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AN ALGEBRAIC FORMULATION OF QUANTUM ELECTRODYNAMICS

by

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SUMMARY

In 1967 Strocchi established that the quantisation of the electromagnetic field using a vector potential is impossible within the context of conventional field theory. Although this result is frequently referred to its significance is largely misunderstood. The fact that the electromagnetic field cannot be described in conventional field theory reflects more upon conventional field theory than theories of the electromagnetic field.

A reappraisal of electromagnetic field theories should therefore be made. It could well be that features of these theories that have been previously regarded as deficiencies are not really deficiencies at all. This thesis is an account of the radiation gauge, Gupta-Bleuler and Fermi methods of quantising the electromagnetic field from that point of view.

The radiation gauge and Gupta-Bleuler methods are well established schemes. Our discussion does not yield any results concerning these methods that cannot be found elsewhere. It does, however, serve to place them in a wider context. The Fermi method is little understood and hence most of this work is concerned with it.

Even though the various formulations of field theory are by no means equivalent, they all eventually reproduce traditional field theory. Thus if we only require that the theory be rigorously formulated for such examples as the neutral scalar field it does not matter which formulation we choose.

The differences are, however, important for applications to the quantisation methods of the electromagnetic field. The formulations have to be modified and the point at which such modifications must be made and their nature depends on both the general formulation and the quantisation method.

The formalism that provides the most suitable framework for a rigorous formulation of the Fermi method turns out to be the C* algebra formulation of Segal. Following Segal, the Weyl algebra of the vector potential is constructed. The Fermi method is then related to a certain representation of the algebra. The representation is specified by a generating functional for a state on the algebra.

Usually, dynamical and kinematical transformations are represented by unitarily implementable automorphisms of the algebra. We prove that this is not always true in the representation given by the Fermi method. The Weyl algebra of the physical field is then constructed as a factor algebra. Difficulties with both the Fermi and Gupta-Bleuler methods can be attributed to the need to use a factor algebra.

The canonical commutation relations $[x_\mu, p_\nu] = - i\, g_{\mu\nu}$ are formulated as a Weyl algebra. We study the Schrödinger representation of the algebra and find that the Fermi method is just the generalisation of this representation to an infinite number of degrees of freedom. Further analogies are also possible. We can construct factor algebras from the Weyl algebra. The mechanics of such procedures can be studied without the additional complications of an infinite number of degrees of freedom. The Schrödinger representation of the Fermi method is then constructed.

We conclude with a discussion of the results that have been obtained and an indication of ways in which the work might be extended.

STATEMENT

This thesis contains no material which has been accepted for the award of any other degree, and to the best of my knowledge and belief, contains no material previously published or written by another person except where due reference is made in the text.

Janice Margaret Gaffney

CONVENTIONS

We adopt the diagonal metric tensor $g_{\lambda\mu}$ ($\lambda,\mu,$ = 0,1,2,3) with $g_{00} = - g_{11} = - g_{22} = - g_{33} = 1$.

CHAPTER 1

DISCUSSION OF THE RADIATION GAUGE, GUPTA-BLEULER AND FERMI QUANTISATION METHODS

1.1 An Historical Perspective

The original formulation of the quantum theory of fields was developed by Dirac, Jordan, Heisenberg and Pauli in the years 1927-1929. The scheme for quantising the electromagnetic field that has come to be known as the radiation gauge method was first used for the quantum theory of the pure radiation field by Dirac[(1)] and Jordan and Pauli[(2)]. Heisenberg and Pauli[(3)] presented the first general account of field theory. As an application of the general principles that they had formulated, they set up the theory of quantum electrodynamics in radiation gauge.

The prescription for quantising a field in the Heisenberg and Pauli approach was to introduce commutation relations into classical field theory by identifying the canonically conjugate coordinates to which commutation rules could be applied. These were determined from a Lagrangian which was formally taken over from the classical theory. From classical electrodynamics the Lagrangian is

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad \text{where} \quad F_{\mu\nu} = \frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu} \; . \qquad (1.1)$$

With that Lagrangian, they found that the momentum conjugate to the zero component of the field $A_o(x)$ was identically zero. Hence, $A_o(x)$ had to be eliminated before the quantisation rules could be applied. The longitudinal and scalar components were therefore removed by a gauge transformation leaving the transverse components to be then quantised.

Fermi[4] suggested a modification to the Lagrangian to enable all four components of the field to be quantised:

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - \tfrac{1}{2} \left(\frac{\partial A^\mu}{\partial x^\mu}\right)^2 \quad . \tag{1.2}$$

The physical states of the theory were then chosen to be those which satisfied the supplementary condition

$$\frac{\partial A^\mu}{\partial x^\mu} |\psi\rangle = 0 \ . \tag{1.3}$$

Both these schemes were considered to be unsatisfactory for the following reasons. In radiation gauge, the procedure of only quantising the transverse parts of the potential destroyed the Lorentz four vector structure of the theory although the theory itself was relativistically invariant. Also, for computational purposes, for example radiation phenomena, it would have been simpler to have been able to treat all four components symmetrically rather than dividing the field into a transverse part and Coulomb interaction.

The undesirable features of the Fermi method that were immediately apparent were,

(1) The Hamiltonian was not positive definite.

(2) The field equations were different from Maxwell's equations.

However, choosing the physical states of the theory to be the solutions of (1.3) seemed to remove these difficulties because it was found that the Hamiltonian became positive definite and that the field equations were reduced to Maxwell's equations when restricted to those states. The physical states, therefore, played a central role in the Fermi method. Hence, the method fell into disrepute when those states became mathematically questionable.

It was discovered, on explicitly introducing a Hilbert space structure to the theory, that the Fermi physical states, being non normalisable, did not lie in the Hilbert space(5).

Gupta and Bleuler(6) suggested a quantisation scheme in which all four components of the vector potential could be quantised. Dirac had observed(7) that for the Fermi method the commutation relations were satisfied by reversing the roles of the emission and absorption operators for the scalar photons. Gupta realized that it was possible to maintain the usual roles for the absorption and emission operators for the scalar photons and satisfy the commutation relations providing that an indefinite metric was used in the state vector space.

He took the physical states of the theory to be those which satisfied the supplementary condition

$$\frac{\partial A^{\mu(+)}}{\partial x^{\mu}} |\psi\rangle = 0 \ . \qquad (1.4)$$

He then showed how a probabilistic interpretation for the physical photons might apply to those physical states even though the original state vector space was not a Hilbert space.

1.2 The Present Evaluation of the Theories

The situation today is that while both the radiation gauge and Gupta-Bleuler methods have become standard procedures for quantising the electromagnetic field, (see for instance Bjorken and Drell(8) and Schweber(9)), the Fermi method is more frequently relegated to an historical foot-note.

Strocchi(10) proved a result which suggests that it is impossible to construct a theory of photons with a four-vector potential within the context of conventional field theory. His result has been widely misunderstood. It only proves that if a Lorentz covariant four vector potential is to be used then the axioms of conventional field theory must necessarily be violated. Strocchi was hence incorrect in claiming that his result showed that the Fermi method had to be inconsistent. He would first have to show that the Fermi method was a quantisation procedure that satisfied the axioms of conventional field theory. Since the Fermi physical states are known to be non normalisable it would seem most likely that the Fermi method would not satisfy the axioms of conventional field theory.

For both the radiation gauge and Gupta-Bleuler methods, however, the developments in the formalism that have occurred have gone a long way in resolving the objections that were originally raised against them. In particular we shall refer to specific developments that have occurred in Fock-space techniques, axiomatic field theory, and the representation theory of the Poincaré group and wave equations.

The Fock-Cook, occupation space representation of field theory was introduced by Fock[11] in 1932. This representation then became the standard representation of field theory. Its mathematical structure was settled by Cook[12] in 1953. Investigations into the foundations of field theory is the concern of axiomatic field theory. The relationship between axiomatic field theory and Fock-space that we shall be concerned with is that the axiomatic schemes that have been proposed for the free field have a Fock-Cook representation. Thus, when we construct a quantum field theory explicitly it is natural to construct many particle states from one particle states and then "second quantise" i.e. construct the Fock-Cook representation.

The Hilbert space of states in the Fock-Cook representation is thus a direct sum of subspaces of states with exactly n particles

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}^{(n)} .$$

Furthermore, by constructing a field theory by this method, we can readily draw upon the great body of work that has been done on the representation theory of the Poincaré group. The one particle states of the Fock-Cook representation are connected with group theory in the following way. For a field of free particles of a given kind they are a representation space for the irreducible unitary representation of the Poincaré group that is associated with the particles under consideration.

The study of these representations is a field in itself. It would be impossible here to survey adequately the many references. Wigner(13) is a general account.

The irreducible representations of the Poincaré group that are relevant for this discussion are therefore those which describe the physical photons. These are the mass zero, ±1 helicity representations. We also need to know the connection between these representations and wave equations because the one particle space will be realized as a Hilbert space of wave functions.

If we want to construct a field theory with a vector potential then one solution is to take one particle wave functions, $\mathbf{A}$, with components A_μ which satisfy the wave equations:

$$\Box A_\mu(x,t) = 0 \quad \mu = 0,1,2,3 \tag{1.5}$$

where

$$\Box = \frac{\partial^2}{\partial t^2} - \underline{\nabla}^2$$

is the D'Alembertian operator.

The representation of the Poincaré group on these wave functions will be given by

$$\boldsymbol{A} \rightarrow \boldsymbol{A}' \quad A_{\mu}'(x) = \Lambda_{\mu}{}^{\nu} A_{\nu}(\Lambda^{-1}(x - a)) \tag{1.6}$$

for the Poincaré transformation with parameters (a,Λ). To establish contact with conventional field theory we then have to construct from this representation the ±1 helicity representations of the Poincaré group.

We can classify the representation of the Poincaré group that (1.6) gives by looking at the representation of the little group $T^4 \circledS E(2)$. That is, we consider that the representation is induced from the representation of $T^4 \circledS E(2)$. The theory of induced representations has been a powerful tool for classifying all the irreducible unitary representations of the Poincaré group. It is implicit in Wigner's original work[14], and the general theory was established by Mackey[15].

In this application, the problem of classifying the representation of the Poincaré group that (1.6) defines, is therefore reduced to determining the representation of an E(2) subgroup of the defining representation of SL(2,C). It turns out that this representation of E(2) is indecomposable i.e. there is an invariant subspace, but the representation is not fully reducible[16]. By contrast, consider massive wave functions. In that case, the little group is $T^4 \circledS SU(2)$. We therefore consider an SU(2) subgroup of SL(2,C). This representation of SU(2) is reducible. It is the direct sum of irreducible unitary representations. Hence we can project onto the irreducible representation of the Poincaré group which is of interest by picking out the appropriate SU(2) representation through the wave equations.

Since E(2) is non compact its finite dimensional irreducible unitary representations will be one dimensional. Thus to construct irreducible unitary representations of the Poincaré group we must develop a procedure for extracting the appropriate irreducible unitary representation of E(2) from the given indecomposable representation.

McKerrell[(17)] gives an account of the construction of a class of covariant massless field theories. The approach that he used was equivalent to developing a method for constructing irreducible unitary representations of the Poincaré group from the representations defined on tensor wave functions. Therefore the techniques that he used can be placed in a more general context from an appreciation of their group theoretic origins. A reformulation of his results in the language of induced representations and indecomposable representations of E(2) has been achieved to a certain extent as people have come to recognize the power of these methods[(18)].

For the particular representation that we are concerned with, the four vector wave function, Shaw[(19)] originally showed how to construct the ±1 helicity representations with an approach which also can be given a group theoretic interpretation.

These considerations have implications for a field theory constructed by Fock-Cook methods. If we start with four vector wave functions for our one particle states then the Gupta-Bleuler method necessarily follows. The indefinite metric, physical states satisfying the Lorentz condition and restricted gauge transformations are therefore necessary from the point of view of representation theory to single out the ±1 helicity representations.

The Gupta-Bleuler method is certainly a description of photons since we finally manage to extract the appropriate representation of the Poincaré group. It is mathematically well defined because once the one particle theory is correct Cook's analysis applies.

A similar analysis is also appropriate for the radiation gauge method. The proofs of the relativistic invariance of the theory are equivalent to establishing that the representation of the Poincaré group that the one particle states carry are the ±1 helicity representations. The Fock-Cook construction therefore can be used.

We have already pointed out that only the transverse components of the field are quantised in the radiation gauge method. This means that the transformation law is not that of a four vector but contains an additional gauge term. Explicitly, for the Poincaré transformation with parameters (a,Λ) the transformation law is

$$\mathbf{A} \to \mathbf{A}'$$

$$A_0' = 0 \qquad \underline{A}' = \underline{A}'' - \frac{1}{\nabla^2} \nabla\, \underline{\nabla} \cdot \underline{A} \tag{1.7}$$

where

$$\frac{1}{\nabla^2} f(x) = -\frac{1}{4\pi} \int dx' \frac{1}{|x - x'|} f(x')$$

and

$$A_\mu''(x) = \Lambda_\mu^{\ \nu} A_\nu (\Lambda^{-1}(x - a)) \ .$$

Thus the theory is not manifestly covariant.

The relativistic invariance of the radiation gauge method means that the mappings given by (1.7) are implemented by operators that are a continuous unitary representation $(a,\Lambda) \to U(a,\Lambda)$ of the Poincaré group. The mapping in (1.7) $\mathbf{A} \to \mathbf{A}'$ can therefore be written as

$$A_\mu(x) \to U(a,\Lambda)\, A_\mu(x)\, U(a,\Lambda)^{-1} \ . \tag{1.8}$$

The fact that the radiation gauge method as a description of photons is just as effective as any other field theory can be taken as evidence that Lorentz covariance as an invariance principle is not as fundamental as relativistic invariance.

The absence of Lorentz covariance does lead to differences in description. In accounts of the radiation gauge method it is sometimes stated that there exists a set of preferred Lorentz frames in which the time component of each polarization vector vanishes. This description is misleading. The time component is actually zero in every Lorentz frame. It is just that since we do not have manifest Lorentz covariance the polarization vectors are not Lorentz four vectors. Statements that require a Lorentz four vector structure cannot be made within a theory that never claims to have that structure in the first place.

The reason Lorentz covariance is demanded is that in practice it often provides useful prescriptions. For example, it is hard to see how interaction could, in general, be introduced without it. Nevertheless, it should be emphasized that the fundamental invariance principle is relativistic invariance. Thus if a theory is constructed without Lorentz covariance but is in every other respect satisfactory then it should not be regarded as deficient. The impression that the radiation gauge method is deficient could easily be gained from reading accounts of it in the literature. For example, manifest covariance is considered to be "abandoned" rather than simply absent.

In fact, it is possible to make Lorentz covariant statements about radiation gauge providing that the class of representations of the Lorentz group which we are willing to consider is broadened to include infinite dimensional representations.

Bender showed[20] that the spatial components A_i can be considered to be components of an infinite component field which transforms as the $(1,1) \oplus (1,-1)$ non unitary representation of the Lorentz group. The notation that we are using to characterize the representations is taken from Gel'fand, Minlos and Shapiro[21]. This infinite component field has only two degrees of freedom and hence its transformation properties are completely specified by (1.8).

An alternative method for arriving at the Lorentz covariance of the potential in radiation gauge has recently been drawn to our attention[22].

The potential in radiation gauge is completely specified. Hence we can solve the equations

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \ . \qquad (1.9)$$

We obtain,

$$A^i = \left(\frac{\partial}{\partial t}\right)^{-1} F^{oi} \ . \qquad (1.10)$$

The representation of the Lorentz group to which A_i belongs can therefore be regarded as the direct product of two representations. The operator $\left(\frac{\partial}{\partial t}\right)^{-1}$, suitably interpreted, is an "expansor". The transformation properties of expansors were first studied by Dirac[23].

Thus from Lorentz covariance considerations it would seem to be more natural to regard the vector components A_i as arising from the electromagnetic field tensor by (1.10) rather than a Lorentz four-vector.

To summarize, therefore, we have indicated the developments that have helped to explain the nature of some of the fundamental difficulties in both the radiation gauge and Gupta-Bleuler methods.

We have seen that these difficulties can be related to Poincaré invariance considerations.

It should be emphasized that by Strocchi's result these methods are necessarily inconsistent with the axioms of conventional field theory. His result should be regarded as fundamental in explaining why difficulties in each quantisation scheme must occur; it cannot however characterize the nature of the difficulties. Thus since all methods will violate the axioms of field theory the choice of one over the other might be governed by some particular advantage to be gained in solving the problem at hand.

Strocchi's result makes the systematic discussion of the electromagnetic field a difficult proposition. It would be an enormous task to discuss first the implications that would result from changing the axiomatic structure of conventional field theory in the fundamental way that seems necessary to cope with the quantisation of the electromagnetic field. Thus the discussion proceeds in the opposite direction. It begins with the quantisation schemes themselves. We have seen how the most fundamental difficulties with both the Gupta-Bleuler and radiation gauge methods have been resolved. We shall attempt to do the same for the Fermi method.

CHAPTER TWO

A WEYL ALGEBRA FOR THE ELECTROMAGNETIC FIELD

The basic mathematical object in an algebraic formulation of field theory is often taken to be an abstract C* algebra. Segal's suggestion for this algebra[24], the Weyl algebra, is only one of a number that have been made[25].

The Hilbert space of traditional field theory, together with the operators that act on it, come out of the algebraic version in the following way. They are the representation space and representors in a certain representation of the basic C* algebra of the algebraic formulation. Thus traditional field theory is obtained from algebraic field theory simply by identifying the appropriate representation of the C* algebra.

The specification of representations of C* algebras is hence highly relevant to the discussion. In this connection we mention the GNS construction[26,27]. By this construction a cyclic representation of the C* algebra can be associated to each positive linear functional on the algebra.

The last remark of a general nature that will be made concerns quantum phenomenology and C* algebras. In the C* algebra approach, the states of the system are regular states (a state being a positive linear functional which maps the identity to 1) on the algebra and the kinematical and dynamical transformations are * automorphisms of the algebra. These automorphisms play the role of the unitary transformations in the Hilbert space approach in that they preserve spectral values and expectation values in states[27].

2.1 The Weyl Algebra Formalism

We shall recall first of all some of the details of the usual formulations of field theory which will enable us to appreciate the significance of the entities which appear in Segal's definition of the Weyl algebra.

Consider the neutral scalar field. In heuristic formulations the field operator, $\mathcal{A}(x)$, is a solution of the equation

$$(\Box - m^2)\mathcal{A}(x) = 0 \; . \tag{2.1}$$

The canonical commutation relations (CCRs), first given by Jordan and Pauli(2), are

$$[\mathcal{A}(x), \mathcal{A}(x')] = i\Delta(x - x') \; . \tag{2.2}$$

In axiomatic field theory, the field operator is found to be meaningful only when smeared. Formally we have,

$$A(f) = \int d^4x \; f(x)\mathcal{A}(x) \; . \tag{2.3}$$

The commutation relations are

$$[A(f), A(g)] = i\sigma(f,g) \; , \tag{2.4}$$

$$\sigma(f,g) = \int d^4x \, d^4x' \; \Delta(x - x') \; f(x) \; g(x') \; . \tag{2.5}$$

f is a test function belonging to $\mathcal{S}$, the set of real-valued, infinitely differentiable functions of fast decrease, defined over space-time. Furthermore, the singular function $\Delta(x - x')$ can be rigorously defined(28) as a distribution on $\mathcal{S}$. We shall specify $\Delta(x - x')$ as the solution of the differential equation,

$$(\Box - m^2)\,\Delta(x - x') = 0 \tag{2.6}$$

which has the Cauchy data at time $t = 0$,

$$\Delta(\underline{x},0) = 0 \; , \tag{2.7}$$

$$\left.\frac{\partial\Delta(\underline{x},t)}{\partial t}\right|_{t=0} = -\,\delta(\underline{x}) \; . \tag{2.8}$$

We now put the CCRs into the Weyl form[29]. We define the Weyl operator,

$$U(f) = e^{iA(f)} . \tag{2.9}$$

The Weyl relation for these operators will therefore be

$$U(f)\,U(g) = e^{-i/2\,\sigma(f,g)}\,U(f+g) . \tag{2.10}$$

Let

$$\phi = \Delta^* f \tag{2.11}$$

where

$$\Delta^* f = \int d^4x\, \Delta(x - x')\, f(x') \tag{2.12}$$

i.e. $\Delta^* f$ is a convolution.

Then since f belongs to $\mathcal{S}$, ϕ will also belong to $\mathcal{S}$ and will be a solution of the wave equation,

$$(\Box - m^2)\phi = 0 . \tag{2.13}$$

Δ is thus a projection operator to the subspace $\mathcal{N}$ of $\mathcal{S}$, of real solutions of (2.13).

Also

$$\sigma(\Delta^* f, \Delta^* g) = \sigma(f,g) . \tag{2.14}$$

In axiomatic field theory, test functions that have the same projection by Δ^* are mapped onto the same operators[30]. We can therefore consider A(f) as the operator value of a mapping from an equivalence class of test functions. The equivalence relation is given by

$$f \equiv g \quad \text{if} \quad \Delta^* f = \Delta^* g . \tag{2.15}$$

Furthermore, we can define a linear 1-1 map from these equivalence classes to $\mathcal{N}$. We therefore can consider $A(f) \equiv A(\phi)$ to be operator-valued distributions over $\mathcal{N}$.

Also, the restriction of the bilinear form σ to $\mathcal{N}$ is non-degenerate, i.e. $(f,g) = 0$ for all $g \in \mathcal{N}$ implies $f = 0$.

We shall now give the basic formalism of the Weyl algebra formulation of field theory. We begin with a generalisation of the definition of the Weyl representation of the CCRs.

Definition 2.1 (Weyl System over (M,B))[31]

Suppose that M is a real, linear, topological vector space and $B(z,z')$ a symplectic form (i.e. B is non degenerate, anti-symmetric, real and bilinear) on M continuous in the given topology. Then we have a Weyl system over (M,B) if to each z,z' belonging to M there corresponds $W(z)$ and $W(z')$ acting on a Hilbert space K and satisfying

$$W(z)\, W(z') = e^{-1/2\, B(z,z')}\, W(z + z') \,. \tag{2.16}$$

Furthermore, we require $W(tz)$ to be a continuous function of t, for any fixed z.

This generalisation was clearly designed to meet the needs of field theory. In that application M is a suitable class of wave functions, B is the commutation relation and W are the Weyl operators.

For finite dimensional M we know that Weyl systems exist because the Schrödinger representation of the CCRs is a Weyl system.

Definition 2.2 (Schrödinger Representation as a Weyl System)

Let the vectors e_i, f_i $(i = 1,\ldots,n)$ be a symplectic basis for the finite dimensional (2n) vector space F and symplectic form σ defined over F. We therefore have that,

$$\sigma(e_i,e_j) = \sigma(f_i,f_j) = 0 \,, \tag{2.17}$$

$$\sigma(e_i,f_j) = -\,\sigma(f_j,e_i) = \delta_{ij} \,. \tag{2.18}$$

Suppose that the coordinates of $z \in F$ with respect to the symplectic basis are (a^i, b^i), i.e.

$$z = \Sigma_i \; a^i e_i + b^i f_i \; . \tag{2.19}$$

Consider the Hilbert space of complex-valued, Lebesgue square integrable functions over R_n. Then the Schrödinger representation will be defined by

$$U(\underline{a}) \; : \; f(\underline{x}) \to f(\underline{x} + \underline{a})$$

$$V(\underline{b}) \; : \; f(\underline{x}) \to e^{i(\underline{b},\underline{x})} \; f(\underline{x})$$

$$W(z) = U(\underline{a}) \; V(\underline{b}) \; e^{-i/2(\underline{a},\underline{b})} \; . \tag{2.20}$$

For finite dimensional M, the Weyl system is unique in the sense of the von Neumann theorem[32].

Theorem 2.3 (Uniqueness of Weyl Systems)

If F is finite dimensional then any Weyl system over (F,B) is unitarily equivalent to a direct sum of copies of the Schrödinger representation over (F,B).

Definition 2.4 (The von Neumann Ring $\mathcal{R}$(F,B))

We define the ring of operators $\mathcal{R}$(F,B) to be the set of all finite linear combinations of W(z), belonging to a Weyl system over (F,B), and their limits in the weak topology.

This definition is independent of the Weyl system chosen because the previous theorem implies that there is a 1-1 correspondence between the weak closures in any two different representations.

Definition 2.5 (C* Algebra)

Let $\mathcal{U}$ be a complex (or real), associative, involutive normed algebra which is complete in the topology induced by its norm. If for all elements $u \in \mathcal{U}$, $\|u\|^2 = \|u^*u\|$, where $\|u\|$ denotes the norm, then $\mathcal{U}$ is a C* algebra.

C* algebras were first characterized by Gel'fand and Naimark[26].

<u>Definition 2.6</u> (Weyl Algebra over (M,B))[24]

Suppose that for (M,B) there exists a system S, of finite-dimensional subspaces such that S is partially ordered, absorbing (i.e. $M = \cup_{F\in S} F$) and the restriction of the symplectic form B to any element of S is non degenerate. Then the Weyl algebra over (M,B) is the C* algebra obtained as the norm completion of the inductive limit[33] of all von Neumann rings $\mathcal{R}(F,B)$ where F is an element of S. We shall denote the Weyl algebra over (M,B) by $\mathcal{W}(M,B)$.

By the GNS construction we can associate a cyclic representation to each state on the algebra. The representation obtained, however, is not necessarily a Weyl system, since $W(\lambda z)$ need not be a continuous function of λ. The following theorem classifies those states which give rise to Weyl systems.

<u>Theorem 2.7</u> (Generating Functional for Regular States)

Suppose that ρ is a complex-valued function on M such that

(i) $\rho(0) = 1$, $\rho(-z) = \overline{\rho(z)}$

(ii) $\rho(z)$ is continuous when restricted to arbitrary finite dimensional subspaces of M

(iii) for arbitrary z_i in M and complex numbers α_i

$$\sum_{i,j\in F} \rho(z_i - z_j)\, \alpha_i\, \bar{\alpha}_j \geq 0$$

where F is a finite index set,

then $\rho(z)$ uniquely determines a state, $E(W(z))$, on $\mathcal{W}(M,B)$. This state, by definition, is regular. Regular states give rise to representations of $\mathcal{W}(M,B)$ such that $W(\lambda z)$ are continuous functions of λ for real λ and fixed z, and conversely. Also in the representation E is a vector state, i.e. $E(W(z)) = (W(z)v,v)$, for some v.

Thus in explicit representations of quantum fields the generating functional is the physical vacuum to physical vacuum expectation value of the operator W(z). We recall that M is the manifold of real, infinitely differentiable solutions of (2.13) for the neutral scalar field. For the Fock-space representation of that field by explicit calculation[34],

$$\rho(\phi) = e^{-\|\phi^+\|/4} \tag{2.21}$$

where $\|\phi^+\|$ is the norm of the positive energy solution ϕ^+ which is associated to each real solution ϕ in the following way. The three-dimensional Fourier transformation, $\mathcal{F}$, is defined by

$$f(\underline{x}) = \left(\frac{1}{2\pi}\right)^{3/2} \int d^3k\ \hat{f}(\underline{k})\ e^{i\ \underline{k}\cdot\underline{x}}\ . \tag{2.22}$$

If $\phi(\underline{x},t)$ is in M then since it is a solution of (2.13) it can be written as

$$\phi(\underline{x},t) = \left(\frac{1}{2\pi}\right)^{3/2} \int \frac{d^3k}{2\sqrt{k}} \left(\phi^+(\underline{k})\ e^{-i(kt-\underline{k}\cdot\underline{x})} + \overline{\phi^+(\underline{k})}\ e^{i(kt-\underline{k}\cdot\underline{x})}\right) \tag{2.23}$$

where $k = \sqrt{\underline{k}^2 + m^2}$.

(2.23) then defines the relationship between real solutions and positive energy solutions. Explicitly, at time t_o,

$$\phi(\underline{x},t_o) \longleftrightarrow \mathcal{F}^{-1}\ (\phi^+(\underline{k})\ e^{-ikt_o})\ . \tag{2.24}$$

In the Segal formalism this connection is taken to be of fundamental significance because it can be exploited to impose a Hilbert space structure $\mathcal{H}^+$ on M which seems relevant and entirely natural.

Let $\phi(\underline{x},t)$ be in M. We define a representation of the Poincaré group on M in the usual fashion

$$\phi \to \phi' : \phi'(x) = \phi(\Lambda^{-1}(x - a)) \tag{2.25}$$

for the transformation with parameters (a,Λ). Then this representation can be thought of as inducing the representation on the positive energy solutions through (2.23). On the positive energy solutions the representation is unitary with respect to the scalar product

$$(\phi^+,\psi^+) = \int d^3k\ \overline{\phi^+(\underline{k})}\ \psi^+(\underline{k}) \ . \tag{2.26}$$

Thus the functions ϕ^+ obtained from elements of M by (2.24) form a pre-Hilbert space with respect to the scalar product given by (2.26).

A convenient way of parametrising M is to specify each solution by its Cauchy data at some time which we will take for convenience to be 0. Suppose that

$$\phi(\underline{x},0) = \phi(\underline{x}) \quad \text{and} \quad \dot{\phi}(\underline{x},0) = \dot{\phi}(\underline{x}) \ . \tag{2.27}$$

Then at time $t_o = 0$ we can write the correspondence given by (2.24) as

$$\phi(\underline{x},t) = (\phi(\underline{x}),\dot{\phi}(\underline{x})) \longleftrightarrow (-\nabla^2)^{\frac{1}{4}}\ \phi(\underline{x}) + i(-\nabla^2)^{-\frac{1}{4}}\ \dot{\phi}(\underline{x}) = \Phi(\underline{x}) \ . \tag{2.28}$$

The operators $(-\nabla^2)^{\pm\frac{1}{4}}$ are defined as the non local operators which multiply the Fourier transforms by $k^{\pm\frac{1}{2}}$. For a discussion of their properties see Kato[35].

The scalar product induced by (2.26) is

$$\begin{aligned}(\phi^+,\psi^+) &= (\Phi,\Psi) \\ &= \int d^3x\ \overline{\Phi(\underline{x})}\ \Psi(\underline{x}) \\ &= \int d^3x\ [\ (-\nabla^2)^{\frac{1}{4}}\ \phi(\underline{x})(-\nabla^2)^{\frac{1}{4}}\ \psi(\underline{x}) \\ &\quad + (-\nabla^2)^{-\frac{1}{4}}\ \dot{\phi}(\underline{x})(-\nabla^2)^{-\frac{1}{4}}\ \dot{\psi}(\underline{x})] \\ &\quad + i[\ \phi(\underline{x})\ \dot{\psi}(\underline{x}) - \dot{\phi}(\underline{x})\ \psi(\underline{x})] \ .\end{aligned} \tag{2.29}$$

The symplectic form σ on M given by (2.4) and (2.5) for the solutions $\phi(\underline{x},t)$ and $\psi(\underline{x},t)$ in terms of their Cauchy data at $t = 0$ is

$$\sigma(\phi(\underline{x},t),\psi(\underline{x},t)) = B((\phi(\underline{x}),\dot{\phi}(\underline{x})), (\psi(\underline{x}),\dot{\psi}(\underline{x})))$$
$$= \int d^3x\ (\phi(\underline{x})\ \dot{\psi}(x) - \dot{\phi}(\underline{x})\ \psi(\underline{x}))\ . \qquad (2.30)$$

Thus to each $(\phi(\underline{x}),\dot{\phi}(\underline{x}))$ in M there corresponds an element $\Phi(\underline{x})$ in $\mathcal{H}^+$ such that the Poincaré transformations defined on M and $\mathcal{H}^+$ by (2.23) and (2.25) are real, symplectic (i.e. leave B invariant) transformations on (M,B) and at the same time unitary transformations on $\mathcal{H}^+$. Note that the symplectic form B is the imaginary part of the scalar product in $\mathcal{H}^+$. The scalar product induces the invariant symmetric form S on M,

$$S((\phi(\underline{x}),\ \dot{\phi}(\underline{x})),\ (\psi(\underline{x}),\ \dot{\psi}(\underline{x}))) =$$
$$\int d^3x\ ((-\ \nabla^2)^{\frac{1}{4}}\ \phi(\underline{x})(-\ \nabla^2)^{\frac{1}{4}}\ \psi(\underline{x})$$
$$+\ (-\ \nabla^2)^{-\frac{1}{4}}\ \dot{\phi}(\underline{x})(-\ \nabla^2)^{-\frac{1}{4}}\ \dot{\psi}(\underline{x})))\ . \qquad (2.31)$$

Thus we can write the generating functional for the free representation as

$$\rho((\phi(\underline{x}),\dot{\phi}(\underline{x})) = e^{-\frac{1}{4}S((\phi(\underline{x}),\dot{\phi}(\underline{x})),(\phi(\underline{x}),\dot{\phi}(\underline{x})))}\ . \qquad (2.32)$$

There exists another invariant Hilbert space structure $\mathcal{H}^-$ for M obtained by putting real solutions and negative energy solutions into correspondence. We then have that

$$\phi(\underline{x},t) = \left(\frac{1}{2\pi}\right)^{3/2} \int \frac{d^3k}{2\sqrt{k}}\ \overline{\phi^-(\underline{k})}\ e^{-i(kt-\underline{k}\cdot\underline{x})} + \phi^-(\underline{k})\ e^{i(kt-\underline{k}\cdot\underline{x})})\ .$$
$$(2.33)$$

The effect therefore is the same as complex conjugation of the vectors in $\mathcal{H}^+$. Hence the scalar product in $\mathcal{H}^-$ will be the complex conjugate of the scalar product in $\mathcal{H}^+$. Therefore B will be the negative of the imaginary part of the scalar product in $\mathcal{H}^-$. This difference in the relationship between M and $\mathcal{H}^+$ and M and $\mathcal{H}^-$ turns out to be crucial in the application of the formalism to the electromagnetic field.

We obtained the generating functional for the Fock-space representation of the scalar field by calculating it. We would hope, however, that the generating functional should come out of the formalism. The first guess at a possible characterisation for the generating functional turned out to be incorrect.

In the Weyl algebra approach, the automorphisms of the algebra representing physical transformations are induced by symplectic transformations in M. On the generating functional the mapping is given by

$$\rho'(z) = \rho(Tz) \ . \tag{2.34}$$

For the scalar field, symplectic transformations on M, unitary transformations on $\mathcal{H}^+$, the invariant scalar product in $\mathcal{H}^+$ and the invariant generating functional are inter-related and hence it was appealing to think that the requirement of invariance of a state on the algebra under the group of all unitary transformations would be sufficient to characterise uniquely the generating functional for the free representation. It turns out, however, that continuously many such states exist[36]. The characterisation of the generating functional that is used in the formalism arises from the following theorem.

Theorem 2.8[37]

Suppose that we have a Weyl system over (M,B). M is also a Hilbert space H with B the imaginary part of the scalar product. Let $U(\alpha)$, for all real α, be a continuous one parameter unitary group on H with positive generator. Then there is only one state E on the Weyl algebra for which $U(\alpha)$ induces a unitary group with positive generator in the representation space associated with E. This state is characterised by the generating functional

$$\rho(z) = e^{-\frac{1}{4}\|z\|^2} .$$

Hence the Fock-space representation is obtained by requiring that some one parameter unitary group has a positive generator. The obvious group to take is the group representing time translations. That group is generated by the energy operator and hence the Fock-space representation can be singled out as the representation for which a positive energy operator is defined. Alternatively, we could require that the number operator be positive.

One final result that we need is the criterion given by Shale[36] for determining whether a class of automorphisms of the Weyl algebra will be implemented by unitary transformations in the Fock-space representation. In general, relatively few of the automorphisms of the Weyl algebra are implemented by unitary transformations in a given representation of the algebra.

Theorem 2.9

Let (M,B) and H be given as in theorem 2.3. Then in the Fock-space representation of $\mathcal{W}(M,B)$, the automorphism of the algebra induced by

$$W'(z) = W(Tz)\, e^{if(z)}$$

is unitarily implementable if and only if

(1) f(z) is a real-valued continuous real-linear functional on H.

(2) T is a symplectic transformation on H (i.e. leaves B invariant) such that $T^*T - I$ = Hilbert-Schmidt operator.

2.2 The Weyl Algebra of the Vector Potential

We now put the usual formalism for the quantisation of the electromagnetic field with a vector potential into the language of Weyl algebras.

We therefore write the commutation relations in the Weyl form as in (2.16). In heuristic formulations the CCRs for the field operators are

$$[\mathcal{A}_\mu(x), \mathcal{A}_\nu(x')] = -i\, g_{\mu\nu}\, D(x - x') \tag{2.35}$$

where $D(x - x')$ is equal to $\Delta(x - x')$ in (2.6) with $m = 0$. Let $\mathbf{F}$ with components f_μ ($\mu = 0,\ldots,3$) be a real test function in $\sum_{i=0}^{3} \oplus \mathcal{J}^{(i)}$. Then the CCRs for the smeared fields $A(\mathbf{F})$ (formally $\int d^4x\, \mathcal{A}^\mu(x) f_\mu(x)$) will be

$$[A(\mathbf{F}), A(\mathbf{F}')] = i\sigma(\mathbf{F},\mathbf{F}') \tag{2.36}$$

$$\sigma(\mathbf{F},\mathbf{F}') = -\int d^4x\, d^4x'\, \Delta(x - x') f^\mu(x) f'_\mu(x') . \tag{2.37}$$

Let

$$U(\mathbf{F}) = e^{iA(\mathbf{F})} \tag{2.38}$$

then the CCRs in Weyl form are

$$U(\mathbf{F})\, U(\mathbf{F}') = e^{-i/2\, \sigma(\mathbf{F},\mathbf{F}')}\, U(\mathbf{F} + \mathbf{F}') . \tag{2.39}$$

The operators $U(\mathbf{F})$, just as for the scalar field, are really mappings from equivalence classes of functions. We therefore identify these equivalence classes with real solutions of the wave equations

$$\Box\phi_\mu = 0 \qquad (\mu = 0,\ldots,3) \tag{2.40}$$

by

$$\mathbf{F} \to \boldsymbol{\phi} \;:\; f_\mu \to \Delta^* f_\mu = \phi_\mu . \tag{2.41}$$

We define

$$W(\boldsymbol{\phi}) = U(\mathbf{F}) . \tag{2.42}$$

The bilinear form σ induces the symplectic form B on the space of real solutions of (2.40)

$$\sigma(\mathbf{F},\mathbf{F}') \to B(\boldsymbol{\phi},\boldsymbol{\phi}')$$

$$= - \int_n dn^\mu(x) \left(\phi^\mu(x) \frac{\partial \phi_\mu'(x)}{\partial x^\mu} - \frac{\partial \phi^\mu(x)}{\partial x^\mu} \phi_\mu'(x)\right) \qquad (2.43)$$

where n is a space-like surface. This form is independent of n and is usually written in terms of the Cauchy data for the solutions. If the real solution $\phi_\mu(\underline{x},t)$ at time t_o has Cauchy data $\phi_\mu(\underline{x})$ and $\dot{\phi}_\mu(\underline{x})$, then (2.32) can be written in the familiar form

$$B(\boldsymbol{\phi},\boldsymbol{\phi}') = - \int d^3x \, (\phi^\mu(\underline{x}) \, \dot{\phi}_\mu'(\underline{x}) - \dot{\phi}^\mu(\underline{x}) \, \phi_\mu'(\underline{x})) \, . \qquad (2.44)$$

We can now define the Weyl algebra of the vector potential for the electromagnetic field. We use definition 2.5. We only need to specify the appropriate M and B. B comes from (2.43) and we take M to be the set of real-valued infinitely differentiable solutions of (2.40), defined over space-time.

A representation of the Poincaré group is defined on the solutions in M by

$$\boldsymbol{\phi} \to \boldsymbol{\phi}' \quad : \quad \phi_\mu'(x) = \Lambda_\mu{}^\nu \phi_\nu(\Lambda^{-1}(x - a)) \, . \qquad (2.45)$$

The symplectic form B is invariant under these transformations and hence the transformation $W \to W'$ induced by $\boldsymbol{\phi} \to \boldsymbol{\phi}'$ is an automorphism of the algebra. We have that $W'(\boldsymbol{\phi}) = W(\boldsymbol{\phi}')$ and hence $W'(\boldsymbol{\phi})W'(\boldsymbol{\psi}) = e^{-1/2 \, B(\boldsymbol{\phi},\boldsymbol{\psi})} \, W'(\boldsymbol{\phi} + \boldsymbol{\psi})$.

Our analysis so far applies equally well to both the Fermi and Gupta-Bleuler methods of quantisation. The same abstract Weyl algebra is associated with each method and furthermore corresponding physical transformations in each theory are given by the same automorphisms of the abstract Weyl algebra.

However, in applying the rest of the Weyl a gebra formalism we shall need to distinguish between the two methods of quantisation. We have yet to consider those aspects of the formalism that refer to representation theory. Once the abstract Weyl algebra has been characterised the traditional theory of free fields is reproduced when the appropriate free field representation of the algebra is given. The characterisation of that representation is therefore an important part of the formalism. We know from Strocchi's result that the usual formalism for free fields cannot be applied without modification. In the part of the Weyl algebra formalism that is concerned with representation theory these modifications are not the same for the two methods of quantisation. The representation of the abstract Weyl algebra is different for the two methods. Thus, in this formalism we would attribute the vastly different appearance of the two theories to the fact that different representations of the Weyl algebra are being used.

The representation of the algebra given by the Gupta-Bleuler method does not arise from a regular state. We can see this immediately from an examination of the Weyl operator. In this case it is not a unitary operator acting on a Hilbert space and hence the Gupta-Bleuler representation is not a Weyl system. We therefore cannot use theorem 2.7 for characterising the representation. For this reason, the formalism is not really suitable for discussing aspects of the Gupta-Bleuler method that depend on an explicit knowledge of the particular representation that is being used.

On the other hand, we find that the formalism is well suited to the Fermi method. In a later chapter, we shall show that in some sense the Fermi method is a more natural representation of the Weyl algebra.

Therefore we shall now calculate the generating functional for the regular state that characterises the Fermi representation. We do this by a treatment which closely parallels the treatment given for the neutral scalar field.

In that case, the generating functional arose from the scalar product in the Hilbert space of positive energy wave functions. So, we begin here by associating real solutions with positive energy solutions in the usual fashion.

$$\phi_\mu(\underline{x},t) = \frac{1}{(2\pi)}^{3/2} \int \frac{d^3k}{2\sqrt{k}} \left(\phi_\mu{}^+(\underline{k})\, e^{-i(kt-\underline{k}\cdot\underline{x})} + \overline{\phi_\mu{}^+(\underline{k})}\, e^{i(kt-\underline{k}\cdot\underline{x})}\right) \tag{2.46}$$

The Poincaré invariant form on $\mathcal{H}^+$ is then

$$(\phi^+,\phi^{+\prime}) = -\int d^3k\, \phi^{\mu+}(\underline{k})\, \phi_\mu{}^+(\underline{k}) \tag{2.47}$$

$B(\phi,\phi')$ is certainly the imaginary part of $(\phi^+,\phi^{+\prime})$ but since this form is indefinite $\mathcal{H}^+$ is not a Hilbert space. Thus the usual formalism breaks down for the quantisation methods of the electromagnetic field at this point. However, for the Fermi method, the formalism needs only to be adapted in a manner which is mathematically quite trivial.

We recall Dirac's observation[7] that the CCRs are satisfied by reversing the roles of the annihilation and creation operators for the zero component of the field operator. This means that the zero component of the one particle wave functions will have negative energy. Hence in the Fermi method we identify the zero component of a real solution of (2.39) with a negative energy solution. The mappings between M and $\mathcal{H}^F$ will therefore be given by

$$\phi_i(\underline{x},t) = \frac{1}{(2\pi)}^{3/2} \int \frac{d^3k}{2\sqrt{k}} \left(\phi_i^{\,+}(\underline{k})\, e^{-i(kt-\underline{k}\circ\underline{x})}\right.$$

$$\left. + \overline{\phi_i^{\,+}(\underline{k})}\, e^{i(kt-\underline{k}\circ\underline{x})}\right) \quad (i=1,2,3)$$

$$\phi_o(\underline{x},t) = \frac{1}{(2\pi)}^{3/2} \int \frac{d^3k}{2\sqrt{k}} \left(\overline{\phi_o^{\,-}(\underline{k})}\, e^{-i(kt-\underline{k}\circ\underline{x})}\right.$$

$$\left. + \phi_o^{\,-}(\underline{k})\, e^{i(kt-\underline{k}\circ\underline{x})}\right) . \tag{2.49}$$

In $\mathcal{H}^F$ we define the inner product

$$<\phi^F,\phi^{F'}> = \int d^3k \sum_{i=1}^{3} \overline{\phi_i^{\,+}(\underline{k})}\, \phi_i^{\,+'}(\underline{k}) + \phi_o^{\,-}(\underline{k})\, \overline{\phi_o^{\,-'}(\underline{k})} . \tag{2.50}$$

Then just as for the scalar field, $B(\phi,\phi')$ is the imaginary part of a scalar product.

The generating functional for the Fermi representation is

$$\rho(\phi) = e^{-\frac{1}{4}<\phi^F,\phi^F>} . \tag{2.51}$$

Poincaré transformations on elements of $\mathcal{H}^F$ will be defined by (2.34) and (2.38). Hence the scalar product in $\mathcal{H}^F$ will not be invariant under Poincaré transformations and therefore the generating functional will also not be invariant. Now the vector state that generates the representation in the GNS construction is the vacuum state Ω_o.

$$\rho(z) = E(W(z)) = (\Omega_o,\, e^{iW(z)}\, \Omega_o) . \tag{2.52}$$

We can distinguish two possibilities for the vacuum that could result from the non-invariance of the generating functional. The first possibility is that the new state can be represented as a vector state in the original representation

$$\rho'(z) = \rho(Tz) = (\Omega_o',\, e^{iW(z)}\Omega_o')$$

Ω_o' is then the new vacuum.

The second possibility is that the new state cannot be represented by a vector state in the original representation. Then the vacuum for the new representation will be orthogonal to the old vacuum. Thus in both cases, the new vacuum will be different from the original one.

We now prove that the second possibility holds in the Fermi representation,

Theorem 2.10

In the representation of the Weyl algebra over (M,B) with M and B given by (2.40) and (2.43) characterised by the generating functional (2.51) the automorphisms of the algebra induced by the Poincaré transformations (2.45) are only implemented by unitary transformations when the transformations are spatial rotations.

Proof

Let $X(a,\Lambda)$ be defined by

$$\phi' = X(a,\Lambda)\phi$$

with ϕ' and ϕ given by (2.45). Then with respect to the scalar product induced in M by the scalar product (2.47). (We shall identify M with $\mathcal{H}^F$)

$$X^*(a,\Lambda)\, X(a,\Lambda)\phi = \Lambda^T\Lambda\phi \ .$$

Suppose that $\Lambda = \Lambda(\ell_1)$ where $\Lambda(\ell_1)$ is a Lorentz transformation in the direction of the x_1 axis. If tanh u is the relative velocity of the two frames the vector $\phi' = (\Lambda^T(\ell_1)\,\Lambda(\ell_1) - I)\phi$ will have components

$$\phi_0^{-\prime}(\underline{k}) = 2\sinh^2 u\ \phi_0^{-}(\underline{k}) - 2\sinh u\cosh u\ \phi_1^{+}(\underline{k})$$

$$\phi_1^{+\prime}(\underline{k}) = 2\sinh u\cosh u\ \phi_0^{-}(\underline{k}) + 2\sinh^2 u\ \phi_1^{+}(\underline{k})$$

$$\phi_2^{+\prime}(\underline{k}) = 0$$

$$\phi_3^{+\prime}(\underline{k}) = 0\ .$$

For an arbitrary basis of M $(e_1, e_2, \ldots)$, the series $\sum_{n=1}^{\infty} \|(\Lambda^T(\ell_1)\Lambda(\ell_1) - I)\, e_n\|^2$ will not be convergent, since we can easily find a basis in which $\|(\Lambda^T(\ell_1)\Lambda(\ell_1) - I)\, e_n\| \not\to 0$ as $n \to \infty$. The operator $(X^*(a,\Lambda(\ell_1))\, X(a,\Lambda(\ell_1)) - I)$ is therefore not Hilbert-Schmidt on M since an operator O is Hilbert-Schmidt if and only if the series

$$\sum_{n=1}^{\infty} \|O e_n\|^2$$

converges for an arbitrary basis $(e_1, e_2, \ldots)$[38].

Then by theorem 2.9, the automorphism induced by $\Lambda(\ell_1)$ is not implementable by a unitary transformation.

If Λ is a spatial rotation, then $(R^T R - I) = 0$ is obviously Hilbert-Schmidt and hence induces an automorphism that is unitarily implementable.

We can parametrize every Lorentz transformation in the form

$$\Lambda = \Lambda(R_1)\Lambda(\ell_1)\Lambda(R_2)$$

where $\Lambda(R_1)$ and $\Lambda(R_2)$ are spatial rotations. Hence every Lorentz transformation that is not purely spatial induces automorphisms that cannot be implemented in the Fermi representation.

Therefore, while observers can construct the free representation by the Fermi method in the manner indicated, only for the case in which the frames of reference are connected by spatial rotations can the representation spaces on which the operators act be connected by a unitary transformation.

This calculation shows that the Fermi representation is significantly different from other free field representations. However, the particular feature just demonstrated results from the representation of the Weyl algebra that is used in the Fermi method. It does not come from the algebraic structure of the theory. We know from other considerations that the Gupta-Bleuler method is a well defined theory of the electromagnetic field. Hence the basic abstract Weyl algebra must be appropriate. The Fermi method is just another representation of this abstract Weyl algebra. We should expect peculiar features in each method because of Strocchi's result. However, in this formalism we can put these features into two categories, those that result from the algebraic structure and those that result from the representation chosen for the algebraic structure. Theorem (2.10) is a feature of the Fermi representation only and is an example of the latter category.

2.3 The Weyl Algebra of Physical Photons

In descriptions of the Gupta-Bleuler and Fermi methods the field operator is often interpreted in terms of annihilation and creation operators for physical and unphysical photons. Such an interpretation is, however, misleading because strictly speaking operators for physical photons only emerge after supplementary conditions and indefinite metric (in the case of the Gupta-Bleuler method) have been applied.

In order to demonstrate this we shall show that the Weyl algebra of the physical photons is a factor algebra of a subalgebra of the original abstract Weyl algebra. The proof in no way depends on the representation of the Weyl algebra and hence the results will apply to both methods. For both these methods it is therefore inappropriate to identify any operators as physical photon operators. This identification should only be used if the algebra of the physical photons were a subalgebra of the original algebra.

The subalgebra of the Weyl algebra associated with the so called "physical photons" will be denoted by $\mathcal{O}$. We shall define $\mathcal{O}$ as an algebra generated (with the term "generated" appropriately generalised) by the elements, $W(\phi)$, of $\mathcal{W}$ where ϕ belongs to the subspace, S, of M consisting of wave functions which satisfy

$$\phi_o = 0, \ \underline{\nabla}\cdot\underline{\phi} = 0 \ . \qquad (2.53)$$

Thus S is the set of wave functions with transverse components only.

For any wave function, ϕ, in M we can project out the transverse components by the following procedure. The Lorentz transformation

$$\Lambda_\mu{}^\nu = \begin{pmatrix} 1 + \frac{k_1{}^2}{k(k_3-k)} & \frac{k_1k_2}{k(k_3-k)} & \frac{k_1}{k} & 0 \\ \frac{k_1k_2}{k(k_3-k)} & 1 + \frac{k_2{}^2}{k(k_3-k)} & \frac{k_2}{k} & 0 \\ \frac{k_1}{k} & \frac{k_2}{k} & \frac{k_3}{k} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (2.54)$$

maps the vector (k_1,k_2,k_3,k) into $(0,0,k,k)$ and is also its own inverse. We therefore apply the projector

$$p' = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$$

to the vector $\Lambda\phi$ and then perform the inverse transformation. Suppose that we call the vector obtained by this method ϕ'. Then we find that ϕ' will have components $\phi_\mu{}'$ given by

$$\phi_o{}'(x) = 0 \ ,$$

$$\underline{\phi}'(x) = \underline{\phi}(x) - \frac{1}{\nabla^2}\nabla\underline{\nabla}\cdot\underline{\phi}(x). \qquad (2.55)$$

An equivalent procedure for obtaining ϕ' is to eliminate the unwanted components from ϕ by a suitable gauge transformation. Then

$$\phi' = U_\Lambda \phi \, U_\Lambda^{-1} \ . \qquad (2.56)$$

This second procedure is usually given as the prescription for obtaining the transverse components.

From a mathematical point of view, both procedures are possible because the transformation $\phi \to \phi'$ can be defined either by (2.56) or

$$\phi' = P\phi \ , \qquad (2.57)$$

where

$$P = \Lambda P'\Lambda \ .$$

This is analogous to the mapping $\phi \to \phi'$ under Poincaré transformations. These two ways of defining the transformation law express the Lorentz covariance and relativistic invariance of the theory. We recall the confusion that has resulted from the failure to distinguish clearly between the two.

In describing the second procedure we use the term "eliminate" rather than "project out" deliberately. The gauge transformation is not a projection operator in the mathematical sense. It does not satisfy $U_\Lambda{}^2 = U_\Lambda$. It is the operator, P, that is the projection operator.

The Weyl algebra, $\mathcal{W}$, was constructed as the norm completion of the inductive limit of the von Neumann rings $\mathcal{R}$ (F,B), where F was an element of a certain set of finite-dimensional subspaces of M. For the construction of the subalgebra, $\mathcal{S}$, we define the subspace, F′, of F such that $F' = S \cap F$. Then if F belongs to a system of finite-dimensional subspaces which spans M with properties given in definition 2.6, F′ will belong to a system which spans S also with those properties.

<u>Definition 2.11</u>

In the weak topology of the von Neumann ring $\mathcal{R}$ (F,B) described in definition 2.6 we form the closure, $\mathcal{R}_S(F)$, of the algebra generated by the elements $W(\phi)$ where ϕ is in F′. Then $\mathcal{S}$

is the norm completion of the inductive limit of the algebras, $\mathcal{A}'(F)$, formed from all rings $\mathcal{R}(F,B)$.

We discussed the transformation properties of the subspace S in Chapter 1 in describing radiation gauge. When Poincaré transformations on S are given by (2.45) then S is not invariant. Hence the automorphisms of $\mathcal{W}$ induced by these Poincaré transformations in M, i.e.

$$W \to W' \quad \text{where} \quad W'(\phi) = W(\phi') \ , \tag{2.58}$$

will not leave $\mathcal{B}$ invariant. Thus we cannot use $\mathcal{B}$ as the algebra of the physical photons and maintain the Lorentz covariance of the theory.

The two subspaces of M that are invariant under the Poincaré transformations (2.45) are the subspace, N, of elements that satisfy

$$\frac{\partial \phi^{\mu}}{\partial x^{\mu}} = 0 \ , \tag{2.59}$$

and the subspace, T, of elements that are derived from scalar functions, i.e. the set ϕ with components $\frac{\partial \Lambda}{\partial x^{\mu}}$. The subalgebras $\mathcal{N}$ and $\mathcal{T}$ are defined in the same way that $\mathcal{B}$ was defined.

Note that it is not strictly necessary for the rings $\mathcal{R}(F,B)$ to be considered in constructing the algebra $\mathcal{B}$. We only needed to consider a system of subspaces F′ because the bilinear form B is still non degenerate when restricted to elements of F′. However, the bilinear form is degenerate in the subspaces N and T and hence we have to construct first the rings $\mathcal{R}(F,B)$ to construct $\mathcal{N}$ and $\mathcal{T}$.

Since S and T are subspaces of N, $\mathcal{B}$ and $\mathcal{T}$ are subalgebras of $\mathcal{N}$. Furthermore, we can write N = S ⊕ T by the following procedure. For all ϕ in N the components in S are the transverse components of ϕ given by (2.55). The scalar component in T is $\frac{1}{\nabla^2}\underline{\nabla}\cdot\underline{\phi}$. Using this decomposition of N we can write the algebra, $\mathcal{R}_N(F)$, as the weak closure of $\mathcal{R}_S \otimes \mathcal{R}_T$. In that sense $\mathcal{N}$ is the "closure" of $\mathcal{S} \otimes \mathcal{Y}$.

Proposition 2.12

$\mathcal{Y}$ is the centre of $\mathcal{N}$.

Proof

Let $\boldsymbol{\phi}$ and $\boldsymbol{\phi}'$ be elements of N. Then

$$[W(\boldsymbol{\phi}),W(\boldsymbol{\phi}')] = -\int d^3x \left(\phi^\mu \frac{\partial\phi'_\mu}{\partial t} - \frac{\partial\phi^\mu}{\partial t}\phi'_\mu\right) .$$

If we require $W(\boldsymbol{\phi})$ to commute with all $W(\boldsymbol{\phi}')$ then choosing $\boldsymbol{\phi}'$ with components:

$$\phi_1' = \frac{\partial\Lambda}{\partial y} , \quad \phi_2' = \frac{\partial\Lambda}{\partial x} , \quad \phi_3' = \phi_o' = 0 \quad \text{gives}$$

$$\int d^3x \left(\phi_1 \frac{\partial^2\Lambda}{\partial t\partial y} - \phi_2 \frac{\partial^2\Lambda}{\partial t\partial x}\right) - \left(\frac{\partial\phi_1}{\partial t}\frac{\partial\Lambda}{\partial y} - \frac{\partial\phi_2}{\partial t}\frac{\partial\Lambda}{\partial x}\right) = 0 ,$$

$$\int d^3x \left(\frac{\partial\phi_1}{\partial y} - \frac{\partial\phi_2}{\partial x}\right)\frac{\partial\Lambda}{\partial t} - \left[\frac{\partial}{\partial t}\left(\frac{\partial\phi_1}{\partial y} - \frac{\partial\phi_2}{\partial x}\right)\right]\Lambda = 0 .$$

Since this must be true for arbitrary Λ

$$\frac{\partial\phi_1}{\partial y} - \frac{\partial\phi_2}{\partial x} = 0 .$$

By suitably choosing $\boldsymbol{\phi}'$ we can show that $\underline{\nabla} \times \underline{\phi} = 0$. Since $\boldsymbol{\phi}$ is also in N it must therefore be a vector with components $\phi_\mu = \frac{\partial\Lambda}{\partial x^\mu}$.
Thus a necessary condition for $[W(\boldsymbol{\phi}),W(\boldsymbol{\phi}')] = 0$ is for $\boldsymbol{\phi}$ to be in T.

It is also sufficient since

$$[W(\phi),W(\phi')] = -\int d^3x \left(\frac{\partial\Lambda}{\partial x^\mu}\frac{\partial\phi'^\mu}{\partial t} - \frac{\partial^2\Lambda}{\partial t\partial x^\mu}\phi'^\mu\right) ,$$

$$= -\int d^3x \left(\frac{\partial\Lambda}{\partial t}\frac{\partial\phi_o'}{\partial t} + \Lambda\frac{\partial}{\partial t}\underline{\nabla}\cdot\underline{\phi}' - \frac{\partial^2\Lambda}{\partial t^2}\phi_o' + \frac{\partial\Lambda}{\partial t}\underline{\nabla}\cdot\underline{\phi}'\right) ,$$

$$= -\int d^3x \left(\Lambda\frac{\partial}{\partial t}\underline{\nabla}\cdot\underline{\phi}' + \underline{\nabla}^2\Lambda\,\phi_o'\right) ,$$

$$= - \int d^3x \left(-\Lambda \frac{\partial^2 \phi_o'}{\partial t} + \Lambda \underline{\nabla}^2 \phi_o' \right) ,$$

$$= 0 \ .$$

Hence $\mathcal{Y}$ will be the centre of $\mathcal{N}$.

$\mathcal{Y}$ will be called the algebra of the supplementary condition operators. For suppose that (formally)

$$W(\phi) = e^{iA(\phi)} \ ,$$

where

$$A(\phi) \equiv \int d^4x \, \mathcal{A}^{\mu}(x) \ f_{\mu}(x) \quad \text{with} \quad \phi_{\mu} = \Delta^* f_{\mu} \ . \tag{2.60}$$

For ϕ in T,

$$A\left(\frac{\partial \Lambda}{\partial x^{\mu}}\right) \equiv \int d^4x \ \mathcal{A}^{\mu}(x)(-) \ \frac{\partial f(x)}{\partial x^{\mu}} \quad \text{with} \quad \Lambda = \Delta^* f \ .$$

On integrating this formal expression by parts we obtain

$$A\left(\frac{\partial \Lambda}{\partial x^{\mu}}\right) \equiv \int d^4x \ \frac{\partial \mathcal{A}^{\mu}(x)}{\partial x^{\mu}} \ f(x) \ ,$$

which is the supplementary condition operator. We therefore relabel the operators in $\mathcal{Y}$. We define

$$X(\Lambda) \equiv A\left(\frac{\partial \Lambda}{\partial x^{\mu}}\right) \ ,$$

and

$$G(\Lambda) \equiv W\left(\frac{\partial \Lambda}{\partial x^{\mu}}\right) \ . \tag{2.61}$$

$\mathcal{Y}$ is then the invariant subalgebra of $\mathcal{W}$ of Lorentz gauge transformations.

Let π be the homomorphism $\mathcal{N} \to \mathcal{B}$ defined for the elements of $\mathcal{N}$ in $\mathcal{B} \otimes \mathcal{Y}$ by $\mathcal{B} \otimes \mathcal{Y} \to \mathcal{B}$. The elements of $\mathcal{N}$ not in $\mathcal{B} \otimes \mathcal{Y}$ are limits in the topology induced by the norm of elements that are limits in the weak topology of the rings $\mathcal{R}(F,B)$ of elements of $\mathcal{B} \otimes \mathcal{Y}$. The homomorphism for those elements is therefore defined by continuity in the relevant topology.

Let the kernel of the mapping, π, be denoted by $\mathcal{J}$. Then for any element, s, in $\mathcal{S}$ the set of $\mathcal{N}$, $\pi^{-1}(s)$, is an equivalence class in $\mathcal{N}$ with respect to the ideal $\mathcal{J}$. The algebra of these equivalence classes is the factor algebra, $\frac{\mathcal{N}}{\mathcal{J}}$, and the identification of s with the equivalence class of elements in $\mathcal{N}$ mapped onto s defines an isomorphism between $\frac{\mathcal{N}}{\mathcal{J}}$ and $\mathcal{S}$.

Corollary 1.8.3 in Dixmier[39], provides a mathematically rigorous justification for these assertions.

<u>Theorem 2.13</u> (Corollary 1.8.3 in Dixmier)

Suppose that A and B are C* algebras, ϕ is a homomorphism $A \to B$ and I the kernel of ϕ. Consider the canonical decomposition of ϕ,

$$A \to \frac{A}{I} \xrightarrow{\psi} \phi(A) \to B .$$

Then I is closed in A, $\phi(A)$ is closed in B and ψ is an isometric isomorphism between the C* algebras $\frac{A}{I}$ and $\phi(A)$.

The equivalence classes in $\mathcal{N}$ can be generated by choosing an element, n, that does not belong to $\mathcal{J}$ and forming the set of elements n+i for all i in $\mathcal{J}$. We denote the equivalence class generated in this way by $\{n + \mathcal{J}\}$. Distinct elements in $\mathcal{S}$ will belong to distinct equivalence classes and from the isomorphism between $\mathcal{S}$ and $\frac{\mathcal{N}}{\mathcal{J}}$ the set of equivalence classes

$$\hat{s} = \{s + \mathcal{J}\} \tag{2.62}$$

will be a dense set in $\frac{\mathcal{N}}{\mathcal{J}}$.

We shall now consider the automorphisms of the algebra, $\mathcal{W}$, given by 2.58, i.e. the automorphisms induced by Poincaré transformations in M. From previous considerations the algebras $\mathcal{N}$ and $\mathcal{T}$ are invariant. Now the elements $s \otimes t$, where s is a fixed element of $\mathcal{S}$ and t is any element of $\mathcal{T}$, form a dense set in the equivalence class $\{s + \mathscr{I}\}$. Hence the ideal, $\mathscr{I}$, will also be invariant under automorphisms of $\mathscr{W}$ given by 2.58.

We can define automorphisms of $\frac{\mathscr{N}}{\mathscr{I}}$ from automorphisms of $\mathscr{W}$ that leave both $\mathscr{N}$ and $\mathscr{I}$ invariant in the following way.

If $n \to n'$ under an automorphism of $\mathscr{W}$ then

$$\{n + \mathscr{I}\} \to \{n + \mathscr{I}\}' = \{n' + \mathscr{I}\} \qquad (2.63)$$

will be an automorphism of $\frac{\mathscr{N}}{\mathscr{I}}$.

Consider the effect of automorphisms of $\frac{\mathscr{N}}{\mathscr{I}}$ defined by 2.63 and 2.58 on the equivalence class,

$$\hat{s} = \{s + \mathscr{I}\} \quad \text{with } \dot{s} \text{ in } \mathscr{S} .$$

The subalgebra, $\mathscr{S}$, will not be invariant. Suppose that $\dot{s} \to s' \otimes t$. Then

$$\begin{aligned} \{s + \mathscr{I}\}' &= \{s' \otimes t + \mathscr{I}\} \quad \text{(from 2.63)} \\ &= \{s' + \mathscr{I}\} \\ &= \hat{s}' . \end{aligned}$$

In particular, this automorphism for the elements,

$$\hat{W}(\phi) = \{W(\phi) + \mathscr{I}\} \text{ where } \phi \text{ is in } S$$

will be given by

$$\begin{aligned} \hat{W}'(\phi) &= \{W'(\phi) + \mathscr{I}\}, \quad \text{(from 2.63)} \\ &= \{W(\phi') + \mathscr{I}\}, \quad \text{(from 2.58)} \end{aligned}$$

In the direct product decomposition of $W(\phi')$, the component in $\mathscr{S}$ will be $W(\phi'')$ where ϕ'' is the component of ϕ' in S, i.e. the transverse components of ϕ'. (see 2.55).

Hence

$$\begin{aligned} \hat{W}'(\phi) &= \{W(\phi'') + \mathscr{I}\} , \\ &= \hat{W}(\phi'') . \end{aligned} \qquad (2.64)$$

Thus by adopting the scheme of labelling the equivalence classes by the element of $\mathcal{D}$ that they contain we can easily identify the algebra $\mathcal{F}$ with the Weyl algebra associated with the radiation gauge method.

From 2.64 we see that the Weyl algebra, $\mathcal{W}'$, associated with the radiation gauge method could be constructed as a Weyl algebra over (M', B') using definition 2.6 with

$$M' = \frac{N}{T} = \frac{S \oplus T}{T}$$

and B' the restriction of B to S. The automorphisms of $\mathcal{W}'$ representing physical transformations will be induced by Poincaré transformations in M'. These transformations in M' will be defined from the Poincaré transformations in M since N and T are invariant subspaces of M.

It was this algebra, $\mathcal{W}'$, that Segal[31] constructed in applying the Weyl algebra formalism to the electromagnetic field.

We have already indicated that the Gupta-Bleuler and Fermi methods are different representations of the Weyl algebra $\mathcal{W}$. We can infer from other considerations[40] that in the Gupta-Bleuler representation of the algebra, $\mathcal{W}$, the representation of the factor algebra $\mathcal{F}$ is constructed on a Hilbert space of states that are equivalence classes of states satisfying the supplementary condition. The indefinite metric is the mechanism for constructing such equivalence classes. We shall show that it is possible to interpret the unorthodox procedures of the Fermi method as also being prescriptions for obtaining a representation of $\mathcal{F}$.

CHAPTER THREE

CANONICAL COMMUTATION RELATIONS FOR FOUR DEGREES OF FREEDOM

To show how the Weyl algebra formalism applies in a familiar situation we shall obtain the abstract Weyl algebra that is associated with the CCRs for four degrees of freedom,

$$[q_\mu, p_\nu] = -i\, g_{\mu\nu} \,. \tag{3.1}$$

This example is particularly relevant to the present discussion as we shall find that various features that we found disturbing in the Weyl algebra formulation of the Gupta-Bleuler and Fermi methods, are also present in a Weyl algebra formulation of this algebra of operators.

3.1 The Abstract Weyl Algebra

We shall discuss first of all the usual Schrödinger representation of (3.1).

Definition 3.1 (Schrödinger Representation)

Consider the Hilbert space, $L^2(R_4, d^4x)$, consisting of equivalence classes of Lebesgue square integrable functions over R_4. Then the Schrödinger representation of the CCRs in (3.1) is

$$q_\mu \;:\; f(\underline{x}) \to x_\mu\, f(\underline{x}) \,,$$

$$p_i \;:\; f(\underline{x}) \to -i\,\frac{\partial f(\underline{x})}{\partial x_i} \,, \qquad i=1,2,3,$$

$$p_o \;:\; f(\underline{x}) \to i\,\frac{\partial f(\underline{x})}{\partial x_o} \,, \tag{3.2}$$

on some domain in L^2. $\underline{x}$ is the vector (x_o, x_1, x_2, x_3) in R_4.

We now consider the operators,

$$U(\underline{a}) = e^{-i\, g_{\mu\nu} a_\mu p_\nu} \,,$$

$$V(\underline{b}) = e^{-i\, g_{\mu\nu} b_\mu q_\nu} \,. \tag{3.3}$$

Their action on functions in L^2 will be

$$U(\underline{a})\, f(\underline{x}) = f(\underline{x} + \underline{a}) \ ,$$

$$V(\underline{b})\, f(\underline{x}) = e^{-i\, g_{\mu\nu} b_\mu x_\nu} f(\underline{x}) \ . \qquad (3.4)$$

The Weyl operator $W(\underline{a},\underline{b})$ is then

$$W(\underline{a},\underline{b}) = U(\underline{a})V(\underline{b})\, e^{i/2\, g_{\mu\nu} a_\mu b_\nu} \ . \qquad (3.5)$$

Therefore, we see that

$$W(\underline{a},\underline{b})W(\underline{a}',\underline{b}') = e^{-i/2\, g_{\mu\nu}(a_\mu b_\nu' - a_\mu' b_\nu)}\, W(\underline{a} + \underline{a}', \underline{b} + \underline{b}') \ . \qquad (3.6)$$

This is the Weyl relation. The symplectic form on R_8 for vectors $\underline{c} = (\underline{a},\underline{b})$ is therefore

$$B(\underline{c},\underline{c}') = g_{\mu\nu}(a_\mu b_\nu' - a_\mu' b_\nu) \ . \qquad (3.7)$$

This form is invariant under the transformations $\underline{c} \to \underline{c}'$ defined by

$$\underline{c}' = (\Lambda\underline{a}, \Lambda\underline{b}) \qquad (3.8)$$

where Λ is a Lorentz transformation. These symplectic transformations induce automorphisms of the algebra which will be given by

$$W(\underline{a},\underline{b}) \to W'(\underline{a},\underline{b}) = W(\Lambda\underline{a}, \Lambda\underline{b}) \ . \qquad (3.9)$$

In terms of the operators q and p these automorphisms are

$$q \to \Lambda^{-1} q \quad \text{and} \quad p \to \Lambda^{-1} p \ , \qquad (3.10)$$

since

$$g_{\mu\nu} a_\mu' p_\nu = g_{\mu\nu} a_\mu p_\nu' \ ,$$

and

$$g_{\mu\nu} b_\mu' q_\nu = g_{\mu\nu} b_\mu q_\nu' \ .$$

Hence we can use definition 2.6 to define the abstract Weyl algebra of CCRs in (3.1). In this case M is R_8 and in a symplectic basis the symplectic form for the vectors $\underline{c} = (\underline{a},\underline{b})$ and $\underline{c}' = (\underline{a}',\underline{b}')$ is given by (3.7).

It is well recognised that the Weyl operators are technically more convenient to handle than the unbounded operators p and q. For the same reason it is easier to discuss the automorphisms of p and q given by (3.10) within the framework of Weyl algebras.

3.2 Representations of the Weyl Algebra

The Schrödinger representation defined in definition (3.1) is obviously a representation of the Weyl algebra. We need only consider the Schrödinger representation to study representations of the algebra that are Weyl systems because of the von Neumann result (theorem 2.3).

We shall calculate the generating functional for the state on the Weyl algebra that generates the Schrödinger representation. It is

$$\rho(W(\underline{a},\underline{b})) = (\Omega_o, W(\underline{a},\underline{b})\Omega_o) \; . \tag{3.11}$$

In the Schrödinger representation we will take the vacuum to be

$$\Omega_o(x_o,x_1,x_2,x_3) = \frac{1}{\pi} e^{-\frac{1}{2}\Sigma_\mu x_\mu{}^2} \; . \tag{3.12}$$

$$\therefore \qquad W(\underline{a},\underline{b})\Omega_o = \frac{1}{\pi} e^{-ig_{\mu\nu}(x_\mu+\frac{1}{2}a_\mu)b_\nu} e^{-\frac{1}{2}\Sigma(x_\mu+a_\mu)^2} ,$$

and

$$(\Omega_o, W(\underline{a},\underline{b})\Omega_o)$$
$$= \frac{1}{\pi^2}\int d^4x \, e^{-ig_{\mu\nu}(x_\mu+\frac{1}{2}a_\mu)b_\nu} e^{-\frac{1}{2}\Sigma_\mu((x_\mu+a_\mu)^2+x_\mu{}^2)} \; .$$

We change the variables to $y_\mu = x_\mu + \frac{1}{2}a_\mu$ to perform this integration. We obtain

$$(\Omega_o, W(\underline{a},\underline{b})\Omega_o) = e^{-\frac{1}{4}\Sigma_\mu(a_\mu{}^2+b_\mu{}^2)} \; . \tag{3.13}$$

This result is hardly surprising. It is another way of looking at the von Neumann uniqueness theorem. If we put

$$d_\mu(\underline{a},\underline{b}) = (\Omega_o, W(\underline{a},\underline{b})\Omega_o) \; , \tag{3.14}$$

then $d_\mu(\underline{a},\underline{b})$ is the only normalised product measure invariant under the group of all unitary transformations on R_8. In fact, von Neumann used this measure in the original proof of his result.

Consider the effect of automorphisms of the algebra on the generating functional. The automorphism of the algebra in which $x \to \Lambda^{-1}x$, is induced by the transformation $(\underline{a},\underline{b}) \to (\Lambda\underline{a},\Lambda\underline{b})$ in the parameter space. Therefore, under these automorphisms, $\rho \to \rho'$ will be given by

$$\begin{aligned}\rho'(W(\underline{a},\underline{b})) &= \rho(W(\Lambda\underline{a},\Lambda\underline{b})) \ , \\ &= e^{-\frac{1}{4}\Sigma_\mu (\Lambda\underline{a})_\mu{}^2 + (\Lambda\underline{b})_\mu{}^2} \ , \\ &\neq \rho(W(\underline{a},\underline{b})) \ .\end{aligned}$$

An analogous situation occurred with the generating functional for the Fermi method. It was also found not to be invariant under transformations induced by automorphisms of the algebra. The analogy is not complete, however, as the representation of the Weyl algebra generated by the new generating functional in this case is necessarily unitarily equivalent to the old representation by the von Neumann uniqueness theorem.

The new generating functional is the new vacuum to new vacuum expectation value of the Weyl operator, i.e.

$$\rho'(W(\underline{a},\underline{b})) = (\Omega_o', W(\underline{a},\underline{b})\Omega_o') \ .$$

The vacuum is not invariant under transformations induced by automorphismsof the algebra. We have that $\Omega_o' = U\Omega_o$. Explicitly, in the Schrödinger representation

$$\begin{aligned}\Omega_o'(\underline{x}) &= \Omega_o(\Lambda^{-1}\underline{x}) \ , \\ &= e^{-\frac{1}{2}\Sigma_\mu (\Lambda^{-1}x)_\mu{}^2} \ , \\ &= U\Omega_o(\underline{x}) \ .\end{aligned}$$

Consider the relationship between the operators x_μ, p_ν and the annihilation and creation operators a_μ, $\bar{a}_\nu$. In the Schrödinger representation with vacuum defined in (3.12) we have that

$$a_\mu = \frac{1}{\sqrt{2}} (x_\mu + \frac{\partial}{\partial x_\mu}) ,$$

$$\bar{a}_\mu = \frac{1}{\sqrt{2}} (x_\mu - \frac{\partial}{\partial x_\mu}) . \qquad (3.15)$$

By combining (3.15) and (3.2) we obtain

$$a_o = \frac{1}{\sqrt{2}} (q_o - ip_o) , \quad a_i = \frac{1}{\sqrt{2}} (q_i + ip_i) ,$$

$$\bar{a}_o = \frac{1}{\sqrt{2}} (q_o + ip_o) , \quad \bar{a}_i = \frac{1}{\sqrt{2}} (q_i - ip_i), \quad i=1,2,3, \qquad (3.16)$$

in the Schrödinger representation. However, because of the von Neumann uniqueness theorem, we must always be able to find operators which satisfy (3.16) in any Weyl system for the algebra.

These annihilation and creation operators transform as a pair of four vectors under automorphisms of the algebra given by (3.10). The two sets of four vectors from (3.16) will be $(a_o,\bar{a}_i)$ and $(\bar{a}_o,a_i)$. If we think of the representation of the CCRs as coming from annihilation and creation operators from (3.16) then we would say that because annihilation and creation operators are mixed by Lorentz transformations the vacuum will not be invariant and hence the generating functional for the representation will not be invariant.

Another way of regarding this analysis is to start with four sets of annihilation and creation operators and to show that the only possible way of combining annihilation and creation operators to form two four vectors is with one vector $(a_o,\bar{a}_i)$ and the other as $(\bar{a}_o,a_i)$. It is impossible to maintain the commutation relations and the adjointness condition under automorphisms of the algebra by Lorentz transformations with any other combinations.

We would then argue that if we require that the operators $\Lambda_\mu{}^\nu q_\nu$ and $\Lambda_\mu{}^\nu p_\nu$ generate unitary groups of transformations for all Lorentz transformations Λ, then we must choose the representation given by (3.16). This line of argument is only a restatement of the von Neumann uniqueness theorem.

From (3.16) we note the difference between the representation of p_0 and p_1 in terms of annihilation and creation operators. We recall, once again Dirac's remark[7] concerning the Fermi method. We see that in the Fermi method the CCRs are satisfied by precisely the same representation of the field operators in terms of annihilation and creation operators. Dirac's remark was that the reality condition for the field was satisfied by reversing the roles of the annihilation and creation operators in the theory for the zero component of the field. We often think of Fock space as being built up of states formed by creation operators acting on a vacuum state. It would therefore be more appropriate to think that the roles of the annihilation and creation operators are maintained and that it is the representation of the field operator that is different so that the commutation relations can be satisfied.

In this context, the non unitary implementability of Lorentz transformations in the Fermi representation is yet another example of the existence of inequivalent representations[41] of the annihilation and creation operators for an infinite number of degrees of freedom.

The representation of the Weyl algebra that is analogous to the Gupta-Bleuler method can also be constructed.

The representation of the operators q_0 and p_0 in terms of annihilation and creation operators is

$$q_o = \frac{1}{\sqrt{2}}(a_o - \bar{a}_o),\ p_o = \frac{-i}{\sqrt{2}}(a_o + \bar{a}_o) \ . \tag{3.17}$$

On the Hilbert space $L^2(R_4, d^4x)$ the operators can be explicitly given by

$$q_o \to i\, x_o \ ,\ p_o \to \frac{\partial}{\partial x_o} \ ,$$

$$q_i \to x_i \quad ,\ p_i \to -i \frac{\partial}{\partial x_i} \ . \tag{3.18}$$

Then

$$a_o \to \frac{1}{\sqrt{2}}(x_o + \frac{\partial}{\partial x_o}) \ ,$$

$$a_i \to \frac{1}{\sqrt{2}}(x_i + \frac{\partial}{\partial x_i}) \ .$$

The vacuum state in this representation will therefore be

$$\Omega_o(\underline{x}) = \frac{1}{\pi} e^{-\Sigma_\mu x_\mu^{\ 2}} \ .$$

In this representation of the Weyl algebra the vacuum will be invariant under automorphisms of the algebra given by $q_\mu \to \Lambda_\mu^{\ \nu} q_\nu$, $p_\mu \to \Lambda_\mu^{\ \nu} p_\nu$ since from

$$\Sigma_\mu x_\mu^{\ 2} = \Sigma_i x_i - (ix_o)^2 \ ,$$

we have

$$e^{-ia_o p_o} f(\underline{x}) = f(x_o - ia_o, x_1, x_2, x_3) \ ,$$

$$e^{-ib_o x_o} f(\underline{x}) = e^{b_o x_o} f(\underline{x}) \ . \tag{3.19}$$

Hence the representation of the Weyl operators $W(\underline{a},\underline{b})$ will be

$$W(\underline{a},\underline{b}) f(\underline{x}) = e^{-i/2\, g_{\mu\nu} a_\mu b_\nu} e^{(b_o x_o + i\Sigma_i b_i x_i)} f(x_o - ia_o, x_1 + a_1, x_2 + a_2, x_3 + a_3) \ . \tag{3.20}$$

This representation is not unitary. The "Gupta-Bleuler" representation is not a Weyl system, i.e. the Weyl operators are not mapped onto unitary operators on $L^2(R_4, d^4x)$.

In discussions of the Gupta-Bleuler method an indefinite metric is used from the outset. The indefinite metric, however, does not arise from the mathematics of the representation of the abstract Weyl algebra. It comes about as part of the prescription for obtaining a representation of the factor algebra.

3.3 Factor Algebras of the Weyl Algebra

The representations of the Euclidean group, E(2), contained in the finite dimensional representations of the Lorentz group are indecomposable[16]. If we choose $E_1 = N_1 - M_2$, $E_2 = N_2 + M_1$ for the generators of E(2) where M_i and N_i are the generators of the pure rotations and boosts respectively, then explicitly on R_8 we shall have

$$E_1 = \begin{pmatrix} E_1' & 0 \\ 0 & E_1' \end{pmatrix}, \quad E_2 = \begin{pmatrix} E_2' & 0 \\ 0 & E_2' \end{pmatrix}, \quad M_3 = \begin{pmatrix} M_3' & 0 \\ 0 & M_3' \end{pmatrix},$$

$$E_1' = \begin{pmatrix} 0 & 0 & -i & i \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad E_2' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & i \\ 0 & i & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \quad M_3' = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.21}$$

Let N, T and S be subspaces of R_8 of vectors of the form $(a_1,a_2,a,a,b_1,b_2,b,b)$, $(0,0,a,a,0,0,b,b)$ and $(a_1,a_2,0,0,b_1,b_2,0,0)$ respectively. Then on vectors in the subspace T, the generators of E(2) are zero. Hence the group E(2) acts trivially on T. The subspace N is also invariant under E(2) but S is not. A similar situation occurred in the parameter space of the Weyl algebra of the potentials.

We define the algebras $\mathcal{N}$, $\mathcal{T}$ and $\mathcal{S}$ to be the set of elements of $\mathcal{W}$ that are finite combinations of $W(\underline{a},\underline{b})$ or their limits in the weak topology of a Weyl system for the algebra with

$(\underline{a},\underline{b})$ in N, T and S respectively. Then $\mathcal{N}$ is the closure in the weak topology of $\mathcal{B} \otimes \mathcal{Y}$. The algebras $\mathcal{N}$ and $\mathcal{Y}$ are invariant under automorphisms of $\mathcal{W}$ induced by the group E(2) in R_8:

$$W(\underline{a},\underline{b}) \rightarrow W'(\underline{a},\underline{b}) = W(G\underline{a},G\underline{b}) \ . \qquad (3.22)$$

Just as before, we can set up the factor algebra $\frac{\mathcal{N}}{\mathcal{J}}$ where $\mathcal{J}$ is the kernel of the homomorphism, π, in which $\mathcal{N} \rightarrow \mathcal{B}$. Since $\mathcal{B} \otimes \mathcal{Y}$ is dense in $\mathcal{N}$ in the topology of $\mathcal{R}$(M,B) the homomorphism is well defined as the mapping $\mathcal{B} \otimes \mathcal{Y} \rightarrow \mathcal{B}$.

For example, under this mapping,

$$W(a_1,a_2,a,a,b_1,b_2,b,b) \rightarrow W(a_1,a_2,o,o,b_1,b_2,o,o) \ . \qquad (3.23)$$

The factor algebra, $\frac{\mathcal{N}}{\mathcal{J}}$, is then invariant under automorphisms induced by E(2) transformations in the parameter space.

We shall consider once again the Schrödinger representation of the Weyl algebra given in definition (3.1). Any realization of a Hilbert space as a space of functions over a locally compact space can be considered as a representation of an abstract Hilbert space in the form of a direct integral. Hence we can think of the Schrödinger representation as the direct integral with respect to the Lebesque measure of one dimensional Hilbert spaces assigned to each point on the real line. Furthermore, to each direct integral of Hilbert spaces there corresponds a commutative weakly closed algebra of bounded linear operators containing the identity operator[42]. The algebra that corresponds to the direct integral decomposition for the Schrödinger representation is the weakly closed algebra generated by $W(o,\underline{b})$. Alternatively, it is a representation of this algebra by essentially bounded measurable functions over the operators q_μ.

Suppose that we decompose a Hilbert space into a direct integral $\mathcal{H} = \int_T d\lambda\, \mathcal{H}_\lambda$. Then, if R is the algebra corresponding to the direct integral decomposition, any operator on $\mathcal{H}$ that commutes with all elements of R is reducible to "diagonal" form, i.e. $T = \{T(\lambda)\}$.

We can apply these considerations to the subalgebras $\mathcal{N}$ and $\mathcal{Y}$. Since $\mathcal{Y}$ is a commutative weakly closed algebra, there will exist a direct integral decomposition of the Hilbert space with respect to $\mathcal{Y}$. Also, since all the elements of $\mathcal{N}$ commute with all the elements of $\mathcal{Y}$ we can reduce elements of $\mathcal{N}$ to "diagonal form".

We shall construct this direct decomposition explicitly.

If

$$W(\underline{a},\underline{b}) = e^{-ig_{\mu\nu}[a_\mu p_\nu + b_\mu q_\nu]}$$

is in $\mathcal{Y}$ then

$$g_{\mu\nu}(a_\mu p_\nu + b_\mu q_\nu) = a(p_o - p_3) + b(q_o - q_3) ,$$

$$= ia \left(\frac{\partial}{\partial x_o} + \frac{\partial}{\partial x_3}\right) + b(x_o - x_3) .$$

Therefore, we seek a unitary mapping of these operators such that

$$i\left(\frac{\partial}{\partial x_o} + \frac{\partial}{\partial x_3}\right) \to \lambda_1 \quad \text{and} \quad (x_o - x_1) \to \lambda_2 . \tag{3.24}$$

This is achieved with the unitary mapping

$$f(x_o,x_1,x_2,x_3) \xrightarrow{U} g(x_1,x_2,\lambda_1,\lambda_2) ,$$

where

$$g(x_1,x_2,\lambda_1,\lambda_2) = \frac{1}{\sqrt{\pi}} \int du\, e^{iu\lambda_1/2} \quad f(\tfrac{1}{2}(u + \lambda_2),x_1,x_2,\tfrac{1}{2}(u - \lambda_2)) . \tag{3.25}$$

For $(\underline{a},\underline{b})$ in N with components $(a_1,a_2,a,a,b_1,b_2,b,b)$

$$W(\underline{a},\underline{b})\, g(x_1,x_2,\lambda_1,\lambda_2)$$

$$= e^{-i(a\lambda_1+b\lambda_2)} \; e^{i(b_1x_1+b_2x_2+\frac{1}{2}(a_1b_1+a_2b_2))} \qquad g(x_1 + a_1,x_2 + a_2,\lambda_1,\lambda_2) . \tag{3.26}$$

We write

$$\mathcal{H} = \int d\lambda_1' \, d\lambda_2' \, \mathcal{H}_{\lambda_1' \lambda_2'} \, .$$

We have therefore represented $\mathcal{H} = L^2(R_4, d^4x)$ as a direct integral of spaces $\mathcal{H}_{\lambda_1' \lambda_2'} = L^2(R_2, d^4x)$. All the elements in $\mathcal{N}$ are in "diagonal" form, i.e. if $T \in \mathcal{N}$ then $T = \{T(\lambda_1', \lambda_2')\}$. If g is in $\mathcal{H}$ then $g = \{g_{\lambda_1' \lambda_2'}\}$ where $g_{\lambda_1' \lambda_2'}$ is the component of g in $\mathcal{H}_{\lambda_1' \lambda_2'}$.

Let the mapping $\Pi_{\lambda_1' \lambda_2'}$ be defined by

$$T \xrightarrow{\Pi_{\lambda_1' \lambda_2'}} T(\lambda_1', \lambda_2')$$

where $T(\lambda_1', \lambda_2')$ is an operator acting on $\mathcal{H}_{\lambda_1' \lambda_2'}$. Then the operators $T(\lambda_1', \lambda_2')$ are an irreducible representation of the algebra $\mathcal{N}$. In this decomposition the fact that $\mathcal{N}$ can be considered as the direct product of the algebras $\mathcal{D}$ and $\mathcal{J}$ is clearly evident. The reducible representation of $\mathcal{N}$ can be considered to be constructed as a direct integral of the direct product of irreducible representations of $\mathcal{D}$ and $\mathcal{J}$.

The mapping $T \to T(0,0)$ is a mapping of a representation of $\mathcal{N}$ to a representation of the factor algebra $\frac{\mathcal{N}}{\mathcal{J}}$. This follows from the way in which the factor algebra $\frac{\mathcal{N}}{\mathcal{J}}$ was defined. It was defined from a homomorphism of $\mathcal{N}$ onto the subalgebra $\mathcal{D}$. The elements of $\frac{\mathcal{N}}{\mathcal{J}}$ are the equivalence classes of elements in $\mathcal{N}$ that are mapped onto a given element of $\mathcal{D}$. By inspection, the mapping $T \to T(0,0)$ is a homomorphism of a representation of $\mathcal{N} \to$ representation of $\mathcal{D}$ and hence a representation of $\frac{\mathcal{N}}{\mathcal{J}}$.

Although the operators T(0,0) act only on $\mathcal{H}_{oo}$ we can identify them with operators acting on $\mathcal{H}$ in the following way. The operators T(0,0) are an irreducible representation of the algebra $\mathcal{O}$. We therefore identify the operator T(0,0) with an element T which belongs to the subalgebra $\mathcal{O}$. Explicitly, consider the element $W(\underline{a},\underline{b})$ given in (3.26). $\mathcal{H}_{oo} = L^2(R_2,d^4x)$ and under the mapping $T \to T(0,0)$, $W(\underline{a},\underline{b}) \to W(\underline{a},\underline{b})(0,0)$, where

$$W(\underline{a},\underline{b})(0,0)g(x_1,x_2) = e^{i(b_1x_1+b_2x_2+\frac{1}{2}(a_1b_1+a_2b_2)}g(x_1 + a_1,x_2 + a_2) \; . \tag{3.27}$$

Therefore, we can identify $W(\underline{a},\underline{b})(0,0)$ with $W(\underline{a}',\underline{b}')$ acting on $\mathcal{H}$ with $(\underline{a}_1,\underline{b}_2) = (a_1,a_2,0,0,b_1,b_2,0,0)$. The operators are of the same form in both $\mathcal{H}$ and $\mathcal{H}_{oo}$. However, while it does not matter whether we think of the operators of the factor algebra $\mathcal{A}$ as acting on either $\mathcal{H}$ or $\mathcal{H}_{oo}$ we cannot identify vectors in $\mathcal{H}_{oo}$ and $\mathcal{H}$. The vectors in $\mathcal{H}_{oo}$ are not normalisable with respect to the measure in $\mathcal{H}$. The function $g(x_1,x_2)$ can be an element of $\mathcal{H}_{oo}$ but it will certainly not be an element of $\mathcal{H}$. It will not be normalisable with respect to the measure d^4x.

Suppose that we solve the eigenvalue problem

$$\begin{aligned} &(p_o - p_3)f = 0 \; , \\ &(q_o - q_3)f = 0 \; , \text{ for functions in } \mathcal{H} . \end{aligned} \tag{3.28}$$

Then the operators $p_o - p_3$ and $q_o - q_3$ will be zero on the solution of (3.28). Therefore, T will be mapped on the subalgebra of T which is isomorphic to T(0,0). The actual form of the operators will be identical. The difference between them is that they act on different Hilbert spaces. Explicitly, the set of solutions of (3.28) will be

$g(x_1,x_2)\delta(\lambda_1)\delta(\lambda_2)$ where g is defined in (3.25).

These vectors are not normalisable and hence do not lie in $\mathcal{H}$. However, we can identify them with vectors in $\mathcal{H}_{\infty}$ by simply dropping the δ functions. We then would interpret the operators as the operators T(0,0) rather than the subalgebra of $\mathcal{T}$ with which they are formally identical.

The operators $p_o - p_3$ and $q_o - q_3$ are the analogues of the supplementary condition operators in the Fermi method. The functions $g(x_1,x_2)\delta(\lambda_1)\delta(\lambda_2)$ are therefore the analogues of the Fermi physical states. Thus, the difficulty of the nonnormalisability of the Fermi physical states can be overcome by re-interpreting them as states in $\mathcal{H}_{\infty}$. We also obtain a representation of the factor algebra $\frac{\mathcal{N}}{\mathcal{J}}$ by this method and it is this algebra that is the algebra of the physical photons.

Alternatively, we can remain in the Hilbert space $\mathcal{H}$ and just map the operators T onto the subalgebra of T which is formally identical to T(0,0).

CHAPTER FOUR

WEYL ALGEBRAS, FOUR-VECTOR AUTOMORPHISMS, THE SCHRÖDINGER REPRESENTATION AND AN INFINITE NUMBER OF DEGREES OF FREEDOM

The principle aim of this chapter is to construct the Schrödinger representation for the Fermi method. The expressions for the Weyl operators that we shall give are natural generalisations of the expressions obtained for four degrees of freedom.

We shall also discuss the example of the CCRs for an infinite number of four vector operators,

$$[q_\mu^k, p_\nu^{k'}] = -i\ \delta_{kk'}\ g_{\mu\nu}. \qquad (4.1)$$

4.1 QUANTUM FIELDS AND ANALYSIS IN FUNCTION SPACE

We recall that the elements of the representation space, K, for the Schrödinger representation of a quantum mechanical system of n degrees of freedom are equivalence classes of functions over R_n that are square integrable with respect to the Lebesgue measure. The canonical operators, q_k, $(k=1,\ldots,n)$ are multiplicative operators in this representation. Each point of R_n is therefore a set of eigenvalues for the operators, q_k, and the elements of K are thus functions over the spectrum of these operators.

The representation space for the Schrödinger representation of a quantum field similarly consists of functions defined over the spectrum of the field operator at a given time. A field has an infinite number of degrees of freedom and hence each point in the spectrum of the field operator will be a vector in some infinite dimensional linear space V. In contrast to the finite dimensional case, the measure on V used for the Schrödinger representation is not translationally invariant.

The general theory of analysis over infinite dimensional spaces is now a sufficiently well developed subject for the conclusion to be drawn that the theory of the translationally invariant measure, i.e. the Lebesque measure, cannot be extended to infinite dimensional spaces. Thus the use of non-translationally invariant measures for the Schrödinger representation of a field is inevitable.

The general properties of the measures that are used for the Schrödinger representation are much weaker than those of the Lebesque measure. They will also depend on the precise mathematical structure of V.

von Neumann(43) was the first to define a representation space for the Schrödinger representation. He showed how to construct a completely additive Gaussian measure on the adjoint of a pre-Hilbert space. The result, however, was not published until 1961 and hence the first mathematically complete account to appear in the literature is Segal's(44). In his formulation, the infinite dimensional space is a real Hilbert space. The Schrödinger representation can also be set up on functions defined on the adjoint space of a nuclear space. For an account of the mathematical properties of nuclear spaces and their adjoints see Gel'fand and Vilenkin[45]. In practice, V is most frequently chosen to be the adjoint space to a real Schwartz space, i.e. the space of tempered distributions.

We shall now briefly discuss some mathematical aspects of infinite dimensional spaces that are relevant to the Schrödinger representation. Firstly, we shall show how the Borel sets are constructed for a certain class of infinite dimensional spaces and then we shall give the definitions for the measures that will be used in the Schrödinger representation. Finally, we show how integration on a Hilbert space can be defined.

Suppose that V' is the adjoint space to a linear topological space V. Then the Borel sets on V' are constructed in the following way[45]. First of all the cylinder sets are constructed.

Definition 4.1 (Cylinder Sets)

Let F be a finite dimensional subspace of V and F^o the subspace of V' consisting of those elements, f, of V' for which $f(v) = 0$ for all v in F. We decompose V' into cosets by putting functionals into the same coset if their difference lies in F^o. A linear mapping, $V' \to \frac{V'}{F^o}$, is defined by mapping each functional to the coset which contains it. If B is a subset of $\frac{V'}{F^o}$ then the set of elements of V' that are mapped into elements of B by the mapping $V' \to \frac{V'}{F^o}$ is called a cylinder set A with base B and generating subspace F^o.

We shall only be considering locally convex topological spaces for which $\frac{V'}{F^o}$ is the adjoint of F. Also, the base B will be a Borel set.

<u>Definition 4.2</u> (Borel Cylinder Set)

Let $(v_1,\ldots,v_n)$ be a basis for a finite dimensional subspace F of V. Then if B is a Borel set in R_n, the set of elements, f, of V′ that are mapped to points of $R_n \in B$ under the mapping,

$$f \to (f(v_1),f(v_2),\ldots,f(v_n)) \ ,$$

is called a Borel cylinder set based on F. (A full discussion of the relationship between this definition and definition 4.1 can be found in Gel'fand and Vilenkin[45].)

For the special case in which V is a real Hilbert space, V and V′ can be identified. The cylinder sets can then be defined in the following way.

<u>Definition 4.3</u> (Cylinder Sets in a Hilbert Space)

Suppose that P is a finite dimensional projection and B a Borel set in the range of P. Then the subset of the real Hilbert space, H, consisting of all elements of H with projection in B is a cylinder set.

Thus the cylinder sets of a Hilbert space are equivalent to the tame subsets, introduced by Segal[44].

The Borel cylinder sets form a ring and hence generate an σ-ring of sets. This ring is defined as the smallest class of sets which contains the Borel cylinder sets and is closed under the operations of countable union and complementation. The members of this σ-ring are the Borel sets in V′.

The most useful measures on infinite dimensional space are those which are reasonably behaved on the Borel sets. The cylinder set measures, providing that they also satisfy mild continuity conditions, are examples of such measures.

Definition 4.4 (Cylinder Set Measure)

A cylinder set measure in the space V′ is a function, $\mu(B)$, defined on the family of all Borel cylinder sets, B, such that

(1) $0 \leq \mu(B) \leq 1$ for all B .

(2) $\mu(V') = 1$.

(3) If B is the union of a sequence $B_1, B_2, \ldots$ of non-intersecting Borel cylinder sets based on the subspace, F, then

$$\mu(B) = \sum_{n=1}^{\infty} \mu(B_n) \ .$$

For a Hilbert space, the "weak distribution" introduced by Segal[44] is an equivalent notion to the cylinder set measure.

We shall now consider the measures, ρ_F, induced in $\frac{V'}{F^o}$ by cylinder set measures, μ, in V′ . A cylinder set measure induces a measure on the Borel sets in every factor space in the following way. If X is a Borel set in $\frac{V'}{F^o}$ then we take the Borel cylinder set, Z, based on F and set

$$\rho_F(X) = \mu(Z) \ . \qquad (4.2)$$

<u>Definition 4.5</u> (Compatibility Condition)

Suppose that $\{p_F\}$ is a system of Borel set measures in the factor spaces $\frac{V'}{F^o}$. Then these measure are said to be compatible if

$$\rho_{F1}(X) = p_{F2}(T^{-1}(X)) \tag{4.3}$$

for every Borel set, X, in $\frac{V'}{F_1^o}$ whenever $F_1 \subset F_2$ where T is the natural mapping of $\frac{V'}{F_2^o}$ onto $\frac{V'}{F_1^o}$.

The measures, ρ_F, on the factor spaces, $\frac{V'}{F^o}$, induced by the cylinder set measure, μ, in V′ by (4.2) are compatible.

Conversely, the compatibility of measures on the factor spaces, $\frac{V'}{F^o}$, is sufficient to ensure the existence of a cylinder set measure on V′.

<u>Theorem 4.6[45]</u>

If $\{\rho_F\}$ is a system of regular, in the sense of Caratheodory, normalised positive measures in the factor spaces, $\frac{V'}{F^o}$, satisfying the compatibility condition (definition 4.5), then

$$\mu(Z) = \rho_F(X)$$

is a cylinder set measure in V′.

This theorem gives a practical method for constructing cylinder set measures.

The measures used for the Schrödinger representation are cylinder set measures. They can be related to certain measures over finite dimensional spaces in the following way.

We first recall the definitions of equivalence and quasi-invariance of measures from measure theory on finite-dimensional spaces.

<u>Definition 4.7</u> (Equivalence of Measures)

Two measures are equivalent or alternatively, mutually absolutely continuous, if they have the same family of null sets.

<u>Definition 4.8</u> (Quasi-Invariance of Measures)

Let V be a linear topological space on which a measure, μ, is defined. Then μ is said to be quasi-invariant if it has the property that every translate of a set, X, of μ-measure zero is also a set of μ-measure zero. That is

$$\mu(X) = 0 \Rightarrow \mu(y + X) = 0$$

for all y in V.

Note that these definitions apply without modification to infinite dimensional spaces.

Now it is not necessary to use the Lebesque measure to set up a representation of the CCRs for a finite number of degrees of freedom on functions over the spectrum of the operators q_k. We can use any regular quasi-invariant measure. The Hilbert space then consists of square integrable functions with respect to that measure.

By the von Neumann uniqueness theorem (theorem 2.3) all these representations are unitarily equivalent. In this context this theorem is equivalent to the result implicit in Plessner's work[46] that all regular quasi-invariant measures are equivalent to the Lebesgue measure.

The measures used for the Schrödinger representation of a field are infinite dimensional analogues of those quasi-invariant measures. The existence of distinct equivalence classes of these measures on infinite dimensional spaces is consistent with the existence of inequivalent representations of the CCRs.

The appropriate measures for the Schrödinger representation on functions are defined on the adjoint space, ϕ', of a nuclear space, ϕ, are the almost quasi-invariant measures.

Definition 4.9[45] (Almost Quasi-Invariant Measure)

A measure, μ, on the conjugate space, ϕ', of the nuclear space, ϕ, is almost quasi-invariant if it has the property that $\mu(\psi + X) = 0$ for every element $\psi \in \phi$ (we have identified elements of ϕ with elements of ϕ') and every set X such that $\mu(X) = 0$.

This definition coincides with the definition of quasi-invariant measures for a Hilbert space.

For our application, the measures will also be Gaussian.

Definition 4.10[45]

Let V be a locally convex linear topological space and $B(u,v)$ a non-degenerate, continuous scalar product. We define the Gaussian measure on the cylinder sets in the following way. Let F be a finite dimensional subspace of V and F^o the subspace of V' of elements, f, of V' for which $f(v) = 0$. If Y is a subset of F then

$$\nu_F(Y) = \left(\frac{1}{\pi}\right)^{\frac{n}{2}} \int_Y e^{-B(v,v)} dv$$

where dv is the Lebesgue measure in F corresponding to the scalar product, $B(v,v)$, is a Gaussian measure on F and $n = \dim F$. Since F and $\frac{V'}{F^o}$ are isomorphic there will correspond to the measure, ν_F, in F the measure, ν_F', in $\frac{V'}{F^o}$. Now the system of measures, $\{\nu_F'\}$, can be shown to be compatible and hence from theorem 4.5 are induced by a measure on the cylinder sets in V'. This measure is the Gaussian measure.

When V is a Hilbert space, H, we can define a Gaussian measure on the cylinder sets of H by the scalar product. For the cylinder set, A, defined as the inverse image of Borel set, B, in the range of a finite dimensional projection the measure will be given by

$$\mu(A) = \left(\frac{1}{\pi}\right)^{\frac{n}{2}} \int_B e^{-(v,v)} dv$$

where (v,v) is the scalar product in the finite dimensional space induced by the product in H and dv is the corresponding Lebesgue measure.

We shall now indicate how a theory of integration over infinite dimensional real Hilbert spaces can be formulated. The approach that we shall give closely follows Gross[47].

The idea on which the formulation is based is to associate a probability space (X,m) with the Hilbert space in an invariant fashion. This association was first suggested by Segal[44].

For example, if H is the square integrable real functions on the interval (0,1) then the probability space may be taken to be Wiener space.

Firstly, the cylinder functions are defined.

Definition 4.11[44] (Cylinder Function)

A function on H is a cylinder function if f(x) = f(Px) where P is a finite dimensional projection on H. The function, f, is then said to be based on the range of P.

Let $\mathcal{B}$ be the algebra of bounded, continuous, complex-valued cylinder functions on H that are measurable with respect to some cylinder set measure or the uniform limit of such functions. With the sup norm this algebra is a commutative C* algebra. Also, the integral of the cylinder functions on H with respect to the Gaussian measure defined by the scalar product is a linear functional. Since this linear functional is continuous in the sup norm it can be extended to all of $\mathcal{B}$ as a continuous linear functional.

Thus the algebra, $\mathcal{B}$, is isomorphic to C(X) where X is a compact Hausdorff space. If we denote the integral of F by I(f) then since I is a positive linear functional on $\mathcal{B}$ it is represented as a countable additive measure on X. We have that

$$I(f) = \int_X \mathfrak{f}\, dm ,$$

where $f \to \mathfrak{f}$ denotes the isomorphism between $\mathcal{B}$ and C(X).

It can be shown that this isomorphism is completely determined by the map on the linear functionals thus establishing contact with Segal's original suggestion for associating a Hilbert space with a measure space through linear mappings of elements of the Hilbert space onto equivalence classes of functions defined on a probability space. Thus Friedrich's space[48] is just a concrete realization of (X,m).

We can therefore identify $L^2(H)$ with $L^2(X,m)$. It should be noted that not all elements of $L^2(H)$ can be represented as square integrable functionals over H but only the limit of them.

4.2 The Schrödinger Representation for the Fermi Method

In section 2.1 we discussed how a complex space is associated with a space of real solutions of the wave equation. We found that for the Fermi method the pre-Hilbert space consisted of both positive energy and negative energy solutions.

For the Fermi method, the complex solutions of the wave equations 2.40 associated with the real solutions are given by 2.49. We shall parametrize this complex space in an analogous fashion to the parametrization of the space of scalar wave functions given by 2.28.

If the Cauchy data for the real solution, $\boldsymbol{\phi}$, of 2.40 at some fixed time, t_o, is denoted by $(\phi_\mu(\underline{x}), \bar{\phi}_\mu(\underline{x}))$ then the complex wave function associated with the real solution, $\boldsymbol{\phi}$, will be given by, $\boldsymbol{\Phi}$, where $\boldsymbol{\Phi}$ has components, Φ_μ, satisfying

$$\Phi_i(\underline{x}) \leftrightarrow (-\nabla^2)^{\frac{1}{4}} \phi_i(\underline{x}) + i(-\nabla^2)^{-\frac{1}{4}} \dot{\phi}_i(\underline{x})$$
$$\Phi_o(\underline{x}) \leftrightarrow (-\nabla^2)^{\frac{1}{4}} \phi_o(\underline{x}) - i(-\nabla^2)^{-\frac{1}{4}} \dot{\phi}_o(\underline{x}) \tag{4.4}$$

The scalar product is then

$$(\Phi,\Psi) = \Sigma_{\mu=o,3} \int d^3x \, \overline{\Phi_\mu(\underline{x})} \, \Psi_\mu(\underline{x}) \; . \tag{4.5}$$

Hence the Hilbert space, H, associated with the real solutions is a space of complex-valued functions over R_3.

The Schrödinger representation can now be set up on $L^2(H',d\mu)$ where H' is the real subspace of H. H' is therefore related to the collection of solutions with $\dot{\phi}_i(\underline{x}) = 0$ at t_o. The measure, μ, is a Gaussian cylinder set measure obtained from the scalar product in H'.

Suppose that A is the cylinder set consisting of all x in H' such that P(x) is in B, where P is a finite dimensional projection and B is a Borel set in the range of P. Then

$$\mu(A) = \left(\frac{1}{\pi}\right)^{\frac{n}{2}} \int_B e^{-\|v\|^2} \, dv \tag{4.6}$$

where dv denotes the Lebesgue measure on the range of P and n is the dimension of the range.

A cylinder function is called a polynomial if it is a polynomial of linear functionals. The polynomials are dense in $L^2(H',d\mu)$. (See, for example, Gross[47].) Therefore, the action of the canonical operators on elements of $L^2(H',d\mu)$ will be defined once it is given for the polynomials.

If f is a polynomial in $L^2(H',\mu)$ then the operators, $U(\psi_1)$ and $V(\psi_2)$, are defined by

$$U(\psi_1): \quad f(\phi) \to f(\psi_1 + \phi)\, e^{-(\psi_1,\phi)-\frac{1}{2}(\psi_1,\psi_1)}$$
$$V(\psi_2): \quad f(\phi) \to f(\phi)\, e^{-i\langle\psi_2,\phi\rangle} \qquad (4.7)$$

where

$$\langle\psi_2,\phi\rangle = \int d^3x\, (\Sigma_{i=1,3}\, \psi_i(\underline{x})\, \phi_i(\underline{x}) - \psi_o(\underline{x})\, \phi_o(\underline{x}))\ .$$

Then if

$$W(\psi_1,\psi_2) \equiv U(\psi_1)\, V(\psi_2)\, e^{\frac{i}{2}\langle\psi_2,\psi_1\rangle} \qquad (4.8)$$

we can check that

$$W(\psi_1,\psi_2)\, W(\psi_1',\psi_2') =$$
$$W(\psi_1 + \psi_1',\psi_2 + \psi_2')\, e^{-\frac{i}{2}(\langle\psi_1,\psi_2'\rangle-\langle\psi_1',\psi_2\rangle)}\ . \qquad (4.9)$$

Now if ψ and ψ' are real solutions with components ψ_μ and ψ_μ' which have Cauchy data at some fixed time, t_o,

$$\psi_\mu(\underline{x} = (-\nabla^2)^{-\frac{1}{4}}\, \psi_{1\mu}(\underline{x})$$
$$\dot{\psi}_\mu(\underline{x}) = (-\nabla^2)^{\frac{1}{4}}\, \psi_{2\mu}(\underline{x}) \qquad (4.10)$$

then

$$\langle\psi_1,\psi_2'\rangle - \langle\psi_1',\psi_2\rangle = \langle\psi,\dot{\psi}'\rangle - \langle\dot{\psi},\psi'\rangle\ .$$

Hence if we define

$$\hat{W}(\psi) \quad W((-\nabla^2)^{\frac{1}{4}}\, \psi(\underline{x}),\ (-\nabla^2)^{-\frac{1}{4}}\, \dot{\psi}(\underline{x})) \qquad (4.11)$$

4.9 can be equivalently written as

$$\hat{W}(\psi)\, \hat{W}(\psi') = \hat{W}(\psi + \psi')\, e^{-\frac{i}{2} B(\psi,\psi')} \qquad (4.12)$$

($B(\psi,\psi')$ is defined in 2.44).

We have therefore verified that the Weyl relation is satisfied and hence that the representation is indeed a representation of the Weyl algebra of the potentials.

For the Weyl algebra for four degrees of freedom, we decomposed the Hilbert space $L^2(R_4, d^4x)$ into a direct integral with respect to an invariant commutative subalgebra. We can do the same for $L^2(H^{\wedge}, d\mu)$. The subalgebra is the algebra, $\mathcal{J}$, given in section 2.3. Then we can reduce the elements of $\mathcal{N}$ (also described in section 2.3) to "diagonal" form.

We proceed in precisely the same way as we did in chapter 3. We must first, however, obtain the transverse components. This can easily be done for each real solution, ϕ, in $\mathcal{J}$ by using the Helmholtz decomposition for the three-vector part of the vector, Φ, in H associated with ϕ by (4.4). Also, by using the Helmholtz decomposition in the form given by Moses[(49)] we can decompose Φ into vectors $\Phi^T_{\pm 1}$, Φ^L and Φ^O that are orthogonal in H.

From Moses, we introduce the vectors, $\chi_\lambda(\underline{x}|\underline{k})$, where,

$$\chi_\lambda(\underline{x}|\underline{k}) = \frac{1}{(2\pi)}^{3/2} e^{i\underline{k}\cdot\underline{x}}\, \underline{Q}_\lambda(\underline{k}) \qquad \lambda=\pm 1, L$$

with

$$\underline{Q}_L(\underline{k}) = -\underline{k}/|\underline{k}|$$

$$\underline{Q}_\lambda(\underline{k}) = -\frac{\lambda}{\sqrt{2}}\left[\frac{k_1(k_1 + i\lambda k_2)}{k(k+k_3)} - 1,\ \frac{k_2(k_1 + i\lambda k_2)}{k(k+k_3)} - i\lambda,\ \frac{k_1 + i\lambda k_2}{k}\right]$$

$$\text{for } \lambda = \pm 1 . \qquad (4.12)$$

Then the three-vector components of the vectors $\underline{\Phi}^T_{\pm 1}$ and $\underline{\Phi}^L$ will be given by

$$\underline{\Phi}^T_{\pm\lambda} = \int d^3k \, \chi_\lambda(\underline{x}|\underline{k}) \; g_\lambda(\underline{k})$$

$$\underline{\Phi}^L = \int d^3k \, \chi_L(\underline{x}|\underline{k}) \; g_L(\underline{k}) \tag{4.13}$$

where

$$g_\lambda(\underline{k}) = \int d^3x \, \chi^*_\lambda \, (\underline{x}|\underline{k}) \cdot \underline{\Phi}(\underline{x}) \; .$$

The time component of these vectors is zero. The only non zero component of the vector, $\underline{\Phi}^O$, is the time component which will be given by

$$\Phi^O_O = \Phi_O \; . \tag{4.14}$$

Now we can write the operators defined in 4.7 in the form

$$U(\psi) = e^{i\langle P,\psi\rangle}$$

$$V(\phi) = e^{-i\langle Q,\phi\rangle} . \tag{4.15}$$

The proof is given in Segal (44).

We write

$$\langle P,\psi\rangle = P^T_1(\psi^T_1) + P^T_{-1}(\psi^T_{-1}) + P^L(\psi^L) - P^O(\psi^O)$$

$$\langle Q,\phi\rangle = Q^T_1(\phi^T_1) + Q^T_{-1}(\phi^T_{-1}) + Q^L(\phi^L) - Q^O(\psi^O) \; . \tag{4.16}$$

We therefore will have on some domain

$$Q^T_\alpha(\phi): \quad f(\psi) \rightarrow (\phi,\psi^T_\alpha) \; f(\psi)$$

$$Q_L(\phi): \quad f(\psi) \rightarrow (\phi,\psi^L) \; f(\psi) \tag{4.17}$$

$$Q_O(\phi): \quad f(\psi) \rightarrow (\phi,\psi^O) \; f(\psi) \; .$$

Now

$$\frac{1}{\sqrt{2}}[Q_L(\psi) - Q_o(\psi)] \;:\; f(\phi) = \frac{1}{\sqrt{2}}(\psi,\phi_L - \phi_o)\, f(\phi)$$

$$\frac{1}{\sqrt{2}}[Q_L(\psi) + Q_o(\psi)] \;:\; f(\phi) = \frac{1}{\sqrt{2}}(\psi,\phi_L + \phi_o)\, f(\phi)\ .$$

Thus under the unitary transformation, U_1, in $L^2(H')$ defined by the change of variables

$$\phi_A = \frac{1}{\sqrt{2}}(\phi_L - \phi_o)$$

$$\phi_B = \frac{1}{\sqrt{2}}(\phi_L + \phi_o) \qquad (4.18)$$

we will have

$$\frac{1}{\sqrt{2}}[Q_L(\psi) - Q_o(\psi)] \to Q_A(\psi)$$

$$\frac{1}{\sqrt{2}}[Q_L(\psi) + Q_o(\psi)] \to Q_B(\psi) \qquad (4.19)$$

$$\frac{1}{\sqrt{2}}[P_L(\psi) - P_o(\psi)] \to P_A(\psi)$$

$$\frac{1}{\sqrt{2}}[P_L(\psi) + P_o(\psi)] \to P_B(\psi)\ .$$

Now we can symbolically write

$$P_L(\psi) \quad \text{as} \quad -i\,\frac{\partial}{\partial Q_L(\psi)} + i\,Q_L(\psi)$$

and (4.20)

$$P_o(\psi) \quad \text{as} \quad i\,\frac{\partial}{\partial Q_o(\psi)} - i\,Q_o(\psi)\ ,$$

then under the change of variables 4.18

$$P_A(\psi) \to -i\,\frac{\partial}{\partial Q_B(\psi)} + i\,Q_B(\psi)$$

$$P_B(\psi) \to -i\,\frac{\partial}{\partial Q_A(\psi)} + i\,Q_A(\psi)\ . \qquad (4.21)$$

Consider now the Wiener transformation, U_2, on $L^2(H')$ (see Segal[44]) by which

$$Q_B(\psi) \to -\left[-i\frac{\partial}{\partial Q_C(\psi)} + i\,Q_C(\psi)\right]$$

and (4.22)

$$-i\frac{\partial}{\partial Q_B(\psi)} + i\,Q_B(\psi) \to Q_C(\psi)\ .$$

Thus if we let $U = U_2U_1$ then the operators in $\mathcal{N}$ will be "diagonalised". The situation is completely analogous to that discussed in chapter 3. We have a direct integral decomposition of $L^2(H',d\mu)$ as before. The discussion of the direct integral decomposition of $L^2(R_4,d^4x)$ applies in this case.

4.3 A Realization of the CCRs for an Infinite Number of Degrees of Freedom

We shall construct a representation of the CCRs

$$[q^k_\mu, p^{k'}_\nu] = -i\,g_{\mu\nu}\,\delta^{kk'}\ , \quad k=1,2,\ldots \tag{4.23}$$

using a Fermi type representation.

The representation space will be constructed from a sector of the infinite tensor product space, $\Pi^{\infty}_{i=1} \otimes H_k$,[50] where H_k is the Hilbert space of complex-valued functions over R_4 that are square integrable with respect to the Gaussian measure,

$$d\mu_k = \frac{1}{\pi^2}\, e^{-[(x^k_0)^2+(x^k_1)^2+(x^k_2)^2+(x^k_3)^2]}\, dx^k_0\, dx^k_1\, dx^k_2\, dx^k_3\ .$$

Let a_k be the vacuum vector in H_k. In the realization of the CCRs that we shall give a_k will be the function identically 1. The tensor product $a = \Pi^{\infty}_{i=1} \otimes a_k$, determines the complete tensor product space in the following way.

Suppose that b_k is in H_k. Then we can form the tensor products,

$$b = (\Pi \bigotimes_{k=1}^{n} b_k) \quad (\Pi \bigotimes_{k=n+1}^{\infty} a_k) \ . \tag{4.24}$$

The scalar product between such tensor products is then defined to be

$$(b,b') = \Pi_k (b_k, b_k') \ .$$

Only a finite number of factors will differ from unity. With respect to this scalar product the set of tensor products given by (4.24) is an incomplete Hilbert space. We then complete it and denote the result as the infinite direct product of (H_k, a_k).

In this case, the tensor products will just be square integrable functions that only depend on a finite number of variables x_μ^k.

The vacuum state of H is the function which is identically 1. Hence on some suitable domain the annihilation operators, a^k, will be represented by

$$a_\mu^k \to \frac{1}{\sqrt{2}} \frac{\partial}{\partial x_\mu^k} \ . \tag{4.25}$$

The creation operators are the adjoint operators of a_μ^k and hence will be given by

$$\bar{a}_\mu^k \to - \frac{1}{\sqrt{2}} \frac{\partial}{\partial x_\mu^k} + \sqrt{2}\, x_\mu^k \ . \tag{4.26}$$

To satisfy the CCRs, (4.23), we use a Fermi construction. We put

$$q_\mu^k = \frac{1}{\sqrt{2}} (a_\mu^k + \bar{a}_\mu^k) = x_\mu^k$$

$$p_i^k = - \frac{i}{\sqrt{2}} (a_i^k - \bar{a}_i^k) = - i \ (\frac{\partial}{\partial x_i^k} - x_i^k) \ \ldots$$

$$p_o^k = \frac{1}{\sqrt{2}} (a_o^k - \bar{a}_o^k) = i \left(\frac{\partial}{\partial x_o^k} - x_o^k\right) . \tag{4.27}$$

Consider now the automorphism of the algebra induced by a Lorentz boost L.

$$\begin{aligned} q'^k_\mu &= L^\nu_\mu \, q^k_\nu \\ p'^k_\mu &= L^\nu_\mu \, p^k_\nu . \end{aligned} \tag{4.28}$$

Let

$$L^\nu_\mu = \begin{bmatrix} \cosh u & -\sinh u & 0 & 0 \\ -\sinh u & \cosh u & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.29}$$

then

$$q'^k_o = \cosh u \; x_o^k - \sinh u \; x_1^k$$

$$q'^k_1 = - \sinh u \; x_o^k + \cosh u \; x_1^k$$

$$p'^k_o = i \left[\cosh u \left(\frac{\partial}{\partial x_o^k} - x_o^k\right) + \sinh u \left(\frac{\partial}{\partial x_1^k} - x_1^k\right)\right]$$

$$p'^k_1 = - i \left[\sinh u \left(\frac{\partial}{\partial x_o^k} - x_o^k\right) + \cosh u \left(\frac{\partial}{\partial x_1^k} - x_1^k\right)\right]$$

$$q'_i = q_i \quad p'_i = p_i \qquad i=2,3 .$$

This representation of the CCRs is inequivalent to the original one. For suppose that we try to determine the vacuum state for this representation. The new annihilation operators will be given by

$$a'^k_0 = \frac{1}{\sqrt{2}}\left[\cosh u \frac{\partial}{\partial x^k_0} + \sinh u \left(\frac{\partial}{\partial x^k_1} - 2x^k_1\right)\right.$$

$$a'^k_1 = \frac{1}{\sqrt{2}}\left[\sinh u \left(\frac{\partial}{\partial x^k_0} - 2x^k_0\right) + \cosh u \frac{\partial}{\partial x^k_1}\right] \qquad (4.31)$$

$$a'^k_2 = \frac{1}{\sqrt{2}}\frac{\partial}{\partial x^k_2} \qquad a'^k_3 = \frac{1}{\sqrt{2}}\frac{\partial}{\partial x^k_3} \quad .$$

The solution of the equations

$$a'^k f(x^k_0, x^k_1, x^k_2, x^k_3) = 0 \qquad \mu = 0,\ldots,3 \qquad (4.32)$$

is

$$f(x^k_0,x^k_1,x^k_2,x^k_3) = A\, e^{2\sinh u \cosh u\, x^k_0 x^k_1 - \sinh^2 u((x^k_0) + (x^k_1)^2)}$$

The vacuum for each H_k is therefore

$$u_k = e^{2\sinh u \cosh u\, x^k_0 x^k_1 - \sinh^2 u((x^k_0)^2 + (x^k_1)^2)}$$

$$(u_k, u_k) = 1 \ .$$

The new vacuum is therefore $u = \Pi_k \otimes u_k$. However, $\Pi_k \otimes u_k$ is not a vector in H. A necessary and sufficient condition for u to lie in H is that the product $\Pi_k (u_k, a_k)$ converges.

Now

$$(u_k, a_k) = \frac{1}{\pi}\int dx^k_0\, dx^k_1\, e^{2\sinh u \cosh u\, x^k_0 x^k_1 - \cosh^2 u((x^k_0)^2 + (x^k_1)^2)}$$

$$= \frac{1}{\cosh u} \ .$$

For $u \neq 0$, $\Pi_k (u_k, a_k)$ therefore will not converge. The new representation does not have a vacuum state. Hence it is inequivalent to the original one.

From the point of view of analysis in function space we would say that the Jacobian for the transformation of variables,

$$x'^k_0 = \cosh u \, x^k_0 - \sinh u \, x^k_1$$
$$x'^k_1 - \sinh u \, x^k_0 + \cosh u \, x^k_1 \qquad (4.33)$$

is zero when k is infinite.

Hence we can attribute the non implementability of automorphisms of the Weyl algebra of the potentials induced by Lorentz boosts, in the Fermi representation to the fact that we have a system with an infinite number of degrees of freedom.

CHAPTER FIVE

DISCUSSION AND CONCLUSIONS

5.1 The Role of the Supplementary Condition Operators in the Fermi Method

We shall begin by re-expressing the formalism that has been given for handling the Fermi method, in a manner which more closely follows the way in which it was actually built up. It should be clearly evident from this reformulation that Hurst's[51] treatment of the supplementary condition in quantum electrodynamics was the source of many of the ideas that contributed to the development of our approach.

The formalism was developed mainly by finding a mathematically rigorous presentation of the heuristic descriptions. However, because the requirements of mathematical rigor may have obscured the underlying motivations, a description of how the formalism actually developed may be helpful. In particular we would like to show how the Fermi method is related to the intuitive pictures we have of conventional field theory.

The starting point of our analysis was the idea that the key to the understanding of the Fermi representation was the recognition of the role played by the supplementary condition operators, $\frac{\partial \mathcal{A}^{\mu}(x)}{\partial x^{\mu}}$, in the theory. The Fermi physical states were considered in order to restrict the algebra of operators acting on the representation space to a smaller algebra of operators (which would be the desired physical operators) acting on a space spanned by these states. The real point of the original approach was to eliminate some superfluous and undesirable quantities that the formulation in terms of vector potentials seems to entail.

If it is desired to restrict an algebra of operators, A, to act on a common eigenspace of a particular set of operators, S, then the only operators which should be considered are those from the subalgebra, B, of A that commutes with S. Such a restriction would be considered as a homomorphism of the subalgebra, B, into itself. The algebra of operators belonging to the range of this homomorphism is then isomorphic to the factor algebra, $\frac{B}{I}$, where I is the kernel of the homomorphism.

This interpretation of the mathematics of the Fermi method was thought to be appropriate not only because it could avoid the difficulties of non normalisability but also because it could be shown to be consistent with considerations of Lorentz invariance. Hence the significance of the Fermi physical states is not their non normalisability but rather that they are the eigenstates of the supplementary condition operators with eigenvalue zero. By restricting the operators to these eigenstates a unitary representation of a factor algebra of operators is obtained.

The well known difficulty is that because the supplementary condition operators in the Fermi representation have a continuous spectrum the eigenstates corresponding to particular eigenvalues cannot be normalised. The Fermi physical states, therefore, cannot belong to a subspace of the original representation space.

Now the original representation space can be expressed as a direct integral of Hilbert spaces over the spectrum of the supplementary condition operators with each Hilbert space in the direct integral decomposition being labelled by some point in the spectrum. Although states in the Hilbert spaces which occur in a direct integral decomposition need not be normalisable with respect to the original representation space nevertheless they can be considered as sensible states in their own right. Hence to resolve the problem of the non normalisability of the Fermi physical states it was suggested that they be interpreted as states in that Hilbert space which is labelled by the eigenvalue zero in a direct integral decomposition of the original representation space, without reference to the larger space in which they are "embedded".

In attempting to find a mathematically rigorous formulation for the direct integral of spaces over the spectrum of the supplementary condition operators, it would seem reasonable that we should consider the Schrödinger representation of the free field. In fact this problem is closely analogous to the problem of an eigenstate of the position operator x. The algebraic formulation of field theory that is most closely associated with the Schrödinger representation is Segal's Weyl algebra formalism and hence it was natural to use the Segal formalism and not some other algebraic formulation. This formalism has proved sufficiently powerful for our purposes, but we do not exclude the possibility of alternative approaches once the consistency of the Fermi method has been demonstrated.

Perhaps the most interesting feature of the Fermi method that was found was the result that in this representation of the algebra of the electromagnetic field, automorphisms of the algebra corresponding to Poincaré transformations are not necessarily implementable by unitary transformations. The corollary to this result is that we cannot define an invariant vacuum state. In all the approaches to quantum electrodynamics that start with a vector potential, there is some price that has to be paid. For Gupta-Bleuler it is the indefinite metric, for radiation gauge it is the lack of manifest covariance, and here it is the non invariant vacuum.

An analogous situation occurs in the example given in section 4.3. In this example, suppose that A and B are observers in two frames of reference connected by the Lorentz transformation (4.29). If we take the observables corresponding to the observer A to be the set, (q_μ^k, p_μ^k), then those corresponding to B will be the set, (q'^k_μ, p'^k_μ), with q'^k_μ and p'^k_μ given by (4.28).

Suppose that A sets up the Schrödinger representation of the CCRs given by (4.27), i.e. the Fock space representation with a vacuum state. If we take the representation of the observables of B to be that given by (4.30) then the representation does not have a vacuum state. In other words, the state that A claims to be the vacuum state would appear to B to contain an infinite number of particles.

Another way of looking at this situation is to consider the set of all elementary Lorentz transformations (4.28) and with each value of the parameter α to associate a separable Hilbert space. Then if we form the direct sum of this continuous infinity of spaces, these elementary Lorentz transformations can be represented by permutation operators which permute the terms in the direct sum according to the rules by which the correspondence has been set up. The vector potentials will be in block diagonal form as a direct sum of inequivalent representations. The spatial rotations and space-time translations will also be in the generalised diagonal form and may be taken to be equivalent. Each Hilbert space in the direct sum will be the space of states as described by an observer at rest in the appropriate frame of reference, and of course can carry a Fock space representation with an associated vacuum state, which the observer could regard as the vacuum. But this vacuum state would not be the one obtained by a Lorentz transformation, because, as already mentioned, the representation of the vector potential is not unitarily implementable and so the vacuum is not Lorentz invariant.

Now although one cannot therefore define a mapping between the constituent Hilbert spaces which provides a unitary representation of the Lorentz group, nevertheless it is possible to do this in a small enough "subspace" namely that corresponding to the supplementary condition. In other words, no matter which representation of the vector potential one starts with, the physical states so obtained are all unitarily equivalent, and so may be regarded as identical. By so identifying them the superstructure of the non-separable Hilbert space may be dispensed with. It is sufficient to start with one of the constituent (separable) spaces, with the knowledge that the final outcome has no trace left of its superficially non invariant origins.

Consider again the example in section 4.3. If we restrict the automorphisms of the operators, $\{q_\mu^k, p_\mu^k\}$, induced by Lorentz transformations to the E(2) subgroup as we did in chapter three, then the analogues of the supplementary condition operators are the operators, $(q_1^k - q_0^k, p_1^k - p_0^k)$ and $(q'^k_1 - q'^k_0, p'^k_1 - p'^k_0)$ respectively.

Suppose that observers A and B each set up Fock space representations over the variables x_μ^k and x'^k_μ respectively as in (4.27). Then the states which satisfy the supplementary condition will be,

$$f_A(x_2^1, x_3^1, \ldots, x_2^k, x_3^k, \ldots) \, \Pi_k \, \delta(x_0^k - x_1^k) \, e^{-ix_0^k x_1^k}$$

and

$$f_B(x'^1_2, x'^1_3, \ldots x'^k_2, x'^k_3, \ldots) \, \Pi_k \, \delta(x'^k_0 - x'^k_1) \, e^{-ix'^k_0 x'^k_1}$$

Hence the physical states for each observer would be the functions $f_A(x_2^1, x_3^1, \ldots, x_2^k, x_3^k, \ldots)$ and $f_B(x'^1_2, x'^1_3, \ldots, x'^k_2, x'^k_3, \ldots)$. The significance of the function, $\Pi_k\ \delta(x_0^k - x_1^k)\ e^{-ix_0^k x_1^k}$, is to ensure that the zero eigenvalue of the supplementary condition operators is taken. The observables of A acting on the original representation space were functions of the operators x^k and $\frac{\partial}{\partial x^k_\mu}$. As we have previously indicated, we can think of the restriction of these operators to the eigenstates of the supplementary condition operators as a representation of a factor algebra of observables. This representation will be in the form of functions of the operators x_2^k, x_3^k, $\frac{\partial}{\partial x^k_2}$ and $\frac{\partial}{\partial x^k_3}$.

In section 4.3 we showed that in general the change of variables $x^k_\mu \rightarrow x'^k_\mu$ cannot be achieved by a unitary mapping. On the other hand, suppose that the restriction of these operators to eigenstates of the supplementary condition operators is given by

$$x'^k_2 = f(x_2^k, x_3^k)$$

and

$$x'^k_3 = g(x_2^k, x_3^k) \ .$$

Then this change of variables can be represented by a unitary transformation. Therefore even though we cannot connect the algebra of observables by a unitary transformation the application of the supplementary condition enables us to do so for a factor algebra.

5.2 The Fermi Representation and Perturbation Theory

The discussion that we have given so far has only referred to the free field. However, calculations have been performed by Belinfante[5] in perturbation theory using the Fermi representation. We shall not reproduce them here as the only comment that needs to be added to his account is the mathematical justification for his handling of the infinite normalisation factor of the wave functions. He quite rightly observed that this factor could be put consistently equal to one. The mathematical justification for this procedure is that this normalisation is appropriate for states belonging to the Hilbert space in the direct integral decomposition of the original Hilbert space with respect to the supplementary condition operators.

Now the physical scattering amplitudes in quantum electrodynamics are invariant under gauge transformations of the photon propagator. This result was first proved by Feynman[52] in the following form. He showed that transition amplitudes obtained directly from Feynman diagrams are not influenced by any terms proportional to either k_μ or k_ν in the photon propagator. However, as Bialynicki-Birula[53] has pointed out the question is more complicated than this and, strictly speaking, Feynman's result is only valid in the lowest order of perturbation theory.

Without going into fine details, the heuristic reason why the Feynman propagator, $g_{\mu\nu}\,D(k^2)$, is equivalent to the exchange of transverse photons only is because we can write

$$g_{\mu\nu} = k_\mu n_\nu + n_\mu k_\nu - m_\mu \bar{m}_\nu - \bar{m}_\mu m_\nu$$

where $k_\mu, n_\mu, m_\mu, \bar{m}_\mu$ are four ortho[redacted]gonal null vectors satisfying

$$k_\mu n^\mu = 1 = - n_\mu \bar{m}_\mu \ ,$$

and all other products zero.

For then, if the propagator describes the exchange of a photon of 4 momentum k between two conserved currents, the terms in $g_{\mu\nu}$ which involve k will disappear because of current conservation $k_\mu j^\mu(k) = 0$.

For the Fermi method, the choice of physical states for the photon states, or equivalently the restriction to the subalgebra of transverse operators (the algebra $\mathcal{\emptyset}$) means that the propagator calculated (a symmetric second rank tensor) cannot contain any terms in n_μ as they would imply a violation of the supplementary condition and any terms in k_μ would disappear for the same reason as above. Hence only terms in m_μ and $\bar{m}_\mu$ survive and these are the usual transverse terms.

This explanation is only intended to be a formal explanation of why such calculations as Belinfante's lead to the usual results, and a complete resolution would require an investigation of all the problems of divergences and their associated inconsistencies.

5.3 Concluding Remarks

The most significant aspect of our work is that it challenges the viewpoint that the Fermi method of quantising the electromagnetic field is inconsistent. This viewpoint is so widely accepted that people never think of considering the Fermi representation in their investigations.

Hence we first remark that the Fermi method should be considered in conjunction with the other quantisation schemes in any investigation of a general nature.

Secondly, we remark that the Fermi method may well prove to be a convenient representation for particular applications. We have shown that the Fermi representation is easily included in the

Segal formalism and hence should be better adapted than the Gupta-Bleuler representation for applications requiring the techniques of C^* algebras.

Finally, we remark that our formulation needs to be extended to include interaction. An indication of the way in which this might be achieved can be found in Hurst's account[51].

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