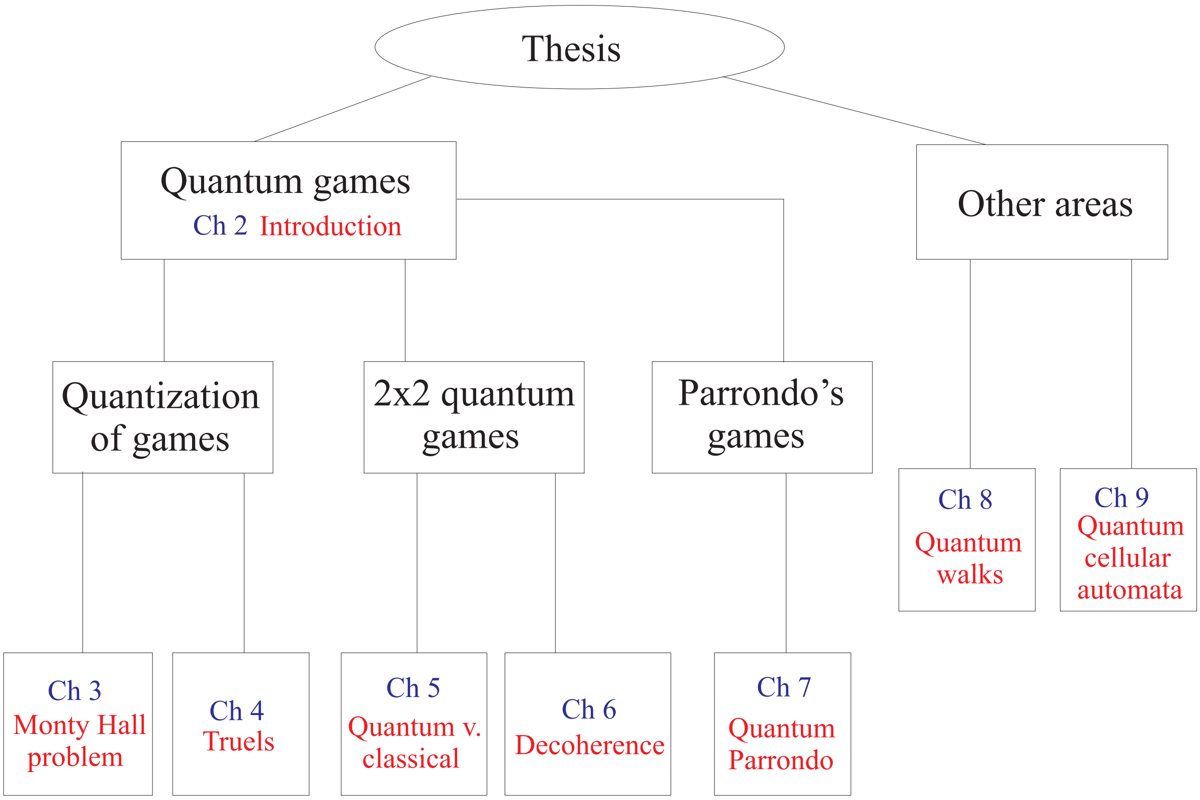


Figure 1.1. Layout of the thesis. A schematic showing the topics covered by this thesis.

Chapter 2

Introduction to Quantum Games

THIS chapter provides a brief overview of classical game theory and a list of definitions of game-theoretic terms that occur elsewhere in the thesis. An introduction to quantum game theory is presented as well as a review of published ideas in the field. The well known scheme of Eisert et al. (1999) for two player, two strategy quantum games with entanglement is discussed. Aspects of this model are the subject of Chapters 5 and 6.

2.1 Game theory

2.1.1 Background

Game theory is a tool for rational decision making in conflict situations. It has long been commonly used in economics, the social sciences and biology to model decision making situations where the outcomes are contingent upon the interacting strategies of two or more agents with conflicting, or at best, self-interested motives. There is now increasing interest in applying game-theoretic techniques in physics. The models are necessarily idealizations of the physical situations. The need for simplification rules out the application of game-theoretic techniques to most situations that lay people would call games, such as chess, where there are simply too many possibilities. The situations of interest to game theory are those where the agents, or players, can select one of a small number of options, or strategies. The results of the game, and the corresponding payoffs to the players, are determined collectively by the strategies of all the agents. The following section gives formal definitions to the terms and gives a simple example.

2.1.2 Basic ideas and terminology

Definition 2.1 Game: a set of players, a set of rules that specify the possible actions of the players, and a set of payoff functions giving the rewards to the players for the various game outcomes, that is, a triple $\{N, \Omega, \Gamma\}$, where N is the number of players, $\Omega = \{S_1, \ldots, S_N\}$ with S_j being the set of strategies available to the jth player and $\Gamma = \{P_1, \ldots, P_N\}$ with P_j being the payoff function for the jth player, $j = 1, \ldots, N$.

Definition 2.2 Payoff *or* **utility:** a number that measures the desirability of a particular game outcome for a player. There is a game outcome associated with each strategy profile $\{s_1, \ldots, s_N\}$, with $s_j \in S_j,\ j = 1, \ldots, N$. Each game outcome is assigned a payoff by each player. A mapping from the set of all possible strategy profiles to the real numbers, $P_k : \{s_1, \ldots, s_N\} \to \mathbb{R}$, is known as the payoff matrix.

Definition 2.3 Action *or* **move:** a choice available to a player.

Definition 2.4 Strategy: a rule that prescribes the action of a player contingent upon the game situation.

Definition 2.5 Pure strategy: a strategy that specifies a unique move in a given game position. Unless otherwise specified, the term "strategy" refers to a pure strategy.

Definition 2.6 Mixed strategy: a strategy that uses a randomizing device, such as a coin, to select amongst alternatives for some or all game positions.

Definition 2.7 Dominant strategy: a strategy that results in a higher payoff than any alternate strategy against all possible strategic choices by the other player(s). That is, s_k is a dominant strategy for player k if

$$\forall s_j,\ j \neq k,\ \ P_k(s_1, \ldots, s_k, \ldots s_N) \geq P_k(s_1, \ldots, s'_k, \ldots s_N)\ \forall s'_k$$

Definition 2.8 $\mathbf{n_1} \times \mathbf{n_2} \times \ldots \times \mathbf{n_N}$ game: an N player game where the jth player has available n_j strategies.

Definition 2.9 Zero sum game: a game in which the sum of all the players' payoffs is zero regardless of the strategies they choose.

Definition 2.10 Game of perfect information: a game where all the information about the position, the strategy sets and the payoff functions of the players is known to all.

Definition 2.11 Symmetric game: one where all agents have the same set of strategies and identical payoff functions, except for the interchange of roles of the players.

Definition 2.12 Nash equilibrium (NE): a game result from which no player can improve their payoff by a unilateral change in strategy (Nash 1950, Nash 1951). That is, the strategy profile $\{s_1, \ldots, s_N\}$ is an NE if

$$\forall k, P_k(s_1, \ldots, s_k, \ldots, s_N) \geq P_k(s_1, \ldots, s'_k, \ldots, s_N)\ \forall s'_k$$

Definition 2.13 Focal point: one amongst several NE that, for psychological reasons, is particularly compelling.

Definition 2.14 Maximin: a game equilibrium where each player maximizes the minimum payoff that they can receive. That is, each player assumes that for any strategy they choose their opponent(s) will respond with the strategy that hurts them the most. With this expected behaviour the optimal choice is the one that provides the maximum of the worst case payoffs. This equilibrium makes sense in zero-sum games where there are purely competitive forces, but fails to take into account possible benefits from cooperation in other situations.

Definition 2.15 Pareto optimal (PO): a game result from which no player can improve their payoff without another player being worse off, that is, if

$$\begin{aligned}\forall k,\ \exists \ell \text{ s.t. } & P_k(s_1,\ldots,s'_k,\ldots,s_\ell,\ldots,s_N) > P_k(s_1,\ldots,s_k,\ldots,s_\ell,\ldots,s_N) \\ & \Rightarrow P_\ell(s_1,\ldots,s'_k,\ldots,s_\ell,\ldots,s_N) < P_\ell(s_1,\ldots,s_k,\ldots,s_\ell,\ldots,s_N)\end{aligned}$$

then the unprimed strategy profile is PO.

Definition 2.16 Evolutionary stable strategy (ESS): Strategy s is **evolutionarily stable** against s' if, $\forall$ small $\epsilon > 0$, s performs better than s' against the mixed strategy $(1-\epsilon)s+\epsilon s'$. An ESS (Maynard Smith and Price 1973) is a strategy that is evolutionarily stable against all other strategies. In practical terms, a population that follows an ESS is resistant against invasion by a small group playing another strategy.

Examples of these definitions in practice can be seen in one of the simplest 2×2 symmetric games: that of Matching Pennies. The players, traditionally referred to as Alice and Bob, each have a coin for which they can select either heads or tails. The choices are revealed simultaneously. Alice wins if the coins show the same face while Bobs wins if they are different. If we assign a value of $+1$ to a win and -1 to a loss, the game can be described by the following payoff matrix:

	Bob: H	Bob: T
Alice: H	$(1,-1)$	$(-1,1)$
Alice: T	$(-1,1)$	$(1,-1)$

(2.1)

Here, and in subsequent examples, the numbers in parentheses refer to Alice's and Bob's payoffs, respectively. Matrix (2.1) is known as the strategic or normal form of the game. Since it includes the identities and strategies of all the players as well as their payoff functions, it is a complete description of the game. In the strategic form, the players' strategies are selected simultaneously. Games where the players make a number of moves sequentially are often better described in extensive form. This is a tree of nodes and branches, the nodes being game positions labeled by the player who has the move and the branches labeled by the possible moves of that player. Examples of the extensive description of a game are given in Chapter 4, however, the strategic form is the one that shall be used in the majority of this thesis.

In Matching Pennies there are two pure strategies: "show heads" or "show tails." A mixed strategy is something like "show heads half the time and tails the other half." A casual examination of the game shows that there is no best move, or dominant strategy, for the players: any option yields a 50% chance of success. For all game results one player wins and the other looses. Thus the game is zero-sum.

2.1.3 An example: the Prisoners' Dilemma

One 2×2 game that has deservedly received much attention is the Prisoners' Dilemma (Rapoport and Chammah 1965). Here, the players' moves are known as cooperation (C) or defection (D), for reasons that shall become clear. The payoff matrix is such that there is a conflict between the NE and the PO outcome. The payoff matrix can be written as

	Bob: C	Bob: D
Alice: C	(3,3)	(0,5)
Alice: D	(5,0)	(1,1)

(2.2)

The game is symmetric and there is a dominant strategy, that of always defecting, since it gives a better payoff if the other player cooperates (five instead of three) or if the other player defects (one instead of zero). Where both players have a dominant strategy, this combination is the NE.

The NE outcome {D,D} is not such a good one for the players, however, since if they had both cooperated they would have both received a payoff of three, the PO result. In the absence of communication or negotiation we have a dilemma between the personal and the mutual good, some form of which is responsible for much of the misery and conflict through out the world. Game theory does not have a solution. In a one-off Prisoners' Dilemma the rational player postulated by the theory should defect. In the real world the opportunity to play the game repeatedly and the ability to negotiate helps to foster some degree of cooperation even in pure Prisoners' Dilemma situations (Axelrod 1981, Axelrod and Hamilton 1984). There is extensive literature on the Prisoners' Dilemma and it is mentioned in any introductory text on game theory (see, for example, Rasmusen (1989)).

2.2 Quantum game theory: the idea

2.2.1 Quantum Penny Flip

One of the simplest gaming devices is that of a two state system such as a coin. If we have a player than can utilize quantum moves, we can demonstrate how the expanded space of possible strategies can be turned to advantage. Meyer, in his seminal work on quantum game theory (Meyer 1999), considered the simple game Penny Flip that consists of the following: Alice prepares a coin in the heads state and places it in a box where neither player can see it. Bob can choose to either flip the coin or leave its state unaltered, and Alice, without knowing Bob's action, can do likewise. Finally, Bob has a second turn at the coin. The coin is now examined and Bob wins if it shows heads. A classical coin clearly gives both players an equal probability of success unless they utilize knowledge of the other's psychological bias, and such knowledge is beyond analysis by game theory.[1]

To quantize this game, replace the coin by a two state quantum system such as a spin one-half particle. Suppose Bob is given the power to make quantum moves while Alice is restricted to classical ones. Can Bob profit from his increased strategic space? Let $|0\rangle$ represent the "heads" state and $|1\rangle$ the "tails" state. Alice initially prepares the system in the $|0\rangle$ state. Bob can proceed by first applying the Hadamard operator,

$$\hat{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{2.3}$$

putting the system into the equal superposition of the two states: $(|0\rangle + |1\rangle)/\sqrt{2}$. Now Alice can leave the "coin" alone or interchange the states $|0\rangle$ and $|1\rangle$. If the coherence of the system is maintained either action will leave the system unaltered, a fact that can be exploited by Bob. In his second move he applies the Hadamard operator again resulting in the pure state $|0\rangle$, thus winning the game with certainty. Bob exploits his ability to apply any unitary operation and the possibility of a superposition to make Alice's strategy irrelevant, as is clear from Figure 2.1. In other cases, quantum entanglement can be exploited by the quantum player, as we shall see particularly in Chapter 5.

[1] The biases of players can be modeled with game theory but additional formalism is required (Rubinstein 1998).

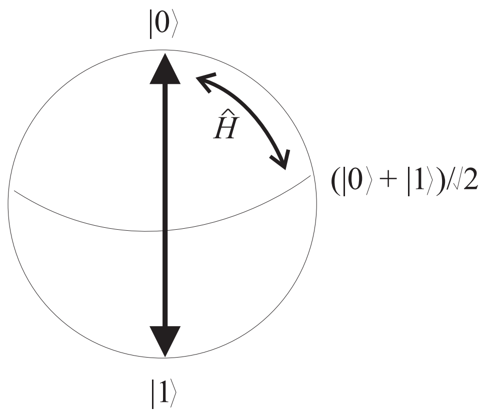


Figure 2.1. Quantum Penny Flip. The Bloch sphere for the quantum coin in Penny Flip. The coin starts in the $|0\rangle$ state. The quantum player (Bob) is able to apply any rotation, while the player restricted to classical moves (Alice) can only apply the identity or a bit-flip (a reflection about the horizontal). Bob exploits his advantage by rotating the qubit to the horizontal using $\hat{H}$ making Alice impotent in her move. Since Bob has certain knowledge of the state of the qubit before his second move, he can again employ $\hat{H}$ to rotate back to $|0\rangle$.

2.2.2 A general prescription

Where a player has a choice of two moves, the choice can be encoded in a single bit. To translate this into the quantum realm the bit is replaced by a quantum bit or qubit, which can be in a linear superposition of the two states. The basis states $|0\rangle$ and $|1\rangle$ correspond to the classical moves. The players' qubits are initially prepared by a referee in a state to be specified later. We suppose the players apply their chosen strategy using a set of instruments that can manipulate their qubit while maintaining coherence of the quantum state. That is, a pure quantum strategy is a local unitary operator acting on the player's qubit. After all players have executed their moves the qubits are returned to the referee who makes a positive operator valued measurement on the set and determines the payoffs according to the outcome of the measurement. The classical strategies "always play 0" and "always play 1" are represented by the identity operator $\hat{I}$ and the bit-flip operator,

$$\hat{F} \equiv i\hat{\sigma}_x = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \tag{2.4}$$

respectively. The resulting quantum game contains the classical one as a subset. A description of the formalism of quantum games is given by Lee and Johnson (2003a).

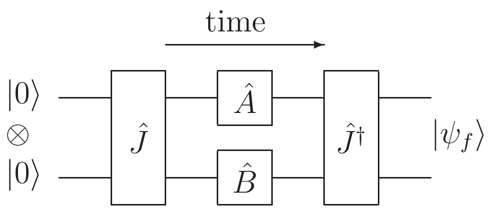


Figure 2.2. Protocol for a two person quantum game. A general protocol for a two person quantum game showing the flow of information (qubits). $\hat{A}$ is Alice's move, $\hat{B}$ is Bob's, and $\hat{J}$ is an entangling gate.

Reviews of quantum games are presented by Flitney and Abbott (2002a) and Piotrowski and Sładowski (2003a).

The list of possible quantum actions can be extended to include any physically realizable action on a player's qubit that is permitted by quantum mechanics. Some of the actions that have been considered include projective measurement (Li et al. 2001) and entanglement with ancillary bits (Benjamin and Hayden 2001b) or qubits (Li et al. 2001, Han et al. 2002a).

2.3 Eisert's model for 2×2 quantum games

In static 2×2 games each player has just a single move. In the absence of entanglement, utilizing a quantum strategy to produce a superposition of alternatives will give the same results as a classical mixed strategy. In order to see non-classical results, Eisert et al. (1999) introduced entanglement between the players' moves using the protocol of Figure 2.2.

The final state is computed by

$$|\psi_f\rangle = \hat{J}^{\dagger}(\hat{A} \otimes \hat{B})\hat{J}|\psi_i\rangle, \tag{2.5}$$

where $|\psi_i\rangle = |00\rangle$ is the initial state of the players' qubits, $|\psi_f\rangle$ the final state, $\hat{J}$ is an operator that entangles the players' qubits, and $\hat{A}$ and $\hat{B}$ represent Alice's and Bob's moves, respectively. A measurement in the computational basis $\{|0\rangle, |1\rangle\}$ is taken on the final state and the payoff is subsequently computed from the classical payoff matrix[2]. The

[2]In terms of the formalism of Sec. 2.2.2, the scheme described is equivalent to a referee preparing the state $(|00\rangle + i|11\rangle)/\sqrt{2}$ to give to the players who then apply a local unitary operator to their qubit, before returning the state to the referee who makes a measurement in the orthonormal basis $\{(|00\rangle - i|11\rangle)/\sqrt{2}, (|01\rangle - i|10\rangle)/\sqrt{2}, (|10\rangle - i|01\rangle)/\sqrt{2}, (|11\rangle - i|00\rangle)/\sqrt{2}\}$.

$\hat{J}^\dagger$ gate is present to ensure that the classical game is a subset of the quantum one. This is achieved by selected an entangling operator that commutes with the direct product of any pair of classical strategies, $\hat{I}$ or $\hat{F}$. In the quantum game it is the expectation value of the players' payoffs that is important. For Alice (Bob) we can write

$$\langle \$ \rangle = \$_{00}|\langle\psi_f|00\rangle|^2 + \$_{01}|\langle\psi_f|01\rangle|^2 + \$_{10}|\langle\psi_f|10\rangle|^2 + \$_{11}|\langle\psi_f|11\rangle|^2 \tag{2.6}$$

where $\$_{ij}$ is the payoff for Alice (Bob) associated with the game outcome $ij,\ i,j \in \{0,1\}$. If both players apply classical strategies the quantum game provides nothing new. However, if the players adopt quantum strategies the entanglement provides the opportunity for the players' moves to interact in ways with no classical analogue.

A maximally entangling operator $\hat{J}$, for an N player two strategy game, may be written, without loss of generality (Benjamin and Hayden 2001b), as

$$\hat{J} = \frac{1}{\sqrt{2}}(\hat{I}^{\otimes N} + i\hat{\sigma}_x^{\otimes N}). \tag{2.7}$$

An equivalent form of the entangling operator that permits the degree of entanglement to be controlled by a parameter $\gamma \in [0, \pi/2]$ is

$$\hat{J} = \exp\left(i\frac{\gamma}{2}\hat{\sigma}_x^{\otimes N}\right), \tag{2.8}$$

with maximal entanglement corresponding to $\gamma = \pi/2$.

Unitary quantum strategies are any $\hat{U} \in \mathrm{SU}(2)$:

$$\hat{U}(\theta,\alpha,\beta) = \begin{pmatrix} e^{i\alpha}\cos(\theta/2) & ie^{i\beta}\sin(\theta/2) \\ ie^{-i\beta}\sin(\theta/2) & e^{-i\alpha}\cos(\theta/2) \end{pmatrix}, \tag{2.9}$$

where $\theta \in [0,\pi]$ and $\alpha, \beta \in [-\pi,\pi]$. The strategies $\tilde{U}(\theta) \equiv \hat{U}(\theta,0,0)$ are equivalent to classical mixtures between the identity and bit-flip operations. When Alice plays $\tilde{U}(\theta_{\mathrm{A}})$ and Bob plays $\tilde{U}(\theta_{\mathrm{B}})$ the payoffs are separable functions of θ_{A} and θ_{B} and we have nothing more than could be obtained from the classical game by employing mixed strategies.

In quantum Prisoners' Dilemma a player with access to quantum strategies can always do at least as well as a classical player. If cooperation is associated with the $|0\rangle$ state and defection with the $|1\rangle$ state, then the strategy "always cooperate" is $\hat{C} \equiv \tilde{U}(0) = \hat{I}$ and the strategy "always defect" is $\hat{D} \equiv \tilde{U}(\pi) = \hat{F}$. Against a classical Alice playing $\tilde{U}(\theta)$ a quantum Bob can play Eisert's "miracle" move[3]

$$\hat{M} = \hat{U}(\frac{\pi}{2}, \frac{\pi}{2}, 0) = \frac{i}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{2.10}$$

that yields a payoff of $\langle \$_{\mathrm{B}} \rangle = 3 + 2\sin\theta$ for Bob while leaving Alice with only $\langle \$_{\mathrm{A}} \rangle = (1 - \sin\theta)/2$. In this case the dilemma is removed in favour of the quantum player. In the partially entangled case, there is a critical value of the entanglement parameter $\gamma = \arcsin(1/\sqrt{5})$, below which the quantum player should revert to the classical dominant strategy $\hat{D}$ to ensure a maximal payoff (Eisert et al. 1999). At the critical level of entanglement there is a phase change-like transition between the quantum and classical domains of the game (Du et al. 2001b, Du et al. 2003c). A detailed examination of critical entanglements in 2×2 quantum games of Eisert's scheme is presented in Chapter 5.

In a space of restricted, or two-parameter, quantum strategies corresponding to setting $\beta = 0$ in Eq. (2.9), Eisert demonstrates that there is a new NE with both players adopting the strategy

$$\hat{Q} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{2.11}$$

The payoff to both players is three, the same as mutual cooperation. This NE has the property of being PO, thus resolving the dilemma. Unfortunately there appears to be no *a priori* justification to restricting the space of quantum operators to those of with $\beta = 0$. Indeed, the two-parameter set is not closed under composition. This has not stopped a number of authors investigating the properties of various quantum games restricted to two-parameter strategies (Iqbal and Toor 2001c, Du et al. 2002a, Özdemir et al. 2003, Shimamura et al. 2003).

With the full set of three-parameter quantum strategies every strategy has a counter strategy that yields the opponent the maximum payoff of five, while the player is left with the minimum of zero (Benjamin and Hayden 2001a). The mathematical interchange symmetry of the Schmidt decomposition of a pure entangled, two qubit state shared between two parties leads to a physical symmetry amongst the actions of the parties (Lo and Popescu 2001). That is, on the maximally entangled state, any local unitary carried out by Alice on her qubit is equivalent to a local unitary that Bob carries out on his. In

[3]There are some notational differences to Eisert et al. (1999). In the current work we select $\hat{D} = \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)$ instead of $\hat{D} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. This necessitates a corresponding change in $\hat{J}$, allowing for an easier generalization of the entanglement operator to multiplayer games. The only affect on the game outcome is a possible rotation of $|\psi_f\rangle$ in the complex plane that is not physically observable.

terms of our notation, $\forall\, \hat{A} = \hat{U}(\theta, \alpha, \beta)\ \exists\, \hat{B} = \hat{U}(\theta, \alpha, -\frac{\pi}{2} - \beta)$ such that

$$(\hat{A} \otimes \hat{I}) \frac{1}{\sqrt{2}}(|00\rangle + i|11\rangle) = (\hat{I} \otimes \hat{B}) \frac{1}{\sqrt{2}}(|00\rangle + i|11\rangle). \tag{2.12}$$

So for any strategy $\hat{U}(\theta, \alpha, \beta)$ chosen by Alice, Bob has the counter $\hat{D}\hat{U}(\theta, -\alpha, \frac{\pi}{2} - \beta)$, essentially "undoing" Alice's move and then defecting. Hence there is no equilibrium in pure quantum strategies.

We still have a (non-unique) NE amongst mixed quantum strategies (Eisert and Wilkins 2000). The idea is that Alice's strategy consists of choosing one of the pair of moves

$$\hat{A}_1 = \hat{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{A}_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{2.13}$$

with equal probability, while Bob counters by selecting one of the corresponding pair of optimal answers

$$\hat{B}_1 = \hat{D} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \hat{B}_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{2.14}$$

with equal probability. The combinations of strategies $\{A_i, B_j\}$ provide Bob with the maximum payoff of five and Alice with the minimum of zero when $i = j$, while the payoffs are reversed when $i \neq j$. The expectation value of the payoffs for each player is then the average of P_{CD} and P_{DC}, or $5/2$. There is a continuous set of NE of this type, where Alice and Bob each play a pair of moves with equal probability, namely

$$\begin{aligned} \hat{A}_1 = \hat{U}(\theta, \alpha, \beta), \ \ \hat{A}_2 = \hat{U}(\theta, \frac{\pi}{2} + \alpha, \frac{\pi}{2} + \beta), \\ \hat{B}_1 = \hat{U}(\pi - \theta, \frac{\pi}{2} + \beta, \alpha), \ \ \hat{B}_2 = \hat{U}(\pi - \theta, \pi + \beta, \frac{\pi}{2} + \alpha). \end{aligned} \tag{2.15}$$

If other values of the payoffs were chosen in Eq. (2.2), while still retaining the conditions for a classical Prisoners' Dilemma[4], the average quantum NE payoff may be below (as is the case here) or above that of mutual cooperation (Benjamin and Hayden 2001a). In the latter case the conflict between the NE and the PO outcome has disappeared, while in the former we have at least an improvement over the classical NE result of mutual defection.

A physical realization of a quantum Prisoners' Dilemma with Eisert's scheme has been achieved on a two qubit nuclear magnetic resonance computer, with various degrees of entanglement ranging from a separable (i.e., classical) game, to a maximally entangled

[4]A Prisoners' Dilemma is characterized by the payoffs for the first player being in the order $\$_{\rm DC} > \$_{\rm CC} > \$_{\rm DD} > \$_{\rm CD}$, and with $\$_{\rm CC} > (\$_{\rm DC} + \$_{\rm CD})/2$, where, for example, the subscript DC means Alice defects and Bob cooperates.

one (Du et al. 2002b). Good agreement between theory and experiment was obtained. There is also a proposed implementation of the game on an optical quantum computer (Zhou and Kuang 2003).

The prescription provided by Eisert et al. is a general one that can be applied to any 2×2 game. Extensions to larger strategic spaces and additional players are considered in Sec. 2.4. A possible classification scheme for 2×2 games in the Eisert model is given by Huertas-Rosero (2004). Issues that have been studied in this model include ESS (Iqbal and Toor 2001c), decoherence (Chen et al. 2003b, Flitney and Abbott 2004a, Flitney and Abbott 2005), quantum versus classical players (Piotrowski and Sładowski 2003c, Flitney and Abbott 2003a, Cheon 2004), and differences between classical and quantum correlations (Özdemir et al. 2003, Shimamura et al. 2003).

A related protocol is that of Marinatto and Weber (2000). Their scheme differs from Eisert's in the omission of the $\hat{J}^\dagger$ gate and by restricting the players' strategies to probabilistic mixtures of the identity and bit-flip operations. Their scheme effectively chooses an initial state of $(|00\rangle + |11\rangle)/\sqrt{2}$, upon which the players act with a mixture of $\hat{I}$ and $\hat{\sigma}_x$. The classical game is reproduced when the initial state is chosen to be $|00\rangle$. Other authors have generalized this model to an arbitrary initial state:

$$|\psi_i\rangle = c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle, \tag{2.16}$$

subject to the normalization condition $\sum |c_{ij}|^2 = 1$. Since a player's strategy can be specified by a single parameter the scheme has the benefit of simplicity, but it does not exploit the full range of quantum operations. A number of authors have used the scheme to study various 2×2 games (Iqbal and Toor 2002a, Mendes 2002, Toyota 2003, Nawaz and Toor 2004a), ESS (Iqbal and Toor 2001a, 2001c, 2002b, 2004c) , three player games (Iqbal and Toor 2002c) and a repeated Prisoners' Dilemma (Iqbal and Toor 2002e). A generalization of Eisert's scheme which includes the model of Marinatto and Weber has been proposed by Nawaz and Toor (2004b). The new scheme has two values of the entanglement parameter γ, with the final state of the players' qubits generated by

$$|\psi_f\rangle = \hat{J}^\dagger(\gamma_2)(\hat{A} \otimes \hat{B})\hat{J}(\gamma_1)|\psi_i\rangle. \tag{2.17}$$

The model of Marinatto and Weber (2000) is reproduced when $\gamma_2 = 0$, while Eisert's scheme results when $\gamma_1 = \gamma_2$.

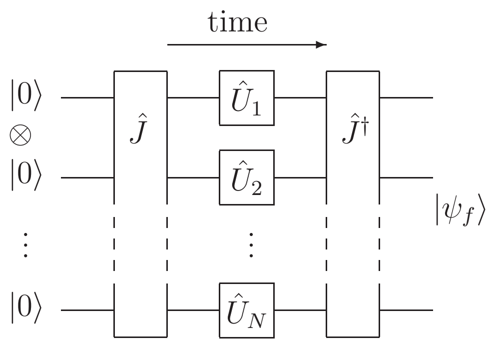


Figure 2.3. Protocol for an N-person quantum game. A protocol for an N-person quantum game, where $\hat{U}_j$ is the move of the jth player and $\hat{J}$ is an entangling gate (Benjamin and Hayden 2001b).

2.4 Larger strategic spaces

The field of quantum games has been extended to multiplayer games and games with more than two classical strategies. As situations become more complex there is more flexibility in the method of quantization. Additional players are easily accommodated in Eisert's protocol by the addition of qubits to the initial state and of extra player operators, as first discussed by Benjamin and Hayden (2001b) in a scheme inspired by N. F. Johnson. The scheme is shown in Figure 2.3. The entanglement operator of Eq. (2.7) creates maximal entanglement between the players' qubits.

Several authors have examined three and four player quantum games (Benjamin and Hayden 2001b, Kay et al. 2001, Du et al. 2002a, Du et al. 2002d, Han et al. 2002b). These offer a greater richness of equilibria than two player games. For example, it is possible to construct a Prisoners' Dilemma-like three handed game that has a NE in pure quantum strategies that is either better or worse than the classical one (Benjamin and Hayden 2001b).

A game where entanglement can be exploited particularly effectively is the multiplayer Minority game. The players have the choice of selecting either zero or one. If they select the least popular choice they are rewarded. No reward is given if the numbers are balanced. Classically the players can do no better than making a random selection, and the situation is not improved in the three player quantum version. In the four player classical game half the time there is no minority, so each player wins on average only one time in eight. However, entanglement in the quantum version allows us to avoid this outcome and provides a NE which rewards each player with probability one quarter, twice the classical average (Benjamin and Hayden 2001b).

A way of implementing multiplayer games with only two particle entanglement has been suggested by Chen et al. (2002). In this model, each pair of players, or just neighbouring players, share a maximally entangled pair of qubits.

The appearance of cooperation in multiplayer games is a feature of classical game theory. Attempts have been made to consider this in the quantum realm (Iqbal and Toor 2002c, Ma et al. 2002).

Games with more than two classical strategies can be modeled by replacing the qubits representing the players' decisions by an n-state quantum system (or qunit) for the n-choice case. The space of unitary quantum strategies is expanded from SU(2) to SU(n).

The childhood game of rock-scissors-paper, where the players have three choices, has been examined by Iqbal and Toor (2002d). However, to make the game amenable to analysis, the authors do not allow the players the full range of unitary operations, but rather restrict the strategies to mixtures of $\hat{I}$ and two operators that involve the interchange of a pair of states. Entanglement still provides for an enrichment over the classical game.

Another three-strategy game that has been examined is the Monty Hall problem, the subject of Chapter 3. There are three distinct quantizations in the literature (Li et al. 2001, Flitney and Abbott 2002c, D'Ariano et al. 2002). Chen et al. (2003a) consider $n_1 \times n_2$ quantum games with a restricted strategic space akin to a generalization of the scheme of Marinatto and Weber (2000) to multiple strategies.

2.5 Other models

Apart from the ideas considered above, there have been a variety of other quantum game-theoretic investigations. These include games that do not involve entanglement (Du et al. 2002c, Grib and Parfionov 2002, Liu and Sun 2002), games of incomplete information (Han et al. 2002a), continuous variable quantum games (Li et al. 2002), and a game that involves EPR-type correlated spins (Iqbal 2004, Iqbal and Weigert 2004) that departs from the models most commonly considered in the literature. A new representation of game theory that encompasses both classical and quantum games (Wu 2004b) has been used to create new quantum versions of the Battle of the Sexes (Wu 2004a) and Prisoners' Dilemma (Wu 2004c).

Some of the mathematical methods of physics have attracted the attention of economists and a new branch of economic mathematics has appeared, known as econophysics. Polish theorists Piotrowski and Sładkowski have proposed a quantum-like approach to economics with its roots in quantum game theory (Piotrowski 2003, Piotrowski and Sładkowski 2001, 2002a, 2002b, 2002c, 2002d, 2002e, 2003b, 2003c, 2004b, Sładkowski 2003). This, of course, must be distinguished from attempts to use the mathematical machinery of quantum field theory to solve *classical* financial market problems (Ilinski 2001, Baaquie 2001, Bonnet et al. 2004). In the new quantum market games, transactions are described in terms of projective operations acting on Hilbert spaces of strategies of traders. A quantum strategy represents a superposition of trading actions and can achieve outcomes not realizable by classical means (Piotrowski and Sładowski 2002d). Furthermore, quantum mechanics has features that can be used to model aspects of market behavior. For example, traders observe the actions of other players and adjust their actions accordingly, so there is non-commutativity of bidding (Piotrowski and Sładowski 2001), maximal capital flow at a given price corresponds to entanglement between buyers and sellers (Piotrowski and Sładowski 2002e), and so on.

Parrondo's paradox, or Parrondo's games, arise when two games that are losing when played in isolation can be played in a combination to form an overall winning game (Harmer and Abbott 1999a, Harmer and Abbott 1999b). There has been much interest in creating quantum versions of Parrondo's games (Meyer 2002, Flitney et al. 2002, Flitney and Abbott 2003c, Ng and Abbott 2004), along with the suggestion that they can possibly be utilized to increase efficiency of quantum algorithms (Lee et al. 2002, Lee and Johnson 2002a). Quantum Parrondo's games are the subject of Chapter 7.

There has been some criticism of quantum games with claims that both Meyer's quantum Penny Flip and Eisert's quantum Prisoners' Dilemma are not truly quantum mechanical (van Enk 2000, van Enk and Pike 2002). In the first case, it is true that the strategy of the quantum player can be implemented classically, however, any classical implementation would scale exponentially with an increase in the size of the Hilbert space, unlike a quantum implementation (Meyer 2000). In the case of the two-parameter quantum Prisoners' Dilemma, van Enk and Pike (2002) consider this equivalent to a new classical game with three strategies C, D and Q, and as a result the {Q, Q} NE does not address the dilemma in the original game. In addition, the sharing of an entangled state blurs the distinction between cooperative and non-cooperative games. While these criticisms have some merit, there is still the issue of efficient implementation of the game and they miss the main reason for studying quantum games, which is not as another model for classical game situations but as a model for competitive scenarios involving quantum information or quantum control.

2.6 Summary

Game theory is the mathematical theory of decision making in competitive situations. The new field of quantum game theory is the extension of game theory into the quantum realm. A protocol for two player, two strategy quantum games has been discussed with indications of how this can be extended to more players and larger strategic spaces. Examples of the various quantum game-theoretic investigations in the literature have been given. In general, the quantum representation of a classical game is not unique, but all contain the original classical game as a subset. The full set of quantum operations can be represented by trace-preserving, completely-positive maps. The possibilities where those operations are not unitary, such as the use of ancillas and the performance of measurements, remain little explored.

Quantization of a game can lead to either the appearance or disappearance of Nash equilibria. The much enhanced strategic space available to the players makes the quantum game more efficient than its classical counter part (Lee and Johnson 2003a). For example, the gap between the Pareto optimal outcome and the Nash equilibrium in the Prisoners' Dilemma is reduced or eliminated, and the average payoff in a multiplayer Minority game is increased, when players are permitted to use (mixed) quantum strategies. There are no NE in the space of pure quantum strategies in an entangled, fair, 2×2 quantum game. However, general results for many player or repeated games remain to be discovered.

Although there is some controversy surrounding the exact nature of quantum games, there is in any case much to learn about the behaviour of the interaction of qubits and quantum information from quantum game-theoretic models.

Chapter 3

Quantum Version of the Monty Hall Problem

THE Monty Hall problem is based around a game show that has a surprising and counterintuitive optimum strategy. The problem has a long history in one form or another but only received public attention in the early 1990s, arousing great passions even amongst faculty mathematicians many of whom were guilty of the same misunderstandings of probability theory as the general public! In this chapter the classical Monty Hall problem is briefly explained. Then we explore how the solution is affected when quantum probability amplitudes are substituted for classical probabilities and player actions are carried out using quantum operators. Without entanglement, the quantum version offers nothing that cannot be achieved in the classical setting using mixed strategies. However, with entanglement one player can gain an advantage by having access to quantum strategies when the other does not. When both players can utilize quantum strategies there is no equilibrium in pure strategies but there is a NE in mixed strategies that gives the same average payoff as the classical game.

The version presented here is one of three distinct quantum versions of the problem to appear in the literature.

Prize behind door	1	2	3	1	2	3
Alice opens door	2 or 3	3	2	2 or 3	3	2
Bob's initial selection	1	1	1	1	1	1
Bob's strategy	switch			do not switch		
Bob's final selection	3 or 2	2	3	1	1	1
Result	lose	win	win	win	lose	lose

Table 3.1. Monty Hall problem. Without loss of generality, suppose Bob's initial choice is door one. Dependent upon the position of the prize, the table shows the actions of Alice and Bob and the result of the game. In the right hand half Bob decides not to switch and in the left hand half he switches.

3.1 The Monty Hall problem

In the Monty Hall game show the host (Alice) secretly selects one door of three behind which to place a prize. The contestant (Bob) chooses a door. Alice then opens a different door showing that the prize is not behind it. Bob now has the option of sticking with his current selection or changing to the untouched door. He wins the prize if he selects the correct door. An early published reference to this problem, presented in terms of three prisoners, one of whom is to be paroled, appeared in the Mathematical Games column of *Scientific American* (Gardner 1959). The optimum strategy for Bob is to alter his choice. Surprisingly this doubles his chance of winning from $\frac{1}{3}$ to $\frac{2}{3}$ (vos Savant 1990, Gillman 1992), as Table 3.1 demonstrates.[5]

The classical Monty Hall problem has generated much interest and controversy (vos Savant 1991, vos Savant 1996) because it is sharply counterintuitive. Also from an informational viewpoint it illustrates the case where an apparent null operation does indeed provide information about the system.

3.2 Quantization scheme

One published attempt at a quantum version of the Monty Hall problem (Li et al. 2001) is briefly described as follows: there is one quantum particle and three boxes $|0\rangle$, $|1\rangle$, and

[5]The seemingly paradoxical nature of the solution is emphasized further in a seven door variant of the problem. Bob chooses three doors. Alice then opens three of the remaining doors to show that the prize is not behind them and offers Bob the choice to switch to the one untouched door or retain his selection of three doors. Switching still improves the odds!

$|2\rangle$. Alice selects a superposition of boxes for her initial placement of the particle and Bob then selects a particular box. The authors make this a fair game by introducing an additional particle entangled with the original one and allowing Alice to make a quantum measurement on this particle as a part of her strategy. If a suitable measurement is taken after a box is opened it can have the result of changing the state of the original particle in such a manner as to "redistribute" the particle evenly between the other two boxes. In the original game Bob has a $\frac{2}{3}$ chance of picking the correct box by altering his choice, but with this change Bob has $\frac{1}{2}$ probability of being correct by either staying or switching.

A second group quantized the Monty Hall problem with the use of an ancillary system, or notepad, used by the host (D'Ariano et al. 2002). In this version the position of the prize is the main quantum variable. It lies in a three-dimensional Hilbert space $\mathcal{H}$, known as the game space. The position of the prize is prepared quantum mechanically and some information about this preparation is recorded in the notepad. Bob's choice of "door" is a one-dimensional projection p on $\mathcal{H}$. Alice then chooses a one-dimensional projection q and makes a von Neumann measurement with projections q and $\mathbb{I} - q$, effectively collapsing the game space to the two-dimensional space $(\mathbb{I} - q)\mathcal{H}$. The constraints on q are that it be orthogonal to p (i.e., a different "door") and that it does not reveal the position of the prize. The notepad is used to ensure the latter. Bob can now choose a one-dimensional projection p' on $(\mathbb{I}-q)\mathcal{H}$ and the corresponding measurement on the game space is carried out to establish whether the prize has been won.

Below, the original Monty Hall problem is quantized directly, without the use of ancillas, and the host and contestant are both permitted access to quantum strategies. The choices of Alice and Bob are represented by qutrits[6] that are initialized in some state to be specified later. Their strategies are operators acting on their respective qutrit. A third qutrit is used to represent the box "opened" by Alice. That is, the the state of the system can be expressed as

$$|\psi\rangle = |oba\rangle, \tag{3.1}$$

where a = Alice's choice of box, b = Bob's choice of box, and o = the box that has been opened. The initial state of the system is designated as $|\psi_i\rangle$. The final state of the system is

$$|\psi_f\rangle = \hat{B}' \, \hat{\mathrm{O}} \, (\hat{I} \otimes \hat{B} \otimes \hat{A})|\psi_i\rangle, \tag{3.2}$$

[6] A qutrit is the three state generalization of a qubit—a system whose state is a member of a three-dimensional Hilbert space (Caves and Milburn 2000).

where $\hat{A}$ = Alice's choice operator, $\hat{B}$ = Bob's initial choice operator, $\hat{\mathrm{O}}$ = the opening box operator, $\hat{B}' = \hat{S}$ (Bob's switch operator) or $\hat{N}$ (Bob's no-switch operator), and $\hat{I}$ = the identity operator. Bob can be permitted a mixed strategy on his second move, that is, selecting $\hat{S}$ with probability $\cos^2\gamma$ and $\hat{N}$ with probability $\sin^2\gamma$, $\gamma \in [0, \frac{\pi}{2}]$. We shall call the final state produced when Bob chooses $\hat{S}$, $|\psi_f^{\mathrm{S}}\rangle$, and when Bob chooses $\hat{N}$, $|\psi_f^{\mathrm{N}}\rangle$. It is necessary for the initial state to contain a designation for an open box, but this should not be taken literally since it does not make sense in the context of the game. The initial state of the open box is fixed as $|0\rangle$.

The open box operator is a unitary operator that can be written

$$\hat{\mathrm{O}} = \sum_{ijk\ell} |\epsilon_{ijk}|\, |njk\rangle\langle \ell jk| \; + \; \sum_{j\ell} |mjj\rangle\langle \ell jj|, \tag{3.3}$$

where $|\epsilon_{ijk}| = 1$, if i, j, k are all different and 0 otherwise, $m = (j + \ell + 1) \pmod 3$, and $n = (i + \ell) \pmod 3$. The second term applies to states where Alice would have a choice of box to open and is one way of providing a unique algorithm for this choice[7]. Here and later the summations are over the range $0, 1, 2$. We should not consider $\hat{\mathrm{O}}$ to be the literal action of opening a box and inspecting its contents, which would constitute a measurement, but rather it is an operator that marks a box by setting the o qutrit in such a way that it is anti-correlated with Alice's and Bob's choices. The coherence of the system is maintained until the final stage when the payoff is determined by a measurement on $|\psi_f\rangle$.

Bob's switch operator can be written as

$$\hat{S} = \sum_{ijk\ell} |\epsilon_{ij\ell}|\, |i\ell k\rangle\langle ijk| \; + \; \sum_{ij} |iij\rangle\langle iij|. \tag{3.4}$$

The second term is not relevant to the mechanics of the game but is added to ensure unitarity. Both $\hat{\mathrm{O}}$ and $\hat{S}$ map each state in the computational basis to a unique basis state.

$\hat{N}$ is the identity operator on the three-qutrit state. The $\hat{A} = (a_{ij})$ and $\hat{B} = (b_{ij})$ operators can be selected by the players to operate on their choice of box (that has some initial value to be specified later) and are restricted to members of SU(3). Bob also selects the parameter γ that controls the mixture of staying or switching.

[7]The operator is written this way to ensure unitarity. However, we are only interested in states where the initial value of the opened box is $|0\rangle$, i.e., $\ell = 0$. The results for the opened box are inconsistent with the rules of the game if $\ell = 1$ or 2.

It is the expectation value of the payoff that is most important. Bob wins if he picks the correct box, hence

$$\langle \$_{\mathrm{B}} \rangle = \cos^2\gamma \sum_{ij} |\langle ijj|\psi_f^{\mathrm{S}}\rangle|^2 + \sin^2\gamma \sum_{ij} |\langle ijj|\psi_f^{\mathrm{N}}\rangle|^2 \tag{3.5}$$

Alice wins if Bob is incorrect, so $\langle \$_{\mathrm{A}} \rangle = 1 - \langle \$_{\mathrm{B}} \rangle$.

3.3 Results

The scheme presented in the previous section is akin to that of Marinatto and Weber (2000) where there is no entangling operator just a specification of an initial state that may involve entanglement. The unentangled and maximally entangled initial states are considered below.

3.3.1 Unentangled initial state

Suppose the initial state of Alice's and Bob's choices is an equal mixture of all possible states with no entanglement:

$$|\psi_i\rangle = |0\rangle \otimes \frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |2\rangle) \otimes \frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |2\rangle). \tag{3.6}$$

Then

$$\begin{aligned}
\hat{\mathrm{O}}(\hat{I} \otimes \hat{B} \otimes \hat{A})|\psi\rangle &= \frac{1}{3}\sum_{ijk} |\epsilon_{ijk}|\,(b_{0j} + b_{1j} + b_{2j})(a_{0k} + a_{1k} + a_{2k})\,|ijk\rangle \\
&\quad + \frac{1}{3}\sum_{j} (b_{0j} + b_{1j} + b_{2j})(a_{0j} + a_{1j} + a_{2j})\,|mjj\rangle; \\
\hat{S}\hat{\mathrm{O}}(\hat{I} \otimes \hat{B} \otimes \hat{A})|\psi_i\rangle &= \frac{1}{3}\sum_{ijk} |\epsilon_{ijk}|\,(b_{0j} + b_{1j} + b_{2j})(a_{0k} + a_{1k} + a_{2k})\,|ikk\rangle \\
&\quad + \frac{1}{3}\sum_{jk} |\epsilon_{jkm}|\,(b_{0j} + b_{1j} + b_{2j})(a_{0j} + a_{1j} + a_{2j})\,|mkj\rangle,
\end{aligned} \tag{3.7}$$

where $m = (j+1) \pmod 3$. This gives

$$\begin{aligned}
\langle \$_{\mathrm{B}} \rangle &= \frac{1}{9}\cos^2\gamma \sum_{jk} (1 - \delta_{jk})\,|b_{0j} + b_{1j} + b_{2j}|^2 |a_{0k} + a_{1k} + a_{2k}|^2 \\
&\quad + \frac{1}{9}\sin^2\gamma \sum_{j} |b_{0j} + b_{1j} + b_{2j}|^2 |a_{0j} + a_{1j} + a_{2j}|^2.
\end{aligned} \tag{3.8}$$

If Alice chooses to apply the identity operator, which is equivalent to her choosing a mixed classical strategy where each of the boxes is chosen with equal probability, Bob's payoff is

$$\langle \$_{\mathrm{B}} \rangle = \left(\frac{2}{9} \cos^2 \gamma + \frac{1}{9} \sin^2 \gamma \right) \sum_j |b_{0j} + b_{1j} + b_{2j}|^2. \tag{3.9}$$

Unitarity of B implies that

$$\begin{aligned} \sum_k |b_{ik}|^2 &= 1 \qquad \text{for } i = 0, 1, 2, \\ \text{and } \sum_k b_{ik}^* b_{jk} &= 0 \qquad \text{for } i, j = 0, 1, 2 \text{ with } i \neq j, \end{aligned} \tag{3.10}$$

which means that the sum in Eq. (3.9) is identically 3. Thus,

$$\langle \$_{\mathrm{B}} \rangle = \frac{2}{3} \cos^2 \gamma + \frac{1}{3} \sin^2 \gamma, \tag{3.11}$$

which is the same as the payoff for a classical mixed strategy where Bob chooses to switch with a probability of $\cos^2 \gamma$ (payoff $\frac{2}{3}$) and not to switch with probability $\sin^2 \gamma$ (payoff $\frac{1}{3}$).

The situation is not changed where Alice uses a quantum strategy and Bob is restricted to applying the identity operator (leaving his choice as an equal superposition of the three possible boxes). Then Bob's payoff becomes

$$\langle \$_{\mathrm{B}} \rangle = \left(\frac{2}{9} \cos^2 \gamma + \frac{1}{9} \sin^2 \gamma \right) \sum_j |a_{0j} + a_{1j} + a_{2j}|^2, \tag{3.12}$$

which, using the unitarity of A, gives the same result as Eq. (3.11).

If both players have access to quantum strategies, Alice can restrict Bob to at most $\langle \$_{\mathrm{B}} \rangle = \frac{2}{3}$ by choosing $\hat{A} = \hat{I}$, while Bob can ensure an average payoff of at least $\frac{2}{3}$ by choosing $\hat{B} = \hat{I}$ and $\gamma = 0$ (switch). Hence this is the NE of the quantum game and it gives the same results as the classical game. The NE is not unique. Bob can also choose either of

$$\hat{R}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{or} \quad \hat{R}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{3.13}$$

for his first move, which amount to a shuffling of his choice, and then switch on his second.

It should not be surprising that the quantum strategies produced nothing new in this case since there was no entanglement in the initial state. This is in keeping with the findings in 2×2 quantum games (Eisert et al. 1999).

3.3.2 Maximally entangled initial state

Suppose

$$|\psi_i\rangle = |0\rangle \otimes \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle), \tag{3.14}$$

representing maximum entanglement between the choices of Alice and Bob. Now

$$\begin{aligned}\hat{\mathrm{O}}(\hat{I}\otimes\hat{B}\otimes\hat{A})|\psi_i\rangle &= \frac{1}{\sqrt{3}}\sum_{ijk\ell}|\epsilon_{ijk}|\, b_{\ell j}a_{\ell k}\,|ijk\rangle + \frac{1}{\sqrt{3}}\sum_{j\ell} b_{\ell j}a_{\ell j}\,|mjj\rangle;\\ \hat{S}\hat{\mathrm{O}}(\hat{I}\otimes\hat{B}\otimes\hat{A})|\psi_i\rangle &= \frac{1}{\sqrt{3}}\sum_{ijk\ell}|\epsilon_{ijk}|\, b_{\ell j}a_{\ell k}\,|ikk\rangle + \frac{1}{\sqrt{3}}\sum_{jk\ell}|\epsilon_{jkm}|\, b_{\ell j}a_{\ell j}\,|mkj\rangle,\end{aligned} \tag{3.15}$$

where again $m = (j+1) \pmod 3$. This gives

$$\begin{aligned}\langle \$_{\mathrm{B}}\rangle &= \frac{1}{3}\sin^2\gamma \sum_j |b_{0j}a_{0j} + b_{1j}a_{1j} + b_{2j}a_{2j}|^2\\ &\quad + \frac{1}{3}\cos^2\gamma \sum_{jk}(1-\delta_{jk})\,|b_{0j}a_{0k} + b_{1j}a_{1k} + b_{2j}a_{2k}|^2.\end{aligned} \tag{3.16}$$

First consider the case where Bob is limited to a classical mixed strategy. For example, setting $\hat{B} = \hat{I}$ is equivalent to the classical strategy of selecting any of the three boxes with equal probability. Bob's payoff is then

$$\begin{aligned}\langle \$_{\mathrm{B}}\rangle &= \frac{1}{3}\sin^2\gamma\,(|a_{00}|^2 + |a_{11}|^2 + |a_{22}|^2)\\ &\quad + \frac{1}{3}\cos^2\gamma\,(|a_{01}|^2 + |a_{02}|^2 + |a_{10}|^2 + |a_{12}|^2 + |a_{20}|^2 + |a_{21}|^2).\end{aligned} \tag{3.17}$$

Alice can then make the game fair by selecting an operator whose diagonal elements all have an absolute value of $\frac{1}{\sqrt{2}}$ and whose off-diagonal elements all have absolute value $\frac{1}{2}$. One such SU(3) operator is

$$\hat{E} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2}\\ -\frac{1}{2} & \frac{3-i\sqrt{7}}{4\sqrt{2}} & \frac{1+i\sqrt{7}}{4\sqrt{2}}\\ \frac{-1-i\sqrt{7}}{4\sqrt{2}} & \frac{-3+i\sqrt{7}}{8} & \frac{5+i\sqrt{7}}{8}\end{pmatrix}. \tag{3.18}$$

This yields a payoff to both players of $\frac{1}{2}$, whether Bob chooses to switch or not.

The situation where Alice is limited to the identity operator (or any other classical strategy) is uninteresting. Bob can achieve a payoff of 1 by setting $\hat{B} = \hat{I}$ and then not switching. The correlation between Alice's and Bob's choice of boxes remains, so Bob is assured of winning. Bob also wins if he applies $\hat{R}_1$ or $\hat{R}_2$ and then switches.

As noted in Sec. 2.3 every quantum strategy has a counterstrategy. That is, for any strategy $\hat{A}$ chosen by Alice, Bob has the counter $\hat{A}^*$:

$$\begin{aligned}(\hat{A}^* \otimes \hat{A})\frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle) &= (\hat{I} \otimes \hat{A}^\dagger \hat{A})\frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle) \\ &= \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle).\end{aligned} \tag{3.19}$$

The correlation between Alice's and Bob's choices remains, so Bob can achieve a unit payoff by not switching.

Similarly for any strategy $\hat{B}$ chosen by Bob, Alice can ensure a win by countering with $\hat{A} = \hat{B}^*$ if Bob has chosen $\gamma = 0$, while a $\gamma = 1$ strategy is defeated by $\hat{B}^*\hat{R}$, where $\hat{R}$ is $\hat{R}_1$ or $\hat{R}_2$ given in Eq. (3.13). As a result there is no NE amongst pure quantum strategies. Note that Alice can also play a fair game, irrespective of the value of γ provided she knows $\hat{B}$, by choosing $\hat{B}^*\hat{E}$, giving an expected payoff of $\frac{1}{2}$ to both players. An NE amongst mixed quantum strategies can still be found. Where both players choose to play $\hat{I}$, $\hat{R}_1$ or $\hat{R}_2$ with equal probabilities neither player can gain an advantage over the classical payoffs. If Bob chooses to switch all the time, when he has selected the same operator as Alice, he loses, but the other two times out of three he wins. Not switching produces the complementary payoff of $\langle \$_\mathrm{B} \rangle = \frac{1}{3}$, so the situation is analogous to the classical game.

3.4 Summary

A quantum version of the interesting game show situation known as the Monty Hall problem has been presented. Direct comparison of results with the other quantum versions in the literature are problematic since the models are quite different. The version presented here is, however, the one most closely resembling the classical version of the problem to which comparisons have been made. In our model, where both participants have access to quantum strategies maximal entanglement in the initial state produces the same payoffs as the classical game for any mixed strategy of switching or not-switching. That is, for the Nash equilibrium strategy the contestant wins two-thirds of the time by switching. If the host, Alice, has access to quantum strategies while the contestant, Bob, does not, the game is fair, since Alice can adopt a strategy with an expected payoff of $\frac{1}{2}$ for each player, while if Bob has access to quantum strategies and Alice does not he can win all the time. Without entanglement the quantum game confirms our expectations by offering nothing more than can be achieved using a mixed strategy in a classical setting.

Chapter 4

Quantum Truel

IN game theory, a popular model of a struggle for survival among three competing agents is a truel, a three person generalization of a duel. Truels contain many interesting game-theoretic problems. In this chapter a quantum scheme for duels and truels is presented. In the classical case, the outcome is sensitive to the precise rules under which the truel is performed and is often counterintuitive. These aspects carry over into our quantum scheme, but interference amongst the players' strategies can arise leading to game equilibria different from the classical case. Extension of the model to the N-player case and to truels with decoherence are discussed.

4.1 Introduction

A situation where there are three competing agents each trying to eliminate the others is described in game-theoretic terms by a truel. Such situations can arise, for example, in biology where there are three species competing for limited resources, or in economics where three companies are competing in a single market place. In the classic wild Western duel, two gunfighters shoot it out and the winner is the one left standing. This situation presents few game theoretic difficulties for the participants: shoot first and calculate the odds later is always the best strategy! When the scenario is generalized to three or more players the situation is more complex and an intelligent use of strategy can be beneficial. For example, consider the case where Alice, Bob and Charles decide to settle their differences with a sequential shoot out, firing in alphabetic order. Suppose Alice has a one-third chance of hitting, Bob two-thirds, and Charles never misses (see Figure 4.1).

It would seem clear that each player should target the opponent they would least like to face in a one on one duel. A superficial examination would suggest that missing a turn by firing in the air would serve no purpose.

Indeed, Bob and Charles are advised to both target their most dangerous opponent: each other. Clearly Alice does not want to hit Bob with her first shot since then she is automatically eliminated by Charles. Surprisingly, Alice is better off abstaining (or firing in the air) in the first round. She then gets the first shot in the resulting duel, a fact that compensates for her poorer marksmanship. Precise results for this case are given below. The paradox of not wanting to fire can been seen most clearly when all three protagonists are perfect shots. Alice is advised not to shoot since after she eliminates one of the others she automatically becomes the target for the third. Unless this is the last round, Bob prefers not to fire as well for the same reason. If there is an unlimited number of rounds, no one wants to be the first to eliminate an opponent. The result is a paradoxical stalemate where all survive.

4.2 The classical truel

In the literature various rules for truels are explored. Firing can be simultaneous or sequential in a fixed or random order, firing into the air can be permitted or not, and the amount of ammunition can be fixed or unlimited. In the current discussion the following assumptions are made:

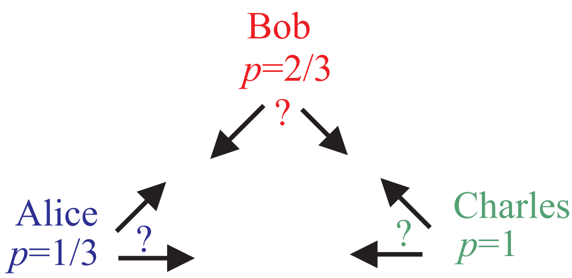


Figure 4.1. Schematic of a truel. A truel between Alice, Bob and Charles. In the game Alice can shoot first, then Bob and then Charles. The firing continues clockwise until only one player survives. The probability of a hit is shown beneath the player names.

- Each player strictly prefers survival over non-survival. Without loss of generality we assign a utility of one to a sole survivor and zero to any eliminated players.
- Each player prefers survival with the fewest co-players. That is, the utility of survival in a pair (u_2) or in a three-some (u_3) will satisfy the relation $0 < u_3 \leq u_2 \leq 1$.
- Alice, Bob and Charles have marksmanship (probability of hitting their chosen target) of $\bar{a} = 1 - a$, $\bar{b} = 1 - b$, $\bar{c} = 1 - c$, respectively, independent of their target and with $0 \leq a, b, c < 1$. There is no probability of hitting a person other than the one chosen.
- The players get no information on the others' strategies apart from knowing who has been hit, and in the quantum model not even that.
- Players fire sequentially in alphabetic order with firing into the air permitted.

An analysis of classical truels is provided by Kilgour for the simultaneous (Kilgour 1972). and the sequential case (Kilgour 1975) A non technical discussion is provided by Kilgour and Brams (1997). To get a flavour of some of Kilgour's results we shall consider the case where the poorest shot fires first and the best last ($\bar{a} < \bar{b} < \bar{c}$) and ammunition is unlimited. First, the expectation value of Alice's payoff in a duel between Alice and Bob, with each having m bullets, is calculated (see Figure 4.2):

$$\langle \$_{\mathrm{A}} \rangle_m = 1 - a \, + \, ab \, \langle \$_{\mathrm{A}} \rangle_{m-1}. \tag{4.1}$$

When $m \to \infty$, $\langle \$_A \rangle_m = \langle \$_A \rangle_{m-1}$, hence

$$\langle \$_{\mathrm{A}} \rangle_\infty = \frac{1 - a}{1 - ab}. \tag{4.2}$$

Note that $\langle \$_\text{B} \rangle = 1 - \langle \$_\text{A} \rangle$. Using this result, the expectation values for each player in a truel can be computed (see Figures 4.3 and 4.4). There are three important strategic mixes to consider depending on Alice's strategy. What ever Alice does, Bob is advised to shoot at Charles, since he is the one that Bob least wants to fight in a duel—and Charles, if he survives, similarly does best by shooting back at Bob. If Alice fires in the air on her first shot (or whenever both other players are alive) Alice is the sole survivor with probability

$$p_0 = \frac{1-a}{1-bc}\left[\frac{1-b}{1-ab} + \frac{b(1-c)}{1-ac}\right]. \tag{4.3}$$

If Alice shoots at Bob or Charles (when she has a choice) her resulting odds of survival are

$$\begin{aligned} p_1 &= \frac{1-a}{1-abc}\left[\frac{a(1-b)}{1-ab} + \frac{c(1-a)+ab(1-c)}{1-ac}\right], \\ p_2 &= \frac{1-a}{1-abc}\left[\frac{a(1-b)+b(1-a)}{1-ab} + \frac{ab(1-c)}{1-ac}\right], \end{aligned} \tag{4.4}$$

respectively. From the fact that $b > c$ it follows that $p_2 > p_1$ so Alice never fires at Bob while Charles is still alive. To make this example concrete, consider the case mentioned earlier: $a = \frac{2}{3}$, $b = \frac{1}{3}$ and $c = 0$. Then $p_0 = 25/63$ which is better than $p_2 = 59/189$ and $p_1 = 50/189$, meaning that Alice is advised to begin by shooting in the air and then to shoot at whoever is left standing after the first round. Surprisingly, even though Alice is the worst shot, this strategy will give her a better than one third probability of survival. Her advantage comes from the fact that she is not targeted until there is only a pair of players left and she gets the first shot in the resulting duel. In contrast, Charles has only a $\frac{2}{9}$ chance of emerging as the sole survivor even though he is a perfect shot! He has the disadvantage of shooting last and being the one that the others most want to eliminate. The results can be sensitive to a minor adjustment of the rules. For example, if the number of rounds is fixed, at some stage Alice may be better served by helping Bob to eliminate Charles, particularly if Bob is a poor marksman, even at the risk of not getting the first shot in a duel with Bob. However, the paradoxical disadvantage of being the best shot and the advantage of being the poorest are common to many truels.

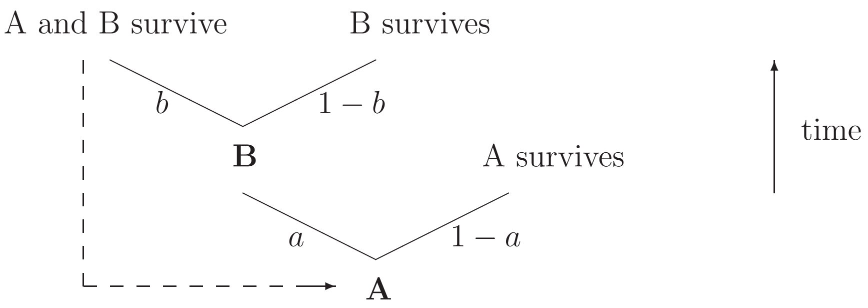


Figure 4.2. Game tree for a duel between Alice and Bob. Extensive form of a duel between Alice (A) and Bob (B). Left hand branches are misses, right hand branches are hits, with lower case letters indicating probabilities. If Alice and Bob both survive and there are further rounds, the tree repeats following the dashed line.

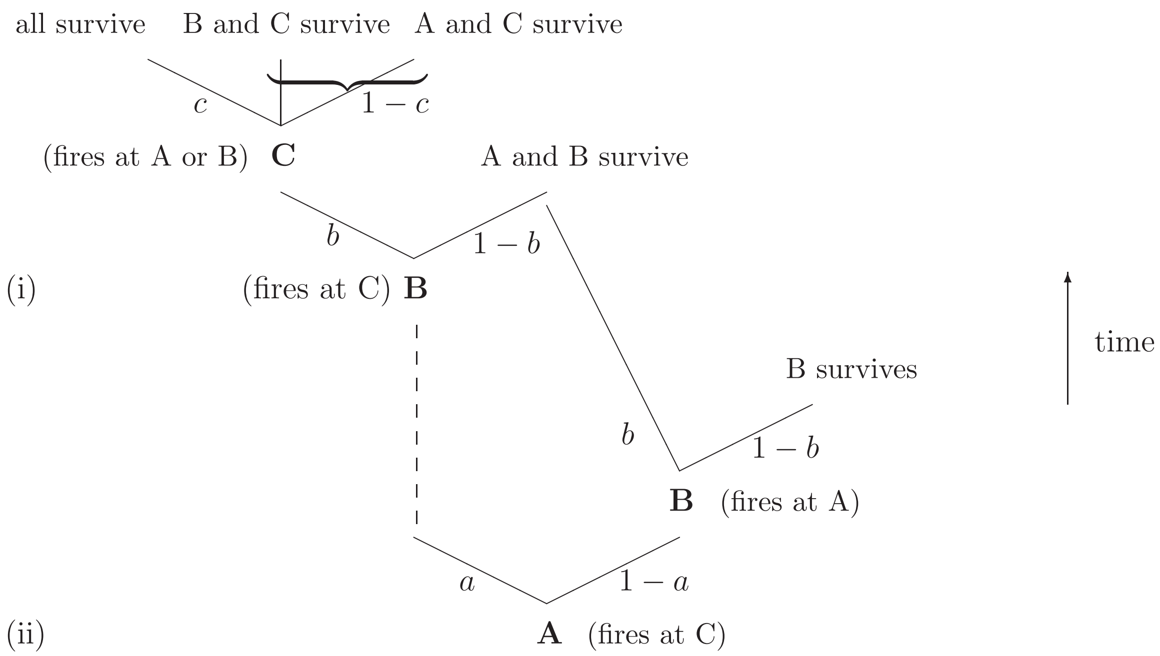


Figure 4.3. Game tree for a one shot truel. Alice (A), Bob (B) and Charles (C) fight a one-shot truel, where Alice fires (i) in the air or (ii) at Charles. Left hand branches are misses, right hand branches are hits, with lower case letters next to branches being probabilities. Case (ii) becomes equivalent to (i) if Alice misses, as indicated by the dashed line. Charles is indifferent as to his target.

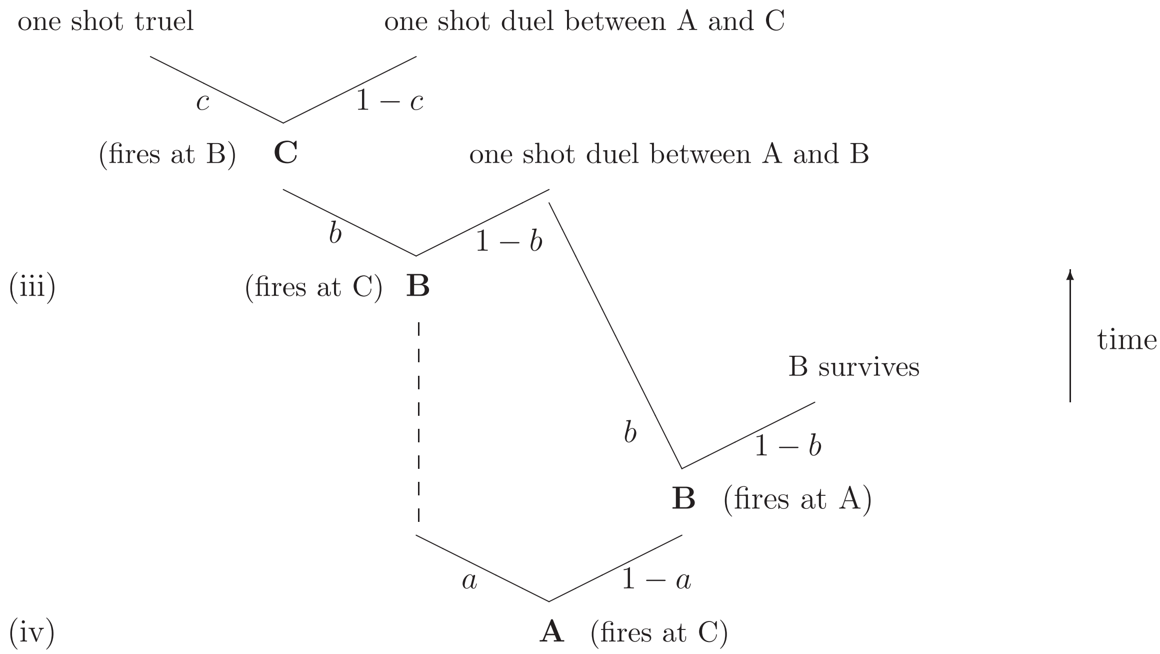


Figure 4.4. Game tree for a two-shot truel. Alice (A), Bob (B) and Charles (C) fight a two-shot truel, where Alice initially fires (iii) in the air or (iv) at Charles. Left hand branches are misses, right hand are hits, with lower case letters next to branches being probabilities. Case (iv) becomes equivalent to (iii) if Alice misses her first shot, as indicated. For a truel of $m > 2$ shots, the one shot duels become $m-1$ shot duels, and the one shot truel becomes an $m-1$ shot truel, with the tree being entered again from the base.

4.3 Quantization scheme

Although the Eisert protocol has become the standard in the literature for 2×2 quantum games, the quantization of more complex situations is less well established and is certainly not unique[8]. The following model of a quantum truel is presented. Each player has a qubit designating their state, with the computational basis states $|0\rangle$ and $|1\rangle$ representing "dead" and "alive," respectively. The combined state of the players is

$$|\psi\rangle = |q_{\rm A}\rangle \otimes |q_{\rm B}\rangle \otimes |q_{\rm C}\rangle = |q_{\rm A}\, q_{\rm B}\, q_{\rm C}\rangle, \tag{4.5}$$

with the initial state being $|\psi_i\rangle = |111\rangle$. In a quantum duel the third qubit is omitted. In a classical truel the players are located separately, however, in the quantum case the qubits

[8]For example, in Chapter 3 the case of the Monty Hall problem was discussed, for which there are three distinct quantum versions in the literature (Li et al. 2001, D'Ariano et al. 2002, Flitney and Abbott 2002c).

representing the states of the players need to be in the one location so that operations can be carried out on the combined state. We envisage, for example, a referee applying operators to $|\psi\rangle$ with the prior instruction of the players. The analogue of firing at an opponent is an attempt to flip an opponent's qubit using a unitary operator acting on $|\psi\rangle$. In a duel between Alice and Bob, the action of Alice "firing" at Bob with a probability of success of $\bar{a} = \sin^2(\theta/2)$ can be represented, with maximum generality, by the operator

$$\begin{aligned} \hat{A}_{\mathrm{B}} = &\left[e^{-i\alpha}\cos(\theta/2)|11\rangle \,+\, ie^{i\beta}\sin(\theta/2)|10\rangle\right]\langle 11| \\ &+ \left[e^{i\alpha}\cos(\theta/2)|10\rangle \,+\, ie^{-i\beta}\sin(\theta/2)|11\rangle\right]\langle 10| \\ &+ |00\rangle\langle 00| + |01\rangle\langle 01|, \end{aligned} \tag{4.6}$$

where $\theta \in [0,\pi]$ determines the marksmanship and $\alpha, \beta \in [-\pi,\pi]$ are arbitrary phase factors. The last two terms of Eq. (4.6) result from the fact that Alice can do nothing if her qubit is in the $|0\rangle$ state. The operator for Bob "firing" at Alice, $\hat{B}_A$, is obtained by reversing the roles of the first and second qubits in Eq. (4.6). For a truel, similar expressions can be obtained with the third qubit being a spectator. For example,

$$\begin{aligned} \hat{A}_{\mathrm{B}} = \sum_j \Big\{ &\left[e^{-i\alpha}\cos(\theta/2)|11j\rangle \,+\, ie^{i\beta}\sin(\theta/2)|10j\rangle\right]\langle 11j| \\ &+ \left[e^{i\alpha}\cos(\theta/2)|10j\rangle \,+\, ie^{-i\beta}\sin(\theta/2)|11j\rangle\right]\langle 10j|\Big\} + \sum_{jk}|0jk\rangle\langle 0jk| \end{aligned} \tag{4.7}$$

is the operation of Alice "firing" at Bob. That is, Alice carries out a control-rotation of Bob's qubit with her qubit being the control (see Figure 4.5). Firing into the air is represented by the identity operator. For α, β and θ the subscripts A, B and C shall be used to refer to Alice, Bob and Charles, respectively. The operators in Eqs. (4.6–4.7) flip between $|0\rangle$ and $|1\rangle$ but do not invert a general $|\psi\rangle$. A general complementing operation in quantum mechanics cannot be achieved unitarily (Bužek et al. 1999, Pati 2001, Pati 2002). The truel shall be of a fixed number of rounds with the coherence of the state being maintained until a measurement is taken on the final state. Partial decoherence at each step, where the players obtain some information about the state of the system, is a possible extension of our scheme, to be considered in Sec. 4.7. Expectation values for the payoffs to Alice, Bob and Charles are, respectively,

$$\begin{aligned} \langle \$_{\mathrm{A}} \rangle &= |\langle 100|\psi_f\rangle|^2 \,+\, u_2(\,|\langle 110|\psi_f\rangle|^2 + |\langle 101|\psi_f\rangle|^2) \,+\, u_3|\langle 111|\psi_f\rangle|^2, \\ \langle \$_{\mathrm{B}} \rangle &= |\langle 010|\psi_f\rangle|^2 \,+\, u_2(\,|\langle 110|\psi_f\rangle|^2 + |\langle 011|\psi_f\rangle|^2) \,+\, u_3|\langle 111|\psi_f\rangle|^2, \\ \langle \$_{\mathrm{C}} \rangle &= |\langle 001|\psi_f\rangle|^2 \,+\, u_2(\,|\langle 101|\psi_f\rangle|^2 + |\langle 011|\psi_f\rangle|^2) \,+\, u_3|\langle 111|\psi_f\rangle|^2. \end{aligned} \tag{4.8}$$

In what follows, we shall take the utility of surviving in a pair to be $u_2 = \frac{1}{2}$ and the utility of surviving in a trio to be $u_3 = \frac{1}{3}$, so that the combined payoff of any outcome is one.

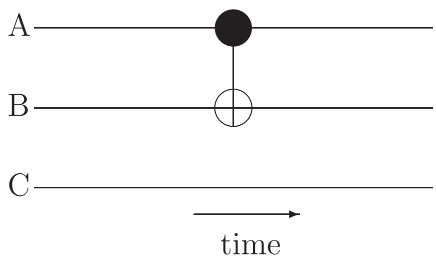


Figure 4.5. Quantum circuit for Alice "firing" at Bob. Diagram representing the operation of Alice "firing" at Bob in a quantum truel. The solid lines indicate the flow of information (qubits) and $\oplus$ is a logical NOT operation that is only applied if the control qubit (filled circle) is $|1\rangle$.

We shall talk of a player being eliminated after a certain number of rounds if there is a probability of one that their qubit is in the $|0\rangle$ state. As distinct from the classical case, however, their qubit may subsequently be flipped back to $|1\rangle$, so the player has not been removed from the game. To play a quantum duel or truel, the players list the operators they are going to use in each round before the game begins. In the classical case, we made the assumption that the players have no information about the others' strategies except to know who has been hit. In the quantum case, since a measurement is not taken until the completion of the final round, the players do not even have this information. Thus there is no loss of generality in deciding at the start of the game the complete set of operators to be used.

4.4 Quantum duels

Consider a quantum duel between Alice and Bob. After m rounds the state of the system will be

$$|\psi_m\rangle = (\hat{B}_{\rm A}\hat{A}_{\rm B})^m|11\rangle. \tag{4.9}$$

After a single round it is easy to see that a measurement taken at this stage will not give results any different from the classical duel with $a = \cos^2(\theta_{\rm A}/2)$ and $b = \cos^2(\theta_{\rm B}/2)$. After two rounds some interference effects can be seen:

$$\begin{aligned}
|\langle 01|\psi_2\rangle|^2 &= (1-b)\Big[ab(1+a) + (1-a)^2 \;+\; 2ab\sqrt{a}\,\cos(\alpha_{\rm A}+2\alpha_{\rm B}) \\
&\qquad -2a(1-a)\sqrt{b}\,\cos(2\alpha_{\rm A}+\alpha_{\rm B}) \;-\; 2(1-a)\sqrt{ab}\,\cos(\alpha_{\rm A}-\alpha_{\rm B})\Big], \\
|\langle 10|\psi_2\rangle|^2 &= a(1-a)\Big(1+b+2\sqrt{b}\,\cos(2\alpha_{\rm A}+\alpha_{\rm B})\Big), \\
|\langle 11|\psi_2\rangle|^2 &= 1 - |\langle 01|\psi_2\rangle|^2 - |\langle 10|\psi_2\rangle|^2.
\end{aligned} \tag{4.10}$$

The last line is a result of the fact that there is no possibility of the $|00\rangle$ state. The expectation value for Alice's payoff can be written as

$$\langle \$_{\mathrm{A}} \rangle = \frac{1}{2}(1 + |\langle 10|\psi_2\rangle|^2 - |\langle 01|\psi_2\rangle|^2), \tag{4.11}$$

with Bob receiving $1 - \langle \$_{\mathrm{A}} \rangle$. The value of a and b will determine which of the cosine terms Alice (or Bob) wishes to maximize. For example, with $a = \frac{2}{3}$ and $b = \frac{1}{2}$ Alice's payoff is maximized for $\alpha_{\mathrm{A}} = \pm\pi/3$, $\alpha_{\mathrm{B}} = \mp 2\pi/3$ or $\alpha_{\mathrm{A}} = \pm\pi$, $\alpha_{\mathrm{B}} = 0$ while Bob's is maximized for $\alpha_{\mathrm{A}} = 0$, $\alpha_{\mathrm{B}} = \pm\pi$ or $\alpha_{\mathrm{A}} = \pm 2\pi/3$, $\alpha_{\mathrm{B}} = \mp\pi/3$ (see Figure 4.6). If the players have discretion over the phase factors, a maximin strategy for the two round duel is for the players to select $\alpha_{\mathrm{A}} = \alpha_{\mathrm{B}} = \pm\pi/3$ in which case the game is balanced. The situation for longer duels is more complex. A classical duel with $a = \frac{2}{3}$ and $b = \frac{1}{2}$ gives each player a one third chance of eliminating their opponent in the first round, with a one-third chance of mutual survival from which the process repeats itself. Hence the duel is fair, irrespective of the number of rounds. Alice's opportunity to fire first compensates for her poorer marksmanship. Figure 4.7 indicates Alice's payoff for the quantum case as a function of the number of rounds. The result is affected by the values of α_{A} and α_{B} but not by β_{A} or β_{B}.

The fact that a measurement is not taken until the completion of the game and that the operators are unitary (hence reversible) means that a $|0\rangle$ state can be unwittingly flipped back to a $|1\rangle$. Thus it may be advantageous for one or other player not to target their opponent. Consider the situation where Alice fires in the air on her second shot:

$$|\psi_2'\rangle = \hat{B}_{\mathrm{A}}\hat{B}_{\mathrm{A}}\hat{A}_{\mathrm{B}}|11\rangle. \tag{4.12}$$

Then

$$\begin{aligned} |\langle 01|\psi_2'\rangle|^2 &= 2ab(1-b)(1+\sin(2\alpha_{\mathrm{B}})), \\ |\langle 10|\psi_2'\rangle|^2 &= 1-a. \end{aligned} \tag{4.13}$$

If a is sufficiently small (i.e., Alice has a high probability of flipping Bob's qubit) then she would prefer this result. A similar effect holds for Bob if b is small. Paradoxically, if Alice is a poor shot (approximately $a > \frac{4}{5}$) and Bob is intermediate ($b \approx \frac{1}{2}$), Alice should refrain from taking a second shot at Bob as indicated in Figure 4.8.

4.5 Quantum truels

In contrast to the classical case, players' decisions are not contingent on the success or otherwise of previous shots. Since coherence of the system is maintained until the

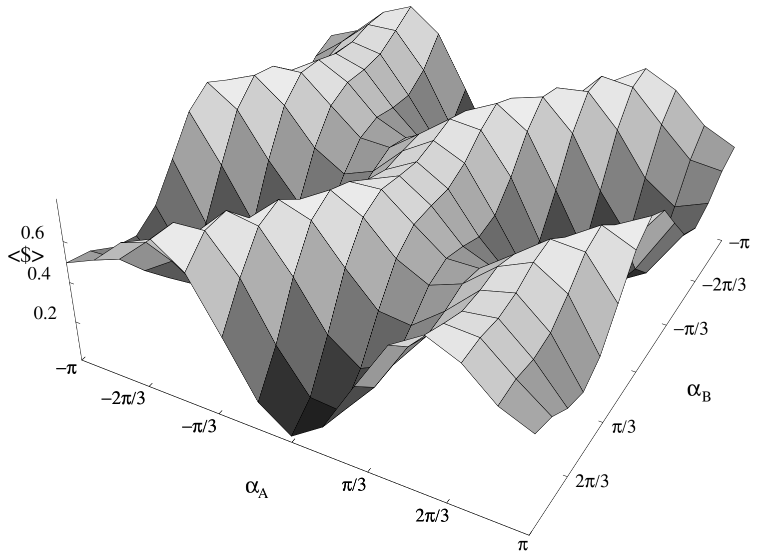


Figure 4.6. Expectation of Alice's payoff in a two shot quantum duel as a function of phases. The expectation value of Alice's payoff in a two shot quantum duel with Bob, as a function of $\alpha_{\rm A}$ and $\alpha_{\rm B}$, when the probability of Alice and Bob missing are $a = \frac{2}{3}$ and $b = \frac{1}{2}$, respectively. The values of $\beta_{\rm A}$ and $\beta_{\rm B}$ have no effect. The α_k and β_k are the phase factors from the operator in Eq. (4.6) with the subscript 1 referring to Alice and 2 to Bob.

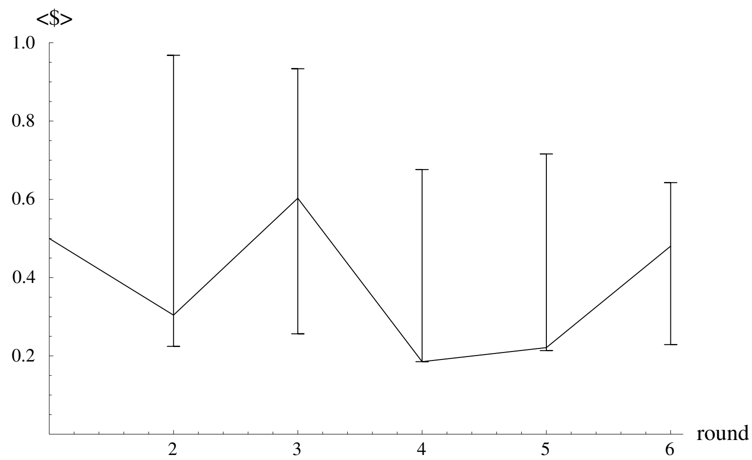


Figure 4.7. Expectation value of Alice's payoff in a repeated quantum duel. The curve shows the expectation value of Alice's payoff in a repeated quantum duel with $a = \frac{2}{3}$, $b = \frac{1}{2}$ and $\alpha_k = \beta_k = 0$. The vertical lines indicate the range of possible payoffs over all values of $\alpha_{\rm A}$ and $\alpha_{\rm B}$. The values of $\beta_{\rm A}$ and $\beta_{\rm B}$ have no effect. For comparison, a classical duel with the same marksmanship gives Alice and Bob equal chances (payoffs are $\frac{1}{2}$).

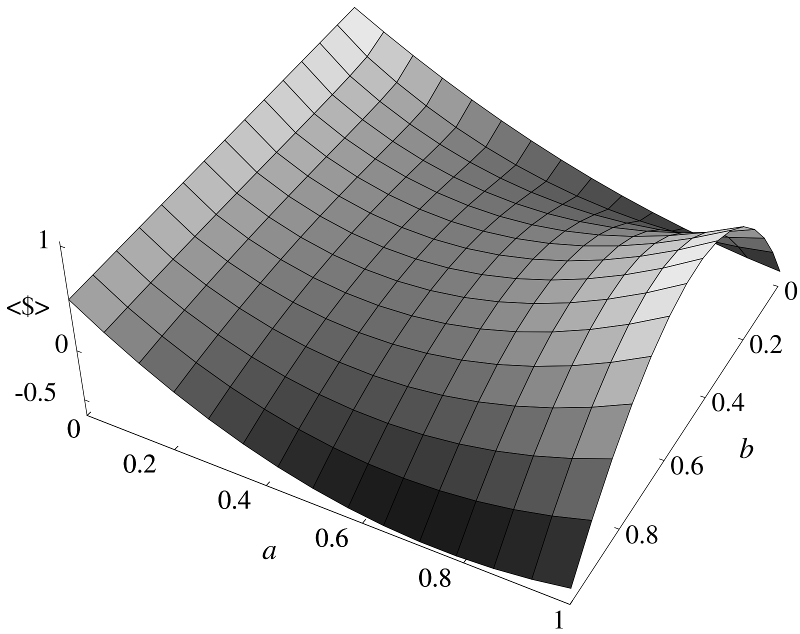


Figure 4.8. Improvement in Alice's payoff in a two shot quantum duel if she chooses to shoot in the air on her second shot. Consider a two shot quantum duel between Alice and Bob with probabilities of a miss of a and b, respectively, and all phase factors zero. The plot shows the improvement in Alice's expected payoff if she chooses to fire in the air on her second shot. When the value is positive Alice does better by adopting this strategy.

completion of the final round, decisions can only be based on the amplitudes of the various states that the players are able to compute under different assumptions as to the others' strategies. The strategies of the other players may be inferred by reasoning that all players are acting in their rational self-interest. This idea will guide the following arguments.

In a quantum truel, interference effects may arise in the first round if two players choose the same target. To make the calculations tractable set $\alpha_k = \beta_k = 0$; $k \in$ A,B,C and consider only the case $a > b > c$. Bob and Charles reason as in the classical case and target each other. Knowing this, what should Alice do? If she targets Charles the resulting state after one round is

$$|\psi_1\rangle = (c_{\rm A}c_{\rm B} - s_{\rm A}s_{\rm B})(c_{\rm C}|111\rangle + s_{\rm C}|101\rangle) + (c_{\rm A}s_{\rm B} + c_{\rm B}s_{\rm A})|110\rangle, \tag{4.14}$$

where $c_k \equiv \cos(\theta_k/2)$ and $s_k \equiv \sin(\theta_k/2)$. The probability that Charles survives the combined attentions of Alice and Bob is $(c_{\rm A}c_{\rm B} - s_{\rm A}s_{\rm B})^2$, compared to the classical case where the probability would be $ab = (c_{\rm A}c_{\rm B})^2$. There is much less incentive for Alice to fire in the air since, unlike the classical case, Bob does not change his strategy (to target Alice) depending on the result of Alice's operation. If $\theta_{\rm A}$ and $\theta_{\rm B}$ are around $\pi/2$ then $c_{\rm A}c_{\rm B} \approx s_{\rm A}s_{\rm B}$ and both Alice and Bob will like the result of Eq. (4.14) since Charles has a high probability of being eliminated.

For example, consider the case mentioned in Sec. 4.2 where $a = (c_{\rm A})^2 = \frac{2}{3}$, $b = (c_{\rm B})^2 = \frac{1}{3}$ and $c = (c_{\rm C})^2 = 0$. If both Alice and Bob target Charles, he is eliminated with certainty in the first round and consequently his strategy is irrelevant! If there are sufficient rounds Alice would appear to be in difficulty in the resulting two person duel since her marksmanship is half that of Bob's. In a repeated quantum duel where both players continue firing this is indeed the case. However, quantum effects come to her rescue if Alice fires in the air on her third shot. The expectation value of her payoff after three rounds is then improved from 0.448 to 0.761. Indeed, Bob's survival chances are diminished to such an extent that he is advised to fire in the air on the second and subsequent rounds. We then reach an equilibrium where it is to the disadvantage of both players to target the other. Alice emerges with the slightly better prospects ($\langle \$_{\rm A}\rangle = 0.554$) since she has had two shots to Bob's one. As a result of being able to "restore" a player to life (i.e., flip $|0\rangle \to |1\rangle$) this quantum example is in marked contrast to classical two person duels where it is never an advantage to fire in the air.

Now, compare this to the option of Alice firing in the air in the first round. With Bob and Charles targeting each other and Charles being a perfect shot, after the first round the amplitude of states where both survive is zero. Since Bob fired first and has better than 50% chance of success, the $|110\rangle$ state will have a larger amplitude than the $|101\rangle$ state, so Alice reasons that it is better for her to target Bob in the second round. Since only Bob or Charles can have survived the first round they each (if alive) target Alice in the second.[9] After two rounds the resulting state is

$$\begin{aligned} |\psi_2\rangle &= (\hat{C}_{\rm A}\hat{B}_{\rm A}\hat{A}_{\rm B})(\hat{C}_{\rm B}\hat{B}_{\rm C})|111\rangle \\ &= \frac{1}{\sqrt{27}}(-\sqrt{6}|001\rangle - \sqrt{8}|010\rangle - \sqrt{6}|100\rangle - i|011\rangle + i\sqrt{4}|110\rangle + \sqrt{2}|111\rangle). \end{aligned} \tag{4.15}$$

Before the start of the game Alice calculates that if she survives the first two rounds there is a 50% chance she is the sole survivor. If she now targets one of the others in the third round she is more likely to flip a $|0\rangle$ state to a $|1\rangle$ than the reverse, hence she fires in the

[9] Recall that coherence of the state is maintained until the completion of the final round, when a measurement reveals who has been hit. This means that all targeting decisions can be made prior to the first round.

air. The argument for Bob and Charles to do likewise for the same reason is even more compelling. Hence, even with a large number of rounds, all players choose to fire in the air after the second round. The resulting payoffs are $\langle \$_{\mathrm{A}} \rangle = 52/162$, $\langle \$_{\mathrm{B}} \rangle = 67/162$ and $\langle \$_{\mathrm{C}} \rangle = 43/162$. Alice clearly prefers to fire at Charles in the first round over this strategy. It is rare in a quantum truel that Alice will opt to fire in the air in the first round. This is in contrast to the classical situation where this is often the weakest player's best strategy.

In situations where one player is not eliminated with certainty, an equilibrium where all three players prefer to fire in the air will generally arise. Each player reasons that to fire at an opponent would increase the amplitude of the $|1\rangle$ state of their target.

4.5.1 One- and two-shot truel

To clarify some of the differences between the classical and quantum truels, consider the simple cases of one- and two-shot truels where Charles is a perfect shot. Where Charles is indifferent as to the choice of target, he uses a fair coin to decide. In the quantum case, Charles will use this method to select his desired operator before any operations are carried out on $|\psi_i\rangle$. For tractability, $\alpha_k = \beta_k = 0$ is assumed.

In the one-shot case, Charles is Bob's only threat so Bob will fire at Charles. Alice may be targeted by Charles so may wish to help Bob, particularly if he is a poor shot. Because of interference, this strategy is more likely to be preferred in the quantum case. The regions of the parameter space (a, b) where Alice should select one strategy over the other are indicated in Figure 4.9—this figure is of interest because it illustrates a case where going from a classical to a quantum regime changes a linear boundary in the probability parameter space into a convex one and such convexity is being intensely studied as it is the basis of Parrondo's paradox (Harmer and Abbott 2002).

The situation is more complex in the two-shot case. When $a > b$, in the first round Bob and Charles again target each other, while Alice either fires in the air or at Charles. Since only one of Bob and Charles survive the first round, they both (if alive) target Alice in the second. In the classical game, Alice's target in the second round is determined since she knows whom of Bob or Charles remains. However, in the quantum case this is unknown and Alice can only base her decision on maximizing the expectation value of her payoff. The regions of the parameter space (a, b) where Alice prefers the different strategies are given in Figure 4.10.

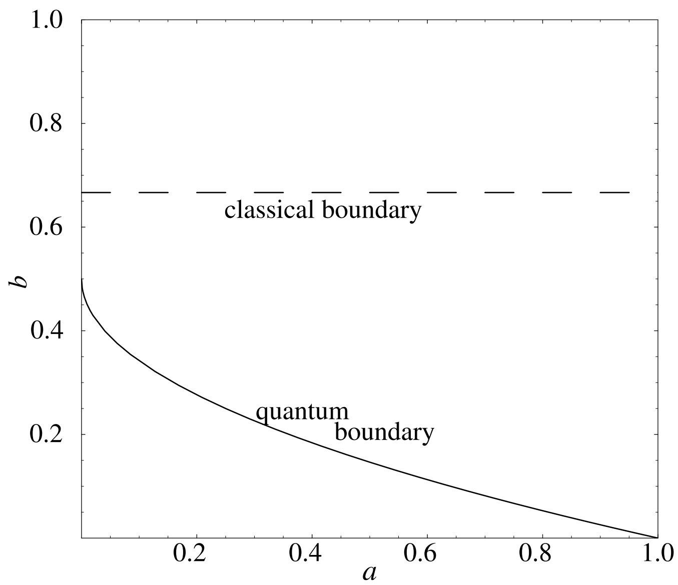


Figure 4.9. Alice's preferred strategy in a one shot quantum truel with Alice being the poorest shot. In a one-shot truel with $c = 0$, Alice's preferred strategy depending on the values of a and b: Alice fires in the air if (a, b) is below the line (solid line for the quantum case, dashed line for the classical case) and at Charles, if above. The quantum boundary is half a parabola whose equation is $a = (1 - 2b)^2$.

If $b > a$, Charles will target Alice in the first round since she is his most dangerous opponent. Likewise, Bob targets Charles. In the second round, reasoning as above, both Alice and Charles (if alive) will target Bob. In the classical case the only strategic choice is whether Alice fires at Charles or into the air in the first round. In the quantum case Bob has a decision to make in the second round since he does not know for certain who was hit in the first. Figure 4.11 shows the regions of parameter space corresponding to the optimal choices of Alice and Bob.

A classical truel where the players do not know which others have been eliminated may be a fairer comparison to the quantum situation. This alters the regions corresponding to the players' optimal strategies, but there are still differences with the quantum truel as a result of interference in the latter case.

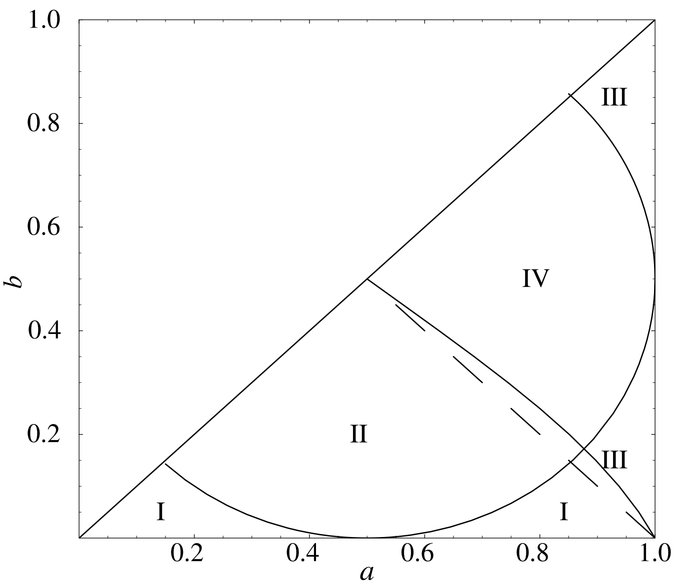


Figure 4.10. Alice's preferred strategy in a two shot quantum truel with Alice being the poorest shot. The figure shows the parameter space (a, b) divided in into regions corresponding to Alice's possible optimal strategies in a two-shot truel with $a > b > c = 0$. The optimal strategy also depends on whether the game is classical or quantum. Classical: I and II, fire into the air and then at the survivor of round one; III and IV, fire at Charles and then at the survivor of round one. Quantum: I, fire into the air and then at Bob; II, fire at Charles both times; III, fire at Charles and then at Bob; IV, fire into the air and then at Charles. The boundary between regions I and III, or II and IV, is the curved line in the classical case and the dashed line in the quantum case.

4.6 Quantum N-uels

A quantum N-uel can be obtained by adding qubits to the state $|\psi\rangle$ in Eq. (4.5):

$$|\psi\rangle = |q_1\rangle \otimes |q_2\rangle \otimes \ldots \otimes |q_N\rangle, \tag{4.16}$$

where $|q_j\rangle$ is the qubit of player j. The players' operators are the same as Eq. (4.7) except with additional spectator qubits. For example, the first player firing at the second is carried out by

$$\begin{aligned}\hat{A}_{\mathrm{B}} = \sum_{j_3,\ldots,j_N} \Big\{ & \big[e^{-i\alpha}\cos(\theta/2)|11j_3\ldots j_N\rangle + ie^{i\beta}\sin(\theta/2)|10j_3\ldots j_N\rangle\big]\langle 11j_3\ldots j_N| \\ & + \big[e^{i\alpha}\cos(\theta/2)|10j_3\ldots j_N\rangle + ie^{-i\beta}\sin(\theta/2)|11j_3\ldots j_N\rangle\big]\langle 10j_3\ldots j_N|\Big\} \\ & + \sum_{j_2,\ldots,j_N} |0j_2\ldots j_N\rangle\langle 0j_2\ldots j_N|,\end{aligned} \tag{4.17}$$

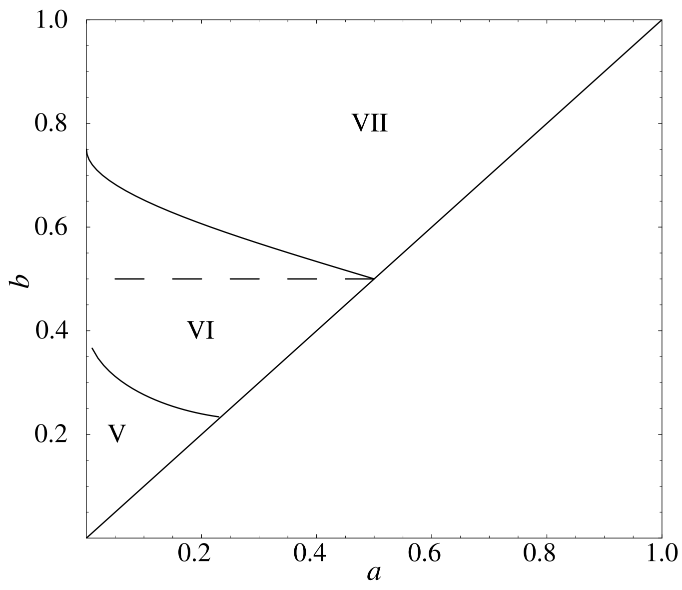


Figure 4.11. Alice and Bob's preferred strategy in a two shot quantum truel with Bob being the poorest shot. The figure shows the parameter space (a, b) divided in into regions corresponding to the possible optimal strategies of Alice and Bob in a two-shot truel with $b > a > c = 0$. The optimal strategy also depends on whether the game is classical or quantum. Classical: in the first round, Alice fires in the air if $b < \frac{1}{2}$ or at Charles if $b > \frac{1}{2}$. Quantum: V, Alice fires into the air in round one and Bob fires at Charles in round two; VI and VII, Alice fires at Charles in round one and Bob fires at Alice (VI) or Charles (VII) in round two.

where the j_i take the values 0 or 1.

The features of the quantum N-uel are the same as those of the quantum truel. Positive and negative interference arising from multiple players choosing a common target is more likely and equilibria where it is to the advantage of all (surviving) players to shoot into the air still arise.

4.7 Classical-quantum correspondence

In the classical case, players are removed from the game once hit. Maintained coherence through out the quantum game weakens the analogy with classical truel, since players can be brought back to "life," that is, have their qubit flipped from $|0\rangle$ to $|1\rangle$. However, there is still a correspondence. During the game, a player can only fire if their qubit is in the $|1\rangle$ state, and they receive a zero payoff at the end of the game if their qubit is in the $|0\rangle$ state. The classical-quantum correspondence can be enhanced by introducing partial decoherence after each move and allowing the players to choose their strategy dynamically depending on the result of previous rounds. In this case, the classical situation is reproduced in the limit of full decoherence. If $\rho = |\psi\rangle\langle\psi|$ is the density operator of the system in state $|\psi\rangle$, one way of effecting partial decoherence is by

$$\rho \to (1-p)\rho \;+\; p\,\mathrm{diag}(\rho), \tag{4.18}$$

where $0 \le p \le 1$. This is equivalent to measuring the state of the system in the computational basis with probability p. When ρ is diagonal, the next player can select their target based on the measurement result. Figure 4.12 shows the regions of the parameter space (a, b) corresponding to Alice's preferred strategy in a one shot truel when Charles is a perfect shot (the situation of Figure 4.9). The boundary between Alice maximizing her expected payoff by firing into the air and targeting Charles depends on the measurement probability p. There is a smooth transition from the quantum case to the classical one as p goes from zero to one. Decoherence in quantum games has been considered[10] by Chen et al. (2003b) and Flitney and Abbott (2005). Chapter 6 considers this issue in detail.

4.8 Summary

A protocol for quantum duels, truels and N-uels has been presented. While the analogy with classical duels is not precise, interesting comparisons can still be made. A one round quantum duel is equivalent to the classical game, but in longer quantum duels the appearance of phase terms in the operators can greatly affect the expected payoff to the players. If players have discretion over the value of their phase factors a maximin choice can in principle be calculated provided the number of rounds is fixed. If one player has a restricted choice the other has a large advantage. The unitary nature of the operators means that the probability of flipping a "dead" state to an "alive" state is the same as that for the reverse, so it can be advantageous for a player to fire in the air rather than target the opponent, something that is never true in a classical duel. Indeed, an equilibrium can be reached where both players forgo targeting their opponent even if there are further rounds to play.

In a quantum truel, strategies are not contingent on earlier results. The players' entire strategy (the list of players to target in different rounds) can be mapped out in advance

[10] Johnson (2001) and Özdemir et al. (2004) consider a quantum game with an initial state corrupted by bit flip errors but do not consider decoherence during the game.

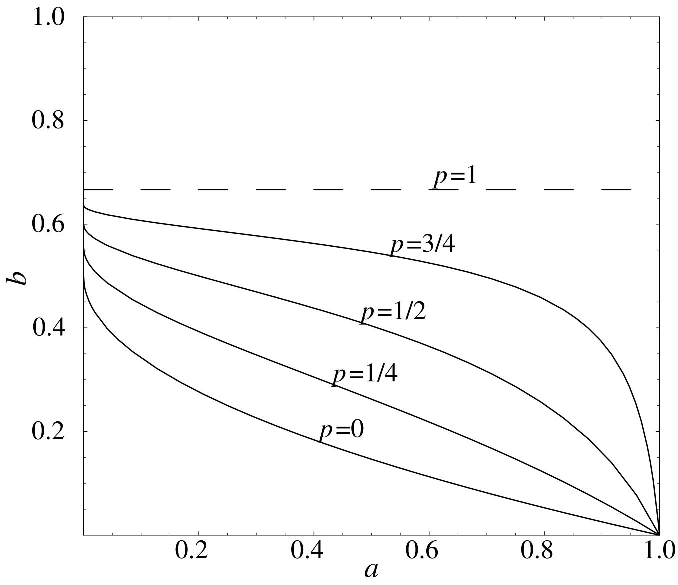


Figure 4.12. Alice's preferred strategy in a one-shot quantum truel with decoherence. In a one shot quantum truel with $c = 0$ and with decoherence, the figure shows the boundaries for different values of the measurement probability p below which Alice maximizes her expected payoff by firing into the air and above which by targeting Charles. There is a smooth transition from the fully quantum case $(p = 0)$ to the classical one $(p = 1)$.

based on the expected amplitudes of the various states resulting from different strategic choices by the players. Interference effects arise where one player is targeted by the other two, and can have dramatic consequences, either enhancing or diminishing the probability of survival of the targeted player compared to the classical case. As with the case of quantum duels, equilibria can arise where it is to the disadvantage of each player to target one of the others. Such equilibria arise only in special cases in a classical truel.

Introducing decoherence in the form of a measurement after each move changes the quantum game. As the measurement probability is increased from zero to one there is a smooth transition from the fully quantum game to the classical one.

Chapter 5

Advantage of a Quantum Player Over a Classical Player

IN the Eisert model of 2×2 quantum games it is known that a player with access to the full set of quantum strategies has an advantage over a player limited to the classical subset. In this chapter we quantify this advantage as a function of the degree of entanglement of the players' qubits. Several well known 2×2 games are considered, including the Prisoners' Dilemma, Chicken and the Battle of the Sexes, giving critical values of the entanglement parameter above which the quantum player's advantage becomes apparent. A list of "miracle" moves, or best moves for the quantum player against a classical player, is provided for arbitrary 2×2 quantum games of the Eisert model.

5.1 Introduction

The games that have generated the most discussion in the literature are those that pose some sort of dilemma, for example, where there is a conflict between multiple NE or where the NE, though a compelling response for the rational player, is less than optimal. We have already seen one such game, the Prisoners' Dilemma, in Chapter 2. A good non-technical discussion of various dilemmas in 2×2 games is given by Poundstone (1992) from which the names of the games treated in this chapter have been taken. Most of the classical games discussed appear in any introductory text on game theory. Table 5.1 gives payoff matrices for various 2×2 games. The payoff for the four possible outcomes are designated a, b, c and d, with $a > b > c > d$. Typical values for (a, b, c, d) for each game are given in the table and these shall be used whenever numerical results are desired. In all the games apart from the Battle of the Sexes, of the two classical strategies, one is helpful to the opposing player and is known as cooperation (C), while the other is damaging to the opposing player and is known as defection (D). In the quantum protocol these moves are represented by the $|0\rangle$ and $|1\rangle$ states, respectively.

5.2 Miracle moves

As noted in Chapter 2, in a maximally entangled 2×2 quantum game of the Eisert scheme, any pure quantum strategy $\hat{U}(\theta, \alpha, \beta)$, that is, a local unitary operation carried out on the player's own qubit, is equivalent to a different pure strategy $\hat{U}(\theta, \alpha, -\frac{\pi}{2} - \beta)$ carried out by the other player, as was seen in Eq. (2.12). Thus, when both players have access to the full set of quantum operators, if one player's strategy $\hat{U}(\theta, \alpha, \beta)$ is known, the other player can undo this operation by selecting $\hat{U}(\theta, \alpha, -\frac{\pi}{2} - \beta)^{-1} = \hat{U}(\theta, -\alpha, \frac{\pi}{2} - \beta)$. Indeed, by choosing a composite strategy $\hat{U}(\theta', \alpha', \beta')\,\hat{U}(\theta, -\alpha, \frac{\pi}{2} - \beta)$ any desired final state can be produced. The consequence of this for maximally entangled 2×2 quantum games is that there can be no NE amongst pure quantum strategies (Eisert and Wilkins 2000, Benjamin and Hayden 2001a).

The situation is more interesting when one player, say Alice, is restricted to $S_{\rm cl} \equiv \{\tilde{U}(\theta) : \theta \in [0, \pi]\}$ while the other, Bob, has access to $S_{\rm q} \equiv \{\hat{U}(\theta, \alpha, \beta) : \theta \in [0, \pi];\ \alpha, \beta \in [-\pi, \pi]\}$. Games with these strategy restrictions shall be referred to as classical–quantum games. Strategies in $S_{\rm cl}$ are "classical" in the sense that the player simply executes two classical moves with fixed probabilities and does not manipulate qubit phase. However, $\tilde{U}(\theta)$ only gives the same results as a classical mixed strategy when both players employ these

game	payoff matrix	NE payoffs	PO payoffs	condition	(a, b, c, d)
PD	$\begin{matrix} (b,b) & (d,a) \\ (a,d) & (c,c) \end{matrix}$	(c, c)	(b, b)	$2b > a + d$	$(5, 3, 1, 0)$
Chicken	$\begin{matrix} (b,b) & (c,a) \\ (a,c) & (d,d) \end{matrix}$	(a, c) or (c, a)	(b, b)	$2b > a + c$	$(4, 3, 1, 0)$
Deadlock	$\begin{matrix} (c,c) & (d,a) \\ (a,d) & (b,b) \end{matrix}$	(b, b)	(b, b)	$2b > a + d$	$(3, 2, 1, 0)$
Stag Hunt	$\begin{matrix} (a,a) & (d,b) \\ (b,d) & (c,c) \end{matrix}$	(a, a) or (c, c)	(a, a)		$(3, 2, 1, 0)$
BoS	$\begin{matrix} (a,b) & (c,c) \\ (c,c) & (b,a) \end{matrix}$	(a, b) or (b, a)	(a, b) or (b, a)		$(2, 1, 0)$

Table 5.1. Payoff matrices for various 2 $\times$ 2 games. A summary of payoff matrices with NE and PO results for various classical games. PD is the Prisoners' Dilemma and BoS is the Battle of the Sexes. In the matrices, the rows correspond to Alices's options of cooperation (C) and defection (D), respectively, while the columns are likewise for Bob's. In the parentheses, the first payoff is Alice's, the second Bob's and $a > b > c > d$. The condition specifies a constraint on the values of $a, b, c,$ and d necessary to create the dilemma. The final column gives standard values for the payoffs.

strategies. If Bob employs a quantum strategy he can exploit the entanglement to his advantage since only he can produce any desired final state by local operations on his qubit. Without knowing Alice's move, Bob's best plan is to play the "miracle" quantum move consisting of assuming that Alice has played $\tilde{U}(\pi/2)$, the median move from $S_{\rm cl}$, undoing this move by

$$\hat{V} = \hat{U}(\frac{\pi}{2}, 0, \frac{\pi}{2}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \tag{5.1}$$

and then preparing his desired final state. The operator

$$\hat{f} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{5.2}$$

has the property

$$(\hat{I} \otimes \hat{f}) \frac{1}{\sqrt{2}}(|00\rangle + i|11\rangle) = (\hat{F} \otimes \hat{I}) \frac{1}{\sqrt{2}}(|00\rangle + i|11\rangle), \tag{5.3}$$

so Bob can effectively flip Alice's qubit as well as adjusting his own.

Suppose we have a general 2×2 game with payoffs

$$\begin{array}{c|cc} & \text{Bob: } 0 & \text{Bob: } 1 \\ \hline \text{Alice: } 0 & (p, p') & (q, q') \\ \text{Alice: } 1 & (r, r') & (s, s') \end{array} \tag{5.4}$$

where the unprimed values refer to Alice's payoffs and the primed to Bob's. Bob has four possible miracle moves depending on the final state that he prefers:

$$\begin{aligned} \hat{M}_{00} &= \hat{V}, \\ \hat{M}_{01} &= \hat{F}\hat{V} = \frac{i}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \\ \hat{M}_{10} &= \hat{f}\hat{V} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \\ \hat{M}_{11} &= \hat{F}\hat{f}\hat{V} = \frac{i}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \end{aligned} \tag{5.5}$$

given a preference for $|00\rangle$, $|01\rangle$, $|10\rangle$, or $|11\rangle$, respectively. In the absence of entanglement, any $\hat{M}_{ij}$ is equivalent to $\tilde{U}(\pi/2)$, that is, the mixed classical strategy of cooperating or defecting with equal probability.

When we use an entangling operator $\hat{J}(\gamma)$ for an arbitrary $\gamma \in [0, \pi/2]$, the expectation value of Alice's payoff if she plays $\tilde{U}(\theta)$ against Bob's four miracle moves of Eq. (5.5) are, respectively,

$$\begin{aligned} \langle \$_{00} \rangle &= \frac{p}{2}(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\sin\gamma)^2 + \frac{q}{2}\cos^2\frac{\theta}{2}\cos^2\gamma \\ &\quad + \frac{r}{2}(\sin\frac{\theta}{2} - \cos\frac{\theta}{2}\sin\gamma)^2 + \frac{s}{2}\sin^2\frac{\theta}{2}\cos^2\gamma, \\ \langle \$_{01} \rangle &= \frac{p}{2}\cos^2\frac{\theta}{2}\cos^2\gamma + \frac{q}{2}(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\sin\gamma)^2 + \frac{r}{2}\sin^2\frac{\theta}{2}\cos^2\gamma \\ &\quad + \frac{s}{2}(\sin\frac{\theta}{2} - \cos\frac{\theta}{2}\sin\gamma)^2, \\ \langle \$_{10} \rangle &= \frac{p}{2}(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\sin\gamma)^2 + \frac{q}{2}\cos^2\frac{\theta}{2}\cos^2\gamma \\ &\quad + \frac{r}{2}(\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\sin\gamma)^2 + \frac{s}{2}\sin^2\frac{\theta}{2}\cos^2\gamma, \\ \langle \$_{11} \rangle &= \frac{p}{2}\cos^2\frac{\theta}{2}\cos^2\gamma + \frac{q}{2}(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\sin\gamma)^2 + \frac{r}{2}\sin^2\frac{\theta}{2}\cos^2\gamma \\ &\quad + \frac{s}{2}(\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\sin\gamma)^2. \end{aligned} \tag{5.6}$$

We add primes to p, q, r, and s to get Bob's payoffs. Although the miracle moves are in some sense best for Bob, in that they guarantee a certain minimum payoff against any classical strategy from Alice, there is not necessarily any NE amongst pure strategies in the classical–quantum game.

5.3 Critical entanglements in 2×2 games

5.3.1 Prisoners' Dilemma

The most famous dilemma in game theory is the Prisoners' Dilemma (Rapoport and Chammah 1965). This may be specified in general by

	Bob: C	Bob: D
Alice: C	(b, b)	(d, a)
Alice: D	(a, d)	(c, c)

(5.7)

In the classical game, the strategy "always defect" dominates since it gives a better payoff than cooperation against any strategy by the opponent. Hence, the NE for the Prisoners' Dilemma is mutual defection, resulting in a payoff of c to both players. However, both players would have done better with mutual cooperation, resulting in the PO payoff of b to each player. The conflict between the NE and PO results gives rise to a dilemma that occurs in many social, biological and political situations. The sizes of the payoffs are generally adjusted so that $2b > a + d$ and are commonly set at $(a, b, c, d) = (5, 3, 1, 0)$.

In the classical–quantum game Bob can help engineer his preferred result[11] of CD or $|01\rangle$ by adopting the strategy $\hat{M}_{01}$. The most important critical value of the entanglement parameter γ is the threshold below which Bob performs worse with his miracle move than he would if he chose the classical dominant strategy of "always defect." This occurs for

$$\sin \gamma = \sqrt{\frac{c - d}{a - d}}, \tag{5.8}$$

which yields the value $\sqrt{1/5}$ for the usual payoffs. As noted in Du et al. (2001b), below this level of entanglement the quantum version of Prisoners' Dilemma behaves classically with a NE of mutual defection. Figure 5.1 shows the expected payoffs in quantum Prisoners' Dilemma as a function of Alice's strategy and the degree of entanglement. When Alice defects the payoffs as a function of entanglement are shown in Figure 5.2 clearly indicating the critical entanglement when Bob should switch his strategy to "always defect."

[11] Recall that CD signifies that Alice cooperates and Bob defects, while DC signifies that Alice defects and Bob cooperates.

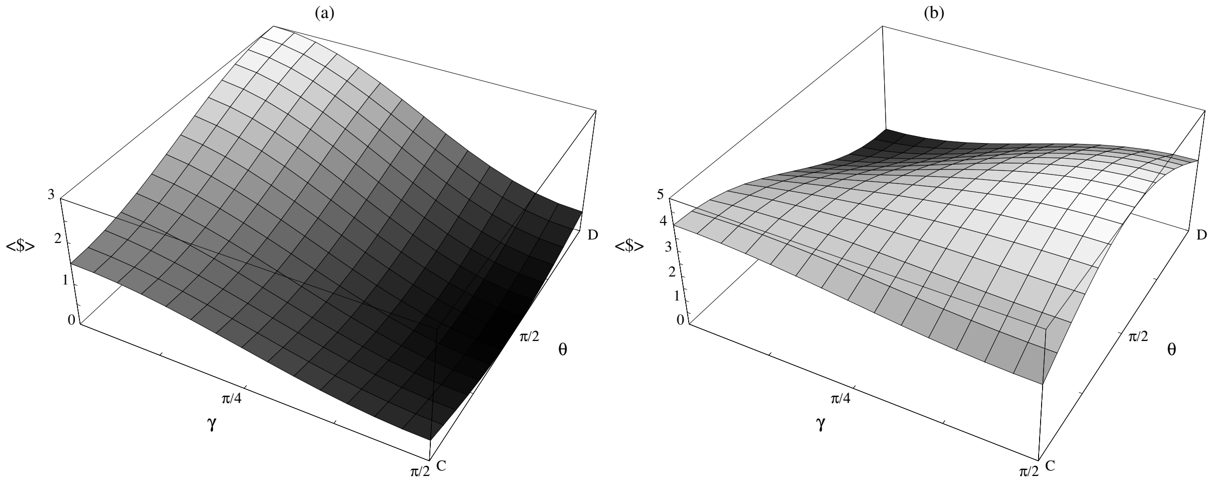


Figure 5.1. Expected payoffs in quantum Prisoners' Dilemma as a function of entanglement. The expected payoffs for (a) Alice, restricted to a classical strategy, and (b) Bob, who plays the quantum miracle move $\hat{M}_{01}$, as a function of Alice's strategy ($\theta = 0$ corresponds to cooperation and $\theta = \pi$ corresponds to defection) and the degree of entanglement γ. The surfaces are drawn for payoffs $(a, b, c, d) = (5, 3, 1, 0)$. Equivalent figures appear in Du et al. (2001b).

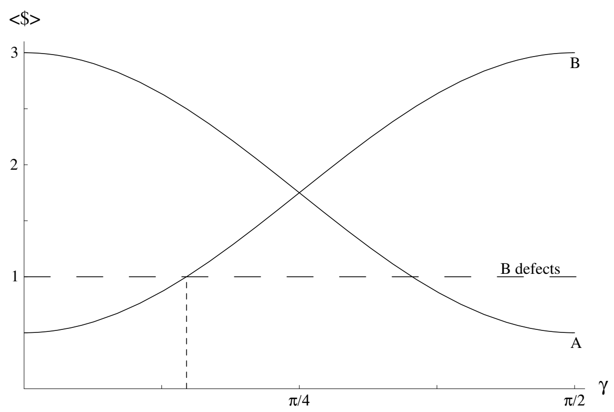


Figure 5.2. Payoffs as a function of entanglement in quantum Prisoners' Dilemma when Alice defects. The expected payoffs for Alice (A) and Bob (B) versus the level of entanglement (γ) with the standard payoffs $(a, b, c, d) = (5, 3, 1, 0)$. The solid lines correspond to the results when Bob plays the quantum move $\hat{M}_{01}$ and the dashed line gives Bob's payoff when he defects. Below an entanglement of $\arcsin(\sqrt{1/5})$ (short dashes) Bob does best, against a defecting Alice, by switching to the strategy "always defect." Figure adapted from Eisert et al. (1999).

5.3.2 Chicken

The archetypal version of Chicken is described as follows:

> *The two players are driving towards each other along the centre of an empty road. Their possible actions are to swerve at the last minute (cooperate) or not to swerve (defect). If only one player swerves he/she is the "chicken" and gets a poor payoff, while the other player is the "hero" and scores best. If both swerve they get an intermediate result but clearly the worst possible scenario is for neither player to swerve.*

Such a situation often arises in the military/diplomatic posturing amongst nations. Each does best if the other backs down against their strong stance, but the mutual worst result is to go to war! The situation is described by the payoff matrix

	Bob: C	Bob: D
Alice: C	(b, b)	(c, a)
Alice: D	(a, c)	(d, d)

(5.9)

The PO result is mutual cooperation. It is usual to impose the condition $2b > a + c$ to ensure that mutual cooperation outperforms alternating results of CD and DC in a repeated game. In the discussion below we shall choose $(a, b, c, d) = (4, 3, 1, 0)$ whenever a numerical example of the payoffs is required. There are two NE in the classical game, CD and DC, from which neither player can improve their result by a unilateral change in strategy. Hence the rational player hypothesized by game theory is faced with a dilemma for which there is no solution: the game is symmetric yet both players want to do the opposite of the other. For the chosen set of numerical payoffs there is a unique NE in mixed classical strategies: each player cooperates or defects with probability one-half. In our protocol this corresponds to both players selecting $\tilde{U}(\pi/2)$.

Quantum versions of Chicken have been discussed in the literature (Eisert and Wilkins 2000, Marinatto and Weber 2000, Benjamin 2000a). The model of Eisert and Wilkins (2000) uses the same protocol as used in this chapter while the that of Marinatto and Weber (2000) differs by the absence of a $J^\dagger$ gate. Both models exhibit quantum effects but vary in the way that the classical game is obtained as a subset of the quantum protocol (see Sec. 2.3).

The preferred outcome for Bob is CD or $|01\rangle$, so he will play $\hat{M}_{01}$. If Alice cooperates, the expected payoffs are

$$\begin{aligned}\langle \$_{\mathrm{A}} \rangle &= \frac{b-d}{2}\cos^2\gamma \, + \, \frac{c+d}{2}, \\ \langle \$_{\mathrm{B}} \rangle &= \frac{b-d}{2}\cos^2\gamma \, + \, \frac{a+d}{2}.\end{aligned} \tag{5.10}$$

Increasing entanglement is bad for the both players. However, Bob out scores Alice by $(a-c)/2$ for all entanglements and does better than the poorer of his two NE results (c) provided

$$\sin\gamma < \sqrt{\frac{a+b-2c}{b-d}} \tag{5.11}$$

which, for the payoffs (4,3,1,0), means that γ can take any value. He performs better than the mutual cooperation result (b) provided

$$\sin\gamma < \sqrt{\frac{a-b}{b-d}} \tag{5.12}$$

which yields a value of $\sqrt{1/3}$ for the chosen payoffs.

Suppose instead that Alice defects. The payoffs are now

$$\begin{aligned}\langle \$_{\mathrm{A}} \rangle &= \frac{a-c}{2}\cos^2\gamma \, + \, \frac{c+d}{2}, \\ \langle \$_{\mathrm{B}} \rangle &= \frac{a-c}{2}\sin^2\gamma \, + \, \frac{c+d}{2}.\end{aligned} \tag{5.13}$$

Increasing entanglement improves Bob's result and worsens Alice's. Bob scores better than Alice provided $\gamma > \pi/4$, regardless of the numerical value of the payoffs. Bob does better than his worst NE result (c) when

$$\sin\gamma > \sqrt{\frac{c-d}{a-c}}\,, \tag{5.14}$$

which yields a value of $\sqrt{1/3}$ for the default payoffs, and better than his PO result (b) when

$$\sin\gamma > \sqrt{\frac{2b-c-d}{a-c}}\,, \tag{5.15}$$

which has no solution for the default values. Thus, except for specially adjusted values of the payoffs, Bob cannot assure himself of a payoff at least as good as that achievable by mutual cooperation. However, Bob escapes from his dilemma for a sufficient degree of entanglement as follows. Against $\hat{M}_{01}$, Alice's optimal strategy from the set S_{cl} is given by

$$\tan\theta = \frac{2(c-d)}{b+c-a-d}\,\frac{\sin\gamma}{\cos^2\gamma}\,. \tag{5.16}$$

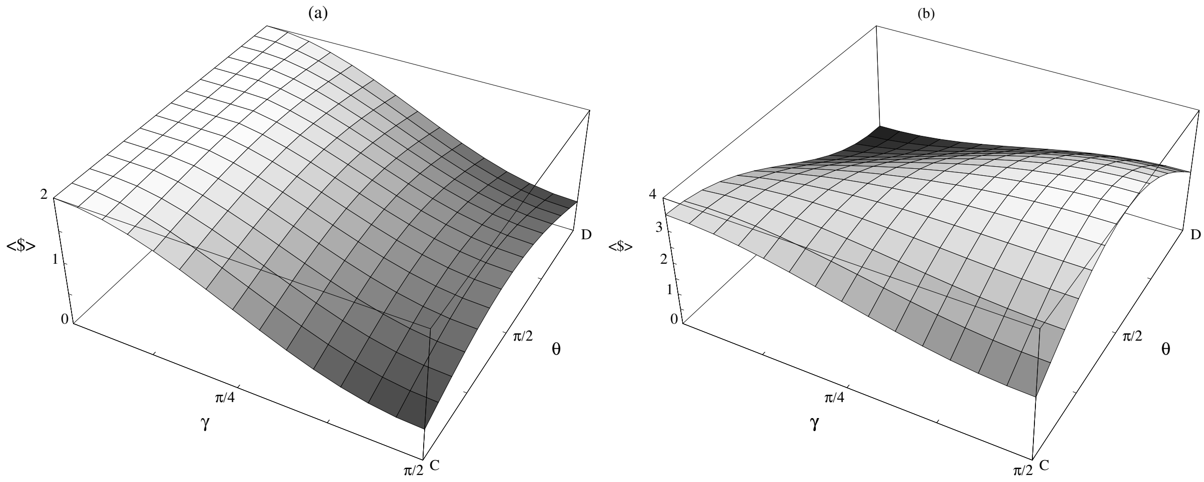


Figure 5.3. Expected payoffs in quantum Chicken as a function of entanglement. The expected payoffs for (a) Alice, restricted to a classical strategy, and (b) Bob, who plays the quantum miracle move $\hat{M}_{01}$, as a function of Alice's strategy ($\theta = 0$ corresponds to cooperation and $\theta = \pi$ corresponds to defection) and the degree of entanglement γ. The surfaces are drawn for payoffs $(a, b, c, d) = (4, 3, 1, 0)$. If Alice knows that Bob is going to play the quantum miracle move, she does best by choosing the crest of the curve, $\theta = \pi/2$, irrespective of the level of entanglement. Against this strategy Bob scores between two and four, an improvement for all $\gamma > 0$ over the payoff he could expect playing a classical strategy.

For $(a, b, c, d) = (4, 3, 1, 0)$ this gives $\theta = \pi/2$. Since $\hat{M}_{01}$ is Bob's best counter to $\tilde{U}(\pi/2)$ these strategies form a NE in the classical–quantum game of Chicken and are the preferred strategies of the players. For this choice, above an entanglement of $\gamma = \pi/6$, Bob performs better than the mutual cooperation result.

Figure 5.3 shows the expected payoffs in quantum Chicken as a function of Alice's strategy and the degree of entanglement. Figure 5.4 demonstrates that if Bob wishes to maximize the minimum payoff he receives, he should alter his strategy from the quantum move $\hat{M}_{01}$ to cooperation, once the entanglement drops below $\arcsin(\sqrt{1/3})$.

5.3.3 Deadlock

Deadlock is characterized by reversing the payoffs for mutual cooperation and defection in the Prisoners' Dilemma:

$$\begin{array}{c|cc} & \text{Bob: C} & \text{Bob: D} \\ \hline \text{Alice: C} & (c, c) & (d, a) \\ \text{Alice: D} & (a, d) & (b, b) \end{array} \tag{5.17}$$

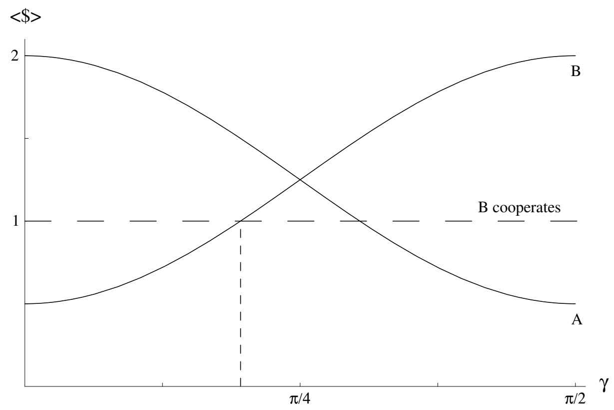


Figure 5.4. Payoffs as a function of entanglement in quantum Chicken when Alice defects. The payoffs for Alice (A) and Bob (B) versus the level of entanglement (γ) with the standard payoffs $(a, b, c, d) = (4, 3, 1, 0)$. The solid lines correspond to the results when Bob plays the quantum move $\hat{M}_{01}$ and the dashed line gives Bob's payoff when he cooperates. Below an entanglement of $\arcsin(\sqrt{1/3})$ (short dashes) Bob does best, against a defecting Alice, by switching to the strategy "always cooperate.'

Defection is again the dominant strategy and there is even less incentive for the players to cooperate in this game than in the Prisoners' Dilemma since the PO result is mutual defection. However, both players would prefer if their opponent cooperated so they could stab them in the back by defecting and achieve the maximum payoff of a. There is no advantage to cooperating so there is no real dilemma in the classical game. In the classical–quantum game Bob can again use his quantum skills to engineer at least partial cooperation from Alice, against any possible strategy from her, by playing $\hat{M}_{01}$. Figure 5.5 gives the payoffs to the players as a function of entanglement when Alice defects and Bob plays $\hat{M}_{01}$. The standard payoffs of (3,2,1,0) are used. From the figure it is clear that for $\gamma < \sqrt{2/3}$ Bob should switch to the classical strategy of "always defect" in order to maximize his payoff.

5.3.4 Stag Hunt

Here, both players prefer the outcome of mutual cooperation since it gives a payoff superior to all other outcomes. However, each are afraid of defection by the other. Although this reduces the defecting player's payoff, it has a more detrimental effect on the cooperator's

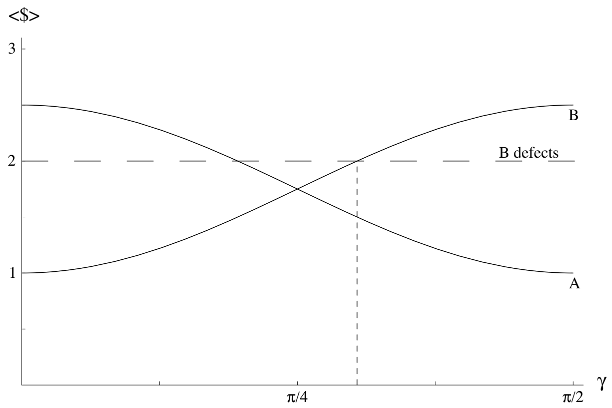


Figure 5.5. Payoffs as a function of entanglement in quantum Deadlock when Alice defects. The payoffs for Alice (A) and Bob (B) versus the level of entanglement (γ) with the standard payoffs $(a, b, c, d) = (3, 2, 1, 0)$. The solid lines correspond to the results when Bob plays the quantum move $\hat{M}_{01}$ and the dashed line gives Bob's payoff when he defects. Below an entanglement of $\arcsin(\sqrt{2/3})$ (short dashes) Bob does best, against a defecting Alice, by switching to the strategy "always defect.'

payoff, as indicated in the payoff matrix below:

	Bob: C	Bob: D
Alice: C	(a, a)	(d, b)
Alice: D	(b, d)	(c, c)

(5.18)

Both mutual cooperation and mutual defection are NE but the former is the PO result. There is no dilemma when two rational players meet. Both recognize the preferred result and have no reason, given their recognition of the rationality of the other player, to defect. Mutual defection will result only if both players allow fear to dominate over rationality. This situation is not changed in the classical–quantum game. However, having the ability to play quantum moves may be of advantage when the classical player is irrational in the sense that they do not try to maximize their own payoff. In that case the quantum player should choose to play the strategy $\hat{M}_{00}$ to steer the result towards the mutual cooperation outcome. The payoffs for this situation as a function of entanglement are displayed in Figure 5.6. Bob is advised to adopt the maximin strategy if he is fearful that Alice is going to try to do him maximum harm by defecting. Below an entanglement of $\gamma = \pi/4$ the maximin strategy is defection, but above this level of entanglement the quantum strategy $\hat{M}_{00}$ is of some advantage, as the figure indicates. An alternative quantization of Stag Hunt, using the scheme of Marinatto and Weber (2000), has been considered by Toyota (2003).

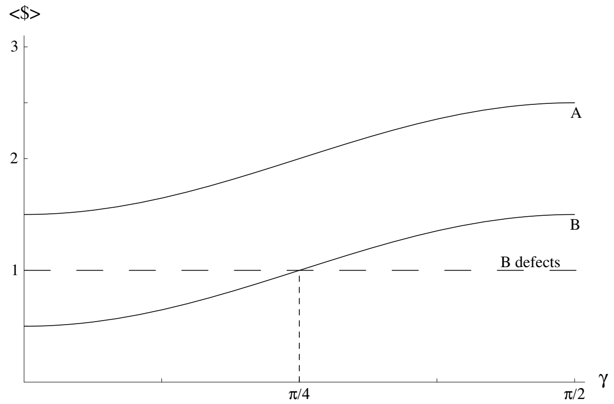


Figure 5.6. Payoffs as a function of entanglement in quantum Stag Hunt when Alice defects. The payoffs for Alice (A) and Bob (B) versus the level of entanglement (γ) with the standard payoffs $(a, b, c, d) = (3, 2, 1, 0)$. The solid lines correspond to the results when Bob plays the quantum move $\hat{M}_{01}$ and the dashed line gives Bob's payoff when he defects. Bob receives a payoff of zero if he cooperates. The strategy that maximizes Bob's minimum payoff is to defect for $\gamma < \pi/4$ and to play $\hat{M}_{00}$ for $\gamma \geq \pi/4$.

5.3.5 Battle of the Sexes

In this game Alice and Bob each want the company of the other in some activity but their preferred activity differs: opera (O) for Alice and television (T) for Bob. If the players end up doing different activities both are punished by a poor payoff. In matrix form this game can be represented as

	Bob: O	Bob: T
Alice: O	(a, b)	(c, c)
Alice: T	(c, c)	(b, a)

(5.19)

The options on the main diagonal are both PO and are NE but there is no way of deciding between them. Bob's quantum strategy will be to choose $\hat{M}_{11}$ to steer the game towards his preferred result of TT. Several quantum versions of the Battle of the Sexes have been discussed in the literature (Marinatto and Weber 2000, Du et al. 2000, Du et al. 2001a, de Farias Neto 2004, Nawaz and Toor 2004a, Wu 2004a) along the lines of the model used here.

With $\hat{M}_{11}$, Bob out scores Alice provided $\gamma > \pi/4$, but is only assured of scoring at least as well as the poorer of his two NE results (b) for full entanglement, and is never certain of bettering it. The quantum move, however, is better than using a fair coin to decide between $\hat{O}$ and $\hat{T}$ for $\gamma > 0$, and equivalent to it for $\gamma = 0$. Hence, even though Bob cannot be assured of scoring greater than b he can improve his worst case payoff for any

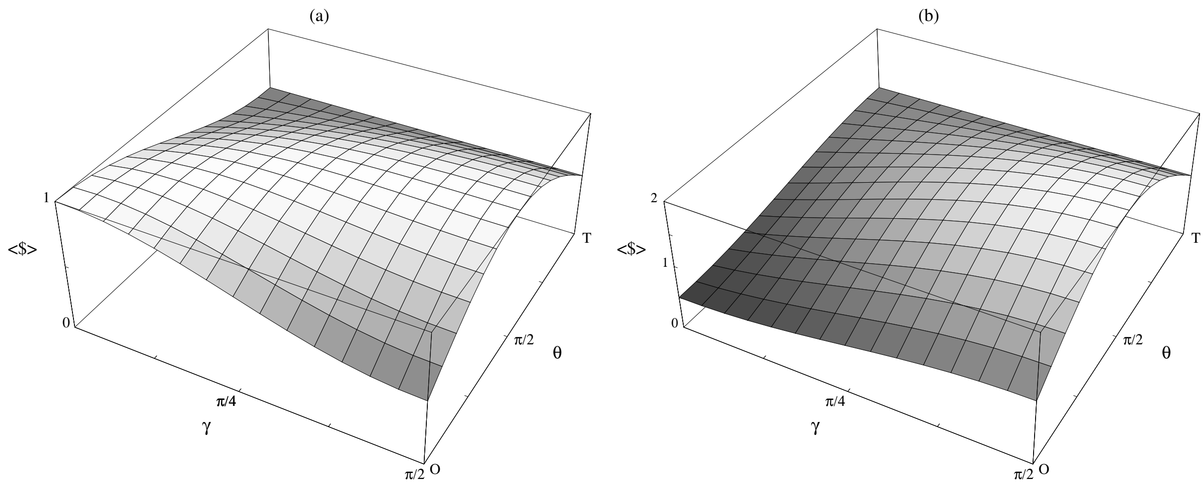


Figure 5.7. Expected payoffs in quantum Battle of the Sexes as a function of entanglement. The expected payoffs for (a) Alice, restricted to a classical strategy, and (b) Bob, who plays the quantum miracle move $\hat{M}_{01}$, as a function of Alice's strategy ($\theta = 0$ corresponds to opera and $\theta = \pi$ corresponds to television) and the degree of entanglement γ. The surfaces are drawn for payoffs $(a, b, c) = (2, 1, 0)$. If Alice knows that Bob is going to play the quantum miracle move, she does best by choosing the crest of the curve, so her optimal strategy changes from O for no entanglement, to $\theta = \pi/2$ for full entanglement. Against this strategy, Bob starts to score better than one for an entanglement exceeding approximately $\pi/5$.

$\gamma > 0$. Figure 5.7 shows the payoffs in quantum Battle of Sexes as a function of the degree of entanglement and Alice's strategy.

5.4 Extensions

The situation is more complex for multiplayer games. No longer can a quantum player playing against classical ones engineer any desired final state, even if the opponents' moves are known. However, a player can never be worse by having access to the quantum domain since this includes the classical possibilities as a subset.

In two player games with more than two pure classical strategies the prospects for the quantum player are better. Some entangled, quantum 3×3 games have been considered in the literature (Iqbal and Toor 2002d, Flitney and Abbott 2002c). Here the full set of quantum strategies is SU(3) and there are nine possible miracle moves (before considering symmetries). The strategies that do not manipulate the phase of the player's qutrit (i.e., classical strategies) can be written as the product of three rotations, each parameterized

game	strategies	$\langle \$_\mathrm{B} \rangle > \langle \$_\mathrm{A} \rangle$	$\langle \$_\mathrm{B} \rangle > \langle \$_\mathrm{B} \rangle_\mathrm{NE}$	$\langle \$_\mathrm{B} \rangle > \langle \$_\mathrm{B} \rangle_\mathrm{PO}$
PD	$\hat{C}$, $\hat{M}_{01}$	always	always	$(a-b)/(c-d)$
	$\hat{D}$, $\hat{M}_{01}$	$d/(2(a-d))$	$(c-d)/(a-d)$	$(2b-c-d)/(a-d)$
Chicken	$\hat{C}$, $\hat{M}_{01}$	always	$< (a+b-2c)/(b-d)$	$< (a-b)/(b-d)$
	$\hat{D}$, $\hat{M}_{01}$	$\frac{1}{2}$	$(c-d)/(a-c)$	$(2b-c-d)/(a-c)$
Deadlock	$\hat{C}$, $\hat{M}_{01}$	always	$(2b-a-c)/(b-c)$	$(2b-a-c)/(b-c)$
	$\hat{D}$, $\hat{M}_{01}$	$\frac{1}{2}$	$(b-d)/(a-d)$	$(b-d)/(a-d)$
Stag Hunt	$\hat{C}$, $\hat{M}_{00}$	$< \frac{1}{2}$	$(c-d)/(a-c)$	never
	$\hat{D}$, $\hat{M}_{00}$	never	$< (a+b-2c)/(b-d)$	never
BoS	$\hat{O}$, $\hat{M}_{11}$	$\frac{1}{2}$	$(b-c)/(a-b)$	$(b-c)/(a-b)$
	$\hat{T}$, $\hat{M}_{11}$	always	if $a+c>2b$	if $a+c>2b$

Table 5.2. Critical entanglements for 2×2 quantum games. Values of $\sin^2\gamma$ above which (or below which where indicated by '$<$') the expected value of Bob's payoff exceeds, respectively, Alice's payoff, Bob's classical NE payoff and, Bob's payoff for the PO outcome. Where there are two NE (or PO) results, the one where Bob's payoff is smallest is used. The strategies are Alice's and Bob's, respectively. In the last line, 'if $a+c>2b$' refers to a condition on the numerical values of the payoffs and not to a condition on γ.

by a rotation angle. Since the form is not unique, it is much more difficult to say what constitutes the median move from this set, so the expressions for the miracle moves are open to debate. Nevertheless, the quantum player will still be able to manipulate the result of the game to increase the probability of his/her favoured result.

5.5 Summary

With a sufficient degree of entanglement, the quantum player in a classical–quantum two player game can use the extra possibilities available to them to help steer the game towards their most desired result, giving a payoff above that achievable by classical strategies alone. The best moves for the quantum player are referred to as "miracle" moves. In this chapter, the four miracle moves in quantum 2×2 game theory are given and their use in several game-theoretic dilemmas is demonstrated. There are critical values of the entanglement parameter γ below (or occasionally above) which it is no longer an advantage to have access to quantum moves, that is, where the quantum player can no longer outscore his/her classical Nash equilibrium result. These represent a phase-like transition in the classical–quantum game, where a switch between the quantum miracle move and the dominant classical strategy is warranted. Table 5.2 summarizes the threshold values of γ. With typical values for the payoffs and a classical player opting for his/her best strategy, the critical value for $\sin\gamma$ is $\sqrt{1/3}$ for Chicken, $\sqrt{1/5}$ for Prisoners' Dilemma and $\sqrt{2/3}$ for Deadlock, while for Stag Hunt there is no particular advantage to the quantum player unless the classical player is adopting a non-optimal strategy. In the Battle of the Sexes there is no clear threshold but for any non-zero entanglement Bob can improve upon the possible worst case result that could arise if he was restricted to classical strategies.

The quantum player's advantage is not as strong in classical–quantum multiplayer games but in multi-strategy, two player games, depending on the level of entanglement, the quantum player would again have access to moves that improve his/her result. The calculation of these moves is problematic because of the larger number of degrees of freedom and has not been attempted here.

Chapter 6

Decoherence in Quantum Games

DECOHERENCE in a quantum system results from the interaction of the system with its environment. The study of the effect of decoherence is necessary in any practical quantum information processing scheme. This chapter presents a scheme for including decoherence in Eisert's model of quantum games. The effect of decoherence is quantified by considering the diminution of the advantage obtainable by a quantum player against a classical one in several well known 2×2 games as decoherence is increased. The current chapter complements Chapter 5 that considers how the quantum player's advantage is affected by the initial degree of entanglement between the players' qubits.

6.1 Introduction

Decoherence can be defined as non-unitary dynamics resulting from the coupling of the system with the environment. In any realistic quantum computer, interaction with the environment cannot be entirely eliminated. Although realization of quantum computers is debated, steady progress towards this ultimate goal continues (Abbott et al. 2003). Decoherence can destroy the special features of quantum computation. A review of the standard mechanisms of quantum decoherence can be found in Zurek (2003). Quantum computing in the presence of noise is possible with the use of quantum error correction (Preskill 1998) or decoherence free subspaces (Lidar and Whaley 2003). These techniques work by encoding the logical qubits in a number of physical qubits. Quantum error correction is successful, provided the error rate is low enough, while decoherence free subspaces control certain types of decoherence. Both have the disadvantage of expanding the number of qubits required for a calculation. Without such measures, the theory of quantum control in the presence of noise and decoherence is little studied. This motivates the study of quantum games, which can be viewed as a game-theoretic approach to quantum control—game-theoretic methods in classical control theory (Carraro and Filar 1995) are well-established and translating them to the quantum realm is a promising area of study.

Johnson (2001) has considered a three player quantum game corrupted by selecting an initial state of $|111\rangle$ instead of $|000\rangle$, with some probability. Above a certain level of corruption it was found that quantum effects impede the players to such a degree that they were better off playing the classical game. The same result was found for various 2×2 quantum games with bit flip errors in the initial state (Özdemir et al. 2004). Chen et al. (2003b) have discussed decoherence in quantum Prisoners' Dilemma. Decoherence was found to have no effect on the NE in this model. The current chapter presents a model for incorporating decoherence in N-player quantum games of the scheme of Benjamin and Hayden (2001b). Results for two player Prisoners' Dilemma, Chicken and the Battle of the Sexes are calculated as examples.

It is most convenient to use the density matrix notation for the state of the system and the operator sum representation for the quantum operators. Decoherence can take many forms including dephasing, which randomizes the relative phases of the quantum states, and dissipation, that modifies the populations of the quantum states. Pure dephasing of a qubit can be expressed by

$$a|0\rangle + b|1\rangle \rightarrow a|0\rangle + b\,e^{i\phi}|1\rangle. \tag{6.1}$$

If we assume that the phase kick ϕ is a random variable with a Gaussian distribution of mean zero and variance 2λ, then the density matrix obtained after averaging over all values of ϕ is (Nielsen and Chuang 2000)

$$\begin{pmatrix} |a|^2 & a\bar{b} \\ \bar{a}b & |b|^2 \end{pmatrix} \rightarrow \begin{pmatrix} |a|^2 & a\bar{b}\, e^{-\lambda} \\ \bar{a}b\, e^{-\lambda} & |b|^2 \end{pmatrix}. \tag{6.2}$$

That is, over time the random phase kicks cause an exponential decay of the off-diagonal elements of the density matrix.

The quantum operator formalism used here is well known to have its limitations in the modeling of decoherence (Royer 1977). For a good description of the quantum operator formalism and an example of its limitations the reader is referred to chapter 8 of Nielsen and Chuang (2000). Other methods for calculating decoherence include using Lagrangian field theory, path integrals, master equations, quantum Langevin equations, short-time perturbation expansions, Monte-Carlo methods, semiclassical methods, and phenomenological methods (Brandt 1998).

In the operator sum representation, the act of making a measurement with probability p in the $\{|0\rangle, |1\rangle\}$ basis on a qubit described by the density matrix ρ is

$$\rho \rightarrow \sum_{j=0}^{2} \mathcal{E}_j\, \rho\, \mathcal{E}_j^\dagger, \tag{6.3}$$

where $\mathcal{E}_0 = \sqrt{p}\,|0\rangle\langle 0|$, $\mathcal{E}_1 = \sqrt{p}\,|1\rangle\langle 1|$ and $\mathcal{E}_2 = \sqrt{1-p}\,\hat{I}$. An extension to N qubits is achieved by applying the measurement to each qubit in turn, resulting in

$$\rho \rightarrow \sum_{j_1,\ldots,j_N=0}^{2} \mathcal{E}_{j_1} \otimes \ldots \otimes \mathcal{E}_{j_N}\, \rho\, \mathcal{E}_{j_N}^\dagger \otimes \ldots \otimes \mathcal{E}_{j_1}^\dagger, \tag{6.4}$$

where ρ is the density matrix of the N qubit system. This process also leads to the decay of the off-diagonal elements of ρ. By identifying $1-p = e^{-\lambda}$, the measurement process has the same results as pure dephasing.

6.2 Decoherence in Meyer's quantum Penny Flip

A simple effect of decoherence can be seen in Meyer's quantum Penny Flip (Meyer 1999) between P, who is restricted to classical strategies, and Q, who has access to quantum operations. In the classical game, P places a coin heads up in a box. First Q, then P, then Q again, have the option of (secretly) flipping the coin or leaving it unaltered, after which the state of the coin is revealed. If the coin shows heads, Q is victorious. Since the players' moves are carried out in secret they do not know the intermediate states of the coin and hence the classical game is balanced.

In the quantum version, the coin is replaced by a qubit prepared in the $|0\rangle$ ("heads") state. Having access to quantum operations, Q applies the Hadamard operator to produce the superposition $(|0\rangle + |1\rangle)/\sqrt{2}$. This state is invariant under the transformation $|0\rangle \leftrightarrow |1\rangle$ so P's action has no effect. On his second move Q again applies the Hadamard operator to return the qubit to $|0\rangle$. Thus Q wins with certainty against any classical strategy by P.

Decoherence can be added to this model by applying a measurement with probability p after Q's first move. Applying the same operation after P's move has the same effect since his move is either the identity or a bit-flip. If the initial state of the coin is represented by the density matrix $\rho_0 = |0\rangle\langle 0|$, the final state can be calculated by

$$\begin{aligned} \rho_f &= \hat{H}\hat{P}\hat{D}\hat{H}\rho_0\hat{H}^\dagger\hat{D}^\dagger\hat{P}^\dagger\hat{H}^\dagger \\ &= \frac{1}{4}\begin{pmatrix} 4-2p & 0 \\ 0 & 2p \end{pmatrix}, \end{aligned} \tag{6.5}$$

where $\hat{H}$ is the Hadamard operator, $\hat{P}$ is P's move ($\hat{I}$ or $\hat{\sigma}_x$), and $\hat{D} = \sqrt{1-p}\,\hat{I} + \sqrt{p}\,(|0\rangle\langle 0| + |1\rangle\langle 1|)$ is a measurement in the computational basis with probability p. Again, the final state is independent of P's move. The expectation of Q winning decreases linearly from one to $\frac{1}{2}$ as p goes from zero to one. Maximum decoherence produces a fair game.

6.3 Decoherence in the Eisert scheme

6.3.1 The model

A quantum game in the Eisert scheme with decoherence can be described in the following manner

$$\begin{aligned}
\rho_i \equiv \rho_0 &= |\psi_0\rangle\langle\psi_0| && \text{(initial state)} \\
\rho_1 &= \hat{J}\rho_0\hat{J}^\dagger && \text{(entanglement)} \\
\rho_2 &= D(\rho_1, p_1) && \text{(partial decoherence)} \\
\rho_3 &= (\otimes_{k=1}^{N}\hat{M}_k)\,\rho_2\,(\otimes_{k=1}^{N}\hat{M}_k)^\dagger && \text{(players' moves)} \\
\rho_4 &= D(\rho_3, p_2) && \text{(partial decoherence)} \\
\rho_5 &= \hat{J}^\dagger\rho_4\hat{J} && \text{(dis-entanglement)},
\end{aligned} \tag{6.6}$$

to produce the final state $\rho_f \equiv \rho_5$ upon which a measurement is taken. The function $D(\rho, p)$ is a completely positive map that applies some form of decoherence to the state ρ controlled by the probability p. The scheme is shown in Figure 6.1. The expectation value of the payoff for the kth player is

$$\langle \$^k \rangle = \sum_{\xi} \hat{\mathcal{P}}_\xi\, \rho_f\, \hat{\mathcal{P}}_\xi^\dagger\, \$_\xi^k, \tag{6.7}$$

where $\hat{\mathcal{P}}_\xi = |\xi\rangle\langle\xi|$ is the projector onto the state $|\xi\rangle$, $\$_\xi^k$ is the payoff to the kth player when the final state is $|\xi\rangle$, and the summation is taken over $\xi = j_1 j_2 \ldots j_N,\ j_i = 0, 1$.

After choosing Eq. (6.4) to represent the function D in Eq. (6.6) we are now in a position to write down the results of decoherence in a 2×2 quantum game. The notation for the players' strategies is given in Eq. (2.9) with, here, the addition of the subscripts A and B to indicate the parameters of the two traditional protagonists, Alice and Bob, respectively. Writing $c_k \equiv \cos(\theta_k/2)$ and $s_k \equiv \sin(\theta_k/2)$ for $k \in \{\text{A,B}\}$, the expectation value of a player's payoff is

$$
\begin{aligned}
\langle \$ \rangle = & \frac{1}{2}(c_A^2 c_B^2 + s_A^2 s_B^2)(\$_{00} + \$_{11}) \; + \; \frac{1}{2}(c_A^2 s_B^2 + s_A^2 c_B^2)(\$_{01} + \$_{10}) \\
& + \frac{1}{2}(1-p_1)^2(1-p_2)^2\{ \\
& \quad [\, c_A^2 c_B^2 \cos(2\alpha_A + 2\alpha_B) \; - \; s_A^2 s_B^2 \cos(2\beta_A + 2\beta_B)](\$_{00} - \$_{11}) \\
& \quad + [\, c_A^2 s_B^2 \cos(2\alpha_A - 2\beta_B) \; - \; s_A^2 c_B^2 \cos(2\alpha_B - 2\beta_A)](\$_{01} - \$_{10})\} \\
& + \frac{1}{4}\sin\theta_A \sin\theta_B \, \big[(1-p_1)^2 \sin(\alpha_A + \alpha_B - \beta_A - \beta_B)(-\$_{00} + \$_{01} + \$_{10} - \$_{11}) \\
& + (1-p_2)^2 \sin(\alpha_A + \alpha_B + \beta_A + \beta_B)(\$_{00} - \$_{11}) \\
& + (1-p_2)^2 \sin(\alpha_A - \alpha_B + \beta_A - \beta_B)(\$_{10} - \$_{01})\big] \,,
\end{aligned} \tag{6.8}
$$

where $\$_{ij}$ is the payoff to the player for the final state $|ij\rangle$. Setting $p_1 = p_2 = 0$ gives the well known result of the Eisert scheme. If in addition, $\alpha_k = \beta_k = 0, \; k \in \{\text{A,B}\}$, a 2×2 classical game results with the mixing between the two classical pure strategies $\hat{I}$ and $\hat{F}$ being determined by θ_A and θ_B for Alice and Bob, respectively. Maximum decoherence with $p_1 = p_2 = 1$ gives a result where the quantum phases α_k and β_k are not relevant:

$$
\langle \$ \rangle = \frac{x}{2}(\$_{00} + \$_{11}) \; + \; \frac{1-x}{2}(\$_{01} + \$_{10}), \tag{6.9}
$$

where $x = c_A^2 c_B^2 + s_A^2 s_B^2$. In this case, a symmetric game yields payoffs to both players that are the identical. The game is not equivalent to the original classical game. Extrema for the payoffs occur when both θ's are 0 or π.

One way of measuring the "quantum-ness" of the game is to consider the known advantage of a player having access to the full set of quantum strategies S_q over a player who is limited to the classical set S_{cl}, as considered in Chapter 5. The classical strategies are those for which the phases α and β vanish. If Alice is restricted in this way then Eq. (6.8) reduces to

$$
\begin{aligned}
\langle \$ \rangle = & \frac{x}{2}(\$_{00} + \$_{11}) \; + \; \frac{1-x}{2}(\$_{01} + \$_{10}) \\
& + \frac{1}{2}(1-p_1)^2(1-p_2)^2 \left\{ c_B^2 \cos 2\alpha_B \, [c_A^2(\$_{00} - \$_{11}) \; + \; s_A^2(\$_{10} - \$_{01})] \right. \\
& \left. - s_B^2 \cos 2\beta_B \, [c_A^2(\$_{10} - \$_{01}) \; + \; s_A^2(\$_{00} - \$_{11})] \right\} \\
& + \frac{1}{4}\sin\theta_A \sin\theta_B \, \big[(1-p_1)^2 \sin(\alpha_B - \beta_B)(-\$_{00} + \$_{01} + \$_{10} - \$_{11}) \\
& + (1-p_2)^2 \sin(\alpha_B + \beta_B)(\$_{00} + \$_{01} - \$_{10} - \$_{11})\big] \,.
\end{aligned} \tag{6.10}
$$

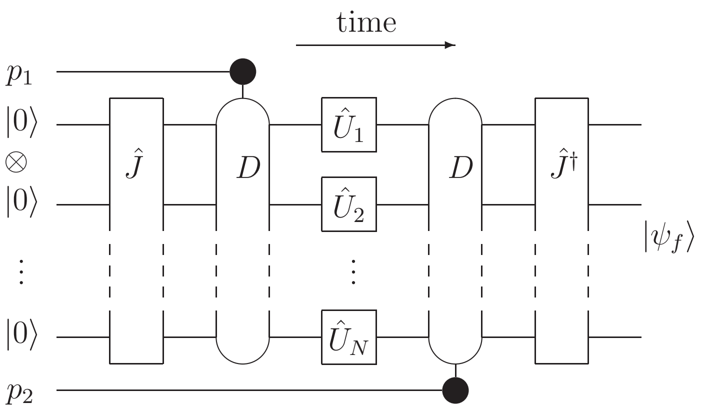


Figure 6.1. Flow of information in a quantum game with decoherence. The flow of information in an N-person quantum game with decoherence, where $\hat{U}_k$ is the move of the kth player and $\hat{J}$ ($\hat{J}^\dagger$) is an entangling (dis-entangling) gate. The central horizontal lines are the players' qubits and the top and bottom lines are classical random bits with a probability p_1 or p_2, respectively, of being 1. Here, D is some form of decoherence controlled by the classical bits.

6.3.2 Prisoners' Dilemma

For Prisoners' Dilemma, the standard payoff matrix is

	Bob: C	Bob: D
Alice: C	$(3, 3)$	$(0, 5)$
Alice: D	$(5, 0)$	$(1, 1)$

(6.11)

where the numbers in parentheses represent payoffs to Alice and Bob, respectively. The classical pure strategies are cooperation (C) and defection (D). Defecting gives a better payoff regardless of the other player's strategy, so it is a dominant strategy, and mutual defection is the NE. The well known dilemma arises from the fact that both players would be better off with mutual cooperation, if this could be engineered. With the payoffs of Eq. (6.11), the best Bob can do from Eq. (6.10) is to select $\alpha_{\rm B} = \pi/2$ and $\beta_{\rm B} = 0$. Bob's choice of $\theta_{\rm B}$ will depend on Alice's choice of $\theta_{\rm A}$. He can do no better than $\theta_{\rm B} = \pi/2$ if he is ignorant of Alice's strategy[12]. Figure 6.2 shows the payoffs in quantum Prisoners' Dilemma as a function of decoherence probability $p \equiv p_1 = p_2$ and Alice's strategy $\theta \equiv \theta_{\rm A}$ when Bob selects his optimal strategy.

[12]See Chapter 5 or Flitney and Abbott (2003a) for details of quantum versus classical players.

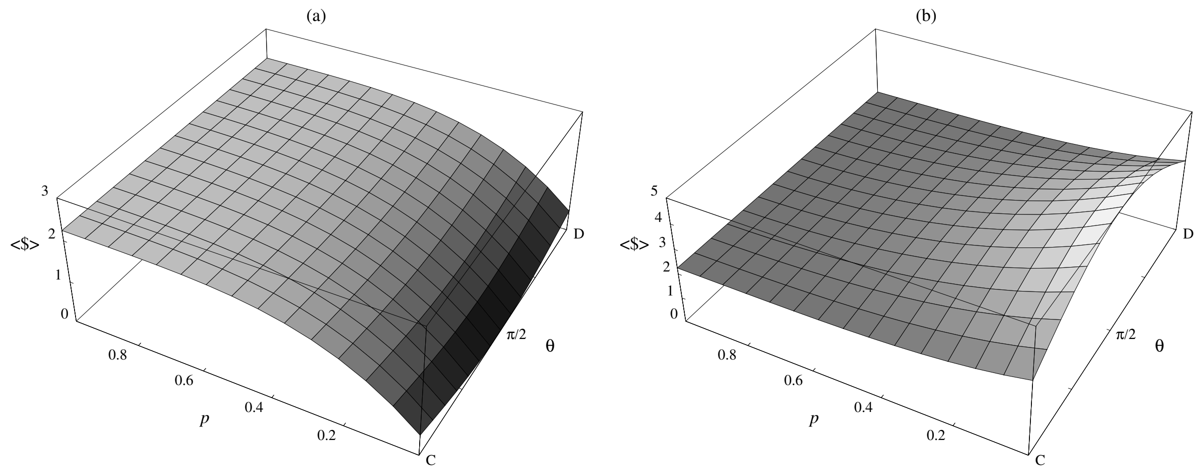


Figure 6.2. Payoffs in quantum Prisoners' Dilemma with decoherence. Payoffs for (a) Alice and (b) Bob as a function of decoherence probability p and Alice's strategy θ (being a measure of the mixing between cooperation (C) and defection (D) with $\theta = 0$ giving C and $\theta = \pi$ giving D), when Bob plays the optimum quantum strategy and Alice is restricted to classical strategies. The decoherence goes from the unperturbed quantum game at $p = 0$ to maximum decoherence at $p = 1$.

6.3.3 Chicken

The standard payoff matrix for the game of Chicken is

	Bob: C	Bob: D
Alice: C	(3,3)	(1,4)
Alice: D	(4,1)	(0,0)

(6.12)

There is no dominant strategy. Both CD and DC are NE, with the former preferred by Bob and the latter by Alice. Again there is a dilemma since the PO result CC is different from both NE. As above, Bob's payoff is optimized by $\alpha_{\rm B} = \pi/2$, $\beta_{\rm B} = 0$ and $\theta_{\rm B} = \pi/2$. Figure 6.3 shows the payoffs in quantum Chicken as a function of decoherence probability p and Alice's strategy θ.

6.3.4 Battle of the Sexes

One form of the payoff matrix for the Battle of the Sexes is

	Bob: O	Bob: T
Alice: O	(2,1)	(0,0)
Alice: T	(0,0)	(1,2)

(6.13)

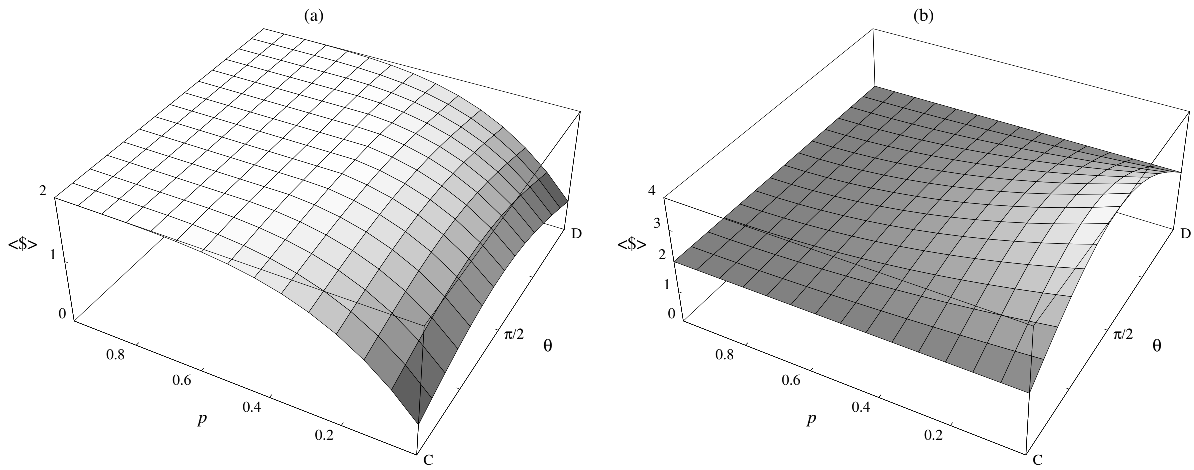


Figure 6.3. Payoffs in quantum Chicken with decoherence. Payoffs for (a) Alice and (b) Bob as a function of decoherence probability p and Alice's strategy θ, when Bob plays the optimum quantum strategy and Alice is restricted to a classical mixed strategy.

Here the two protagonists must decide on an evening's entertainment. Alice prefers opera (O) and Bob television (T), but their primary concern is that they do an activity together. In the absence of communication there is a coordination problem. A quantum Bob maximizes his payoff in a competition with a classical Alice by choosing $\alpha_{\mathrm{B}} = -\pi/2$, $\beta_{\mathrm{B}} = 0$ and $\theta_{\mathrm{B}} = \pi/2$. By doing so he achieves at least partial coordination irrespective of Alice's strategy. Figure 6.4 shows the resulting payoffs for Alice and Bob as a function of decoherence probability p and Alice's strategy θ.

6.3.5 General remarks on 2×2 games

The optimal strategy for Alice in the three games considered is $\theta = \pi$ (or 0) for Prisoners' Dilemma, or $\theta = \pi/2$ for Chicken and Battle of the Sexes. Figure 6.5 shows the expectation value of the payoffs to Alice and Bob as a function of the decoherence probability p for each of the games when Alice chooses her optimal classical strategy. In all cases considered, Bob out scores Alice and performs better than his classical NE result[13] provided $p < 1$. The advantage of having access to quantum strategies decreases as p increases, being minimal above $p \approx 0.5$, but is still present for all levels of decoherence up to the maximum. At maximum decoherence ($p = 1$), with the selected strategies, the game result is randomized and the expectation of the payoffs are simply the average over the four possible results. The results presented in Figures 6.2, 6.3 and 6.4 are comparable

[13] In the case of Chicken or the Battle of the Sexes there are two NE. The one with the lower payoff has been chosen.

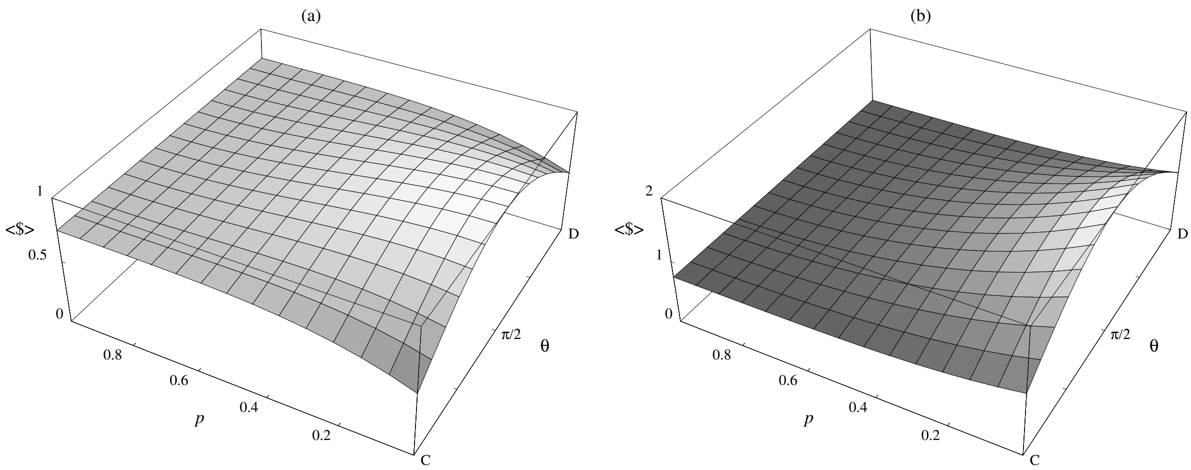


Figure 6.4. Payoffs in quantum Battle of the Sexes with decoherence. Payoffs for (a) Alice and (b) Bob as a function of decoherence probability p and Alice's strategy θ, when Bob plays the optimum quantum strategy and Alice is restricted to a classical mixed strategy.

to the results for different levels of entanglement (Flitney and Abbott 2003a). They are also consistent with the results of Chen et al. (2003b) who show that with increasing decoherence the payoffs to both players approach the average of the four payoffs in a quantum Prisoners' Dilemma.

6.4 Summary and open questions

A method of introducing decoherence into quantum games has been presented. One measure of the "quantum-ness" of a quantum game subject to decoherence is the advantage a quantum player has over a player restricted to classical strategies. As expected, increasing the amount of decoherence degrades the advantage of the quantum player. However, in the model considered, this advantage does not entirely disappear until the decoherence is a maximum. When this occurs in a 2×2 symmetric game, the results of the players are equal. The classical game is not reproduced. The loss of advantage to the quantum player is very similar to that which occurs when the level of entanglement between the players' qubits is reduced.

In multi-player quantum games it is known that new Nash equilibria can arise (Benjamin and Hayden 2001b). The effect of decoherence on the existence of the new equilibria is an interesting open question. There has been some work on continuous-variable quantum games (Li et al. 2002) involving an infinite dimensional Hilbert space. The study of

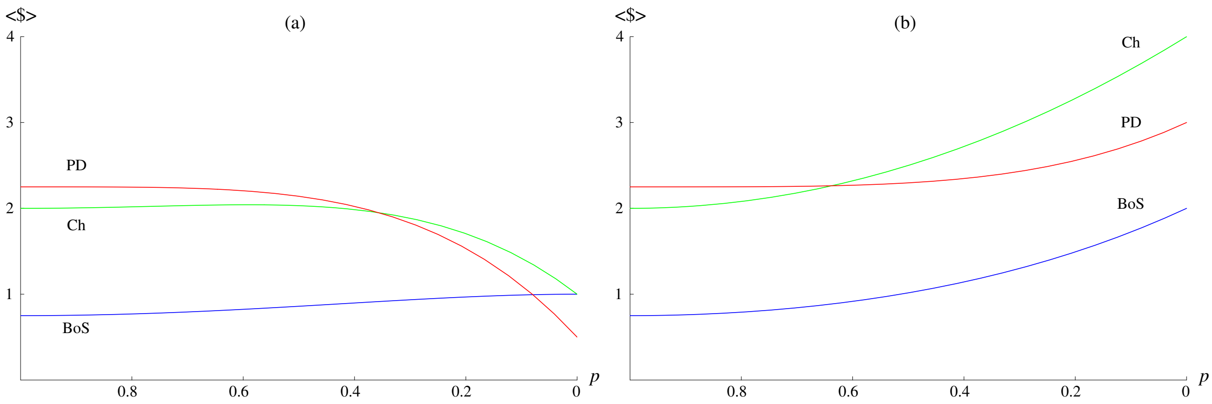


Figure 6.5. Payoffs with optimal strategies as a function of decoherence in Prisoners' Dilemma, Chicken and Battle of the Sexes. Payoffs as a function of decoherence probability p, going from fully decohered on the left ($p = 1$) to fully coherent on the right ($p = 0$), for (a) Alice and (b) Bob for the quantum games Prisoners' Dilemma (PD), Chicken (Ch) and the Battle of the Sexes (BoS). Bob plays the optimum quantum strategy and Alice her best classical counter strategy. As expected, the payoff to the quantum player, Bob, increases with increasing coherence while Alice performs worse except in the case of Battle of the Sexes. This game is a coordination game—both players do better if they select the same move—and Bob can increasingly engineer coordination as coherence improves, helping Alice as well as himself.

decoherence in infinite dimensional Hilbert space quantum games would need to go beyond the simple quantum operator method presented in this chapter and is yet to be considered.

This chapter has focused on *static* quantum games and so future work on game-theoretic methods for dynamic quantum systems with different types of decohering noise will be of great interest. A particular open question will be to compare the behaviour of such quantum games for (a) the non-Markovian case, where the quantum system is coupled to a dissipative environment with memory, (b) the Markovian (memoryless) limit where the correlation times, in the decohering environment, are small compared to the characteristic time scale of the quantum system.

Chapter 7

Quantum Parrondo's Games

PARRONDO games involve an apparent paradox where the mixture of two losing games creates a winning game. Positive results can be obtained with either periodic or random mixtures. Classical Parrondo's games and their relationship to Brownian ratchets have generated much interest. Meyer and Blumer (2002a) introduced a Parrondo game in the quantum sphere using the discretized Brownian motion of a particle in one dimension under the influence of a position-dependent potential. A quantum Parrondo model with history dependence was formulated in 2000 by Ng and Abbott (2004). Independently, Lee and Johnson (2002a) have suggested a method of exploiting Parrondo-like effects to generate quantum algorithms. This chapter presents a summary of the classical Parrondo games and details of the quantum versions. Some new calculations for both quantum models are presented. For the position-dependent quantum Parrondo game, the net gain resulting from various periodic sequences of the two games and for different parameter values is presented. Short sequences in the history-dependent quantum Parrondo game are studied and comparisons with the equivalent classical sequences are made.

7.1 Introduction

Parrondo's game is the name given to an apparent paradox that can arise from the mixing of two games of chance (Parrondo 1996). The defining feature of a Parrondo game is that a homogeneous sequence of either game gives rise to a losing process, while a random mixed sequence, or various periodic sequences, of the two games results in a winning process. The effect was first explored as a combination of two gambling games involving the use of biased coins (Harmer and Abbott 1999a).

The classical Parrondo game consists of two sub-games A and B. In the usual scenario, game A is the toss of a single biased coin, while game B utilizes two or more biased coins, the choice of which depends on the game situation. To obtain Parrondian behaviour, a form of feedback from the current game state is required. This can take the form of a dependence on the total capital (Harmer and Abbott 1999b), on past results (Parrondo et al. 2000), on the spatial neighbourhood (Toral 2001), or on spatial extension (Masuda and Kondo 2004). The capital- and history-dependent Parrondo games are the most intensely studied and the basic formalism is described in a number of papers (Harmer et al. 2000, Harmer and Abbott 2002, Johnson et al. 2003, Kay and Johnson 2003, Harmer et al. 2004).

A ratchet and pawl driven in one direction by the random thermal motion of the surrounding particles is discussed by Feynman et al. (1963). At thermal equilibrium, this is ruled out by the second law of thermodynamics, however, with a Brownian ratchet (Reimann 2002) or flashing ratchet (Doering 1995) directed motion can be obtained from random fluctuations or noise in the absence of systematic macroscopic forces. The flashing ratchet, consisting of a Brownian particle under the influence of a potential that is switched on or off either periodically or stochastically, provides a physical model for Parrondo's games (Harmer and Abbott 2001, Allison and Abbott 2002, Toral et al. 2003b). Classical Parrondo's games appear to be ubiquitous[14] and there is speculation that they may arise in many areas including population genetics (McClintock 1999), spin systems (Moraal 2000), control systems (Allison and Abbott 2001), biological systems at the molecular level (Astumian 2001), biogenesis (Davies 2001), and evolutionary processes (Abbott et al. 2002). There is even a suggestion of a profitable stockmarket trading strategy exploiting Parrondo's games (Klarreich 2001a). In all these examples the combination of processes leads to counterintuitive dynamics.

[14] Furthermore see Costa et al. (2004) where it is argued that ubiquity is the rule rather than the exception.

A quantum Parrondo game is a translation of the Parrondo effect into the quantum world. Quantum interference effects provide a mechanism, in additional to the classical feedback, that can enhance or inhibit the Parrondo effect. Quantum systems that are analogous to the capital- and the history-dependent Parrondo games have been proposed (Flitney et al. 2002, Meyer and Blumer 2002a, Ng and Abbott 2004) as well as other uniquely quantum variants (Lee et al. 2002).

7.2 Classical Parrondo's games

7.2.1 Capital-dependent games

In capital-dependent Parrondo's games, game A is the toss of a single biased coin with winning probability $p = 1/2 - \epsilon$, for some small $\epsilon > 0$, while game B employs two coins whose use depends on the total capital of the player: coin B_1 with winning probability p_1 is used if the capital is divisible by three, otherwise B_2 is used with winning probability p_2. This situation is shown schematically in Figure 7.1. By choosing, for example,

$$p_1 = 1/10 - \epsilon, \quad p_2 = 3/4 - \epsilon, \quad \epsilon > 0, \tag{7.1}$$

a net loss over time is generated (Harmer and Abbott 1999b). Although the weighted average of the winning probabilities in Eq. (7.1) is positive for small ϵ, the "bad" coin B_1 is used more often than the one-third of the time that might naively be expected. By mixing games A and B this effect is broken and the combination can now be winning provided the net positive effect of game B exceeds the negative bias of game A. Figure 7.2 shows the expected results over 100 coin tosses for some deterministic sequences of games A and B as well as the random sequence where the choice of game to be played at each step is determined by a fair coin.

7.2.2 History-dependent games

In the classical history dependent Parrondo's paradox, game A is as above, while game B is a collection of biased coins, the selection of which is dependent on the results of previous games as indicated in Figure 7.3. In order to obtain Parrondian behaviour, a dependence on at least the previous two results is necessary. An analysis of game B shows it to be losing for $\epsilon > 0$ when we choose (Parrondo et al. 2000)

$$p_1 = 9/10 - \epsilon, \quad p_2 = p_3 = 1/4 - \epsilon, \quad p_4 = 7/10 - \epsilon. \tag{7.2}$$

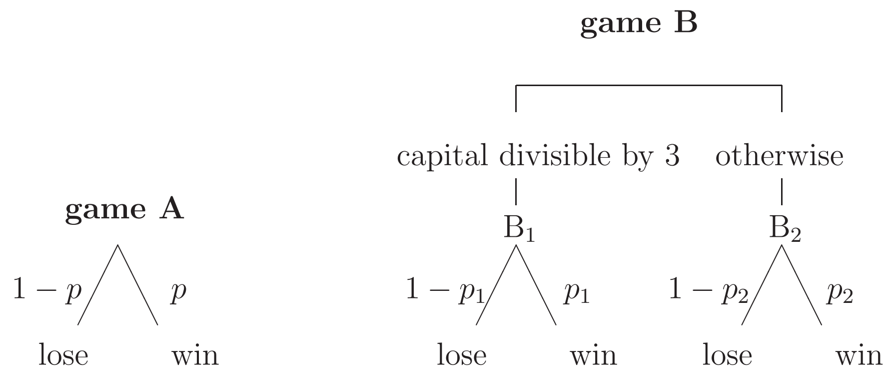


Figure 7.1. Classical capital-dependent Parrondo's game. Winning and losing probabilities for game A and the capital-dependent game B.

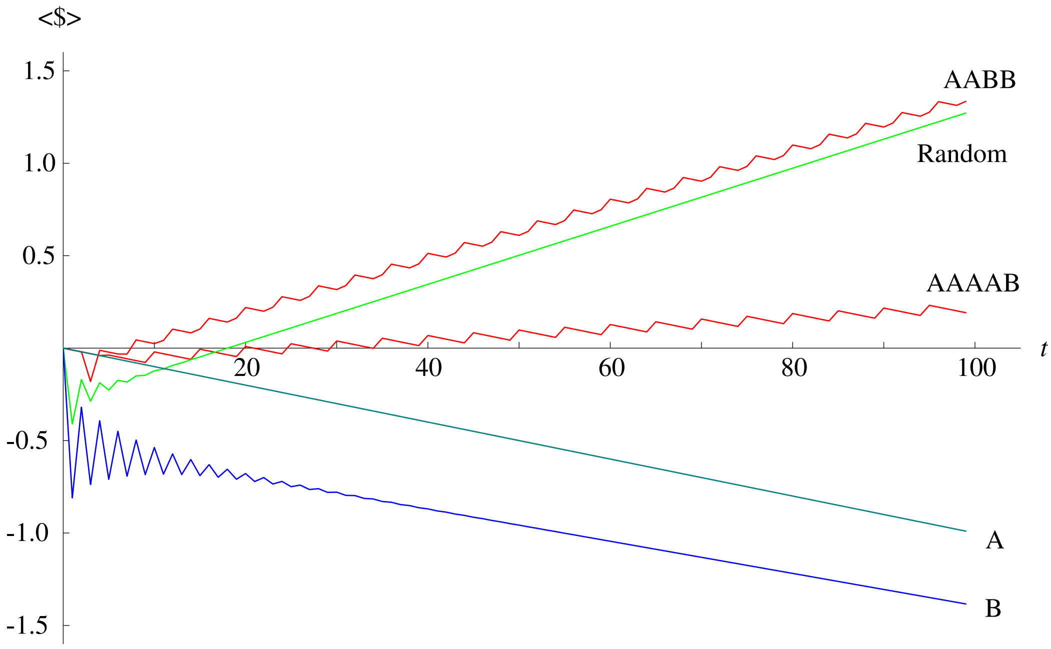


Figure 7.2. Results for a classical capital-dependent Parrondo game for various sequences. The expected gain from playing various sequences of games A and B with the winning probabilities $p = 1/2 - \epsilon$ for game A, and $p_1 = 1/10 - \epsilon$ and $p_2 = 3/4 - \epsilon$ for game B, where $\epsilon = 0.005$. The capital changes by one unit per game. The red line are the results for repeatedly playing the indicated sequence of games. The sequence marked random results from selecting A or B at each play using a fair coin. Time t is a measure of the number of games played. The curves are exact expectation values of the net gain. Equivalent curves based on the average over a large number of simulations are given by Harmer and Abbott (1999b).

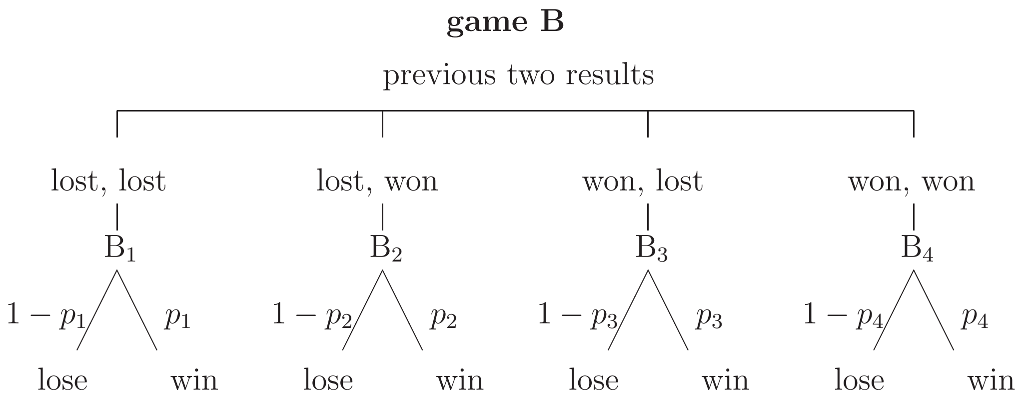


Figure 7.3. History-dependent Parrondo's games. Winning and losing probabilities for the history-dependent game B.

However, various sequences of A and B, including the random mixed sequence where a fair coin is used to select the game to be played at each step, produce a positive expected payoff provided $\epsilon < 1/168$ (Parrondo et al. 2000). Examples of the expected net gain versus the number of games played for various sequences of A and B are given in Figure 7.4. The effect can be generalized by replacing game A with another history dependent game (Kay and Johnson 2003). Indeed, game A with a single coin is a special case of a history-dependent game, where the same biased coin is used for all histories. *Mathematica* code for the capital- and the history-dependent Parrondo games is given in Appendix A.

7.2.3 Other classical Parrondo's games

Parrondo game models have been developed with multiple players. The model of Toral (2001) is a history-dependent model within a one-dimensional line of players. In game B the choice of coin depends upon the previous results of the two neighbouring players. Results are comparable to the one player history-dependent model. In another multiplayer model by the same author, game A is replaced by a redistribution of one unit of capital between two randomly selected players (Toral 2002). This model can be made equivalent to the original capital-dependent model by considering the new game A as two original (fair) game A's, a winning game played by the receiver, and a losing one played by the giver, of the capital in the redistribution.

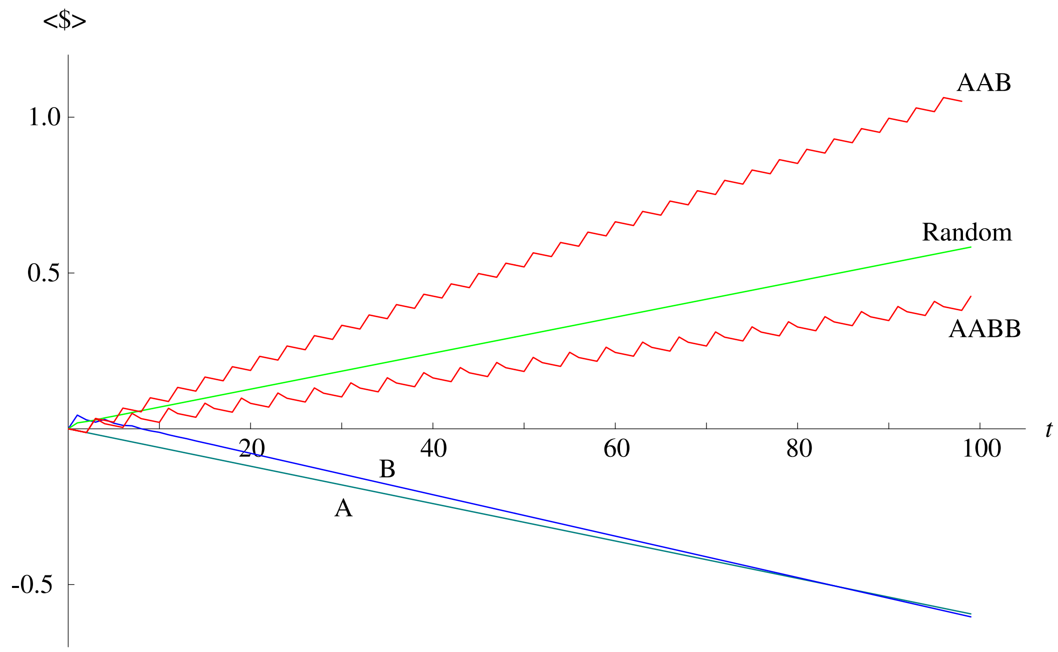


Figure 7.4. Results for a classical history-dependent Parrondo game for various sequences. The expected gain from playing various sequences of games A and B with $p = 1/2 - \epsilon$ for game A, and $p_1 = 9/10 - \epsilon$, $p_2 = p_3 = 1/4 - \epsilon$ and $p_4 = 7/10 - \epsilon$ for game B, where $\epsilon = 0.003$. The red lines are the results for repeatedly playing the indicated sequence of games. The sequence marked random results from selecting A or B at each play using a fair coin. The curves are exact expectation values of the net gain averaged over the four possible initial conditions. Note that the curves in Harmer and Abbott (2002) are for the starting condition 'loss-loss' and not an average over the four possibilities as stated. The resulting curves are parallel to the ones given here since only the initial transient behaviour is different.

Dinis and Parrondo (2002) show there is a risk in attempting to optimize using a short time horizon within a multiplayer capital-dependent Parrondo game. By choosing a game for the group that gives the maximum short-term return a net loss over time results while, in Parrondian fashion, a periodic or random mixed sequence of games yields a steady increase in capital for the group.

7.3 Quantum Parrondo's games

7.3.1 Position-dependent games

In the quantum analogue of the capital-dependent Parrondo game by Meyer and Blumer (2002a) the capital corresponds to a discretization of the position of a particle undergoing

Brownian motion in one dimension. The application of an appropriate potential produces the effect of games A or B. A potential uniformly increasing with x is the analogy of game A, while game B corresponds to a tilted sawtooth potential. The quantum "coin" is a two state system such as a spin–$\frac{1}{2}$ particle, in a superposition of the $|R\rangle$ and $|L\rangle$ states,[15] the eigenstates of σ_z. The quantum analogue of an unbiased coin flip is a unitary transformation represented by the matrix $\frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & i \\ i & 1 \end{smallmatrix}\right)$. Let $|x\rangle$ correspond to the gambling capital, and $|R\rangle$ and $|L\rangle$ indicate a win or a loss, respectively. Motion towards increasingly positive x corresponds to a winning process. An unbiased "coin" flip is effected by the unitary transformation

$$\begin{aligned} |x, L\rangle &\rightarrow \frac{1}{\sqrt{2}}\left(|x-1, L\rangle + i|x+1, R\rangle\right), \\ |x, R\rangle &\rightarrow \frac{1}{\sqrt{2}}\left(i|x-1, L\rangle + |x+1, R\rangle\right). \end{aligned} \tag{7.3}$$

The initial state is chosen to be $\frac{1}{\sqrt{2}}(|0, R\rangle - |0, L\rangle)$ so the particle begins with no particular momentum bias and an unbiased game A produces no net drift[16]. The effect of the potentials are implemented by multiplication by an x-dependent phase factor (Meyer 1997). The quantum version of the games is the unbiased transition in Eq. (7.3) multiplied by a phase $e^{-iV(x)}$ where

$$V_A(x) = \alpha x, \tag{7.4a}$$

$$V_B(x) = \alpha x + \beta(1 - \frac{1}{2}(x \bmod 3)), \tag{7.4b}$$

for games A and B, respectively. The potential $V_B(x)$ is indicated in Figure 7.5.

The analogy with the classical capital-dependent Parrondo game is not exact. In the quantum case, for game A $\langle x\rangle$ is periodic with period $2\pi/\alpha$. However, game A is losing in the sense that for $\alpha > 0$, $\langle x\rangle \leq 0$ for all times. The situation is similar for game B. For detail refer to Meyer and Blumer (2002a).

Choosing $\alpha = 2\pi/5000$ and $\beta = \pi/3$ gives results for the individual games comparable (within a factor of two) to the classical games with the probabilities of Eq. (7.1). Repeating the sequence[17] AAAAB produces one of the greatest positive movements of the particle,

[15] The notation $|R\rangle$ and $|L\rangle$ is used in preference to $|\uparrow\rangle$ and $|\downarrow\rangle$ for consistency with Chapter 8.

[16] In Travaglione and Milburn (2002) an unbiased quantum walk is created by using the Hadamard operator, represented by $\frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$, and starting with the initial state $\frac{1}{\sqrt{2}}(|0, L\rangle + i|0, R\rangle)$. The two schemes are equivalent.

[17] Meyer and Blumer (2002a) indicate that the sequence is BAAAA and this was repeated in the review by Flitney and Abbott (2003c). This was an error that has now been cleared up: four games of A are played first (Meyer 2003)—in other words AAAAB is the correct sequence. In fact, the sequence BAAAA produces net motion in the negative direction.

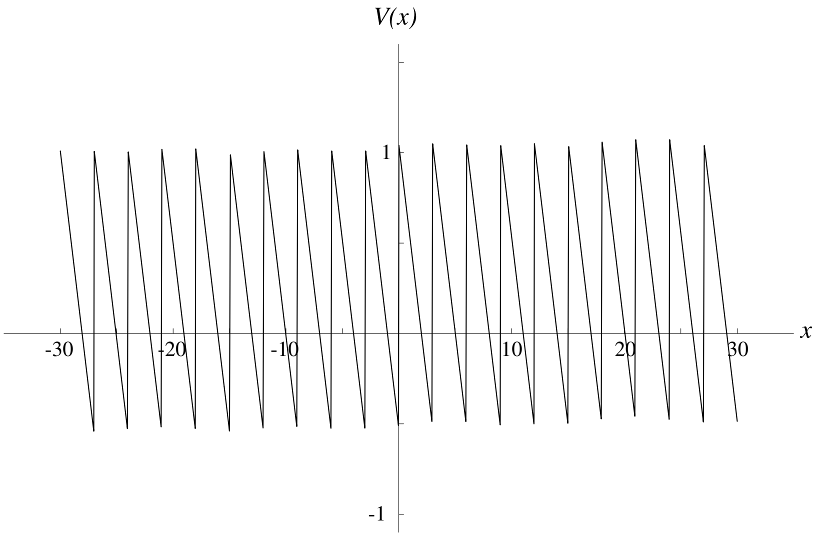


Figure 7.5. Tilted sawtooth potential. The tilted sawtooth potential $V_B(x)$ from Eq. (7.4) with $\alpha = 2\pi/5000$ and $\beta = \pi/3$.

as indicated in Figures 7.6 and 7.7. There is an extreme sensitivity to initial conditions that is not present in the classical situation. For example, playing B first prior to the sequences AAAAB or AABB yields a negative expectation for x after 100 plays, instead of a positive one. Playing the single game B can be considered as a change in the initial state of the particle. The following game sequences are then unaltered. In the classical scenario, such a change merely has the effect of an initial offset for the expected gain versus time curve, without influencing the long term trend. The effect of varying the parameters α and β in the potentials can be observed in Figure 7.8 for the (mostly) winning periodic sequences AABB and AAAAB.

7.3.2 History-dependent games

The history-dependent Parrondo game has been quantized directly by replacing the rotation of a bit, representing a toss of a classical coin, by an SU(2) operation on a qubit (Flitney et al. 2002, Ng and Abbott 2004) where 1 represents a win and 0 a loss:

$$\hat{U}(\theta, \alpha, \beta) = \begin{pmatrix} e^{i\alpha}\cos(\theta/2) & ie^{i\beta}\sin(\theta/2) \\ ie^{-i\beta}\sin(\theta/2) & e^{-i\alpha}\cos(\theta/2) \end{pmatrix}, \tag{7.5}$$

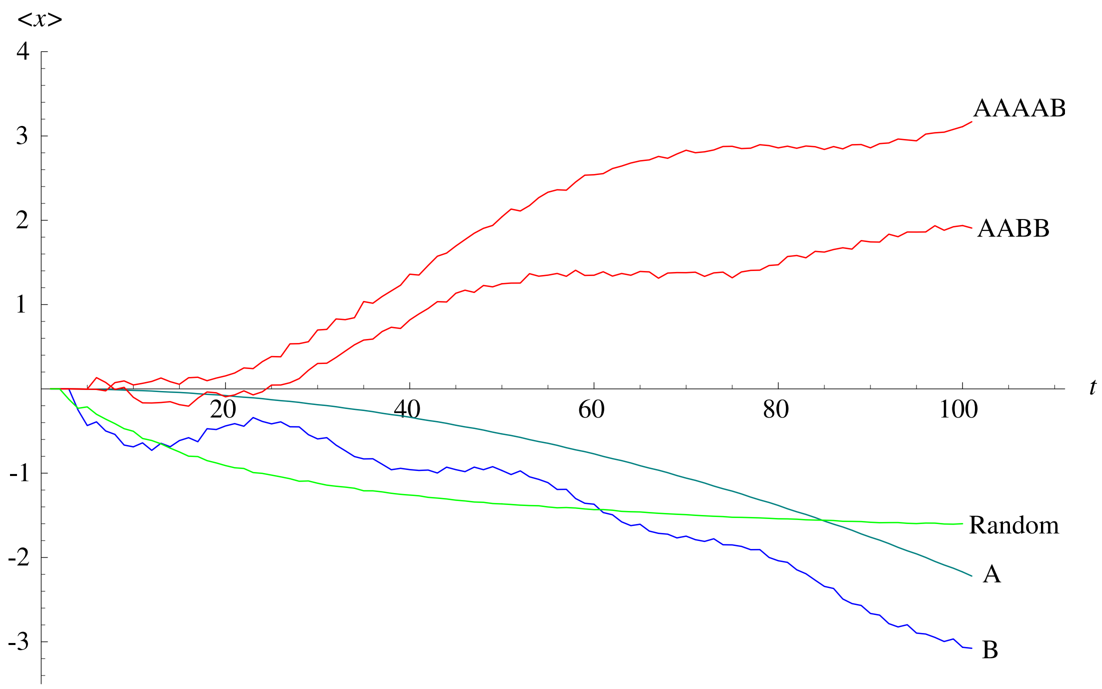


Figure 7.6. Expected gain for a quantum position-dependent Parrondo game for various sequences. The expectation value of the gain for the quantum games A and B, some periodic mixed sequences of A and B and a random mixed sequence. The random curve is obtained by selecting game A or B at each step using a fair coin and is the average over 500 trials. In the quantum game, the position x corresponds to the capital \$ in the classical game. A win in the quantum game moves the particle one unit in the positive x direction, while a loss moves it one unit in the negative x direction. The parameters in Eq. (7.4) have been set to $\alpha = 2\pi/5000$ and $\beta = \pi/3$.

where $\theta \in [0, \pi]$ and $\alpha, \beta \in [0, 2\pi]$. Game A is carried out by the operation $\hat{A} = \hat{U}(\theta_0, \alpha_0, \beta_0)$. The $\hat{B}$ operator consists of four control-control SU(2) operations (as indicated in Figure 7.9):

$$\hat{B}(\theta_1, \alpha_1, \beta_1, \theta_2, \alpha_2, \beta_2, \theta_3, \alpha_3, \beta_3, \theta_4, \alpha_4, \beta_4) = \begin{bmatrix} \hat{U}(\theta_1, \alpha_1, \beta_1) & 0 & 0 & 0 \\ 0 & \hat{U}(\theta_2, \alpha_2, \beta_2) & 0 & 0 \\ 0 & 0 & \hat{U}(\theta_3, \alpha_3, \beta_3) & 0 \\ 0 & 0 & 0 & \hat{U}(\theta_4, \alpha_4, \beta_4) \end{bmatrix}. \tag{7.6}$$

This acts on the three-qubit state $|q_{t-2}\rangle \otimes |q_{t-1}\rangle \otimes |q\rangle$, where $|q_{t-1}\rangle$ and $|q_{t-2}\rangle$ represent the results of the two previous games and $|q\rangle$ is the initial state of the target qubit. That is,

$$\hat{B}|q_{t-2}\, q_{t-1}\, q\rangle = |q_{t-2}\, q_{t-1}\, q_t\rangle, \tag{7.7}$$

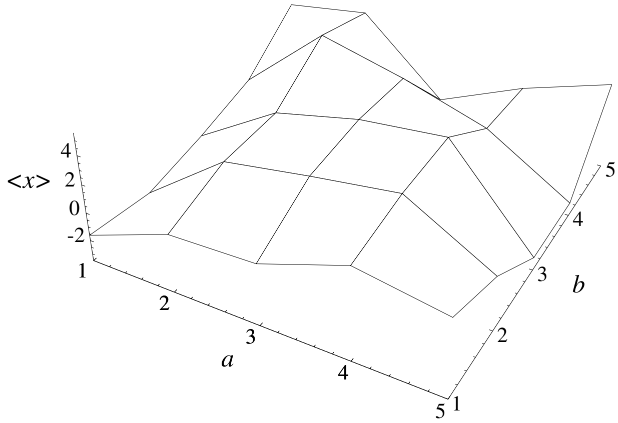


Figure 7.7. Expected gain for a quantum position-dependent Parrondo game as a function of game mixture. The expectation value of the gain after 100 games of a periodic mixed sequence created by repeating a games of A followed by b games of B, with $\alpha = 2\pi/5000$ and $\beta = \pi/3$. For example, $a = 4$, $b = 1$ represents the sequence AAAAB repeated 20 times, giving a total of 100 games.

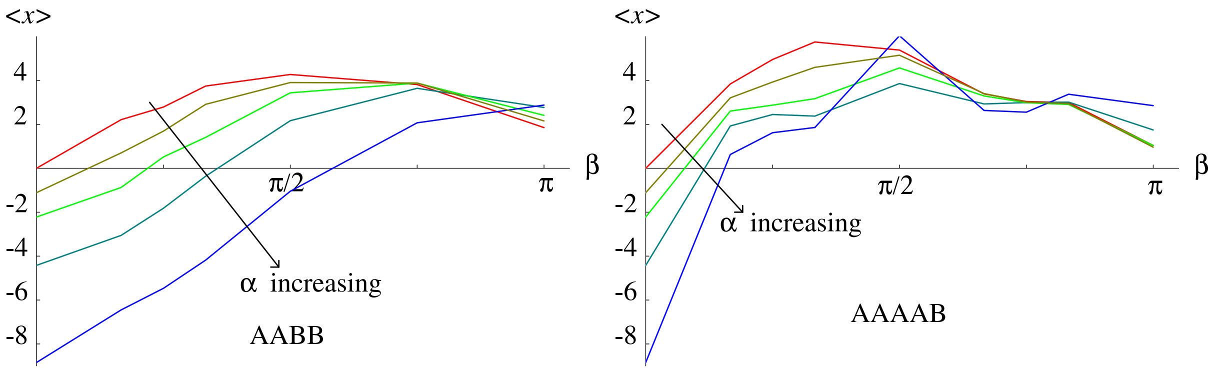


Figure 7.8. Expected gain for a quantum position-dependent Parrondo game for various parameter values in the potentials. The expectation value of the gain after 100 games using the sequence (a) AABB or (b) AAAAB, for various values of α and β. The curves are for $\alpha = 0$, $\alpha = \pi/5000$, $\alpha = \pi/2500$, $\alpha = \pi/1250$, and $\alpha = \pi/625$, increasing in the direction indicated.

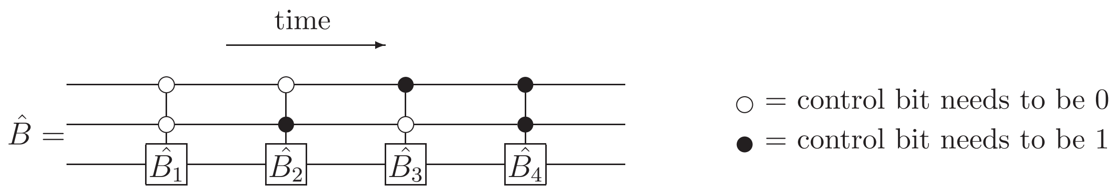


Figure 7.9. Quantum circuit for a history-dependent Parrondo game. In the history-dependent quantum Parrondo game, $\hat{B}$ consists of four control-control rotations depending on the four possible states of the two control qubits.

where q_t is the result of the game B.

The initial state $|\psi_i\rangle$ consists of one qubit for each game to be played, equivalent to a pile of coins each of which is tossed in succession. The payoff is the excess of the number of 1's over 0's in the final state $|\psi_f\rangle$. The quantum system analogous to classical games A and B with a given set of probabilities has $\sin^2(\theta_0/2) = p$ and $\sin^2(\theta_i/2) = p_i,\ i = 1, \ldots, 4$. The corresponding classical history-dependent Parrondo game is reproduced when $|\psi_i\rangle = |00\ldots0\rangle$. Quantum effects begin to appear when the initial state is a superposition of computational basis states. For example,

$$|\psi_i\rangle = (\,|00\ldots0\rangle\,+\,|11\ldots1\rangle\,)/\sqrt{2}, \tag{7.8}$$

leads to interference, effectively between two different games, those with initial states $|00\ldots0\rangle$ and $|11\ldots1\rangle$. The payoff is then dependent on the phase angles α_i and β_i in the A and B operators. By judicious selection of the phases, the extent of the interference can be controlled, either enhancing or diminishing the payoff.

The results of n successive games of B can be computed by

$$\begin{aligned}|\psi_f\rangle = (\hat{I}^{\otimes n-1} \otimes \hat{B})(\hat{I}^{\otimes n-2} \otimes \hat{B} \otimes \hat{I})(\hat{I}^{\otimes n-3} \otimes \hat{B} \otimes \hat{I}^{\otimes 2})\\ \ldots(\hat{I} \otimes \hat{B} \otimes \hat{I}^{\otimes n-2})(\hat{B} \otimes \hat{I}^{\otimes n-1})\,|\psi_i\rangle,\end{aligned} \tag{7.9}$$

with $|\psi_i\rangle$ being an initial state of $n+2$ qubits. The first two qubits of $|\psi_i\rangle$ are left unchanged and are only necessary as an input to the first game of B. In this and Eq. (7.10), $\hat{I}$ is the identity operator for a single qubit. The flow of information in this protocol is shown in Figure 7.10(a). The result of other game sequences can be computed in a similar manner. Figure 7.10(b) shows the information flow for an alternating sequence of A and B. The simplest case to study is that of two games of A followed by one game of B, since the results of one set of games do not feed into the next. The sequence AAB played n

times results in the state

$$
\begin{aligned}
|\psi_f\rangle = & \left(\hat{I}^{\otimes 3n-3} \otimes (\hat{B}(\hat{A} \otimes \hat{A} \otimes \hat{I}))\right) \\
& \left(\hat{I}^{\otimes 3n-6} \otimes (\hat{B}(\hat{A} \otimes \hat{A} \otimes \hat{I})) \otimes \hat{I}^{\otimes 3}\right) \\
& \ldots \left((\hat{B}(\hat{A} \otimes \hat{A} \otimes \hat{I})) \otimes \hat{I}^{\otimes 3n-3}\right) |\psi_i\rangle \\
= & \hat{G}^{\otimes n} |\psi_i\rangle,
\end{aligned}
\tag{7.10}
$$

where $\hat{G} = \hat{B}(\hat{A} \otimes \hat{A} \otimes \hat{I})$ and $|\psi_i\rangle$ is an initial state of $3n$ qubits. The information flow for this sequence is shown in Figure 7.10(c).

To determine the expected gain from a sequence of games let the payoff for a $|1\rangle$ state be one and for a $|0\rangle$ state be negative one. If the final state is $|\psi_f\rangle$, the expected gain can be computed by

$$
\langle \$ \rangle = \sum_{j=0}^{n} \left((2j - n) \sum_{j'} \left| \langle \psi_j^{j'} | \psi_f \rangle \right|^2 \right), \tag{7.11}
$$

where the second summation is taken over all basis states $\langle \psi_j^{j'} |$ with j ones and $n - j$ zeros.

7.4 New results for a quantum history-dependent game

Consider the game sequence AAB. With an initial state of $|000\rangle$ this yields a payoff of

$$
\begin{aligned}
\langle \$^0_{\mathrm{AAB}} \rangle = & \sin^4(\theta_0/2)\,(2 - \cos\theta_4) \;-\; \cos^4(\theta_0/2)\,(2 + \cos\theta_1) \\
& - \frac{1}{4} \sin^2\theta_0\,(\cos\theta_2 + \cos\theta_3),
\end{aligned}
\tag{7.12}
$$

which is the same as the classical result. In order to obtain interference there needs to be two different ways of arriving at the same state. We need only choose an initial state that is some superposition of the computational basis states, not necessarily the maximally entangled state, however, it is this that is the most interesting to study. Choosing $|\psi_i^{\mathrm{m}}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ the resulting expected payoff for AAB is

$$
\begin{aligned}
\langle \$^{\mathrm{m}}_{\mathrm{AAB}} \rangle = & \frac{1}{2} \cos\theta_0\,(\cos\theta_4 - \cos\theta_1) \\
& - \frac{1}{4} \sin^2\theta_0\,[\sin(2\alpha_0 + \alpha_1 - 2\beta_0 - \beta_1)\,\sin\theta_1 \\
& - \sin(2\alpha_0 + \alpha_2 - 2\beta_0 - \beta_2)\,\sin\theta_2 \;-\; \sin(2\alpha_0 + \alpha_3 - 2\beta_0 - \beta_3)\,\sin\theta_3 \\
& + \sin(2\alpha_0 + \alpha_4 - 2\beta_0 - \beta_4)\,\sin\theta_4]\,.
\end{aligned}
\tag{7.13}
$$

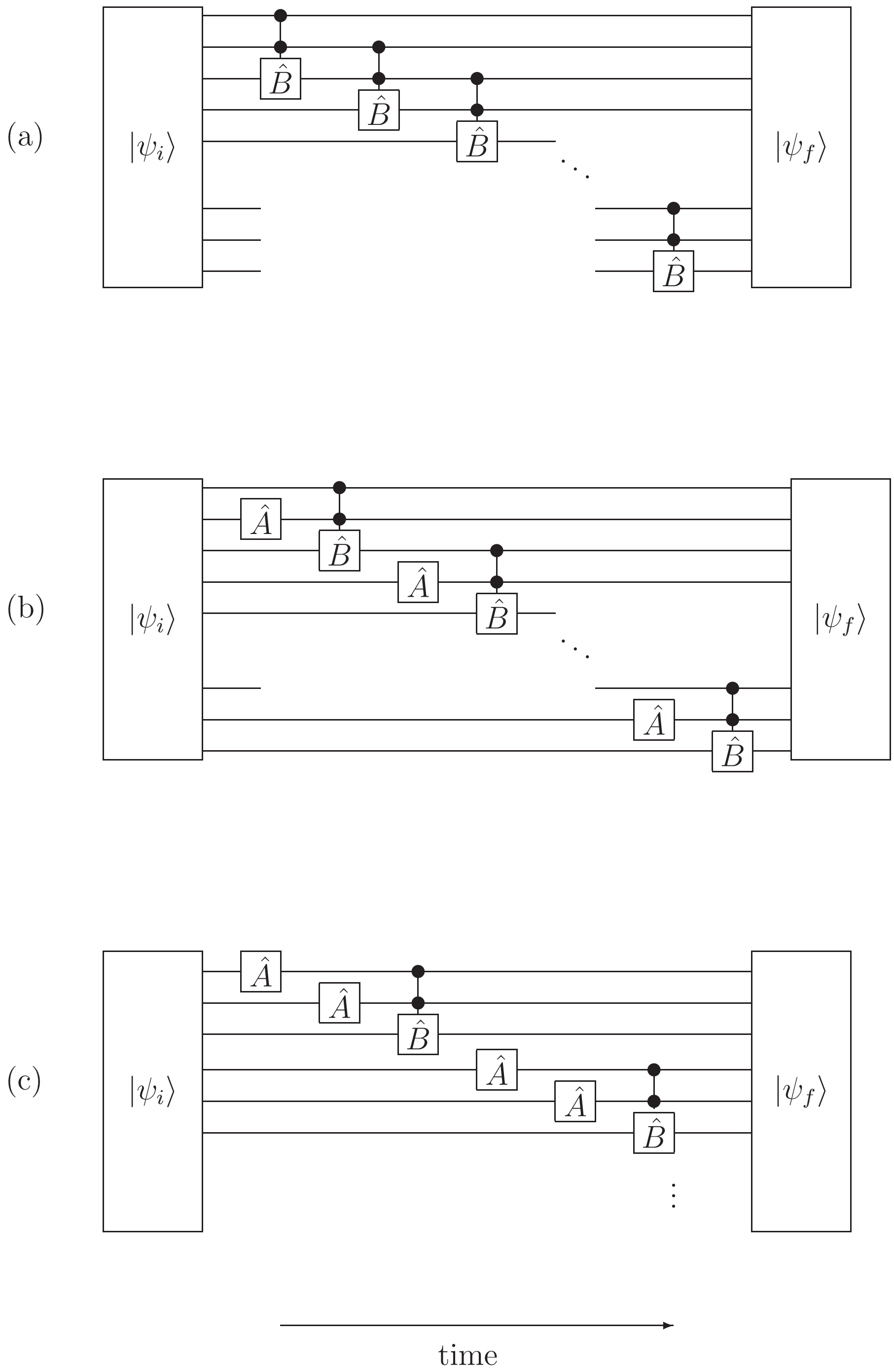


Figure 7.10. Quantum circuits for various periodic sequences of games A and B in a history-dependent Parrondo game. The information flow in qubits (solid lines) in a series of (a) B, (b) an alternating sequence of A and B, and (c) two games of A followed by one of B. In each case a measurement on $|\psi_f\rangle$ is taken on completion of the series to determine the payoff. Note in (c) that the output of one set of AAB does not feed into the next, so that each set of three games decouple from the remainder.

It is the dependence on the phase angles α_i and β_i that permit a result that cannot be obtained from the classical games. In the quantum case, a range of payoffs can be obtained for a given set of θ's (that is, for a given set of probabilities for games A and B) by adjusting these phase factors.

After choosing θ_i's corresponding to the probabilities in Eq. (7.2), the expectation value of the payoff [to $O(\epsilon)$] in the quantum system for a single sequence of AAB can vary between $0.812 + 0.24\epsilon$ and $-0.812 + 0.03\epsilon$. The maximum result is obtained by setting

$$\begin{aligned} \alpha_2 - \beta_2 = \quad \alpha_3 - \beta_3 = \quad & \pi/2 - 2\alpha_0 + 2\beta_0; \\ \alpha_1 - \beta_1 = \quad \alpha_4 - \beta_4 = \quad & 3\pi/2 - 2\alpha_0 + 2\beta_0, \end{aligned} \tag{7.14}$$

while the minimum is obtained by

$$\begin{aligned} \alpha_1 - \beta_1 = \quad \alpha_4 - \beta_4 = \quad & \pi/2 - 2\alpha_0 + 2\beta_0; \\ \alpha_2 - \beta_2 = \quad \alpha_3 - \beta_3 = \quad & 3\pi/2 - 2\alpha_0 + 2\beta_0. \end{aligned} \tag{7.15}$$

Classically, AAB is a winning sequence provided $\epsilon < 1/112$. This and other results for short sequences of games are given in Table 7.1.

The average payoff for the classical game sequence AAB_1 (that is, AAB where each branch of B is the best branch B_1) is $\frac{4}{5} - 6\epsilon$ which is less than the greatest value of $\langle \$^{\mathrm{m}}_{\mathrm{AAB}} \rangle$. Thus the entanglement and the resulting interference can make game B in the sequence AAB better than its best branch taken alone! Indeed the expectation value for the payoff of a quantum AAB_1 on the maximally entangled initial state vanishes due to destructive interference. (This can be seen from Eq. (7.13) by setting θ_2, θ_3 and θ_4 equal to θ_1 and similarly for the α's and β's.)

The quantum enhancement disappears when we play a series of AAB's on the maximally entangled initial state. In this case the phase dependent terms undergo destructive interference and we are left with a gain per qubit of order ϵ as indicated in the last line of Table 7.1.

A sequence of B's leaves the first two qubits unaltered while a sequence of AB's leaves the first qubit unaffected. In these cases the final states that arise from $|\psi_i\rangle = |000\rangle$ and $|\psi_i\rangle = |111\rangle$ are distinct, so a superposition of these two states produces no interference. For these sequences, a different superposition for the initial state is required to give rise to interference effects.

sequence	classical payoff	quantum payoff	
AA ... A	-2ϵ	0	
B	$1/60 - 2\epsilon/3$	$1/15$	
BB	$1/75 - 19\epsilon/15$	$13/400 + \epsilon/20$	
BBB	$0.008 - 1.1\epsilon$	$0.017 + 0.03\epsilon$	
AB	$1/60 - 19\epsilon/15$	$1/30 + \epsilon/15$	
ABAB	$0.032 - 2.5\epsilon$	$0.019 + 0.08\epsilon$	
AAB	$1/60 - 28\epsilon/15$	$-0.271 + 0.03\epsilon$;	$0.271 + 0.24\epsilon$
AAB ... AAB	$1/60 - 28\epsilon/15$	$2\epsilon/15$	

Table 7.1. Expected payoffs per qubit for various sequences in a history-dependent Parrondo game. The classical payoffs are the average over the four possible initial conditions, while the quantum payoffs are calculated for the maximally entangled initial state, $\frac{1}{\sqrt{2}}(|00\ldots0\rangle + |11\ldots1\rangle)$. For the sequence AAB the two values given for the quantum payoff are the minimum and maximum that can be obtained by adjusting the phase factors in $\hat{A}$ and $\hat{B}$ (see text). All payoffs are given to $O(\epsilon)$.

7.5 Other quantum Parrondian behaviour

Lee and Johnson consider how decoherence in a quantum system can be suppressed by a Parrondo-like effect (Lee et al. 2002, Lee and Johnson 2002a). In addition, the authors approach the construction of Grover's search algorithm with a view to exploiting Parrondian behaviour. In their model "games" A and B represent a partitioning of the steps in Grover's search algorithm. Neither step alone is efficient, but by randomly combining A and B the original algorithm is recreated, thus giving a constructive role to randomness in the creation of quantum algorithms.

7.6 Summary

Parrondian behaviour arises in the mixing of two games when the surface dividing the winning and losing regions of the parameter space of the games is convex (Harmer and Abbott 2002, Harmer et al. 2004). This means that a convex linear combination of two losing games can become a winning game, or vice versa. Classical Parrondo's games have a physical analogue in Brownian or flashing ratchets. A summary of the classical Parrondo effect has been presented in order to motivate the study of quantum versions. Results for both the capital- and history-dependent games have been presented, in the latter case correcting earlier published results.

A position-dependent quantum Parrondo game has been described (Meyer and Blumer 2002a, Meyer and Blumer 2002b) and new results for various periodic sequences of games and different parameter values in the biasing potentials have been given. A quantum version of a history-dependent Parrondo game has been detailed (Flitney et al. 2002, Ng and Abbott 2004) and the results of short sequences of games presented. If the initial state is a superposition of basis states, payoffs different from the corresponding classical game can be obtained as a result of interference. In some cases payoffs can be considerably altered by adjusting the phase factors associated with the operators without altering the amplitudes (and hence the associated classical probabilities). If the initial state is simply $|00\ldots0\rangle$ the payoffs are independent of the phases and are no different from the classical payoffs. In other cases it is possible to obtain either much larger or smaller payoffs, provided the initial state involves a superposition that gives the possibility of interference for that particular game sequence.

In classical gambling games there is a random element. In a Parrondo game the results of the random process are used to alter the evolution of the game through a form of feedback. The quantum mechanical model is deterministic until a measurement is made at the end of the process. The element of chance that is necessary in the classical game is replaced by a superposition that represents all the possible results in parallel. New behaviour can arise by the addition of phase factors in the operators and by interference between states. Another random element, warranting further study, can be introduced by perturbing the system with noise or decoherence (Meyer 2003).

Classical Parrondo games can be extracted by discretizing the classical Fokker-Planck equation (Allison and Abbott 2003, Toral et al. 2003a, Amengual et al. 2004). It is therefore an interesting open question whether the quantum Fokker-Planck equation (Banik et al. 2002) can be used to generate quantum Parrondo games. In future work, it will also be interesting to consider whether entanglement alone can provide the coupling between the games and give rise to quantum Parrondian behaviour.

Chapter 8

Quantum Walks with History Dependence

QUANTUM walks are the quantum analogue of classical random walks and display some interesting features that make them a plausible candidate for use in quantum computation. In this chapter a brief overview of quantum walks and their main features is presented, before detailing new work on a history-dependent quantum walk that can give rise to another quantum Parrondo's game. A multi-coin discrete-time quantum walk is introduced where the amplitude for a coin flip depends upon previous results. Although the corresponding history-dependent classical random walk is unbiased, a bias can be introduced into the quantum walk by varying the history dependence. By mixing a biased and an unbiased quantum walk, the direction of the bias can be reversed leading to a new quantum version of Parrondo's paradox. Two-, three- and four-coin history-dependent quantum walks, the effect of the biasing parameters, and the new quantum Parrondo effect are discussed.

8.1 Introduction

8.1.1 Motivation

Classical random walks have long been a powerful tool in mathematics, have a number of applications in theoretical computer science (Papadimiriou 1994, Motwani and Raghavan 1995) and form the basis for much computational physics, such as Monte Carlo simulations. This has inspired significant interest in quantum walks,[18] both in continuous-time (Farhi and Gutman 1998, Childs et al. 2002) and in discrete-time (Aharanov et al. 1993, Meyer 1996, Aharonov et al. 2001, Ambainis et al. 2001, Watrous 2001). Meyer has shown that a discrete-time, discrete-space, quantum walk requires an additional degree of freedom (Meyer 1996), or quantum "coin," and can be modeled by a quantum lattice gas automaton (Meyer and Blumer 2002a). However, an approximately unitary quantum walk can be modeled without a coin state, leading to very similar behaviour (Patel et al. 2004).

Quantum walks reveal a number of startling differences to their classical counterparts. In particular, diffusion on a line is quadratically faster (Nayak and Vishwanath 2000, Travaglione and Milburn 2002), while propagation across some graphs is exponentially faster (Childs et al. 2002, Childs et al. 2003). Quantum walks show promise as a means of implementing quantum algorithms. A discrete-time, coined quantum walk is able to find a specific item in an unsorted database with a quadratic speedup over the best classical algorithm (Shenvi et al. 2003), a performance equal to Grover's algorithm. A spatial search by a continuous-time quantum walk on a $d > 4$ dimensional lattice also shows significant speed-up over its classical counter part (Childs and Goldstone 2004). Several methods for implementing quantum walks have been proposed, including on an ion trap computer (Travaglione and Milburn 2002), on an optical lattice (Dür et al. 2002), and using cavity quantum electrodynamics (Sanders et al. 2003). A simple continuous-time quantum walk has been experimentally demonstrated on a two qubit nuclear magnetic resonance machine (Du et al. 2003b). An overview of quantum walks is given by Kempe (2003).

[18] The word "random" has been dropped from the name since in the quantum case the time evolution is deterministic, the system evolving into a superposition of all possible states. Randomness is only introduced if a measurement is taken on the final state. In the literature both the terms "quantum random walk" and "quantum walk" are used and the meanings are identical. However, "quantum walk" is increasingly recognized as the preferred term.

8.1.2 Single coin quantum walk

A direct translation of a classical discrete random walk into the quantum domain is not possible. If a quantum particle moving in one-dimension along a line is updated at each step, in superposition, to the left and right, the global process is necessarily non-unitary. However, the addition of a second degree of freedom, the chirality, taking values L and R, allows interesting quantum walks to be constructed. Consider a particle whose position is discretized in one-dimension. Let $\mathcal{H}_{\rm P}$ be the Hilbert space of particle positions, spanned by the basis $\{|x\rangle : x \in \mathbb{Z}\}$. In each time-step the particle will move either to the left or right depending on its chirality. Let $\mathcal{H}_{\rm C}$ be the Hilbert space of chirality, or "coin" states, spanned by the orthonormal basis $\{|L\rangle, |R\rangle\}$. A simple quantum walk in the Hilbert space $\mathcal{H}_{\rm P} \otimes \mathcal{H}_{\rm C}$ consists of a quantum mechanical "coin toss," a unitary operation $\hat{U}$ on the coin state, followed by the updating of the position to the left or right:

$$\hat{E} = (\hat{S} \otimes \hat{\mathcal{P}}_R \; + \; \hat{S}^{-1} \otimes \hat{\mathcal{P}}_L)(\hat{I}_{\rm P} \otimes \hat{U}), \tag{8.1}$$

where $\hat{S}$ is the shift operator in position space, $\hat{S}|x\rangle = |x{+}1\rangle$, $\hat{I}_{\rm P}$ is the identity operator in position space, and $\hat{\mathcal{P}}_R$ and $\hat{\mathcal{P}}_L$ are projection operators on the coin space with $\hat{\mathcal{P}}_R{+}\hat{\mathcal{P}}_L = \hat{I}_{\rm C}$, the coin identity operator. For example, a walk controlled by an unbiased quantum coin is carried out by the transformations

$$\begin{aligned} |x, L\rangle &\to \frac{1}{\sqrt{2}}\left(|x-1, L\rangle \; + \; i|x+1, R\rangle\right), \\ |x, R\rangle &\to \frac{1}{\sqrt{2}}\left(i|x-1, L\rangle \; + \; |x+1, R\rangle\right). \end{aligned} \tag{8.2}$$

Figure 8.1 shows the distribution of probability density after 100 steps of Eq. (8.2) with the initial state $|\psi_0\rangle = (|0, L\rangle - |0, R\rangle)/\sqrt{2}$. Notice that the scheme of Eq. (8.2) is equivalent to the Hadamard quantum walk with initial state $(|0, L\rangle + i|0, R\rangle)/\sqrt{2}$. The initial state $|\psi_0\rangle$ is chosen so that a symmetrical distribution results. In fact the states $|0, R\rangle$ and $|0, L\rangle$ evolve independently. This can be seen since any flip $|R\rangle \leftrightarrow |L\rangle$ involves multiplication by a factor of i. Thus, any $|x, L\rangle$ state that started from $|0, R\rangle$ will be multiplied by an odd power of i and is orthogonal to any $|x, L\rangle$ state that originated from $|0, L\rangle$, and similarly for the $|x, R\rangle$ states. Figure 8.1 contrasts sharply with a classical coined random walk, which gives rise to a Gaussian distribution spreading in proportion to $\sqrt{t}$.

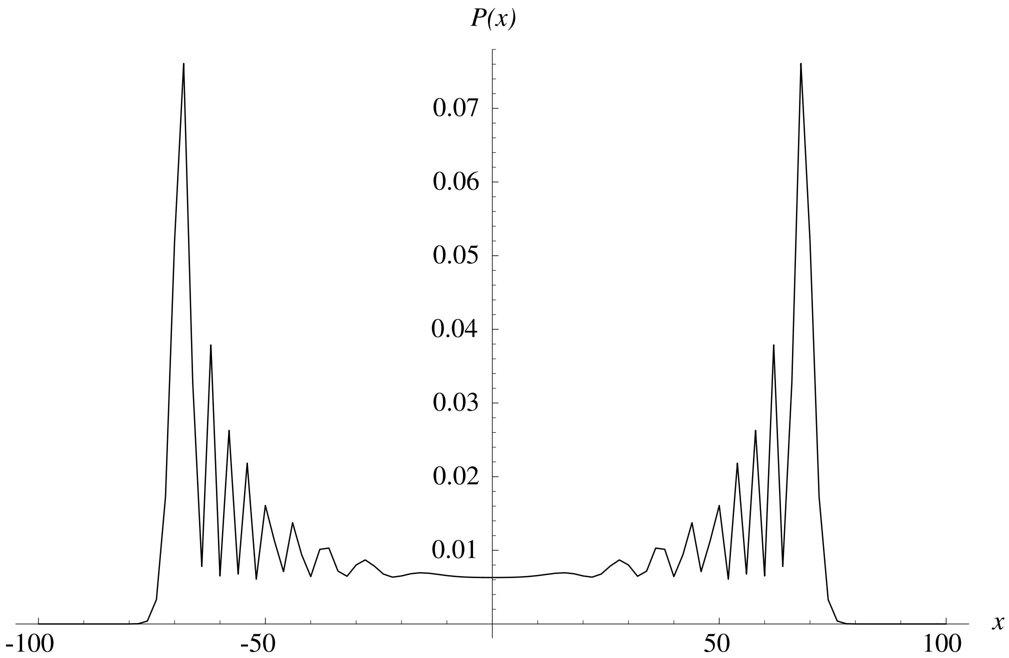


Figure 8.1. Probability density distribution for an unbiased quantum walk. The distribution of probability density $P(x) = |\psi(x)|^2$ at toss $t = 100$ for an unbiased, single coin quantum walk with $|\psi_0\rangle = (|0, L\rangle - |0, R\rangle)/\sqrt{2}$. Only even positions are plotted since $\psi(x)$ is zero for odd x at $t = 100$. The graph is the same as Figure 7 in Meyer and Blumer (2002a) except for the smoothing technique. The total area under the graph is equal to one.

8.2 History-dependent multi-coin quantum walk

To construct a quantum walk with history dependence requires an extension of the Hilbert space by additional coin states. Where there is a dependence on the last $M - 1$ results, the total system Hilbert space is a direct product between the particle position in one dimension and M coin states:

$$\mathcal{H} = \mathcal{H}_{\mathrm{P}} \otimes ({\mathcal{H}_{\mathrm{C}}}^{\otimes M}). \tag{8.3}$$

The M coins represent the results of tosses at times $t-1, t-2, \ldots, t-M$. A single step in the walk consists of tossing the Mth coin, updating the position depending on the result of the toss, and then re-ordering the coins so that the newly tossed coin is in the first (most recent) position. In general, the unitary coin operator $\hat{U}$ can be specified, up to an overall phase that is not observable, by three parameters, two of which are phases. In the single coin case the effect of the phases can be completely mimicked by changes to $|\psi_0\rangle$ (Ambainis et al. 2001, Tregenna et al. 2003). This does not carry over to our multi-coin history-dependent scheme. However, for the sake of simplicity the phases shall be omitted, giving

$$\hat{U}(\rho) = \begin{pmatrix} \sqrt{\rho} & i\sqrt{1-\rho} \\ i\sqrt{1-\rho} & \sqrt{\rho} \end{pmatrix}, \tag{8.4}$$

where $1-\rho$ is the classical probability that the coin changes state, with $\rho = \frac{1}{2}$ being an unbiased coin. To allow for history dependence, ρ will depend upon the results of the last $M-1$ coin tosses, so that a single step is effected by the operator

$$\begin{aligned} \hat{E} = &\left(\hat{S} \otimes \hat{I}_{\mathrm{C}}{}^{\otimes(M-1)} \otimes \hat{\mathcal{P}}_R \,+\, \hat{S}^{-1} \otimes \hat{I}_{\mathrm{C}}{}^{\otimes(M-1)} \otimes \hat{\mathcal{P}}_L\right) \\ &\times \left(\hat{I}_{\mathrm{P}} \otimes \sum_{j_1,\ldots,j_{M-1} \in \{L,R\}} \hat{\mathcal{P}}^*_{j_1 \cdots j_{M-1}} \otimes \hat{U}(\rho_{j_1 \cdots j_{M-1}})\right), \end{aligned} \tag{8.5}$$

where $\hat{\mathcal{P}}_j,\ j \in \{L,R\}$ is the projection operator of the Mth coin onto the state $|j\rangle$ and $\hat{\mathcal{P}}^*_{j_1 \cdots j_{M-1}},\ j_k \in \{L,R\}$ is the projection operator of the first $M-1$ coins onto the state $|j_1 \ldots j_{M-1}\rangle$. The second parenthesized term in Eq. (8.5) flips the Mth coin with a parameter ρ that depends upon the state of the first $M-1$ coins, while the first term updates the particle position depending on the result of the flip. Re-ordering of the coins is then achieved by

$$\hat{O} = \hat{I}_{\mathrm{P}} \otimes \sum_{j_1,\ldots,j_M \in \{L,R\}} |j_M\, j_1 \ldots j_{M-1}\rangle\langle j_1 \ldots j_{M-1}\, j_M|. \tag{8.6}$$

This scheme is distinct from that of Brun et al. (2003b) on quantum walks with multiple coins, where the walk is carried out by cycling through a given sequence of M coins, $\hat{U}(\rho_1), \ldots, \hat{U}(\rho_M)$. In Brun's scheme, a coin toss is performed by

$$\hat{E} = (\hat{S} \otimes \hat{I}_{\mathrm{C}}{}^{\otimes(M-1)} \otimes \hat{\mathcal{P}}_R \,+\, \hat{S}^{-1} \otimes \hat{I}_{\mathrm{C}}{}^{\otimes(M-1)} \otimes \hat{\mathcal{P}}_L)\left(\hat{I}_{\mathrm{P}} \otimes \hat{I}_{\mathrm{C}}{}^{\otimes(M-1)} \otimes \hat{U}(\rho_k)\right), \tag{8.7}$$

where $k = (t \bmod M)$, and the step is completed by the $\hat{O}$ operator as before. The scheme has memory but not the dependence on history of the current method. The two schemes are only equivalent when all the ρ_k and $\rho_{j_1 \cdots j_{M-1}}$ are equal, for example, when all the coins are unbiased. This amounts to asserting that the scheme of Brun et al. (2003b) does not display Parrondian behaviour.

The probability density distributions for unbiased 2, 3, and 4 coin history-dependent quantum walks, with initial states that are an equal superposition of the possible $L \leftrightarrow R$ antisymmetric coin states[19] are shown in Figure 8.2. These distributions are essentially symmetric versions of the graphs of Brun et al. (2003b) that result from an initial state $|\psi_0\rangle = |R\rangle^{\otimes M}$.

[19] For example, with $M = 2$, the initial state is $|\psi_0\rangle = (|0,LL\rangle - |0,LR\rangle - |0,RL\rangle + |0,LL\rangle)/2$. For the purposes of this thesis an initial state that is symmetrical for $L \leftrightarrow R$ could equally well have been chosen.

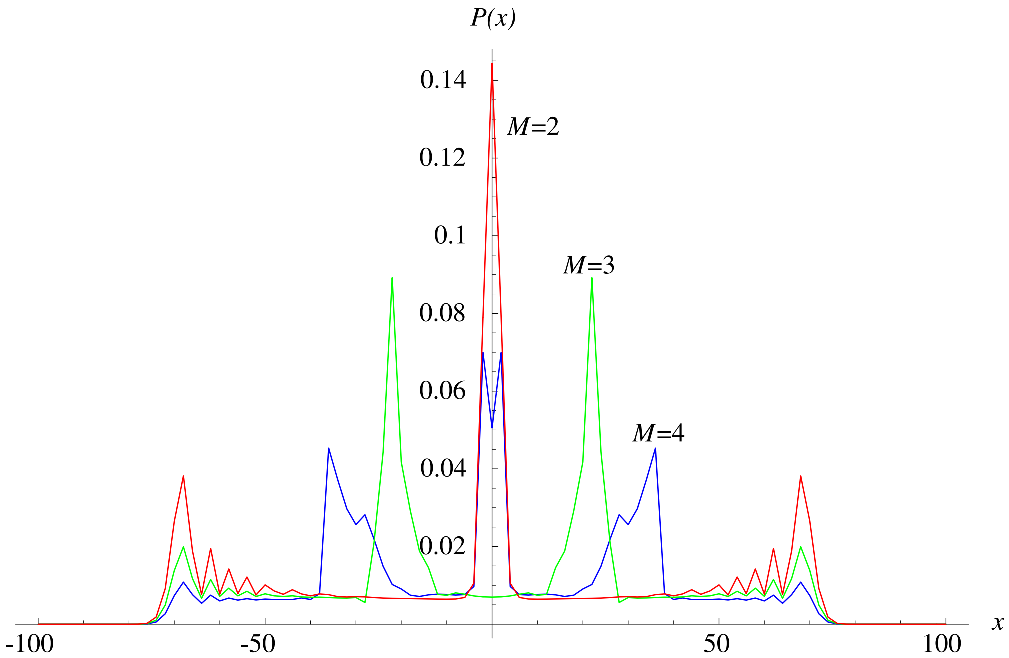


Figure 8.2. Probability density distributions for 2-, 3- and 4-coin unbiased quantum walks. The probability density distributions $P(x) = |\psi(x)|^2$ at toss $t = 100$, for the 2- (red), 3- (green) and 4- (blue) coin unbiased, symmetrical, quantum walks. Only even positions are plotted since $\psi(x)$ is zero for odd x at $t = 100$. The area under each curve is equal to one.

8.3 Results and discussion

For arbitrary M we have, as for the $M = 1$ case, two parts of the initial state that evolve without interacting. Thus, for $M = 2$ for example, states arising from $|0, LL\rangle$ and $|0, RR\rangle$ will interfere, as will states arising from $|0, LR\rangle$ and $|0, RL\rangle$, but the two groups evolve into states that are orthogonal, for any given x. For the M coin quantum walk there are $M + 1$ peaks with even values of M having a central peak, the others necessarily being symmetrically placed around $x = 0$ by our choice of initial state. The outer most pair of peaks are in the same position as the peaks for $M = 1$ (Figure 8.1) at $x(t) \approx 0.68t$. All the peaks are interference phenomenon, the central one being the easiest to understand. It arises since there are states centred on $x = 0$ that cycle back to themselves (i.e., that are stationary states over a certain time period). With $M = 2$, the simplest cycle over

However, the antisymmetric starting state is the one that gives the correct behaviour in the presence of a potential. The state $|\psi_0\rangle$ is the quantum equivalent of the average over past histories that is taken in the classical history-dependent Parrondo game.

$t = 2$ is proportional to

$$\begin{aligned}(|0, LR\rangle \; - \; |0, RL\rangle)/\sqrt{2} &\rightarrow (|+1, RL\rangle \; + \; i|-1, LL\rangle \; - \; |-1, LR\rangle \; - \; i|+1, RR\rangle)/2 \\ &\rightarrow (|0, LR\rangle \; - \; |0, RL\rangle)/\sqrt{2}.\end{aligned} \tag{8.8}$$

At the second step, complete destructive interference occurs for the states with $x = \pm 2$, so that there is no probability flux leaving the central three x values. In practice, the central region asymptotically approaches a more complex stationary cycle than Eq. (8.8), such as the $t = 2$ cycle

$$\begin{aligned}|\psi_{\text{centre}}\rangle \propto \; &(ai - b)(|-2, LL\rangle \; + \; |+2, RR\rangle) \\ &+ \; (1 - a - i + bi)(|-2, LR\rangle \; + \; |+2, RL\rangle) \\ &+ \; (i - 1)(|-2, RL\rangle \; + \; |+2, LR\rangle) \\ &+ \; (b - ai)(|0, LL\rangle \; + \; |0, RR\rangle) \; + \; (a + bi)(|0, LR\rangle \; + \; |0, RL\rangle),\end{aligned} \tag{8.9}$$

where a and b are real.

Adjusting the values of the various ρ can introduce a bias into the walk. To create a quantum walk analogous to the history-dependent game B of Sec. 7.2.2 requires $M = 3$, giving four parameters, ρ_{RR}, ρ_{RL}, ρ_{LR} and ρ_{LL}. Figure 8.3 shows the affect of individual variations in these parameters on the expectation value and standard deviation of the position after 100 time-steps. Some examples of the probability density distribution for biased 3-coin quantum walks are given in Figure 8.4. As ρ_{RR} increases, the right-most peak moves further towards positive x as a consequence of the increased probability of consecutive R results. The behaviour resulting from changes in ρ_{RL} is more complex. The effect of variations in ρ_{LL} or ρ_{LR} is the mirror image of that for ρ_{RR} or ρ_{RL}, respectively.

8.4 Quantum Parrondo effect

It is useful to consider the classical limit to our quantum scheme. That is, the random walk that would result if the scattering amplitudes were replaced by classical probabilities. As an example consider the $M = 2$ case, with winning probabilities $1 - \rho_L$ and $1 - \rho_R$. The analysis below follows that of Parrondo et al. (2000). Markov chain methods cannot be used directly because of the history dependence of the scheme. If, however, the vector

$$y(t) = [x(t-1) - x(t-2), \; x(t) - x(t-1)] \tag{8.10}$$

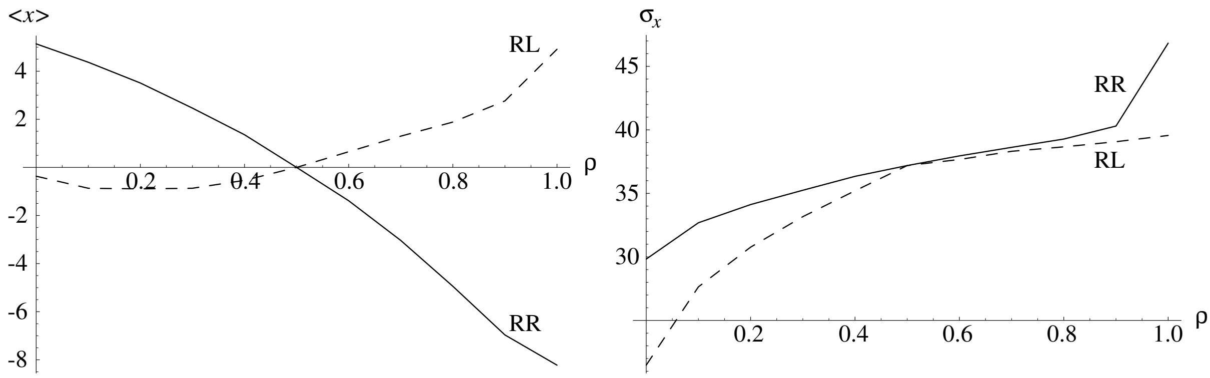


Figure 8.3. Expectation value and standard deviation of position for a 3-coin quantum walk for various parameter values. For the $M = 3$ quantum history-dependent walk, $\langle x \rangle$ and σ_x at time-step $t = 100$ as a function of ρ_{RR} (solid line) or ρ_{RL} (dashed line) while the other ρ_{ij} are kept constant at $\frac{1}{2}$. Varying ρ_{LL} has the opposite effect on $\langle x \rangle$ and the same on σ_x as varying ρ_{RR}. Similarly for ρ_{LR} compared to ρ_{RL}

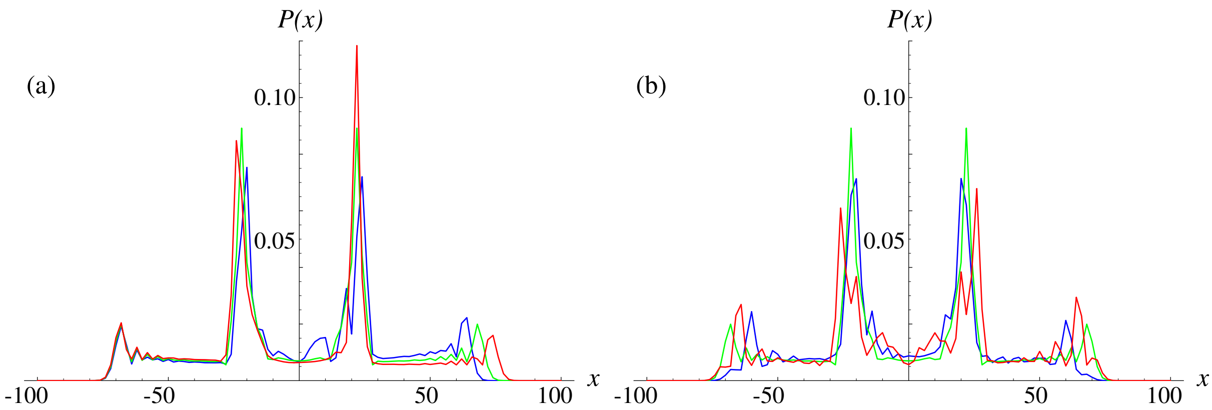


Figure 8.4. Probability density distribution for biased 3-coin quantum walks. The probability density distributions $P(x) = |\psi(x)|^2$ at toss $t = 100$, for biases (a) $\rho_{RR} = 0.4$ (blue) and 0.6 (red), and (b) $\rho_{RL} = 0.3$ (blue) and 0.7 (red), with all the other $\rho_{ij} = 0.5$. The unbiased distribution is shown in green in both figures. The distributions for biases in ρ_{LL} and ρ_{LR} are reflections about $x = 0$ of those for ρ_{RR} and ρ_{RL}, respectively. Only even positions are plotted since $\psi(x)$ is zero for odd x at $t = 100$. The area under each curve is equal to one.

is formed, where $x(t)$ is the position at time t, then $y(t)$ forms a discrete-time Markov chain between the states $[-1,-1]$, $[-1,+1]$, $[+1,-1]$ and $[+1,+1]$ with a transition matrix

$$T = \begin{pmatrix} \rho_L & 1-\rho_L & 0 & 0 \\ 0 & 0 & \rho_R & 1-\rho_R \\ 1-\rho_L & \rho_L & 0 & 0 \\ 0 & 0 & 1-\rho_R & \rho_R \end{pmatrix}. \tag{8.11}$$

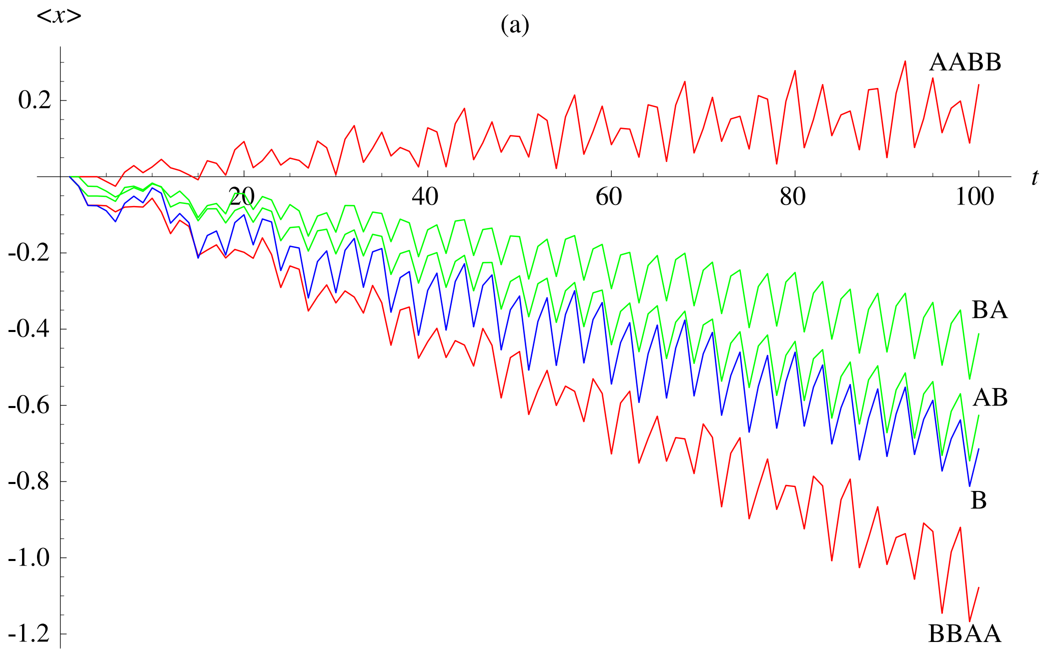
Define $\pi_{ij}(t)$ to be the probability of $y(t) = [i,j],\ i,j \in \{-1,+1\}$. A state is now transformed by $T\boldsymbol{\pi}$ at each time-step. Having represented the history-dependent game as a discrete-time Markov chain, the standard Markov techniques can be applied. The equilibrium distribution is found by solving $T\boldsymbol{\pi}_\mathrm{s} = \boldsymbol{\pi}_\mathrm{s}$. This yields $\boldsymbol{\pi}_\mathrm{s} = [1,1,1,1]/4$, giving a process with no net bias to the left or right irrespective of the values of ρ_L and ρ_R. The same analysis holds for $M > 2$. The probability density distributions are approximately Gaussian, centred on zero. However, interference in the quantum case presents an entirely different picture.

The comparison with the classical history-dependent Parrondo game requires $M = 3$. For game A, we select the unbiased game, $\rho_{LL} = \rho_{LR} = \rho_{RL} = \rho_{RR} = 0.5$. For game B, we choose, for example, $\rho_{RR} = 0.55$ or $\rho_{LR} = 0.6$ to produce a suitable bias (see Figure 8.3). The operators associated with A and B are applied repeatedly, in some fixed sequence, to the state $|\psi\rangle$. For example, the results of the game sequence AABB after t time-steps is

$$|\psi_t\rangle = (\hat{B}\hat{B}\hat{A}\hat{A})^{t/4}|\psi_0\rangle, \tag{8.12}$$

where t is a multiple of four. Figure 8.5 displays $\langle x\rangle$ for various sequences. Of sequences up to length four, with game B biased by $\rho_{RR} > 0.5$ only AABB and AAB give a positive expectation, while when game B is biased by $\rho_{LR} > 0.5$ only AAAB is positive. These results hold for ρ up to approximately 0.6, above which there are no positive sequences of length less than or equal to four.

The sequences AABB and BBAA can be considered the same but with different initial states. That is, if instead of $|\psi_0\rangle$, we start with $|\psi_0'\rangle = \hat{A}\hat{A}|\psi_0\rangle$, BBAA gives the same results (displaced by two time-steps) as AABB does with the original starting state. In the classical case, altering the order of the sequence results in the same trend but with a small offset, as one might expect. However, as Figure 8.5 indicates, the change of order in the quantum case can produce radically different results. This feature also appears in the quantum Parrondo model of Meyer and Blumer (2002a)—recall their model is based on a position-dependent scheme rather than a history-dependent one as in the present case. Details of their scheme are given in Chapter 7.



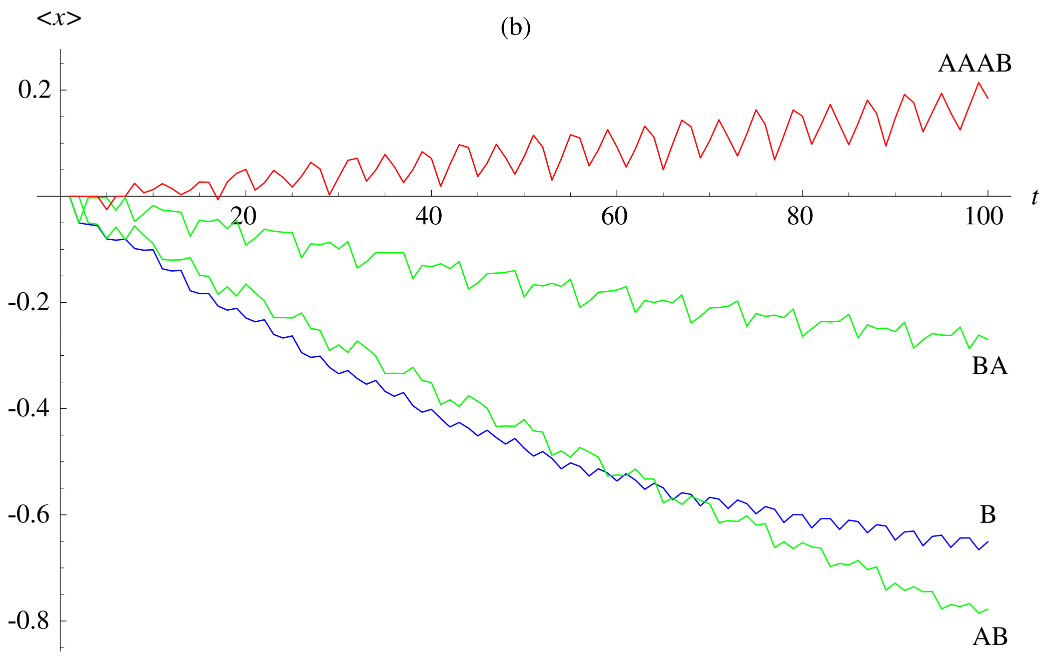


Figure 8.5. An example of a Parrondo effect in a 3-coin history-dependent quantum walk. Parrondian behaviour occurs in the $M = 3$ history-dependent quantum walk where game B has (a) $\rho_{RR} = 0.55$ or (b) $\rho_{LR} = 0.6$, with the other $\rho_{ij} = 0.5,\ i,j \in \{L,R\}$. Game A has all $\rho_{ij} = 0.5$ (unbiased). The letters next to each curve represent the sequence of games played repeatedly. For example, AB means apply $\hat{A}$ and then $\hat{B}$ to the state, repeating this sequence 50 times to get to $t = 100$.

8.5 Summary

A scheme for a discrete-time quantum walk with history dependence has been presented, involving the use of multiple quantum coins. By suitable selection of the amplitudes for coin flips dependent on certain histories, the walk can be biased to give positive or negative $\langle x \rangle$. In common with many other properties of quantum walks, the bias results from interference, since the classical equivalent of our walks are unbiased. With a starting state averaged over possible histories, the average spread of probability density in the multi-coin scheme is slower than in the single coin case, with the appearance of multiple peaks in the distribution. For even numbers of coins there is a substantial probability of $x \approx 0$. However, the positions of the outer most peaks are the same as those of a single coin quantum walk. As the memory effect increases, the dispersion of the quantum walk decreases. One may speculate that this feature may be relevant to an understanding of decoherence, here considered as loss of coherence within the central portion of the graph around $x \approx 0$. In particular, the dispersion in the wavefunction decreases as we move from a first-order Markov system to a non-first-order Markov system, that is, one with memory. This is consistent with the idea that the Markovian approximations tend to over-estimate the decoherence of the system (Blum 1981). Indeed, the form of a classical distribution is quickly approached as the quantum coins decohere (Brun et al. 2003a).

The scheme presented in this chapter is a quantum analog of the history-dependent game in the form of Parrondo's paradox presented in Sec. 7.2.2. The quantum history-dependent walk also exhibits a Parrondo effect, where the disruption of the history dependence in a biased walk by mixing with a second, unbiased walk can reverse the bias. In distinction to the classical case, the effect seen here is very sensitive to the exact sequence of operations, a quality it shares with other forms of quantum Parrondo's games, as discussed in Sec. 7.3.1. This sensitivity is consistent with the idea that the effect relies on full coherence over space and in time.

Only quantum walks on a line have been considered. The effect of memory driven quantum walks on networks with different topologies and whether the memory structure can be chosen to optimize the path in such networks, are fascinating open questions.

Chapter 9

Some Ideas on Quantum Cellular Automata

CELLULAR automata provide a means of obtaining complex behaviour from a simple array of cells and a deterministic updating rule. They supply a method of computation that dispenses with the need for manipulation of individual cells. Classical cellular automata have proved of great interest to computer scientists but the construction of quantum cellular automata pose particular difficulties. This chapter is a brief introduction to quantum cellular automata and presents a version of John Conway's famous two-dimensional classical cellular automata Life that has some quantum-like features, including interference effects. Some basic structures in the new automata are given and comparisons are made with Conway's game of Life.

9.1 Background and motivation

9.1.1 Classical cellular automata

A cellular automaton (CA) consists of an infinite array of identical cells, the states of which are simultaneously updated in discrete time steps according to a deterministic rule. Formally, they consist of a quadruple (d, Q, N, f), where $d \in \mathbb{Z}^+$ is the dimensionality of the array, Q is a finite set of possible states for a cell, $N \subset \mathbb{Z}^d$ is a finite neighbourhood, and $f : Q^{|N|} \to Q$ is a local mapping that specifies the transition rule of the automaton. The simplest cellular automata are constructed from a one-dimensional array of cells taking binary values, with a nearest neighbour transition function, as indicated in Figure 9.1. Such CA were studied intensely by Wolfram (1983) in a publication that lead to a resurgence of interest in the field. Wolfram classified cellular automata into four classes. The classes showed increasingly complex behaviour, culminating in class four automata that exhibited self-organization, that is, the appearance of order from a random initial state.

In general, information is lost during the evolution of a CA. Knowledge of the state at a given time is not sufficient to determine the complete history of the system. However, reversible CA are of particular importance, for example, in the modeling of reversible phenomena. Furthermore, it has been shown that there exists a one-dimensional reversible CA that is computationally universal (Morita and Harao 1989). Toffoli (1977) demonstrated that any d-dimensional CA could be simulated by a $(d+1)$-dimensional reversible CA and later Morita (1995) found a method using partitioning (see Figure 9.2) where by any one-dimensional CA can be simulated by a reversible one-dimensional CA. There is an algorithm for deciding on the reversibility of a one-dimensional CA (Amoroso and Patt 1972), but in dimensions greater than one, the reversibility of a CA is, in general, undecidable (Kari 1990).

9.1.2 Conway's game of Life

John Conway's game of Life (Gardner 1970) is a well known two-dimensional CA where cells are arranged in a square grid and have binary values generally known as "dead" or "alive." The status of the cells change in discrete time steps known as "generations." The new value depends upon the number of living neighbours, the general idea being that a cell dies if there is either overcrowding or isolation. There are many different rules that can be applied for birth or survival of a cell and a number of these give rise to interesting

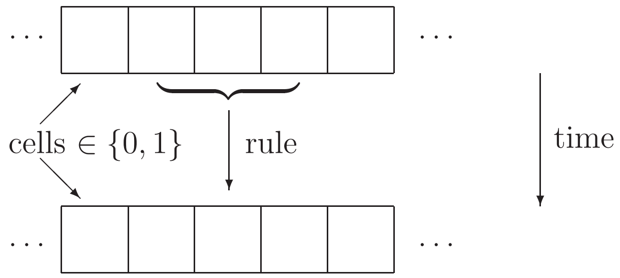


Figure 9.1. One-dimensional cellular automaton. A schematic of a one-dimensional, nearest neighbour, classical cellular automaton showing the updating of one cell in an infinite array.

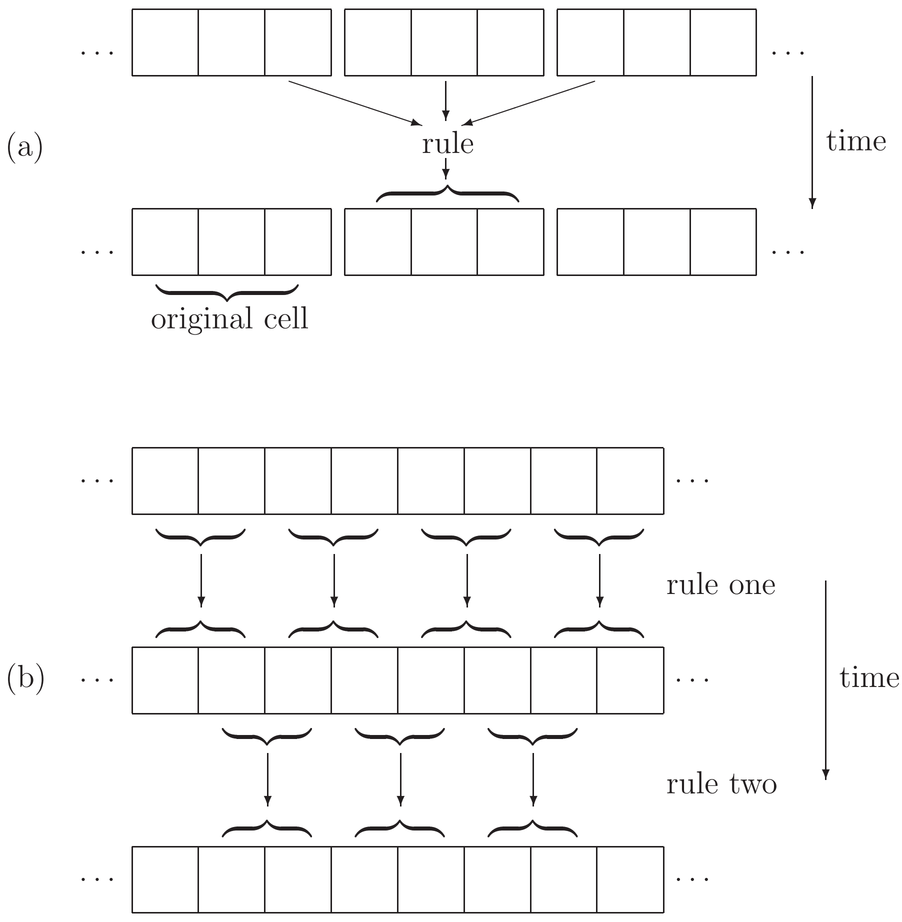


Figure 9.2. One-dimensional partitioned cellular automata. A schematic of a one-dimensional, nearest neighbour, classical (a) partitioned cellular automaton (Morita 1995) and (b) block (or Margolus) partitioned cellular automata. In (a), each cell is initially duplicated across three cells and a new transition rule $f : Q^3 \rightarrow Q^3$ is used. In (b), a single step of the automata is carried out over two clock cycles, each with its own rule $f : Q^2 \rightarrow Q^2$.

properties such as still lives (stable patterns), oscillators (patterns that periodically repeat), spaceships or gliders (fixed shapes that move across the Life universe), glider guns, and so on (Gardner 1971, Gardner 1983, Berlekamp et al. 1982). Conway's original rules are one of the few that are balanced between survival and extinction of the Life "organisms." In this version a dead (or empty) cell becomes alive if it has exactly three living neighbours, while an alive cell survives if and only if it has two or three living neighbours. Much literature on the game of Life and its implications exists and a search on the world wide web reveals numerous resources. For a discussion on the possibilities of this and other CA the interested reader is referred to Wolfram (2002).

The simplest still lives and oscillators are given in Figure 9.3, while Figure 9.4 shows a glider, the simplest and most common moving form. A large enough random collection of alive and dead cells will, after a period of time, usually decay into a collection of still lives and oscillators like those shown here, while firing a number of gliders off towards the outer fringes of the Life universe.

9.1.3 Quantum cellular automata

The idea of generalizing classical cellular automata to the quantum domain was already considered by Feynman (1982). Grössing and Zeilinger made the first serious attempts to consider quantum cellular automata (QCA) (Grössing and Zeilinger 1988a, Grössing and Zeilinger 1988b), though their ideas are considerably different from modern approaches. Quantum cellular automata are a natural model of quantum computation where the well developed theory of classical CA might be exploited. Quantum computation using optical lattices (Mandel et al. 2003) or with arrays of microtraps (Dumke et al. 2002) are possible candidates for the experimental implementation of useful quantum computing. It is typical of such systems that the addressing of individual cells is more difficult than a global change made to the environment of all cells (Benjamin 2000b) and thus they become natural candidates for the construction of QCA. An accessible discussion of QCA is provided by Gruska (1999).

The simple idea of quantizing existing classical CA by making the local translation rule unitary is problematic: the global rule on an infinite array of cells is rarely described by a well defined unitary operator. One must decide whether a given local unitary rule leads to "well-formed" unitary QCA (Durr and Santha 2002) that properly transform probabilities by preserving their sum squared to one. One construction method to achieve the necessary reversibility of a QCA is to partition the system into blocks of cells and apply blockwise

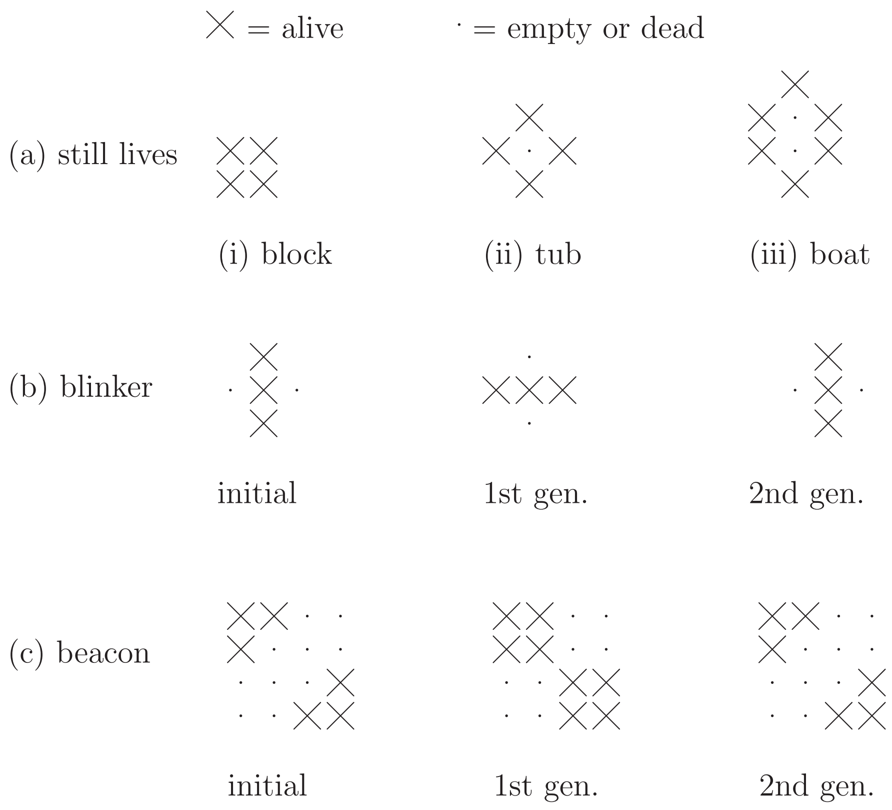


Figure 9.3. Simple patterns in Conway's Life. A small sample of the simplest structures in Conway's Life: (a) the simplest still-lives (stable patterns) and (b)–(c) the simplest period two oscillators (periodic patterns). A number of these forms will normally evolve from any moderate sized random collection of alive and dead cells.

unitary transformations. This is the quantum generalization to the scheme shown in Figure 9.2(b)—indeed, all QCA, even those with local irreversible rules, can be obtained in such a manner (Schumacher and Werner 2004). Formal rules for the realization of QCA using a transition rule based on a quasi-local algebra on the lattice sites is described by Schumacher and Werner (2004). In this formalism, a unitary operator for the time evolution is not necessary. The authors demonstrate that all nearest neighbour one-dimensional QCA arise by a combination of a single qubit unitary, a possible left- or right-shift, and a control-phase gate,[20] as indicated in Figure 9.5.

A *Mathematica* package to implement the scheme of Figure 9.5 is given in Sec. A.4.1. Reversible one-dimensional nearest neighbour classical CA are a subset of the quantum ones. In the classical case, the single qubit unitary can only be the identity or a bit-flip,

[20] A control-phase gate is a two-qubit gate that multiplies the target qubit by $\begin{pmatrix}1 & 0\\ 0 & \exp(i\phi)\end{pmatrix}$ if the control qubit is 1.

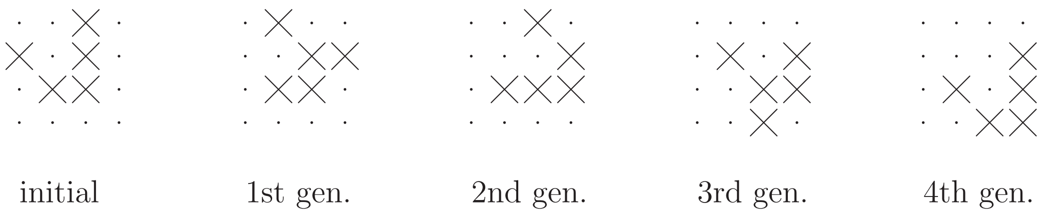


Figure 9.4. A Life glider. In Conway's Life, the simplest spaceship (a pattern that moves continuously through the Life universe), the glider. The figure shows how the glider moves one cell diagonally over a period of four generations.

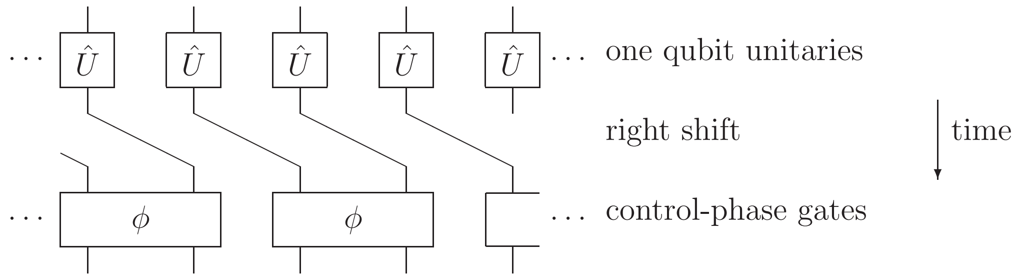


Figure 9.5. One-dimensional quantum cellular automaton. A schematic of a one-dimensional nearest neighbour quantum cellular automaton according to the scheme of Schumacher and Werner (2004) (from Figure 10 of that publication). The right-shift may be replaced by a left-shift or no shift.

while the control-phase gate is absent. This leaves just six classical CA, all of which are trivial.

9.2 Semi-quantum Life

9.2.1 The idea

Conway's Life is irreversible while, in the absence of a measurement, quantum mechanics is reversible. In particular, operators that represent measurable quantities must be unitary. A full quantum Life on an infinite array would be impossible given the known difficulties of constructing unitary QCA (Meyer 1996). Interesting behaviour is still obtained in a version of Life that has some quantum mechanical features. Cells are representing by classical sine-wave oscillators with a period equal to one generation, an amplitude between zero and one, and a variable phase. The amplitude of the oscillation represents the coefficient of the alive state so that the square of the amplitude gives the probability of finding the cell in the alive state when a measurement of the "health" of the cell is taken. If the initial state of the system contains at least one cell that is in a superposition of eigenstates the neighbouring cells will be influenced according to the coefficients of the respective eigenstates, propagating the superposition to the surrounding region.

If the coefficients of the superpositions are restricted to positive real numbers, qualitatively new phenomena are not expected. By allowing the coefficients to be complex, that is, by allowing phase differences between the oscillators, qualitatively new phenomena such as interference effects, may arise. The interference effects seen are those due to an array of classical oscillators with phase shifts and are not fully quantum mechanical.

9.2.2 A first model

To represent the state of a cell introduce the following notation:

$$|\psi\rangle = a|\text{alive}\rangle + b|\text{dead}\rangle, \tag{9.1}$$

subject to the normalization condition

$$|a|^2 + |b|^2 = 1. \tag{9.2}$$

The probability of measuring the cell as alive or dead is $|a|^2$ or $|b|^2$, respectively. If the values of a and b are restricted to non-negative real numbers, destructive interference does not occur. The model still differs from a classical probabilistic mixture, since here it is the amplitudes that are added and not the probabilities. In our model $|a|$ is the amplitude of the oscillator. Restricting a to non-negative real numbers corresponds to the oscillators all being in phase.

The birth, death and survival operators have the following effects:

$$\begin{aligned}\hat{B}|\psi\rangle &= |\text{alive}\rangle,\\ \hat{D}|\psi\rangle &= |\text{dead}\rangle,\\ \hat{S}|\psi\rangle &= |\psi\rangle.\end{aligned} \tag{9.3}$$

A cell can be represented by the vector $\binom{a}{b}$. The $\hat{B}$ and $\hat{D}$ operators are not unitary. Indeed they can be represented in matrix form by

$$
\begin{aligned}
\hat{B} &\propto \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \\
\hat{D} &\propto \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix},
\end{aligned}
\tag{9.4}
$$

where the proportionality constant is not relevant for our purposes. After applying $\hat{B}$ or $\hat{D}$ (or some mixture) the new state will require (re-)normalization so that the probabilities of being dead or alive still sum to unity.

A new generation is obtained by determining the number of living neighbours each cell has and then applying the appropriate operator to that cell. The number of living neighbours in our model is the amplitude of the superposition of the oscillators representing the surrounding eight cells. This process is carried out on all cells effectively simultaneously. When the cells are permitted to take a superposition of states, the number of living neighbours need not be an integer. Thus a mixture of the $\hat{B}$, $\hat{D}$ and $\hat{S}$ operators may need to be applied. For consistency with standard Life the following conditions will be imposed upon the operators that produce the next generation:

- If there are an integer number of living neighbours the operator applied must be the same as that in standard Life.

- The operator that is applied to a cell must continuously change from one of the basic forms to another as the sum of the a coefficients from the neighbouring cells changes from one integer to another.

- The operators can only depend upon this sum and not on the individual coefficients.

If the sum of the a coefficients of the surrounding eight cells is

$$A = \sum_{i=1}^{8} a_i \tag{9.5}$$

then the following set of operators, depending upon the value of A, is the simplest that has the required properties

$$\begin{aligned}
&0 \leq A \leq 1;\ \hat{G}_0 = \hat{D},\\
&1 < A \leq 2;\ \hat{G}_1 = (\sqrt{2}+1)(2-A)\hat{D} \ +\ (A-1)\hat{S},\\
&2 < A \leq 3;\ \hat{G}_2 = (\sqrt{2}+1)(3-A)\hat{S} \ +\ (A-2)\hat{B},\\
&3 < A < 4;\ \hat{G}_3 = (\sqrt{2}+1)(4-A)\hat{B} \ +\ (A-3)\hat{D},\\
&\quad A \geq 4;\ \hat{G}_4 = \hat{D}.
\end{aligned} \tag{9.6}$$

For integer values of A, the $\hat{G}$ operators are the same as the basic operators of standard Life, as required. For non-integer values in the range $(1, 4)$, the operators are a linear combination of the standard operators. The factors of $\sqrt{2}+1$ have been inserted to give more appropriate behaviour in the middle of each range. For example, consider the case where $A = 3 + 1/\sqrt{2}$, a value that may represent three neighbouring cells that are alive and one the has a probability of one-half of being alive. The operator in this case is

$$\hat{G} = \frac{1}{\sqrt{2}}\hat{B} + \frac{1}{\sqrt{2}}\hat{D} \propto \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}. \tag{9.7}$$

Applying this to either a cell in the alive, $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ or dead, $\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$ states will produce the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}|\text{alive}\rangle + \frac{1}{\sqrt{2}}|\text{dead}\rangle \tag{9.8}$$

which represents a cell with a 50% probability of being alive. That is, $\hat{G}$ is an equal combination of the birth and death operators, as might have been expected given the possibility that A represents an equal probability of three or four living neighbours. Of course the same value of A may have been obtained by other combinations of neighbours that do not lie half way between three and four living neighbours, but one of our requirements is that the operators can only depend on the sum of the a coefficients of the neighbouring cells and not on how the sum was obtained.

In general the new state of a cell is obtained by calculating A, applying the appropriate operator $\hat{G}$:

$$\begin{pmatrix}a'\\ b'\end{pmatrix} = \hat{G}\begin{pmatrix}a\\ b\end{pmatrix}, \tag{9.9}$$

and then normalizing the resulting state so that $|a'|^2 + |b'|^2 = 1$. It is this process of normalization that means that multiplying the operator by a constant has no effect. Hence, for example, $\hat{G}_2$ for $A = 3$ has the same effect as $\hat{G}_3$ in the limit as $A \to 3$, despite differing by the constant factor $(\sqrt{2}+1)$.

9.2.3 A semi-quantum model

To get qualitatively different behaviour from classical Life we need to introduce a phase associated with the coefficients, that is, a phase difference between the oscillators. We require the following features from this version of Life:

- It must smoothly approach the classical mixture of states as all the phases are taken to zero.
- Interference, that is, partial or complete cancellation between cells of different phases, must be possible.
- The overall phase of the Life universe must not be measurable, that is, multiplying all cells by $e^{i\phi}$ for some real ϕ will have no measurable consequences.
- The symmetry between $(\hat{B}, |\text{alive}\rangle)$ and $(\hat{D}, |\text{dead}\rangle)$ that is a feature of the original game of Life should be retained. This means that if the state of all cells is reversed ($|\text{alive}\rangle \longleftrightarrow |\text{dead}\rangle$) and the operation of the $\hat{B}$ and $\hat{D}$ operators is reversed the system will behave in the same manner.

In order to incorporate complex coefficients, while keeping the above properties, the basic operators are modified in the following way:

$$\begin{aligned} \hat{B}|\text{dead}\rangle &= e^{i\phi}|\text{alive}\rangle, \\ \hat{B}|\text{alive}\rangle &= |\text{alive}\rangle, \\ \hat{D}|\text{alive}\rangle &= e^{i\phi}|\text{dead}\rangle, \\ \hat{D}|\text{dead}\rangle &= |\text{dead}\rangle, \\ \hat{S}|\psi\rangle &= |\psi\rangle, \end{aligned} \tag{9.10}$$

where the superposition of the surrounding oscillators is

$$\alpha = \sum_{i=1}^{8} a_i = Ae^{i\phi}, \tag{9.11}$$

A and ϕ being real positive numbers. That is, the birth and death operators are modified so that the new alive or dead state has the phase of the sum of the surrounding cells. The operation of the $\hat{B}$ and $\hat{D}$ operators on the state $\binom{a}{b}$ can be written as

$$\begin{aligned} \hat{B}\begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} a + |b|e^{i\phi} \\ 0 \end{pmatrix}, \\ \hat{D}\begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} 0 \\ |a|e^{i\phi} + b \end{pmatrix}, \end{aligned} \tag{9.12}$$

with $\hat{S}$ leaving the cell unchanged. The modulus of the sum of the neighbouring cells A determines which operators apply, in the same way as before [see Eq. (9.6)]. The addition of the phase factors for the cells allows for interference effects since the coefficients of alive cells may not always reinforce in taking the sum, $\alpha = \sum a_i$. A cell with $a = -1$ still has a unit probability of being measured in the alive state but its effect on the sum will cancel that of a cell with $a = 1$. A phase for the dead cell is retained in order to maintain the alive $\longleftrightarrow$ dead symmetry, however, it has no effect. Such an effect would conflict with the physical model presented earlier and would be inconsistent with Conway's Life, where the empty cells have no influence.

A useful notation to represent semi-quantum Life is to use an arrow whose length represents the amplitude of the a coefficient and whose angle with the horizontal is a measure of the phase of a. That is, the arrow represents the phaser of the oscillator at the beginning of the generation. For example

$$\begin{aligned} \longrightarrow &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ \uparrow &= e^{i\pi/2}\begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix} = \begin{pmatrix} i/2 \\ i\sqrt{3}/2 \end{pmatrix}, \\ \nearrow &= e^{i\pi/4}\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} (1+i)/2 \\ (1+i)/2 \end{pmatrix}, \end{aligned} \tag{9.13}$$

etc. In this picture α is the vector sum of the arrows. This notation includes no information about the b coefficient. The magnitude of this coefficient can be determined from a and the normalization condition. The phase of the b coefficient has no effect on the evolution of the game state so it is not necessary to represent this.

9.2.4 Discussion

The above rules have been implemented in the computer algebra language *Maple* (see Sec. A.4.2). All the structures of standard Life can be recreated by making the phase of all the alive cells equal. The interest lies in whether there are new effects in the semi-quantum model or whether existing effects can be reproduced in simpler or more generalized structures. The most important aspect not present in standard Life is interference. Two live cells can work against each other as indicated in Figure 9.6 that shows an elementary example in a block still life with one cell out of phase with its neighbours. In standard Life there are linear structures called wicks that die or "burn" at a constant rate.

The simplest such structure is a diagonal line of live cells as indicated in Figure 9.7(a). In this, it is not possible to stabilize an end without introducing other effects. In the new model a line of cells of alternating phase ($\ldots \longrightarrow\longleftarrow \ldots$) is a generalization of this effect since it can be in any orientation and the ends can be stabilized easily. Figure 9.7(b)–(c) shows some examples. A line of alternating phase live cells can be used to create other structures such as the loop in Figure 9.8. This is a generalization of the boat still life, Figure 9.3(a)(iii), in the standard model that is of a fixed size and shape. The stability of the line of $\longrightarrow\longleftarrow$'s results from the fact that while each cell in the line has exactly two living neighbours, the cells above or below this line have a net of zero (or one at a corner) living neighbours due to the canceling effect of the opposite phases. No new births around the line will occur, unlike the case where all the cells are in phase.

Oscillators (Figure 9.3) and spaceships (Figure 9.4) cannot be made simpler than the minimal examples presented for standard Life. Figure 9.9 shows a stable boundary that results from the appropriate adjustment of the phase differences between the cells. The angles have been chosen so that each cell in the line has between two and three living neighbours, while the empty cells above and below the line have either two or four living neighbours and so remain life-less. Such boundaries are known in standard Life but require a more complex structure.

In Conway's Life interesting effects can be obtained by colliding gliders. In the semi-quantum model additional effects can be obtained from colliding gliders and "anti-gliders," where all the cells have a phase difference of π with those of the original glider. For example, a head-on collision between a glider and an anti-glider, as indicated in Figure 9.10, causes annihilation, where as the same collision between two gliders leaves a block. However, there is no consistency with this effect since other glider-antiglider collisions produce alternative effects, sometimes being the same as those from the collision of two gliders.

9.3 Summary

John Conway's game of Life is a two-dimensional cellular automaton where the new state of a cell is determined by the sum of neighbouring states that are in one particular state generally referred to as "alive." A modification to this model is proposed where the cells may be in a superposition of the alive and dead states with the coefficient of the alive state being represented by an oscillator having a phase and amplitude. The equivalent of evaluating the number of living neighbours of a cell is to take the superposition of the

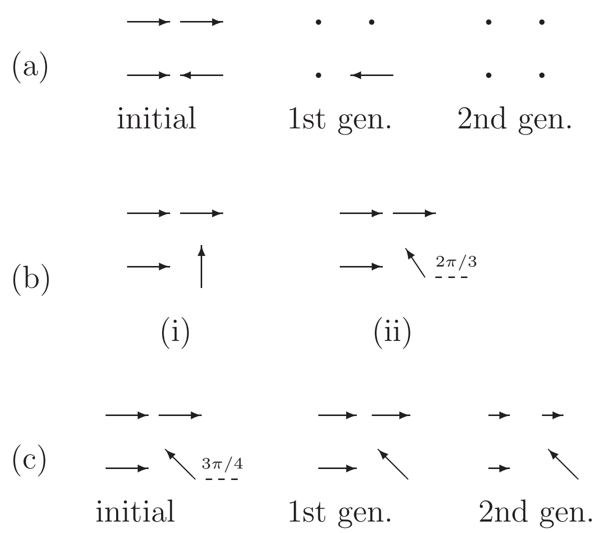


Figure 9.6. Destructive interference in semi-quantum Life. (a) A simple example of destructive interference in semi-quantum Life: a block with one cell out of phase by π dies in two generations. (b) Blocks where the phase difference of the fourth cell is insufficient to cause complete destructive interference; each cell maintains a net of at least two living neighbours and so the patterns are stable. In the second of these, the fourth cell is at a critical angle. Any greater phase difference causes instability resulting in eventual death as seen in (c), which dies in the fourth generation.

oscillators of the surrounding states. The amplitude of this superposition will determine which operator(s) to apply to the central cell to determine its new state, while the phase gives the phase of any new state produced. Such a system show some quantum-like aspects such as interference.

Some of the results that can be obtained with this new scheme have been touched on in this chapter. New effects and structures occur and some of the known effects in Conway's Life can occur in a simpler manner. However, the scheme described should not be taken to be a full quantum analogue of Conway's Life and does not satisfy the definition of a QCA.

The field of quantum cellular automata is still in its infancy. The protocol of Schumacher and Werner (2004) provides a construction method for the simplest QCA. Exploration and classification of these automata is an important unsolved task and may lead to developments in the quantum domain comparable to those in the classical field that followed

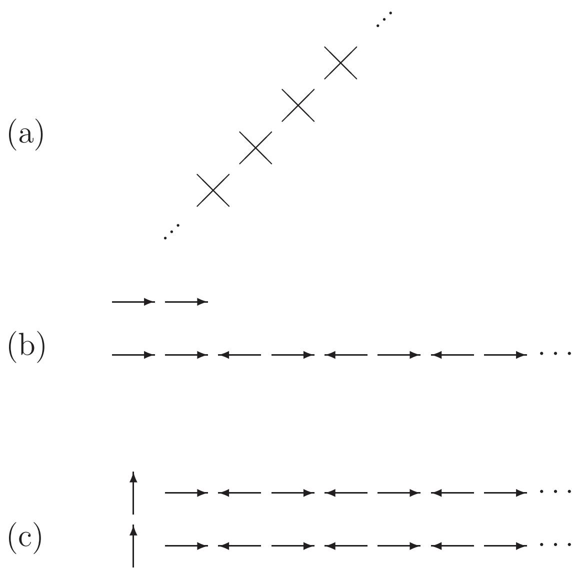


Figure 9.7. Wicks in semi-quantum Life. (a) A wick (an extended structure that dies, or "burns", at a constant rate) in standard Life that burns at the speed of light (one cell per generation), in this case from both ends. It is impossible to stabilize one end without giving rise to other effects. (b) In semi-quantum Life an analogous wick can be in any orientation. The block on the left-hand end stabilizes that end; a block on both ends would give a stable line; the absence of the block would give a wick that burns from both ends. (c) Another example of a light-speed wick in semi-quantum Life showing one method of stabilizing the left-hand end.

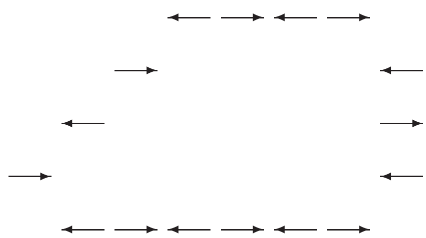


Figure 9.8. A stable loop in semi-quantum Life. An example of a stable loop made from cells of alternating phase. Above a certain minimum, such structures can be made of arbitrary size and shape compared with a fixed size and limited orientations in Conway's scheme.

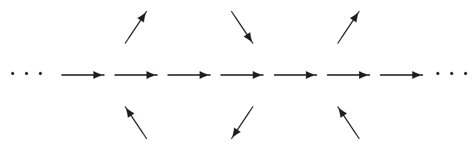


Figure 9.9. A stable boundary in semi-quantum Life. A boundary utilizing appropriate phase differences to produce stability. The upper cells are out of phase by $\pm\pi/3$ and the lower by $\pm 2\pi/3$ with the central line.

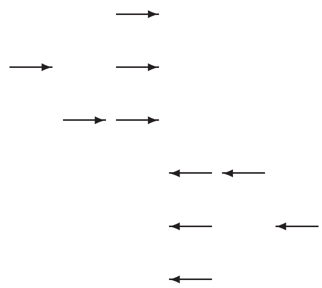


Figure 9.10. A collision between a glider and an anti-glider in semi-quantum Life. A head on collision between a glider and its phase reversed counter part, an anti-glider, produces annihilation in six generations.

the exploration of classical CA. Quantum cellular automata are a viable candidate for achieving useful quantum computing.

Chapter 10

Conclusions and Future Directions

QUANTUM game theory is an exciting new tool for the study of conflict or competition situations in the quantum domain. The theory contributes to our understanding of quantum information and has the potential for application in quantum control, quantum algorithms, quantum communication, and other quantum computing tasks. Quantum walks and quantum cellular automata present promising protocols for the implementation of useful quantum computing. This thesis has presented new ideas in the above fields. The concluding chapter presents an extended summary of this work, detailing the original contributions, and indicating possible future directions.

10.1 New quantum models of classical games

The starting point for quantum game theory are the existing classical game-theoretic problems. Existing scenarios are translated into the quantum domain by changing classical probabilities into quantum probability amplitudes, permitting superpositions of classical strategies and, possibly, by introducing entanglement between the options of different players. The original problem remains as a subset of the quantum game. By quantizing a game, the nature of the game is changed and in that sense quantum games do not address the original problem. Nevertheless, the quantum models demonstrate what can be achieved when the game's domain is expanded into the quantum realm. More importantly, when dealing with problems in quantum computing and quantum communication, where the information *is* quantum, the new theory is necessary to deal with competitive or conflict situations that may arise. In this thesis, new quantizations of two interesting game-theoretic problems have been presented.

10.1.1 Monty Hall problem—Chapter 3

The Monty Hall problem originated in a TV game show where the competitor has the task of guessing behind which of three doors the host has hidden a prize. After an initial selection by the competitor, the host opens a different door showing that the prize is not behind it. The player is then given the option of switching their selection to the untouched door or remaining with their initial choice. "Common sense" seems to suggest that, now the player knows the prize lies behind one of two doors, both options should yield a 50/50 chance of securing the prize. The counterintuitive idea that lies behind this simple problem is this: switching doors yields a two-thirds chance of winning, while retaining the initial choice of door results in a win only one time in three. This can easily be verified by referring to Table 3.1 but the ease of this observation did not stop the Monty Hall problem from generating much interest and controversy when it captured the attention of the public and of mathematicians in the early 1990s (vos Savant 1991).

Three distinct quantizations of the Monty Hall problem have appeared in the literature, that described in this thesis being the second (Flitney and Abbott 2002c). Those of Li et al. (2001) and D'Ariano et al. (2002) are briefly described in the introductory paragraphs of Sec. 3.2. The quantization scheme presented in this thesis is the one that most directly follows the classical version. There is a quantum particle and three boxes $|0\rangle$, $|1\rangle$, and $|2\rangle$. The choices of the contestant (Bob) and the host (Alice) are represented by qutrits that are initialized in some specified state. The initial selections of Alice and Bob are carried out by operators acting on their qutrit. A third qutrit is used to represent the box "opened" by Alice. The system is represented by the state $|\psi\rangle = |oba\rangle$ [Eq. (3.1)], where a = Alice's choice of box, b = Bob's choice of box, and o = the box that has been opened. After the initial choices, Alice applies an "opening box" operator that sets the o qutrit so that it is different from the choices of both Alice and Bob. This does not represent the physical opening of a box, which would constitute a measurement; the coherence of the system is maintained until the completion of the game. Bob then has the option of applying a "switch box" operator or the identity operator, or a probabilistic mixture of both. Finally, a measurement is made on the system to determine whether the boxes selected by Alice and Bob are the same. Bob's average payoff is the expectation value of this correlation. The final state prior to the measurement is obtained by Eq. (3.2).

If the initial state of the players' qutrits are an equal superposition of the three possibilities with no entanglement, the new scenario offers nothing more than can be achieved using a mixed strategy in a classical setting: Bob wins $\frac{2}{3}$ of the time by switching or $\frac{1}{3}$ of the time by not switching regardless of the strategies employed by the players. Maximal entanglement of the initial state alters the situation. Now, if either player is restricted to classical operations—the identity operator or permutations among boxes—the other player benefits substantially from having access to the full set of unitary strategies. If the host, Alice has access to quantum strategies while the contestant, Bob does not, the game is fair, since Alice can adopt a strategy [Eq. (3.18)] with an expected payoff of $\frac{1}{2}$ for each player, while if Bob has access to quantum strategies and Alice does not he can win all the time. Where both participants have access to quantum strategies, maximal entanglement in the initial state produces the same payoffs as the classical game for any mixed strategy of switching or not-switching. That is, for the Nash equilibrium strategy the contestant wins $\frac{2}{3}$ of the time by switching.

10.1.2 Duels and truels—Chapter 4

A situation where there are three competing agents each trying to eliminate the others is described in game-theoretic terms by a truel, or three person generalization of a duel. The extension to N players is called an N-uel. It is a popular model of a struggle for survival among multiple competing agents, for example, companies in a market place, or species competing for the same limited resource. The optimal play in a truel can be counterintuitive: it is sometimes better for a player to forgo their option of shooting, rather than risk eliminating an opponent only to become the target for the third player. The optimal strategy is sensitive to the exact conditions under which the truel is carried out. A non-technical discussion of classical truels is provided by Kilgour and Brams (1997) with detailed analysis of the case of simultaneous firing (Kilgour 1972) and sequential firing (Kilgour 1975) provided by the same author. An introduction to classical truels and an example of the seemingly paradoxical nature of the optimal play has been presented in Sec. 4.2.

In this thesis a novel quantization scheme for this problem is presented. Each player has a qubit designating their state, with the computational basis states $|0\rangle$ and $|1\rangle$ representing "dead" and "alive," respectively. The combined state of the players Alice, Bob, and Charles is $|\psi\rangle = |q_{\rm A}\, q_{\rm B}\, q_{\rm C}\rangle$ [Eq. (4.5)] with the initial state being $|\psi_i\rangle = |111\rangle$. The system is easily extended to more players by the addition of further qubits. The analogue of firing at an opponent is an attempt to flip an opponent's qubit using a unitary operator acting on $|\psi\rangle$. An action can only be carried out if the player is alive, so the appropriate unitaries are control-rotations, or more generally control-SU(2) operations, where the player's qubit is the control and the target's qubit is the subject of the rotation. Equations (4.6) and (4.7) represent the actions of Alice firing at Bob in a duel or a truel, respectively. The game consists of a number of rounds of sequential firing. Coherence of the system is maintained until the completion of the final round, where upon a measurement in the computational basis is taken on the final state and payoffs are awarded to the players still living. The formalism for carrying out quantum duels and truels is detailed in Sec. 4.3 with extensions to the case of N players given in Sec. 4.6.

The game differs from the classical scenario in allowing player states to be a superposition of alive and dead, in permitting dead players to be bought back to life by having their qubit flipped from $|0\rangle \to |1\rangle$, and by the fact that the players get no information about the state of the system in intermediate rounds. This latter fact means that, in contrast to the classical case, players' decisions are not contingent on the success or otherwise of previous actions. A player can select the operators they wish to apply for each round at the beginning of the game, based on their forward estimates of who will be alive at that stage, rather than making dynamic choices during the game.

A one round, two player quantum duel offers nothing different from the classical game, but in longer quantum duels phase terms in the player operators can greatly affect the expected payoffs (see Figures 4.6 and 4.7). If players have discretion over the value of their phase factors a maximin choice can in principle be calculated provided the number of rounds is fixed. If one player has a restricted choice the other has a large advantage. The unitary nature of the operators means that the probability of flipping a dead state to an alive state is the same as that for the reverse, so it can be advantageous for a player to fire in the air rather than target the opponent, something that is never true in a classical duel, and this can result in an equilibrium where both players forgo targeting their opponent even if there are further rounds to play (see Figure 4.8).

In a quantum truel, interference effects arise when one player is targeted by the other two, and can have dramatic consequences, either enhancing or diminishing the probability of survival of the targeted player compared to the classical case. Such interference effects can occur as early as the first round. As with the case of quantum duels, equilibria can arise where it is to the disadvantage of each player to target one of the others. Such equilibria occur more frequently in the quantum case than in the classical where they depend upon exceptional circumstances.

The optimal play depends on the marksmanship of the players. Section 4.5.1 summarizes the optimal play for one and two round truels where the third player is a perfect shot. The regions of parameter space that correspond to the various preferred strategies of the first two players differ from those of the classical game (see Figures 4.9–4.11).

The analogy with the classical scenario can be made closer by introducing decoherence in the form of a measurement in the computational basis with probability p after each move. In the case of a measurement, players can alter their decisions dependent on the result of the measurement. As the measurement probability is increased from zero to one there is a smooth transition from the fully quantum game to the classical one as described in Sec. 4.7.

10.1.3 Future directions

The time when it was exciting to produce quantizations of abstract game theory problems is mostly past. Future work in this area needs to demonstrate novel effects, in addition to the interference phenomena that is expected in the quantum world, or have relevance to particular practical problems in the areas of quantum computing, quantum control or quantum communication. Game theory is the natural language for competitive scenarios such as communication in the presence of an eavesdropper. The important task of applying quantum game theory to the problems of quantum communication remain to be explored. Classical game theory is regularly applied to problems in queuing theory. The possibility of utilizing quantum game theory to produce a quantum theory of queuing has been mooted (Pati 2003).

10.2 Quantum 2×2 games

10.2.1 A quantum player versus a classical player—Chapter 5

The work by Eisert et al. (1999) was one of the two seminal papers on quantum game theory, detailing a protocol for two player, two strategy quantum games with entanglement. In this work it was noted that, in the well known game of Prisoners' Dilemma, if one player has access to quantum strategies while the other player is restricted to the classical strategy subset, the quantum player could achieve a considerably greater payoff than they could have achieved playing classically, where the best that could be hoped for is the (classical) NE result. By selecting the move dubbed by Eisert as the "miracle" move, the quantum player can partially direct the result of the game towards their preferred result, regardless of the classical player's strategy. Indeed, if the strategy of the classical player is known the quantum player can exploit the entanglement between the players' qubits to create any desired final state.

Du et al. (2001b) generalized the Eisert result to Prisoners' Dilemma with a variable payoff matrix. The extent of the quantum advantage is dependent on the degree of entanglement between the qubits that represent the players' strategies. Below a certain level of entanglement, the advantage of having access to the full set of quantum strategies disappears, and the NE solution to the quantum game is identical to that of the classical game.

In this thesis, a study of the effect of the degree of entanglement on the quantum player's advantage in a variety of 2×2 games is detailed for the first time. Depending on the relative values of the entries in the payoff matrix, the quantum player has a preference for one of the four possible game results[21] each having a corresponding "miracle" move given by Eq. (5.5). Quantum versions of the games of Prisoners' Dilemma, Chicken, Deadlock, Stag Hunt and the Battle of the Sexes are considered, with Bob having access to the full range of unitary strategies while Alice is restricted to classical strategies. For each game there are critical values of the entanglement parameter γ, with $\gamma = \pi/2$ corresponding to

[21] It is possible that two or more results are equally preferred, but this does not change the general argument.

maximal entanglement while $\gamma = 0$ corresponds to no entanglement—refer to Eq. (2.8)—below which it is no longer an advantage to have access to quantum moves. Section 5.3 presents calculations for the various threshold values of the entanglement for these games with generalized payoff matrices. The results are summarized in Table 5.2. With typical values in the payoff matrix and the classical player, Alice opting for her best strategy,[22] the critical value for $\sin\gamma$ is $\sqrt{1/3}$ for Chicken, $\sqrt{1/5}$ for Prisoners' Dilemma and $\sqrt{2/3}$ for Deadlock, while for Stag Hunt there is no advantage to the quantum player unless the classical player is adopting a non-optimal strategy. There is no clear threshold in the Battle of the Sexes, but for any non-zero entanglement Bob can improve upon his possible worst case result of the classical game.

10.2.2 Decoherence in quantum games—Chapter 6

Decoherence in a quantum system is caused by the coupling of the system to the environment and results in non-unitary dynamics. Since interaction with the environment cannot be totally eliminated it is important to consider decoherence in any quantum application. Decoherence destroys the interesting features of quantum games. Decoherence and noise in quantum games was little studied in the literature (Johnson 2001, Chen et al. 2003b, Özdemir et al. 2004) prior to the publication of the work in this thesis (Flitney and Abbott 2004a, Flitney and Abbott 2005). Section 6.3 presents a model for incorporating decoherence in quantum games of the Eisert scheme. The simplest model of decoherence is used, that of a measurement in the computation basis with probability p on any of the players' qubits. Decoherence is incorporated both before and after the players make their moves, possibly with different values for p (see Figure 6.1). The operator product expansion is employed for the calculation of the final density matrix. The expectation value for the players' payoffs in a general 2×2 quantum game is given by Eq. (6.8).

As a measure of the "quantum-ness" of the game, the advantage that a player with access to the full set of quantum strategies has over a player restricted to classical strategies is examined. Chapter 6 extends these results to games with decoherence. The games of Prisoners' Dilemma, Chicken and the Battle of the Sexes are considered. The advantage of having access to quantum strategies is reduced as p increases, as expected, becoming marginal above $p \approx 0.5$. However, in all cases the quantum player retains some advantage

[22]That is, the strategy that maximizes her payoff given that Bob is selecting the appropriate miracle move.

until the decoherence is maximum. When this occurs for 2×2 symmetric games such as Prisoners' Dilemma and Chicken, the payoffs to the two players are equal.

Although the results presented on decoherence are comparable to those given for different levels of entanglement, there are no threshold values of the measurement probability p corresponding to the thresholds for the entanglement parameter γ, beyond which the advantage pertaining to the quantum player disappears.

10.2.3 Future directions

When the Eisert scheme is extended to multiple players, Nash equilibria not present in the corresponding classical game can arise, for example, in a four player Minority[23] game (Benjamin and Hayden 2001b). It is expected that the advantage a quantum player can obtain against a group of classical players in a quantum multiplayer game would not be as strong as that in the two player case, since the quantum player no longer has the ability to produce any desired final state even if he/she know the other players' moves. In multiplayer games, no study of the optimal moves of a quantum player against classical players has been carried out. This is made problematic by the increased computational difficulty that multiplayer games present, but is a worthwhile future task. There must be some threshold value of the entanglement parameter below which the new equilibria disappear.

The type of three-partite entanglement is known to be important in three player quantum Prisoners' Dilemma (Han et al. 2002b). In future work, quantum games could provide an avenue for exploring multi-partite entanglement through its influence on the game equilibria, the advantage of a quantum player over a classical player, and the like.

The model of decoherence in quantum games in the Eisert scheme presented in this thesis can serve as a starting point for a more general exploration of decoherence in quantum games. Decoherence in general multiplayer games has not been considered. The effect of decoherence on the presence of the new NE that arise in some multiplayer games is an interesting open question. As above, there must be some level of decoherence that would eliminate the new NE but is this level less than maximum decoherence? Furthermore, the consideration of decoherence in infinite dimensional Hilbert space games is an interesting

[23] Recall that a Minority game is one where the players are rewarded if they select the least popular choice from the two available alternatives.

open question that would involve calculation techniques that go beyond those considered here.

This thesis has considered only static quantum games. Future work on the application of game-theoretic methods to dynamic quantum systems with various types of decohering noise will be of great interest. It will be interesting to consider both the behaviour of quantum games for (a) non-Markovian noise, where the quantum system is coupled to a dissipative environment with memory, and (b) the Markovian, or memoryless, limit where the time scales for decoherence are small compared to the characteristic time scale of the quantum system.

10.3 Quantum Parrondo's games—Chapter 7

10.3.1 Capital- or position-dependent Parrondo's games

A Parrondo game is the name given to the apparent paradox that arises when a homogeneous sequence of either of two games are losing, but a random mixed sequence, or certain periodic sequences, of the games are winning. Classical Parrondo games have traditionally been formulated from two gambling games, A and B, involving biased coins. One or both games has a form of feedback from the game state. The most intensely studied models involve game B being a set of biased coins, the selection of which is dependent on the total gambling capital (Harmer and Abbott 1999b) or on the results of the two previous games (Parrondo et al. 2000). Details are given in Figures 7.1 and 7.3, respectively. There are many examples where mechanisms akin to Parrondo's games may arise in nature. A list of possibilities discussed in the literature is given in Sec. 7.1.

Game B is designed to be a winning game in the absence of feedback, but with the feedback in place yields a net loss over time. Game A is the toss of a single biased coin. When the two games are mixed, game A acts like noise to break the feedback in game B, and the combination of the two games can then be winning. A summary of the main results of the capital- and history-dependent Parrondo games are given in Secs. 7.2.1 and 7.2.2, respectively.

A quantum analogue to the capital-dependent Parrondo game was introduced by Meyer and Blumer (2002a). In this model, a quantum particle undergoes Brownian motion along a one-dimension lattice under the influence of some potential. The discretized position, x of the particle in the lattice corresponds to the capital. The quantum "coin" takes values $|L\rangle$ or $|R\rangle$, representing the direction of motion of the particle along the line, and the equivalent of an unbiased coin toss is carried out by the unitary operation given in Eq. (7.3). The equivalent of the capital-dependent game B can be created by applying a tilted sawtooth potential (see Figure 7.5), while the equivalent of game A is a potential uniformly increasing with x. The potentials are described by Eq. (7.4).

With the appropriate choice of parameters, this model exhibits Parrondian behaviour: potentials A or B applied homogeneously over time move the particle towards negative x, while periodic switching between the potentials can produce motion towards positive x, as indicated in Figure 7.6. Note, however, that a random mixed sequence of A and B still gives rise to net motion in the negative direction, in contrast to the equivalent classical situation, a fact not mentioned in Meyer and Blumer (2002a). The expectation value of x after 100 time steps of various periodic sequences of A and B is systematically studied in this thesis with the results displayed in Figure 7.7. The effect of varying the strengths of the potentials is shown in Figure 7.8 for two of the periodic sequences that generally give rise to positive motion. The Parrondo effect in the quantum game is a result of interference and is sensitive to the exact initial state of the particle.

10.3.2 History-dependent Parrondo's games

In 2000, a history-dependent quantum Parrondo game was introduced by Ng and Abbott (2004). The model is a close analogue of the classical history-dependent Parrondo game. Game B is a three qubit operator consisting of four control-control-rotations (see Figure 7.9), one of which is executed depending on the possible states of the two control qubits. The control qubits represent the results of the previous two games. For a series of games, the initial state consists of one qubit for each game to be played; the result of each game is recorded by a fresh qubit. The coupling between successive games, as shown in Figure 7.10(a)–(b), make the scheme computationally cumbersome for longer series. The periodic sequence AAB is the only one for which the results of each group of three games decouple from the remainder, as indicated in Eq. (7.10) and Figure 7.10(c).

In this thesis, the expected payoff for short sequences of games in this model are computed (see Table 7.1). Interference effects can arise when the initial state is taken to be a superposition of states of the computational basis, and can result in payoffs either larger or smaller than the corresponding classical situation. In some examples, payoffs can be considerably altered by varying the phase factors in the rotation operators without changing the rotation angles (and hence the associated classical probabilities). If the initial state is $|00\ldots0\rangle$ the payoffs are independent of the phases and are no different from the classical case.

10.3.3 Future directions

Gambling games rely on a random element. By quantizing Parrondo's games the random element is replaced by a superposition over all possible results. New behaviour arises through interference—and such interference can be modified by the introduction of phase factors in the quantum operators—that has no classical analogue. A random element, warranting further study, can be introduced by perturbing the system with noise or decoherence. A first consideration of decoherence in the position-dependent quantum Parrondo game has already been made (Meyer 2003). The constructive use of decoherence (Lee and Johnson 2002a) is another area of interest that deserves more attention.

Discretizing the classical Fokker-Planck equation can be used to generate classical Parrondo games (Allison and Abbott 2003, Amengual et al. 2004, Toral et al. 2003a). It is therefore an interesting open question whether the quantum Fokker-Planck equation can do the same for quantum Parrondo games.

Parrondo's games require a form of coupling between the system state and the winning probabilities. Quantum entanglement offers a method of coupling with no classical analogue. It is an interesting open question whether coupling through entanglement alone can give rise to quantum Parrondian behaviour.

Lee et al. (2002) have examined a quantum Parrondo-like construction of Grover's algorithm. It will be interesting to consider, in future work, the application of quantum Parrondo games to the construction of new quantum algorithms.

10.4 Quantum walks—Chapter 8

10.4.1 History-dependent quantum walk

Classical random walks have long been a powerful computational tool in many branches of mathematics and science. Consequently, significant attention has been focused on quantum walks, the analogue in the quantum domain of classical random walks. The fact that quantum walks diffuse quadratically, or in some cases exponentially, faster than their classical counter parts (Nayak and Vishwanath 2000, Childs et al. 2003) suggests they are promising candidates for implementing quantum algorithms (Shenvi et al. 2003, Childs and Goldstone 2004).

This thesis has introduced a quantum walk with history dependence analogous to the history-dependent game B in Parrondo's games. Unitary quantum walks require an extension to the Hilbert space of the particle by the addition of a "coin" state representing the direction of motion of the particle. A quantum walk dependent upon the previous $M-1$ results requires M coins states. With an initial state that is an equal superposition of all the possible coin states, this scheme gives rise to a probability density distribution with $M+1$ peaks, a central peak for even numbers of coins with the other peaks being symmetrically distributed around the origin. The outer most peaks are in the same positions for all M but are reduced in size as M increases. Examples of the distributions for two-, three- and four-coin, unbiased quantum walks are shown in Figure 8.2. Adjusting the amplitude for a coin flip can introduce a bias into the walk resulting in positive or negative $\langle x \rangle$. In common with other properties of quantum walks, the bias is the result of quantum interference, since the corresponding classical walks are unbiased. Figure 8.3 quantifies the bias as a function of the walk parameters for the three coin case, while Figure 8.4 shows some examples of the probability density distributions for biased three-coin quantum walks.

By mixing biased and unbiased steps, a quantum Parrondo effect can be observed, as demonstrated in Figure 8.5. The effect is not strong and is restricted to a small number of the possible periodic sequences of biased and unbiased steps. Along with earlier quantum Parrondo models, this new effect shows a sensitivity to initial conditions and to the exact sequence of operations. This is consistent with the idea that the effect is dependent upon full coherence over space and time.

Our scheme is distinct from the multi-coin quantum walk introduced by Brun et al. (2003a) in that this model has no history dependence, and as a result is not able to give rise to Parrondian behaviour. The behaviour of unbiased walks in the two schemes is, however, identical.

10.4.2 Future directions

For even numbers of coins there is a substantial peak in the probability density distribution around $x \approx 0$. This peak grows with increasing M at the expense of the outer dispersion peaks. That is, as the memory effect increases, the dispersion of the quantum walk decreases. One may speculate that this feature may be relevant to an understanding of decoherence, here considered as loss of coherence within the central portion of the graph around $x \approx 0$. In particular, the dispersion in the wavefunction decreases as we move from a first-order Markov system to a non-first-order Markov system—one with memory—consistent with the idea that the Markovian approximations tend to over-estimate the decoherence of the system.

In this thesis, only history-dependent quantum walks on a line have been considered. Future work could consider such walks on networks with different topologies. Whether the memory structure of the walk can be chosen to optimize the path on such networks is a fascinating open question.

10.5 Quantum cellular automata—Chapter 9

10.5.1 One-dimensional quantum cellular automata

A cellular automaton consists of an infinite array of identical cells which are simultaneously updated in a discrete-time fashion by a deterministic transition function. Cellular automata have generated significant interest since they can generate complex behaviour from simple rules (Wolfram 1983) and are computationally universal (Morita and Harao 1989). The best known CA is John Conway's game of Life played on a two-dimensional grid of states taking binary values generally referred to as "alive" and "dead" (or empty). Birth, death or survival of cells is determined by the number of neighbouring cells that are in the alive state.

With the interest in quantum computing, the generalization of CA to the quantum domain has assumed great importance. Quantum cellular automata provide a model of quantum computation that dispenses with the need to address individual qubits. The theory of QCA is yet to be fully developed. The idea of quantizing existing CA by simply making the local transition function unitary is problematic since the global transition function over all cells—recall the cells are updated simultaneously—is rarely described by a unitary function. Schumacher and Werner (2004) have developed formal rules for generating QCA and have demonstrated that all one-dimensional QCA can be constructed from a combination of a single qubit unitary, a possible left- or right-shift, and a control-phase gate (see Figure 9.5).

10.5.2 Semi-quantum version of the game of Life

This thesis has presented a modification of Conway's game of Life that exhibits some quantum properties, notably interference effects. Cells can be in a superposition of the alive and dead states and the local transition functions can be linear combinations of the classical ones. The physical picture of the alive cells is as oscillators with an amplitude and phase. The squared modulus of the amplitude represents the probability of the cell being alive. Taking the superposition of the oscillators of surrounding cells replaces the task of summing the number of living neighbours. The model is termed "semi-quantum" since it makes no effort to be a fully quantum model. In particular, neither local nor global transition functions are unitary.

The literature on Conway's game of Life is large and it would be impractical to make a full comparison of the new scheme with the known structures of the classical game. Some new structures in the semi-quantum scheme are presented in this thesis (see Figures 9.7–9.9).

10.5.3 Future directions

Mathematica code for executing a general one-dimensional QCA according to the scheme of Schumacher and Werner (2004) is given in Sec. A.4.1. Exploration of such QCA is an important task for the future.

QCA in more than one dimension will be more difficult to construct, but may hold interesting undiscovered properties. Since QCA may provide a mechanism for the construction of quantum computers, a significant unsolved task is the writing of quantum algorithms as quantum cellular automata. It is reasonable to expect that QCA will be at least as important in quantum computation as their classical counterparts have been in classical computation.

10.6 Final comments

The work in this thesis has covered various aspects of quantum game theory, presenting new models of interesting classical game-theoretic problems and extending the theory of 2×2 quantum games. New results have been presented for quantum Parrondo's games and a quantum walk with history dependence has been detailed. Finally, quantum cellular automata have been touched upon.

The theory of quantum games remains fragmented and much work is still to be done, particularly in multiplayer games. The hope is that quantum games will prove a valuable tool in developing useful quantum algorithms. Quantum walks and quantum cellular automata both provide possible mathematical machinery for quantum computation. Could their practical realization provide the easiest route to the construction of a scalable quantum computer? This tantalizing question and the theoretical understanding of quantum walks and quantum cellular automata is of significant interest for future study.

Appendix A

Software routines

COMPUTER algebra packages in *Mathematica* (Wolfram 1988) for many of the calculations presented in this thesis are listed in this appendix. Packages to carry out the following are given:

- 2×2 quantum games in the Eisert scheme
- classical capital-dependent Parrondo's games
- classical history-dependent Parrondo's games
- one-dimensional nearest neighbour quantum cellular automata
- semi-quantum Life (written in *Maple*)
- quantum walks and quantum position-dependent Parrondo's games.

The packages are written in a functional programming style. Commands are carried out by functions returning the desired value or array. The functions are nested so that the "guts" of the calculations are carried out at the deepest levels. All packages are commented and usage statements are provided for the main commands. Function names begin with a capital letter, as is standard in *Mathematica*, while variable names are lower case. For the definition of terms and details of the calculations refer to the appropriate chapters.

A.1 Quantum 2×2 games—Chapters 5 and 6

The following are a few functions that can be used, together with standard *Mathematica* commands for matrix manipulation, to compute the final state of a 2×2 quantum game with decoherence as described in Chapter 6. The function `Play`$[A, B, \rho]$ executes the strategies A and B of the two players on a two qubit state described by the density matrix ρ. The strategies A and B are 2×2 complex matrices as described by Eq. (2.9). Values for the density matrices $\rho_{ij} = |ij\rangle\langle ij|$, $i, j \in \{0, 1\}$ are set and one can be used as an initial state for the players' qubits. The function `J`$[\gamma]$ is a two qubit entangling matrix. The function `Decohere`$[\rho, p]$ returns the two qubit density matrix ρ after a measurement in the computational basis with probability p on either qubit. `DirectProduct`$[A, B]$ returns the direct product of the matrices A and B, while `Dag`$[A]$ returns the Hermitian conjugate of A.

```
Dag::usage = "Dag[A] returns the complex conjugate transpose of the matrix A."

Decohere::usage = "Decohere[rho, p] returns the density matrix for a two qubit state
rho after a measurement in the computational basis with probability p on either qubit."

DirectProduct::usage = "DirectProduct[A, B] returns the direct product of the matrices
A and B. DirectProduct[A, B, C] returns the direct product of the three matrices.
DirectProduct[A, n] where n is an integer returns the direct product of A with itself
n times."

(* Possible initial density matrices *)
rho00 = {{1, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}
rho01 = {{0, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}
rho10 = {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 0}}
rho11 = {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 1}}


(* Entangling operator *)
J[g_] := {{Cos[g/2], 0, 0, I Sin[g/2]}, {0, Cos[g/2], I Sin[g/2], 0},
        {0, I Sin[g/2], Cos[g/2], 0}, {I Sin[g/2], 0, 0, Cos[g/2]}}


(* Executes the strategies A and B on the game state rho *)
Play[A_List, B_List, rho_List] := DirectProduct[A,B].rho.Dag[DirectProduct[A,B]]


(* Return the density matrix for a two qubit state rho after a measurement
in the computational basis with probability p on either qubit *)
Decohere[ rho_List, p_ ] := (1-p)^2 rho + p^2 Diag[rho] +
                            p(1-p) (Diag1[rho] + Diag2[rho])
```

```
(* Various functions of a square (density) matrix A *)
Diag[ A_ ] := Table[ If[ i == j, A[[i,j]], 0 ], {i, Length[A]}, {j, Length[A]} ]
Diag1[ A_ ] := Table[ If[ Floor[(i+1)/2] == Floor[(j+1)/2], A[[i,j]], 0 ],
                    {i, Length[A]}, {j, Length[A]} ]
Diag2[ A_ ] := Table[ If[ (i == j)||(Abs[i-j] == 2), A[[i,j]], 0 ],
                    {i, Length[A]}, {j, Length[A]} ]
Dag[ A_ ] := Transpose[ A /.  I -> ZZ /.  {-I -> I, ZZ -> -I} ]

(* Rules for the direct product *)
DirectProduct[ A_List ] := A
DirectProduct[ A_List, 1 ] := A
DirectProduct[ A_List, n_Integer ] :=
    Module[
        { AA = A },
        Do[ AA = DirectProduct[AA, A], {i,2,n} ];
        AA
    ]
DirectProduct[ A_List, B_List ] :=
    Module[
        { M,
            nr=Length[A], nc=Length[ A[[1]] ],
            mr=Length[B], mc=Length[ B[[1]] ] },
        Do[
            M[ (i-1)mr+k, (j-1)mc+l ] = A[[i,j]] B[[k,l]],
            {l,mc}, {k,mr}, {j,nc}, {i,nr}
        ];
        Array[ M, {nr mr, nc mc} ]
    ]
DirectProduct[ A_List, B_List, CC_List ] := DirectProduct[ DirectProduct[A,B], CC ]
```

A.2 Classical Parrondo's games—Chapter 7

A.2.1 Capital-dependent game—Section 7.2.1

This package is used to generate the expected return versus time for a mixture of game A and the capital-dependent game B. A one-dimensional array is required to hold the probability of having capitals from $-n$ to n, where n is the maximum number of games to be played. This is set up by the `Start`$[n]$ command for an initial capital of 0. The command `Results`$[c, p, \ldots, n]$ generates a list of the expected payoffs for n steps of the specified game (A if one probability is given, B if two) while Results$[c, p, p_1, p_2, n_A, n_B, n]$ does the same for the periodic mixed sequence of n_A games of A followed by n_B games of B. ResultsRandom$[c, p, p_1, p_2, \gamma, n]$ does the same for a random mixed sequence with probability γ of selecting A and probability $1-\gamma$ of selecting B, at each step.

```
BeginPackage["ParrondoCap`"]

Expect::usage = "Expect[cap] returns the expectation value ⟨$⟩ of the array of capital
probabilities."

InitializeArray::usage = "InitializeArray[cap] returns the array of capital
probabilities cap for initial capital = 0."

MakeEmpty::usage = "MakeEmpty[n] returns an empty array to hold the probabilities for
capitals from −n to n."

Results::usage = "Results[cap, p, n] returns the expected payoffs for a sequence of n
games A with winning probability of p.  Results[cap, p1, p2, n] returns the expected
payoffs for a sequence of n games B with winning probabilities of p1 and p2 for coins
B1 and B2, respectively.  Results[cap, p, p1, p2, na, nb, n] returns the expected
payoffs for the periodic sequence of na games A followed by nb games B, for a total of
n games.  An initialized array of capital probabilities is held in cap."

Results2::usage = "Results2[cap, p, p1, p2, na, nb, n] returns the expected payoffs for
the periodic sequence of nb games of B (with winning probabilities p1 and p2) followed
by na games of A (with winning probability p), for a total of n games; cap is an
initialized array of capital probabilities."

ResultsRandom::usage = "ResultsRandom[cap, p, p1, p2, gamma, n] returns the expected
payoffs for n games of a random sequence of A and B with probability gamma of choosing
A at each step.  The winning probability of games A, B1 and B2 are p, p1 and p2,
respectively, and cap is an initialized array of capital probabilities."

Start::usage = "Start[n] return an initialized array to hold the probability of a
particular capital, where n is the maximum number of games to be played.  Initial
capital is 0."

Begin["Private`"]

(* Return an initialized array to hold the probability of a particular capital:
Start[n][[x]] is the probability of having capital = x − (n+1) where n is the maximum
number of games to be played.  Initial capital is 0.  *)
Start[n_Integer] := InitializeArray[ MakeEmpty[n] ]

(* Return an empty array to hold the probabilities for capitals from −n to n *)
MakeEmpty[ n_Integer ] := Table[0, {i,−n,n}]


(* Return the array of capital probabilities with initial capital set to 0 *)
```

```
InitializeArray[ cap_List ] :=
    Module[
        { nc = cap },
        nc[[ (Length[nc] + 1)/2 ]] = 1;
        nc
    ]

(* Return the array of capital probabilities with initial capital set to -1,0,+1
each with probability 1/3 *)
InitializeArray2[ cap_List ] :=
    Module[
        { nc = cap, n = (Length[nc]-1)/2 },
        nc[[n]] = 1/3;
        nc[[n+1]] = 1/3;
        nc[[n+2]] = 1/3;
        nc
    ]

(* Return a list of expected payoffs for n steps of game A *)
Results[ cap_List, p_, n_Integer ] := Results[cap, p, p, n]

(* Return a list of expected payoffs for n steps of game B *)
Results[ cap_List, p1_, p2_, n_Integer ] :=
    Module[
        { results = Table[0, {i,n}], nc = cap },
        Do[ nc = NextStep[ nc, p1, p2];
            results[[i]] = Expect[nc],
            {i,n}
        ];
        results
    ]

(* Return a list of expected payoffs for n steps of a periodic sequence
of na games of A followed by nb games of B for a total of n games *)
Results[ cap_List, p_, p1_, p2_, na_Integer, nb_Integer, n_Integer ] :=
    Module[
        { results = Table[0, {i,n}],
            nseries = Floor[n/(na+nb)],
            nc=cap },
        Do[
            Do[ nc = NextStep[nc, p, p];
                results[[i+j]] = Expect[nc],
```

```
                {j,na}
            ];
            Do[ nc = NextStep[nc, p1, p2];
                results[[i+na+k]] = Expect[nc],
                {k,nb}
            ],
            {i, 0, (nseries-1)(na+nb), (na+nb)}
        ];
        results
    ]

(* Return a list of expected payoffs for n steps of a periodic sequence
of nb games of B followed by na games of A for a total of n games *)
Results2[ cap_List, p_, p1_, p2_, na_Integer, nb_Integer, n_Integer ] :=
    Module[
        { results = Table[0, {i,n}],
            nseries = Floor[n/(na+nb)],
            nc=cap },
        Do[
            Do[ nc = NextStep[nc, p1, p2];
                results[[i+j]] = Expect[nc],
                {j,nb}
            ];
            Do[ nc = NextStep[nc, p, p];
                results[[i+nb+k]] = Expect[nc],
                {k,na}
            ],
            {i, 0, (nseries-1)(na+nb), (na+nb)}
        ];
        results
    ]

(* Return a list of expected payoffs for n steps of a random mixed sequence
of games A and B, with a probability gamma of selecting A at each step *)
ResultsRandom[ cap_, p_, p1_, p2_, gamma_, n_Integer ] :=
    Module[
        { results = Table[0, {i,n}], nc = cap },
        Do[ nc = gamma * NextStep[nc, p] + (1-gamma) * NextStep[nc, p1, p2];
            results[[i]] = Expect[nc],
            {i,n}
        ];
        results
    ]
```

```
(* Return the capital array after a randomly chosen game, choosing game A
(with winning probability p) with probability gamma and game B
(with winning probabilities p1 and p2) with probability 1-gamma *)
NextStepRandom[ c_List, p_, p1_, p2_, gamma_ ] :=
    gamma * NextStep[c, p, p] + (1-gamma) * NextStep[c, p1, p2]

(* Return the capital array after a step(s) of the capital-dependent game B
with probabilities p1 and p2.  Game A is a special case with p1=p2.  *)
NextStep[ c_List, p1_, p2_, m_Integer ] :=
    Module[ { nc=c }, Do[ nc = NextStep[nc, p1, p2], {j,m} ]; nc ]
        NextStep[ c_List, p1_, p2_ ] := Table[ Step[c, p1, p2, i], {i,Length[c]} ]
        Step[ c_List, p1_, p2_, x_ ] :=
        N[
        Which[
            (x == 1),
                If[ Mod[2 - (Length[c] + 1)/2, 3] == 0,
                    c[[2]] (1-p1),
                    c[[2]] (1-p2)
                ],
            (x == Length[c]),
                If[ Mod[(x-1) - (Length[c] + 1)/2, 3] == 0,
                    c[[x-1]] p1,
                    c[[x-1]] p2
                ],
            True,
                Which[
                    Mod[ (x+1) - (Length[c] + 1)/2, 3] == 0,
                        c[[x+1]] (1-p1) + c[[x-1]] p2,
                    Mod[(x-1) - (Length[c] + 1)/2, 3] == 0,
                        c[[x+1]] (1-p2) + c[[x-1]] p1,
                    True,
                        c[[x+1]] (1-p2) + c[[x-1]] p2
                ]
        ]
    ]

(* Return the expectation value ⟨$⟩ of the capital array c *)
Expect[ c_List ] :=
    Module[
        { pay = 0, n = (Length[c]-1)/2 },
        Do[ pay += c[[i]] (i-n-1), {i,Length[c]} ];
        pay
    ]
```

```
End[]

EndPackage[]
```

A.2.2 History-dependent game—Section 7.2.2

This package is used to generate the expected return versus time for a mixture of game A and the history-dependent game B, or for a mixture of two history-dependent games (see Sec. 7.2.2). The commands are very similar to the capital-dependent game except with the addition of extra probabilities in game B. The command `Results` can be used as before to return a list of expected payoffs for game A, B or a mixture. By specifying two sets of probabilities for the history-dependent game, `Results`$[c, p_1, p_2, p_3, p_4, p'_1, p'_2, p'_3, p'_4, n_{\mathrm{B}}, n_{\mathrm{B'}}, n]$ returns a list of the expected payoffs for n steps of the periodic sequence of n_{B} games of B followed $n_{\mathrm{B'}}$ games of B$'$. Similarly, by specifying two sets of history-dependent probabilities in `ResultsRandom` a random mixture of the two history-dependent games can be considered.

```
BeginPackage["ParrondoHist`"]

Expect::usage = "Expect[cap] returns the expectation value ⟨$⟩ of the array of capital
probabilities."

InitializeArray::usage = "InitializeArray[cap] returns the array of capital
probabilities cap for initial capital = 0."

MakeEmpty::usage = "MakeEmpty[n] returns an empty array to hold the probabilities for
capitals from -n to n."

Results::usage = "Results[cap, p, n] returns the expected payoffs for a sequence of
game A with winning probability p.  Results[cap, p1, p2, p3, p4, n] returns the
expected payoffs for a sequence of game B with winning probabilities p1, p2, p3 and p4,
for coins B1, B2, B3 and B4, respectively.  Results[cap, p, p1, p2, p3 p4, na, nb, n]
returns the expected payoffs for the periodic sequence of na games of A followed by nb
games of B. Results[cap, q1, q2, q3, q4, p1, p2, p3, p4, n1, n2, n] returns the
expected payoffs for a periodic sequence of n1 games of B with winning probabilities q1,
q2, q3 and q4, followed by n2 game of B with winning probabilities p1, p2, p3 and p4.
The total number of games is n and cap is an initialized array of capital
probabilities."

Results2::usage = "Results2[cap, p, p1, p2, p3, p4, na, nb, n] returns the expected
payoffs for the periodic sequence of nb games of B (with winning probabilities p1, p2,
p3 and p4) followed by na games of A (with winning probability p).  The total number of
games is n and cap is an initialized array of capital probabilities."
```

```
ResultsRandom::usage = "ResultsRandom[cap, p, p1, p2, p3, p4, gamma, n] returns the
expected payoffs for n games of a random sequence of A and B with probability gamma of
choosing A at each step.  The winning probabilities are p for game A, and for p1, p2,
p3 and p4 for game B; cap is an initialized array of capital probabilities.
ResultsRandom[cap, q1, q2, q3, q4, p1, p2, p3, p4, gamma, n] returns the expected
payoffs for a random mixed sequence of two games B (with probabilities q1, q2, q3, and
q4, or p1, p2, p3, and p4), selecting the first game with probability gamma at each
step."

Start::usage = "Start[n] returns an initialized array to hold the probability of a
particular capital, where n is the maximum number of games to be played.  Initial
capital is 0."

Begin["Private`"]

(* Return an initialized array to hold the probability of a particular capital:
Start[n][[x]] is the probability of having capital = x - (n+1) where n is the
maximum number of games to be played.  Initial capital is 0 *)
Start[ n_Integer ] := Table[ If[ i == n+1, 1, 0 ], {i,2n+1} ]

(* Set up a history vector holding the results of the last two games:  preferred
starting point is to take an equal mixture of the four possible histories *)
inithist = {{1/4},{1/4},{1/4},{1/4}}

(* History-dependent game B with winning probabilities p1 after (loss, loss),
p2 after (loss, win), p3 after (win, loss), p4 after (win, win) *)
B[ p1_, p2_, p3_, p4_ ] :=
    {{1-p1, 0, 1-p3, 0}, {p1, 0, p3, 0}, {0, 1-p2, 0, 1-p4}, {0, p2, 0, p4}}

(* Game A is a special case of game B with all the probabilities the same *)
A[ p_ ] := B[p, p, p, p]

(* Return a list of expected payoff versus t for a sequence of n games A *)
Results[ cap_List, hist_List, p_, n_Integer ] := Results[cap, hist, p, p, p, p, n]

(* Return a list of expected payoff versus t for a sequence of n games B *)
Results[ cap_List, hist_List, p1_, p2_, p3_, p4_, n_Integer ] :=
    Module[ { results = Table[0,{i,n}], nc = cap, h = hist },
        Do[ h = B[p1, p2, p3, p4].h;
            nc = NextStep[nc, Win[h]];
            results[[i]] = Expect[nc],
            {i,n}
        ];
        results
    ]
```

```
(* Return a list of expected payoff versus t from playing a periodic sequence
of na games of A followed by nb games of B for a total of n games *)
Results[ cap_List, hist_List, p_, p1_, p2_, p3_, p4_,
        na_Integer, nb_Integer, n_Integer ] :=
    Results[cap, hist, p, p, p, p, p1, p2, p3, p4, na, nb, n]


(* Return a list of expected payoff versus t from playing a periodic sequence
of na games of B (with probabilities q1, q2, q3, q4) followed by nb games of B
(with probabilities p1, p2, p3, p4), for a total of n games *)
Results[ cap_List, hist_List, q1_, q2_, q3_, q4_, p1_, p2_, p3_, p4_,
        n1_Integer, n2_Integer, n_Integer ] :=
    Module[
        { results = Table[0,{i,n}],
            nseries = Floor[n/(n1+n2)],
            nc=cap, h=hist },
        Do[
            Do[ h = B[q1, q2, q3, q4].h;
                nc = NextStep[nc, Win[h]];
                results[[i+j]] = Expect[nc],
                {j,n1}
            ];
            Do[ h = B[p1, p2, p3, p4].h;
                nc = NextStep[nc, Win[h]];
                results[[i+n1+k]] = Expect[nc],
                {k,n2}
            ],
            {i, 0, (nseries-1)(n1+n2), (n1+n2)}
        ];
    results
    ]


(* Return a list of expected payoff versus t from playing a periodic sequence
of nb games of B followed by na games of A for a total of n games *)
Results2[ cap_List,hist_List, p_, p1_, p2_, p3_, p4_,
                    na_Integer, nb_Integer, n_Integer ] :=
    Results[cap, hist, p1, p2, p3, p4, p, p, p, p, nb, na, n]


(* Return a list of expected payoff versus t from playing a random sequence of n games
of A and B, with a probability gamma of selecting A at each step *)
ResultsRandom[ cap_List, hist_List, p_, p1_, p2_, p3_, p4_, gamma_, n_Integer ] :=
    ResultsRandom[ cap, hist, p, p, p, p, p1, p2, p3, p4, gamma, n ]
```

```
(* Return a list of expected payoff versus t from playing a random mixed sequence of
two different games B (with probs q1, q2, q3, q4 chosen with frequency gamma, or with
probs p1, p2, p3, p4 chosen with frequency 1-gamma) for a total of n games *)
ResultsRandom[ cap_List, hist_List, q1_, q2_, q3_, q4_, p1_, p2_, p3_, p4_,
            gamma_, n_Integer ] :=
    Module[
        { results = Table[0,{i,n}],
            qp1 = gamma q1 + (1-gamma) p1,
            qp2 = gamma q2 + (1-gamma) p2,
            qp3 = gamma q3 + (1-gamma) p3,
            qp4 = gamma q4 + (1-gamma) p4,
            nc=cap, h=hist },
        Do[ h = B[qp1, qp2, qp3, qp4].h;
            nc = NextStep[nc, Win[h]];
            results[[i]] = Expect[nc],
            {i,n}
        ];
        results
    ]

(* Update the array of capital probabilities c according to the winning probs w *)
NextStep[ c_List, w_, m_Integer ] :=
    Module[ {nc=c}, Do[ nc = NextStep[nc, w], {j,m} ]; nc ]
NextStep[ c_List, w_ ] := Table[ Step[c, w, i], {i,Length[c]} ]
Step[ c_List, w_, x_ ] :=
    N[
        Which[
            (x == 1),
                c[[x+1]] (1-w),
            (x == Length[c]),
                c[[x-1]] w,
            True,
                c[[x+1]] (1-w) + c[[x-1]] w
        ]
    ]

(* Return the probability of a win based on the history vector h *)
Win[ h_List ] := h[[2,1]] + h[[4,1]]

(* Return the expectation value <$> of the capital array c *)
Expect[ c_List ] :=
    Module[
```

```
        { pay = 0, n = (Length[c]−1)/2 },
        Do[ pay += c[[i]] (i−n−1), {i,Length[c]} ];
        pay
    ]

End[]

EndPackage[]
```

A.3 Quantum walks—Section 7.3.1 and Chapter 8

The package below contains functions that execute the history-dependent quantum walk or the quantum walk with a position-dependent potential. A combination of two walks to create a quantum Parrondo's game can also be executed. An array to contain the x amplitudes is created by the functions `SymStart`[n] or `AntStart`[n], for a starting start that is, respectively, symmetric or antisymmetric under the interchange of $L \leftrightarrow R$. The function `Results`[c, ρ, n] returns a list of expectation values of x for n steps of the quantum walk starting with the state c and governed by the list of probabilities ρ (e.g., for $M = 3$, ρ is a list of the four probabilities $\{\rho_{\text{LL}}, \rho_{\text{LR}}, \rho_{\text{RL}}, \rho_{\text{RR}}\}$). The position-dependent potentials of Eq. (7.4) with parameters α and β can be applied by `Results`[$c, \rho, \alpha, \beta, n$]. A periodic mixture of two different walks is carried out by `Results`[$c, \rho_1, \rho_2, n_1, n_2, n$]. The package includes several plotting functions to display various features of the quantum walks, of which the most useful is `PlotProbs`[c] that plots the probability density distribution for the state c.

```
BeginPackage["QuantumWalk‘"]

AntStart::usage = "AntStart[n, m] returns an initialized array (with starting state
anti-symmetric) to hold the amplitudes for positions from −n to n for m coins."

SymStart::usage = "SymStart[n, m] returns an initialized array (with symmetric starting
state) to hold the amplitudes for positions from −n to n for m coins."

MakeEmpty::usage = "MakeEmpty[n, m] returns an empy array to hold the amplitudes for
positions from −n to n for m coins."

ExtendArray::usage = "ExtendArray[cap, n] extends the array of amplitudes cap by an
additional n positions (+ and −)."

Start::usage = "Start[n, st] returns an initialized array for m=Log[2, Length[st]]
coins for positions from −n to n with a normalized starting state at x=0 whose
relative amplitudes and phases are specified by the list st."
```

```
Results::usage = "Results[cap, p, n] returns a list of ⟨x⟩ for a sequence of n steps
with probabilities specified by the list p.  Results[cap, p, a, b, n] is the same but
with a biased sawtooth potential depending on a and b, as per Meyer's scheme.
Results[cap, q, p, na, nb, n] does na steps
with probabilities q, followed by nb steps with probabilities p, repeating the sequence
n times.  The p an q are lists of history-dependent probabilities; cap is a an
initialized array of x amplitudes."

ResultsRandom::usage = "ResultsRandom[cap, gamma, q, p, n] returns a list of ⟨x⟩ for a
random mixture of steps with probabilities specified by the lists q or p, the first
being chosen with probability gamma and the second with probability 1−gamma.  The total
number of steps in n and cap is an initialized array of x amplitudes."

NextStep::usage = "NextStep[cap, p] returns the position array cap after one step with
probabilities specified by the list p.  NextStep[cap, p, n] returns the array after
n steps.  NextStep[cap, p, a, b, n] returns the array after n steps with the addition
of a biased sawtooth potential specified by a and b."

Expect::usage = "Expect[cap] returns ⟨x⟩ for the array of position amplitudes cap."

PlotSmooth::usage = "PlotSmooth[cap, range, color] plots the distribution of position
probabilities for the array cap, smoothing by (p(x−1) + 2 p(x) + p(x+1))/4.
Specifying a range {ymin,ymax} or color
(e.g., Red, RGBColor[0,0,1]) is optional."

PlotProb::usage = "PlotProb[cap, range, color] plots the distribution of probabilities
for the array of position amplitudes cap.  Specifying a range {ymin,ymax} or color (e.g.
Red, RGBColor[0,0,1] etc.)  is optional."

ProbDist::usage = "ProbDist[cap, p, n, int] plots the distribution of position
probabilities after every int moves for a starting array cap and probabilities
specified by the list p for a total of n moves.  ProbDist[cap, p, a, n, int] adds a
biasing potential V(x) = a x."

ProbRange::usage = "ProbRange[cap, x, y] returns the probability that the particle is
located between x and y."

Begin["Private`"]

(* Create an empy array to hold amplitudes for positions from −n to +n *)
MakeEmpty[ n_Integer, m_Integer ] := Table[0, {i,−n,n}, {j, 2^m}]


(* Return a starting array symmetric in past histories *)
SymStart[ n_Integer, m_Integer ]:=
    Table[ If[ i == 0, (1/Sqrt[2])^m, 0 ], {i,−n,n},{j, 2^m} ]


(* Return a starting array anti-symmetric in the momentum direction for each
```

```
momentum in the history *)
AntStart[ n_Integer, m_Integer ] :=
    Table[
        If[ i == 0,
            Antisym[m]/Sqrt[2]^m,
            Table[0, {i, 2^m}]
        ],
        {i,-n,n}
    ]
Antisym[ m_Integer ] := Table[ (-1)^Parit[i-1], {i, 2^m} ]
Parit[ m_Integer ] := Sum[ IntegerDigits[m, 2][[j]], {j, Length[IntegerDigits[m,2]]} ]

(* Return an initialized starting array for capitals from -n to n with an arbitrary
starting combination at x=0 specified by the list st of 2^m elements *)
Start[ n_Integer, st_List ] :=
    Table[
        If[ i == 0,
            st/Sqrt[ Sum[ Abs[st[[i]]]^2, {i,Length[st]} ] ],
            Table[ 0, {i,Length[st]} ]
        ],
        {i,-n,n}
    ]

(* Extend an existing array of capitals *)
ExtendArray[ cap_List, n_Integer ] :=
    Table[
        If[ (j <= n) || (j > n + Length[cap]),
            Table[ 0, {i,Length[cap[[1]]]} ],
            cap[[j - n]]
        ],
        {j, Length[cap] + 2 n}
    ]

(* Return the expectation values of x for a sequence of n steps *)
Results[ cap_List, p_List, n_Integer ] :=
    Module[ { results = Table[0,i,n], nc=cap, j },
        Do[ nc = NextStep[nc, p];
            results[[j]] = Expect[nc],
            {j,n}
        ];
        results
    ]
```

```
(* Add a potential specified by a and b *)
Results[ cap_List, p_List, a_, b_, n_Integer ] :=
    Module[ { results = Table[0,{i,n}], nc=cap, j },
        Do[ nc = NextStep[nc, p, a, b];
            results[[j]] = Expect[nc],
            {j,n}
        ];
        results
    ]

(* Return <x> for a periodic mixed sequence of two walks, with na steps with
probabilities q followed by nb steps with probabilities p, for a total of n steps *)
Results[ cap_List, q_List, p_List,
        na_Integer, nb_Integer, n_Integer ] :=
    Module[
        { results = Table[0,{i,n}],
            nseries = Floor[n/(na+nb)],
            nc=cap },
        Do[
            Do[
                nc = NextStep[nc, q];
                results[[(i-1)(na+nb)+j]] = Expect[nc],
                {j,na}
            ];
            Do[
                nc = NextStep[nc, p];
                results[[(i-1)(na+nb)+na+k]] = Expect[nc],
                {k,nb}
            ],
            {i,nseries}
        ];
        results
    ]

(* Add potentials specified by a1, b1 and a2, b2 *)
Results[ cap_List, q_List, a1_, b1_, p_List, a2_, b2_,
        na_Integer, nb_Integer, n_Integer ] :=
    Module[
        { results = Table[0,{i,n}],
            nseries = Floor[n/(na+nb)],
            nc=cap },
        Do[
```

```
            Do[
                nc = NextStep[nc, q, a1, b1];
                results[[(i-1)(na+nb)+j]] = Expect[nc],
                {j,na}
            ];
            Do[ nc = NextStep[nc, p, a2, b2];
                results[[(i-1)(na+nb)+na+k]] = Expect[nc],
                {k,nb}
            ],
            {i,nseries}
        ];
        results
    ]

(* Return <x> for a random mixed sequence of walks with probabilities q or p,
selecting the 1st with probability gamma and the 2nd with probability 1-gamma) *)
ResultsRandom[ cap_, q_List, p_List, gamma_ n_Integer ] :=
    Module[
        { results = Table[0, {i,n}],
            nc=cap },
        Do[
            If[
                Random[] < gamma,
                nc = NextStep[nc, q],
                nc = NextStep[nc, p]
            ];
            results[[j]] = Expect[nc],
            {j,n}
        ];
        results
    ]


(* Add potentials specified by a1, b1 and a2, b2 *)
ResultsRandom[ cap_, q_List, a1_, b1_, p_List, a2_, b2_, gamma_, n_Integer ] :=
    Module[
        { results = Table[0, {i,n}], nc=cap },
        Do[
            If[
                Random[] < gamma,
                nc = NextStep[nc, q, a1, b1],
                nc = NextStep[nc, p, a2, b2]
```

```
            ];
            results[[j]] = Expect[nc],
            {j,n}
        ];
        results
    ]


(* Take a step(s) in array c with probability list p and (optional) potential
specified by a and b *)
NextStep[ c_List, p_List, m_Integer ] :=
    Module[ {nc=c}, Do[ nc = NextStep[nc, p, 0, 0], {k,m} ]; nc ]
NextStep[ c_List, p_List, a_, b_, m_Integer ] :=
    Module[ {nc=c}, Do[ nc = NextStep[nc, p, a, b], {k,m} ]; nc ]
NextStep[ c_List, p_List] := Table[ Step[c, p, 0, 0, i], {i,Length[c]} ]
NextStep[ c_List, p_List, a_, b_] := Table[ Step[c, p, a, b, i], {i,Length[c]} ]
Step[ c_, p_, x_ ] :=
    Module[
        { m = Length[c[[x]]] },
        N[
            Table[
                If[ j <= m/2,
                    If[ x == Length[c],
                        0,
                        Sqrt[p[[j]]] c[[x+1, 2j-1]] +
                            I Sqrt[1-p[[j]]] c[[x+1, 2j]]
                    ],
                    If[ x == 1,
                        0,
                        I Sqrt[1-p[[j-m/2]]] c[[x-1, 2j-m-1]] +
                            Sqrt[p[[j-m/2]]] c[[x-1, 2j-m]]
                    ]
                ],
                {j,m}
            ]
        ]
    ]
Step[ c_, p_, a_, b_, x_ ] :=
    Module[
        { m = Length[c[[x]]],
            v1 = Exp[-I V[c, x+1, a, b] ],
            v2 = Exp[-I V[c, x-1, a, b] ] },
        N[
```

```
            Table[
                If[ j <= m/2,
                    If[ x == Length[c],
                        0,
                        v1 (Sqrt[p[[j]]] c[[x+1, 2j-1]] +
                            I Sqrt[1-p[[j]]] c[[x+1, 2j]])
                    ],
                    If[ x == 1,
                        0,
                        v2 (I Sqrt[1-p[[j-m/2]]] c[[x-1, 2j-m-1]] +
                            Sqrt[p[[j-m/2]]] c[[x-1, 2j-m]])
                    ]
                ],
                {j,m}
            ]
        ]
    ]

(* Return the value of the potential specified by a and b, at a position z:
b=0 gives VA (linear x-dependent potential), b>0 gives VB (sawtooth potential),
b<0 gives V(x) = a P(x) *)
V[ c_, z_, a_, b_ ] :=
    If[ b == -1,
        If[ (z > 0)&&(z <= Length[c]),
            a Sum[ Abs[c[[z,k]]^2], {k,Length[c[[1]]]} ],
            0
        ],
        a (z-(Length[c]+1)/2) + b (1-Mod[z-(Length[c]+1)/2, 3]/2)
    ]

(* Additional potentials *)
(* Step function at x=a of height b (Pi/2 is infinite?)  *)
V1[ c_, z_, a_, b_ ] := If[ z >= (Length[c]+1)/2 + a, b, 0 ]

(* Barrier of width 10 at x=a of height b *)
V2[ c_, z_, a_, b_ ] :=
    If[ (z >= (Length[c]+1)/2 + a) && (z <= (Length[c]+1)/2 + a + 10), b, 0 ]

(* Return <x> for array c, assumed to be numerical *)
Expect[ c_List ] :=
    Sum[
        Abs[ c[[i,j]]^2 ] (i-(Length[c]+1)/2),
```

```
        {j,Length[c[[1]]]}, {i,Length[c]}
    ]

(* Return the variance for array c, assumed numerical *)
Variance[c_List] := Expect[c,2] - Expect[c]^2

(* Return the k-th moment of c *)
Expect[ c_List, k_Integer ] :=
    Sum[
        Abs[ c[[i,j]]^2 ] (i-(Length[c]+1)/2)^k,
        {j,Length[c[[1]]]}, {i,Length[c]}
    ]
Moment[ p_List, k_Integer ] :=
    Sum[ p[[i,2]] p[[i,1]]^k, {i,Length[p]} ]/Length[p]

(* Return the probability that the particle is located between x and y *)
ProbRange[ c_, x_, y_ ] :=
    Sum[ Abs[ c[[j,k]]^2 ], {k,Length[c[[1]]]},
        {j, (Length[c]+1)/2 + x, (Length[c]+1)/2 + y} ]

(* Plot the probability densities for the range of x-values smoothing by a weighted
average; can also plot two distributions on the same graph in different colors *)
<<Graphics`MultipleListPlot`
PlotSmooth[ c_ ] := PlotSmooth[ c, All, Black ]
PlotSmooth[ c_ , range_List ] := PlotSmooth[ c, range, Black ]
PlotSmooth[ c_ , color_ ] := PlotSmooth[ c, All, color ]
PlotSmooth[ c_ , range_ , color_ ] :=
    ListPlot[
        Smooth[ Probs[c] ],
        PlotJoined -> True,
        AxesLabel -> {"x", "P"},
        PlotRange -> range,
        PlotStyle -> color
    ]
PlotSmooth[ c1_, c2_, range_, color1_, color2_ ] :=
    MultipleListPlot[
        Smooth[ Probs[c1] ], Smooth[ Probs[c2] ],
        PlotJoined -> True,
        AxesLabel -> {"x", "P"},
        SymbolShape -> None,
        PlotRange -> range,
        PlotStyle -> {color1, color2}
    ]
```

```
(* Plot the probability densities for the range of x values,
plotting odd or even points depending on which are non-zero *)
PlotProb[ c_List ] := PlotProb[ c, All, Black ]
PlotProb[ c_List, range_List ] := PlotProb[ c, range, Black ]
PlotProb[ c_List, color_ ] := PlotProb[ c, All, color ]
PlotProb[ c_List, range_, color_ ] :=
    ListPlot[
        If[ ProbRange[c,0,0] == 0, EvenList[Probs[c]], OddList[Probs[c]] ],
        PlotJoined -> True,
        AxesLabel -> {"x", "P"},
        PlotRange -> range,
        PlotStyle -> color
    ]
PlotProb[ c1_List, c2_List, range_, color1_, color2_ ] :=
    MultipleListPlot[
        If[ ProbRange[c1,0,0] == 0,
            EvenList[Probs[c1]],
            OddList[Probs[c1]]
        ],
        If[ ProbRange[c2,0,0] == 0,
            EvenList[Probs[c2]],
            OddList[Probs[c2]]
        ],
        PlotJoined -> True,
        AxesLabel -> {"x", "P"},
        SymbolShape -> None,
        PlotRange -> range,
        PlotStyle -> {color1, color2}
    ]


(* Show successive probability distributions as they evolve, plotting in colors
successively changing from red to blue *)
ProbDist[ c_List, p_List, n_, int_ ] := ProbDist[ c, p, 0, 0, n, int, 0.3 ]
ProbDist[ c_List, p_List, n_, int_, maxy_ ] := ProbDist[ c, p, 0, 0, n, int, maxy ]
ProbDist[ c_List, p_List, a_, b_, n_, int_ ] := ProbDist[ c, p, a, b, n, int, 0.3 ]
ProbDist[ c_List, p_List, a_, b_, n_, int_, maxy_ ] :=
    Module[ { nc=c },
        Do[
            nc = NextStep[nc, p, a, b, int];
            PlotProb[ nc, {{-(Length[nc]-1)/2, (Length[nc]-1)/2},
            {0,maxy}}, Red ];
            Print["t = ", j*int],
```

```
            {j,Floor[n/int]}
        ];
        nc
    ]

(* Return a list of probabilities *)
Probs[ c_ ] := Table[ {j-(Length[c]+1)/2, Sum[ Abs[c[[j,k]]^2], {k,Length[c[[1]]]} ]},
            {j,Length[c]} ]

(* Return the odd or even points in a list---i.e.  the non-zero points *)
OddList[ l_ ] := Table[ l[[i]], {i,1,Length[l],2} ]
EvenList[ l_ ] := Table[ l[[i]], {i,2,Length[l],2} ]

(* Return a list of smoothed probabilities *)
Smooth[ c_ ] := Table[ (c[[j-1]] + 2 c[[j]] + c[[j+1]])/4, {j,2,Length[c]-1} ]

(* Return the position of a peak between x=n1 and x=n2 in the prob distribution p *)
PeakPosn[ p_List, n1_, n2_ ] :=
    p[[ Ordering[ Table[ p[[j, 2]], {j, n1 + (Length[p] + 1)/2,
        n2 + (Length[p] + 1)/2}], -1] + n1 + (Length[p] - 1)/2 , 1]]

(* Plot the position of a peak between n1*t/10 and n2*t/10 as a function of t *)
PlotPeakPosn[ c_List, n1_, n2_, color_ ] :=
    ListPlot[
        Flatten[ Table[ PeakPosn[Probs[c[[i]]], i*n1, i*n2], {i, Length[c]} ] ],
        PlotJoined -> True, AxesLabel -> {"t", "x"}, PlotStyle -> color,
        Ticks -> { {{1, ""}, {2, ""}, {3, ""}, {4, ""}, {5, "50"},
            {6, ""}, {7, ""}, {8, ""}, {9, ""}, {10, "100"}, {11, ""},
            {12, ""}, {13, ""}, {14, ""}, {15, "150"}, {16, ""}, {17, ""},
            {18, ""}, {19, ""}, {20, "200"}}, Automatic }
    ]

(* Return x position of the peak and summed probability under the peak *)
GetPeak[ xp_List, n1_Integer, n2_Integer, cut_ ] :=
    Module[
        { m1 = n1 + (Length[xp]+1)/2,
            m2 = n2 + (Length[xp]+1)/2,
            posnmax = Ordering[ Table[ xp[[j,2]], j,m1,m2 ], -1 ][[1]] + m1 },
        { posnmax-(Length[xp]-1)/2,
            Sum[ xp[[j, 2]], { j, posnmax + GetCutOff[xp, posnmax, cut, -2],
                posnmax + GetCutOff[xp, posnmax, cut, +2]} ] }
    ]
```

```
(* Return the cutoff for the peak in the direction specified by step with a cutoff
ratio of c *)
GetCutOff[ p_List, pmax_, c_, step_ ] :=
    Module[
        { width, m },
        For[
            width=0; m=(Length[p]+1)/2,
            p[[pmax-width,2]]/p[[pmax,2]] > c,
            width += step
        ];
        width
    ]

(* Quantify the difference between two probability distributions ---
the second function is chi squared between the two curves *)
ProbDiff[ p1_List, p2_List ] :=
    Sum[ Abs[p1[[i,2]]-p2[[i,2]]], {i, Min[Length[p1], Length[p2]]} ]
ChiSq[ p1_List, p2_List ] :=
    Sum[ (p1[[i,2]]-p2[[i,2]])^2, {i, Min[Length[p1], Length[p2]]} ]



End[]
EndPackage[]
```

A.4 Quantum cellular automata—Chapter 9

A.4.1 One-dimensional QCA—Section 9.1.3

This package implements the one-dimensional quantum cellular automata described in Schumacher and Werner (2004) and shown schematically in Figure 9.5. The function `MakeEmpty`$[n]$ sets up an empty complex vector to hold a line of n cells. The function `SetQubit`$[Q,\ p,\ v]$ sets the pth qubit in the state Q to the value v, while the states specified by the p_i are set to the values v_i by the function `SetStates`$[Q,\ \{p_1, v_1, p_2, v_2, \ldots\}]$. It is simplest to specify the p_i in binary, e.g., 2^^1011 for the state $|1011\rangle$. The function `ApplyRule`$[Q,\ \theta,\ \alpha,\ \beta,\ \phi,\ S,\ m]$ applies the transition function specified by the angles θ, α, β, and ϕ to the state Q a total of m times (default 1 if m is omitted). The variable S is an optional character that can be L or R to apply an intermediate left- or right-shift. Alternately, the rule can be specified by giving the two matrices: the one qubit unitary $U(\theta, \alpha, \beta)$ and a control-phase gate $P(\phi)$. These can be set by the commands SetU[θ, α, β] and SetP[ϕ], respectively. PrintStates[Q] prints a list of the amplitudes of all possible states of Q. See Sec. 9.1.3 for details.

```
BeginPackage["QCA`"]

ApplyRule::usage = "ApplyRule[Q, theta, alpha, beta, phi, sh, m] returns the value of Q
after applying the rule specified by the one-qubit unitary U(theta, alpha, beta) and
the control-phase gate P(phi).  An optional character sh can be set to L or R for an
intermediate left- or right-shift.  If omitted no shift is performed.  The integer m is
an optional specification of the number of iterations to perform (default 1).
ApplyRule[Q, U, P, sh, m] returns the the value of Q after m iterations (default 1 if m
omitted) of the rule specified by the one-qubit unitary U, the control-phase gate P
(specified as matrices), and the optional shift sh."

MakeEmpty::usage = "MakeEmpty[n] returns an empty configuration of n qubits."

PrintStates::usage = "PrintStates[Q] prints the amplitudes of all the states of the
configuration Q."

SetP::usage = "SetP[phi] returns a two qubit control-phase gate with phase phi."

SetU::usage = "SetU[theta, alpha, beta] returns a one qubit unitary specified by the
rotation angle theta and the phases alpha and beta."

SetQubit::usage = "SetQubit[Q, n, value] returns a list of cells with the qubit in
position n set to value."

SetStates::usage = "SetStates[Q, p1, v1, p2, v2, ...]  returns Q with the states
specified by the pi set to the values vi.  It is simplest to specify the pi in binary,
e.g., 2^^1011 for the state |1011⟩."

Begin["Private`"]

(* Set up initial Q vector *)
MakeEmpty[n_Integer] := Table[ 0.  I + 0., i,1,2^n ]

(* Set the nth qubit of Q to val *)
SetQubit[ Q_List, n_Integer, val_ ] :=
    Module[ newQ = Q, newQ[[2^(n-1) + 1]] = val; newQ ]

(* Return a configuration with the list of states set to the specified values:  e.g.
{2^^011, 0.5, 2^^101, 0.4 I} sets the |011⟩ state to 0.5 and the |101⟩ state to 0.4 I *)
SetStates[ Q_List, vals_List ] :=
    Module[
        { newQ = Q },
        Do[ newQ[[vals[[i]]+1]] = vals[[i+1]], {i,1,Length[vals],2} ];
        newQ
    ]
```

```
(* Return Q after applying the specified rule m times (default 1).
The rule can be specified as two matrices U (single qubit unitary) and P (control-phase
gate), or as angles theta, alpha, beta (single qubit unitary with rotation angle theta
and phases alpha and beta) and phi (phase gate).  A following argument "L" ("R")
performs a left- (right-) shift in the intermediate stage, otherwise no shift is
performed.  A final (optional) integer argument specifies a number of repetitions.  *)
ApplyRule[ Q_List, t_, a_, b_, p_, m_Integer ] :=
    Module[ { newQ=Q },
        Do[ newQ = ApplyRule[newQ, t, a, b, p], {i,m} ]; newQ ]
ApplyRule[ Q_List, t_, a_, b_, p_, sh_, m_Integer ] :=
    Module[ { newQ=Q },
        Do[ newQ = ApplyRule[newqca, t, a, b, p, sh], {i,m} ]; newQ ]
ApplyRule[ Q_List, t_, a_, b_, p_ ] :=
    ApplyRule[ Q, SetU[t,a,b], SetP[p] ]
ApplyRule[ Q_List, t_, a_, b_, p_, sh_ ] :=
    ApplyRule[ Q, SetU[t,a,b], SetP[p], sh ]
ApplyRule[ Q_List, U_, P_, sh_, m_Integer ] :=
    Module[ { newQ=Q },
        Do[ newQ = ApplyRule[ newQ, U, P, sh ], {i,m} ]; newQ ]


(* These next two definitions actually perform the rule *)
ApplyRule[ Q_List, U_, P_ ] :=
    Module[
        { n = Floor[ Log[2, Length[Q]] ] },
        pgate = DirectProduct[P, Floor[n/2]];
        ShiftR[ pgate.ShiftL[ pgate.DirectProduct[U,n].Q ] ]
    ]
ApplyRule[ Q_List, U_, P_, s_ ] :=
    Which[
        IntegerQ[s],
            Module[ {newQ=Q},
                Do[ newQ = ApplyRule[newQ, U, P], {i,s} ]; newQ ],
        s == "L",
            Module[
                { n = Floor[ Log[2, Length[Q]] ] },
                pgate = DirectProduct[P, Floor[n/2]];
                ShiftR[ pgate.ShiftL[ pgate.ShiftL[ DirectProduct[U,n].Q ] ] ]
            ],
        s == "R",
            Module[ { n = Floor[ Log[2, Length[Q]] ] },
                pgate = DirectProduct[P, Floor[n/2]];
                ShiftR[ pgate.ShiftL[ pgate.ShiftR[ DirectProduct[U,n].Q ] ] ]
```

```
        ]
    ]

(* Return the matrices U or phase gate P given from the given arguments *)
SetU[ theta_, alpha_, beta_ ] :=
    { { Exp[I alpha] Cos[theta], I Exp[I beta] Sin[theta] },
        { I Exp[-I beta] Sin[theta], Exp[-I alpha] Cos[theta] } }
SetP[ phi_ ] := {{1,0,0,0},{0,1,0,0},{0,0,1,0},{0,0,0, Exp[I phi]}}

(* Do the shift left or right *)
ShiftR[ x_ ] := Table[ If[ i <= Length[x]/2, x[[2i-1]], x[[2i-Length[x]]] ],
                {i,Length[x]} ]
ShiftL[ x_ ] := Table[ If[ Mod[i,2] == 0, x[[i + (Length[x]/2 - Floor[i/2])]],
                x[[i - Floor[i/2]]] ], {i,Length[x]} ]

(* Prints the state of the specified configuration *)
PrintStates[ psi_ ] :=
    Module[
        { x, threshold=10^(-15), newpsi=psi },
        Do[ If[ Abs[psi[[i]]] < threshold, newpsi[[i]]=0 ], {i,Length[psi]} ];
        Do[ Print[ " |", IntegerDigits[ i-1, 2, Floor[Log[2,Length[psi]]] ], "> ",
            newpsi[[i]] /.
                { Complex[0.`, 1.`] -> I, Complex[0.`, -1.`]-> -I,
                    Complex[0.`, 0.`] -> 0, Complex[1.`, 0.`] -> 1,
                    Complex[-1.`, 0.`] -> -1,
                    Complex[x_, 0.`] -> x, Complex[0.`, x_] -> x I,
                    1.  -> 1, 0.  -> 0, -1.  -> -1 } ],
            {i,Length[psi]}
        ]
    ]

(* Rules for the direct product *)
DirectProduct[ A_List ] := A
DirectProduct[ A_List, 1 ] := A
DirectProduct[ A_List, n_Integer ] :=
    Module[
        { i, AA = A },
        Do[ AA = DirectProduct[AA, A], {i,2,n} ];
        AA
    ]
DirectProduct[ A_List, B_List ] :=
    Module[
```

```
        { M,
            nr=Length[A], nc=Length[ A[[1]] ],
            mr=Length[B], mc=Length[ B[[1]] ] },
        Do[
            M[ (i-1)mr+k, (j-1)mc+l ] = A[[i,j]] B[[k,l]],
            {l,mc}, {k,mr}, {j,nc}, {i,nr}
        ];
        Array[ M, {nr mr, nc mc} ]
    ]
DirectProduct[ A_List, B_List, CC__List ] := DirectProduct[ DirectProduct[A,B], CC ]



End[]

EndPackage[]
```

A.4.2 Semi-quantum Life—Section 9.2

The following code is a *Maple* routine for running the semi-quantum version of the game of Life. It is a basic routine lacking in sophisticated screen display. The procedure runs four generations of an 8×8 universe with an initial state set by the procedure `inputuniverse` that is currently set to the example given in Figure 9.8.

```
# Rudimentary Maple version of semi-quantum Life---Main program

with(linalg);

# Set the size of the universe
n := 8;

# Set maximum number of generations
maxgen := 4;


# Set up an empty universe
# A[i,j,1] is alive coefficient; A[i,j,2] is dead coefficient
A := array(0..n+1,0..n+1,1..2);
B := array(0..n+1,0..n+1,1..2);
for i from 1 to n do
    for j from 1 to n do
        A[i,j,1] := 0;
        A[i,j,2] := 1;
        B[i,j,1] := 0;
        B[i,j,2] := 1;
```

```
    od;
od;

# Surround the Universe by a series of null cells to avoid boundary problems
for i from 0 to n+1 do
    A[i,0,1] := 0;
    A[i,0,2] := 0;
    A[i,n+1,1] := 0;
    A[i,n+1,2] := 0;
    A[0,i,1] := 0;
    A[0,i,2] := 0;
    A[n+1,i,1] := 0;
    A[n+1,i,2] := 0;
od;

# Set up an empty print-out array
pa := array(1..2*n, 1..n);
for i from 1 to n do
    for j from 1 to n do
        B[i,j,1] := evalf( A[i,j,1] );
        B[i,j,2] := evalf( A[i,j,2] );
        pa[2*i-1, j] := evalf( round( abs(B[i,j,1])*100) );
        pa[2*i, j] := evalf( round( arctan(Im(B[i,j,1]), Re(B[i,j,1]))/Pi*100) );
    od;
od;

# Produce next generation
for gen from 1 to maxgen do
    showuniverse(A,n);
    print("generation=", gen);
    for i from 1 to n do
        for j from 1 to n do
            # Calculate surrounding amplitude and phase
            surrounds :=
                evalf( A[i-1,j-1,1] + A[i-1,j,1] + A[i-1,j+1,1] + A[i,j-1,1] +
                    A[i,j+1,1] + A[i+1,j-1,1 ] + A[i+1,j,1] + A[i+1,j+1,1] );
            amp := evalf( abs(surrounds) );
            phi := evalf( arctan( Im(surrounds), Re(surrounds) ) );
            # Calculate new status of cell
            if (amp <= 1) or (amp > 4) then
                B[i,j,1] := 0;
                B[i,j,2] := evalf(exp(I * phi) * abs(A[i,j,1]) + A[i,j,2]);
```

```
            else if (amp > 1) and (amp <= 2) then
                B[i,j,1] := evalf((sqrt(2) + 1) * (amp-1) * A[i,j,1]);
                B[i,j,2] := evalf((sqrt(2) + 1) * (amp-1) * A[i,j,2] +
                                (2-amp) * (exp(I * phi) * abs(A[i,j,1]) + A[i,j,2]))
            else if (amp > 2) and (amp <= 3) then
                B[i,j,1] := evalf((sqrt(2) + 1) * (amp-2) * ( A[i,j,1] +
                                exp(I * phi) * abs(A[i,j,2])) + (3-amp) * A[i,j,1]);
                B[i,j,2] := evalf((3-amp) * A[i,j,2])
            else if (amp > 3) and (amp <= 4) then
                B[i,j,1] := evalf((4-amp) * (A[i,j,1] + exp(I * phi) * abs(A[i,j,2])));
                B[i,j,2] := evalf( (sqrt(2) + 1) * (amp-3) * ( exp(I * phi)
                                * abs(A[i,j,1]) + A[i,j,2]) )
            fi; fi; fi; fi;
            # Normalize the resulting vector
            normalize := evalf( sqrt( abs( B[i,j,1] )^2 + abs( B[i,j,2] )^2 ));
            B[i,j,1] := evalf( B[i,j,1]/normalize );
            B[i,j,2] := evalf( B[i,j,2]/normalize );
            # evaluate amplitude and phase (as a fraction of Pi) for display
            pa[2*i-1, j] := evalf( round( abs(B[i,j,1])*100) );
            pa[2*i, j] := evalf( round( arctan(Im(B[i,j,1]), Re(B[i,j,1]))/Pi*100) );
        od;
    od;
    # Update Universe
    for i from 1 to n do
        for j from 1 to n do
            A[i,j,1] := evalf( B[i,j,1] );
            A[i,j,2] := evalf( B[i,j,2] );
        od;
    od;
od;
# End of new generation loop --- end of main program


# Set up an initial Universe with some example data --- replace by desired structure
inputuniverse := proc(A,n)
    A[2,2,1] := evalf( exp(I * phi) );
    A[2,2,2] := 0;
    A[4,2,1] := evalf( exp(2 * I * phi) );
    A[4,2,2] := 0;
    A[3,2,1] := -1;
    A[3,2,2] := 0;
    A[3,3,1] := 1;
    A[3,3,2] := 0;
```

```
    A[3,4,1] := -1;
    A[3,4,2] := 0;
    A[3,5,1] := 1;
    A[3,5,2] := 0;
    A[3,6,1] := -1;
    A[3,6,2] := 0;
    A[3,7,1] := 1;
    A[3,7,2] := 0;
    A[2,7,1] := evalf( exp(-I * phi) );
    A[2,7,2] := 0;
    A[4,7,1] := evalf( exp(-2 * I * phi) );
    A[4,7,2] := 0;
    RETURN(A)
end;

# Operators on cells
death := proc(B, i::integer, j::integer, phi)
    B[i,j,1] := 0;
    B[i,j,2] := exp(I * phi);
    RETURN(B)
end;

birth := proc(B, i::integer, j::integer, phi)
    B[i,j,1] := exp(I * phi);
    B[i,j,2] := 0;
    RETURN(B)
end;

survival := proc(B, i,j)
    RETURN(B)
end;

normalize := proc(B, i::integer, j::integer)
    local norm;
    norm := sqrt( abs(B[i,j,1])^2 + abs(B[i,j,2])^2 );
    B[i,j,1] := B[i,j,1]/norm;
    B[i,j,2] := B[i,j,2]/norm
    RETURN(B)
end;

# Display the Universe --- simply print the alive parts of the cells
```

```
showuniverse := proc(A,n)
    local i,j;
    for i from 1 to n do
        for j from 1 to n do
            print(A[i,j,1], A[i,j,2]);
        od;
        print();
    od;
end;
```

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Acronyms

BoS Battle of the Sexes (game)

CA Cellular automaton/automata

Ch (game of) Chicken

ESS Evolutionary stable strategy

NE Nash equilibrium/equilibria

PD (game of) Prisoners' Dilemma

PO Pareto optimal

QCA Quantum cellular automaton/automata

Symbols Used

The symbols used in the thesis and their meanings are given in the table below. Of necessity some symbols have multiple meanings dependent upon the chapter. Where necessary, the domain of applicability is noted in the last column. Some trivial one off uses of symbols have been omitted.

Symbol	Meaning	Chapter(s)
a, b	the choice of box for Alice, Bob	3
a, b	coefficients in a superposition;	6, 8
	the coefficients of $\lvert\text{alive}\rangle$ or $\lvert\text{dead}\rangle$, respectively	9
a, b, c	the probabilities of a miss for Alice, Bob, Charles	4
$\bar{a}, \bar{b}, \bar{c}$	the probabilities of a hit for Alice, Bob, Charles	4
a, b, c, d	entries in the payoff matrix for 2×2 games; $a > b > c > d$	5
A	the amplitude of the sum of surrounding cells	9
A, $\hat{A}$	Alice, Alice's move	2–6
B, $\hat{B}$	Bob, Bob's move	2–6
$\hat{A}, \hat{B}$	operators for games A and B	7
$\hat{B}$	birth operator (in semi-quantum Life)	9
c_k	$\cos(\theta_k/2)$, where $k \in \{\text{A,B,C}\}$ designates a player	4
C, $\hat{C}$	cooperation, cooperation operator	2–6
d	dimensionality	9
D, $\hat{D}$	defection, defection operator	2–6
D, $\hat{D}$	decoherence function, decoherence operator	6
$\hat{D}$	death operator (in semi-quantum Life)	9
$\hat{E}$	special move in the quantum Monty Hall problem	3
$\mathcal{E}_j$	measurement operators	6
f	transition function for a cellular automaton	9
$\hat{f}$	special move $= \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$	5
$\hat{F}$	(complex) bit-flip operator $= i\hat{\sigma}_x$	
$\hat{G}, \hat{G}_0, \hat{G}_1, \ldots$	operators for a particular game step or game sequence	7, 9
$\hat{H}$	Hadamard operator	
$\mathcal{H}, \mathcal{H}_\text{C}, \ldots$	Hilbert space, Hilbert space for system C, etc.	
i, j, k, ℓ	indices	
$\mathbb{I}, \hat{I}$	identity, identity operator	

$\hat{J}$	entangling operator	
L	coin state in a random walk indicating motion to the left	7, 8
m	number of rounds in a duel or truel	4
M	number of coins in a multi-coin quantum walk	8
$\hat{M}, \hat{M}_{ij}$	miracle moves in a 2×2 quantum game	2, 5
n	number of games, qubits etc.	
n_j	number of strategies available to the jth player	
N	number of players	2–6
N	neighbourhood size for a cellular automaton	9
$\hat{N}$	no-switch operator	3
o	opened box	3
O	choice of opera in the Battle of the Sexes	5, 6
$\hat{O}$	opening door operator	3
$\hat{O}$	re-ordering operator for coin states	8
$p, p_1, p_2, \ldots$	probabilities	
p, q	projections onto a Hilbert space	3
p, q, r, s	entries in the payoff matrix for 2×2 games	5
P_j	payoff function or payoff matrix for the jth player	
$P(x)$	Probability density	8
$\hat{\mathcal{P}}_j$	projection operator onto state $\|j\rangle$	
$q_1, q_2, \ldots, q_{\mathrm{A}}, q_{\mathrm{B}}, \ldots$	qubits	
Q, $\hat{Q}$	special quantum move in Prisoner's Dilemma	2
Q	set of possible states for a cell in a cellular automaton	9
R	coin state in a random walk indicating motion to the right	7, 8
$\hat{R}, \hat{R}_1, \hat{R}_2$	operator on a qutrit that rotates among three choices	3
$\mathbb{R}$	set of real numbers	
s_k	$\sin(\theta_k/2)$, where $k \in \{\mathrm{A,B,C}\}$ designates a player	4
s_j	strategy chosen by the jth player $\in S_j$	
S_j	strategy set of jth player	
S_{cl}	set of classical strategies	
S_{q}	full set of unitary quantum strategies	
$\hat{S}$	switch door operator	3
$\hat{S}$	particle shift operator for a quantum walk	8
$\hat{S}$	survival operator (in semi-quantum Life)	9
t	time (number of steps or number of games)	
T	the choice of television in the Battle of the Sexes	5, 6
T	transition matrix	8

u_2, u_3	utility of survival in a pair or three-some in a truel	4
$\hat{U}$	unitary operator, generally $\in \mathrm{SU}(2)$	
$\hat{U}_j$	(unitary) move of player j	
$\tilde{U}$	unitary operator in (classical) subset of SU(2)	
$\hat{V}$	special move $= \left(\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix}\right)/\sqrt{2}$	5
$V_A(x), V_B(x)$	potentials	7
x	$\cos^2(\theta_\mathrm{A}/2)\cos^2(\theta_\mathrm{B}/2) + \sin^2(\theta_\mathrm{A}/2)\sin^2(\theta_\mathrm{A}/2)$	6
x	position in a one-dimensional array of lattice sites	8, 9
$y(t)$	vector representing the results at times $t-1$ and $t-2$	8
$\mathbb{Z}$	the set of integers	
α	superposition of the surrounding cells	9
α, β	parameters in potentials	7
$\alpha, \beta, \alpha_j, \beta_j$	phase factors (in an SU(2) operator)	
γ	entangling parameter	2, 4–6
γ	controls the mix of switching or not-switching	3
Γ	set of payoff functions $= \{P_1, \ldots, P_N\}$	
ϵ	a small positive number, $0 < \epsilon \ll 1$	
θ, θ_j	rotation angle (in an SU(2) operator)	
$\pi_{ij}, \boldsymbol{\pi}, \boldsymbol{\pi}_\mathrm{s}$	probability, probability vector, same for stationary state	8
ρ	density matix	6
$\rho, \rho_k, \rho_{ij}, \ldots$	coin probabilities in a history-dependent quantum walk	8
$\sigma_x, \sigma_y, \sigma_z$	Pauli spin matrices	
ϕ	phase factor	
ψ, ξ	quantum system	
Ω	set of strategy sets $= \{S_1, \ldots, S_N\}$	
$\$, \$_\mathrm{A}, \$_\mathrm{B}$	payoff, payoff to Alice, Bob	
$\$_{ij}$	payoff for the game result $\lvert ij\rangle$	
$\$^k_\xi$	payoff to the kth player for game result $\lvert\xi\rangle$	6
$\$_\mathrm{AAB}$ etc.	payoff for the sequence of games AAB etc. in a Parrondo game	7

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Résumé

Adrian Paul Flitney completed a Bachelor of Science with first class honours in theoretical physics at the University of Tasmania in 1983. He was on the Dean of Science roll of excellence in each undergraduate year. He worked in the field of ionospheric physics and high frequency radio communication, for the Department of Science in 1984–5 and for Andrew Antennas Corporation in 1988. During 1987–92 Flitney was a researcher in quantum field theory at the Department of Physics and Mathematical Physics, The University of Adelaide. During this period, and subsequently, he worked as a tutor for the department and privately. In 2001 he began a PhD in the field of quantum game theory at the Department of Electrical and Electronic Engineering, The University of Adelaide under the supervision of Associate Professor Derek Abbott. He has authored ten peer reviewed publications and has presented five conference papers, including the ISDG best student paper at the 9th International Symposium on Dynamic Games and Applications, Adelaide, 2000. He was awarded an Australian Research Council (ARC) Postdoctoral Fellowship to study quantum games and decoherence at the School of Physics, University of Melbourne for the period 2005–7.

Flitney's non-academic interests include chess, where he was actively involved in administration for many years. He continues to be a regular competitor and has won a number of events including four Tasmanian state titles.