THE UNIVERSITY OF ADELAIDE
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Quadrals and their Associated Subspaces

PhD Thesis

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Contents

Abstract

This thesis concerns sets of points in the finite projective space $\mathrm{PG}(n,q)$ that are combinatorially identical to quadrics. A *quadric* is the set of points of $\mathrm{PG}(n,q)$ whose coordinates satisfy a quadradic equation, and the term *quadral* is used in this thesis to mean a set of points with all the combinatorial properties of a quadric.

Most of the thesis concerns the characterisation of certain sets of subspaces associated with quadrals. Characterisations are proved for the external lines of an oval cone in $\mathrm{PG}(3,q)$, of a non-singular quadric in $\mathrm{PG}(4,q)$, q even, and of a large class of cones in $\mathrm{PG}(n,q)$, q even. Characterisations are also proved for the planes meeting the non-singular quadric of $\mathrm{PG}(4,q)$ in a non-singular conic, and for the tangents and generator lines of this quadric for q odd.

The second part of the thesis is concerned with the intersection of ovoids of $\mathrm{PG}(3,q)$. A new bound is proved on the number of points two ovoids can share, and configurations of secants and external lines that two ovoids can share are determined. The structure of ovoidal fibrations is discussed, and this is used to prove new results on the intersection of two ovoids sharing all of their tangents.

Signed Statement

This work contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text.

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Introduction

The tension between algebra and combinatorics is central to the study of finite geometry. The finite projective space $\mathrm{PG}(n, q)$ itself (for $n > 2$) can be equally well defined algebraically using the underlying field $\mathrm{GF}(q)$ or combinatorially using the axioms of incidence. This connection between algebra and combinatorics in the very definition of projective space leads to the belief that similar connections should exist for each object contained in $\mathrm{PG}(n, q)$. That is, an object with an algebraic definition is expected to have an equally valid combinatorial definition; and vice versa an object with a combinatorial definition is expected to have a simple algebraic description.

The simplest algebraic description is a linear equation, and the points satisfying linear equations are exactly the subspaces. The next simplest algebraic description is a quadratic equation and the set of points defined by a quadratic equation is called a *quadric*. The connection between algebra and combinatorics leads to the expectation that quadrics should also have a valid combinatorial definition. That is, it should be possible to show that if a set of points has some key combinatorial properties of a quadric, then it must in fact *be* a quadric. Such a result is called a *characterisation* result – the quadrics would be characterised by these key properties.

Unfortunately, it is not possible to characterise certain quadrics using only combinatorial properties. In certain cases, there exist objects with all the combinatorial properties of quadrics, but which are *not* quadrics. That is, their points do not all satisfy a quadratic equation. In this thesis, the term *quadral* is used to mean a set of points combinatorially identical to a quadric.

A large part of this thesis is concerned with characterisation, but not characterisation of the quadrals themselves. Instead the focus is on characterising the families of subspaces associated with the quadral. One of the key properties of quadrals is that subspaces can only meet a quadral in certain ways. The subspaces of $\mathrm{PG}(n,q)$ can be grouped into families based on how they meet a particular quadral, and then each family has its own combinatorial properties. For example, consider the lines that contain no point of the quadral $\mathscr{Q}$ – the *external lines* of $\mathscr{Q}$. This set of lines has many combinatorial properties. Given a set of lines $\mathscr{L}$ with some of these properties, the aim is to prove that $\mathscr{L}$ is in fact the family of external lines to a quadral like $\mathscr{Q}$. In a sense, the family of lines is used to reconstruct the quadral $\mathscr{Q}$.

This leads to the remaining part of the thesis. Given that the external lines of $\mathscr{Q}$ can be used to reconstruct $\mathscr{Q}$, it is natural to ask how much information is required to fully determine $\mathscr{Q}$. For example, how many external lines are required to fully determine $\mathscr{Q}$? Another way to ask this is to ask how many external lines two quadrals may share. The second part of the thesis concerns this type of problem for a particular family of quadrals – the ovoids of $\mathrm{PG}(3,q)$.

Outline of Thesis

The thesis begins with the necessary background information. Chapter 0 contains the basic definitions and results of projective geometry that are needed to discuss quadrals. Chapter 1 describes quadrals, which are the objects of interest in this thesis. The definitions and important properties of quadrics, ovals, ovoids and their cones are included here.

Chapter 2 discusses the characterisation of quadrals. Several results are stated characterising quadrals by their intersections with lines and planes, including the characterisations of Tallini, Buekenhout and Thas. These characterisations will be used in later chapters to prove characterisations of the families of subspaces associated with quadrals.

Chapter 3 discusses the existing characterisations of the families of subspaces associated with quadrals. Of particular note are those characterising the external lines of an ovoid and of a hyperbolic quadric in $\mathrm{PG}(3,q)$.

The next three chapters present original results characterising certain families of subspaces associated with quadrals. Chapter 4 presents characterisations of the external lines of an oval cone in $\mathrm{PG}(3,q)$ for q odd and q even. Chapter 5 presents characterisations of the planes meeting a parabolic quadric of $\mathrm{PG}(4,q)$ in a non-singular conic, and of the external lines of this quadric when q is even. Chapter 6 presents characterisations of the external lines of an oval cone in $\mathrm{PG}(n,q)$, and also the external lines of a cone over any parabolic quadric.

Chapter 7 is concerned with the question of how much information is required to fully determine a quadral, in particular for ovals and ovoids. This question is related to the number of points that two distinct ovals or ovoids may share. The intersection of ovals and ovoids is discussed, and several new results for the case q even are proved. The number of secant or external lines that two ovoids can share is also investigated.

The thesis concludes with a short discussion on what has been achieved and the direction of future research. A list of the main original results in the thesis is given in this concluding chapter.

Note on Original Work

This thesis presents original results and proofs, but it also reviews existing results by other authors. Results and proofs due to other authors will be clearly marked with a citation such as [1]. Any result not marked in this way is an original result whose proof is due to the author of this thesis.

Chapter 0

Preliminaries

This chapter provides an introduction to the basic concepts required to understand the content of this thesis. Most of the definitions and results used in the proofs later in the thesis are collected here. Very few proofs will be given, and the citation given for each result refers to a reference book where the proof can be found. For further information on the topics in this chapter, the reader is referred to the following good books [35, 21, 36, 37, 39, 26, 7].

0.1 Algebra

In this section the definitions and basic results about groups, fields and vector spaces are collected. All the definitions and results here can be found in an algebra reference such as [35]. This information is necessary to discuss projective spaces. Finite groups are discussed first.

Definition 0.1.1 *Let G be a set and let $*$ be a binary operation on G. The set G is called a* group *under $*$ if*

- $a * b \in G$ *for all* a, $b \in G$,
- $a * (b * c) = (a * b) * c$ *for all* a, b, $c \in G$,
- *There exists an element* $e \in G$ *such that* $a * e = e * a = a$ *for all* $a \in G$,

- *For all $a \in G$, there exists $a' \in G$ such that $a' * a = a * a' = e$.*

The group G is called abelian *if also*

- *$a * b = b * a$ for all a, $b \in G$.*

In the groups considered in this thesis, the operation $*$ is either addition $(+)$, multiplication $(\cdot)$ or composition of functions. The element e in the definition above is unique and is called the *identity* of G. The unique element a' such that $a * a' = e$ is called the *inverse* of a.

For a group under addition, the identity is denoted by 0, and the inverse of a is denoted by $-a$. The element $a + (-b)$ is written $a - b$. The element $a + a + \cdots + a$ (n times) is denoted by na, and the element $-a - a - \cdots - a$ (n times) is denoted by $-na$.

For a group under multiplication, the identity is denoted by 1, and the inverse of a is denoted by a^{-1}. The element $a \cdot b$ is often written ab. The element $aa \ldots a$ (n times) is denoted by a^n, and the element $a^{-1}a^{-1} \ldots a^{-1}$ is denoted by a^{-n}. It is a convention to write $a^0 = 1$. These elements are called the *powers of* a. Composition of functions is generally written as multiplication, with the same terminology and notation. The identity function is usually denoted by ι.

If the set G is finite, then the group is called a *finite group*. All groups considered in this thesis are finite groups. If G is a finite group, then the size $|G|$ of G is called the *order* of G.

Definition 0.1.2 *Let G be a group under the operation $*$ and let H be a subset of G. The subset H is called a* subgroup *of G if H is a group under $*$.*

An important result about subgroups is Lagrange's theorem.

Theorem 0.1.3 [35] *Let H be a subgroup of the finite group G. Then the order of H divides the order of G.*

Given a particular element g of a group G, it is possible to form a subgroup using g. To describe this subgroup, it is convenient to assume G is a group under multiplication. Then the *subgroup generated by* g is $\langle g \rangle = \{g^m \mid m \in \mathbb{N}\}$. This is indeed a subgroup, since $g^m g^n = g^{m+n}$. The order of this subgroup is the smallest number n such that $g^n = 1$ and is called the *order* of g. This leads to the following corollary of Lagrange's theorem:

Corollary 0.1.4 [35] *Let g be an element of the finite group G. Then the order of g divides $|G|$.*

The subgroup generated by an element is a special kind of group called a cyclic group.

Definition 0.1.5 *The group G is called* cyclic *if $G = \langle g \rangle$ for some $g \in G$. The element g is called a generator of G.*

A cyclic group is necessarily abelian, and it has easily determined subgroups.

Lemma 0.1.6 [35] *Let G be a finite cyclic group. Then for each natural number d dividing $|G|$, there is a unique subgroup of G of order d.*

If $G = \langle g \rangle$ has order n, then the unique subgroup of order d is $\langle g^{\frac{n}{d}} \rangle$.

This completes the discussion on finite groups. Finite fields are discussed next.

Definition 0.1.7 *A set F with binary operations $+$ and $\cdot$ is called a* field *if*

- *F is an abelian group under $+$,*
- *$F \setminus \{0\}$ is an abelian group under $\cdot$ (where 0 is the identity of F under $+$),*
- *$a(b + c) = ab + ac$ and $(a + b)c = ac + bc$ for all a, b, $c \in F$.*

The field F is called a *finite field* if F is a finite set. Only finite fields are considered in this thesis. The size of F is called the *order* of F. In addition to the conventions listed for groups, the element $n1$ is denoted simply by n. The smallest number n such that $n = 0$ is called the *characteristic* of F. For finite fields, the characteristic is always a prime number.

Definition 0.1.8 *Let E be a field and let F be a subset of E. The subset F is called a* subfield *of E if F is a field under the same operations as E. The field E is called an* extension *of F.*

An important theorem about subfields is the following.

Theorem 0.1.9 [35] *Let E be a field and let F be a subfield of E. Then E is a vector space over F, with scalar multiplication the multiplication in E.*

For the definition of vector space, see Definition 0.1.17. The set of elements $\{n \mid n \in \mathbb{N}\}$ is the smallest possible subfield of a finite field F, and necessarily has prime order. It follows that a finite field has prime power order.

At this point, it is appropriate to describe how different fields are distinguished.

Definition 0.1.10 *Let F_1 and F_2 be two fields. An* isomorphism *between F_1 and F_2 is a one-to-one function $\phi : F_1 \to F_2$ such that $\phi(a + b) = \phi(a) + \phi(b)$ and $\phi(ab) = \phi(a)\phi(b)$ for all a, $b \in F_1$. If there exists an isomorphism between F_1 and F_2, the two fields are said to be* isomorphic*. An isomorphism $\phi : F \to F$ is called an* automorphism*.*

Theorem 0.1.11 [35] *If q is a prime power, then all finite fields of order q are isomorphic.*

The above theorem implies that there is essentially a unique finite field of each prime power order. This field is denoted by $\mathrm{GF}(q)$. Some properties of $\mathrm{GF}(q)$ are listed below.

Lemma 0.1.12 [35] *The group* $GF(q) \setminus \{0\}$ *under* $\cdot$ *is cyclic.*

Corollary 0.1.13 [35] *If* a *is an element of* $\mathrm{GF}(q)$*, then* $a^q = a$.

Lemma 0.1.14 [35] *The set of elements* a *of* $\mathrm{GF}(q^2)$ *such that* $a^q = a$ *is the subfield* $\mathrm{GF}(q)$.

Lemma 0.1.15 [35] *Let* $f(x) = a_0 + a_1 x + \cdots + a_r x^r$ *be a polynomial with coefficients* $a_0, a_1, \ldots, a_r \in \mathrm{GF}(q)$. *Then* f *has at most* r *zeros in* $\mathrm{GF}(q)$ *unless* $a_0 = \cdots = a_r = 0$*, in which case every element of* $\mathrm{GF}(q)$ *is a zero of* F.

Lemma 0.1.16 [35] *Let* $f(x) = a_0 + a_1 x + \cdots + a_r x^r$ *be a polynomial with coefficients* $a_0, a_1, \ldots, a_r \in \mathrm{GF}(q)$ *such that* $r > 0$ *and* $a_r \neq 0$. *There exists an extension field* F *of* $\mathrm{GF}(q)$ *such that* $f(x)$ *has* r *zeros in* F.

If the zeros of f are $z_1, \ldots, z_r$, then it is possible to write $f(x) = (x - z_1) \ldots (x - z_r)$. The polynomial f is said to factorise *over* F. In this thesis, the particular interest is in quadratic polynomials of the form $f(x) = a_0 + a_1 x + a_2 x^2$. If f has no zeros in $\mathrm{GF}(q)$, then the extension field containing the zeros of f is isomorphic to $\mathrm{GF}(q^2)$.

Similar terminology is applied to quadratic equations in several variables. Consider the *homogeneous quadratic equation*

$$\phi(x_0, \ldots, x_n) = a_{00} x_0^2 + \cdots + a_{nn} x_n^2 + \sum_{i<j} a_{ij} x_i x_j = 0$$

where $a_{ij} \in \mathrm{GF}(q)$ for all $0 \leq i, j \leq n$ and not all a_{ij} are equal to 0

The expression ϕ may factorise into two linear expressions. If this is possible using coefficients from $\mathrm{GF}(q)$, then ϕ is said to factorise *over* $\mathrm{GF}(q)$. If this is only possible using coefficients from an extension field, then this extension field is isomorphic to $\mathrm{GF}(q^2)$ and ϕ is said to factorise *over* $\mathrm{GF}(q^2)$. If it is not possible to factorise ϕ even over $\mathrm{GF}(q^2)$, then ϕ is said to be *irreducible.*

Homogeneous quadratic equations will be used to define quadrics in Chapter 1.

This completes the discussion of finite fields. Finite vector spaces are discussed next.

Definition 0.1.17 *A set V with a binary operation $+$ is a* vector space *over the field K if the following axioms hold.*

- *V is an abelian group under $+$,*
- *$a\mathbf{v} \in V$ for all $a \in K$ and $\mathbf{v} \in V$,*
- *$a(b\mathbf{v}) = (ab)\mathbf{v}$ for all a, $b \in K$ and $\mathbf{v} \in V$,*
- *$(a+b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$ for all a, $b \in K$ and $\mathbf{v} \in V$,*
- *$a(\mathbf{v}+\mathbf{w}) = a\mathbf{v} + a\mathbf{w}$ for all $a \in K$ and $\mathbf{v}$, $\mathbf{w} \in V$,*
- *$0\mathbf{v} = \mathbf{0}$ and $1\mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in V$, where 0 is the identity of K under $+$, 1 is the identity of K under $\cdot$ and $\mathbf{0}$ is the identity of V under $+$.*

The elements of V are called *vectors* and the elements of K are called *scalars*. The operation $a\mathbf{v}$ for $a \in K$ and $\mathbf{v} \in V$ is called *scalar multiplication*. Note that the symbol $+$ is used to represent addition in both K and V, and juxtaposition is used to represent both multiplication in K and scalar multiplication.

A *linear combination* of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$ is a vector of the form $a_1\mathbf{v}_1 + \cdots + a_m\mathbf{v}_m$ for $a_1, \ldots, a_m \in K$. The vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$ are *linearly independent* if the only solution to the equation $x_1\mathbf{v}_1 + \cdots + x_m\mathbf{v}_m = \mathbf{0}$ is $x_1 = \cdots = x_m = 0$. The set of all linear combinations of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$ is called the *span* of $\mathbf{v}_1, \ldots, \mathbf{v}_m$ and is denoted by $\langle \mathbf{v}_1, \ldots, \mathbf{v}_m \rangle$. The span of a set of vectors forms a subspace of a vector space, which is now defined.

Definition 0.1.18 *Let V be a vector space over the field K. A subset W is called a* subspace *of V if W is a vector space over K under the same operations.*

To show that the subset W is a subspace, it is sufficient to show that any linear combination of vectors in W is also a vector in W. If the subspace $W = \langle \mathbf{v}_1, \ldots, \mathbf{v}_m \rangle$ for vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$, then these vectors are said to *span* W. A *basis* for the vector space V is a set of linearly independent vectors B such that B spans V. The following theorem describes the important properties of a basis.

Theorem 0.1.19 [35] *Let V be a vector space over the field K and suppose $B = \{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ is a basis for V. Then every basis of V has m elements and every element $\mathbf{v} \in V$ can be written uniquely as $\mathbf{v} = a_1\mathbf{u}_1 + \cdots + a_m\mathbf{u}_m$ for some $a_1, \ldots, a_m \in K$.*

The number of vectors in a basis for V is called the *dimension* of V over K. The vector spaces considered in this thesis are always finite, so a finite basis always exists. Note that the dimension of V is the smallest number m such that m vectors span V.

The above theorem implies that if V has dimension m over $\mathrm{GF}(q)$, then $|V| = q^m$. In fact, all vector spaces of the same dimension over $\mathrm{GF}(q)$ are essentially the same vector space.

Apart from finite fields themselves, the vector spaces considered in this thesis are sets of n-tuples with coordinates from $\mathrm{GF}(q)$. Here, addition and scalar multiplication are defined in the natural way as follows.

$$(a_1, \ldots, a_m) + (b_1, \ldots, b_m) = (a_1 + b_1, \ldots, a_m + b_m)$$
$$k(a_1, \ldots, a_m) = (ka_1, \ldots, ka_m)$$

If the set of vectors is written as $V = \{(x_1, \ldots, x_m) \mid x_1, \ldots, x_m \in \mathrm{GF}(q)\}$, then it is possible to describe subspaces using linear equations in $x_1, \ldots, x_m$. In fact, the following theorem is true.

Theorem 0.1.20 [35] *Let $V = \{(x_1, \ldots, x_m) \mid x_1, \ldots, x_m \in \mathrm{GF}(q)\}$ be the vector space of m-tuples with coordinates from $\mathrm{GF}(q)$. A subset W of V is a subspace of V if and only if W is the set of solutions to some number of homogeneous linear equations in $x_1, \ldots, x_m$.*

For example, a single linear equation $a_1x_1 + \cdots + a_mx_m = 0$ describes a subspace of dimension $m - 1$.

This completes the discussion of finite vector spaces.

0.2 Projective Spaces

This section presents definitions and results for finite projective spaces.

Definition 0.2.1 *Let $\bar{\Sigma}$ be a finite set of elements called* points *and let n be an integer with $n \geq 2$. Distinguish certain subsets of $\bar{\Sigma}$ called* subspaces *and for each subspace Π define an integer* $\dim(\Pi)$*, called the* dimension of Π*, such that* $-1 \leq \dim(\Pi) \leq n$*. The set $\bar{\Sigma}$, together with its subspaces, is called a* finite projective space of dimension n *if the following axioms hold.*

- *There exists a subspace of dimension d for every integer d with $-1 \leq d \leq n$.*
- *The set $\bar{\Sigma}$ is the unique subspace of dimension n.*
- *The empty set is the unique subspace of dimension -1.*
- *The subspaces of dimension 0 are the points of $\bar{\Sigma}$.*
- *The intersection of two subspaces is a subspace.*
- *If Π and Λ are subspaces with $\Pi \subseteq \Lambda$, then $\dim(\Pi) \leq \dim(\Lambda)$ and $\dim(\Pi) = \dim(\Lambda)$ if and only if $\Pi = \Lambda$.*
- *If Π and Λ are subspaces and $\Pi \oplus \Lambda$ is the intersection of all the subspaces containing both Π and Λ, then $\dim(\Pi \oplus \Lambda) + \dim(\Pi \cap \Lambda) = \dim(\Pi) + \dim(\Lambda)$.*
- *There exists a subspace of dimension 1 containing at least 3 points.*

Let $\bar{\Sigma}$ be a finite projective space of dimension n. A subspace of $\bar{\Sigma}$ of dimension m is called an m-space. An $(n-1)$-space is called a *hyperplane*, a 1-space is called a *line* and a 2-space is called a *plane*. Note that for $m \geq 2$, an m-space is a finite projective space of dimension m.

Let Π and Λ be subspaces of $\bar{\Sigma}$. The subspace $\Pi \oplus \Lambda$ is called the *span* of Π and Λ. The subspaces Π and Λ are said to *span* $\Pi \oplus \Lambda$.

If $\Pi \subseteq \Lambda$, then Λ is said to *pass through* Π. If P is a point in Π, then P is said to *lie in* or *lie on* Π. If the line ℓ is contained in Π, it is said to *lie in* Π.

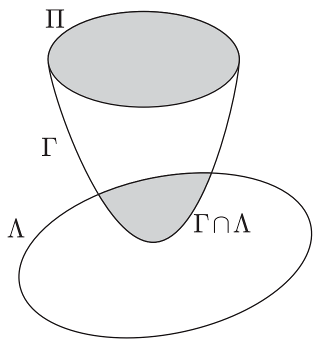
A set of points is said to be *collinear* if they lie on the same line, and *coplanar* if they lie in the same plane. A set of lines are said to be *coplanar* if they lie in the same plane, and *concurrent* if they pass through the same point.

If $\Pi \cap \Lambda$ is a d-space, then Π and Λ are said to *meet* in a d-space. If $\Pi \cap \Lambda$ is the empty set, then Π and Λ are said to be *skew*. A set of subspaces is called *mutually skew* if every pair of them are skew.

If Π and Λ are skew subspaces and also $\Pi \oplus \Lambda = \bar{\Sigma}$, then Π and Λ are said to be *complementary subspaces*. The subspace Λ is said to be a *complementary subspace of* Π. Note that if Π and Λ are complementary subspaces, then $\dim(\Pi)+\dim(\Lambda) = n-1$.

If P and Q are distinct points, then the subspace $P \oplus Q$ is a line, which is usually denoted by PQ. The line PQ is said to *join* P and Q.

Some important results about subspaces are now given.



Lemma 0.2.2 [7] *Let Π be a d-space of $\bar{\Sigma}$ and let Λ be a complementary subspace of Π. Suppose Γ is a $(d+m)$-space through Π for $m \geq 0$. Then Γ meets Λ in an $(m-1)$-space.*

Corollary 0.2.3 [7] *Let Σ be a hyperplane and let Π be a d-space that is not contained in Σ. Then Π meets Σ in a $(d-1)$-space. In particular, every line not contained in Σ meets Σ in a unique point and every hyperplane other than Σ meets Σ in an $(n-2)$-space.*

Lemma 0.2.4 [7] *Suppose Π and Λ are distinct non-empty subspaces of $\bar{\Sigma}$. Then $\Pi \oplus \Lambda$ is the set of all points on the lines joining a point of Π to a point of Λ.*

Lemma 0.2.5 [7] *A subset A of $\bar{\Sigma}$ is a subspace if and only if every line containing two points of A is contained in A.*

This completes the basic properties of projective spaces and their subspaces. A particular example of a finite projective space will now be given using a vector space.

Definition 0.2.6 *Let V be a vector space with dimension $n+1$ over* $\mathrm{GF}(q)$*, for $n \geq 2$. Then* $\mathrm{PG}(n,q)$ *is the finite projective space whose d-spaces are the $(d+1)$-dimensional subspaces of V for $-1 \leq d \leq n$.*

It is important to note that the vector $\mathbf{0}$ is the 0-dimensional subspace of V and corresponds to the empty subspace in $\mathrm{PG}(n,q)$. Also note that it is possible to define $\mathrm{PG}(0,q)$ and $\mathrm{PG}(1,q)$ in a similar way, but these are not technically projective spaces according to Definition 0.2.1. In this thesis, $\mathrm{PG}(n,q)$ always has dimension $n \geq 2$.

The number q is called the *order* of $\mathrm{PG}(n,q)$, and $\mathrm{PG}(n,q)$ is a finite projective space of dimension n. In fact, for $n \geq 3$, it is the *only* finite projective space.

Theorem 0.2.7 [26] *Let $\bar{\Sigma}$ be a finite projective space of dimension n. If $n \geq 3$, then $\bar{\Sigma}$ is isomorphic to* $\mathrm{PG}(n,q)$ *for some q.*

For the definition of isomorphism between projective spaces, see Definition 0.3.2. The above theorem means that projective spaces (of dimension at least 3) can be defined equally well axiomatically using Definition 0.2.1 or algebraically using Definition 0.2.6. Note that while there exist projective planes other than $\mathrm{PG}(2,q)$, these are not considered in this thesis. Some notational conventions are now introduced.

Let V be the vector space of $(n+1)$-tuples with coordinates from $\mathrm{GF}(q)$ and with addition and scalar multiplication defined in the natural way. The points of $\mathrm{PG}(n,q)$ are the 1-dimensional subspaces of V. That is, a point of $\mathrm{PG}(n,q)$ is of the form $\langle \mathbf{v} \rangle = \{\lambda \mathbf{v} \mid \lambda \in \mathrm{GF}(q)\}$ for some $\mathbf{v} \in V$. If points P and Q are represented by $\langle \mathbf{v} \rangle$ and $\langle \mathbf{w} \rangle$, then the line PQ is the vector subspace $\langle \mathbf{v}, \mathbf{w} \rangle$.

It is more convenient to speak of a point of $\mathrm{PG}(n,q)$ as an $(n+1)$-tuple. That is, $\mathrm{PG}(n,q)$ is the set of points $(x_0, \ldots, x_n)$ with $x_0, \ldots, x_n \in \mathrm{GF}(q)$, not all zero, with the identification that for $\lambda \in \mathrm{GF}(q) \setminus \{0\}$, the vector $(\lambda x_0, \ldots, \lambda x_n)$ represents the same point as $(x_0, \ldots, x_n)$. It P is a point, then the symbol P is often used to represent both the point and also any particular coordinates representing P. With this convention, if P and Q are distinct points of $\mathrm{PG}(n, q)$, then the line PQ consists of the points $\lambda P + \mu Q$ for λ, $\mu \in \mathrm{GF}(q)$, not both zero.

The points $P_0, \ldots, P_m$ are linearly independent if their coordinate vectors are linearly independent. A $(d+1)$-dimensional subspace of V is the span of $d+1$ linearly independent vectors, so a d-space of $\mathrm{PG}(n, q)$ is the span of $d+1$ linearly independent points.

A hyperplane of $\mathrm{PG}(n, q)$ is a subspace of V of dimension n. It is therefore the span of n linearly independent points. The whole space $\mathrm{PG}(n, q)$ is itself the span of $n+1$ linearly independent points. A hyperplane can also be defined as the set of solutions to a homogeneous linear equation $a_0x_0 + \cdots + a_nx_n = 0$. The vector $[a_0, \ldots, a_n]$ is called the coordinates of the hyperplane. Note that any scalar multiple of these coordinates represents the same hyperplane. A vector subspace of V is the set of solutions to several linear equations, so a subspace of $\mathrm{PG}(n, q)$ is also the set of solutions to several linear equations.

The fact that points and hyperplanes have similar representations leads to the concept of duality for a projective space.

Definition 0.2.8 *The* dual space *of* $\mathrm{PG}(n, q)$ *is denoted by* $\mathrm{PG}(n, q)^*$ *and is the projective space whose points are the hyperplanes of* $\mathrm{PG}(n, q)$*. For each integer* d *with* $-1 \leq d \leq n$*, the* d*-spaces of* $\mathrm{PG}(n, q)^*$ *are the* $(n-1-d)$*-spaces of* $\mathrm{PG}(n, q)$*. If* Π *and* Λ *are two subspaces of* $\mathrm{PG}(n, q)^*$*, then* $\Pi \subseteq \Lambda$ *as subspaces of* $\mathrm{PG}(n, q)^*$ *if and only if* $\Lambda \subseteq \Pi$ *as subspaces of* $\mathrm{PG}(n, q)$*.*

The dual space $\mathrm{PG}(n, q)^*$ is indeed a projective space. In fact, $\mathrm{PG}(n, q)^*$ is isomorphic to $\mathrm{PG}(n, q)$. If π is a projective plane other than $\mathrm{PG}(2, q)$, then the dual plane π^* is the projective plane whose points are the lines of π and whose lines are the points of π. It is important to note that π^* is not necessarily isomorphic to π, although $\mathrm{PG}(2, q)^*$ is isomorphic to $\mathrm{PG}(2, q)$.

The above theorem implies that for any statement true of d_1-spaces, $\ldots$, d_k-spaces of $\mathrm{PG}(n, q)$, there is a corresponding dual statement true of $(n-1-d_1)$-spaces, $\ldots$, $(n-1-d_k)$-spaces of $\mathrm{PG}(n, q)$. This is the *principle of duality* for projective spaces.

The final results in this section concern the numbers of various subspaces in $\mathrm{PG}(n, q)$ and relationships between these numbers.

Lemma 0.2.9 [36] *The number of points in an m-space of* $\mathrm{PG}(n,q)$ *is* $\dfrac{q^{m+1}-1}{q-1}$.

In particular, a line contains $q+1$ *points and a plane contains* q^2+q+1 *points.*

The number $\dfrac{q^{m+1}-1}{q-1}$ is denoted by θ_m. The value of q is to be understood by context. The following identities will be useful.

Lemma 0.2.10 [36]

- $\theta_{-1}=0$, $\theta_0=1$, $\theta_1=q+1$ *and* $\theta_m=q^m+q^{m-1}+\cdots+1$ *for* $m\geq 2$.
- $\theta_m(q-1)=q^{m+1}-1$.
- $q\theta_m+1=\theta_{m+1}$.
- θ_k *divides* θ_m *if and only if* $k+1$ *divides* $m+1$. *In particular,* $q+1$ *divides* θ_m *if and only if* m *is odd.*
- *If* q *is even, then* θ_m *is odd for* $m\geq 0$. *If* q *is odd, then* θ_m *is odd for* m *even and even for* m *odd.*

The following lemmas give some results involving various numbers of subspaces in $\mathrm{PG}(n,q)$.

Lemma 0.2.11 [36]

$$\textit{In } \mathrm{PG}(n,q) \textit{ the number of lines is } \frac{\theta_n\theta_{n-1}}{q+1}$$
$$\textit{and the number of planes is } \frac{\theta_n\theta_{n-1}\theta_{n-2}}{(q+1)(q^2+q+1)}.$$

Lemma 0.2.12 [36] *In* $\mathrm{PG}(n,q)$, *the number of d-spaces is the same as the number of* $(n-d-1)$*-spaces for each dimension d. Also, for each dimension d and for each integer m with* $0\leq m\leq n-d$, *the number of* $(d+m)$*-spaces through a d-space is the same as the number of* $(m-1)$*-spaces in an* $(n-d-1)$*-space.*

Corollary 0.2.13 [36]

$$\textit{In } \mathrm{PG}(n,q) \textit{ the number of hyperplanes is } \theta_n$$
$$\textit{and the number of } (n-2)\textit{-spaces is } \frac{\theta_n\theta_{n-1}}{q+1}.$$

Corollary 0.2.14 [36] *In* PG(n,q) *the number of* $(m+1)$*-spaces through an* m *space is* θ_{n-m-1}*. In particular, the number of lines through a point is* θ_{n-1}*, the number of planes through a line is* θ_{n-2}*, the number of 3-spaces through a plane is* θ_{n-3} *and the number of hyperplanes through an* $(n-2)$*-space is* $q+1$*.*

The following corollaries calculate the numbers of various subspaces in PG(n,q) for $n \leq 4$.

Corollary 0.2.15 [36] *In* PG$(2,q)$,

- *The number of points is* q^2+q+1,
- *The number of lines is* q^2+q+1,
- *The number of lines through a point is* $q+1$.

Corollary 0.2.16 [36] *In* PG$(3,q)$,

- *The number points is* $q^3+q^2+q+1 = (q+1)(q^2+1)$,
- *The number of lines is* $q^4+q^3+2q^2+q+1 = (q^2+1)(q^2+q+1)$,
- *The number of planes is* q^3+q^2+q+1,
- *The number of lines through a point is* q^2+q+1,
- *The number of planes through a point is* q^2+q+1,
- *The number of planes through a line is* $q+1$.

Corollary 0.2.17 [36] *In* PG$(4, q)$,

- *The number of points is* $q^4 + q^3 + q^2 + q + 1$,
- *The number of lines is* $q^6+q^5+2q^4+2q^3+2q^2+q+1 = (q^2+1)(q^4+q^3+q^2+q+1)$,
- *The number of planes is* $q^6 + q^5 + 2q^4 + 2q^3 + 2q^2 + q + 1$,
- *The number of 3-spaces is* $q^4 + q^3 + q^2 + q + 1$,
- *The number of lines through a point is* $q^3 + q^2 + q + 1$,
- *The number of planes through a point is* $q^4 + q^3 + 2q^2 + q + 1$,
- *The number of 3-spaces through a point is* $q^3 + q^2 + q + 1$,
- *The number of planes through a line is* $q^2 + q + 1$,
- *The number of 3-spaces through a line is* $q^2 + q + 1$,
- *The number of 3-spaces through a plane is* $q + 1$.

Finally, the following characterisation of subspaces is given.

Theorem 0.2.18 [10] *Let A be a set of points of* PG(n, q) *such that every m-space meets A in at least one point. Then $|A| \geq \theta_{n-m}$ and $|A| = \theta_{n-m}$ if and only if A is an $(n - m)$-space. In particular, a set of size θ_{n-1} that meets every line of* PG(n, q) *is a hyperplane.*

0.3 Collineations and Polarities

Definition 0.3.1 *A* collineation *of* PG(n, q) *is a one-to-one map g that sends subspaces of* PG(n, q) *to subspaces of* PG(n, q) *and such that if Π and Λ are subspaces with $\Pi \subseteq \Lambda$, then $g(\Pi) \subseteq g(\Lambda)$.*

Lemma 0.2.5 states that a subspace is a set of points A such that if P and Q are distinct points of A, then the line PQ is contained in A. So a collineation may be defined as a one-to-one map from points of $\mathrm{PG}(n, q)$ to points of $\mathrm{PG}(n, q)$ such that if ℓ is a line of $\mathrm{PG}(n, q)$, then $g(\ell) = \{g(P) \mid P \in \ell\}$ is also a line of $\mathrm{PG}(n, q)$.

The set of all collineations of $\mathrm{PG}(n, q)$ forms a group under composition of functions. The identity of this group is the map that sends every point of $\mathrm{PG}(n, q)$ to itself. This map is called the *identity collineation* and is denoted by ι.

It is also possible to define a collineation from one projective space to another.

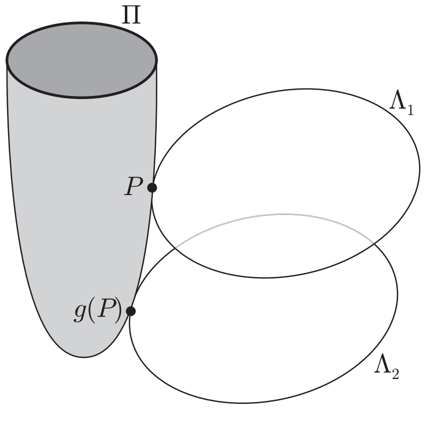
Definition 0.3.2 *Let $\bar{\Sigma}_1$ and $\bar{\Sigma}_2$ be finite projective spaces. A* collineation *from $\bar{\Sigma}_1$ to $\bar{\Sigma}_2$ is a one-to-one map g sending subspaces of $\bar{\Sigma}_1$ to subspaces of $\bar{\Sigma}_2$ such that if Π and Λ are subspaces of $\bar{\Sigma}_1$ with $\Pi \subseteq \Lambda$, then $g(\Pi) \subseteq g(\Lambda)$. If there exists a collineation from $\bar{\Sigma}_1$ to $\bar{\Sigma}_2$, then $\bar{\Sigma}_1$ and $\bar{\Sigma}_2$ are said to be* isomorphic

Again, a collineation may be defined as a one-to-one map g from points of $\bar{\Sigma}_1$ to points of $\bar{\Sigma}_2$ such that if ℓ is a line of $\bar{\Sigma}_1$, then $g(\ell)$ is a line of $\bar{\Sigma}_2$. A collineation between projective spaces is analogous to an isomorphism of fields, which is why two projective spaces are called isomorphic when there is a collineation between them. A collineation of $\mathrm{PG}(n, q)$ is analogous to an automorphism of fields.

If a collineation exists between two sets of points in $\mathrm{PG}(n, q)$, then any theorem true of one will also be true of the other. That is, if X and Y are sets of points in $\mathrm{PG}(n, q)$ and there exists a collineation g such that $g(X) = \{g(P) \mid P \in X\} = Y$, then a theorem true of X will also be true of Y. The sets X and Y are said to be *projectively equivalent*. The concept of projective equivalence can be used to assign coordinates to $\mathrm{PG}(n, q)$ in a convenient way.

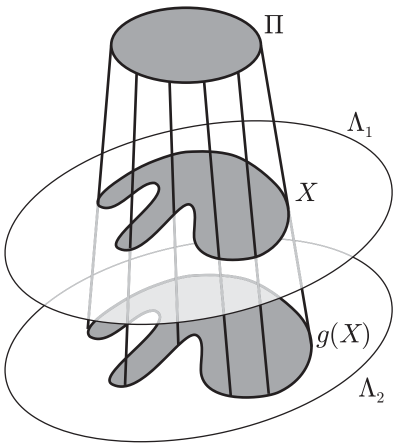
Theorem 0.3.3 [36] *Let $\{P_0, \ldots, P_{n+1}\}$ be an ordered set of $n+2$ points in $\mathrm{PG}(n, q)$ such that any $n+1$ are linearly independent. Let $\{Q_0, \ldots, Q_{n+1}\}$ be another ordered set of $n + 2$ points in $\mathrm{PG}(n, q)$ such that any $n + 1$ are linearly independent. Then there exists a collineation h such that $h(P_i) = Q_i$ for $i = 0, \ldots, n + 1$.*

Using the above theorem, the set of points $\{P_0, \ldots, P_{n+1}\}$ may be chosen without loss of generality to be *any* set of $n + 2$ points such that any $n + 1$ are linearly independent. This assignment of coordinates makes it possible to use coordinates to prove theorems in a straightforward manner.



Some particular collineations will now be described. The first are collineations between the subspaces of $\mathrm{PG}(n,q)$. Let Π be a subspace of $\mathrm{PG}(n,q)$, for $n \geq 3$, with dimension at most $(n-3)$ and let Λ_1 and Λ_2 be distinct complementary subspaces of Π. Let P be a point of Λ_1. Then the subspace $\Pi \oplus P$ has dimension one greater than Π. By Lemma 0.2.2, the subspace $\Pi \oplus P$ meets Λ_2 in a unique point. Define the map $g : \Lambda_1 \to \Lambda_2$ by $g(P) = (\Pi \oplus P) \cap \Lambda_2$.

If ℓ is a line in Λ_1, then the image of ℓ is the intersection of the subspace $\Pi \oplus \ell$ with Λ_2, which is a line of Λ_2 by Lemma 0.2.2. So the map g is a collineation from Λ_1 to Λ_2. This collineation is called a *projection* from Π.



If X is a set of points in Λ_1, then the set of points $g(X) = \{(\Pi \oplus P) \cap \Lambda_2 \mid P \in X\}$ is called a *projection* of X. The projection map is also defined if Λ_1 and Λ_2 are lines or points, but the map is not technically a collineation in this case. However, the two sets X and $g(X)$ are still projectively equivalent when considered as sets in $\mathrm{PG}(n,q)$. These observations will be important in the discussion of cones in Section 1.3.

Before the next collineations are discussed, some terminology is introduced.

If P is a point of $\mathrm{PG}(n,q)$, then a collineation g is said to *fix* P if $g(P) = P$. The point P is then called a *fixed point* of g. If X is any set of points in $\mathrm{PG}(n,q)$, then the collineation g is said to *fix* X if $g(X) = X$. That is, $g(P) \in X$ for every $P \in X$. This is not the same as saying that $g(P) = P$ for every point $P \in X$. If this is the case, then X is said to be *fixed pointwise* by g.

Given a group G of collineations, the set of images of a point under the collineations of G is also of interest.

Definition 0.3.4 *Let G be a group of collineations of* $\mathrm{PG}(n,q)$ *and let P be a point of* $\mathrm{PG}(n,q)$*. The set of points $G(P) = \{g(P) \mid g \in G\}$ is called an* orbit of G.

Theorem 0.3.5 [36, Chapter 2] *Let G be a group of collineations of* $\mathrm{PG}(n,q)$*. The orbits of G partition the points of* $\mathrm{PG}(n,q)$*. If X is an orbit of G, then $X = G(P)$ for every point $P \in X$. Also, $|X|$ divides $|G|$.*

With the above terminology in hand, the next type of collineation will be discussed.

Definition 0.3.6 *Let S be a cyclic group of collineations of* $\mathrm{PG}(n,q)$ *such that for every pair of points P, $Q \in \mathrm{PG}(n,q)$ there exists a unique collineation $g \in S$ with $g(P) = Q$. Then S is called a* cyclic Singer group *of* $\mathrm{PG}(n,q)$*.*

There do exist cyclic Singer groups for every $\mathrm{PG}(n,q)$. This is proved by representing $\mathrm{PG}(n,q)$ as the $(n+1)$-dimensional vector space $\mathrm{GF}(q^{n+1})$ over $\mathrm{GF}(q)$. Then $\mathrm{GF}(q^{n+1}) \setminus \{0\}$ is a cyclic group under multiplication. Writing $\mathrm{GF}(q^{n+1}) \setminus \{0\} = \langle \beta \rangle$, the map $g : P \mapsto \beta \cdot P$ is a collineation and $\langle g \rangle$ is a cyclic Singer group. Note that a cyclic Singer group necessarily has order θ_n and fixes no points of $\mathrm{PG}(n,q)$.

Since a cyclic Singer group S is cyclic, it has a unique subgroup of order d for every d dividing θ_n. In particular, a cyclic Singer group of $\mathrm{PG}(3,q)$ has unique subgroups of orders $(q+1)$ and (q^2+1), since $\theta_3 = q^3+q^2+q+1 = (q+1)(q^2+1)$. The orbits of these subgroups are described by Lemmas 0.4.11 and 7.5.3. Note that if G is a subgroup of a cyclic Singer group of $\mathrm{PG}(n,q)$ and P is a point of $\mathrm{PG}(n,q)$, then the orbit $G(P)$ has size $|G|$, since each collineation in G sends P to a distinct point. In fact, if X is a set of size $|G|$ such that $g(X) = X$ for each $g \in G$, then X contains $G(P)$ for any point $P \in X$, so X is an orbit of G.

The last types of collineations to be discussed are collineations between $\mathrm{PG}(n,q)$ and its dual space $\mathrm{PG}(n,q)^*$. Such maps are called *correlations*.

Definition 0.3.7 *Let σ be a one-to-one map from subspaces of* $\mathrm{PG}(n,q)$ *to subspaces of* $\mathrm{PG}(n,q)$ *such that if Π and Λ are subspaces with $\Pi \subseteq \Lambda$, then $\sigma(\Lambda) \subseteq \sigma(\Pi)$. Then σ is called a* correlation *of* $\mathrm{PG}(n,q)$*.*

A correlation σ necessarily interchanges points and hyperplanes, and in general if Π is a d-space then $\sigma(\Pi)$ is an $(n-1-d)$-space. So, a correlation can be thought of as a collineation between $\mathrm{PG}(n,q)$ and $\mathrm{PG}(n,q)^*$. If Π and Λ are subspaces of $\mathrm{PG}(n,q)$ and σ is a correlation, then the definition of correlation also implies that $\sigma(\Pi \oplus \Lambda) = \sigma(\Pi) \cap \sigma(\Lambda)$ and $\sigma(\Pi \cap \Lambda) = \sigma(\Pi) \oplus \sigma(\Lambda)$.

Using correlations of $\mathrm{PG}(n,q)$ it is possible to construct collineations of $\mathrm{PG}(n,q)$, as the following lemma states.

Lemma 0.3.8 [24] *Let σ and ρ be two correlations of $\mathrm{PG}(n,q)$. Then the map $\sigma\rho : \mathrm{PG}(n,q) \to \mathrm{PG}(n,q)$ is a collineation.*

Proof Let Π and Λ be subspaces of $\mathrm{PG}(n,q)$ with $\Pi \subseteq \Lambda$. Then $\rho(\Pi)$ and $\rho(\Lambda)$ are subspaces of $\mathrm{PG}(n,q)$ such that $\rho(\Lambda) \subseteq \rho(\Pi)$. Thus $\sigma\rho(\Pi)$ and $\sigma\rho(\Lambda)$ are subspaces of $\mathrm{PG}(n,q)$ such that $\sigma\rho(\Pi) \subseteq \sigma\rho(\Lambda)$. Thus $\sigma\rho$ is a collineation of $\mathrm{PG}(n,q)$. □

Certain special correlations are of particular interest.

Definition 0.3.9 *Let σ be a correlation of $\mathrm{PG}(n,q)$ such that $\sigma^2 = \iota$. That is, $\sigma(\sigma(P)) = P$ for every point $P \in \mathrm{PG}(n,q)$. Then σ is called a* polarity *of* $\mathrm{PG}(n,q)$.

The definition of a polarity implies the following result.

Lemma 0.3.10 [36] *Let σ be a polarity of $\mathrm{PG}(n,q)$ and let P and Q be points of $\mathrm{PG}(n,q)$. Then $P \in \sigma(Q)$ if and only if $Q \in \sigma(P)$.*

The image of a point P under the polarity σ is called the *polar* of P under σ. If $P \in \sigma(P)$ for some point P, then P is called an *absolute point* of σ. Similarly, if Π is any subspace such that $\Pi \subseteq \sigma(\Pi)$, then Π is called *absolute*. A polarity such that every point is absolute is called a *null polarity*. The set of absolute lines of a null polarity in $\mathrm{PG}(3,q)$ is described in Section 0.4.

0.4 Special sets of lines

In this section, various special sets of lines in $\mathrm{PG}(3,q)$ are discussed. In order to do this, the Klein correspondence is introduced. For a more complete discussion of the Klein correspondence, and for proofs of the following statements, see Hirschfeld [37, Chapter 15].

Let $P = (p_0, p_1, p_2, p_3)$ and $Q = (q_0, q_1, q_2, q_3)$ be distinct points of $\mathrm{PG}(3,q)$ and let ℓ be the line PQ. The *Plücker coordinates* of the line PQ are

$$(p_0q_1 - p_1q_0,\ p_0q_2 - p_2q_0,\ p_0q_3 - p_3q_0,\ p_1q_2 - p_2q_1,\ p_3q_1 - p_1q_3,\ p_2q_3 - p_3q_2).$$

These coordinates can be interpreted as a point in $\mathrm{PG}(5,q)$. If the points P and Q on ℓ are chosen differently, then the resulting coordinates are a scalar multiple of the ones listed above and so represent the same point of $\mathrm{PG}(5,q)$. The above correspondence between lines of $\mathrm{PG}(3,q)$ and points of $\mathrm{PG}(5,q)$ is called the *Klein correspondence*.

If ℓ is a line of $\mathrm{PG}(3,q)$ with Plücker coordinates $(a_0, a_1, a_2, a_3, a_4, a_5)$, it is possible to show that $a_0a_5 + a_1a_4 + a_2a_3 = 0$. That is, the point $(a_0, a_1, a_2, a_3, a_4, a_5)$ of $\mathrm{PG}(5,q)$ satisfies the homogeneous quadratic equation $x_0x_5 + x_1x_4 + x_2x_3 = 0$. The set of points of $\mathrm{PG}(5,q)$ satisfying this equation is called the *Klein quadric* and there is a one-to-one correspondence between the points of the Klein quadric and the lines of $\mathrm{PG}(3,q)$.

The Klein quadric is a set of $q^4 + q^3 + 2q^2 + q + 1$ points in $\mathrm{PG}(5,q)$, which contains lines and planes. A plane contained in the Klein quadric corresponds either to the set of lines of $\mathrm{PG}(3,q)$ in a plane or to the set of lines of $\mathrm{PG}(3,q)$ through a point. A line contained in the Klein quadric corresponds to the set of $q+1$ lines of $\mathrm{PG}(3,q)$ through a point in a plane. Such a set of lines is called a *pencil* of lines.

Of particular interest are the sets of lines whose Plücker coordinates satisfy a homogeneous linear equation. These correspond to the intersection of a hyperplane of $\mathrm{PG}(5,q)$ with the Klein quadric. Such sets of lines are called *linear complexes*, and there are two distinct types.

A *special linear complex* is the set of lines of $\mathrm{PG}(3,q)$ meeting a particular line ℓ, including ℓ itself.

A *general linear complex* is a set $\mathscr{T}$ of lines of $\mathrm{PG}(3,q)$ such that every point of $\mathrm{PG}(3,q)$ lies on $q+1$ lines of $\mathscr{T}$, and every plane contains $q+1$ lines of $\mathscr{T}$. Also, the $q+1$ lines of $\mathscr{T}$ through a point are coplanar, and the $q+1$ lines of $\mathscr{T}$ in a plane are concurrent.

If σ is the map sending a point P to the plane formed by the lines of $\mathscr{T}$ through P, then σ is a null polarity. The lines of $\mathscr{T}$ are the absolute lines of σ. That is, if $\ell \in \mathscr{T}$, then $\sigma(\ell) = \ell$. In fact, the set of absolute lines of a null polarity is always a general linear complex, and σ is the unique null polarity with $\mathscr{T}$ as its set of absolute lines. The polarity σ is called the polarity *associated with* $\mathscr{T}$.

The intersection of two linear complexes is called a *linear congruence*. A linear congruence corresponds to the intersection of a 3-space of $\mathrm{PG}(5,q)$ with the Klein quadric. There are four distinct types of linear congruence.

A *degenerate linear congruence* is the set of lines either in a plane π or through a point P, where $P \in \pi$. A degenerate linear congruence is contained in $q+1$ special linear complexes and no general linear complexes.

An *elliptic linear congruence* is a special set of q^2+1 mutually skew lines of $\mathrm{PG}(3,q)$. These lines necessarily partition the points of $\mathrm{PG}(3,q)$. An elliptic linear congruence is contained in $q+1$ general linear complexes and no special linear complexes. It is important to note that every general linear complex contains many elliptic linear congruences.

A *parabolic linear congruence* is a special set of $q+1$ pencils, all sharing a common line. A parabolic linear congruence is contained in q general linear complexes and one special linear complex.

Finally, a *hyperbolic linear congruence* is the set of lines meeting both of two skew lines. That is, the set of *transversals* of two skew lines. A hyperbolic linear congruence is contained in $q-1$ general linear complexes and two special linear complexes.

There are some sets of lines related to those above that are also of interest.

A hyperbolic linear congruence is the set of transversals of two skew lines. The set of transversals of *three* mutually skew lines is called a *regulus*. If ℓ, m and n are three mutually skew lines and $\mathscr{R}$ is the regulus of their transversals, then $\mathscr{R}$ is a set of

$q+1$ mutually skew lines [37]. These lines also have $q+1$ transversals, which form another regulus $\mathscr{R}^*$, called the *opposite regulus* of $\mathscr{R}$. This is the unique regulus containing ℓ, m and n [37].

An elliptic linear congruence is an example of a set of lines called a *spread*.

Definition 0.4.1 *Let $\mathscr{S}$ be a set of q^2+1 mutually skew lines of* $\mathrm{PG}(3,q)$*. Then $\mathscr{S}$ is called a* spread *of* $\mathrm{PG}(3,q)$*.*

A regular spread *is a spread $\mathscr{S}$ such that if ℓ, m and n are distinct lines of $\mathscr{S}$, then each line of the regulus containing ℓ, m and n is also contained in $\mathscr{S}$.*

The lines of a spread necessarily partition the points of $\mathrm{PG}(3,q)$. That is, every point of $\mathrm{PG}(3,q)$ lies on exactly one line of the spread. The following result describes the regular spreads of $\mathrm{PG}(3,q)$.

Lemma 0.4.2 [37, Chapter 17] *A spread $\mathscr{S}$ is regular if and only if it is an elliptic linear congruence.*

It is worth noting that for most values of q, there exist spreads which are not regular. In fact, for many q there exist spreads which contain no reguli at all [13].

Spreads and general linear complexes are both examples of the following special set of lines.

Definition 0.4.3 *An m-cover of* $\mathrm{PG}(3,q)$ *is a set of lines $\mathscr{M}$ such that every point of* $\mathrm{PG}(3,q)$ *lies on m lines of $\mathscr{M}$.*

With this definition, a spread is a 1-cover and a general linear complex is a $(q+1)$-cover. The set of tangents and generator lines of a non-singular quadric of $\mathrm{PG}(3,q)$, q odd, is an example of a $(q+1)$-cover that is not a general linear complex (see Lemma 1.5.33). It is possible to partition the lines of $\mathrm{PG}(3,q)$ into q^2+q+1 disjoint spreads [37, Chapter 17], so for every $m \le q^2+q+1$, an m-cover can be constructed as the union of m disjoint spreads. The following result can be proved about m-covers.

Lemma 0.4.4 [43] *Let $\mathscr{M}$ be an m-cover of* PG$(3,q)$. *Then every plane of* PG$(3,q)$ *contains m lines of $\mathscr{M}$, and $|\mathscr{M}| = m(q^2+1)$.*

For more information about m-covers, see [43].

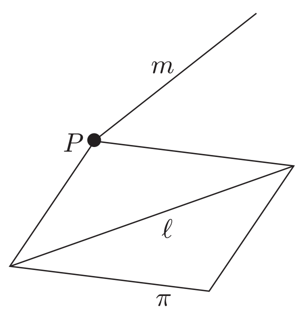
This section is completed with some results about the collineations that fix each line of a regular spread. In Chapter 7, these collineations will be used to investigate the intersection of ovoids.

Let $\mathscr{S}$ be a spread of PG$(3,q)$. The set of all collineations that fix each line of $\mathscr{S}$ is called the *kernel of $\mathscr{S}$*. The following four results describe the kernel of $\mathscr{S}$. The analogous results for regular spreads are proved in [12], but the proofs given here are more elementary and apply to any spread.

Lemma 0.4.5 *The kernel of $\mathscr{S}$ is a group under composition of functions.*

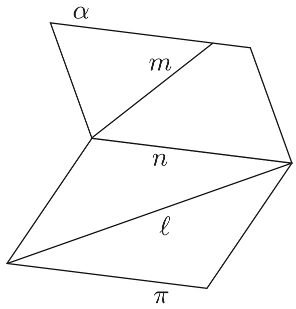
Proof Let g and h be collineations in the kernel of $\mathscr{S}$ and let ℓ be a line of $\mathscr{S}$. The collineation g fixes ℓ, so the collineation g^{-1} also fixes ℓ. Also, $gh(\ell) = g(\ell) = \ell$, so gh is in the kernel of $\mathscr{S}$. Thus $\mathscr{S}$ is a subgroup of the group of collineations. □

Lemma 0.4.6 *Let g be a collineation in the kernel of $\mathscr{S}$. If g fixes any point or plane, then g is the identity.*



Proof Suppose g fixes the plane π and let ℓ be the unique line of $\mathscr{S}$ in π. Let P be a point of π not on ℓ. Then P lies on a unique line m of $\mathscr{S}$. Since ℓ is the only line of $\mathscr{S}$ in π and P is not on ℓ, the line m is not in π. Thus $P = m \cap \pi$ and $g(P) = g(m) \cap g(\pi)$. But g fixes π and every line of $\mathscr{S}$, so $g(P) = m \cap \pi = P$. Thus every point of π not on ℓ is fixed by g.

Let n be a line of π other than ℓ, and let P and Q be two points of n not lying on ℓ. Then $n = PQ$, so $g(n) = g(P)g(Q)$. Since every point of π not on ℓ is fixed by g, $g(n) = PQ = n$. So every line of π is fixed.



Let α be a plane not through ℓ, meeting π in the line $n \neq \ell$. Also let m be the unique line of $\mathscr{S}$ in α. Then $\alpha = m \oplus n$, so $g(\alpha) = g(m) \oplus g(n)$. But g fixes every line in π and every line of $\mathscr{S}$, so $g(\alpha) = m \oplus n = \alpha$. So every plane not through ℓ is fixed.

However, since g fixes the plane α through the line m of $\mathscr{S}$, a similar argument to that above will show that every plane not through m is fixed. Thus every plane through ℓ is fixed. Hence g fixes every plane, and so g is the identity collineation.

A dual argument shows that if g fixes any point, then g is the identity. □

Lemma 0.4.7 *Let $\mathscr{S}$ be a spread of* $\mathrm{PG}(3,q)$ *and let g_1 and g_2 be distinct collineations fixing each line of $\mathscr{S}$. Then, for any point P, $g_1(P) \neq g_2(P)$.*

Proof Suppose $g_1(P) = g_2(P)$ for some point P. Then $g_2^{-1}g_1(P) = P$. Now $g_2^{-1}g_1$ is a collineation fixing each line of $\mathscr{S}$, and also fixing the point P. So by Lemma 0.4.6, $g_2^{-1}g_1 = \iota$, the identity collineation. Thus $g_1 = g_2$. □

Corollary 0.4.8 *Let $\mathscr{S}$ be a spread of* $\mathrm{PG}(3,q)$ *and let G be the group of collineations fixing each line of $\mathscr{S}$. Then $|G| \leq q+1$.*

Proof Let P be a point of $\mathrm{PG}(3,q)$ and let ℓ be the unique line of $\mathscr{S}$ through P. If g is a collineation in G, then g fixes ℓ, so $g(P) \in \ell$. That is, each collineation in G sends P to a point of ℓ. By Lemma 0.4.7, no two of these collineations sends P to the same point. Thus, since there are $q+1$ points of ℓ, there are at most $q+1$ collineations in G. □

The next few lemmas describe the kernel of $\mathscr{S}$ when $\mathscr{S}$ is a *regular* spread. The first follows directly from the fact that a regular spread is an elliptic linear congruence (Lemma 0.4.2).

Lemma 0.4.9 [37, Chapter 15] *Suppose $\mathscr{S}$ is a regular spread. Then there are $q+1$ general linear complexes containing $\mathscr{S}$, each defining a distinct null polarity.*

Lemma 0.4.10 *Suppose $\mathscr{S}$ is a regular spread and let R be the set of $q+1$ polarities defined by the general linear complexes containing $\mathscr{S}$. Let G be the kernel of $\mathscr{S}$ and let ρ be a fixed polarity in R. Then $|G| = q+1$ and $G = \{\rho\sigma \mid \sigma \in R\}$.*

Proof By Lemma 0.3.8, each of the maps $\rho\sigma$ for $\sigma \in R$ is a collineation. Moreover, each of the polarities $\sigma \in R$ fixes the lines of its general linear complex, so in particular, each of the polarities σ fixes the lines of $\mathscr{S}$. Thus the collineation $\rho\sigma$ also fixes the lines of $\mathscr{S}$.

Finally, $\rho\sigma_1 = \rho\sigma_2$ if and only if $\sigma_1 = \sigma_2$. Thus the set $\{\rho\sigma \mid \sigma \in R\}$ contains $q+1$ distinct collineations fixing each line of $\mathscr{S}$. By Corollary 0.4.8, this is the maximum number of such collineations, so $G = \{\rho\sigma \mid \sigma \in R\}$. □

The next result gives another way to represent the kernel of $\mathscr{S}$. It will not be proved here.

Lemma 0.4.11 [33] *Suppose $\mathscr{S}$ is a regular spread and G is the kernel of $\mathscr{S}$. Then there exists a cyclic Singer group S such that G is the unique subgroup of S of order $q+1$, and the orbits of G are the lines of $\mathscr{S}$.*

The final result in this section gives some technical details on the composition of the collineations in the kernel of $\mathscr{S}$ with the polarities associated with $\mathscr{S}$.

Corollary 0.4.12 *Suppose $\mathscr{S}$ is a regular spread and let R be the set of $q+1$ polarities defined by the general linear complexes containing $\mathscr{S}$. Let G be the kernel of $\mathscr{S}$. Then $g\rho = \rho g^{-1}$ for any $g \in G$ and $\rho \in R$.*

Proof Let $g \in G$ and $\rho \in R$. By Lemma 0.4.10, $g = \rho\sigma$ for some $\sigma \in R$. First, g^{-1} is determined.

For points P and Q of $\mathrm{PG}(3, q)$,

$$\begin{aligned} g^{-1}(P) = Q &\iff P = g(Q) \\ &\iff P = \rho\sigma(Q) \\ &\iff \rho(P) = \sigma(Q) \\ &\iff \sigma\rho(P) = Q. \end{aligned}$$

Thus $g^{-1} = \sigma\rho$. It follows that $\rho g^{-1} = \rho(\sigma\rho) = \rho\sigma\rho = g\rho$. □

Chapter 1

Quadrals

This thesis is concerned with the families of subspaces associated with quadrics of PG(n, q). The main aim is to characterise these families of subspaces by their combinatorial properties. In order to achieve such characterisations, the combinatorial properties of the quadrics and their subspaces must first be determined. The purpose of this chapter is to present detailed combinatorial information about the families of subspaces associated with different quadrics and some related objects. These related objects are combinatorially identical to quadrics, but may not have the same algebraic properties. They are the ovals, the ovoids and the cones over them. The quadrics, ovals, ovoids and the cones over them are collectively called *quadrals*.

In this chapter each type of quadral is carefully defined. Appropriate notation and terminology is introduced and this is used to describe the important properties of the quadrals. Since this chapter is introductory in nature, very few proofs will be given. The citations refer to an appropriate reference where the proof can be found.

1.1 Ovals

The discussion of quadrals begins with ovals in PG$(2, q)$. Ovals are included in the list of quadrals because they are combinatorially identical to non-singular conics (see Section 1.5). Since ovals are examples of arcs, this section begins with a discussion of the properties of arcs.

Definition 1.1.1 *If k is a positive integer, then a set $\mathscr{K}$ of k points in* $\mathrm{PG}(2,q)$ *is called a k-*arc *if every line meets $\mathscr{K}$ in* 0, 1 *or* 2 *points. For simplicity, a k-arc will be called an* arc.

In order to make the discussion of arcs easier, special names are given to each type of line.

Definition 1.1.2 *Let $\mathscr{K}$ be an arc in* $\mathrm{PG}(2,q)$ *and let ℓ be a line of* $\mathrm{PG}(2,q)$*. If ℓ contains no point of $\mathscr{K}$, it is called an* external line *of $\mathscr{K}$. If ℓ contains exactly one point of $\mathscr{K}$, it is called a* tangent *of $\mathscr{K}$. If ℓ contains exactly two points of $\mathscr{K}$, it is called a* secant *of $\mathscr{K}$.*

If it is clear that $\mathscr{K}$ is the arc under consideration, an external line of $\mathscr{K}$ may be referred to simply as an external line, and similarly for tangents and secants. The first result gives a relationship between the size of an arc and the number of lines of each type.

Lemma 1.1.3 [36, Chapter 8] *Let $\mathscr{K}$ be a k-arc of* $\mathrm{PG}(2,q)$ *and let P be a point of $\mathscr{K}$. Through P there are $k-1$ secants and $q+2-k$ tangents. Also, $\mathscr{K}$ has in total $k(q+2-k)$ tangents, $\frac{1}{2}k(k-1)$ secants and $\frac{1}{2}q(q-1)+\frac{1}{2}(q+1-k)(q+2-k)$ external lines.*

The relationships given above make it possible to calculate the maximum number of points in an arc.

Theorem 1.1.4 [36, Chapter 8] *Let $\mathscr{K}$ be an arc of* $\mathrm{PG}(2,q)$*. If q is odd, then $|\mathscr{K}| \le q+1$. If q is even, then $|\mathscr{K}| \le q+2$.*

These arcs of largest size have special names, which are now defined.

Definition 1.1.5 *Let $\mathscr{K}$ be an arc in* $\mathrm{PG}(2,q)$*. If $|\mathscr{K}| = q+1$, then $\mathscr{K}$ is called an* oval*. If $|\mathscr{K}| = q+2$, then $\mathscr{K}$ is called a* hyperoval.

Because of their similarity to certain conics (see Theorem 1.5.23), ovals are included in the list of quadrals (see Definition 1.6.1). Hyperovals are not quadrals, but are very closely related to ovals. The combinatorial properties of ovals and hyperovals are now described. First, Lemma 1.1.3 is restated for ovals and hyperovals.

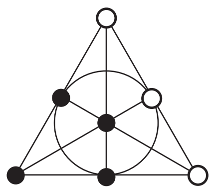
Lemma 1.1.6 [36, Chapter 8] *Let $\mathscr{O}$ be an oval of* $\mathrm{PG}(2,q)$ *and let P be a point of $\mathscr{O}$. Through P there are q secants and one tangent. Also $\mathscr{O}$ has in total $q+1$ tangents, $\frac{1}{2}q(q+1)$ secants and $\frac{1}{2}q(q-1)$ external lines.*

Let $\bar{\mathscr{O}}$ be a hyperoval of $\mathrm{PG}(2,q)$*. Then in total, $\bar{\mathscr{O}}$ has no tangents, $\frac{1}{2}(q+1)(q+2)$ secants and $\frac{1}{2}q(q-1)$ external lines.*

Using Lemma 1.1.3, ovals and hyperovals can be described using tangents and external lines.

Corollary 1.1.7 [36, Chapter 8] *Let $\mathscr{K}$ be an arc of* $\mathrm{PG}(2,q)$*.*

- *$\mathscr{K}$ is a hyperoval if and only if $\mathscr{K}$ has no tangents.*
- *$\mathscr{K}$ is an oval if and only if there is exactly one tangent at each point of $\mathscr{K}$.*
- *If $\mathscr{K}$ has $\frac{1}{2}q(q-1)$ external lines, then $\mathscr{K}$ is either an oval or a hyperoval.*

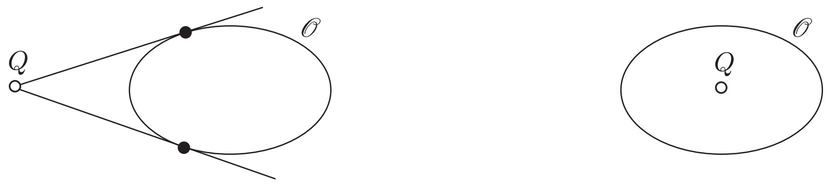


It is important to note the structure of a hyperoval in $\mathrm{PG}(2,2)$. In $\mathrm{PG}(2,2)$, a hyperoval has 4 points, and 1 external line. Since there are 7 points in $\mathrm{PG}(2,2)$, a hyperoval is the set of 4 points not on the 1 external line.

The properties of ovals in $\mathrm{PG}(2,q)$, for q odd are now described.

Lemma 1.1.8 [36, Chapter 8] *Let $\mathscr{O}$ be an oval of* $\mathrm{PG}(2,q)$*, q odd, and let Q be a point not on $\mathscr{O}$. Then Q lies on 0 or 2 tangents. If Q lies on 0 tangents, then it lies on $\frac{1}{2}(q+1)$ secants and $\frac{1}{2}(q+1)$ external lines. If Q lies on 2 tangents, then it lies on $\frac{1}{2}(q-1)$ secants and $\frac{1}{2}(q-1)$ external lines.*

In light of the above lemma, the following definition is made.



Definition 1.1.9 *Let $\mathscr{O}$ be an oval of* $\mathrm{PG}(2, q)$*, q odd, and let Q be a point not on $\mathscr{O}$. If Q lies on no tangents, then Q is called an* interior point *of $\mathscr{O}$. If Q lies on 2 tangents, then Q is called an* exterior point *of $\mathscr{O}$.*

It is possible to count the number of interior and exterior points of $\mathscr{O}$ in total, and on each line.

Lemma 1.1.10 [36, Chapter 8] *Let $\mathscr{O}$ be an oval of* $\mathrm{PG}(2, q)$*, q odd. Then there are $\frac{1}{2}q(q+1)$ exterior points and $\frac{1}{2}q(q-1)$ interior points. A tangent contains 1 point of $\mathscr{O}$ and q exterior points; a secant contains 2 points of $\mathscr{O}$, $\frac{1}{2}(q-1)$ interior points and $\frac{1}{2}(q-1)$ exterior points; and an external line contains $\frac{1}{2}(q+1)$ interior points and $\frac{1}{2}(q+1)$ exterior points.*

Using the above properties, the set of tangents of an oval can be described as a set of points in the dual plane.

Lemma 1.1.11 *Let $\mathscr{O}$ be an oval of* $\mathrm{PG}(2, q)$*, q odd. Denote by $\mathscr{O}^*$ the set of $q+1$ tangents of $\mathscr{O}$. Then $\mathscr{O}^*$ is an oval of the dual plane* $\mathrm{PG}(2, q)^*$.

Proof By Lemma 1.1.8, no three lines of $\mathscr{O}^*$ are concurrent. Thus $\mathscr{O}^*$ is an oval of $\mathrm{PG}(2, q)^*$. □

Using the notation of the above lemma, a point of $\mathscr{O}$ lies on one tangent of $\mathscr{O}$, and so a point of $\mathscr{O}$ corresponds to a tangent of $\mathscr{O}^*$. An exterior point of $\mathscr{O}$ lies on two tangents, and so corresponds to a secant of $\mathscr{O}^*$. An interior point of $\mathscr{O}$ lies on no tangents, and so corresponds to an external line of $\mathscr{O}^*$.

This completes the basic combinatorial properties of an oval when q is odd. The properties of an oval when q is even are now described.

Lemma 1.1.12 [36, Chapter 8] *Let $\mathscr{O}$ be an oval of* $\mathrm{PG}(2,q)$*, q even. Then the tangents of $\mathscr{O}$ all pass through a single point.*

In light of the above lemma, the following definition is made.

Definition 1.1.13 *Let $\mathscr{O}$ be an oval of* $\mathrm{PG}(2,q)$*, q even, and let N be the unique point through which all the tangents of $\mathscr{O}$ pass. The point N is called the* nucleus *of $\mathscr{O}$.*

Using the notation of the above lemma, since $\mathscr{O}$ has $q+1$ tangents, every line through N meets $\mathscr{O}$ in one point. Thus every line meets the set of points $\mathscr{O} \cup \{N\}$ in at most two points. That is, $\mathscr{O} \cup \{N\}$ is a hyperoval. Thus every oval is contained in a hyperoval. On the other hand, each hyperoval contains $q+2$ ovals, each obtained by removing one point of the hyperoval.

An analogue of Lemma 1.1.8 can now be given.

Lemma 1.1.14 [36, Chapter 8] *Let $\mathscr{O}$ be an oval of* $\mathrm{PG}(2,q)$*, q even, with nucleus N. Then every point not on $\mathscr{O}$ other than N lies on 1 tangent, $\frac{1}{2}q$ secants and $\frac{1}{2}q$ external lines.*

Let $\bar{\mathscr{O}}$ be a hyperoval of $\mathrm{PG}(2,q)$*, q even. Then every point not on $\bar{\mathscr{O}}$ lies on no tangents, $\frac{1}{2}q+1$ secants and $\frac{1}{2}q$ external lines.*

Before the discussion of ovals is complete, it is appropriate to give the following result. The statement for q odd was proved by Segre in 1955 [52]. (For the definition of a conic, see Section 1.5.)

Theorem 1.1.15 [36, Chapter 8] *For q odd, $q=2$ and $q=4$, every oval in* $\mathrm{PG}(2,q)$ *is a non-singular conic. For q even, $q>4$, there exist ovals in* $\mathrm{PG}(2,q)$ *that are not conics. For $q = 2$, 4 or 8, the only hyperovals are those formed by a conic plus its nucleus. For q even, $q>8$, there exist hyperovals that do not contain conics.*

This completes the discussion of arcs, ovals and hyperovals.

1.2 Ovoids

The next quadrals to be discussed are the ovoids. Ovoids are included in the list of quadrals because they are combinatorially identical to elliptic quadrics of $\mathrm{PG}(3, q)$ (see Theorem 1.5.26). Since ovoids are examples of caps, this section begins with a discussion of the properties of caps.

Definition 1.2.1 *If k is a positive integer, then a set $\mathscr{K}$ of k points in $\mathrm{PG}(n, q)$ is called a* k-cap *if every line meets $\mathscr{K}$ in 0, 1 or 2 points. For simplicity, a k-cap will be called a* cap.

Note that a cap in $\mathrm{PG}(2, q)$ is an arc. The lines associated with arcs were given special names, and the same names are given to the lines associated with caps.

Definition 1.2.2 *Let $\mathscr{K}$ be a cap of $\mathrm{PG}(n, q)$ and let ℓ be a line of $\mathrm{PG}(n, q)$. If ℓ contains no point of $\mathscr{K}$, it is called an* external line *of $\mathscr{K}$. If ℓ contains exactly one point of $\mathscr{K}$, it is called a* tangent *of $\mathscr{K}$. If ℓ contains exactly two points of $\mathscr{K}$, it is called a* secant *of $\mathscr{K}$.*

If it is clear that $\mathscr{K}$ is the cap under consideration, an external line of $\mathscr{K}$ may be referred to simply as an external line, and similarly for tangents and secants.

The first result is an analogue of Lemma 1.1.3.

Lemma 1.2.3 [37, Chapter 16] *Let $\mathscr{K}$ be a k-cap of $\mathrm{PG}(n, q)$ and let P be a point of $\mathscr{K}$. Through P there are $k - 1$ secants and $\theta_{n-1} + 1 - k$ tangents. Also, $\mathscr{K}$ has in total $k(\theta_{n-1} + 1 - k)$ tangents and $\frac{1}{2}k(k - 1)$ secants.*

It can be shown that a cap has size at most $\theta_{n-1} + 1$, which is analogous to Theorem 1.1.4 for arcs. However, the following lemma shows that this case is only possible for $q = 2$.

Lemma 1.2.4 [37, Chapter 27] *A cap with no tangents has size $\theta_{n-1} + 1$. A cap of size $\theta_{n-1} + 1$ in $\mathrm{PG}(n, q)$ only exists when $q = 2$, in which case it is the set of points not in a hyperplane.*

There is a much smaller upper bound for the size of a cap when $q \neq 2$. The theorem below was proved by Bose [9] for $\mathrm{PG}(3,q)$, q odd, by Qvist [45] for $\mathrm{PG}(3,q)$, q even, $q > 2$, and by Tallini [53] for $\mathrm{PG}(n,q)$, $n \geq 4$, $q > 2$.

Theorem 1.2.5 [9, 45, 53] *Let $\mathscr{K}$ be a cap in $\mathrm{PG}(n,q)$, $n \geq 3$, $q > 2$. Then $|\mathscr{K}| \leq q^{n-1} + 1$.*

The caps of particular interest in this thesis are ovoids, which were first defined by Tits [56].

Definition 1.2.6 *An* ovoid *of $\mathrm{PG}(n,q)$ is a cap Ω such for any point P of Ω, the union of the tangents of Ω through P is a hyperplane. This hyperplane is called the* tangent hyperplane of Ω at P.

The first result shows that ovoids only exist in $\mathrm{PG}(2,q)$ and $\mathrm{PG}(3,q)$.

Theorem 1.2.7 [26] *There are no ovoids in $\mathrm{PG}(n,q)$ for $n \geq 4$.*

Note that in $\mathrm{PG}(2,q)$, an ovoid is an arc such that there is a unique tangent through each point of the arc. That is, an ovoid of $\mathrm{PG}(2,q)$ is an oval. In light of this fact and the theorem above, the term ovoid will always be used to mean an ovoid of $\mathrm{PG}(3,q)$.

The properties of ovoids are now discussed. If Ω is an ovoid, then the tangents through each point of Ω form a plane, so there are $q+1$ tangents through each point of Ω. Lemma 1.2.3 implies the following result.

Lemma 1.2.8 *An ovoid of $\mathrm{PG}(3,q)$ has q^2+1 points.*

The converse of the above lemma is also true for $q > 2$. This was proved independently by Barlotti [5] and Panella [44].

Lemma 1.2.9 [5, 44] *Let $\mathscr{K}$ be a (q^2+1)-cap in $\mathrm{PG}(3,q)$. Then $\mathscr{K}$ is an ovoid.*

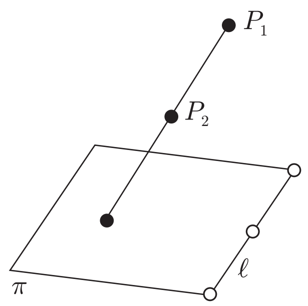
For $q = 2$, an alternative can be given using the number of points of the cap in a plane.

Lemma 1.2.10 *Let $\mathscr{K}$ be a cap of* $\mathrm{PG}(3,2)$ *such that every plane meets $\mathscr{K}$ in at least one point. Then $\mathscr{K}$ is an ovoid of* $\mathrm{PG}(3,2)$.

Proof Let ℓ be a line containing no points of $\mathscr{K}$. There are 3 planes through ℓ, each containing at least one point of $\mathscr{K}$, so $|\mathscr{K}| \geq 3$.

Let α be a plane containing three points of $\mathscr{K}$, and let m be a line in π containing no points of $\mathscr{K}$. The other two planes through m each contain at least one point of $\mathscr{K}$, so $|\mathscr{K}| \geq 3 + 2 = 5$.

Suppose π is a plane meeting $\mathscr{K}$ in a hyperoval, which is the set of points of π not on a line ℓ. There are 3 planes through ℓ, including π. Denote the other two planes through ℓ by π_1 and π_2.



Suppose P_1 is a point of $\mathscr{K}$ in π_1 and P_2 is a point of $\mathscr{K}$ in π_2. Consider the line P_1P_2. This line contains points of π_1 and π_2 not in ℓ, so it is skew to ℓ. Hence, it meets π in a point not on ℓ.

However, every point of π not on ℓ is a point of $\mathscr{K}$, so the line P_1P_2 contains three points of $\mathscr{K}$. This is a contradiction, so the planes π_1 and π_2 cannot both contain points of $\mathscr{K}$. That is, there exists a plane containing no points of $\mathscr{K}$. This is also a contradiction, so there exist no planes meeting $\mathscr{K}$ in a hyperoval.

Let s be a line containing two points of $\mathscr{K}$. Each of the three planes through s meets $\mathscr{K}$ in at most 3 points, so $|\mathscr{K}| \leq 2 + 3 = 5$. Thus $|\mathscr{K}| = 5$ and each plane through s contains 3 points of $\mathscr{K}$. It follows that every plane meets $\mathscr{K}$ in 0, 1 or 3 points.

Let P be a point of $\mathscr{K}$. By Lemma 1.2.3, the point P lies on 3 tangents of $\mathscr{K}$. Let t be a tangent through P. The planes through t meet $\mathscr{K}$ in 1 or 3 points. Since there are 5 points of $\mathscr{K}$, it follows that exactly one plane through t meets $\mathscr{K}$ in only P. The 3 lines through P in this plane are tangents of $\mathscr{K}$. Thus the 3 tangents through a point of $\mathscr{K}$ form a plane. That is, $\mathscr{K}$ is an ovoid. □

Further properties of ovoids are now discussed. First, Lemma 1.2.3 is restated for ovoids.

Lemma 1.2.11 [37, Chapter 16] *Let* Ω *be an ovoid of* $\mathrm{PG}(3, q)$. *Then* Ω *has* $q^3 + q^2 + q + 1$ *tangents,* $\frac{1}{2}q^2(q^2 + 1)$ *secants and* $\frac{1}{2}q^2(q^2 + 1)$ *external lines. Also, through each point of* Ω *there are* $q + 1$ *tangents, which form a plane, and* q^2 *secants.*

Next, the possible intersections of planes with an ovoid may be determined.

Lemma 1.2.12 [37, Chapter 16] *Let* Ω *be an ovoid of* $\mathrm{PG}(3, q)$. *Then every plane meets* Ω *in one point or an oval.*

In light of the above lemma, the following definitions are made.

Definition 1.2.13 *Let* Ω *be an ovoid of* $\mathrm{PG}(3, q)$. *A plane meeting* Ω *in one point is called a* tangent plane. *A plane meeting* Ω *in an oval is called a* secant plane.

It is possible to count the number of each type of plane through each type of line.

Lemma 1.2.14 [37, Chapter 16] *Let* Ω *be an ovoid of* $\mathrm{PG}(3, q)$. *Then* Ω *has* $q^2 + 1$ *tangent planes and* $q^3 + q$ *secant planes. An external line lies on* 2 *tangent planes and* $q - 1$ *secant planes. A tangent lies on* 1 *tangent plane and* q *secant planes. A secant lies on* 0 *tangent planes and* $q + 1$ *secant planes.*

Using the above information, the number of each type of line through each point not on an ovoid can be determined.

Lemma 1.2.15 [37, Chapter 16] *Let* Ω *be an ovoid of* $\mathrm{PG}(3, q)$ *and let* P *be a point not in* Ω. *Through* P *there are* $\frac{1}{2}q(q+1)$ *external lines,* $q+1$ *tangents and* $\frac{1}{2}q(q-1)$ *secants. Also, through* P *there are* $q + 1$ *tangent planes and* q^2 *secant planes.*

This concludes the numerical information on the lines and planes associated with ovoids. The remaining part of this section deals with other geometrical properties. First, the tangent planes of an ovoid may be described using the dual space.

Lemma 1.2.16 [37, Chapter 16] *Let Ω be an ovoid of* $\mathrm{PG}(3,q)$ *and denote the set of tangent planes of Ω by Ω^*. Then Ω^* is an ovoid of the dual space* $\mathrm{PG}(3,q)^*$.

Proof By Lemma 1.2.14, every line of $\mathrm{PG}(3,q)$ is contained in 0, 1 or 2 tangent planes. Thus, each line of the dual space $\mathrm{PG}(3,q)^*$ contains 0, 1 or 2 points of the set Ω^*. Since Ω^* has q^2+1 points of $\mathrm{PG}(3,q)^*$, Lemma 1.2.9 implies that Ω^* is an ovoid of $\mathrm{PG}(3,q)^*$ for $q>2$. For $q=2$, every point of $\mathrm{PG}(3,2)$ is contained in a tangent plane of Ω, so every plane of $\mathrm{PG}(3,2)^*$ contains a point of Ω^*. Thus Lemma 1.2.10 implies that Ω^* is an ovoid of $\mathrm{PG}(3,2)^*$. □

Using the notation of the above lemma, a point of Ω lies on 1 tangent plane, and so is a tangent plane of Ω^*. A point not in Ω lies on $q+1$ tangent planes, and so is a secant plane of Ω^*. An external line of Ω lies on 2 tangent planes, and so is a secant of Ω^*. A secant of Ω lies on no tangent planes, and so is an external line of Ω^*. Finally, a tangent of Ω lies on one tangent plane, and so is a tangent of Ω^*.

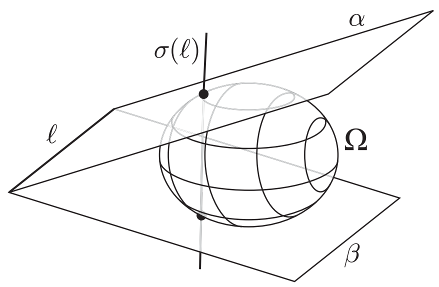
The structure of the tangents of an ovoid for q even is now described. The following lemma is required first.

Lemma 1.2.17 [37, Chapter 16] *Let Ω be an ovoid of* $\mathrm{PG}(3,q)$, *q even, and let P be a point not on Ω. The $q+1$ tangents of Ω through P are coplanar.*

Lemma 1.2.18 [37, Chapter 16] *Let Ω be an ovoid of* $\mathrm{PG}(3,q)$, *q even, and let $\mathscr{T}$ be the set of tangents of Ω. Then $\mathscr{T}$ is a general linear complex.*

The fact that the tangents of an ovoid form a general linear complex when q is even makes it easier to describe the set of lines that are tangents of two ovoids. This will be seen in Chapter 7. It also means that there is a polarity associated with the ovoid.

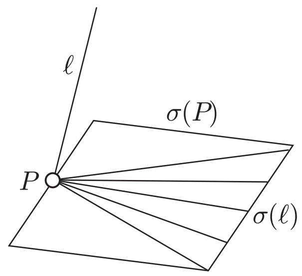
Let Ω be an ovoid of $\mathrm{PG}(3, q)$, q even, and let $\mathscr{T}$ be the general linear complex formed by the tangents of Ω. There is a unique polarity σ which fixes all the lines of $\mathscr{T}$ (see Section 0.4). This polarity interchanges a plane π with the point where the tangents in π meet. That is, if π is a tangent plane meeting Ω in the point P, then σ interchanges π with P; and if π is a secant plane meeting Ω in the oval $\mathscr{O}$, then σ interchanges π with the nucleus of $\mathscr{O}$. The polarity σ is called the *polarity associated with* Ω.



The polarity σ fixes the tangents of Ω. It is important to describe the action of σ on secants and external lines. Let ℓ be an external line contained in the tangent planes α and β. Then $\sigma(\ell) = \sigma(\alpha \cap \beta) = \sigma(\alpha) \oplus \sigma(\beta)$. Thus $\sigma(\ell)$ is the line joining the two points $\sigma(\alpha)$ and $\sigma(\beta)$. These points are points of Ω, so $\sigma(\ell)$ is a secant. Thus σ sends an external line to a secant.

Let s be a secant meeting Ω in the points P and Q. Then $\sigma(s) = \sigma(PQ) = \sigma(P) \cap \sigma(Q)$. Thus $\sigma(s)$ is the line where the two tangent planes $\sigma(P)$ and $\sigma(Q)$ meet. This line is necessarily an external line. Thus σ sends a secant to an external line. It is important to note that in both cases, a non-tangent line and its image are skew. The final result in this section describes the tangents of an ovoid meeting a line.

Lemma 1.2.19 *Let* Ω *be an ovoid of* $\mathrm{PG}(3, q)$*,* q *even, with associated polarity* σ*. Let* ℓ *be a secant or external line of* Ω*. Then the tangents meeting* ℓ *are the transversals of the two skew lines* ℓ *and* $\sigma(\ell)$*.*



Proof Let P be a point of ℓ. Then the tangents of Ω through P are precisely the lines through P in the plane $\sigma(P)$. The line ℓ is not a tangent of Ω, so ℓ does not lie in $\sigma(P)$. Thus $\sigma(\ell)$ does not pass through P. Hence, the transversals of ℓ and $\sigma(\ell)$ through P are the lines through P in the plane $\sigma(\ell) \oplus P$.

Since ℓ is not in $\sigma(P)$, it follows that $P = \ell \cap \sigma(P)$, and so $\sigma(P) = \sigma(\ell) \oplus P$. Hence, the tangents of Ω through P are the transversals of ℓ and $\sigma(\ell)$ through P. Since this is true for each point P on ℓ, the tangents of Ω meeting ℓ are the transversals of ℓ and $\sigma(\ell)$. □

The set of tangents of an ovoid when q is odd do not form a general linear complex. However, there is still a polarity associated with the ovoid. This will be described in Section 1.5 on quadrics, because all ovoids for q odd are elliptic quadrics. This result was proved independently by Barlotti [5] and Panella [44]. (For the definition of an elliptic quadric, see Section 1.5.)

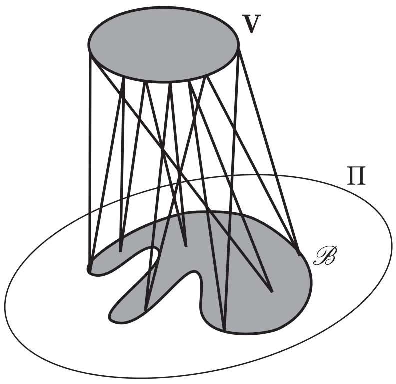
Theorem 1.2.20 [37, Chapter 16] *Every ovoid in* $\mathrm{PG}(3, q)$, *q odd, is an elliptic quadric.*

The above result is not true in general for q even (although it has been proved true for $q = 2$, $q = 4$ and $q = 16$ [42, 41]). To date, one other type of ovoid has been discovered, which is known as the Tits ovoid. This ovoid was first discovered by Segre [50] in $\mathrm{PG}(3, 8)$, but Tits [57] constructed an infinite family. Fellegara [31] proved that Segre's example in $\mathrm{PG}(3, 8)$ was part of the family constructed by Tits. Tits ovoids exist when q is a non-square (and $q > 2$). It is conjectured that all ovoids are either elliptic quadrics or Tits ovoids.

This completes the discussion of ovoids.

1.3 Cones

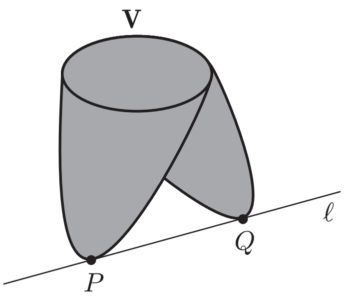
The quadrals defined thus far are the ovals of $\mathrm{PG}(2,q)$ and the ovoids of $\mathrm{PG}(3,q)$. From these quadrals, other quadrals in higher-dimensional spaces may be constructed. This section gives details on this construction.



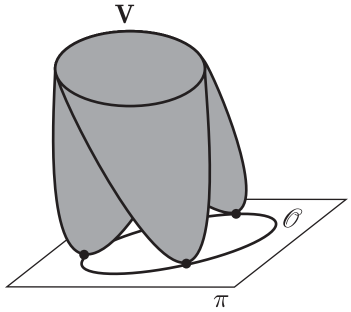
Definition 1.3.1 *Let* Π *be a subspace of* $\mathrm{PG}(n,q)$ *of dimension at most* $(n-1)$ *and let* $\mathscr{B}$ *be a set of points in* Π. *Let* $\mathbf{V}$ *be a complementary subspace of* Π. *The* cone *with* vertex $\mathbf{V}$ *and* base $\mathscr{B}$ *is denoted by* $\mathbf{V}\mathscr{B}$ *and is defined to be the set of points either in* $\mathbf{V}$, *or in* $\mathscr{B}$, *or on a line joining a point of* $\mathbf{V}$ *to a point of* $\mathscr{B}$. *A cone with base* $\mathscr{B}$ *is also called a* cone over $\mathscr{B}$.

Note that it is possible to define a cone more generally by not assuming that the vertex $\mathbf{V}$ is a complementary subspace of Π, but only assuming that it is skew to Π. However, the only cones considered in this thesis are those satisfying Definition 1.3.1.

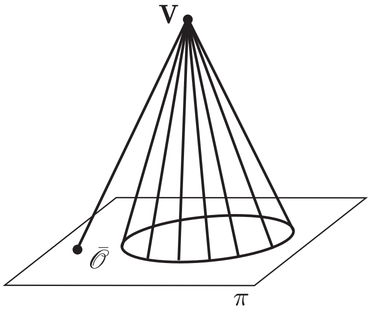
The set of points $\mathscr{B}$ may be empty, in which case the cone $\mathbf{V}\mathscr{B} = \mathbf{V}\emptyset$ is just the subspace $\mathbf{V}$. It is more usual to consider cones where the base is non-empty. If $P \in \mathscr{B}$, then the set of points on the lines joining a point of $\mathbf{V}$ to P is the subspace $\mathbf{V} \oplus P$. Thus if $\mathscr{B}$ is non-empty, then $\mathbf{V}\mathscr{B}$ is the union of the subspaces $\mathbf{V} \oplus P$ for $P \in \mathscr{B}$.



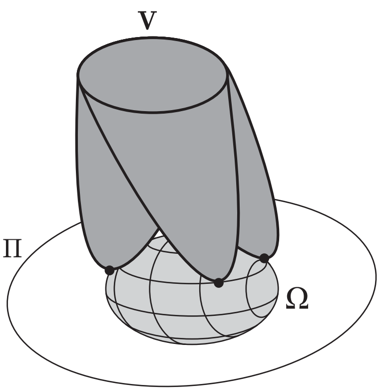
Some examples of cones are now given. Let ℓ be a line of $\mathrm{PG}(n,q)$, $n \geq 2$, and let $\mathscr{B} = \{P, Q\}$ be a pair of points of ℓ. Let $\mathbf{V}$ be an $(n-2)$-space skew to ℓ. Then the cone $\mathbf{V}\mathscr{B}$ is the pair of hyperplanes $\mathbf{V} \oplus P$ and $\mathbf{V} \oplus Q$.



Let π be a plane of $\mathrm{PG}(n,q)$, $n \geq 3$, and let $\mathscr{O}$ be an oval in π. Let $\mathbf{V}$ be an $(n-3)$-space skew to π. Then the cone $\mathbf{V}\mathscr{O}$ is called an *oval cone*. It consists of the $q+1$ $(n-2)$-spaces formed by joining $\mathbf{V}$ to each point of $\mathscr{O}$. In particular, in $\mathrm{PG}(3,q)$, an oval cone consists of $q+1$ lines joining the point $\mathbf{V}$ to $\mathscr{O}$.

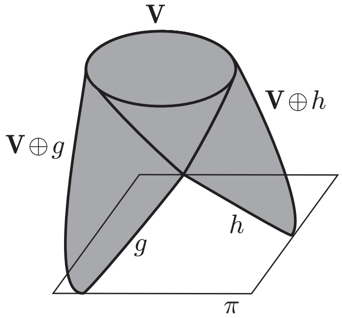


Let π be a plane of $\mathrm{PG}(n,q)$, $n \geq 3$, q even, and let $\bar{\mathscr{O}}$ be a hyperoval in π. Let $\mathbf{V}$ be an $(n-3)$-space skew to π. Then the cone $\mathbf{V}\bar{\mathscr{O}}$ is called a *hyperoval cone* and consists of the $q+2$ $(n-2)$-spaces formed by joining $\mathbf{V}$ to each point of $\bar{\mathscr{O}}$. (A hyperoval cone in $\mathrm{PG}(3,q)$ is pictured to the left.) Let N be a point of $\bar{\mathscr{O}}$ and let $\mathscr{O}$ be the oval obtained by removing the point N from $\bar{\mathscr{O}}$. Then the oval cone $\mathbf{V}\mathscr{O}$ is contained in the hyperoval cone $\mathbf{V}\bar{\mathscr{O}}$.

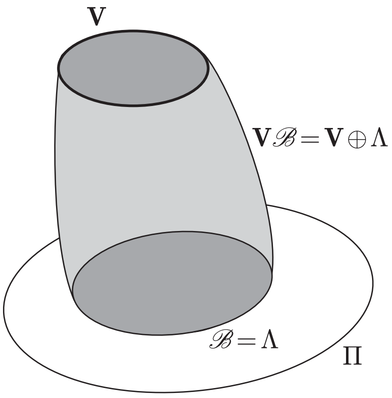


Let Π be a 3-space of $\mathrm{PG}(n,q)$, $n \geq 4$, and let Ω be an ovoid of Π. Let $\mathbf{V}$ be an $(n-4)$-space skew to Π. Then the cone $\mathbf{V}\Omega$ is called an *ovoid cone* and consists of the q^2+1 $(n-3)$-spaces formed by joining $\mathbf{V}$ to a point of Ω.

In the examples presented so far, the set $\mathscr{B}$ contains no lines, but this is not necessarily always the case, as the following examples illustrate.



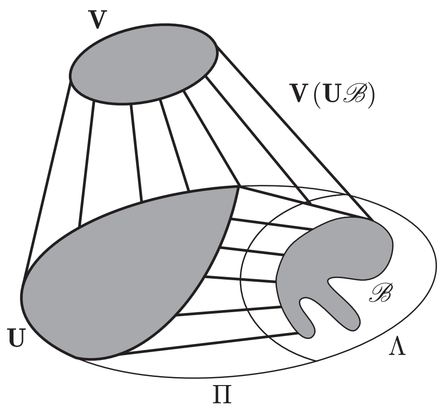
Let π be a plane of $\mathrm{PG}(n,q)$, $n \geq 3$ and let $\mathscr{B} = g \cup h$ be a pair of lines in π. Let $\mathbf{V}$ be an $(n-3)$-space skew to π. Then the cone $\mathbf{V}\mathscr{B}$ consists of the $(n-2)$-spaces $\mathbf{V} \oplus P$ for $P \in g \cup h$. However, the union of the $(n-2)$-spaces $\mathbf{V} \oplus P$ for $P \in g$ is the hyperplane $\mathbf{V} \oplus g$, and the union of the $(n-2)$-spaces $\mathbf{V} \oplus P$ for $P \in h$ is the hyperplane $\mathbf{V} \oplus h$. Thus the cone $\mathbf{V}\mathscr{B}$ is the pair of hyperplanes $\mathbf{V} \oplus g$ and $\mathbf{V} \oplus h$.

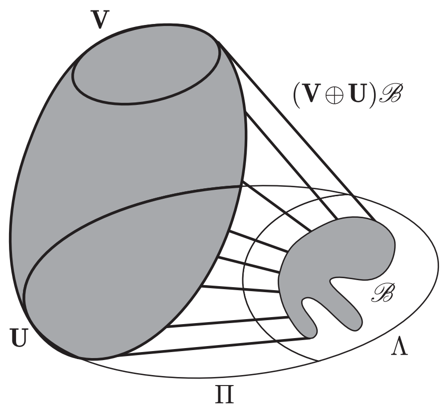


Let Π be a subspace of PG(n, q) of dimension at most $n - 1$, and let $\mathscr{B} = \Lambda$ be a subspace of Π. Let **V** be a complementary subspace of Π. Then the cone $\mathbf{V}\mathscr{B}$ is the subspace $\mathbf{V} \oplus \Lambda$.

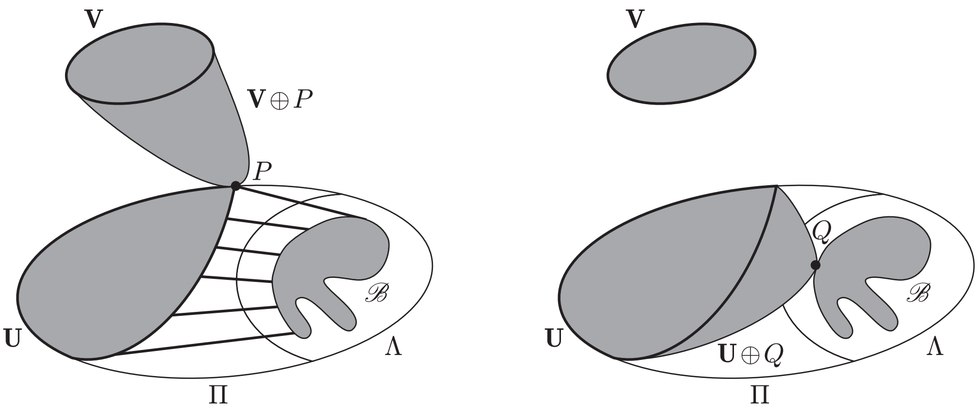
Given the examples above, it is reasonable to ask what form a cone $\mathbf{V}\mathscr{B}$ takes when the set $\mathscr{B}$ is itself a cone. This is the subject of the following lemma.

Lemma 1.3.2 *Let* **V** *be a non-empty subspace of* PG(n, q) *and let* Π *be a complementary subspace of* PG(n, q). *Suppose* **U** *is a non-empty subspace of* Π *and let* Λ *be a complementary subspace of* **U** *in* Π. *Let* $\mathscr{B}$ *be a set of points in* Λ. *Then* $\mathbf{V}(\mathbf{U}\mathscr{B}) = (\mathbf{V} \oplus \mathbf{U})\mathscr{B}$.

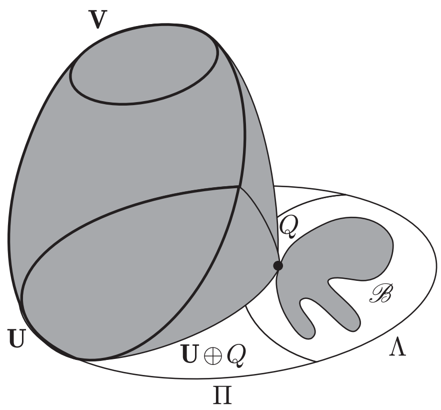




Proof Denote the dimensions of **V** and **U** by d and c respectively. Then the dimension of Π is $n - d - 1$ and the dimension of Λ is $n - d - c - 2$. Also, since **V** and **U** are skew, the dimension of $\mathbf{V} \oplus \mathbf{U}$ is $d + c + 1$. In order for the notation $(\mathbf{V} \oplus \mathbf{U})\mathscr{B}$ to make sense, the subspaces $\mathbf{V} \oplus \mathbf{U}$ and Λ must be complementary. Now $n-d-c-2+d+c+1 = n-1$, and $(\mathbf{V} \oplus \mathbf{U}) \oplus \Lambda = \mathbf{V} \oplus (\mathbf{U} \oplus \Lambda) = \mathbf{V} \oplus \Pi = \mathrm{PG}(n, q)$, so $\mathbf{V} \oplus \mathbf{U}$ and Λ are complementary subspaces.



If $\mathscr{B}$ is empty, then $\mathbf{V}(\mathbf{U}\mathscr{B}) = (\mathbf{V} \oplus \mathbf{U})\mathscr{B} = \mathbf{V} \oplus \mathbf{U}$. Assume that $\mathscr{B}$ is non-empty. Then $\mathbf{V}(\mathbf{U}\mathscr{B})$ is the union of the subspaces $\mathbf{V} \oplus P$ for all points $P \in \mathbf{U}\mathscr{B}$. Also, $\mathbf{U}\mathscr{B}$ is the union of the subspaces $\mathbf{U} \oplus Q$ for all points $Q \in \mathscr{B}$.



Consider a subspace $\mathbf{U} \oplus Q$ for $Q \in \mathscr{B}$. The union of the subspaces $\mathbf{V} \oplus P$ for $P \in \mathbf{U} \oplus Q$ is the subspace $\mathbf{V} \oplus (\mathbf{U} \oplus Q) = (\mathbf{V} \oplus \mathbf{U}) \oplus Q$. Thus $\mathbf{V}(\mathbf{U}\mathscr{B})$ is the union of the subspaces $(\mathbf{V} \oplus \mathbf{U}) \oplus Q$ for $Q \in \mathscr{B}$. That is $\mathbf{V}(\mathbf{U}\mathscr{B}) = (\mathbf{V} \oplus \mathbf{U})\mathscr{B}$. $\square$

Further examples of cones will appear later in this chapter, and will be used to describe many of the quadrals, and quadrics in particular. It is appropriate to include information that decides if a set of points is a cone. In order to do this, a special type of point is defined.

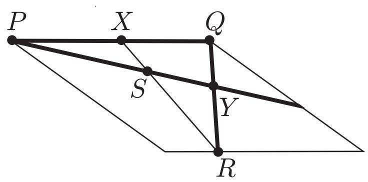
Definition 1.3.3 *Let $\mathscr{C}$ be a set of points in* $\mathrm{PG}(n, q)$ *and let V be a point of $\mathscr{C}$. The point V is called a* singular point of $\mathscr{C}$ *if every line through V meets $\mathscr{C}$ in* 1 *or* $q + 1$ *points. Otherwise, V is called a* non-singular point.

In a cone $\mathbf{V}\mathscr{B}$, the points of the vertex $\mathbf{V}$ are all singular points of $\mathscr{B}$, because of the way that the cone $\mathbf{V}\mathscr{B}$ is constructed. In fact, the set of singular points of any set always forms a subspace. The citation for the following lemma refers to a proof when every line meets $\mathscr{C}$ in 0, 1, 2 or $q + 1$ points. However, the result is true for any set of points.

Lemma 1.3.4 [39, Chapter 22] *Let* $\mathscr{C}$ *be a set of points of* $\mathrm{PG}(n,q)$. *Then the set of singular points of* $\mathscr{C}$ *forms a subspace.*

Proof Denote the set of singular points of $\mathscr{C}$ by $\mathbf{V}$. Suppose P and Q are distinct points of $\mathbf{V}$. If will be shown that the line joining PQ is contained in $\mathbf{V}$.

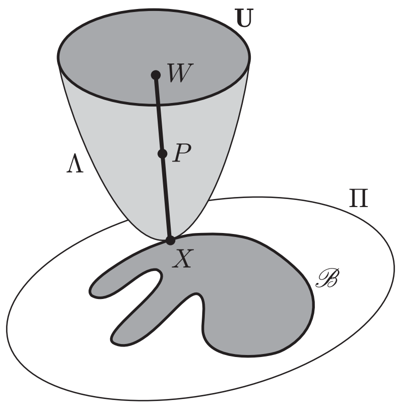
Let X be a point of PQ other than P or Q. Since P is a singular point, and the line PQ contains two points of $\mathscr{C}$, it follows that the line PQ is contained in $\mathscr{C}$. Thus $X \in \mathscr{C}$. For X to be a singular point, every line through X must meet $\mathscr{C}$ in 1 or $q+1$ points. That is, if a line through X contains a further point of $\mathscr{C}$, then every point on it must be a point of $\mathscr{C}$.



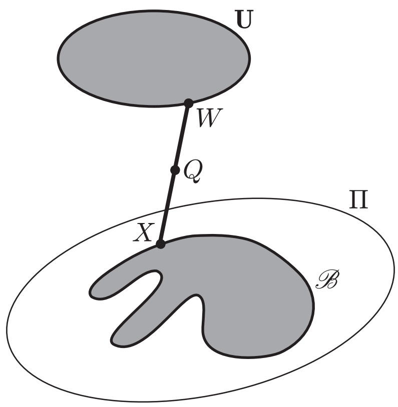
Suppose R is a point of $\mathscr{C}$ not on PQ and let S be a point on XR other than X and R. Let $Y = PS \cap QR$. The line QR passes through the singular point Q and contains the point $R \in \mathscr{C}$, so QR is contained in $\mathscr{C}$. Thus $Y \in \mathscr{C}$. The line PY passes through the singular point P and contains the point $Y \in \mathscr{Q}$, so PY is contained in $\mathscr{C}$. Thus $S \in \mathscr{C}$. It follows that every point of XR is a point of $\mathscr{C}$. So every line through X with a further point of $\mathscr{C}$ is contained in $\mathscr{C}$, and hence X is a singular point of $\mathscr{C}$. Thus $PQ \subseteq \mathbf{V}$ and $\mathbf{V}$ is a subspace by Lemma 0.2.5. $\square$

In light of the above lemma, the set of singular points of a set $\mathscr{C}$ is called the *singular space* of $\mathscr{C}$. Using the singular space of $\mathscr{C}$, it is possible to show that $\mathscr{C}$ is a cone. The following result is implicit in many proofs concerning quadrics and quadratic sets, which are discussed later. For examples of such proofs, see [39, Chapter 22].

Lemma 1.3.5 *Let* $\mathscr{C}$ *be a set of points of* $\mathrm{PG}(n,q)$ *and suppose* $\mathbf{U}$ *is a non-empty subspace contained in the singular space of* $\mathscr{C}$. *Let* Π *be a complementary subspace of* $\mathbf{U}$ *meeting* $\mathscr{C}$ *in the set of points* $\mathscr{B}$. *Then* $\mathscr{C}$ *is the cone* $\mathbf{U}\mathscr{B}$.



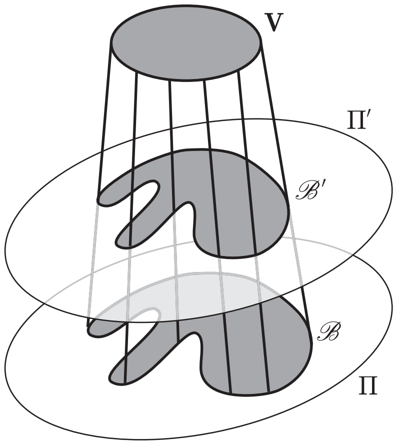
Proof Let P be a point of $\mathscr{C}$. If $P \in \mathbf{U}$, then P is a point of $\mathbf{U}\mathscr{B}$. Suppose $P \notin \mathbf{U}$ and consider the subspace $\Lambda = P \oplus \mathbf{U}$. Since $\mathbf{U}$ is complementary to Π, the subspace Λ meets Π in exactly one point. Denote this point by X. The line PX is contained in Λ and so meets $\mathbf{U}$ in a point, W say. The line PW passes through the singular point W of $\mathscr{C}$ and contains a point $P \in \mathscr{C}$, so PX is contained in $\mathscr{C}$. Thus X is a point of $\mathscr{C}$ in Π. That is, $X \in \mathscr{B}$. Thus P lies on a line joining a point of $\mathbf{U}$ to a point of $\mathscr{B}$. That is, P is a point of the cone $\mathbf{U}\mathscr{B}$. So every point of $\mathscr{C}$ is a point of $\mathbf{U}\mathscr{B}$ and $\mathscr{C} \subseteq \mathbf{U}\mathscr{B}$.



Let Q be a point of $\mathbf{U}\mathscr{B}$. If $Q \in \mathbf{U}$, then Q is a point of $\mathscr{C}$. Suppose $Q \notin \mathbf{U}$. Then Q lies on a line joining a point W of $\mathbf{U}$ to a point X of $\mathscr{B}$. Now the line WX contains two points of $\mathscr{C}$, and passes through the singular point W of $\mathscr{C}$. Thus the line WX is contained in $\mathscr{C}$ and so $Q \in WX$ is a point of $\mathscr{C}$. Hence, every point of $\mathbf{U}\mathscr{B}$ is a point of $\mathscr{C}$ and $\mathbf{U}\mathscr{B} \subseteq \mathscr{C}$. Since also $\mathscr{C} \subseteq \mathbf{U}\mathscr{B}$, it follows that $\mathscr{C} = \mathbf{U}\mathscr{B}$. □

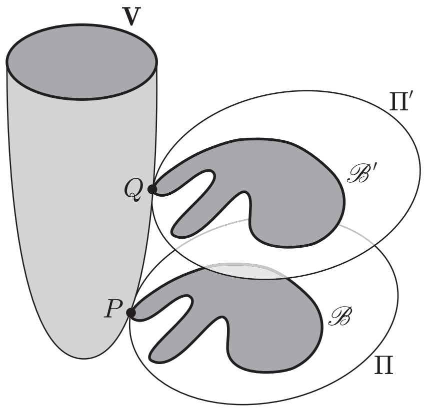
The above lemma will be used several times in the course of the thesis, both to describe quadrals and to show that a set of points is a quadral. Next, several combinatorial properties of cones are given. Many of these results have been used implicitly by other authors, but no formal statements or proofs seem to exist in the literature. The proofs given here are the author's.

Let $\mathbf{V}$ be a non-empty subspace of $\mathrm{PG}(n, q)$ and let Π be a complementary subspace of $\mathbf{V}$. Let $\mathscr{B}$ be a set of points in Π. Denote the dimension of Π by b and the dimension of $\mathbf{V}$ by d. Note that since Π and $\mathbf{V}$ are complementary subspaces, $b + d = n - 1$. Denote the cone $\mathbf{V}\mathscr{B}$ by $\mathscr{C}$. The following results all concern the cone $\mathscr{C}$. The first results concern the intersection of a subspace with $\mathscr{C}$.



Lemma 1.3.6 *Let* Π' *be any b-space of* $\mathrm{PG}(n,q)$ *skew to* $\mathbf{V}$. *Then* Π' *meets the cone* $\mathbf{V}\mathscr{B}$ *in a set of points* $\mathscr{B}'$ *projectively equivalent to* $\mathscr{B}$. *Moreover,* $\mathbf{V}\mathscr{B} = \mathbf{V}\mathscr{B}'$.

Proof Since Π' is skew to $\mathbf{V}$, it follows that Π' and $\mathbf{V}$ are complementary subspaces. If $\mathscr{B}$ is the empty set, then $\mathbf{V}\mathscr{B}$ is the subspace $\mathbf{V}$, so every b-space skew to $\mathbf{V}$ meets the cone in the empty set.



If $\mathscr{B}$ is non-empty, then $\mathbf{V}\mathscr{B}$ is the union of the $(d+1)$-spaces $\mathbf{V}\oplus P$ for the points $P \in \mathscr{B}$. Thus the set of points where $\mathbf{V}\mathscr{B}$ meets Π' is the set of points where these $(d+1)$-spaces meet Π'. Since Π' and $\mathbf{V}$ are complementary subspaces, each $(d+1)$-space through $\mathbf{V}$ meets Π' in a unique point. Moreover, each point not in $\mathbf{V}$ lies in a unique $(d+1)$-space through $\mathbf{V}$, so the $(d+1)$-spaces $\mathbf{V}\oplus P$ for $P \in \mathscr{B}$ each define a distinct point of $\mathbf{V}\mathscr{B}$ in Π'. Thus the intersection of Π' and $\mathbf{V}\mathscr{B}$ is a projection of $\mathscr{B}$ from $\mathbf{V}$ and so $\mathscr{B}'$ is projectively equivalent to $\mathscr{B}$.

The cone $\mathbf{V}\mathscr{B}$ is the union of the $(d+1)$-spaces $\mathbf{V}\oplus P$ for $P \in \mathscr{B}$. But each of these subspaces defines a distinct point of $\mathscr{B}'$. So $\mathbf{V}\mathscr{B}$ is also the union of the $(d+1)$-spaces $\mathbf{V}\oplus Q$ for $Q \in \mathscr{B}'$. That is $\mathbf{V}\mathscr{B} = \mathbf{V}\mathscr{B}'$. □

The above information about the structure of a cone makes it possible to count the number of points in a cone, and the number of each different type of line associated with the cone.

Lemma 1.3.7 *Denote the size of* $\mathscr{B}$ *by* B. *Then the number of points of the cone* $\mathbf{V}\mathscr{B}$ *is* $q^{d+1}B + \theta_d$.

Proof If $\mathscr{B}$ is empty, then $\mathbf{V}\mathscr{B} = \mathbf{V}$, so $|\mathbf{V}\mathscr{B}| = |\mathbf{V}| = \theta_d = q^{d+1} \cdot 0 + \theta_d$.

If $\mathscr{B}$ is non-empty, then $\mathbf{V}\mathscr{B}$ is the union of the B $(d+1)$-spaces $\mathbf{V} \oplus P$ for $P \in \mathscr{B}$. These $(d+1)$-spaces each contain q^{d+1} points not in $\mathbf{V}$, so $|\mathbf{V}\mathscr{B}| = q^{d+1}B + \theta_d$. □

Lemma 1.3.8 *Denote by $L_{m\Pi}$ the number of lines in Π meeting $\mathscr{B}$ in m points, where $m \neq 1,\ q+1$. Then the number of lines of* $\mathrm{PG}(n, q)$ *meeting the cone $\mathbf{V}\mathscr{B}$ in m points is $q^{2d+2}L_{m\Pi}$.*

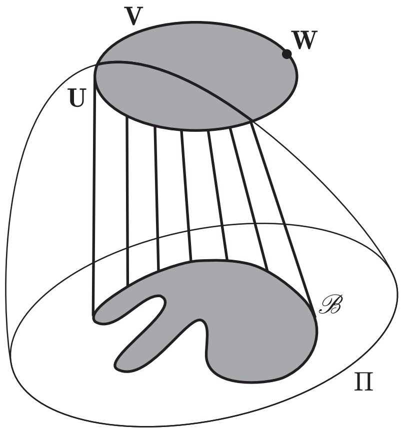
Proof The statement will be proved by induction on the dimension d of $\mathbf{V}$. Suppose $d = 0$. That is, $\mathbf{V}$ is a point and Π is a hyperplane.

Consider the set of pairs $X = \{(\ell, \Sigma) \mid \ \ell$ is a line meeting $\mathbf{V}\mathscr{B}$ in m points, Σ is a hyperplane through ℓ but not through $\mathbf{V}$ $\}$. The size of X will be counted in two ways.

Let ℓ be a line meeting $\mathbf{V}\mathscr{B}$ in m points. Denote the number of such lines by L_m. The cone $\mathbf{V}\mathscr{B}$ is either just the point $\mathbf{V}$ or is the union of lines through $\mathbf{V}$. So every line of $\mathrm{PG}(n, q)$ through $\mathbf{V}$ meets $\mathbf{V}\mathscr{B}$ in 1 or $q+1$ points. Thus ℓ does not pass through $\mathbf{V}$. There are θ_{n-2} hyperplanes through ℓ, and there are θ_{n-3} hyperplanes through the plane $\ell \oplus \mathbf{V}$. So there are q^{n-2} hyperplanes through ℓ but not $\mathbf{V}$. This gives $|X| = q^{n-2}L_m$.

Let Σ be a hyperplane not through $\mathbf{V}$. Then Σ meets $\mathbf{V}\mathscr{B}$ in a set of points projectively equivalent to $\mathscr{B}$ by Lemma 1.3.6. So Σ contains $L_{m\Pi}$ lines meeting $\mathbf{V}\mathscr{B}$ in m points. The number of hyperplanes through V is θ_{n-1}, so the number of hyperplanes not through $\mathbf{V}$ is q^n. This gives $|X| = q^n L_{m\Pi}$.

Hence $q^{n-2}L_m = q^n L_{m\Pi}$ and so $L_m = q^2 L_{m\Pi}$.



Suppose $d > 0$ and assume that the result is true when $\mathbf{V}$ has dimension $d-1$. Let $\mathbf{U}$ be a $(d-1)$-space in $\mathbf{V}$ and let W be a point of $\mathbf{V}$ not in $\mathbf{U}$. Then $\mathbf{V} = W \oplus \mathbf{U}$, so $\mathbf{V}\mathscr{B} = (W \oplus \mathbf{U})\mathscr{B} = W(\mathbf{U}\mathscr{B})$. Since $\mathbf{U}$ is a $(d-1)$-space in $\mathbf{U} \oplus \Pi$, the number of lines of $\mathbf{U} \oplus \Pi$ meeting $\mathbf{U}\mathscr{B}$ in m points is $q^{2d}L_{m\Pi}$.

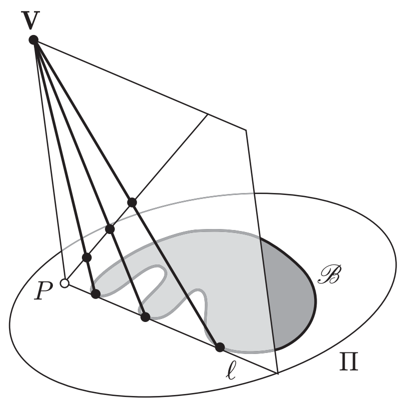
Since the dimension of W is zero, it follows that the number of lines of $\mathrm{PG}(n,q)$ meeting $W(\mathbf{U}\mathscr{B})$ in m points is $q^2 \cdot q^{2d}L_{m\Pi} = q^{2d+2}L_{m\Pi}$.

□

This section is completed with some results concerning the number of each type of line through a point. The following terminology is introduced for the next three lemmas. If P is a point of $\mathrm{PG}(n,q)$, denote by L_{mP} the number of lines through the point P meeting $\mathscr{C}$ in m points. If Λ is a subspace through P, then denote by $L_{mP\Lambda}$ the number of lines of Λ through P meeting $\mathscr{C}$ in m points.

Lemma 1.3.9 *Let P be a point of Π. Then $L_{mP} = q^{d+1}L_{mP\Pi}$, for $m \neq 1,\ q+1$.*

Proof The statement is proved by induction on the dimension d of $\mathbf{V}$. Suppose $d = 0$. That is, suppose $\mathbf{V}$ is a point, and Π is a hyperplane.



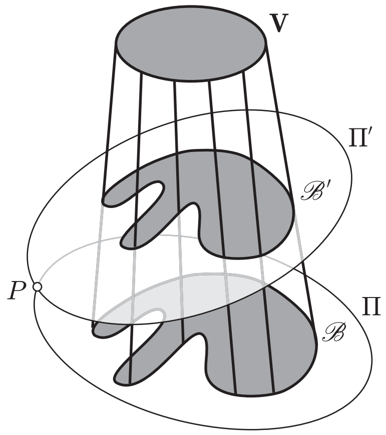
The lines through P are contained the planes through $P\mathbf{V}$, so consider a plane π through $P\mathbf{V}$. The plane π meets Π in a line through P. Denote this line by ℓ and denote the number of points of $\mathscr{B}$ on ℓ by k.

The cone $\mathbf{V}\mathscr{B}$ is the union of the lines $\mathbf{V}Q$ for $Q \in \mathscr{B}$. Thus $\mathbf{V}\mathscr{B}$ meets π in the union of the lines $\mathbf{V}Q$ for $Q \in \ell \cap \mathscr{B}$. Since ℓ has k points of $\mathscr{B}$, the plane π meets $\mathbf{V}\mathscr{B}$ in the union of k lines through V. (If $k = 0$, then π meets $\mathbf{V}\mathscr{B}$ in just the point V.) Thus in π the line $P\mathbf{V}$ meets $\mathbf{V}\mathscr{B}$ in one point or $q+1$ points, and the remaining q lines of π through P each meet $\mathbf{V}\mathscr{B}$ in k points.

So the number of lines through P meeting $\mathscr{C}$ in m points in each plane through $P\mathbf{V}$ is q or 0, depending on whether the line $\pi \cap \Pi$ meets $\mathscr{B}$ in m points or not. There are $L_{mP\Pi}$ lines of Π meeting $\mathscr{B}$ in m points, so the number of lines of $\mathrm{PG}(n, q)$ through P meeting $\mathbf{V}\mathscr{B}$ in m points is $qL_{mP\Pi}$.

Suppose $d > 0$ and assume that the result is true when $\mathbf{V}$ has dimension $d-1$. Let $\mathbf{U}$ be a $(d-1)$-space in $\mathbf{V}$ and let W be a point of $\mathbf{V}$ not in $\mathbf{U}$. Then $\mathbf{V} = W \oplus \mathbf{U}$, so $\mathbf{V}\mathscr{B} = (W \oplus \mathbf{U})\mathscr{B} = W(\mathbf{U}\mathscr{B})$. Since $\mathbf{U}$ is a $(d-1)$-space in $\mathbf{U} \oplus \Pi$, the number of lines of $\mathbf{U} \oplus \Pi$ through P meeting $\mathbf{U}\mathscr{B}$ in m points is $q^d L_{mP\Pi}$.

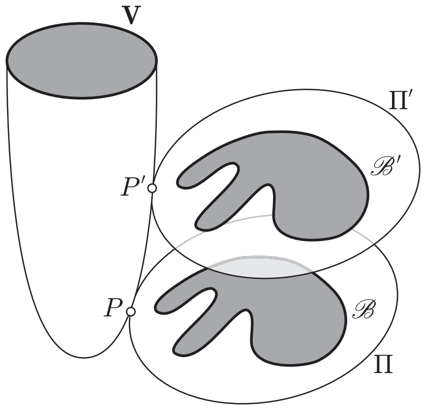
Since the dimension of W is zero, it follows that the number of lines of $\mathrm{PG}(n, q)$ through P meeting $W(\mathbf{U}\mathscr{B})$ in m points is $q \cdot q^d L_{mP\Pi} = q^{d+1} L_{mP\Pi}$. That is, $L_{mP} = q^{d+1} L_{mP\Pi}$. □



Lemma 1.3.10 *Let P be a point of Π. If Π' is a b-space through P skew to $\mathbf{V}$, then $L_{mP\Pi'} = L_{mP\Pi}$, for $m \neq 1,\ q+1$.*

Proof Denote by $\mathscr{B}'$ the set of points $\Pi' \cap \mathscr{C}$. By Lemma 1.3.9, $L_{mP} = q^{d+1} L_{mP\Pi}$. However, by Lemma 1.3.6, $\mathbf{V}\mathscr{B} = \mathbf{V}\mathscr{B}'$, so also $L_{mP} = q^{d+1} L_{mP\Pi'}$. Equating these two expressions for L_{mP} gives $L_{mP\Pi} = L_{mP\Pi'}$. □

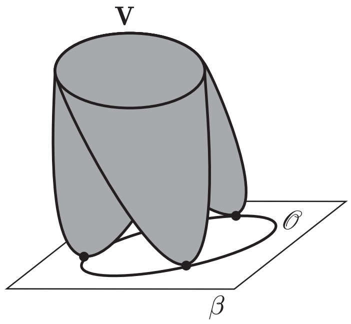
Lemma 1.3.11 *Let P be a point of Π and let P' be a point of $\mathbf{V} \oplus P$ not in $\mathbf{V}$. Then $L_{mP'} = L_{mP}$, for $m \neq 1,\ q+1$.*



Proof Let Π' be a b-space through P' skew to $\mathbf{V}$ and meeting $\mathscr{C}$ in the set of points $\mathscr{B}'$. Then $\mathscr{B}$ and $\mathscr{B}'$ are projectively equivalent and $\mathbf{V}\mathscr{B} = \mathbf{V}\mathscr{B}'$. This equivalence is defined by projecting Π' onto Π from $\mathbf{V}$. Under this projection, $\mathscr{B}$ maps to $\mathscr{B}'$ and P maps to P'. Thus the number of lines of Π through P meeting $\mathscr{B}$ in m points is the number of lines of Π' through P' meeting $\mathscr{B}'$ in m points. Lemma 1.3.10 implies that $L_{mP} = L_{mP'}$. □

1.4 Oval and Ovoid Cones

The previous section described the properties of cones. In this section, these properties are described in particular for oval and ovoid cones, which are included in the list of quadrals. First, the oval cones are dealt with.



Let β be a plane of $\mathrm{PG}(n, q)$, $n \geq 3$, and let $\mathscr{O}$ be an oval in β. Let $\mathbf{V}$ be an $(n-3)$-space skew to β. Denote the cone $\mathbf{V}\mathscr{O}$ by $\mathscr{C}$. The following lemmas concern the oval cone $\mathscr{C}$. Many are restatements of the lemmas in the previous section on general cones. The first lemma follows from Lemma 1.3.6.

Lemma 1.4.1 *Let π be a plane of $\mathrm{PG}(n, q)$ skew to $\mathbf{V}$ meeting $\mathscr{C}$ in the set of points $\mathscr{O}'$. Then $\mathscr{O}'$ is an oval projectively equivalent to $\mathscr{O}$ and $\mathscr{C} = \mathbf{V}\mathscr{O}'$.*

A plane meeting $\mathscr{C}$ in an oval is called a *secant plane* of $\mathscr{C}$. Using the secant planes, it is possible to determine how lines meet $\mathscr{C}$.

Lemma 1.4.2 *Every line meets $\mathscr{C}$ in 0, 1, 2 or $q+1$ points.*

Proof Let ℓ be a line of $\mathrm{PG}(n,q)$. If ℓ meets the vertex $\mathbf{V}$, it necessarily meets $\mathscr{C}$ in 1 or $q+1$ points. If ℓ is skew to $\mathbf{V}$, it is contained in a plane skew to $\mathbf{V}$, which meets $\mathscr{C}$ in an oval. Thus ℓ meets $\mathscr{C}$ in 0, 1 or 2 points. □

In light of the above lemma, the following names are given to the lines of $\mathrm{PG}(n,q)$.

Definition 1.4.3 *Let ℓ be a line of $\mathrm{PG}(n,q)$. If ℓ contains no point of $\mathscr{C}$, it is called an* external line *of $\mathscr{C}$. If ℓ contains exactly one point of $\mathscr{C}$, it is called a* tangent *of $\mathscr{C}$. If ℓ contains exactly two points of $\mathscr{C}$, it is called a* secant *of $\mathscr{C}$. If ℓ contains $q+1$ points of $\mathscr{C}$, it is called a* generator line *of $\mathscr{C}$, or a* line of $\mathscr{C}$*.*

It is also possible to determine the intersection of a plane with $\mathscr{C}$.

Lemma 1.4.4 *Every plane meets $\mathscr{C}$ in one point, a line, a pair of lines, an oval, or else is contained in $\mathscr{C}$.*

Proof Let π be a plane of $\mathrm{PG}(n,q)$. If π is skew to $\mathbf{V}$, then $\pi \cap \mathscr{C}$ is an oval by Lemma 1.4.1.

Suppose $\pi \cap \mathbf{V}$ contains the point W and let ℓ be a line in π not through W, meeting $\mathscr{C}$ in the set of points $\mathscr{C}_\ell$. The point W is a singular point of the set $\pi \cap \mathscr{C}$ so by Lemma 1.3.5, $\pi \cap \mathscr{C}$ is the cone $W\mathscr{C}_\ell$. By Lemma 1.4.2, the line ℓ meets $\mathscr{C}$ in 0, 1, 2 or $q+1$ points, so $W\mathscr{C}_\ell$ is the point W, a line through W, a pair of lines through W, or the whole of π. □

The above result is the main reason why oval cones are included in the list of quadrals. This will become clear in the following chapter, where quadrals are characterised by their plane sections. The following lemma completes the results on the intersection of a subspace with an oval cone.

Lemma 1.4.5 *Suppose $n \geq 4$ and let Λ be a 3-space of $\mathrm{PG}(n,q)$. Then Λ meets $\mathscr{C}$ in a line, a plane, a pair of planes, an oval cone or the whole of Λ.*

Proof Since $\mathbf{V}$ is an $(n-3)$-space, the 3-space Λ meets $\mathbf{V}$ in at least one point. Let W be a point of $\mathbf{V}$ in Λ and let π be a plane in Λ not through W meeting $\mathscr{C}$ in the set of points $\mathscr{C}_\pi$. The point W is a singular point of the set $\Lambda \cap \mathscr{C}$, so by Lemma 1.3.5, $\Lambda \cap \mathscr{C}$ is the cone $W\mathscr{C}_\pi$.

By Lemma 1.4.4, the plane π meets $\mathscr{C}$ in a point, a line, a pair of lines, an oval or the whole of π. Thus $W\mathscr{C}_\pi$ is a line through W, a plane through W, a pair of planes through W, an oval cone with vertex W or the whole of Λ. □

Next, the numbers of various points and lines are counted. These results follow from Lemmas 1.3.7 to 1.3.11 and the properties of ovals listed in Lemmas 1.1.6, 1.1.8 and 1.1.14.

Lemma 1.4.6 *The oval cone $\mathscr{C}$ has θ_{n-1} points, $\frac{1}{2}q^{2n-3}(q-1)$ external lines and $\frac{1}{2}q^{2n-3}(q+1)$ secants. A point of $\mathscr{C}$ not in $\mathbf{V}$ lies on q^{n-1} secants.*

Lemma 1.4.7 *Suppose q is odd and let P be a point not in $\mathscr{C}$. Then one of the following occurs.*

- *The point P lies on $\frac{1}{2}q^{n-2}(q-1)$ secants and $\frac{1}{2}q^{n-2}(q-1)$ external lines, and is an exterior point of the oval $\mathscr{C} \cap \pi$ for every secant plane π through P. Every point in $\mathbf{V} \oplus P$ not in $\mathbf{V}$ also lies on $\frac{1}{2}q^{n-2}(q-1)$ secants and $\frac{1}{2}q^{n-2}(q-1)$ external lines.*
- *The point P lies on $\frac{1}{2}q^{n-2}(q+1)$ secants and $\frac{1}{2}q^{n-2}(q+1)$ external lines, and is an interior point of the oval $\mathscr{C} \cap \pi$ for every secant plane π through P. Every point in $\mathbf{V} \oplus P$ not in $\mathbf{V}$ also lies on $\frac{1}{2}q^{n-2}(q+1)$ secants and $\frac{1}{2}q^{n-2}(q+1)$ external lines.*

In light of the above lemma, the following definition is made.

Definition 1.4.8 *Suppose q is odd. A point P not in $\mathscr{C}$ lying on $\frac{1}{2}q^{n-2}(q-1)$ secants of $\mathscr{C}$ is called an* exterior point *of $\mathscr{C}$. A point P not in $\mathscr{C}$ lying on $\frac{1}{2}q^{n-2}(q+1)$ secants of $\mathscr{C}$ is called an* interior point *of $\mathscr{C}$.*

The case where q is even is dealt with now.

Lemma 1.4.9 *Suppose q is even and let P be a point not in $\mathscr{C}$. Then one of the following occurs.*

- *The point P lies on $\frac{1}{2}q^{n-1}$ secants and $\frac{1}{2}q^{n-1}$ external lines, and is not the nucleus of the oval $\mathscr{C} \cap \pi$ for any secant plane π through P. Every point of $\mathbf{V} \oplus P$ not in $\mathbf{V}$ also lies on $\frac{1}{2}q^{n-1}$ secants and $\frac{1}{2}q^{n-1}$ external lines.*
- *The point P lies on no secants and no external lines. That is P lies only on tangents of $\mathscr{C}$. The point P is the nucleus of the oval $\mathscr{C} \cap \pi$ for every secant plane π through P. Every point of $\mathbf{V} \oplus P$ also lies on no secants and no external lines.*

In light of the above lemma, the following definition is made.

Definition 1.4.10 *Suppose q is even and let N be the nucleus of the oval $\mathscr{O}$ in β. The subspace $\mathbf{V} \oplus N$ is called the* nuclear space *of $\mathscr{C}$. If $n = 3$, then the nuclear space is called the* nuclear line*.*

By Lemma 1.4.9, the points of the nuclear space not in the vertex are the points of $\mathrm{PG}(n, q)$ lying only on tangents of $\mathscr{C}$. This implies that the tangents and generator lines of the cone $\mathscr{C}$ are the lines meeting the nuclear space. If $n = 3$, then this means that the tangents and generator lines of the cone are the lines meeting the nuclear line, and so form a special linear complex.

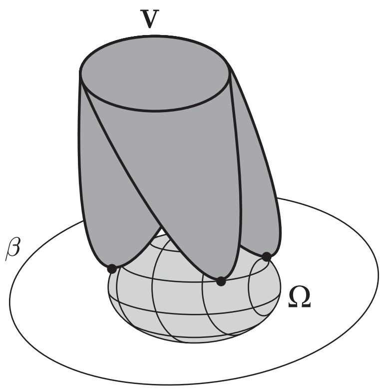
The set of points obtained by adjoining the nuclear space to $\mathscr{C}$ is a *hyperoval cone*. Similar statements to those above also apply to hyperoval cones. In particular, every line meets a hyperoval cone in 0, 1, 2 or $q + 1$ points, and the same terms are used for lines depending on how they meet the hyperoval cone. The following lemma will be useful later.

Lemma 1.4.11 *Suppose q is even and let N be the nucleus of the oval $\mathscr{O}$ in β. Denote the hyperoval $\mathscr{O} \cup \{N\}$ by $\bar{\mathscr{O}}$ and the hyperoval cone $\mathbf{V}\bar{\mathscr{O}}$ by $\bar{\mathscr{C}}$. The set of external lines of $\mathscr{C}$ is the same as the set of external lines of $\bar{\mathscr{C}}$.*

Proof Let ℓ be an external line of $\bar{\mathscr{C}}$. Since $\mathscr{C}$ is contained in $\bar{\mathscr{C}}$, ℓ is also an external line of $\mathscr{C}$.

Let ℓ be an external line of $\mathscr{C}$. The points of the nuclear space $\mathbf{V} \oplus N$ of $\mathscr{C}$ lie on no external lines, and so ℓ contains no point of $\mathbf{V} \oplus N$. Thus ℓ contains no point of $\mathscr{C} \cup (\mathbf{V} \oplus N) = \bar{\mathscr{C}}$. That is, ℓ is an external line of $\bar{\mathscr{C}}$. □

Note that if $\mathscr{O}'$ is an oval obtained by removing any point of the hyperoval $\bar{\mathscr{O}}$ in the above lemma, then the oval cone $\mathbf{V}\mathscr{O}'$ also has the same set of external lines as $\mathbf{V}\bar{\mathscr{O}}$. This completes the discussion on the properties of oval cones.



The properties of ovoid cones will now be discussed. Let β be a 3-space of $\mathrm{PG}(n,q)$, $n \geq 4$, and let Ω be an ovoid in β. Let $\mathbf{V}$ be an $(n-4)$-space skew to β. Denote the cone $\mathbf{V}\Omega$ by $\mathscr{D}$. The following lemmas concern the ovoid cone $\mathscr{D}$. Many are restatements of the lemmas in the previous section on general cones. The first results follows from Lemma 1.3.6.

Lemma 1.4.12 *Let Π be a 3-space of $\mathrm{PG}(n,q)$ skew to $\mathbf{V}$ meeting $\mathscr{D}$ in the set of points Ω'. Then Ω' is an ovoid projectively equivalent to Ω and $\mathscr{D} = \mathbf{V}\Omega'$.*

Lemma 1.4.13 *Every line meets $\mathscr{D}$ in 0, 1, 2 or $q+1$ points.*

Proof Let ℓ be a line of $\mathrm{PG}(n,q)$. If ℓ meets the vertex $\mathbf{V}$, it necessarily meets $\mathscr{D}$ in 1 or $q+1$ points. If ℓ is skew to $\mathbf{V}$, it is contained in a 3-space skew to $\mathbf{V}$, which meets $\mathscr{D}$ in an ovoid. Thus ℓ meets $\mathscr{D}$ in 0, 1 or 2 points. □

In light of the above lemma, the following names are given to the lines of $\mathrm{PG}(n,q)$.

Definition 1.4.14 *Let ℓ be a line of $\mathrm{PG}(n,q)$. If ℓ contains no point of $\mathscr{D}$, it is called an* external line *of $\mathscr{D}$. If ℓ contains exactly one point of $\mathscr{D}$, it is called a* tangent *of $\mathscr{D}$. If ℓ contains exactly two points of $\mathscr{D}$, it is called a* secant *of $\mathscr{D}$. If ℓ contains $q+1$ points of $\mathscr{D}$, it is called a* generator line *of $\mathscr{D}$, or a* line *of $\mathscr{D}$.*

Lemma 1.4.15 *Every plane meets $\mathscr{D}$ in one point, a line, a pair of lines, an oval, or else is contained in $\mathscr{D}$.*

Proof Let π be a plane of $\mathrm{PG}(n, q)$. If π is skew to $\mathbf{V}$, then π is contained in a 3-space skew to $\mathbf{V}$. This 3-space meets $\mathscr{D}$ in an ovoid, and so π meets $\mathscr{D}$ in a point or an oval.

Suppose $\pi \cap \mathbf{V}$ contains the point W and let ℓ be a line in π not through W, meeting $\mathscr{D}$ in the set of points $\mathscr{D}_\ell$. The point W is a singular point of the set $\pi \cap \mathscr{D}$ so by Lemma 1.3.5, $\pi \cap \mathscr{D}$ is the cone $W\mathscr{D}_\ell$. By Lemma 1.4.2, the line ℓ meets $\mathscr{D}$ in 0, 1, 2 or $q+1$ points, so $W\mathscr{D}_\ell$ is the point W, a line through W, a pair of lines through W, or the whole of π. □

This completes the investigation of the intersection of subspaces with ovoid cones. The following lemma counts the numbers of various points and lines. This lemma follows from Lemmas 1.3.7 to 1.3.11 and the properties of ovoids listed in Lemmas 1.2.11 and 1.2.15.

Lemma 1.4.16 *The ovoid cone $\mathscr{D}$ has $\theta_{n-1} - q^{n-2}$ points, $\frac{1}{2}q^{2n-4}(q^2+1)$ secants and $\frac{1}{2}q^{2n-4}(q^2+1)$ external lines. A point of $\mathscr{D}$ not in $\mathbf{V}$ lies on q^{n-1} secants. A point not in $\mathscr{D}$ lies on $\frac{1}{2}q^{n-2}(q-1)$ secants and $\frac{1}{2}q^{n-2}(q+1)$ external lines.*

This completes the discussion of ovoid cones.

1.5 Quadrics

The quadrals defined so far are the ovals, the ovoids, the oval cones and the ovoid cones. The remaining quadrals are all quadrics, so this section deals with the properties of quadrics.

Definition 1.5.1 *A* quadric *of* PG(n, q) *is the set of points of* PG(n, q) *satisfying a homogeneous quadratic equation of degree 2. That is, an equation of the form:*

$$\phi(x_0, \ldots, x_n) = a_{00}x_0^2 + \cdots + a_{nn}x_n^2 + \sum_{i<j} a_{ij}x_ix_j = 0$$

$$\textit{where } a_{ij} \in \mathrm{GF}(q) \textit{ for } 0 \leq i, j \leq n \textit{ and not all } a_{ij} \textit{ are equal to } 0$$

A quadric of PG$(2, q)$ *is called a* conic.

In some situations, points satisfying the equation in PG(n, q^2) are also considered to be points of the quadric. However, in this thesis only those points in PG(n, q) satisfying the equation are considered to be points of the quadric. That is, a quadric is identified with its set of points in PG(n, q).

Before any of the particular types of quadric are described, it is appropriate to state some combinatorial properties of quadrics. Let $\mathscr{Q}$ be a quadric of PG(n, q) with equation $\phi(x_0, \ldots, x_n) = 0$ as given above. The following discussion concerns the quadric $\mathscr{Q}$. Firstly, the interesection of subspaces with $\mathscr{Q}$ are considered. The intersection of an m-space with $\mathscr{Q}$ is called an *m-space section of $\mathscr{Q}$*. The following lemma asserts that in general, an m-space section of $\mathscr{Q}$ is a quadric of PG(m, q).

Lemma 1.5.2 [21] *Let* Π *be a subspace of* PG(n, q) *of dimension at least* 2. *Then either* Π *is contained in* $\mathscr{Q}$ *or* $\Pi \cap \mathscr{Q}$ *is a quadric of* Π.

The above lemma implies in particular that if $n \geq 3$, then a plane of PG(n, q) is either contained in $\mathscr{Q}$, or else meets $\mathscr{Q}$ in a conic. The converse of this statement is also true. That is, if $\mathscr{K}$ is a set of points such that every plane is either contained in $\mathscr{K}$ or else meets $\mathscr{K}$ in a conic, then $\mathscr{K}$ is a quadric. This is Theorem 2.2.5 in Chapter 2.

A similar argument to that used to prove the above theorem can be used to describe how a line meets $\mathscr{Q}$.

Lemma 1.5.3 [21] *Every line of* PG(n, q) *meets* $\mathscr{Q}$ *in* 0, 1, 2 *or* $q + 1$ *points.*

In light of the above result, the following special names are given to the lines of PG(n, q).

Definition 1.5.4 *Let* ℓ *be a line of* $\mathrm{PG}(n, q)$*. If* ℓ *contains no point of* $\mathscr{Q}$*, it is called an* external line *of* $\mathscr{Q}$*. If* ℓ *contains exactly one point of* $\mathscr{Q}$*, it is called a* tangent *of* $\mathscr{Q}$*. If* ℓ *contains exactly two points of* $\mathscr{Q}$*, it is called a* secant *of* $\mathscr{Q}$*. If* ℓ *contains* $q + 1$ *points of* $\mathscr{Q}$*, it is called a* generator line *of* $\mathscr{Q}$*, or a* line of $\mathscr{Q}$*.*

As for caps and their cones, if it is clear that $\mathscr{Q}$ is the quadric under consideration, an external line of $\mathscr{Q}$ may be referred to simply as an external line, and similarly for tangents, secants and generator lines.

Using the equation $\phi = 0$ of $\mathscr{Q}$, it is possible to define algebraic conditions to decide when a line is a tangent, secant, external line or generator line. Using this algebraic condition, the following results may be proved.

Lemma 1.5.5 [39, Chapter 22] *Let* Q *be a point of the quadric* $\mathscr{Q}$*. Denote the set of points lying on the tangents and generator lines of* $\mathscr{Q}$ *through* Q *by* $\mathbf{T}_Q$*. Then* $\mathbf{T}_Q$ *is either a hyperplane or the whole of* $\mathrm{PG}(n, q)$*.*

The subspace $\mathbf{T}_Q$ of all the points lying on a tangent or generator line of $\mathscr{Q}$ through Q is called the *tangent space* of $\mathscr{Q}$ at Q. If $\mathbf{T}_Q$ is a hyperplane, then it is called the *tangent hyperplane* of $\mathscr{Q}$ at Q.

In light of the above discussion, the following definition is made.

Definition 1.5.6 *Let* Q *be a point of* $\mathscr{Q}$*. The point* Q *is called a* singular point *of* $\mathscr{Q}$ *if* $\mathbf{T}_Q = \mathrm{PG}(n, q)$*. Otherwise the point* Q *is called a* non-singular point*. A quadric* $\mathscr{Q}$ *is called a* singular quadric *if it contains a singular point. Otherwise the quadric is called* non-singular*.*

From Lemma 1.5.5, a point Q of the quadric $\mathscr{Q}$ is a singular point if and only if every line through Q is a tangent or generator line of $\mathscr{Q}$. Thus the definition of singular point for a quadric matches with the definition of singular point given in Section 1.3. Lemma 1.3.4 implies that the set of singular points of $\mathscr{Q}$ is a subspace, which is called the *singular space* of $\mathscr{Q}$. Using the singular space, it is possible to construct the quadric $\mathscr{Q}$ as a cone. The following lemma restates Lemma 1.3.5 for quadrics.

Lemma 1.5.7 *Suppose* $\mathbf{U}$ *is a non-empty subspace contained in the singular space of* $\mathscr{Q}$*. Let* Π *be a complementary subspace of* $\mathbf{U}$ *meeting* $\mathscr{Q}$ *in the set of points* $\mathscr{Q}_\Pi$*. Then* $\mathscr{Q}$ *is the cone* $\mathbf{U}\mathscr{Q}_\Pi$*.*

If the subspace $\mathbf{U}$ is chosen to be the whole singular space of $\mathscr{Q}$, then more can be said about the structure of the singular quadric. First, the quadrics with the singular spaces of largest dimension are considered.

Lemma 1.5.8 [39, Chapter 22] *Suppose the singular space of* $\mathscr{Q}$ *has dimension at least* $n-2$*. Then* $\mathscr{Q}$ *is an* $(n-2)$*-space, a hyperplane or a pair of hyperplanes.*

Proof Let $\mathbf{U}$ be an $(n-2)$-space contained in the singular space of $\mathscr{Q}$ and let ℓ be a line skew to $\mathbf{U}$ meeting $\mathscr{Q}$ in the set of points $\mathscr{Q}_\ell$. Then $\mathscr{Q}$ is the cone $\mathbf{U}\mathscr{Q}_\ell$. If $\mathscr{Q}_\ell$ is non-empty, this cone is the union of the hyperplanes $\mathbf{U} \oplus P$ for $P \in \mathscr{Q}_\ell$. Since ℓ meets $\mathscr{Q}$ in 0, 1, 2 or $q+1$ points, this implies that $\mathscr{Q}$ is the $(n-2)$-space $\mathbf{U}$, a hyperplane through $\mathbf{U}$, a pair of hyperplanes through $\mathbf{U}$ or the whole of $\mathrm{PG}(n,q)$. This last case is not a quadric, so the result follows. □

The three quadrics described in the above lemma are called the *reducible quadrics*. The quadric $\mathscr{Q}$ is reducible if and only if the homogeneous quadratic expression ϕ factorises into two homogeneous linear expressions over $\mathrm{GF}(q)$ or $\mathrm{GF}(q^2)$ [39, Chapter 22]. If $\mathscr{Q}$ is not reducible, it is called *irreducible*. The irreducible singular quadrics are now described.

Lemma 1.5.9 [39, Chapter 22] *Suppose* $\mathscr{Q}$ *is an irreducible singular quadric and denote its singular space by* $\mathbf{V}$*. Let* Π *be a complementary subspace of* $\mathbf{V}$ *meeting* $\mathscr{Q}$ *in the set of points* $\mathscr{Q}_\Pi$*. Then* $\mathscr{Q}_\Pi$ *is a non-singular quadric of* Π *and* $\mathscr{Q}$ *is the cone* $\mathbf{V}\mathscr{Q}_\Pi$*.*

Proof By the previous lemma, $\mathscr{Q} = \mathbf{V}\mathscr{Q}_\Pi$. The set of points $\mathscr{Q}_\Pi$ is a quadric by Lemma 1.5.2, so it remains to show that $\mathscr{Q}_\Pi$ is a *non-singular* quadric.

Suppose $\mathscr{Q}_\Pi$ contains the singular point W as a quadric of Π. Then every line in Π through W is a tangent or generator line of $\mathscr{Q}_\Pi$. That is, the number of secants of $\mathscr{Q}_\Pi$ through W is zero. Lemma 1.3.9 implies that the number of secants of $\mathbf{V}\mathscr{Q}_\Pi$ through W is also zero. Thus W is a singular point of $\mathscr{Q}$. This is a contradiction, so $\mathscr{Q}_\Pi$ is a non-singular quadric of Π. □

The converse of the above lemma is also true. That is, a cone over a quadric is a quadric.

Lemma 1.5.10 [39, Chapter 22] *Let $\mathbf{V}$ and Π be complementary subspaces of $\mathrm{PG}(n, q)$ such that the dimension of Π is at least 2. Let $\mathscr{Q}_0$ be a quadric of Π. Then the cone $\mathbf{V}\mathscr{Q}_0$ is a quadric of $\mathrm{PG}(n, q)$.*

The above lemmas imply that in order to describe the combinatorial properties of irreducible quadrics, it is reasonable to consider only non-singular quadrics, since every irreducible singular quadric is a cone over a non-singular quadric.

It is appropriate to mention here the following combinatorial properties of non-singular quadrics and their tangent spaces. The first three results may be proved using the structure of conics (described below) and the fact that every plane meets a quadric in a conic.

Lemma 1.5.11 [39, Chapter 22] *Suppose $\mathscr{Q}$ is non-singular and $n \geq 3$. Let Σ be a hyperplane that is not a tangent hyperplane of $\mathscr{Q}$ at any point of $\mathscr{Q}$. Then Σ meets $\mathscr{Q}$ in a non-singular quadric of Σ.*

Lemma 1.5.12 [39, Chapter 22] *Suppose $\mathscr{Q}$ is non-singular and $n \geq 3$. Let P be a point of $\mathscr{Q}$ with tangent hyperplane $\mathbf{T}_P$. Then P is the unique singular point of the quadric $\mathscr{Q} \cap \mathbf{T}_P$ of $\mathbf{T}_P$.*

Lemma 1.5.13 [39, Chapter 22] *Suppose $\mathscr{Q}$ is non-singular and $n \geq 4$. Let P be a point of $\mathscr{Q}$. The tangent hyperplane $\mathbf{T}_P$ meets $\mathscr{Q}$ in a cone with vertex P and base a non-singular quadric in an $(n-2)$-space.*

Lemma 1.5.14 [39, Chapter 22] *Suppose $\mathscr{Q}$ is non-singular and $n \geq 3$. Then every tangent hyperplane of $\mathscr{Q}$ meets $\mathscr{Q}$ in the same number of points.*

Proof Let P be a point of $\mathscr{Q}$ and denote the number of points of $\mathscr{Q}$ in the tangent hyperplane $\mathbf{T}_P$ by T. The points of $\mathbf{T}_P$ are the points of $\mathrm{PG}(n,q)$ on the tangents and generator lines of $\mathscr{Q}$ through P. The q^{n-1} lines through P not in $\mathbf{T}_P$ are secants, each containing one further point of $\mathscr{Q}$. Thus $|\mathscr{Q}| = q^{n-1} + T$. In this equation, T is determined by $|\mathscr{Q}|$, so every tangent hyperplane meets $\mathscr{Q}$ in the same number of points. □

The discussion above concerned the structure of the tangents and generator lines of $\mathscr{Q}$ through a point of $\mathscr{Q}$, and the consequences of this information. The structure of the tangents through a point *not* on $\mathscr{Q}$ will now be discussed.

Lemma 1.5.15 [39, Chapter 22] *Let Q be a point not on $\mathscr{Q}$. Either every point of $\mathscr{Q}$ lies on a tangent through Q or there exists a hyperplane Σ such that $\Sigma \cap \mathscr{Q}$ is the set of points of $\mathscr{Q}$ lying on tangents through Q.*

If q is even, more can be said about the tangents through a point not on $\mathscr{Q}$.

Lemma 1.5.16 [39, Chapter 22] *Suppose q is even and let Q be a point not on $\mathscr{Q}$. The tangents of $\mathscr{Q}$ through Q form a subspace, which is either a hyperplane or the whole of* $\mathrm{PG}(n,q)$.

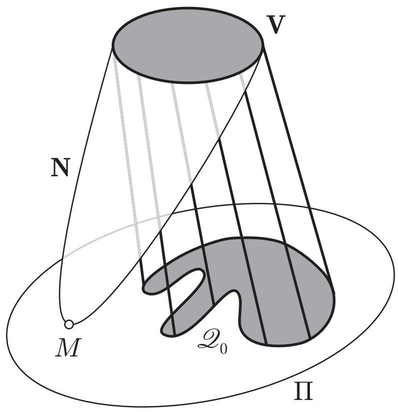
According to the above lemma, it is possible when q is even for a point not on $\mathscr{Q}$ to lie only on tangents. If q is odd, this is only possible if $\mathscr{Q}$ is a hyperplane [39, Chapter 22].

If q is even, a point N not on $\mathscr{Q}$ such that every line through N is a tangent is called a *nucleus* of $\mathscr{Q}$. The set of nuclei and singular points of $\mathscr{Q}$ is the set of points P of $\mathrm{PG}(n,q)$ such that every line through P meets $\mathscr{Q}$ in 1 or $q+1$ points. Using arguments similar to those used to prove Lemma 1.3.4, it can be proved that this set of points is a subspace. This subspace is called the *nuclear space* of $\mathscr{Q}$.

The following lemmas describe the structure of the nuclear space of $\mathscr{Q}$ for q even, depending on the the dimension n.

Lemma 1.5.17 [39, Chapter 22] *Suppose q is even and $\mathscr{Q}$ is non-singular. Then $\mathscr{Q}$ has no nucleus if n is odd, and a unique nucleus if n is even.*

Corollary 1.5.18 [39, Chapter 22] *Suppose q is even and $\mathscr{Q}$ has non-empty singular space $\mathbf{V}$ with dimension at most $(n-3)$. Let Π be a complementary subspace of $\mathbf{V}$ meeting $\mathscr{Q}$ in the non-singular quadric $\mathscr{Q}_0$. If $\mathscr{Q}_0$ has nucleus N, then the nuclear space of $\mathscr{Q}$ is $\mathbf{V} \oplus N$. If $\mathscr{Q}_0$ has no nucleus, then $\mathscr{Q}$ has no nucleus.*



Proof Denote the nuclear space of $\mathscr{Q}$ by $\mathbf{N}$. Every point of $\mathbf{N}$ not in $\mathbf{V}$ lies only on tangents of $\mathscr{Q}$. Thus every point of $\mathbf{N} \cap \Pi$ lies only on tangents of $\mathscr{Q}_0$. That is, any point of $\mathbf{N} \cap \Pi$ is a nucleus of $\mathscr{Q}_0$. On the other hand, if M is a nucleus of $\mathscr{Q}_0$, then M lies on no secants or external lines of $\mathscr{Q}_0$ in Π. Lemma 1.3.9 then implies that there are no secants or external lines of $\mathbf{V}\mathscr{Q}_0 = \mathscr{Q}$ through M in $\mathrm{PG}(n, q)$. That is, M is a nucleus of $\mathscr{Q}$. Hence, $\mathbf{N} \cap \Pi$ is the set of nuclei of $\mathscr{Q}_0$ in Π.

Suppose $\mathscr{Q}_0$ has no nucleus, this implies that $\mathbf{N} \cap \Pi$ is empty. Since $\mathbf{N}$ contains the singular space $\mathbf{V}$, and $\mathbf{V}$ is a complementary subspace of Π, this implies $\mathbf{N} = \mathbf{V}$.

Suppose $\mathscr{Q}$ has a unique nucleus M. Then $\mathbf{N} \cap \Pi = M$. Now Π is a complementary subspace of $\mathbf{V}$ and $\mathbf{N}$ contains $\mathbf{V}$ and meets Π in a unique point. Thus $\mathbf{N}$ has dimension one larger than $\mathbf{V}$. Hence $\mathbf{N} = \mathbf{V} \oplus M$. □

This completes the discussion of the nuclear space of a quadric. If $\mathscr{Q}$ has no singular points and no nuclei, then Lemmas 1.5.15 and 1.5.5 imply that for every point P there is a unique hyperplane Σ_P containing all the points of $\mathscr{Q}$ on the tangents and generator lines through P. If $P \in \mathscr{Q}$, then the hyperplane Σ_P is the tangent hyperplane of $\mathscr{Q}$ at P.

Denote by σ the map sending P to the hyperplane Σ_P for each P in $\mathrm{PG}(n, q)$.

Lemma 1.5.19 [39, Chapter 22] *The map σ defined above is a polarity of* $\mathrm{PG}(n, q)$.

The polarity σ is called the *polarity associated with* $\mathscr{Q}$ or simply the *polarity of* $\mathscr{Q}$. Given a subspace Π, its image $\sigma(\Pi)$ is called the *polar of* Π. The polar of a point P on $\mathscr{Q}$ is the tangent hyperplane of $\mathscr{Q}$ at P. The polar of a point P not on $\mathscr{Q}$ is a non-tangent hyperplane of $\mathscr{Q}$. The polarity associated with $\mathscr{Q}$ will be described in more detail later for particular quadrics. Three basic properties are be given now.

Lemma 1.5.20 [39, Chapter 22] *Suppose q is odd and $\mathscr{Q}$ is non-singular. Then the set of points of* $\mathrm{PG}(n,q)$ *lying on their own polar is the set of points of $\mathscr{Q}$.*

Lemma 1.5.21 [39, Chapter 22] *Suppose q is even, n is odd and $\mathscr{Q}$ is non-singular. Then every point lies on its own polar, so σ is a null polarity. Also, if P is a point not on $\mathscr{Q}$, then P is the nucleus of the quadric $\sigma(P)\cap\mathscr{Q}$.*

Proof Let P be a point of $\mathrm{PG}(n,q)$. If $P\in\mathscr{Q}$, then $\sigma(P)$ is the tangent hyperplane at P, so $P\in\sigma(P)$. If $P\notin\mathscr{Q}$, then $\sigma(P)$ is the hyperplane containing all points of $\mathscr{Q}$ on tangents through P. By Lemma 1.5.5, the tangents through P form a subspace, so P is contained in $\sigma(P)$. Since P lies only on tangents in $\sigma(P)$, it is the nucleus of the quadric $\sigma(P)\cap\mathscr{Q}$. Note that $\sigma(P)$ is not a tangent hyperplane, so $\sigma(P)\cap\mathscr{Q}$ is a non-singular quadric in a hyperplane and so has a unique nucleus by Lemma 1.5.17. □

Lemma 1.5.22 [39, Chapter 22] *Suppose $\mathscr{Q}$ is non-singular and either q is odd or q is even and n is odd. Denote the set of tangent hyperplanes of $\mathscr{Q}$ by $\mathscr{Q}^*$. Then $\mathscr{Q}^*$ is a non-singular quadric of the dual space* $\mathrm{PG}(n,q)^*$.

Proof Denote the polarity of $\mathscr{Q}$ by σ. Then σ is a collineation from $\mathrm{PG}(n,q)$ to $\mathrm{PG}(n,q)^*$. The image of a point P of $\mathscr{Q}$ under σ is the tangent hyperplane of $\mathscr{Q}$ at P. So $\mathscr{Q}^*$ is the image of $\mathscr{Q}$ under σ. Hence $\mathscr{Q}^*$ is a non-singular quadric of $\mathrm{PG}(n,q)^*$. □

This completes the discussion of the properties of general quadrics. The various quadrics in certain dimension will now be described. First, the conics in $\mathrm{PG}(2,q)$ are considered.

Theorem 1.5.23 [36, Chapter 7] *Let $\mathscr{Q}$ be a conic of* $\mathrm{PG}(2,q)$. *If $\mathscr{Q}$ is non-singular, then $\mathscr{Q}$ is an oval. If $\mathscr{Q}$ is singular, then $\mathscr{Q}$ is a point, a line or a pair of lines.*

Proof Suppose $\mathscr{Q}$ is singular. That is, suppose the singular space has dimension at least 0. By Lemma 1.5.8, $\mathscr{Q}$ is a point, a line or a pair of lines.

Suppose $\mathscr{Q}$ is non-singular and non-empty, and let P be a point of $\mathscr{Q}$. The tangent space of $\mathscr{Q}$ at P is a hyperplane of $\mathrm{PG}(2,q)$ – a line. The remaining q lines through P are secants with one further point of $\mathscr{Q}$. Thus $\mathscr{Q}$ has $q+1$ points, and also a line containing a point of $\mathscr{Q}$ meets $\mathscr{Q}$ in 1 or 2 points. Thus $\mathscr{Q}$ is an oval.

Hence, every conic is either the empty set, a point, a line, a pair of lines or an oval. The number of conics that are a point, a line, a pair of lines or an oval may be counted. The sum of these numbers is the total number of conics in $\mathrm{PG}(n,q)$ (see [21]). Thus every conic contains at least one point. Hence, a non-singular conic is an oval. □

Using Lemma 1.5.2, the above lemma implies that every plane meets a quadric in a point, a line, a pair of lines, an oval or the whole plane. As stated in Lemmas 1.2.12, 1.4.4 and 1.4.15, this property is also true of the other quadrals. In the next chapter, this property will be shown to characterise the quadrals. Also, a singular conic in $\mathrm{PG}(2,q)$ is a reducible conic and vice versa.

Before the quadrics in $\mathrm{PG}(n,q)$, $n \geq 3$ are described, the following results are stated.

Theorem 1.5.24 [36, Chapter 7] *All non-singular conics of* $\mathrm{PG}(2,q)$ *are projectively equivalent.*

Lemma 1.5.25 [36, Chapter 7] *Let $\{P_1, P_2, P_3, P_4, P_5\}$ be a set of five distinct points of* $\mathrm{PG}(2,q)$ *with no three collinear. Then there is a unique non-singular conic containing P_1, P_2, P_3, P_4 and P_5.*

The quadrics in $\mathrm{PG}(3,q)$ are now described.

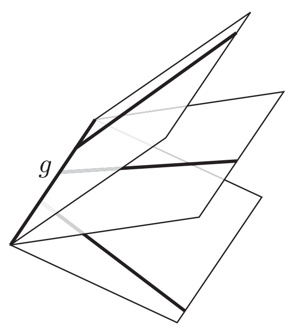
Theorem 1.5.26 [37, Chapter 15] *A non-singular quadric of* $\mathrm{PG}(3, q)$ *is either an ovoid or the set of points on the lines of a regulus.*

Proof Let $\mathscr{Q}$ be a non-singular quadric in $\mathrm{PG}(3, q)$ and let P be a point of $\mathscr{Q}$. Let π be the tangent plane of $\mathscr{Q}$ at P and denote by $\mathscr{Q}_\pi$ the intersection of π with $\mathscr{Q}$.

By Lemma 1.5.12, the point P is the only singular point of $\mathscr{Q}_\pi$. Thus $\mathscr{Q}_\pi$ is either the single point P or a pair of lines through P. By Lemma 1.5.14, either every tangent plane meets $\mathscr{Q}$ in a single point or every tangent plane meets $\mathscr{Q}$ in a pair of lines.

Suppose $\mathscr{Q}$ contains no lines. Then every line meets $\mathscr{Q}$ in 0, 1 or 2 points, and the tangents at each point of $\mathscr{Q}$ form a plane. Thus $\mathscr{Q}$ is an ovoid.

Suppose $\mathscr{Q}$ contains a line g and let Q be a point of g. The tangent plane $\mathbf{T}_Q$ of $\mathscr{Q}$ at Q contains all points on tangents or generator lines through Q, so $\mathbf{T}_Q$ contains g. Thus the plane $\mathbf{T}_Q$ meets $\mathscr{Q}$ in a pair of lines, and so *every* tangent plane meets $\mathscr{Q}$ in a pair of lines. Since all the generator lines through Q lie in the tangent plane at Q, this implies that there are exactly two generator lines of $\mathscr{Q}$ through each point of $\mathscr{Q}$.



Consider the $q + 1$ tangent planes at the points of g. These planes are all distinct and each defines a further line of $\mathscr{Q}$. Denote this set of lines by $\mathscr{R}$. Each line of $\mathscr{R}$ is contained in a different plane through g and passes through a different point of g. Thus the lines of $\mathscr{R}$ are mutually skew.

By Lemma 1.5.14, there are $q^2 + q + 1$ points on $\mathscr{Q}$. Thus $\mathscr{Q}$ is the set of points on the lines of $\mathscr{R}$. However, choosing a line of $\mathscr{R}$ in place of g shows that $\mathscr{Q}$ is also the set of points on another set of $q + 1$ mutually skew lines. Thus $\mathscr{R}$ is a regulus. $\square$

In light of the above theorem, the following definitions are made.

Definition 1.5.27 *Let* $\mathscr{Q}$ *be a non-singular quadric of* $\mathrm{PG}(n, q)$. *If* $\mathscr{Q}$ *is an ovoid, then* $\mathscr{Q}$ *is called an* elliptic quadric. *If* $\mathscr{Q}$ *is the set of points on the lines of a regulus, then* $\mathscr{Q}$ *is called a* hyperbolic quadric.

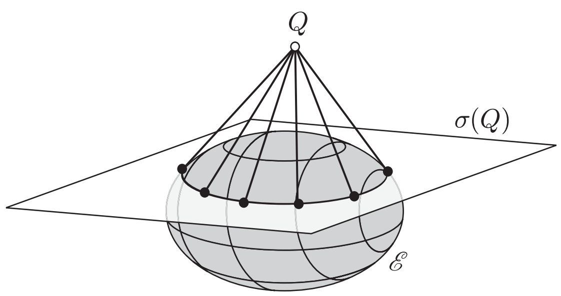
Using these definitions, it is possible to describe all the non-singular quadrics of $\mathrm{PG}(3, q)$.

Lemma 1.5.28 [37, Chapter 15] *All hyperbolic quadrics of* $\mathrm{PG}(3, q)$ *are projectively equivalent. All elliptic quadrics of* $\mathrm{PG}(3, q)$ *are projectively equivalent.*

The above lemma can be proved by assigning coordinates to $\mathrm{PG}(3, q)$ in such a way that the equation of the quadric is fully determined. Also of interest is the following partial converse.

Theorem 1.5.29 [37, Chapter 15] *The set of points on the lines of a regulus in* $\mathrm{PG}(3, q)$ *is always a hyperbolic quadric.*

Unfortunately, not every ovoid is an elliptic quadric if q is even, which is why ovoids are included in the list of quadrals, separately from elliptic quadrics. The combinatorial properties of elliptic quadrics are the combinatorial properties of ovoids and were given in Section 1.2. Some special properties of elliptic quadrics in $\mathrm{PG}(3, q)$, q odd, are mentioned here.



Let $\mathscr{E}$ be an elliptic quadric in $\mathrm{PG}(3, q)$, q odd, and denote the polarity associated with $\mathscr{E}$ by σ. If P is a point of $\mathscr{E}$, then $\sigma(P)$ is the tangent plane of $\mathscr{E}$ at P. If Q is a point not on $\mathscr{E}$, then $\sigma(Q)$ is a secant plane of $\mathscr{E}$. The points of $\mathscr{E}$ on the tangents through Q are the points of the non-singular conic $\sigma(Q) \cap \mathscr{E}$. By Lemma 1.5.20, the point Q does not lie in the plane $\sigma(Q)$, so the tangents through Q form a quadric cone. Thus the tangents of $\mathscr{E}$ form a $(q + 1)$-cover that is not a general linear complex.

The polarity σ sends secants of $\mathscr{E}$ to external lines of $\mathscr{E}$ and vice versa. Also, it sends tangents of $\mathscr{E}$ to tangents of $\mathscr{E}$, although it does not fix any of these tangents. Finally, if $\mathscr{E}^*$ is the set of tangent planes of $\mathscr{E}$, then $\mathscr{E}^*$ is an elliptic quadric of the dual space $\mathrm{PG}(3, q)^*$ by Lemma 1.5.22.

The properties of hyperbolic quadrics will now be described. First, the number of each type of lines is counted.

Lemma 1.5.30 [37, Chapter 15] *Let* $\mathscr{H}$ *be a hyperbolic quadric of* $\mathrm{PG}(3, q)$*. Then* $\mathscr{H}$ *has* $(q-1)(q+1)^2$ *tangents,* $\frac{1}{2}q^2(q+1)^2$ *secants,* $\frac{1}{2}q^2(q-1)^2$ *external lines and* $2(q+1)$ *generator lines.*

Next, the possible plane sections of a hyperbolic quadric are considered.

Lemma 1.5.31 [37, Chapter 15] *Let* $\mathscr{H}$ *be a hyperbolic quadric of* $\mathrm{PG}(3, q)$*. Then every plane meets* $\mathscr{H}$ *in a pair of lines or an oval.*

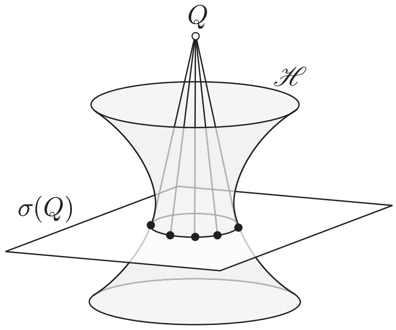
Proof In Theorem 1.5.26, it was determined that the tangent planes of $\mathscr{H}$ meet $\mathscr{H}$ in a pair of lines. By Lemma 1.5.11, every non-tangent plane meets $\mathscr{H}$ in a non-singular conic. □

A plane meeting a hyperbolic quadric $\mathscr{H}$ in a conic is called a *secant plane* of $\mathscr{H}$. The number of each type of plane through each type of line is given in the next lemma.

Lemma 1.5.32 [37, Chapter 15] *Let* $\mathscr{H}$ *be a hyperbolic quadric of* $\mathrm{PG}(3, q)$*. Then* $\mathscr{H}$ *has* q^2+2q+1 *tangent planes and* q^3-q *secant planes. An external line lies on no tangent planes and* $q+1$ *secant planes. A generator line lies on* $q+1$ *tangent planes and no secant planes. A tangent lies on one tangent plane and* q *secant planes. A secant lies on two tangent planes and* $q-1$ *secant planes.*

Using the above information, the number of each type of line through each point on and off a hyperbolic quadric can be determined.

Lemma 1.5.33 [37, Chapter 15] *Let* $\mathscr{H}$ *be a hyperbolic quadric of* $\mathrm{PG}(3, q)$*. Through a point on* $\mathscr{H}$ *there are 2 generator lines,* $q-1$ *tangents,* q^2 *secants,* $2q+1$ *tangent planes and* q^2-q *secant planes. Through a point not on* $\mathscr{H}$ *there are* $q+1$ *tangents,* $\frac{1}{2}q(q+1)$ *secants,* $\frac{1}{2}q(q-1)$ *external lines,* $q+1$ *tangent planes and* q^2 *secant planes.*



Note that when q is even, the $q+1$ tangents through a point not on $\mathscr{H}$ form a plane and the set of tangents and generator lines of $\mathscr{H}$ is a general linear complex. When q is odd, the $q+1$ tangents of $\mathscr{H}$ form a quadric cone (pictured to the left), so they are not coplanar. Thus the tangents and generator lines form a $(q+1)$-cover that is not a general linear complex.

The final property of a hyperbolic quadric of $\mathrm{PG}(3, q)$ to be discussed is the structure of the set of tangent planes.

Let $\mathscr{H}$ be a hyperbolic quadric of $\mathrm{PG}(3, q)$. Denote by $\mathscr{H}^*$ the set of tangent planes of $\mathscr{H}$. Then by Lemma 1.5.22, $\mathscr{H}^*$ is a quadric of $\mathrm{PG}(3, q)^*$. In fact, $\mathscr{H}^*$ is a hyperbolic quadric of $\mathrm{PG}(3, q)^*$, since it has $q^2 + 2q + 1$ points.

A secant of $\mathscr{H}$ lies on 2 tangent planes, and so is a secant of $\mathscr{H}^*$ in $\mathrm{PG}(3, q)^*$. An external line of $\mathscr{H}$ lies on no tangent planes, and so is an external line of $\mathscr{H}^*$ in $\mathrm{PG}(3, q)^*$. A tangent of $\mathscr{H}$ is a tangent of $\mathscr{H}^*$ in $\mathrm{PG}(3, q)^*$, and a generator line of $\mathscr{H}$ is a generator line of $\mathscr{H}^*$ in $\mathrm{PG}(3, q)^*$.

This completes the list of properties of the non-singular quadrics in $\mathrm{PG}(3, q)$. It is appropriate to mention here the *singular* quadrics of $\mathrm{PG}(3, q)$. If a quadric has a line contained in its singular space, then it is reducible and forms a line, a plane or a pair of planes by Lemma 1.5.8. One important property of a reducible quadric in $\mathrm{PG}(3, q)$ is listed here.

Lemma 1.5.34 *Let ℓ be a line of* $\mathrm{PG}(3, q)$*. Then there are q^4 lines of* $\mathrm{PG}(3, q)$ *not meeting ℓ.*

If a quadric has a unique singular point V, then it forms a cone $V\mathscr{C}$, where $\mathscr{C}$ is a non-singular conic in a plane not through V. The quadric $V\mathscr{C}$ is called a *quadric cone* and is an example of an oval cone. The quadric cone is the singular irreducible quadric in $\mathrm{PG}(3, q)$. Its properties are described in Section 1.4.

The non-singular quadrics in $\mathrm{PG}(n, q)$ for $n \geq 4$ will now be discussed. In $\mathrm{PG}(2, q)$ there is one type of non-singular conic, and in $\mathrm{PG}(3, q)$, there are two types of non-singular quadric. This pattern continues for the higher dimensions.

Theorem 1.5.35 [36, Chapter 5] *In* PG(n, q)*, n even, all non-singular quadrics are projectively equivalent. In* PG(n, q)*, n odd, there are two distinct non-singular quadrics, up to projective equivalence.*

The above theorem is proved by applying successive collineations until the equation of the quadric is of a particular form. For these equations see Hirschfeld and Thas [39, Chapter 22]. For the purposes of this thesis, the two types of non-singular quadric in PG(n, q), n odd, will be distinguished by their combinatorial properties, just as was done for in Theorem 1.5.26 for PG$(3, q)$. The distinguishing property is the intersection of a tangent hyperplane with the quadric, which is described in Lemma 1.5.13. This property is used to give the following recursive definition.

Definition 1.5.36 *Let $\mathscr{Q}$ be a non-singular quadric of* PG(n, q)*, n odd, $n > 3$, and let P be a point of $\mathscr{Q}$. The quadric $\mathscr{Q}$ is called a* hyperbolic quadric *if $\mathbf{T}_P$ meets $\mathscr{Q}$ in a cone $P\mathscr{H}$ for $\mathscr{H}$ a hyperbolic quadric in* PG$(n-2, q)$*. The quadric $\mathscr{Q}$ is called an* elliptic quadric *if $\mathbf{T}_P$ meets $\mathscr{Q}$ in a cone $P\mathscr{E}$ for $\mathscr{E}$ an elliptic quadric in* PG$(n-2, q)$.

Definition 1.5.37 *A non-singular quadric in* PG(n, q)*, n even, is called a* parabolic quadric.

The different types of non-singular quadric can also be distinguished by their size, as shown by the following lemma.

Lemma 1.5.38 [36, Chapter 5] *Let $\mathscr{Q}$ be a non-singular quadric of* PG(n, q)*. If $\mathscr{Q}$ is a parabolic quadric, then $|\mathscr{Q}| = \theta_{n-1}$. If $\mathscr{Q}$ is a hyperbolic quadric, then $|\mathscr{Q}| = \theta_{n-1} + q^{\frac{1}{2}(n-1)}$. If $\mathscr{Q}$ is an elliptic quadric, then $|\mathscr{Q}| = \theta_{n-1} - q^{\frac{1}{2}(n-1)}$.*

Proof The proof is by induction on the dimension n. Each case is treated separately.

A parabolic quadric in PG$(2, q)$ is a non-singular conic with $q + 1 = \theta_1$ points. Suppose a parabolic quadric in PG$(n-2, q)$ has θ_{n-3} points and let $\mathscr{Q}$ be a parabolic quadric in PG(n, q). Consider the tangent hyperplane $\mathbf{T}_P$ at a point $P \in \mathscr{Q}$. By Lemma 1.5.13, $\mathbf{T}_P$ meets $\mathscr{Q}$ in a cone $P\mathscr{P}$, where $\mathscr{P}$ is a parabolic quadric in an $(n-2)$-space. Thus $|\mathbf{T}_P \cap \mathscr{Q}| = q\theta_{n-3} + 1 = \theta_{n-2}$. By Lemma 1.5.14, $|\mathscr{Q}| = q^{n-1} + \theta_{n-2} = \theta_{n-1}$.

An elliptic quadric in $\mathrm{PG}(3,q)$ has $q^2+1 = \theta_2 - q$ points. Suppose an elliptic quadric in $\mathrm{PG}(n-2,q)$ has $\theta_{n-3} - q^{\frac{1}{2}(n-3)}$ points and let $\mathscr{Q}$ be an elliptic quadric in $\mathrm{PG}(n,q)$. Consider the tangent hyperplane $\mathbf{T}_P$ at a point $P \in \mathscr{Q}$. By Definition 1.5.36, $\mathbf{T}_P$ meets $\mathscr{Q}$ in a cone $P\mathscr{E}$, where $\mathscr{E}$ is an elliptic quadric in an $(n-2)$-space. Thus $|\mathbf{T}_P \cap \mathscr{Q}| = q(\theta_{n-3} - q^{\frac{1}{2}(n-3)}) + 1 = \theta_{n-2} - q^{\frac{1}{2}(n-1)}$. By Lemma 1.5.14, $|\mathscr{Q}| = q^{n-1} + \theta_{n-2} - q^{\frac{1}{2}(n-1)} = \theta_{n-1} - q^{\frac{1}{2}(n-1)}$.

A hyperbolic quadric in $\mathrm{PG}(3,q)$ has $q^2+2q+1 = \theta_2 + q$ points. A similar argument to that above shows that a hyperbolic quadric in $\mathrm{PG}(n,q)$ has $\theta_{n-1} + q^{\frac{1}{2}(n-1)}$ points. $\square$

Given the above results, it is possible to calculate the number of each type of line and plane associated with a quadric, and to describe the incidence between them. This will only be shown here for selected lines and points of certain quadrics. First, the parabolic quadric in $\mathrm{PG}(4,q)$ will be dealt with.

Let $\mathscr{P}$ be a parabolic quadric in $\mathrm{PG}(4,q)$. The following results concern the quadric $\mathscr{P}$. The first result is a restatement of Lemmas 1.5.11 and 1.5.12.

Lemma 1.5.39 [39, Chapter 22] *The quadric $\mathscr{P}$ has q^3+q^2+q+1 points. A tangent hyperplane of $\mathscr{P}$ meets $\mathscr{P}$ in a quadric cone of the hyperplane. A non-tangent hyperplane meets $\mathscr{P}$ either in an elliptic quadric or a hyperbolic quadric of the hyperplane.*

Since an irreducible quadric in $\mathrm{PG}(3,q)$ contains no planes, the above result implies that $\mathscr{P}$ contains no planes. A plane meeting $\mathscr{P}$ in a non-singular conic is called a *secant plane* of $\mathscr{P}$. The number of the different types of lines through a point on $\mathscr{P}$ can easily be calculated.

Lemma 1.5.40 [39, Chapter 22] *Through a point of $\mathscr{P}$ there pass q^3 secants, $q+1$ generator lines and q^2 tangents.*

Proof Let P be a point of $\mathscr{P}$ and consider the tangent hyperplane $\mathbf{T}_P$ at P. This hyperplane is the union of the tangents and generator lines through P and meets $\mathscr{P}$ in a quadric cone. Thus through P there pass $q+1$ generator lines and q^2 tangents. The remaining q^3 lines through P are secants. □

Using the above information, the number of secant planes through each point and each line can be calculated.

Lemma 1.5.41 *A generator line of $\mathscr{P}$ lies on no secant planes. Every other line of* $\mathrm{PG}(4,q)$ *lies on q^2 secant planes. A point on $\mathscr{P}$ lies on q^4 secant planes. A point not on $\mathscr{P}$ lies on q^4+q^2 secant planes.*

Proof A secant plane contains no generator lines, so a generator line is contained in no secant planes.

Let t be a tangent meeting $\mathscr{P}$ in the point P. Denote the number of secant planes through t by S_t and let $X = \{(\pi, s) \mid \pi$ is a plane through t, s is a secant in π through $P\}$. The size of X will be counted in two ways. Note that the planes through a tangent meet $\mathscr{P}$ in a single point, a line, a pair of lines, or a non-singular conic. Of these types of planes, only the secant planes contain both a secant and a tangent through P. So, counting π then s, there are S_t planes through t containing q secants through P, and the remaining planes through t contain no secant through P. So $|X| = qS_t$. Counting s then π, there are q^3 secants through P by Lemma 1.5.40, and each defines one plane through t. So $|X| = q^3$. Equating these two expressions for $|X|$ gives $S_t = q^2$.

Let s be a secant and denote by S_s the number of secant planes through s. Every point of $\mathscr{P}$ lies in a plane through s, so consider the planes through s. Each of these planes is either a secant plane or meets $\mathscr{P}$ in a pair of lines, since the other possible intersections have no secants. Thus $|\mathscr{P}|-2 = (q+1-2)S_s+(2q+1-2)(q^2+q+1-S_s)$. Solving this equation gives $S_s = q^2$.

Let ℓ be an external line and denote by S_ℓ the number of secant planes through ℓ. Each plane through ℓ is either a secant plane or meets $\mathscr{P}$ in one point. An argument similar that above shows that $S_\ell = q^2$.

Let P be a point of $\mathscr{P}$ and denote by S_P the number of secant planes through P. Consider the set of pairs $Y = \{(\pi, s) \mid \pi \text{ is a secant plane through } P, s \text{ is a secant in } \pi \text{ through } P\}$. The size of Y will be counted in two ways. Counting π then s, there are S_P secant planes through P and q secants through P in each. So $|Y| = qS_P$. Counting s then π, there are q^3 secants through P and q^2 secant planes through each, so $|Y| = q^3q^2 = q^5$. Equating these two expressions for $|Y|$ gives $S_P = q^4$.

Let Q be a point not in $\mathscr{P}$ and denote by S_Q the number of secant planes through Q. Consider the set of pairs $Z = \{(\pi, \ell) \mid \pi \text{ is a secant plane through } Q, \ell \text{ is a line of } \pi \text{ through } Q\}$. The size of Z is counted in two ways. Counting π then ℓ, there are S_Q secant planes through P and $q+1$ lines through P in each. So $|Z| = (q+1)S_Q$. Counting ℓ then π, there are q^3+q^2+q+1 lines through Q and q^2 secant planes through each, since there are no generator lines through Q. So $|Z| = q^2(q^3+q^2+q+1) = q^2(q^2+1)(q+1)$. Equating these two expressions for $|Z|$ gives $S_Q = q^4 + q^2$. □

The number of each type of line through a point not on $\mathscr{P}$ can also be calculated, but the result is different for q odd and q even.

Lemma 1.5.42 *Suppose q is odd and P is a point not on $\mathscr{P}$. Either P lies on q^2+1 tangents, $\frac{1}{2}q(q^2+1)$ secants and $\frac{1}{2}q(q^2+1)$ external lines, or P lies on q^2+2q+1 tangents, $\frac{1}{2}q(q^2-1)$ secants and $\frac{1}{2}q(q^2-1)$ external lines.*

Proof The polar of P is a non-tangent hyperplane of $\mathscr{P}$, and so meets $\mathscr{P}$ in either an elliptic quadric or a hyperbolic quadric. The points of $\mathscr{P}$ lying on tangents through P are the points where the polar of P meets $\mathscr{P}$. Thus the number of tangents through P is either q^2+1 or q^2+2q+1. The remaining points of $\mathscr{P}$ lie on secants through P, and this gives the corresponding numbers of secants through P. The remaining lines through P are external lines, and this gives the corresponding numbers of external lines through P. □

Lemma 1.5.43 *Suppose q is even and P is a point not on $\mathscr{P}$ other than the nucleus of $\mathscr{P}$. Then P lies on q^2+q+1 tangents, $\frac{1}{2}q^3$ secants and $\frac{1}{2}q^3$ external lines.*

Proof Since P is not the nucleus of $\mathscr{P}$, there exists a secant s through P. Let π be a plane through s. Since π contains a secant of $\mathscr{P}$, it meets $\mathscr{P}$ either in a pair of lines or a non-singular conic, since the other conics have no secants. If π meets $\mathscr{P}$ in a pair of lines, P lies on one tangent in π. If π meets $\mathscr{P}$ in a non-singular conic $\mathscr{C}$, then P is not the nucleus of $\mathscr{C}$, since it lies on a secant. Thus P lies on one tangent in π. Since there are q^2+q+1 planes through s, each containing one tangent through P, it follows that there are q^2+q+1 tangents through P. The remaining q^3 points of $\mathscr{P}$ lie on secants through P, so there are $\frac{1}{2}q^3$ secants through P. The remaining $\frac{1}{2}q^3$ lines through P are external lines. □

For q odd, the polarity of $\mathscr{P}$ makes it possible to discuss the set of tangent hyperplanes of $\mathscr{P}$ as a quadric of the dual space $\mathrm{PG}(4,q)^*$. Then each plane of $\mathrm{PG}(4,q)$ corresponds to a line of $\mathrm{PG}(4,q)^*$. Of particular interest are the secant planes of $\mathscr{P}$.

Lemma 1.5.44 [39, Chapter 22] *Suppose q is odd and denote by $\mathscr{P}^*$ the set of tangent hyperplanes of $\mathscr{P}$. The set of secant planes of $\mathscr{P}$ in* $\mathrm{PG}(4,q)$ *is the set of external lines and secants of the quadric $\mathscr{P}^*$ in* $\mathrm{PG}(4,q)^*$.

Proof It is possible to count the number of each type of 3-space through each type of plane using arguments similar to those above. From this information it follows that a plane meeting $\mathscr{P}$ in one point is contained in one tangent hyperplane, a plane meeting $\mathscr{P}$ in a pair of lines is contained in one tangent hyperplane, a plane meeting $\mathscr{P}$ in one line is contained in $q+1$ tangent hyperplanes, and a secant plane is contained in either 0 or 2 tangent hyperplanes. Thus, as a line of $\mathrm{PG}(4,q)^*$, a secant plane of $\mathscr{P}$ contains 0 or 2 points of $\mathrm{PG}(4,q)^*$. That is, the set of secant planes of $\mathscr{P}$ in $\mathrm{PG}(3,q)$ is the set of external lines and secants of the quadric $\mathscr{P}^*$ in $\mathrm{PG}(4,q)^*$. □

This completes the discussion on the parabolic quadric $\mathscr{P}$ in $\mathrm{PG}(4,q)$. For non-singular quadrics in higher dimensions, only one combinatorial property will be considered apart from size. This is the number of each type of line through a point not on the quadric. For parabolic quadrics, this will only be calculated for q even.

Lemma 1.5.45 *Let $\mathscr{P}$ be a parabolic quadric of* $\mathrm{PG}(n,q)$*, n even, $n \geq 4$, q even. Let P be a point not on $\mathscr{P}$ other than the nucleus of $\mathscr{P}$. Then P lies on θ_{n-2} tangents, $\frac{1}{2}q^{n-1}$ secants and $\frac{1}{2}q^{n-1}$ external lines of $\mathscr{P}$.*

Proof Since P is not the nucleus of $\mathscr{P}$, there exists a secant s through P. Let π be a plane through s. Since π contains a secant of $\mathscr{P}$, it meets $\mathscr{P}$ either in a pair of lines or a non-singular conic, since the other conics have no secants. If π meets $\mathscr{P}$ in a pair of lines, P lies on one tangent in π. If π meets $\mathscr{P}$ in a non-singular conic $\mathscr{C}$, P is not the nucleus of $\mathscr{C}$, since it lies on a secant. Thus P lies on one tangent in π. Since there are θ_{n-2} planes through s, each containing one tangent through P, it follows that there are θ_{n-2} tangents through P. The remaining q^{n-1} points of $\mathscr{P}$ lie on secants through P, so there are $\frac{1}{2}q^{n-1}$ secants through P. The remaining $\frac{1}{2}q^{n-1}$ lines through P are external lines. □

Lemma 1.5.46 *Let $\mathscr{H}$ be a hyperbolic quadric of* $\mathrm{PG}(n,q)$*, n odd, $n \geq 3$. Let P be a point not on $\mathscr{H}$. Then P lies on θ_{n-2} tangents, $\frac{1}{2}(q^{n-1} + q^{\frac{1}{2}(n-1)})$ secants and $\frac{1}{2}(q^{n-1} - q^{\frac{1}{2}(n-1)})$ external lines.*

Let $\mathscr{E}$ be an elliptic quadric of $\mathrm{PG}(n,q)$*, n odd, $n \geq 3$. Let Q be a point not on $\mathscr{E}$. Then Q lies on θ_{n-2} tangents, $\frac{1}{2}(q^{n-1} - q^{\frac{1}{2}(n-1)})$ secants and $\frac{1}{2}(q^{n-1} + q^{\frac{1}{2}(n-1)})$ external lines.*

Proof The points of $\mathscr{H}$ on the tangents through P are the points of $\mathscr{H}$ in the polar of P. Since P is not on $\mathscr{H}$, its polar is a non-tangent hyperplane, which meets $\mathscr{H}$ in a non-singular quadric. This quadric has size θ_{n-2} by Theorem 1.5.38, so there are θ_{n-2} tangents through P. The remaining points of $\mathscr{H}$ lie on secants through P. The number of points of $\mathscr{H}$ is $\theta_{n-1} + q^{\frac{1}{2}(n-1)}$, so there are $\frac{1}{2}(q^{n-1} + q^{\frac{1}{2}(n-1)})$ secants through P. The remaining $\frac{1}{2}(q^{n-1} - q^{\frac{1}{2}(n-1)})$ lines through P are external lines.

The remaining part of the theorem is proved using a similar argument. □

This completes the discussion of non-singular quadrics. The singular quadrics of $\mathrm{PG}(n,q)$, $n \geq 2$, will now be considered. The following terminology will be useful.

Definition 1.5.47 *A quadric is* of parabolic type *if it is either a parabolic quadric, a cone over a parabolic quadric, or consists of a hyperplane. A quadric is* of elliptic type *if it is either an elliptic quadric, a cone over an elliptic quadric, or consists of an $(n-2)$-space. A quadric is* of hyperbolic type *if it is either a hyperbolic quadric, a cone over a hyperbolic quadric, or consists of a pair of hyperplanes.*

The inclusion of the reducible quadrics in the above lists is justified by the basic combinatorial property of their sizes, as seen in the following result.

Lemma 1.5.48 [39, Chapter 22] *Let $\mathscr{Q}$ be a quadric in $\mathrm{PG}(n,q)$, $n \geq 2$. If $\mathscr{Q}$ is of parabolic type, then $|\mathscr{Q}| = \theta_{n-1}$. If $\mathscr{Q}$ is of elliptic type, then $|\mathscr{Q}| = \theta_{n-1} - q^m$ for some m. If $\mathscr{Q}$ is of hyperbolic type, then $|\mathscr{Q}| = \theta_{n-1} + q^m$ for some m.*

Proof Suppose $\mathscr{Q}$ is of parabolic type. If $\mathscr{Q}$ is a hyperplane, then $|\mathscr{Q}| = \theta_{n-1}$. If $\mathscr{Q}$ is a parabolic quadric, then $|\mathscr{Q}| = \theta_{n-1}$ by Lemma 1.5.38. If $\mathscr{Q}$ is singular and irreducible, then $\mathscr{Q}$ is a cone $\mathbf{V}\mathscr{P}$ for some subspace $\mathbf{V}$ of dimension $d \geq 0$ and some parabolic quadric $\mathscr{P}$ in a complementary subspace of $\mathbf{V}$. By Lemma 1.3.7, $|\mathscr{Q}| = q^{d+1}\theta_{n-2-d} + \theta_d = \theta_{n-1}$.

Suppose $\mathscr{Q}$ is of elliptic type. If $\mathscr{Q}$ is an $(n-2)$-space, then $|\mathscr{Q}| = \theta_{n-2} = \theta_{n-1} - q^{n-1}$. If $\mathscr{Q}$ is an elliptic quadric, then $|\mathscr{Q}| = \theta_{n-1} - q^{\frac{1}{2}(n-1)}$ by Lemma 1.5.38. If $\mathscr{Q}$ is singular and irreducible, then $\mathscr{Q}$ is a cone $\mathbf{V}\mathscr{E}$ for some subspace $\mathbf{V}$ of dimension $d \geq 0$ and some elliptic quadric $\mathscr{E}$ in a complementary subspace of $\mathbf{V}$. By Lemma 1.3.7, $|\mathscr{Q}| = q^{d+1}(\theta_{n-2-d} - q^{\frac{1}{2}(n-2-d)}) + \theta_d = \theta_{n-1} - q^{\frac{1}{2}(n+d)}$.

Finally suppose $\mathscr{Q}$ is of hyperbolic type. If $\mathscr{Q}$ is a pair of hyperplanes, then $|\mathscr{Q}| = \theta_{n-1} + q^{n-1}$. If $\mathscr{Q}$ is a hyperbolic quadric, then $|\mathscr{Q}| = \theta_{n-1} + q^{\frac{1}{2}(n-1)}$ by Lemma 1.5.38. If $\mathscr{Q}$ is singular and irreducible, then $\mathscr{Q}$ is a cone $\mathbf{V}\mathscr{H}$ for some subspace $\mathbf{V}$ of dimension $d \geq 0$ and some elliptic quadric $\mathscr{H}$ in a complementary subspace of $\mathbf{V}$. By Lemma 1.3.7, $|\mathscr{Q}| = q^{d+1}(\theta_{n-2-d} + q^{\frac{1}{2}(n-2-d)}) + \theta_d = \theta_{n-1} + q^{\frac{1}{2}(n+d)}$. □

The final result in this section is a calculation of the number of external lines through a point not on a each type of quadric for q even.

Lemma 1.5.49 *Let $\mathscr{Q}$ be a quadric in* $\mathrm{PG}(n,q)$*, q even, and suppose P is a point not on $\mathscr{Q}$ lying on at least one external line of $\mathscr{Q}$. The number of external lines through P is less than $\frac{1}{2}q^{n-1}$ if $\mathscr{Q}$ is of hyperbolic type, greater than $\frac{1}{2}q^{n-1}$ if $\mathscr{Q}$ is of elliptic type and exactly $\frac{1}{2}q^{n-1}$ if $\mathscr{Q}$ is of parabolic type.*

Proof If $\mathscr{Q}$ is a hyperplane or pair of hyperplanes, then $\mathscr{Q}$ has no external lines. If $\mathscr{Q}$ is an $(n-2)$-space then there are θ_{n-2} lines through P meeting $\mathscr{Q}$, and so there are q^{n-1} lines through P not meeting $\mathscr{Q}$. Thus the result is true when $\mathscr{Q}$ is reducible. The result is also true when $\mathscr{Q}$ is non-singular by Lemmas 1.5.45 and 1.5.46.

Assume $\mathscr{Q}$ is singular and irreducible and denote the singular space of $\mathscr{Q}$ by $\mathbf{V}$. Let ℓ be an external line through P and let Π be a complementary subspace of $\mathbf{V}$ through ℓ, meeting $\mathscr{Q}$ in the non-singular quadric $\mathscr{Q}_0$. Denote the dimension of $\mathbf{V}$ by d and the dimension of Π by b. Note that $b+d=n-1$, since $\mathbf{V}$ and Π are complementary subspaces.

Denote by E_0 be the number of external lines of $\mathscr{Q}_0$ in Π through P, and denote by E the number of external lines of $\mathscr{Q}$ through P. By Lemma 1.3.9, $E=q^{d+1}E_0$. If $\mathscr{Q}$ is of hyperbolic type, then $\mathscr{Q}_0$ is a hyperbolic quadric and $E_0<\frac{1}{2}q^{b-1}$ by Lemma 1.5.46. Thus $E<\frac{1}{2}q^{d+1}q^{b-1}=\frac{1}{2}q^{n-1}$. If $\mathscr{Q}$ is of elliptic type, then $\mathscr{Q}_0$ is an elliptic quadric and $E_0>\frac{1}{2}q^{b-1}$ by Lemma 1.5.46. Thus $E>\frac{1}{2}q^{d+1}q^{b-1}=\frac{1}{2}q^{n-1}$. Finally, if $\mathscr{Q}$ is of parabolic type, then $\mathscr{Q}_0$ is a parabolic quadric and $E_0=\frac{1}{2}q^{b-1}$ by Lemma 1.5.45. Thus $E=\frac{1}{2}q^{d+1}q^{b-1}=\frac{1}{2}q^{n-1}$. □

This completes the discussion of quadrics.

1.6 Quadrals

In this section, a quadral is formally defined by making reference to the definitions given so far.

Definition 1.6.1 *A quadral of* $\mathrm{PG}(2,q)$ *is an oval or a conic. A quadral of* $\mathrm{PG}(3,q)$ *is an ovoid, an oval cone or a quadric. A quadral of* $\mathrm{PG}(n,q)$*, $n\geq 4$ is an oval cone, an ovoid cone or a quadric.*

The reasoning behind this definition is that oval cones and ovoid cones are combinatorially identical to quadrics. That is, using only combinatorial information, it is not possible to distinguish between a non-quadric quadral and a quadric. It is reasonable to ask why there are no other objects in the list of quadrals. That is, are there any more sets of points with all the same combinatorial properties as quadrics? The answer is no, and this is proved by the characterisations given in the following chapter.

At this point it is appropriate to define some further terminology. Since an oval is combinatorially identical to a conic, oval cones are combinatorially identical to certain quadrics of parabolic type. In a similar way, ovoid cones are combinatorially identical to certain quadrics of elliptic type. Thus the following terminology is introduced.

Definition 1.6.2 *A* parabolic quadral *is an oval in* $\mathrm{PG}(2,q)$ *or a parabolic quadric in* $\mathrm{PG}(n,q)$*,* $n \geq 4$*. An* elliptic quadral *is an ovoid in* $\mathrm{PG}(3,q)$ *or an elliptic quadric in* $\mathrm{PG}(n,q)$*,* $n \geq 5$*. A* hyperbolic quadral *is a hyperbolic quadric.*

A quadral of parabolic type *is an oval in* $\mathrm{PG}(2,q)$*, an oval cone in* $\mathrm{PG}(n,q)$ *for* $n \geq 3$*, or a quadric of parabolic type. A* quadral of elliptic type *is an ovoid of* $\mathrm{PG}(3,q)$*, an ovoid cone in* $\mathrm{PG}(n,q)$ *for* $n \geq 4$*, or a quadric of elliptic type. A* quadral of hyperbolic type *is a quadric of hyperbolic type.*

In order to distinguish the quadrals that are cones, the following terminology is introduced.

Definition 1.6.3 *Let* $\mathscr{Q}$ *be a quadral of* $\mathrm{PG}(n,q)$ *and let* P *be a point of* $\mathscr{Q}$*. Then* P *is called a* singular point *of* $\mathscr{Q}$ *if every line through* P *is either a tangent or a generator line of* $\mathscr{Q}$*. Otherwise,* P *is called a* non-singular point*.*

The quadral $\mathscr{Q}$ *is called* singular *if it has a singular point, and* non-singular *otherwise.*

This definition matches with the definition of singular points given in Definition 1.3.3 for a general set. Note that if the quadral $\mathscr{Q}$ is a cone with vertex $\mathbf{V}$, then every point of $\mathbf{V}$ is a singular point, by the definition of a cone. Thus the non-singular quadrals are the ovals of $\mathrm{PG}(2,q)$, the ovoids of $\mathrm{PG}(3,q)$ and the non-singular quadrics. Also note that in $\mathrm{PG}(n,q)$, $n \geq 4$, a non-singular quadral is always a quadric.

If the quadral $\mathscr{Q}$ in $\mathrm{PG}(n,q)$, q even, is of parabolic type, then the singular points of $\mathscr{Q}$ are not the only points of $\mathrm{PG}(n,q)$ lying on only tangents and generator lines. There also exist points not on $\mathscr{Q}$ lying only on tangents. These points, along with the set of singular points of $\mathscr{Q}$ form the *nuclear space* of $\mathscr{Q}$. The nuclear space was defined separately for quadrics and for oval cones.

Finally, a quadral will be called *reducible* if it is an $(n-2)$-space, a hyperplane or a pair of hyperplanes. That is, a quadral is reducible if it is a reducible quadric. All other quadrals are called *irreducible* quadrals. That is, the irreducible quadrals are the irreducible quadrics, the ovals of $\mathrm{PG}(2,q)$, the ovoids of $\mathrm{PG}(3,q)$, the oval cones and the ovoid cones.

This chapter is completed with the following results, which combine some of the information given in the previous sections. The first theorem combines Definitions 1.1.1 and 1.2.1 and Lemmas 1.4.2, 1.4.13 and 1.5.3.

Theorem 1.6.4 [21] *Let $\mathscr{Q}$ be a quadral in $\mathrm{PG}(n,q)$. Then every line of $\mathrm{PG}(n,q)$ meets $\mathscr{Q}$ in 0, 1, 2 or $q+1$ points.*

The next theorem combines Lemmas 1.2.12, 1.4.4, 1.4.15 and 1.5.2.

Theorem 1.6.5 *Let $\mathscr{Q}$ be a quadral in $\mathrm{PG}(n,q)$. Then every plane of $\mathrm{PG}(n,q)$ meets $\mathscr{Q}$ in a point, a line, a pair of lines, an oval or else is contained in $\mathscr{Q}$.*

The next theorem combines Lemmas 1.4.6, 1.4.16 and 1.5.48.

Theorem 1.6.6 *Let $\mathscr{Q}$ be a quadral in $\mathrm{PG}(n,q)$. If $\mathscr{Q}$ is of parabolic type, then $|\mathscr{Q}| = \theta_{n-1}$. If $\mathscr{Q}$ is of elliptic type, then $|\mathscr{Q}| = \theta_{n-1} - q^m$ for some m. If $\mathscr{Q}$ is of hyperbolic type, then $|\mathscr{Q}| = \theta_{n-1} + q^m$ for some m.*

The final theorem combines Lemmas 1.4.9, 1.4.16 and 1.5.49.

Theorem 1.6.7 *Let $\mathscr{Q}$ be a quadral in* $\mathrm{PG}(n, q)$, *q even, and suppose P is a point not on $\mathscr{Q}$ lying on at least one external line of $\mathscr{Q}$. The number of external lines through P is less than $\frac{1}{2}q^{n-1}$ if $\mathscr{Q}$ is of hyperbolic type, greater than $\frac{1}{2}q^{n-1}$ if $\mathscr{Q}$ is of elliptic type and exactly $\frac{1}{2}q^{n-1}$ if $\mathscr{Q}$ is of parabolic type.*

Chapter 2

Characterisations of the set of points of a quadral

In the previous chapter, the objects that are the focus of this thesis were introduced. These are the ovals, the ovoids, the quadrics and the cones over them, which are collectively called *quadrals*. Detailed information about the quadrals and their associated subspaces was presented. In later chapters, this information will be used to prove characterisations of the families of subspaces associated with quadrics.

This chapter discusses existing characterisations of the quadrics themselves as sets of points. Given a set of points $\mathscr{K}$ in $\mathrm{PG}(n, q)$, different properties are assumed for $\mathscr{K}$, and the consequences of these properties are investigated. It will become clear that for quadrals, the most useful properties are the different ways that lines and planes meet the set of points.

2.1 Tallini Sets

One of the basic combinatorial properties of a quadric is the way that lines meet it: all lines meet a quadric in 0, 1, 2 or $q+1$ points. In this section, sets of points with this property will be investigated. Lefévre called such sets *Tallini sets* in honour of G. Tallini, who was first to publish detailed results about them. The same convention is followed here.

Definition 2.1.1 *A* Tallini set *of* $\mathrm{PG}(n, q)$ *is a set of points* $\mathscr{K}$ *such that every line meets* $\mathscr{K}$ *in* 0, 1, 2 *or* $q + 1$ *points.*

Quadrals are examples of Tallini sets by Theorem 1.6.4. Any subspace is a Tallini set, and so is any cap. Also, if q is even and $\mathscr{P}$ is a quadral of parabolic type, then any point in the nuclear space of $\mathscr{P}$ lies only on tangents of $\mathscr{P}$. So $\mathscr{P} \cup \Pi$ is a Tallini set for any subspace Π contained in the nuclear space of $\mathscr{P}$. For further examples of Tallini sets see [40]. When $q = 2$, there are $q + 1 = 3$ points on a line, so *any* set of points $\mathscr{K}$ has the property that every line meets $\mathscr{K}$ in 0, 1, 2 or $q + 1$ points. That is, any set of points is a Tallini set for $q = 2$.

In order to make the following discussion easier, special names are given to each type of line. These names match those given to the lines associated with each of the quadrals in Chapter 1.

Definition 2.1.2 *Let* $\mathscr{K}$ *be a Tallini set of* $\mathrm{PG}(n, q)$ *and let* ℓ *be a line of* $\mathrm{PG}(n, q)$. *If* ℓ *contains no point of* $\mathscr{K}$, *it is called an* external line *of* $\mathscr{K}$. *If* ℓ *contains exactly one point of* $\mathscr{K}$, *it is called a* tangent *of* $\mathscr{K}$. *If* ℓ *contains exactly two points of* $\mathscr{K}$, *it is called a* secant *of* $\mathscr{K}$. *If* ℓ *contains* $q + 1$ *points of* $\mathscr{K}$ *(that is,* ℓ *is contained in* $\mathscr{K}$*), it is called a* generator line *of* $\mathscr{K}$, *or a* line of $\mathscr{K}$.

If only one Tallini set $\mathscr{K}$ is under consideration, an external line of $\mathscr{K}$ may be referred to simply as an external line, and similarly for tangents, secants and generator lines.

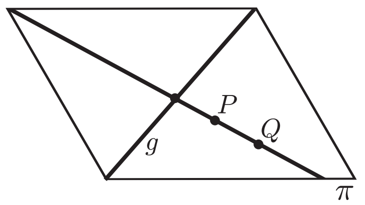
Some basic results about Tallini sets can now be stated. The first result gives information about the sections of a Tallini set.

Lemma 2.1.3 [53] *Let* $\mathscr{K}$ *be a Tallini set of* $\mathrm{PG}(n, q)$ *and let* Π *be a subspace of* $\mathrm{PG}(n, q)$ *meeting* $\mathscr{K}$ *in the set of points* $\mathscr{K}_\Pi$. *Then* $\mathscr{K}_\Pi$ *is a Tallini set.*

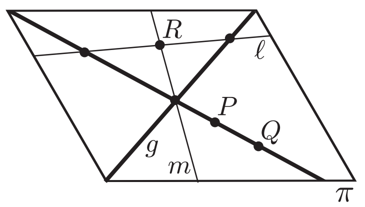
In particular, a plane meets a Tallini set in a Tallini set. It is appropriate then, to describe the possible Tallini sets in the plane.

Lemma 2.1.4 [53] *Let $\mathscr{K}$ be a Tallini set of* PG$(2,q)$*,* $q > 2$*. Then $\mathscr{K}$ is one of the following: the empty set, a point, a pair of points, an arc, a line, a line plus a point, a pair of lines, or the whole of* PG$(2,q)$*.*

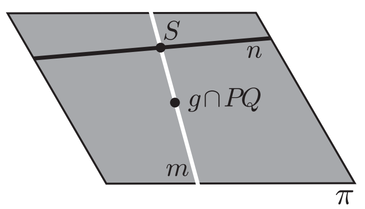
Proof If $\mathscr{K}$ contains no lines, it is the empty set, a point, a pair of points or an arc. Suppose $\mathscr{K}$ contains a line g. There may be no further points of $\mathscr{K}$, in which case $\mathscr{K}$ is a line. Suppose this is not the case and let P be a point of $\mathscr{K}$ not on g. There may be no further points of $\mathscr{K}$, in which case $\mathscr{K}$ is a line plus a point.



Suppose this is not the case and Q is a point of $\mathscr{K}$ not on g and distinct from P. The line PQ meets g in a point of $\mathscr{K}$ and so contains three points of $\mathscr{K}$. Thus PQ is a generator line of $\mathscr{K}$. There may be no further points of $\mathscr{K}$, in which case $\mathscr{K}$ is a pair of lines.



Suppose this is not the case and R is a point of $\mathscr{K}$ not on g or PQ. Let m be the line $R \oplus (g \cap PQ)$. Let ℓ be any other line of π through R. The line ℓ meets g and PQ in distinct points, and so contains three points of $\mathscr{K}$. Thus ℓ is a generator line of $\mathscr{K}$ and any point of π not on m is a point of $\mathscr{K}$.



Let S be a point of m and let n be a line of π through S other than m. The line n has q points of π not on m, which are all points of $\mathscr{K}$. Thus the line n contains at least three points of $\mathscr{K}$ and so is a generator line of $\mathscr{K}$. Thus $S \in \mathscr{K}$ and every point of PG$(2,q)$ is a point of $\mathscr{K}$. $\square$

In order to investigate the structure of a Tallini set in more detail, the singular points of a Tallini set are discussed.

Definition 2.1.5 *Let $\mathscr{K}$ be a Tallini set of* $\mathrm{PG}(n, q)$ *and let P be a point of $\mathscr{K}$. The point P is called a* singular point *of $\mathscr{K}$ if it lies on only tangents and generator lines of $\mathscr{K}$. That is, there are no secants of $\mathscr{K}$ through P.*

If the Tallini set $\mathscr{K}$ contains a singular point, it is called a singular *Tallini set. Otherwise, it is called* non-singular

The above definition agrees with that for each type of quadral and for general sets as given in Definition 1.3.3. By Lemma 1.3.4, the set of all singular points of a Tallini set $\mathscr{K}$ forms a subspace, which is called the *singular space* of $\mathscr{K}$. The following lemmas describe the structure of a Tallini set using its singular space.

Lemma 2.1.6 *Let $\mathscr{K}$ be a singular Tallini set of* $\mathrm{PG}(n, q)$ *and let* $\mathbf{U}$ *be any subspace contained in the singular space of $\mathscr{K}$. Let Π be a complementary subspace of* $\mathbf{U}$ *meeting $\mathscr{K}$ in the set of points $\mathscr{K}_\Pi$. Then $\mathscr{K}$ is the cone* $\mathbf{U}\mathscr{K}_\Pi$.

If the subspace $\mathbf{U}$ is chosen to be the whole singular space of $\mathscr{K}$, then more can be said about the structure of the Tallini set. The following lemmas are analogues of Lemmas 1.5.8 and 1.5.9, and are proved by the same arguments.

Lemma 2.1.7 [53] *Let $\mathscr{K}$ be a Tallini set of* $\mathrm{PG}(n, q)$ *and suppose the singular space of $\mathscr{K}$ has dimension at least $n-2$. Then $\mathscr{K}$ is an $(n-2)$-space, a hyperplane, a pair of hyperplanes or the whole of* $\mathrm{PG}(n, q)$.

Lemma 2.1.8 [53] *Let $\mathscr{K}$ be a singular Tallini set of* $\mathrm{PG}(n, q)$ *with singular space* $\mathbf{V}$. *Suppose* $\mathbf{V}$ *has dimension at most $(n-3)$ and let Π be a complementary subspace of* $\mathbf{V}$ *meeting $\mathscr{K}$ in the set of points $\mathscr{K}_\Pi$. Then $\mathscr{K}_\Pi$ is a non-singular Tallini set of Π and $\mathscr{K}$ is the cone* $\mathbf{V}\mathscr{K}_\Pi$.

The above lemma implies that in order to describe all Tallini sets (other than the reducible quadrics), only the non-singular Tallini sets need to be considered.

To complete this section, one more concept for Tallini sets is introduced. This is the concept of a *tangent set*. The concept of tangent set has been used by various authors [17, 18, 38, 55] to prove characterisations similar to that of Buekenhout discussed in Section 2.2.

Definition 2.1.9 *Let $\mathscr{K}$ be a Tallini set of* PG(n, q) *and let P be a point of $\mathscr{K}$. The* tangent set *of $\mathscr{K}$* at *P is the set of all points of* PG(n, q) *lying on either a tangent or generator line of $\mathscr{K}$ through P, plus the point P itself. The tangent set of $\mathscr{K}$ at P is denoted by $\mathbf{T}_P(\mathscr{K})$, or simply by $\mathbf{T}_P$.*

If P is a singular point of the Tallini set $\mathscr{K}$, then every point of PG(n, q) lies on a tangent or generator line of $\mathscr{K}$ through P, so $\mathbf{T}_P(\mathscr{K}) = \mathrm{PG}(n, q)$. If $\mathscr{K}$ is a quadric and P is a non-singular point of $\mathscr{K}$, then $\mathbf{T}_P(\mathscr{K})$ is the tangent hyperplane of $\mathscr{K}$ at P.

For quadrics and ovoids, the tangent set is always a hyperplane or the whole of PG(n, q) (see Chapter 1). It is appropriate to mention that this is not true of every Tallini set. For example, consider a hyperoval $\bar{\mathscr{O}}$ in PG$(2, q)$, and a point $P \in \bar{\mathscr{O}}$. A hyperoval has no tangents, so $\mathbf{T}_P(\bar{\mathscr{O}})$ is just the point P. For a further example, consider a cap $\mathscr{K}$ of size q^2 in PG$(3, q)$. If Q is a point of $\mathscr{K}$, then there are $q + 2$ tangents of $\mathscr{K}$ through Q by Lemma 1.2.3. Since $\mathbf{T}_Q(\mathscr{K})$ is the union of these tangents, it cannot be a subspace. Tallini sets whose tangent sets do form subspaces are discussed in the next section.

The last results in this section concerns Tallini sets with a large number of points.

Theorem 2.1.10 [53] *Let $\mathscr{K}$ be a Tallini set of* PG(n, q)*, $q > 2$, with at least θ_{n-1} points. Then $\mathscr{K}$ is one of the following:*

- *The whole of* PG(n, q)*,*
- *A hyperplane, plus a further subspace of dimension d, $-1 \leq d \leq n - 1$,*
- *A quadral of parabolic type,*
- *A quadral of parabolic type plus a subspace contained in its nuclear space (if q is even),*
- *A quadric of hyperbolic type.*

The above theorem was proved by Tallini [53] in 1955. The same proof can be applied to the case $q = 2$, given some assumptions on the plane sections of the set.

Theorem 2.1.11 *Let $\mathscr{K}$ be a set of at least θ_{n-1} points in* $\mathrm{PG}(n,2)$ *such that no plane meets $\mathscr{K}$ in* 0, 2 *or* 6 *points. Then $\mathscr{K}$ is one of the following:*

- *The whole of* $\mathrm{PG}(n,2)$,
- *A hyperplane, plus a further subspace of dimension d, $-1 \leq d \leq n-1$,*
- *A quadral of parabolic type,*
- *A quadral of parabolic type plus a subspace contained in its nuclear space,*
- *A quadric of hyperbolic type.*

The proof of the above theorem follows Tallini's proof of Theorem 2.1.10 exactly. The assumptions that no plane meets $\mathscr{K}$ in 0, 2 or 6 points are necessary to make various arguments in Tallini's proof applicable to the case $q = 2$. Of particular note is the condition that every plane meets $\mathscr{K}$. Using Lemma 1.2.10 and the size of an ovoid, this condition ensures that $\mathscr{K}$ is not a cap.

2.2 Quadratic Sets

In this section a characterisation of quadrics that Buekenhout published in 1969 [16] is presented (for an English translation see [39]). Buekenhout considered sets of points with two basic properties of quadrics. He called these objects *quadratic sets*.

Definition 2.2.1 *A* quadratic set *of* $\mathrm{PG}(n,q)$ *is a set of points $\mathscr{K}$ such that*

- *Every line meets $\mathscr{K}$ in* 0, 1, 2 *or* $q+1$ *points,*
- *For any point $P \in \mathscr{K}$, the set of points lying on tangents or generator lines of $\mathscr{K}$ through P forms either a hyperplane or all of* $\mathrm{PG}(n,q)$.

Using the terminology of the previous section, a quadratic set is a Tallini set such that the tangent set at any point is either a hyperplane or all of $\mathrm{PG}(n,q)$. In light of this, the tangent set at each point is called a *tangent space*. A point P of the quadratic set $\mathscr{K}$ such that the tangent space at P is $\mathrm{PG}(n,q)$ is called a *singular point* of $\mathscr{K}$. A quadratic set is called *singular* if it has a singular point and non-singular otherwise.

For any quadric $\mathscr{Q}$, the tangent set at each non-singular point is a hyperplane, and at each singular point the tangent set is $\mathrm{PG}(n,q)$. So all quadrics are quadratic sets. The set of tangents at each point of an ovoid is a plane, so ovoids are also quadratic sets. In fact, the definition of an ovoid (Definition 1.2.6) is essentially a non-singular quadratic set with no generator lines. Finally, if Π is a subspace, then every point of Π lies on only tangents and generator lines of Π, so any subspace is a quadratic set. The oval and ovoid cones are also quadratic sets, and this will be shown shortly.

Before Buekenhout's characterisation is given, some properties of quadratic sets will be stated. First, the sections of a quadratic set will be considered.

Lemma 2.2.2 [16] *Let $\mathscr{K}$ be a quadratic set and let Π be a subspace meeting $\mathscr{K}$ in the set of points $\mathscr{K}_\Pi$. Then $\mathscr{K}_\Pi$ is a quadratic set of Π.*

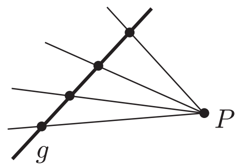
In particular, every plane meets a quadratic set in a quadratic set of the plane. In fact, this is all that is needed to show that a set of points is a quadratic set.

Lemma 2.2.3 [16] *Let $\mathscr{K}$ be a set of points in $\mathrm{PG}(n,q)$ such that every plane meets $\mathscr{K}$ in a quadratic set of the plane. Then $\mathscr{K}$ is a quadratic set.*

The above result means that a quadratic set can be defined as a set of points whose plane sections are quadratic sets of the plane. In light of this, it is appropriate at this point to describe the possible quadratic sets of the plane.

Lemma 2.2.4 [16] *The quadratic sets of $\mathrm{PG}(2,q)$ are the empty set, a single point, a line, a pair of lines, an oval, or the whole of $\mathrm{PG}(2,q)$.*

Proof Suppose $\mathscr{K}$ contains no lines. If $\mathscr{K}$ has a singular point V, then every line through V is a tangent (since there are no generator lines), so $\mathscr{K}$ is just the point V. If $\mathscr{K}$ has no singular points, then it may have no points at all. That is, $\mathscr{K}$ is the empty set. Suppose this is not the case and let P be a point of $\mathscr{K}$. The tangent space of $\mathscr{K}$ at P is a hyperplane of $\mathrm{PG}(2, q)$, so it is a line. Since there are no generator lines, this line is a tangent. That is, at each point of $\mathscr{K}$ there is a unique tangent. Thus $\mathscr{K}$ is a set of points such that every line meets $\mathscr{K}$ in 0, 1 or 2 points and there is a unique tangent at each point. That is, $\mathscr{K}$ is an *oval*.



Suppose $\mathscr{K}$ contains a line g. There may be no further points of $\mathscr{K}$, in which case $\mathscr{K}$ is a line. Suppose this is not the case and let P be a point of $\mathscr{K}$ not on g. If P is singular, then every line through it is a generator line or a tangent. But every line through P meets g in a further point of $\mathscr{K}$, so every line through P is a generator line and every point of $\mathrm{PG}(2, q)$ is in $\mathscr{K}$. That is $\mathscr{K} = \mathrm{PG}(2, q)$. If P is non-singular, then exactly one line through P is a generator line or tangent, but again every line through P contains a further point of $\mathscr{K}$, so there are no tangents through P and $\mathscr{K}$ consists of two lines.

Thus $\mathscr{K}$ is the empty set, a point, a line, a pair of lines, an oval or the whole of $\mathrm{PG}(2, q)$. □

By Lemmas 1.4.4 and 1.4.15, every plane meets an oval cone or an ovoid cone in a point, a line, a pair of lines, an oval or the whole plane. It follows that oval and ovoid cones are quadratic sets, and so every quadral is a quadratic set.

The plane sections of a quadratic set can also be used to show that particular quadratic sets are quadrics. In order to state this result, some terminology is introduced. Let $\mathscr{K}$ be a quadratic set and let π be a plane meeting $\mathscr{K}$ in an oval $\mathscr{O}$. Then the oval $\mathscr{O}$ is called an *oval section* of $\mathscr{K}$.

Theorem 2.2.5 [16] *Let $\mathscr{K}$ be a quadratic set of $\mathrm{PG}(n, q)$ that is not a subspace. If every oval section of $\mathscr{K}$ is a conic, then $\mathscr{K}$ is a quadric.*

Note that the above result was proved for ovoids by Barlotti [5] and Panella [44] before Buekenhout.

The singular quadratic sets are now considered. The following lemmas are Lemmas 2.1.7 and 2.1.8 reworded for quadratic sets.

Lemma 2.2.6 [16] *Let* $\mathscr{K}$ *be a quadratic set of* $\mathrm{PG}(n,q)$ *and suppose the singular space of* $\mathscr{K}$ *has dimension at least* $n-2$. *Then* $\mathscr{K}$ *is an* $(n-2)$*-space, a hyperplane, a pair of hyperplanes or the whole of* $\mathrm{PG}(n,q)$.

Lemma 2.2.7 [16] *Let* $\mathscr{K}$ *be a singular quadratic set of* $\mathrm{PG}(n,q)$ *with singular space* $\mathbf{V}$. *Suppose* $\mathbf{V}$ *has dimension at most* $(n-3)$ *and let* Π *be a complementary subspace of* $\mathbf{V}$ *meeting* $\mathscr{K}$ *in the set of points* $\mathscr{K}_\Pi$. *Then* $\mathscr{K}_\Pi$ *is a non-singular quadratic set of* Π *and* $\mathscr{K}$ *is the cone* $\mathbf{V}\mathscr{K}_\Pi$.

The above lemma implies that in order to classify all quadratic sets, it is only necessary to consider the *non-singular* quadratic sets. The fundamental result about non-singular quadratic sets is the following.

Theorem 2.2.8 [16] *Let* $\mathscr{K}$ *be a non-singular quadratic set of* $\mathrm{PG}(n,q)$ *containing a line. Then* $\mathscr{K}$ *is a quadric.*

Note that in Buekenhout's original proof of the above result, the quadratic sets in $\mathrm{PG}(3,q)$ were classified first, and their properties were used to complete the theorem. In particular, Buekenhout used the fact that the points on the lines of a regulus form a hyperbolic quadric (Theorem 1.5.29). However, the theorem can be proved directly without this fact, and then Theorem 1.5.29 is a corollary of Theorem 2.2.8.

The non-singular quadratic sets can now be classified.

Lemma 2.2.9 [16] *Let* $\mathscr{K}$ *be a non-empty, non-singular quadratic set in* $\mathrm{PG}(n,q)$. *Then* $\mathscr{K}$ *is a non-singular quadric, or* $n=2$ *and* $\mathscr{K}$ *is an oval, or* $n=3$ *and* $\mathscr{K}$ *is an ovoid.*

Proof If $\mathscr{K}$ contains a line, then by Theorem 2.2.8, $\mathscr{K}$ is a (non-singular) quadric.

If $\mathscr{K}$ contains no lines, then it satisfies the definition of an ovoid. By Theorem 1.2.7, there are no ovoids in $\mathrm{PG}(n, q)$ for $n \geq 4$, so either $n = 2$ and $\mathscr{K}$ is an oval or $n = 3$ and $\mathscr{K}$ is an ovoid. □

Using Lemma 2.2.9 and 2.2.7, it is now possible to completely classify all quadratic sets.

Theorem 2.2.10 [16] *Let $\mathscr{K}$ be a quadratic set of* $\mathrm{PG}(n, q)$, $n \geq 2$. *Then $\mathscr{K}$ is either a subspace or a quadral.*

Proof Assume $\mathscr{K}$ is not a subspace and suppose $\mathscr{K}$ is non-singular. Then by Lemma 2.2.9, $\mathscr{K}$ is a non-singular quadric, or $n = 2$ and $\mathscr{K}$ is an oval, or $n = 3$ and $\mathscr{K}$ is an ovoid. That is, $\mathscr{K}$ is a quadral.

Suppose $\mathscr{K}$ is singular with singular space $\mathbf{V}$. If the dimension of $\mathbf{V}$ is at least $n-2$, then $\mathscr{K}$ is a reducible quadric by Lemma 2.2.6. If the dimension of $\mathbf{V}$ is at most $n-3$, let Π be a subspace complementary to $\mathbf{V}$ meeting $\mathscr{K}$ in the set of points $\mathscr{K}_\Pi$. Then by Lemma 2.2.7, $\mathscr{K}_\Pi$ is a non-singular quadratic set and $\mathscr{K}$ is the cone $\mathbf{V}\mathscr{K}_\Pi$.

By the previous argument, Π is a plane and $\mathscr{K}_\Pi$ is an oval, or Π is a 3-space and $\mathscr{K}_\Pi$ is an ovoid, or $\mathscr{K}_\Pi$ is a non-singular quadric. Thus $\mathscr{K}_\Pi$ is either an oval cone, an ovoid cone, or a quadric.

Hence $\mathscr{K}$ is either a subspace or a quadral. □

2.3 Characterisation of quadrals by their plane sections

In the previous section it was shown that a quadratic set is either a quadral or a subspace. It was also proved that a set of points is a quadratic set if its plane sections are quadratic sets. In this section, these two ideas are drawn together to give characterisations of quadrals by their plane sections. Some useful characterisations of ovoids in $\mathrm{PG}(3, q)$ are also listed.

First, Theorem 2.2.10 and Lemma 2.2.3 are combined into one result characterising quadrals.

Theorem 2.3.1 *Let $\mathscr{K}$ be a set of points in $\mathrm{PG}(n, q)$ such that every plane meets $\mathscr{K}$ in one point, a line, a pair of lines, an oval or the whole plane. If $\mathscr{K}$ is not the whole of $\mathrm{PG}(n, q)$, then $\mathscr{K}$ is a quadral.*

Proof Every plane meets $\mathscr{K}$ in a quadratic set of the plane, so by Lemma 2.2.3, $\mathscr{K}$ is a quadratic set. Theorem 2.2.10 implies that $\mathscr{K}$ is either a subspace or a quadral.

Suppose $\mathscr{K}$ is a subspace of dimension $d \leq n-3$. Then a complementary subspace of $\mathscr{K}$ has dimension at least 2. Thus there exist planes skew to $\mathscr{K}$. This is a contradiction. So if $\mathscr{K}$ is a subspace, then it has dimension $n-2$ or $n-1$ and is a (reducible) quadral. $\square$

Plane sections can also be used to distinguish between singular and non-singular quadrals.

Corollary 2.3.2 *Let $\mathscr{K}$ be a quadral of $\mathrm{PG}(n, q)$ such that every point of $\mathscr{K}$ is contained in a plane meeting $\mathscr{K}$ in an oval. Then $\mathscr{K}$ is a non-singular quadral. If $n \geq 4$ then $\mathscr{K}$ is a non-singular quadric.*

Proof Every point of $\mathscr{K}$ lies in a plane meeting $\mathscr{K}$ in an oval, and every point of an oval lies on a secant. Thus every point of $\mathscr{K}$ lies on a secant, and so $\mathscr{K}$ is a non-singular quadral. By Lemma 2.2.9, a non-singular quadral is a quadric for $n \geq 4$, so the result follows. □

As well as the above characterisation provided by Buekenhout, there are some other useful characterisations of ovoids and elliptic quadrics in $\mathrm{PG}(3, q)$ by their plane sections. The following result was proved by Thas in 1973.

Theorem 2.3.3 [54] *Let $\mathscr{K}$ be a set of points in $\mathrm{PG}(3, q)$ such that every plane meets $\mathscr{K}$ in 1 or $k > 1$ points. Then $\mathscr{K}$ is either a line or an ovoid.*

For the purposes of combinatorial characterisation, it is not possible to distinguish between an elliptic quadric and any other ovoid. However, in some other situations, the distinction is very important. Theorem 2.2.5 implies that if all the oval sections of an ovoid are conics, then the ovoid is an elliptic quadric. In fact, it is only necessary to know that *one* oval section is a conic. The following result is due to Brown [11] in 2000.

Theorem 2.3.4 [11] *Let Ω be an ovoid of $\mathrm{PG}(3, q)$ such that one secant plane meets Ω in a conic. Then Ω is an elliptic quadric.*

The above theorem implies that in order to show a quadral is a quadric, it is only necessary to show that one oval section is a conic. That is, the following corollary is true.

Corollary 2.3.5 *Let $\mathscr{K}$ be a set of points in $\mathrm{PG}(n, q)$, $n \geq 3$, such that every plane meets $\mathscr{K}$ in a point, a line, a pair of lines, an oval or the whole plane. Suppose that there exists a plane meeting $\mathscr{K}$ in a non-singular conic. Then $\mathscr{K}$ is a quadric.*

Proof The set $\mathscr{K}$ is a quadral by Theorem 2.3.1. If $\mathscr{K}$ is reducible, then $\mathscr{K}$ is an $(n-2)$-space, a hyperplane or a pair of hyperplanes, so there are no planes meeting $\mathscr{K}$ in an oval. Thus $\mathscr{K}$ is an irreducible quadral.

Denote the singular space of $\mathscr{K}$ by $\mathbf{V}$. Let π be a plane meeting $\mathscr{K}$ in a non-singular conic. The plane π has no singular points and so is skew to $\mathbf{V}$. Let Π be a complementary subspace of $\mathbf{V}$ through π meeting $\mathscr{K}$ in the set of points $\mathscr{K}_\Pi$. Then $\mathscr{K}_\Pi$ is a non-singular quadral of Π and $\mathscr{K} = \mathbf{V}\mathscr{K}_\Pi$.

If Π has dimension at least 4, then Lemma 2.2.9 implies that $\mathscr{K}_\Pi$ is a non-singular quadric, and so $\mathscr{K}$ is a quadric.

If Π is a 3-space, then $\mathscr{K}_\Pi$ is either a hyperbolic quadric or an ovoid. If $\mathscr{K}_\Pi$ is a hyperbolic quadric, then $\mathscr{K} = \mathbf{V}\mathscr{K}_\Pi$ is a quadric. Suppose $\mathscr{K}_\Pi$ is an ovoid. Then the plane π meets $\mathscr{K}_\Pi$ in a non-singular conic, and so $\mathscr{K}_\Pi$ is an elliptic quadric by Theorem 2.3.4. Thus $\mathscr{K} = \mathbf{V}\mathscr{K}_\Pi$ is a quadric.

Finally, if Π is a plane, then $\Pi = \pi$ and $\mathscr{K}_\Pi$ is a non-singular conic. Thus $\mathscr{K} = \mathbf{V}\mathscr{K}_\Pi$ is a quadric. □

Chapter 3

Existing characterisations of the subspaces associated with quadrals

In the previous chapters, the properties of quadrals have been discussed, and characterisations of their sets of points have been presented. It is now possible to discuss characterisations of the families of subspaces associated with quadrals. In this chapter, the past research in this area is outlined. Of particular note are the characterisations of the external lines of the non-singular quadrals in $\mathrm{PG}(3, q)$ provided by Durante and Olanda [29] and Di Gennaro, Durante and Olanda [27].

3.1 Lines associated with quadrals in $\mathrm{PG}(2, q)$

In this section, some results are presented which characterise sets of lines associated with quadrals in $\mathrm{PG}(2, q)$. The reducible quadrals are considered first.

Suppose $\mathscr{K}$ is a quadral in $\mathrm{PG}(2, q)$ with exactly one singular point V. Then $\mathscr{K}$ is either the point V or a pair of lines through V. If $\mathscr{K}$ is the point V, then $\mathscr{K}$ has no secants and the external lines of $\mathscr{K}$ are the lines not through V. If $\mathscr{K}$ is a pair of lines through V, then $\mathscr{K}$ has no external lines and the secants of $\mathscr{K}$ are the lines not through V. The following results characterise the set of lines in $\mathrm{PG}(2, q)$ not through a point. The first result was proved in [32] but with greater restriction on the numbers of lines that pass through a point.

Lemma 3.1.1 *Let $\mathscr{L}$ be a set of q^2 lines in* $\mathrm{PG}(2,q)$ *such that no point of* $\mathrm{PG}(2,q)$ *lies on $q+1$ lines of $\mathscr{L}$. Then every point of* $\mathrm{PG}(2,q)$ *lies on 0 or q lines of $\mathscr{L}$.*

Proof Let ℓ be a line of $\mathscr{L}$. Since every line meets ℓ, the lines of $\mathscr{L}$ may be counted by counting the lines of $\mathscr{L}$ through each point on ℓ. For $i = 1, \ldots, q$, denote by a_i the number of points of ℓ lying on i lines of $\mathscr{L}$. Counting this way, the line ℓ itself has been included $q+1$ times – once for each point on ℓ. Thus

$$a_1 \cdot 1 + \cdots + a_{q-1} \cdot (q-1) + a_q \cdot q = q^2 + q. \tag{3.1}$$

Also,

$$a_1 + \cdots + a_{q-1} + a_q = q + 1. \tag{3.2}$$

Subtracting Equation (3.1) from q times Equation (3.2) gives

$$(q-1) \cdot a_1 + \cdots + 1 \cdot a_{q-1} = 0. \tag{3.3}$$

Now $q-1, \ldots, 1 > 0$ and $a_1, \ldots, a_{q-1} \geq 0$, so Equation 3.3 is only possible if $a_1 = \cdots = a_{q-1} = 0$.

Hence, $a_q = q+1$ and so every point on a line of $\mathscr{L}$ lies on q lines of $\mathscr{L}$. That is, every point of $\mathrm{PG}(2,q)$ lies on 0 or q lines of $\mathscr{L}$. □

Lemma 3.1.2 [36] *Let $\mathscr{L}$ be a non-empty set of lines in* $\mathrm{PG}(2,q)$ *such that every point of* $\mathrm{PG}(2,q)$ *lies on 0 or q lines of $\mathscr{L}$. Then $\mathscr{L}$ is the set of lines not through a single point.*

Proof Let ℓ be a line of $\mathscr{L}$. Each point on ℓ lies on a line of $\mathscr{L}$ and so lies on q lines of $\mathscr{L}$. Since every line of $\mathscr{L}$ in $\mathrm{PG}(2,q)$ meets ℓ, it follows that $|\mathscr{L}| = (q+1)(q-1) + 1 = q^2$.

Denote by W the number of points in $\mathrm{PG}(2,q)$ lying on q lines of $\mathscr{L}$. Consider the set $X = \{(\ell, P) \mid \ \ell \text{ is a line of } \mathscr{L}, P \text{ is a point on } \ell\ \}$. The size of X will be counted in two ways. Counting ℓ then P, there are q^2 choices for ℓ and $q+1$ choices for P on ℓ. So, $|X| = q^2(q+1)$. Counting P then ℓ, there are W points lying on q lines of $\mathscr{L}$, and the remaining points lie on no lines of $\mathscr{L}$. So $|X| = qW$. Equating these two expressions for $|X|$ gives $W = q^2 + q$. Thus there is one point on no lines of $\mathscr{L}$. Denote this point by V. There are q^2 lines not through V, and there are q^2 lines of $\mathscr{L}$, none of which pass through V. So $\mathscr{L}$ is the set of lines not through V. □

Next, the lines associated with an oval in $\mathrm{PG}(2, q)$, q even, are considered.

Lemma 3.1.3 [46] *Let $\mathscr{L}$ be a set of $\frac{1}{2}q(q+1)$ lines in $\mathrm{PG}(2, q)$, q even, such that every point of $\mathrm{PG}(2, q)$ lies on 0, $\frac{1}{2}q$ or q lines of $\mathscr{L}$. Then $\mathscr{L}$ is the set of secants of an oval.*

The following theorem was proved by Di Gennaro, Durante and Olanda in [27] as part of their proof of Theorem 3.2.7.

Lemma 3.1.4 [27] *Let $\mathscr{L}$ be a non-empty set of lines in $\mathrm{PG}(2, q)$, q even, such that every point of $\mathrm{PG}(2, q)$ lies on 0 or $\frac{1}{2}q$ lines of $\mathscr{L}$. Then $\mathscr{L}$ is the set of external lines of a hyperoval.*

Proof Let ℓ be a line of $\mathscr{L}$. Every point of ℓ lies on $\frac{1}{2}q$ lines of $\mathscr{L}$, and every line of $\mathscr{L}$ meets ℓ. Thus $|\mathscr{L}| = (q+1)(\frac{1}{2}q - 1) + 1 = \frac{1}{2}q(q-1)$.

Let m be a line not in $\mathscr{L}$ and let B be the number of points on m lying on no lines of $\mathscr{L}$. The remaining $q + 1 - B$ points on m lie on $\frac{1}{2}q$ lines of $\mathscr{L}$ each. Every line of $\mathscr{L}$ passes through a point of m, so $\frac{1}{2}q(q-1) = \frac{1}{2}q(q+1-B)$. Thus $B = 2$.

Denote the set of points lying on no lines of $\mathscr{L}$ by $\mathscr{B}$. Then every line meets $\mathscr{B}$ in 0 or 2 points. That is, $\mathscr{B}$ is an arc with no tangents. Hence $\mathscr{B}$ is a hyperoval and the lines of $\mathscr{L}$ are its external lines. □

Finally, the lines associated with a non-singular conic in $\mathrm{PG}(2, q)$, q odd, are considered. The original results were stated in terms of sets of points of $\mathrm{PG}(2, q)$. The lemmas below are the dual statements.

Lemma 3.1.5 [32] *Let $\mathscr{L}$ be a set of $\frac{1}{2}q(q+1)$ lines in $\mathrm{PG}(2, q)$, q odd, such that every point of $\mathrm{PG}(2, q)$ lies on $\frac{1}{2}(q-1)$, $\frac{1}{2}(q+1)$ or q lines of $\mathscr{L}$. Then $\mathscr{L}$ is the set of secants of a non-singular conic.*

Lemma 3.1.6 [25] *Let $\mathscr{L}$ be a set of lines in $\mathrm{PG}(2, q)$, q odd, $q > 3$, such that every point of $\mathrm{PG}(2, q)$ lies on 0, $\frac{1}{2}(q-1)$ or $\frac{1}{2}(q+1)$ lines of $\mathscr{L}$. Then $\mathscr{L}$ is the set of external lines of a non-singular conic.*

3.2 Lines associated with Quadrals in $\mathrm{PG}(n, q)$

Below are listed several results characterising special sets of lines associated with quadrals. The first few results concern the tangents and generator lines of a quadric. This first result is due to Venezia in 1983.

Theorem 3.2.1 [58] *Let $\mathscr{L}$ be a set of lines in $\mathrm{PG}(3, q)$, q odd, such that*

(I) In every plane of $\mathrm{PG}(3, q)$, there are 0, 1, 2 or $q+1$ lines of $\mathscr{L}$ through each point,

(II) Every point of $\mathrm{PG}(3, q)$ lies on $q+1$ lines of $\mathscr{L}$,

(III) Every plane of $\mathrm{PG}(3, q)$ contains $q+1$ lines of $\mathscr{L}$.

Then $\mathscr{L}$ is either a general linear complex or the set of tangents and generator lines of a non-singular quadric in $\mathrm{PG}(3, q)$.

The proof of this theorem made heavy use of the Klein correspondence, and the original statement of the theorem listed the conditions in terms of the correspondence. That is, $\mathscr{L}$ is represented by a set of points $\mathscr{K}$ on the Klein quadric such that every line of the quadric meets $\mathscr{K}$ in 0, 1, 2 or $q+1$ points and every plane on the Klein quadric meets $\mathscr{K}$ in $q+1$ points. Using the same ideas, Bichara and Zanella [8] proved a similar characterisation for the tangents and generator lines of a quadric in $\mathrm{PG}(n, q)$, $n \geq 3$, q odd.

Theorem 3.2.2 [8] *Let $\mathscr{L}$ be a set of lines in $\mathrm{PG}(n, q)$, $n \geq 3$, q odd such that*

(I) Every plane of $\mathrm{PG}(n, q)$ contains $q+1$ or q^2+q+1 lines of $\mathscr{L}$,

(II) In every plane of $\mathrm{PG}(n, q)$, there are 0, 1, 2 or $q+1$ lines of $\mathscr{L}$ through any point,

(III) For every line $\ell \in \mathscr{L}$, there is a point P on ℓ and a hyperplane Σ of $\mathrm{PG}(n, q)$ through ℓ such that every line in Σ through P is a line of $\mathscr{L}$.

Then $\mathscr{L}$ is either a linear complex, or the set of tangents and generator lines of a quadric in $\mathrm{PG}(n,q)$.

In 1985, de Resmini proved a characterisation of the tangents and generator lines of a parabolic quadric in $\mathrm{PG}(4,q)$ using purely combinatorial arguments.

Theorem 3.2.3 [47] *Let $\mathscr{L}$ be a set of lines in* $\mathrm{PG}(4,q)$, *q odd, such that:*

(I) Every point of $\mathrm{PG}(4,q)$ *lies on* q^2+1, q^2+q+1 *or* q^2+2q+1 *lines of* $\mathscr{L}$,

(II) In every plane of $\mathrm{PG}(4,q)$, *there are 0, 1, 2 or* $q+1$ *lines of $\mathscr{L}$ through each point,*

(III) If a point P of $\mathrm{PG}(4,q)$ *lies on n lines of $\mathscr{L}$ in some plane, then every line of $\mathscr{L}$ through P is contained in a plane with n lines of $\mathscr{L}$ through P,*

(IV) For any hyperplane of $\mathrm{PG}(4,q)$, *there are at most two possible choices for the number of lines of $\mathscr{L}$ in each of its planes.*

Then $\mathscr{L}$ is the set of tangents and generator lines of a parabolic quadric in $\mathrm{PG}(4,q)$.

The proof of this theorem made use of the characterisation of large Tallini sets given in Theorem 2.1.10.

The above characterisations were only proved for q odd. When q is even, the tangents and generator lines of a non-singular quadral in $\mathrm{PG}(3,q)$ form a general linear complex, and the tangents and generator lines of an oval cone form a special linear complex. The following theorem was proved by de Resmini in 1984 and characterises the linear complexes in $\mathrm{PG}(3,q)$.

Theorem 3.2.4 [48] *Let $\mathscr{L}$ be a set of lines in* $\mathrm{PG}(3,q)$ *such that*

(I) Every point of $\mathrm{PG}(3,q)$ *lies on m or n lines of* $\mathscr{L}$,

(II) In every plane of $\mathrm{PG}(3,q)$, *the number of lines of $\mathscr{L}$ through each point is* 1 *or m.*

Then either $\mathscr{L}$ is a special linear complex (in which case $m = q+1$ and $n = q^2+q+1$), or $\mathscr{L}$ is a general linear complex (in which case $m = n = q+1$).

The above results concern the tangents and generator lines of a quadral. Similar results have been obtained for some other types of lines associated with quadrals. In the 1980s, Ferri and Tallini [32] and de Resmini [49] proved characterisations of the secants of an ovoid for q odd and q even respectively. These results were improved recently. In 2005, Durante and Olanda published the following characterisations of the secants and external lines of an ovoid.

Theorem 3.2.5 [29] *Let $\mathscr{L}$ be a set of lines in $\mathrm{PG}(3, q)$, $q > 2$, such that*

(I) Every point of $\mathrm{PG}(3, q)$ lies on q^2 or $\frac{1}{2}q(q-1)$ lines of $\mathscr{L}$,

(II) Every plane of $\mathrm{PG}(3, q)$ contains 0 or $\frac{1}{2}q(q+1)$ lines of $\mathscr{L}$.

Then $\mathscr{L}$ is the set of secants of an ovoid.

Theorem 3.2.6 [29] *Let $\mathscr{L}$ be a set of lines in $\mathrm{PG}(3, q)$, $q > 2$, such that:*

(I) Every point of $\mathrm{PG}(3, q)$ lies on 0 or $\frac{1}{2}q(q+1)$ lines of $\mathscr{L}$,

(II) Every plane of $\mathrm{PG}(3, q)$ contains q^2 or $\frac{1}{2}q(q-1)$ lines of $\mathscr{L}$.

Then $\mathscr{L}$ is the set of external lines of an ovoid.

In their article, the characterisation of the secants is proved first, and then the characterisation of the external lines follows using duality. Note that while the above theorems are stated for $q > 2$, it is also possible to prove them in the case $q = 2$.

In 2004, Di Gennaro, Durante and Olanda published characterisations of the external lines of a hyperbolic quadric of $\mathrm{PG}(3, q)$ [27]. In their proof, an assumption was made that was not stated in their theorems. Also, a possibility was omitted in the case $q = 2$. These omissions have been rectified in the statements below.

Theorem 3.2.7 [27] *Let $\mathscr{L}$ be a non-empty set of lines in* $\mathrm{PG}(3,q)$*, q odd, such that*

(I) Every point of $\mathrm{PG}(3,q)$ *lies on 0 or* $\frac{1}{2}q(q-1)$ *lines of* $\mathscr{L}$*,*

(II) Every plane of $\mathrm{PG}(3,q)$ *contains 0 or* $\frac{1}{2}q(q-1)$ *lines of* $\mathscr{L}$*,*

(III) In every plane of $\mathrm{PG}(3,q)$*, there are 0,* $\frac{1}{2}(q-1)$ *or* $\frac{1}{2}(q+1)$ *lines of* $\mathscr{L}$ *through any point,*

(IV) There exists a point of $\mathrm{PG}(3,q)$ *lying on no lines of* $\mathscr{L}$*.*

Then the set of points lying on no lines of $\mathscr{L}$ forms a line, a pair of lines or a hyperbolic quadric. In the last case, $\mathscr{L}$ is the set of external lines of this quadric.

Theorem 3.2.8 [27] *Let $\mathscr{L}$ be a non-empty set of lines in* $\mathrm{PG}(3,q)$*, q even, such that*

(I) In every plane of $\mathrm{PG}(3,q)$*, there are 0 or* $\frac{1}{2}q$ *lines of* $\mathscr{L}$ *through any point,*

(II) There exists a point of $\mathrm{PG}(3,q)$ *lying on no lines of* $\mathscr{L}$*.*

If $q > 2$, then the set of points lying on no lines of $\mathscr{L}$ forms a line, a pair of lines or a hyperbolic quadric. In the last case, $\mathscr{L}$ is the set of external lines of this quadric. If $q = 2$, then $\mathscr{L}$ may also be a single line.

This completes the list of existing characterisations of subspaces associated with quadrals. The next three chapters present original results characterising sets of lines and planes associated with quadrals. In Chapter 4, the external lines of an oval cone in $\mathrm{PG}(3,q)$ are characterised for q odd and q even. In Chapter 5, characterisations are proved for sets of subspaces associated with a parabolic quadric in $\mathrm{PG}(4,q)$. The external lines are characterised for q even, the secant planes are characterised for q even and q odd, and the tangents and generator lines are characterised for q odd. Finally in Chapter 6, characterisations are proved for the external lines of an oval cone in $\mathrm{PG}(n,q)$, q even, and for the external lines of a quadral of parabolic type in $\mathrm{PG}(n,q)$, q even. The methods used to prove these results are inspired by those used to prove the theorems above.

Chapter 4

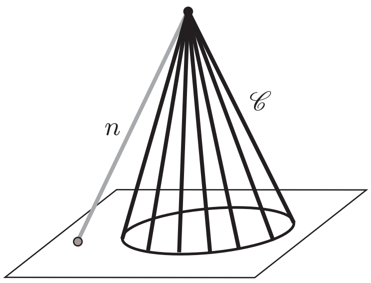
Characterisation of the external lines of an oval cone in $\mathrm{PG}(3, q)$

In the previous chapter, several existing characterisations of line sets related to quadrics were listed. In particular, Theorems 3.2.6, 3.2.7 and 3.2.8 were stated, which characterised the external lines of the ovoid and the hyperbolic quadric in $\mathrm{PG}(3, q)$. The ovoid and the hyperbolic quadric are the non-singular irreducible quadrals in $\mathrm{PG}(3, q)$. In $\mathrm{PG}(3, q)$, there also exists a *singular* irreducible quadral – the oval cone. Since it is possible to characterise the external lines of the other irreducible quadrals, it is reasonable to attempt to characterise the external lines of an oval cone.

This chapter presents original results characterising the external lines of an oval cone in $\mathrm{PG}(3, q)$ for q even and q odd.

4.1 The external lines of an oval cone in $\mathrm{PG}(3, q)$, q even

In this section, a characterisation of the set of external lines of an oval cone in $\mathrm{PG}(3, q)$, q even, will be proved. Before this is done, it is appropriate to highlight some important features of this set of lines.



Let $\mathscr{C}$ be an oval cone of $\mathrm{PG}(3,q)$, q even, with nuclear line n. Then every point of $\mathscr{C}$ lies on no external lines. However, these are not the only points with this property. The points on the nuclear line n lie only on tangents of $\mathscr{C}$, and so these points lie on no external lines. Every other point of $\mathrm{PG}(3,q)$ lies on $\frac{1}{2}q^2$ external lines (Lemma 1.4.9).

This means that the set of external lines of the oval cone $\mathscr{C}$ is the same as the set of external lines of the hyperoval cone $\mathscr{C} \cup n$. Indeed, if an oval cone is formed by removing any of the generator lines of $\mathscr{C} \cup n$, then this cone also has the same set of external lines (Lemma 1.4.11). In light of this observation, the following theorem is proved.

Theorem 4.1.1 *Let $\mathscr{L}$ be a non-empty set of lines in* $\mathrm{PG}(3,q)$, *q even, such that*

(I) Every plane of $\mathrm{PG}(3,q)$ *contains 0, q^2 or $\frac{1}{2}q(q-1)$ lines of $\mathscr{L}$,*

(II) Every point of $\mathrm{PG}(3,q)$ *lies on 0 or $\frac{1}{2}q^2$ lines of $\mathscr{L}$.*

Then $\mathscr{L}$ is the set of external lines of a hyperoval cone $\mathscr{C}$. Further, $\mathscr{L}$ is the set of external lines of each of the $q+2$ different oval cones contained in $\mathscr{C}$.

The proof of Theorem 4.1.1

Let $\mathscr{L}$ be a set of lines as described in Theorem 4.1.1. By a series of lemmas, it will be proved that $\mathscr{L}$ is the set of external lines of a hyperoval cone. In order to make the argument clearer, some terminology is now introduced.

- A point on 0 lines of $\mathscr{L}$ is called a *black point*; a point on $\frac{1}{2}q^2$ lines of $\mathscr{L}$ is called a *white point*.
- A plane containing 0 lines of $\mathscr{L}$ is called a *null plane*.
- A plane containing q^2 lines of $\mathscr{L}$ is called a *V-plane*.

- A plane containing $\frac{1}{2}q(q-1)$ lines of $\mathscr{L}$ is called a *secant plane*.

It will be shown that the set of black points is a hyperoval cone $\bar{\mathscr{C}}$. The null planes are those planes meeting $\bar{\mathscr{C}}$ in a pair of lines, the V-planes are those planes meeting $\bar{\mathscr{C}}$ in only its vertex, and the secant planes are the planes meeting $\bar{\mathscr{C}}$ in a hyperoval. The first lemma gives the number of each type of plane through a line of $\mathscr{L}$.

Lemma 4.1.2 *There are q secant planes and one V-plane through any line of $\mathscr{L}$.*

Proof Let ℓ be a line of $\mathscr{L}$ and let v_ℓ be the number of V-planes through ℓ. Denote by L_ℓ the number of lines of $\mathscr{L}$ meeting ℓ in one point. The number L_ℓ will be counted in two ways.

Each point on ℓ is a white point, and so lies on $\frac{1}{2}q^2$ lines of $\mathscr{L}$ (including ℓ). Counting this way, ℓ itself has been included $q+1$ times. So $L_\ell = \frac{1}{2}q^2(q+1) - (q+1)$.

On the other hand, every line of $\mathscr{L}$ meeting ℓ lies in a plane through ℓ, so L_ℓ may be counted by considering the lines in each plane through ℓ. There are v_ℓ V-planes through ℓ, and no null planes. This gives $(q+1-v_\ell)$ secant planes through ℓ. Each V-plane contains q^2 lines of $\mathscr{L}$ (including ℓ) and each secant plane contains $\frac{1}{2}q(q-1)$ lines of $\mathscr{L}$ (including ℓ). Again ℓ itself has been counted $q+1$ times, so

$$\begin{aligned} L_\ell &= q^2 \cdot v_\ell + \tfrac{1}{2}q(q-1)\cdot(q+1-v_\ell) - (q+1) \\ &= \tfrac{1}{2}q(q+1)v_\ell + \tfrac{1}{2}q^2(q+1) - \tfrac{1}{2}q(q+1) - (q+1). \end{aligned}$$

Equating these two expressions for L_ℓ gives

$$\begin{aligned} \tfrac{1}{2}q^2(q+1) - (q+1) &= \tfrac{1}{2}q(q+1)v_\ell + \tfrac{1}{2}q^2(q+1) - \tfrac{1}{2}q(q+1) - (q+1) \\ v_\ell &= 1. \end{aligned}$$

That is, there is one V-plane through ℓ, and hence there are q secant planes through ℓ.

□

The above lemma ensures the existence of both secant planes and V-planes since $\mathscr{L}$ is non-empty. The next few lemmas determine the number of black points in each type of plane.

Lemma 4.1.3 *There is at most one black point in a V-plane*

Proof Let π be a V-plane and suppose P and Q are two black points in π. Then P and Q lie on no lines of $\mathscr{L}$. There are $q^2 - q$ lines in π not through P and Q, so there are at most $q^2 - q$ lines of $\mathscr{L}$ in π. But π is a V-plane and contains q^2 lines of $\mathscr{L}$. This is a contradiction, so there is at most one black point in π. □

Lemma 4.1.4 *All secant planes have the same number of black points, all V-planes have the same number of black points, and all null planes have the same number of black points.*

Proof Let π be a plane containing L_π lines of $\mathscr{L}$. Denote the number of black points in π by B_π. Thus there are $(q^2 + q + 1 - B_\pi)$ white points in π. The size of $\mathscr{L}$ will be counted by considering the lines of $\mathscr{L}$ through each point in π.

Through each black point there are no lines of $\mathscr{L}$, and through each white point there are $\frac{1}{2}q^2$ lines of $\mathscr{L}$. Counting this way, each line of $\mathscr{L}$ in π has been included $q + 1$ times – once for each white point on it. Thus

$$\begin{aligned} |\mathscr{L}| &= 0 \cdot B_\pi + \tfrac{1}{2}q^2 \cdot (q^2 + q + 1 - B_\pi) - qL_\pi \\ &= \tfrac{1}{2}q^2(q^2 + q + 1 - B_\pi) - qL_\pi. \end{aligned}$$

In the above equation, $|\mathscr{L}|$ is a constant, so B_π is only dependent on L_π. That is, two planes with the same number of lines of $\mathscr{L}$ have the same number of black points. □

Lemma 4.1.5 *There are $\frac{1}{2}q^3(q-1)$ lines of $\mathscr{L}$ and $q^2 + 2q + 1$ black points. A null plane contains $2q + 1$ black points, a V-plane contains 1 black point, and a secant plane contains $q + 2$ black points.*

Proof Denote by B_0 be the number of black points in a null plane, by B_V the number of black points in a V-plane, and by B_s the number of black points in a secant plane. From the proof of Lemma 4.1.4, if π is a plane containing B_π black points and L_π lines of $\mathscr{L}$, then

$$|\mathscr{L}| = \tfrac{1}{2}q^2(q^2 + q + 1 - B_\pi) - qL_\pi. \tag{4.1}$$

If π is a V-plane, then $L_\pi = q^2$ and Equation (4.1) becomes

$$|\mathscr{L}| = \tfrac{1}{2}q^2(q^2 + q + 1 - B_V) - q^3.$$

By Lemma 4.1.3, there is at most one black point in a V-plane, so $B_V = 0$ or 1. If $B_V = 0$, then $|\mathscr{L}| = \frac{1}{2}q^2(q^2 + q + 1) - q^3 = \frac{1}{2}q^2(q^2 - q + 1)$ and if $B_V = 1$, then $|\mathscr{L}| = \frac{1}{2}q^2(q^2 + q) - q^3 = \frac{1}{2}q^3(q - 1)$.

Let $X = \{(\ell, \pi) \mid \ \ell$ is a line of $\mathscr{L}$, π is a V-plane through ℓ $\}$. The size of X will be counted in two ways.
Count ℓ first, then π: There are $|\mathscr{L}|$ choices for ℓ, and then 1 choice for π through ℓ by Lemma 4.1.2. So, $|X| = |\mathscr{L}|$.
Count π first, then ℓ: Let v be the total number of V-planes. Then there are v choices for π, and q^2 choices for ℓ within π. So, $|X| = v \cdot q^2$.
Thus, $|\mathscr{L}| = q^2 v$, and so $v = \frac{|\mathscr{L}|}{q^2}$.

If $|\mathscr{L}| = \frac{1}{2}q^2(q^2 - q + 1)$, then $v = \frac{1}{2}(q^2 - q + 1)$. But $q^2 - q + 1$ is an odd number, which implies that v is not a whole number. This is a contradiction, since v is the number of V-planes, so $|\mathscr{L}| = \frac{1}{2}q^3(q - 1)$, and hence $B_V = 1$.

If π is a secant plane, then $L_\pi = \frac{1}{2}q(q - 1)$. Substituting this and the value of $|\mathscr{L}|$ into equation (4.1) gives

$$\begin{aligned}\tfrac{1}{2}q^3(q - 1) &= \tfrac{1}{2}q^2(q^2 + q + 1 - B_s) - \tfrac{1}{2}q^2(q - 1)\\ B_s &= q + 2.\end{aligned}$$

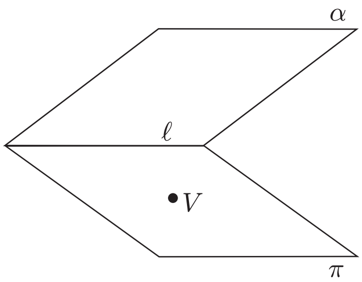
If π is a null plane, then $L_\pi = 0$. Substituting this and the value of $|\mathscr{L}|$ into equation (4.1) gives

$$\begin{aligned}\tfrac{1}{2}q^3(q - 1) &= \tfrac{1}{2}q^2(q^2 + q + 1 - B_0)\\ B_0 &= 2q + 1.\end{aligned}$$

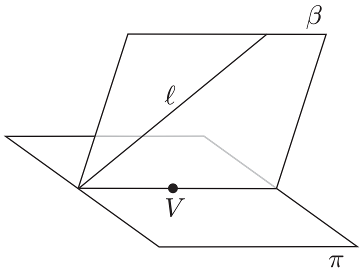
Finally, let ℓ be a line of $\mathscr{L}$ and consider the planes through ℓ. The line ℓ has one V-plane through it and q secant planes by Lemma 4.1.2. The V-plane has one black point, and the secant planes each have $q+2$ black points. Thus the number of black points is $1 \cdot 1 + q \cdot (q+2) = q^2 + 2q + 1$. □

Lemma 4.1.6 *There exists a unique (black) point V such that all null planes and V-planes pass through V. The secant planes are the planes not containing V.*

Proof Let π be a V-plane. By Lemma 4.1.5, π contains a single black point, V. The q^2 lines of $\mathscr{L}$ in π do not pass through V. There are q^2 lines of π not through V, so every line of π not through V is a line of $\mathscr{L}$.



Suppose α is a plane not through V. Then α meets π in a line ℓ not through V. This line is a line of $\mathscr{L}$, so α is not a null plane. By Lemma 4.1.2, a line of $\mathscr{L}$ is contained in one V-plane. Since ℓ is contained in the V-plane π, the plane α is a secant plane. Thus every plane not through V is a secant plane.



Suppose β is a secant plane through V and let ℓ be a line of $\mathscr{L}$ in β. By Lemma 4.1.2, a line of $\mathscr{L}$ is contained in exactly q secant planes. Consider the planes through ℓ. One of these planes is β, which contains V. The remaining planes do not contain V and so are secant planes. Thus there are $q+1$ secant planes through ℓ. This is a contradiction, so there exist no secant planes through V. Thus the secant planes are the planes not through V and all V-planes and null planes pass through V. □

Lemma 4.1.7 *Let m be a line not in $\mathscr{L}$. If m passes through V, then m contains 1 or $q+1$ black points. If m does not pass through V, then m contains 2 black points.*

Proof Suppose m passes through V and contains a further black point P. Let π be a plane through m. By Lemma 4.1.6, π is not a secant plane, since it contains V. By Lemma 4.1.5, π is not a V-plane, since it contains at least two black points. Thus π is a null plane, and so every plane through m is a null plane. Since there are no lines of $\mathscr{L}$ in any null plane, there are no lines of $\mathscr{L}$ meeting m. Thus every point on m is a black point. So a line through V either has V as its only black point, or contains $q+1$ black points.

Suppose m does not pass through V and denote by B_m the number of black points on m. The total number of all black points will be counted by considering the black points in the planes through m.

Exactly one plane passes through both m and V. , If this plane is a V-plane, then m is a line of $\mathscr{L}$ (as noted in the proof of Lemma 4.1.6). This is a contradiction, so the plane $m \oplus V$ is a null plane and contains $2q+1$ black points by Lemma 4.1.5. The other q planes through m are secant planes and each contains $q+2$ black points. Counting this way, the black points on m have been counted $q+1$ times, once for each plane through m. Thus

$$q^2 + 2q + 1 = 2q + 1 + q(q+2) - qB_m$$
$$B_m = 2.$$

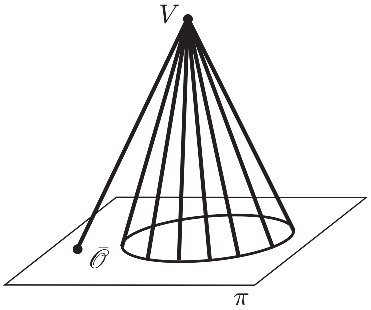
Hence a line not through V is either a line of $\mathscr{L}$ or contains exactly 2 black points.

□

It is now possible to show that the set of black points is a hyperoval cone.

Lemma 4.1.8 *Denote the set of black points by $\bar{\mathscr{C}}$. Then $\bar{\mathscr{C}}$ is a hyperoval cone and $\mathscr{L}$ is its set of external lines.*

Proof Every line meets $\bar{\mathscr{C}}$ in 0, 1, 2 or $q+1$ points by Lemma 4.1.7, so $\bar{\mathscr{C}}$ is a Tallini set. By Lemma 4.1.7, the point V lies only on tangents and generator lines of $\bar{\mathscr{C}}$, so V is a singular point of $\bar{\mathscr{C}}$.



Let π be a secant plane and let $\bar{\mathscr{O}}$ be the set of black points in π. The plane π does not pass through V by Lemma 4.1.6, so each line in π contains 0, 1 or 2 black points by Lemma 4.1.7. Thus $\bar{\mathscr{O}}$ is an arc, which has $q+2$ points by Lemma 4.1.5. That is, $\bar{\mathscr{O}}$ is a hyperoval. Since V is a singular point of $\bar{\mathscr{C}}$, it follows from Lemma 2.1.6 that $\bar{\mathscr{C}}$ is the hyperoval cone $V\bar{\mathscr{O}}$.

By Lemma 4.1.7, a line not in $\mathscr{L}$ contains a black point, so the lines of $\mathscr{L}$ are the external lines of $\bar{\mathscr{C}}$. □

Finally, note that if g is a generator line of the hyperoval cone $\bar{\mathscr{C}}$, then the set of points obtained by removing all the points of g except V is an *oval* cone. Since $\bar{\mathscr{C}}$ has $q+2$ generator lines, there are $q+2$ different oval cones contained in $\bar{\mathscr{C}}$. If $\mathscr{C}$ is one of these cones, then $\mathscr{L}$ is the set of external lines of $\mathscr{C}$ by Lemma 1.4.11. This completes the proof of Theorem 4.1.1.

4.2 The external lines of an oval cone in PG$(3, q)$, q odd

In this section, a characterisation of the set of external lines of an oval cone in PG$(3, q)$, q odd, is proved. Before the result is stated, the combinatorial properties of this set of lines will be discussed.

Let $\mathscr{C}$ be a quadric cone in PG$(3, q)$, q odd, and let $\mathscr{L}$ be its set of external lines. By Lemmas 1.4.4 and 1.4.7, the set of lines $\mathscr{L}$ has the following properties.

(I) Every point of PG$(3, q)$ lies on 0, $\frac{1}{2}q(q-1)$ or $\frac{1}{2}q(q+1)$ lines of $\mathscr{L}$;

(II) Every plane of PG$(3, q)$ contains 0, $\frac{1}{2}q(q-1)$ or q^2 lines of $\mathscr{L}$.

There are several different sets of lines with these properties. For example, the set of external lines of a hyperbolic quadric has the above properties (Lemma 1.5.33), as does the set of external lines of an ovoid (Lemma 1.2.15). Thus further conditions must be placed on a set of lines in order to prove that it is the set of external lines of a quadric cone.

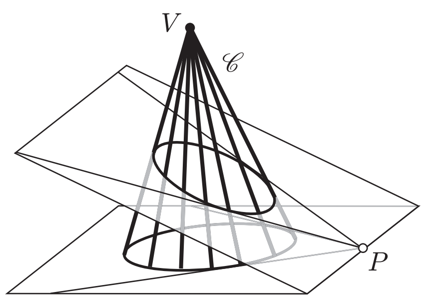
A possibility is to require that all the possible types of point and plane do exist. However, using only properties (I) and (II) and this assumption, it does not seem possible to show that the set of points on no lines of $\mathscr{L}$ is a quadric cone, or even a Tallini set.

Another possibility is to use the possible numbers of lines through a point in a plane. That is, to use the following property in addition to properties (I) and (II) above.

(III) In every plane of $\mathrm{PG}(3, q)$, there are 0, $\frac{1}{2}(q-1)$, $\frac{1}{2}(q+1)$ or q lines of $\mathscr{L}$ through any point.

However, the sets of external lines of a hyperbolic quadric or an ovoid also satisfy this condition. It is necessary to find a property unique to the set of external lines of a quadric cone.

Let P be a point not in the quadric cone $\mathscr{C}$ and let π be a secant plane through P. The plane π meets $\mathscr{C}$ in a non-singular conic $\mathscr{O}$, and P is either an interior point or an exterior point of $\mathscr{O}$.



By Lemma 1.4.7, if P is an interior point in π, then P is an interior point in every secant plane through P; and if P is an exterior point in π, then P is an exterior point in every secant plane through P. In other words, if P is on some number of lines of $\mathscr{L}$ in a secant plane, it is on the same number of lines of $\mathscr{L}$ in every secant plane through P. It is this property that will be used to characterise the external lines of the quadric cone. That is, the following theorem is proved.

Theorem 4.2.1 *Let $\mathscr{L}$ be a non-empty set of lines in* $\mathrm{PG}(3,q)$*, q odd, such that*

(I) Every point of $\mathrm{PG}(3,q)$ *lies on* 0*,* $\frac{1}{2}q(q+1)$ *or* $\frac{1}{2}q(q-1)$ *lines of* $\mathscr{L}$*,*

(II) Every plane of $\mathrm{PG}(3,q)$ *contains* 0*,* q^2 *or* $\frac{1}{2}q(q-1)$ *lines of* $\mathscr{L}$*,*

(III) For any point P of $\mathrm{PG}(3,q)$*, if P lies on two planes which contain the same number of lines of $\mathscr{L}$, then P lies on the same number of lines of $\mathscr{L}$ in both planes.*

Then $\mathscr{L}$ is the set of external lines of a quadric cone.

The Proof of Theorem 4.2.1

Let $\mathscr{L}$ be a set of lines as described in Theorem 4.2.1. By a series of lemmas, it will be proved that $\mathscr{L}$ is the set of external lines of a quadric cone. In order to make the argument clearer, some terminology is now introduced.

- A point on 0 lines of $\mathscr{L}$ is called a *black point*; all other points are called *white points*.
- A (white) point on $\frac{1}{2}q(q-1)$ lines of $\mathscr{L}$ is called an *exterior point* and a (white) point on $\frac{1}{2}q(q+1)$ lines of $\mathscr{L}$ is called an *interior point*.
- A plane containing 0 lines of $\mathscr{L}$ is called a *null plane*.
- A plane containing q^2 lines of $\mathscr{L}$ is called a *V-plane*.
- A plane containing $\frac{1}{2}q(q-1)$ lines of $\mathscr{L}$ is called a *secant plane*.

It will be shown that the set of black points is a quadric cone $\mathscr{C}$, and that $\mathscr{L}$ is the set of external lines of $\mathscr{C}$. The null planes are those planes containing a generator line of $\mathscr{C}$, the V-planes are those planes that meet $\mathscr{C}$ in only its vertex, and the secant planes are those planes that meet $\mathscr{C}$ in a non-singular conic. The first two lemmas concern the number of each type of plane through a line of $\mathscr{L}$.

Lemma 4.2.2 *Let P be a white point. Then every line of $\mathscr{L}$ through P lies on the same number of V-planes.*

Proof Denote by L_P the number of lines of $\mathscr{L}$ through P. By Condition (III) of Theorem 4.2.1, P lies on the same number of lines of $\mathscr{L}$ in every secant plane through P. Denote this number of lines by L_{Ps}. If there are no secant planes through P, set $L_{Ps} = 0$. Similarly, P lies on the same number of lines of $\mathscr{L}$ in every V-plane through P. Denote this number by L_{Pv}. If there are no V-planes through P, set $L_{Pv} = 0$.

Let ℓ be a line of $\mathscr{L}$ through P and denote by v_ℓ the number of V-planes through ℓ. Since a null plane contains no lines of $\mathscr{L}$, there are no null planes through ℓ. So the number of secant planes through ℓ is $(q + 1 - v_\ell)$. The number of lines of $\mathscr{L}$ through P will be counted by considering the lines of $\mathscr{L}$ through P in each plane through ℓ.

Each V-plane through ℓ contains L_{Pv} lines of $\mathscr{L}$ through P, including ℓ. Each secant plane through ℓ contains L_{Ps} lines of $\mathscr{L}$ through P, including ℓ. Counting this way, the line ℓ itself has been included $q + 1$ times, so

$$L_P = v_\ell L_{Pv} + (q + 1 - v_\ell)L_{Ps} - q. \tag{4.2}$$

In the above equation, L_P, L_{Pv} and L_{Ps} are constants, so v_ℓ is uniquely determined by P. That is, every line of $\mathscr{L}$ through P lies on the same number of V-planes. □

Lemma 4.2.3 *A line of $\mathscr{L}$ lies on at most two V-planes.*

Proof Let ℓ be a line of $\mathscr{L}$. Denote by v_ℓ the number of V-planes through ℓ and by I_ℓ the number of interior points on ℓ. Since ℓ contains no black points, there are $(q + 1 - I_\ell)$ exterior points on ℓ; and since ℓ lies on no null planes, there are $(q + 1 - v_\ell)$ secant planes through ℓ. Let L_ℓ be the number of lines of $\mathscr{L}$ meeting ℓ in one point. The number L_ℓ will be counted in two ways.

Each interior point lies on $\frac{1}{2}q(q+1)$ lines of $\mathscr{L}$ (including ℓ), and each exterior point lies on $\frac{1}{2}q(q - 1)$ lines of $\mathscr{L}$ (including ℓ). Counting this way, the line ℓ itself has been included $q + 1$ times, so $L_\ell = \frac{1}{2}q(q + 1)I_\ell + \frac{1}{2}q(q - 1)(q + 1 - I_\ell) - (q + 1)$.

On the other hand, every line meeting ℓ is contained in a plane through ℓ, so L_ℓ may be counted by considering the lines of $\mathscr{L}$ in the planes through ℓ. Each V-plane through ℓ contains q^2 lines of $\mathscr{L}$ (including ℓ), and each secant plane through ℓ

contains $\frac{1}{2}q(q-1)$ lines of $\mathscr{L}$ (including ℓ). Again, ℓ itself has been included $(q+1)$ times, so $L_\ell = q^2 v_\ell + \frac{1}{2}q(q-1)(q+1-v_\ell) - (q+1)$.

Equating the above two expressions for L_ℓ gives

$$\frac{1}{2}q(q+1)I_\ell + \frac{1}{2}q(q-1)(q+1-I_\ell) - (q+1) = q^2 v_\ell + \frac{1}{2}q(q-1)(q+1-v_\ell) - (q+1)$$

$$(q+1)v_\ell = 2I_\ell. \tag{4.3}$$

Now $I_\ell \leq q+1$, so $(q+1)v_\ell \leq 2(q+1)$. Thus $v_\ell \leq 2$. □

The above lemma ensures the existence of secant planes, since there are at most two V-planes through a line of $\mathscr{L}$. There is no guarantee at this stage that a V-plane exists. However, if a V-plane does exist, the following lemma gives information about the structure of the set of lines of $\mathscr{L}$ in it.

Lemma 4.2.4 *Suppose π is a V-plane. Then each point in π lies on* 0 *or q lines of $\mathscr{L}$ in π.*

Proof First it is shown that every point of π lies on at most q lines of $\mathscr{L}$ in π. Suppose that P is a point of π on $q+1$ lines of $\mathscr{L}$ in π. Denote by L_P the total number of lines of $\mathscr{L}$ through P and by v_P the number of V-planes through P. By Condition (III) of Theorem 4.2.1, every V-plane through P contains the same number of lines of $\mathscr{L}$ through P. That is, every V-plane through P contains $q+1$ lines of $\mathscr{L}$ through P. Also, by Lemma 4.2.2, every line of $\mathscr{L}$ through P lies on the same number of V-planes. Denote this number by $v_{P\ell}$. By Lemma 4.2.3, $v_{P\ell} \leq 2$. However, since P lies on lines of $\mathscr{L}$ in the V-plane π, every line of $\mathscr{L}$ through P is contained in at least one V-plane. That is, $v_{P\ell} = 1$ or 2. An equation relating L_P, v_P and $v_{P\ell}$ will be formed by counting a set of pairs.

Let $X = \{(\ell, \alpha) \mid \ \ell$ is a line of $\mathscr{L}$ through P, α is a V-plane through $\ell\ \}$. The size of X will be counted in two ways.
Count ℓ first, then α: There are L_P choices for a line ℓ of $\mathscr{L}$ through P, and $v_{P\ell}$ choices for a V-plane α through each. So $|X| = L_P v_{P\ell}$.
Count α first, then ℓ: There are v_P choices for a V-plane α through P, and $(q+1)$ choices for a line ℓ of $\mathscr{L}$ through P in each. So $|X| = (q+1)v_P$. Thus

$$(q+1)v_P = L_P v_{P\ell}. \tag{4.4}$$

Suppose $v_{P\ell} = 1$. That is, suppose that there is exactly one V-plane through each line of $\mathscr{L}$ containing P. Any V-plane α through P other than π meets π in a line through P. Since all lines through P in π are lines of $\mathscr{L}$, the line $\alpha \cap \pi$ is a line of $\mathscr{L}$ with two V-planes through it. However, each line of $\mathscr{L}$ through P lies on exactly one V-plane. So, P lies on no V-plane other than π. That is $v_P = 1$. Equation 4.4 now becomes $q + 1 = L_P$. Now $L_P = \frac{1}{2}q(q-1)$ or $\frac{1}{2}q(q+1)$, and neither of these are equal to $q + 1$ for odd integer q. Thus $v_{P\ell} \neq 1$ and hence $v_{P\ell} = 2$.

Since every line of $\mathscr{L}$ through P lies on two V-planes, the $q+1$ lines of $\mathscr{L}$ in π define $q + 1$ further V-planes through P. There are no further V-planes through P since any plane through P other than π meets π in a line through P. Thus $v_P = q + 2$. Equation 4.4 now becomes $(q + 1)(q + 2) = 2L_P$. But $2L_P = q(q + 1)$ or $q(q - 1)$. Again this is a contradiction, so there cannot be $q + 1$ lines of $\mathscr{L}$ through P in π. That is, every point of π lies on at most q lines of $\mathscr{L}$ in π. Since there are q^2 lines of $\mathscr{L}$ in π, Lemma 3.1.1 implies that every point of π lies on 0 or q lines of $\mathscr{L}$ in π.

$\square$

Lemma 4.2.5 *A line of $\mathscr{L}$ lies on one V-plane and q secant planes. A line of $\mathscr{L}$ contains $\frac{1}{2}(q+1)$ interior points and $\frac{1}{2}(q+1)$ exterior points.*

Proof Let ℓ be a line of $\mathscr{L}$ lying on v_ℓ V-planes and containing I_ℓ interior points. Equation 4.3 states that $2I_\ell = (q + 1)v_\ell$. Also, by Lemma 4.2.3, $v_\ell \leq 2$. The cases of $v_\ell = 0$, 2 will be ruled out by considering the lines through one point on ℓ.

Let P be a point on ℓ lying on L_P lines of $\mathscr{L}$ in total and L_{Ps} lines of $\mathscr{L}$ in each secant plane. If π is a V-plane through ℓ, then P lies on at least one line of $\mathscr{L}$ in π. Lemma 4.2.4 implies that P lies on q lines of $\mathscr{L}$ in π, so by Condition (III) of Theorem 4.2.1, P lies on q lines of $\mathscr{L}$ in every V-plane through P. Equation 4.2 gives

$$L_P = v_\ell \cdot q + (q + 1 - v_\ell)L_{Ps} - q.$$

If $v_\ell = 0$, then from Equation 4.3 it follows that $I_\ell = 0$, so all points on ℓ are exterior points. Thus $L_P = \frac{1}{2}q(q-1)$. Hence,

$$\tfrac{1}{2}q(q-1) = (q+1)L_{Ps} - q$$
$$L_{Ps} = \tfrac{1}{2}q.$$

But q is odd, so $\frac{1}{2}q$ is not an integer. This is a contradiction, so $v_\ell \neq 0$.

If $v_\ell = 2$, then from Equation 4.3, it follows that $I_\ell = q+1$, so all points on ℓ are interior points. Thus $L_P = \frac{1}{2}q(q+1)$. Hence,

$$\begin{aligned} \tfrac{1}{2}q(q+1) &= 2q + (q-1)L_{Ps} - q \\ L_{Ps} &= \tfrac{1}{2}q. \end{aligned}$$

This is a contradiction as before, so $v_\ell \neq 2$.

Hence $v_\ell = 1$ and $I_\ell = \frac{1}{2}(q+1)\cdot 1 = \frac{1}{2}(q+1)$. It follows that there are q secant planes through ℓ and $\frac{1}{2}(q+1)$ exterior points on ℓ. $\square$

The above lemma ensures the existence of V-planes, interior points and exterior points since $\mathscr{L}$ is non-empty. It is now possible to calculate the number of lines of $\mathscr{L}$ through any point in any plane.

Lemma 4.2.6 *Let P be a white point. If P is an interior point, then P lies on q lines of $\mathscr{L}$ in every V-plane through P and $\frac{1}{2}(q+1)$ lines of $\mathscr{L}$ in every secant plane through P. If P is an exterior point, then P lies on q lines of $\mathscr{L}$ in every V-plane through P and $\frac{1}{2}(q-1)$ lines of $\mathscr{L}$ in every secant plane through P.*

Proof Let P be a white point and let ℓ be a line of $\mathscr{L}$ through P. By Lemma 4.2.5, ℓ is contained in a unique V-plane. Denote this plane by π. In the plane π, P lies on at least one line of $\mathscr{L}$, and so by Lemma 4.2.4, P lies on q lines of $\mathscr{L}$ in π. By Condition (III) of Theorem 4.2.1, the point P lies on q lines of $\mathscr{L}$ in every V-plane through P.

Denote by L_{Ps} the number of lines of $\mathscr{L}$ through P in a secant plane and let L_P be the total number of lines of $\mathscr{L}$ through P. Through ℓ there are q secant planes and one V-plane, and the V-plane contains q lines of $\mathscr{L}$ through P. From Equation 4.2, it follows that $L_P = qL_{Ps} + 1\cdot q - q = qL_{Ps}$. If P is an interior point, then $L_P = \frac{1}{2}q(q+1)$, and so $L_{Ps} = \frac{1}{2}(q+1)$. If P is an exterior point, then $L_P = \frac{1}{2}q(q-1)$, and so $L_{Ps} = \frac{1}{2}(q-1)$. $\square$

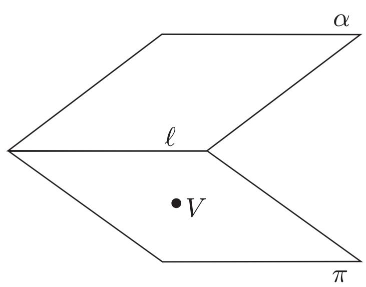
Lemma 4.2.7 *A V-plane contains exactly one black point, and the lines of $\mathscr{L}$ in the plane are the lines not through this black point.*

Proof Let π be a V-plane. Then every point of π lies on 0 or q lines of $\mathscr{L}$ in π. By Lemma 3.1.2, there exists a point V in π such that the lines of $\mathscr{L}$ in π are the lines not through V. By Lemma 4.2.6, a white point lies on a line of $\mathscr{L}$ in every V-plane through it, so V is a black point, and every other point of π is white. □

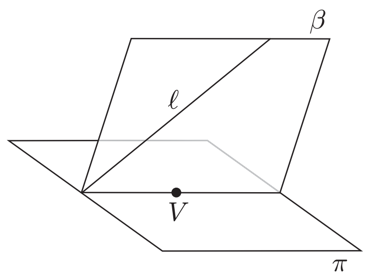
Note that since V-planes exist, the above lemma ensures that there exists a black point.

Lemma 4.2.8 *There exists a unique (black) point V through which all null planes and V-planes pass. The secant planes are the planes not containing V.*

Proof Let π be a V-plane. By Lemma 4.2.7, π contains a single black point, V, and the q^2 lines of π not through V are the lines of $\mathscr{L}$ in π.



Suppose α is a plane not through V. Then α meets π in a line ℓ not through V. This line is a line of $\mathscr{L}$, so α is not a null plane. By Lemma 4.2.5, a line of $\mathscr{L}$ is contained in one V-plane. Since ℓ is contained in the V-plane π, the plane α is a secant plane. Thus every plane not through V is a secant plane.



Suppose β is a secant plane through V and let ℓ be a line of $\mathscr{L}$ in β. By Lemma 4.2.5, a line of $\mathscr{L}$ is contained in exactly q secant planes. Consider the planes through ℓ. One of these planes is β, which contains V. The remaining planes do not contain V and so are secant planes. Thus there are $q+1$ secant planes through ℓ. This is a contradiction, so there are no secant planes through V. Thus the secant planes are the planes not through V and all V-planes and null planes pass through V. □

The next three lemmas complete the proof of Theorem 4.2.1.

Lemma 4.2.9 *Let m be a line not in $\mathscr{L}$. If m passes through V, then m contains 1 or $q+1$ black points. If m does not pass through V, then m contains 1 or 2 black points.*

Proof Suppose m contains V and a further black point P. Let π be a plane through m. By Lemma 4.2.8, π is not a secant plane, since it contains V. By Lemma 4.2.7, π is not a V-plane, since it contains at least two black points. Thus π is a null plane, and so every plane through m is a null plane. Since there are no lines of $\mathscr{L}$ in any null plane, there are no lines of $\mathscr{L}$ meeting m. Thus every point on m is a black point. So a line through V either has 1 or $q+1$ black points.

Suppose m does not contain V. Then exactly one plane through m contains V and the remaining q planes through m do not. These q planes are secant planes by Lemma 4.2.8. In light of this, let π be a secant plane through m.

Denote by B_m the number of black points on m, by E_m the number of exterior points on m, and by I_m the number of interior points on m. The number of lines of $\mathscr{L}$ in π will be counted by considering the lines of $\mathscr{L}$ in π through each point on m. There are no lines of $\mathscr{L}$ in π through each black point, $\frac{1}{2}(q+1)$ through each interior point and $\frac{1}{2}(q-1)$ through each exterior point by Lemma 4.2.6. Thus

$$\begin{aligned} \tfrac{1}{2}q(q-1) &= \tfrac{1}{2}(q+1)I_m + \tfrac{1}{2}(q-1)E_m \\ \tfrac{1}{2}(q-1)(q-E_m) &= \tfrac{1}{2}(q+1)I_m. \end{aligned} \tag{4.5}$$

Now $\frac{1}{2}(q+1)$ and $\frac{1}{2}(q-1)$ are coprime, so $\frac{1}{2}(q+1)$ divides $q-E_m$. That is, $q-E_m \equiv 0 \pmod{\frac{1}{2}(q+1)}$, and so $E_m \equiv q \pmod{\frac{1}{2}(q+1)}$. Since $0 \leq E_m \leq q+1$, this implies that $E_m = \frac{1}{2}(q-1)$ or q.

If $E_m = q$, then by Equation 4.5, $I_m = 0$ and so $B_m = 1$. If $E_m = \frac{1}{2}(q-1)$, then by Equation 4.5, $I_m = \frac{1}{2}(q-1)$ and so $B_m = 2$. Thus if m does not pass through V, it contains 1 or 2 black points. □

Denote the set of black points by $\mathscr{C}$. The above lemma shows that $\mathscr{C}$ is a Tallini set, and that V is a singular point of $\mathscr{C}$. This means that the set of black points is a cone. The following lemmas show that $\mathscr{C}$ is a quadric cone.

Lemma 4.2.10 *The set of black points in a secant plane forms a non-singular conic.*

Proof Let π be a secant plane and denote by $\mathscr{B}_\pi$ the set of black points in π. By Lemma 4.2.8, the plane π does not contain V, so by Lemma 4.2.9, a line of π contains at most two black points. Thus $\mathscr{B}_\pi$ is an arc. The lines of $\mathscr{L}$ in π are the external lines of $\mathscr{B}_\pi$ by Lemma 4.2.9. Since there are $\frac{1}{2}q(q-1)$ lines of $\mathscr{L}$ in π, Corollary 1.1.7 implies that $\mathscr{B}_\pi$ is an oval. By Theorem 1.5.24, every oval in $\mathrm{PG}(2,q)$, q odd, is a non-singular conic, so the set of black points in π forms a non-singular conic. □

Lemma 4.2.11 *The set of black points $\mathscr{C}$ is a quadric cone and $\mathscr{L}$ is the set of external lines of $\mathscr{C}$.*

Proof By Lemma 4.2.9, every line meets $\mathscr{C}$ in 0, 1, 2 or $q+1$ points, so $\mathscr{C}$ is a Tallini set. By Lemma 4.2.9, the point V lies only on tangents and generator lines of $\mathscr{C}$. That is, V is a singular point of $\mathscr{C}$.

Let π be a secant plane and denote by $\mathscr{O}$ the non-singular conic of black points in π. Since $V \notin \pi$, $\mathscr{C}$ is the quadric cone $V\mathscr{O}$ by Lemma 2.1.6.

By Lemma 4.2.9, a line not in $\mathscr{L}$ contains a black point, so the lines of $\mathscr{L}$ are the external lies of $\mathscr{C}$. □

This completes the proof of Theorem 4.2.1.

Chapter 5

Characterisation of subspaces associated with a parabolic quadric in $\mathrm{PG}(4, q)$

The previous chapters contained characterisations of the external lines of the three irreducible quadrals in $\mathrm{PG}(3, q)$. That is, characterisations of the external lines of an ovoid, a hyperbolic quadric, and an oval cone (the characterisations for the oval cone are new). The next natural step is to investigate the external lines of the quadrals in $\mathrm{PG}(4, q)$. A reasonable place to begin is with the parabolic quadric in $\mathrm{PG}(4, q)$. This chapter presents an original result characterising the external lines of this quadric for q even.

In $\mathrm{PG}(4, q)$, there are families of planes associated with a parabolic quadric which can also be characterised by their combinatorial properties. This chapter presents a new characterisation result for the set of secant planes of a parabolic quadric of $\mathrm{PG}(4, q)$. Using this result, an alternative to de Resmini's characterisation (Theorem 3.2.3) of the tangents and generator lines of this quadric is proved for q odd.

5.1 The external lines of a parabolic quadric in $\mathrm{PG}(4,q)$, q even

In this section, a new characterisation of the external lines of the non-singular quadric in $\mathrm{PG}(4,q)$, q even, is proved. In $\mathrm{PG}(3,q)$, it was possible to characterise the external lines of the irreducible quadrals using the number of the lines through each point and in each plane. For the external lines of the parabolic quadric in $\mathrm{PG}(4,q)$, it does not seem possible to complete the characterisation using only these properties. To complete the characterisation, it is necessary to also use the number of external lines in each 3-space. Hence, the following theorem is proved.

Theorem 5.1.1 *Let $\mathscr{L}$ be a set of lines in $\mathrm{PG}(4,q)$, q even, such that*

(I) Every point of $\mathrm{PG}(4,q)$ lies on 0 or $\frac{1}{2}q^3$ lines of $\mathscr{L}$,

(II) Every plane of $\mathrm{PG}(4,q)$ contains 0, q^2 or $\frac{1}{2}q(q-1)$ lines of $\mathscr{L}$,

(III) Every 3-space of $\mathrm{PG}(4,q)$ contains $\frac{1}{2}q^2(q^2+1)$, $\frac{1}{2}q^2(q-1)^2$ or $\frac{1}{2}q^3(q-1)$ lines of $\mathscr{L}$.

Then $\mathscr{L}$ is the set of external lines of a parabolic quadric of $\mathrm{PG}(4,q)$.

The above theorem only applies for q even. The case of q odd is much more difficult and is a topic for future research.

The Proof of Theorem 5.1.1

Let $\mathscr{L}$ be a set of lines of $\mathrm{PG}(4,q)$, q even, with the properties described in Theorem 5.1.1. By a series of lemmas, it is proved that $\mathscr{L}$ is the set of external lines of a parabolic quadric in $\mathrm{PG}(4,q)$. In order to make the argument clearer, some terminology is introduced.

- A point on 0 lines of $\mathscr{L}$ is called a *black* point. A point on $\frac{1}{2}q^3$ lines of $\mathscr{L}$ is called a *white* point.
- A plane containing 0 lines of $\mathscr{L}$ is called an *null plane*. A plane containing q^2 lines of $\mathscr{L}$ is called a *tangent plane*. A plane containing $\frac{1}{2}q(q-1)$ lines of $\mathscr{L}$ is called a *secant plane*.
- A 3-space containing $\frac{1}{2}q^2(q^2+1)$ lines of $\mathscr{L}$ is called an *elliptic 3-space*. A 3-space containing $\frac{1}{2}q^2(q-1)^2$ lines of $\mathscr{L}$ is called a *hyperbolic 3-space*. A 3-space containing $\frac{1}{2}q^3(q-1)$ lines of $\mathscr{L}$ is called a *tangent 3-space*.

It will be shown that the set of black points is a parabolic quadric $\mathscr{Q}$ of $\mathrm{PG}(4,q)$, plus its nucleus. The null planes are the planes containing a line of $\mathscr{Q}$; the tangent planes are the planes meeting $\mathscr{Q}$ in a single point. The secant planes are the planes meeting $\mathscr{Q}$ in a non-singular conic. Finally, an elliptic 3-space is a 3-space meeting $\mathscr{Q}$ in an elliptic quadric; a hyperbolic 3-space is a 3-space meeting $\mathscr{Q}$ in a hyperbolic quadric; and a tangent 3-space is a 3-space meeting $\mathscr{Q}$ in a quadric cone.

The first lemma concerns the planes through a line of $\mathscr{L}$. Note that there exist lines of $\mathscr{L}$ because every 3-space contains lines of $\mathscr{L}$.

Lemma 5.1.2 *Each line of $\mathscr{L}$ is contained in $q+1$ tangent planes and q^2 secant planes.*

Proof Let ℓ be a line of $\mathscr{L}$ contained in T_ℓ tangent planes. Since there are no lines of $\mathscr{L}$ in a null plane, ℓ is contained in no null planes, so the number of secant planes through ℓ is $(q^2+q+1-T_\ell)$. Let L_ℓ be the number of lines of $\mathscr{L}$ meeting ℓ in one point. The number L_ℓ will be counted in two ways.

Each point on ℓ is a white point and lies on $\frac{1}{2}q^3$ lines of $\mathscr{L}$ including ℓ. Counting this way, the line ℓ itself has been included $q+1$ times, so

$$\begin{aligned} L_\ell &= \tfrac{1}{2}q^3(q+1) - (q+1) \\ &= \tfrac{1}{2}q^4 + \tfrac{1}{2}q^3 - q - 1. \end{aligned}$$

On the other hand, every line of $\mathscr{L}$ meeting ℓ is contained in a plane through ℓ. Each tangent plane through ℓ contains q^2 lines of $\mathscr{L}$ including ℓ and each secant plane through ℓ contains $\frac{1}{2}q(q-1)$ lines of $\mathscr{L}$ including ℓ. The line ℓ itself has been included q^2+q+1 times, so

$$
\begin{aligned}
L_\ell &= q^2 T_\ell + \tfrac{1}{2}q(q-1)(q^2+q+1-T_\ell) - (q^2+q+1) \\
&= \tfrac{1}{2}q(q+1)T_\ell + \tfrac{1}{2}q^4 - \tfrac{1}{2}q - q^2 - q - 1.
\end{aligned}
$$

Equating these two expressions for L_ℓ gives

$$
\begin{aligned}
\tfrac{1}{2}q(q+1)T_\ell + \tfrac{1}{2}q^4 - \tfrac{1}{2}q - q^2 - q - 1 &= \tfrac{1}{2}q^4 + \tfrac{1}{2}q^3 - q - 1 \\
T_\ell &= q+1.
\end{aligned}
$$

So ℓ is contained in $q+1$ tangent planes. Hence, ℓ is contained in q^2 secant planes. □

The above lemma ensures the existence of both tangent planes and secant planes since lines of $\mathscr{L}$ exist. The next several lemmas determine the numbers of each type of 3-space through each plane, and the number of black points in each type of plane. In order to do this, the following notation is introduced. If π is a plane, then the number of black points in π is denoted by B_π, the number of hyperbolic 3-spaces through π is denoted by H_π, the number of elliptic 3-spaces through π is denoted by E_π and the number of tangent 3-spaces through π is denoted by C_π.

Lemma 5.1.3 *For any plane π, $C_\pi = 1$ or $q+1$.*

Proof Let π be a plane. Denote by L_π the number of lines of $\mathscr{L}$ in π and by $L_{1\pi}$ the number of lines of $\mathscr{L}$ meeting π in exactly one point. The number $L_{1\pi}$ is counted in two ways.

There are B_π black points in π, each lying on no lines of $\mathscr{L}$, and $q^2+q+1-B_\pi$ white points in π, each lying on $\frac{1}{2}q^3$ lines of $\mathscr{L}$ (including those in π). Counting this way, any line of $\mathscr{L}$ in π has been included $(q+1)$ times. So,

$$
\begin{aligned}
L_{1\pi} &= \tfrac{1}{2}q^3(q^2+q+1-B_\pi) - (q+1)L_\pi \\
&= \tfrac{1}{2}q^3(q^2+q+1) - \tfrac{1}{2}q^3 B_\pi - (q+1)L_\pi.
\end{aligned}
$$

On the other hand, every line meeting π in one point is contained in a 3-space through π, so consider the 3-spaces through π. In each elliptic 3-space there are

$\frac{1}{2}q^2(q^2+1)$ lines of $\mathscr{L}$, in each hyperbolic 3-space there are $\frac{1}{2}q^2(q-1)^2$ lines of $\mathscr{L}$, and in each tangent 3-space there are $\frac{1}{2}q^3(q-1)$ lines of $\mathscr{L}$. Counting this way, the lines of $\mathscr{L}$ in π have been included $(q+1)$ times. So

$$L_{1\pi} = \tfrac{1}{2}q^2(q^2+1)E_\pi + \tfrac{1}{2}q^2(q-1)^2H_\pi + \tfrac{1}{2}q^3(q-1)C_\pi - (q+1)L_\pi.$$

Equating these two expressions for $L_{1\pi}$ and simplifying gives

$$\begin{aligned} (q^2+1)E_\pi + (q-1)^2H_\pi + q(q-1)C_\pi &= q(q^2+q+1) - qB_\pi \\ (q^2+1)(E_\pi+H_\pi+C_\pi) - 2qH_\pi - (q+1)C_\pi &= q^3+q^2+q-qB_\pi \\ (q^2+1)(q+1) - 2qH_\pi - (q+1)C_\pi &= q^3+q^2+q-qB_\pi \\ qB_\pi + 1 &= 2qH_\pi + (q+1)C_\pi. \end{aligned} \qquad (5.1)$$

Reducing the above equation mod q gives $1 \equiv C_\pi \pmod{q}$. Since $C_\pi \leq q+1$, this implies that $C_\pi = 1$ or $q+1$. □

The above lemma ensures the existence of tangent 3-spaces, since every plane is contained in at least one tangent 3-space. The following lemma gives a useful relationship between the number of hyperbolic 3-spaces through a plane and the number of black points in the plane. This relationship makes it possible to determine the structure of the black points in a tangent plane.

Corollary 5.1.4 *Let π be a plane. If π is contained in $q+1$ tangent 3-spaces, then $B_\pi = q+2$. If π is contained in 1 tangent 3-space, then $B_\pi = 2H_\pi + 1$.*

Proof Equation 5.1 states that $qB_\pi + 1 = 2qH_\pi + (q+1)C_\pi$. If $C_\pi = q+1$, then $H_\pi = 0$, so $qB_\pi + 1 = q^2 + 2q + 1$. Hence $B_\pi = q+2$.

If $C_\pi = 1$ then $qB_\pi + 1 = 2qH_\pi + q + 1$, which implies that $B_\pi = 2H_\pi + 1$. □

Corollary 5.1.5 *Let π be a tangent plane. Then π is contained in 1 tangent 3-space and q elliptic 3-spaces. Also, there is exactly one black point in π, and the lines of $\mathscr{L}$ in π are the lines of π not through this point.*

Proof Firstly note that the lines of $\mathscr{L}$ in π cannot pass through any black points in π. Thus π contains at most one black point, since otherwise there would not be q^2 lines of $\mathscr{L}$ in π. If π is contained in $q+1$ tangent 3-spaces, then $B_\pi = q+2$ by Corollary 5.1.4. But $B_\pi \leq 1$, so π is contained in exactly one tangent 3-space. Corollary 5.1.4 implies that $B_\pi = 2H_\pi + 1 \geq 1$. Since π contains at most one black point, it follows that $B_\pi = 1$. Thus $H_\pi = 0$ and the remaining q 3-spaces through π are elliptic 3-spaces.

Let V be the unique black point in π. There are q^2 lines of $\mathscr{L}$ in π, none of which contains V. On the other hand, there are q^2 lines in π not passing through V. Thus the lines of $\mathscr{L}$ in π are the lines of π not through V. □

The above lemma ensures the existence of elliptic 3-spaces, since there exist tangent planes. In order to fully determine the structure of the black points in any plane, the number of black points on a line not in $\mathscr{L}$ is determined.

Lemma 5.1.6 *Every line contains 0, 1, 2 or $q+1$ black points. The lines of $\mathscr{L}$ are precisely those lines with no black points.*

Proof Let m be a line not in $\mathscr{L}$ and let W_m be the number of white points on $\mathscr{L}$. Denote by T_m the number of tangent planes through m and by S_m the number of secant planes through m. Finally, denote by L_m the number of lines of $\mathscr{L}$ meeting m. The number L_m is counted in two ways.

There are W_m white points on m, each lying on $\frac{1}{2}q^3$ lines of $\mathscr{L}$. The remaining points on m are black and lie on no lines of $\mathscr{L}$, so $L_m = \frac{1}{2}q^3 W_m$.

On the other hand, there are T_m tangent planes through m, each containing q^2 lines of $\mathscr{L}$, and there are S_m secant planes through m, each containing $\frac{1}{2}q(q-1)$ lines of $\mathscr{L}$. The remaining planes through m are null planes and contain no lines of $\mathscr{L}$. So $L_m = q^2 T_m + \frac{1}{2}q(q-1)S_m$.

Equating the above two expressions for L_m gives

$$\begin{aligned} \tfrac{1}{2}q^3 W_m &= q^2 T_m + \tfrac{1}{2}q(q-1)S_m \\ q^2 W_m &= 2qT_m + (q-1)S_m. \end{aligned} \tag{5.2}$$

Suppose m is contained in a tangent plane π. By Corollary 5.1.5, π contains exactly one black point and the lines of π not through this point are lines of $\mathscr{L}$. Since m is not a line of $\mathscr{L}$, it therefore contains exactly one black point.

Suppose m is contained in no tangent planes. That is, suppose $T_m = 0$. Then Equation 5.2 becomes $q^2 W_m = (q-1)S_m$. The numbers q^2 and $q-1$ are coprime, so this implies that q^2 divides S_m. The number of planes through m is $q^2+q+1 < 2q^2$, so $S_m < 2q^2$. Thus $S_m = 0$ or q^2. If $S_m = 0$, then $W_m = 0$ and m contains $q+1$ black points. If $S_m = q^2$, then $W_m = q-1$ and m contains 2 black points. Thus a line not in $\mathscr{L}$ contains 1, 2 or $q+1$ black points. □

The above lemma shows that the set of black points is a Tallini set and that $\mathscr{L}$ is its set of external lines. The next lemma determines the structure of black points in each type of plane, depending on the number of tangent 3-spaces through the plane.

Lemma 5.1.7 *Let π be a plane and let $\mathscr{B}_\pi$ be the set of black points in π.*

(i) Suppose π is a secant plane. If π is contained in $q+1$ tangent 3-spaces, then $\mathscr{B}_\pi$ is a hyperoval. If π is contained in 1 tangent 3-space, then $\mathscr{B}_\pi$ is an oval. In this case, π is contained in $\frac{1}{2}q$ hyperbolic 3-spaces and $\frac{1}{2}q$ elliptic 3-spaces.

(ii) Suppose π is a null plane. If π is contained in $q+1$ tangent 3-spaces, then $\mathscr{B}_\pi$ is a line plus one point. If π is contained in 1 tangent 3-space, then $\mathscr{B}_\pi$ is either a line or a pair of lines. If $\mathscr{B}_\pi$ is a line, then π is contained in $\frac{1}{2}q$ hyperbolic 3-spaces and $\frac{1}{2}q$ elliptic 3-spaces. If $\mathscr{B}_\pi$ is a pair of lines, then π is contained in q hyperbolic 3-spaces and no elliptic 3-spaces.

Proof By Corollary 5.1.4, if $C_\pi = q+1$ then $B_\pi = q+2$; and if $C_\pi = 1$, then $B_\pi = 2H_\pi + 1$.

(i) Suppose π is a secant plane and let ℓ be a line of $\mathscr{L}$ in π. Then every line of π meets ℓ and so contains a white point. Lemma 5.1.6 implies that every line of π contains 0, 1 or 2 black points. That is, the set $\mathscr{B}_\pi$ of black points in π is an *arc*. The lines of $\mathscr{L}$ in π are the external lines of $\mathscr{B}_\pi$ by Lemma 5.1.6, so $\mathscr{B}_\pi$ has $\frac{1}{2}q(q-1)$ external lines. Corollary 1.1.7 implies that $\mathscr{B}_\pi$ is an oval or a hyperoval.

If π is contained in $q+1$ tangent 3-spaces, then $B_\pi = q+2$, so $\mathscr{B}_\pi$ is a hyperoval. If π is contained in exactly one tangent 3-space, then $B_\pi = 2H_\pi + 1$, which is an odd number. So $\mathscr{B}_\pi$ is an oval. Since $2H_\pi + 1 = B_\pi = q + 1$, it follows that $H_\pi = \frac{1}{2}q$. That is, there are $\frac{1}{2}q$ hyperbolic 3-spaces through π. Hence there are $\frac{1}{2}q$ elliptic 3-spaces through π.

(ii) Suppose π is a null plane. If there are no lines in π with $q+1$ black points, then $\mathscr{B}_\pi$ is an arc and has an external line. However, line with no black points is a line of $\mathscr{L}$ by Lemma 5.1.6 and a null plane contains no lines of $\mathscr{L}$. Thus there exists a line in π with $q+1$ black points. Let g be such a line.

Suppose π is contained in $q+1$ tangent 3-spaces. Then $B_\pi = q+2$, so $\mathscr{B}_\pi$ is the line g, plus one further point.

Suppose π is contained in exactly one tangent 3-space, then $B_\pi = 2H_\pi + 1$. The set $\mathscr{B}_\pi$ may consist of just the line g, in which case $B_\pi = q+1$ and $H_\pi = \frac{1}{2}q$. Thus, π is contained in $\frac{1}{2}q$ hyperbolic 3-spaces, and $\frac{1}{2}q$ elliptic 3-spaces.

Suppose $\mathscr{B}_\pi$ contains a point P not on g. Then it contains another point Q as B_π is an odd number. The line PQ meets m in a further black point and so contains at least three black points. Since any line contains 0, 1, 2 or $q+1$ black points, this implies that $\mathscr{B}_\pi$ contains two lines and $B_\pi \geq 2q+1$. However, note that $H_\pi \leq q$ as $C_\pi = 1$, so $B_\pi \leq 2q+1$. Thus $B_\pi = 2q+1$ and $\mathscr{B}_\pi$ is a pair of lines. It follows that $H_\pi = q$. That is, π is contained in q hyperbolic 3-spaces, and no elliptic 3-spaces. $\square$

With the above information, the structure of the black points in each 3-space can now be determined. In order to do this, the following notation is introduced. If Σ is a 3-space, then the set of black points in Σ is denoted by $\mathscr{B}_\Sigma$ and the number of black points in Σ is denoted by B_Σ.

Lemma 5.1.8 *Let Σ be an elliptic 3-space. Then $\mathscr{B}_\Sigma$ is an ovoid of Σ.*

Proof By Corollary 5.1.5 and Lemma 5.1.7, the planes in the elliptic 3-space Σ are tangent planes containing one black point, secant planes containing $q+1$ black points forming an oval, or null planes containing $q+1$ black points forming a line. Thus every plane of Σ meets $\mathscr{B}_\Sigma$ in 1 or $q+1$ points. By Theorem 2.3.3, $\mathscr{B}_\Sigma$ is either a line or an ovoid.

The set of lines of $\mathscr{L}$ in Σ is the set of external lines of $\mathscr{B}_\Sigma$. If $\mathscr{B}_\Sigma$ is a line, there are q^4 lines of Σ not meeting $\mathscr{B}_\Sigma$ by Lemma 1.5.34. Since lines containing no black points are lines of $\mathscr{L}$, this implies that there are q^4 lines of $\mathscr{L}$ in Σ. However, the number of lines of $\mathscr{L}$ in Σ is $\frac{1}{2}q^2(q^2+1)$, which is not equal to q^4, so $\mathscr{B}_\Sigma$ is not a line. Thus $\mathscr{B}_\Sigma$ is an ovoid. □

If π is a null plane whose black points form a line, then Lemma 5.1.7 implies that π is contained in an elliptic 3-space. However, by Lemma 5.1.8, the set of black points in an elliptic 3-space is an ovoid and contains no lines. Thus null planes whose black points form a single line do not exist.

Lemma 5.1.9 *Let Γ be a tangent 3-space. Then $\mathscr{B}_\Gamma$ is an oval cone plus one point on its nuclear line.*

Proof Let Σ be any 3-space. Denote by L_Σ the number of lines of $\mathscr{L}$ in Σ and by L be the total number of lines of $\mathscr{L}$. Since every line of $\mathrm{PG}(4,q)$ meets Σ, the number L can be counted by considering the lines of $\mathscr{L}$ through each point of Σ.

There are B_Σ black points in Σ, each lying on no lines of $\mathscr{L}$, and there are $q^3+q^2+q+1-B_\Sigma$ white points in Σ, each lying on $\frac{1}{2}q^3$ lines of $\mathscr{L}$. Counting this way, the lines of $\mathscr{L}$ in Σ have been included $q+1$ times. So

$$L = \tfrac{1}{2}q^3(q^3+q^2+q+1-B_\Sigma) - qL_\Sigma. \qquad (5.3)$$

If Σ is an elliptic 3-space, then $B_\Sigma = q^2+1$ and $L_\Sigma = \frac{1}{2}q^2(q^2+1)$. Substituting these values into the above equation gives

$$\begin{aligned} L &= \tfrac{1}{2}q^3(q^3+q^2+q+1-q^2-1) - q\cdot\tfrac{1}{2}q^2(q^2+1) \\ &= \tfrac{1}{2}q^3(q^3-q^2+q-1). \end{aligned}$$

If Σ is a tangent 3-space, then $L_\Sigma = \frac{1}{2}q^3(q-1)$. Substituting this and the value of L into Equation 5.3 gives

$$\begin{aligned}\tfrac{1}{2}q^3(q^3-q^2+q-1) &= \tfrac{1}{2}q^3(q^3+q^2+q+1-B_\Sigma) - q\cdot\tfrac{1}{2}q^3(q-1)\\ B_\Sigma &= q^2+q+2.\end{aligned}$$

That is, a tangent 3-space contains q^2+q+2 black points. In particular, $B_\Gamma = q^2+q+2$.

Let ℓ be a line of $\mathscr{L}$ in the tangent 3-space Γ. By Corollary 5.1.5 and Lemma 5.1.7 the planes through ℓ in the tangent 3-space Γ are tangent planes with one black point, secant planes with an oval of black points, or secant planes with a hyperoval of black points. Denote by S_1 the number of secant planes in Γ through ℓ with $q+1$ black points, and by S_2 be the number of secant planes in Γ through ℓ with $q+2$ black points. The remaining $q+1-S_1-S_2$ planes through ℓ in Γ are tangent planes, each containing one black point. Since every black point of Γ lies in exactly one plane through ℓ, it follows that

$$\begin{aligned}(q+1)S_1+(q+2)S_2+q+1-S_1-S_2 &= q^2+q+2\\ qS_1+(q+1)S_2 &= q^2+1 \qquad (5.4)\\ S_2 &\equiv 1 \pmod{q}.\end{aligned}$$

Since $S_2 \leq q+1$, this implies that $S_2 = 1$ or $q+1$. There are $q+1$ planes through a line in $\mathrm{PG}(3,q)$, so if $S_2 = q+1$, then $S_1 = 0$ and Equation 5.4 becomes $(q+1)(q+1) = q^2+1$. This is a contradiction, so $S_2 = 1$, and Equation 5.4 implies that $S_1 = q-1$. Thus the number of tangent planes in Γ through ℓ is 1.

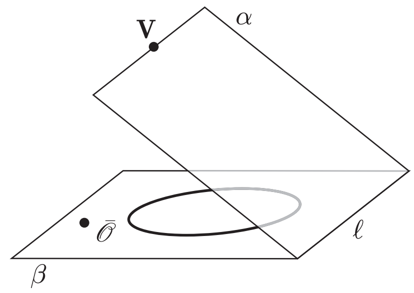
Let P be a point on ℓ. The lines of $\mathscr{L}$ through P in Γ are contained in the planes through ℓ. In the tangent plane through ℓ, P lies on q lines of $\mathscr{L}$. In each secant plane through ℓ (whether the plane contains an oval or a hyperoval of black points), P lies on $\frac{1}{2}q$ lines of $\mathscr{L}$. Counting this way, the line ℓ itself has been included $q+1$ times, so the number of lines of $\mathscr{L}$ through P in Γ is $q+q\cdot\frac{1}{2}q-q = \frac{1}{2}q^2$. Thus, every point in Γ on a line of $\mathscr{L}$ in Γ lies on $\frac{1}{2}q$ lines of $\mathscr{L}$ in Γ.

Denote by $\mathscr{L}_\Gamma$ the set of lines of $\mathscr{L}$ in Γ. Then $\mathscr{L}_\Gamma$ has these properties:

(I) Every point of Γ lies on 0 or $\frac{1}{2}q^2$ lines of $\mathscr{L}_\Gamma$,

(II) Every plane of Γ contains 0, q^2 or $\frac{1}{2}q(q-1)$ lines of $\mathscr{L}_\Gamma$.

By Theorem 4.1.1, $\mathscr{L}_\Gamma$ is the set of external lines of a hyperoval cone $\bar{\mathscr{C}}$ of Γ. That is, the set of points of Γ on no lines of $\mathscr{L}_\Gamma$ is $\bar{\mathscr{C}}$.



Denote by α the unique tangent plane in Γ through the line ℓ and by β the secant plane in Γ through ℓ containing a hyperoval of black points. Let V be the unique black point in α and let $\bar{\mathscr{O}}$ be the hyperoval of black points in β. Then $\bar{\mathscr{C}}$ is the cone $V\bar{\mathscr{O}}$.

Consider a black point P in Γ not in α or β. The line PV is a line of the cone $\bar{\mathscr{C}}$ and so meets $\bar{\mathscr{O}}$ in a point. This is a black point, so the line PV contains three black points, and hence contains $q+1$ black points by Lemma 5.1.6. Thus every black point in Γ not in α or β lies on a line of black points through V. Since there are q^2+q+2 black points in Γ, it follows that the set of black points in Γ consists of $q+1$ lines of the hyperoval cone $\bar{\mathscr{C}}$, plus one further point of $\bar{\mathscr{C}}$. That is $\mathscr{B}_\Gamma$ is an oval cone, plus one point on its nuclear line. $\square$

In light of the above lemma, if Γ is a tangent 3-space whose black points form an oval cone $\mathscr{C}$ plus the point N, then the point N is called the *special point* of Γ. The following lemma shows that the tangent 3-spaces all contain the same special point.

Lemma 5.1.10 *There exists a point N such that N is the special point in every tangent 3-space. The set of tangent 3-spaces is the set of 3-spaces through N.*

Proof Let Γ be a tangent 3-space whose black points form the oval cone $\mathscr{C}$, plus the special point N. The planes in Γ through N meet $\mathscr{C}$ in a single line or an oval, since N is on the nuclear line of $\mathscr{C}$. Thus each plane in Γ through N contains $q+2$ black points. By Lemma 5.1.7, each plane in Γ through N is contained in $q+1$ tangent 3-spaces. Since any 3-space through N meets Γ in a plane through N, this implies that every 3-space through N is a tangent 3-space.

Let Σ be an elliptic 3-space. Then no plane in Σ is contained $q+1$ tangent 3-spaces, so by Lemma 5.1.3, every plane in Σ is contained in exactly one tangent 3-space. Thus the number of tangent 3-spaces is the number of planes in Σ, which is

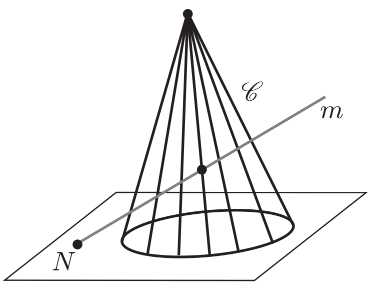
$q^3 + q^2 + q + 1$. But this is also the number of 3-spaces through a point in $\mathrm{PG}(4, q)$, so the set of tangent 3-spaces is the set of 3-spaces through N.

Finally, let Γ' be a tangent 3-space with special point N'. The same argument as above shows that the tangent 3-spaces are the 3-spaces through N'. Thus $N' = N$ and every tangent 3-space has N as its special point. □

It is now possible to complete the proof of Theorem 5.1.1.

Lemma 5.1.11 *Let $\mathscr{Q}$ be the set of black points, excluding the point N. Then $\mathscr{Q}$ is a parabolic quadric of $\mathrm{PG}(4, q)$ and $\mathscr{L}$ is the set of external lines of $\mathscr{Q}$.*

Proof First it is shown that $\mathscr{L}$ is the set of lines containing no point of $\mathscr{Q}$.



Let m be a line through N. Since every 3-space through N is a tangent 3-space, it follows that any 3-space through m is a tangent 3-space. Let Γ be a tangent 3-space through m. The set of black points in Γ consists of an oval cone $\mathscr{C}$, plus the point N, which is on the nuclear line of $\mathscr{C}$. Since N is on the nuclear line of $\mathscr{C}$, every line through N meets $\mathscr{C}$ in exactly one point. In particular, m contains one black point other than N. Hence, every line through N meets $\mathscr{Q}$ in one point. This implies that the set of lines containing no black points is the same as the set of lines containing no point of $\mathscr{Q}$. That is, $\mathscr{L}$ is the set of external lines of $\mathscr{Q}$.

Now it is shown that $\mathscr{Q}$ is a non-singular quadric. Let π be a plane and let Γ be a tangent 3-space through π. The black points in Γ form an oval cone $\mathscr{C}$, plus the point N, which lies on the nuclear line of $\mathscr{C}$. Thus Γ meets $\mathscr{Q}$ in the oval cone $\mathscr{C}$. The plane π meets $\mathscr{C}$ in a point, a line, a pair of lines, or an oval. Thus every plane meets $\mathscr{Q}$ in a point, a line, a pair of lines or an oval. By Theorem 2.3.1, $\mathscr{Q}$ is a quadral.

Let P be a point of $\mathscr{Q}$ and consider the line PN. By Lemma 5.1.10, P and N are the only black points on PN. Using Equation 5.2, the number of secant planes through PN is q^2. Thus P is contained in a secant plane.

Hence, $\mathscr{Q}$ is a quadral such that every point of $\mathscr{Q}$ lies in a plane meeting $\mathscr{Q}$ in an oval. Corollary 2.3.2 implies that $\mathscr{Q}$ is a parabolic quadric of $\mathrm{PG}(4, q)$. □

5.2 The secant planes of a parabolic quadric of PG$(4,q)$

In the previous section, a set of lines associated with a parabolic quadric in PG$(4,q)$ was characterised. In PG$(4,q)$, it is also reasonable to consider the sets of *planes* associated with this quadric. The secant planes of a parabolic quadic (the planes that meet the quadric in a non-singular conic) have particularly amenable properties for characterisation (see Lemma 1.5.41), which are the same for all q. Using these properties, the following new characterisation result is proved.

Theorem 5.2.1 *Let $\mathscr{S}$ be a set of planes in* PG$(4,q)$ *such that*

(I) Every point of PG$(4,q)$ *lies on q^4 or q^4+q^2 planes of $\mathscr{S}$,*

(II) Every line of PG$(4,q)$ *lies on* 0 *or q^2 planes of $\mathscr{S}$.*

Then either $\mathscr{S}$ is the set of secant planes of a parabolic quadric of PG$(4,q)$, *or there exists a 3-space Σ and a $(q+1)$-cover $\mathscr{M}$ in Σ such that $\mathscr{S}$ is the set of planes not containing any line of $\mathscr{M}$.*

In the proof of the above theorem, Lemma 5.2.14 provides a third condition to separate the two possible cases for $\mathscr{S}$. Thus the proof of Theorem 5.2.1 also proves the following theorem.

Theorem 5.2.2 *Let $\mathscr{S}$ be a set of planes in* PG$(4,q)$ *such that*

(I) Every point of PG$(4,q)$ *lies on q^4 or q^4+q^2 planes of $\mathscr{S}$,*

(II) Every line of PG$(4,q)$ *lies on* 0 *or q^2 planes of $\mathscr{S}$,*

(III) Every 3-space of PG$(4,q)$ *contains at least one plane of $\mathscr{S}$.*

Then $\mathscr{S}$ is the set of secant planes of a parabolic quadric of PG$(4,q)$.

The proof of these two theorems follows.

The Proof of Theorems 5.2.1 and 5.2.2

Let $\mathscr{S}$ be a set of planes of $\mathrm{PG}(4,q)$ with the properties described in Theorem 5.2.1. By a series of lemmas, it will be shown that if every 3-space contains a plane of $\mathscr{S}$, then $\mathscr{S}$ is the set of secant planes of a parabolic quadric of $\mathrm{PG}(4,q)$. Otherwise, there exists a 3-space Σ and a $(q+1)$-cover $\mathscr{M}$ in Σ such that $\mathscr{S}$ is the set of planes not containing any line of $\mathscr{M}$. This will prove both Theorem 5.2.1 and Theorem 5.2.2.

In order to make the proof clearer, some terminology is introduced.

- A point on q^4 planes of $\mathscr{S}$ is called a *black point*. A point on q^4+q^2 planes of $\mathscr{S}$ is called a *white point*. The total number of black points is denoted by B.
- A line on no planes of $\mathscr{S}$ is called a *dark line*. A line on q^2 planes of $\mathscr{S}$ is called a *light line*. The total number of dark lines is denoted by D.
- Finally, the number of planes of $\mathscr{S}$ is denoted by S.

The first lemma can now be stated.

Lemma 5.2.3 *The number of dark lines through a black point is $q+1$. There are no dark lines through a white point.*

Proof Let P be a point lying on D_P dark lines and S_P planes of $\mathscr{S}$. Consider the set of pairs $X = \{(\ell,\pi) \mid \ \ell \text{ is a line through } P,\ \pi \text{ is a plane of } \mathscr{S} \text{ through } \ell\ \}$. The size of X is counted in two ways.
Count ℓ, then π: There are D_P dark lines through P, each lying on no planes of $\mathscr{S}$ and $(q^3+q^2+q+1-D_P)$ light lines through P, each lying on q^2 planes of $\mathscr{S}$. So $|X| = 0\cdot D_P + q^2(q^3+q^2+q+1-D_P) = q^5+q^4+q^3+q^2-q^2D_P$.
Count π, then ℓ: There are S_P planes of $\mathscr{S}$ through P and $q+1$ lines through P in each. So $|X| = (q+1)S_P$.
Hence, $(q+1)S_P = q^5+q^4+q^3+q^2-q^2D_P$.

If P is a black point then $S_P = q^4$, so

$$(q+1) \cdot q^4 = q^5 + q^4 + q^3 + q^2 - q^2 D_P$$
$$D_P = q + 1.$$

So there are $(q+1)$ dark lines through a black point.

If P is a white point then $S_P = q^4 + q^2$, so

$$(q+1)(q^4 + q^2) = q^5 + q^4 + q^3 + q^2 - q^2 D_P$$
$$D_P = 0.$$

So there are no dark lines through a white point. □

This lemma has two immediate corollaries.

Corollary 5.2.4 *A dark line contains only black points. A line containing a white point is a light line.*

Proof A white point lies on no dark lines, so a dark line contains no white points. That is, a dark line contains only black points. Equivalently, a line containing a white point is a light line. □

Note that a line containing only black points is not necessarily a dark line.

Corollary 5.2.5 *The number of dark lines is the same as the number of black points. That is, $D = B$.*

Proof Consider the set of pairs $X = \{(P, \ell) \mid \; P \text{ is a black point, } \ell \text{ is a dark line through } P\}$. By Lemma 5.2.3, each black point lies on $q+1$ dark lines, so $|X| = B(q+1)$. By Corollary 5.2.4, each dark line contains $q+1$ black points, so $|X| = D(q+1)$. Hence $B(q+1) = D(q+1)$ and so $B = D$. □

The above results make it possible to find the number of black points in a plane of $\mathscr{S}$.

Lemma 5.2.6 *A plane of $\mathscr{S}$ contains $q+1$ black points and q^2 white points.*

Proof Let π be a plane of $\mathscr{S}$ containing B_π black points. Note that there are no dark lines contained in π as dark lines are contained in no planes of $\mathscr{S}$. Every plane of $\mathscr{S}$ (other than π) meets π in a line or in a single point. So the number S of planes of $\mathscr{S}$ may be counted by considering the planes of $\mathscr{S}$ through each point of π. To do this, it is necessary to first find how many planes of $\mathscr{S}$ meet π in a line and how many meet π in a point.

Consider the lines of π. There are q^2+q+1 lines in π and they are all light lines lying on q^2 planes of $\mathscr{S}$ each, including π itself. Hence, the number of planes of $\mathscr{S}$ meeting π in a line is $(q^2+q+1)(q^2-1) = q^4+q^3-q-1$. Thus, the number of planes of $\mathscr{S}$ meeting π in a point is $S-1-(q^4+q^3-q-1) = S-q^4-q^3+q$.

Let $X_1 = \{(P,\sigma) \mid\ P \text{ is a point of } \pi,\ \sigma \text{ is a plane of } \mathscr{S} \text{ through } P\ \}$. The size of X_1 is counted in two ways.

Count P, then σ: There are B_π black points in π, each lying on q^4 planes of $\mathscr{S}$. There are $q^2+q+1-B_\pi$ white points in π, each lying on q^4+q^2 planes of $\mathscr{S}$. So

$$\begin{aligned}|X_1| &= B_\pi q^4 + (q^2+q+1-B_\pi)(q^4+q^2)\\ &= q^6+q^5+2q^4+q^3+q^2-q^2B_\pi.\end{aligned}$$

Count σ, then P: There is one plane of $\mathscr{S}$ (π itself) meeting π in q^2+q+1 points. There are (q^4+q^3-q-1) planes of $\mathscr{S}$ meeting π in $q+1$ points. Finally, there are $S-q^4-q^3+q$ planes of $\mathscr{S}$ meeting π in one point. So

$$\begin{aligned}|X_1| &= q^2+q+1+(q^4+q^3-q-1)(q+1)+S-q^4-q^3+q\\ &= q^5+q^4+S.\end{aligned}$$

Equating the above two expressions for $|X_1|$ gives:

$$q^6+q^5+2q^4+q^3+q^2-q^2B_\pi = q^5+q^4+S$$

$$S = q^6+q^4+q^3+q^2-q^2B_\pi \tag{5.5}$$

Since S is a constant, it follows that B_π is the same for all choices of π in $\mathscr{S}$. Denote by B_S the number of black points in a plane of $\mathscr{S}$. Several sets of pairs will be counted in order to form an equation that can be solved for B_S.

Let $X_2 = \{(P, \pi) \mid \; P \text{ is a black point}, \pi \text{ is a plane of } \mathscr{S} \text{ through } P\;\}$.
Count P, then π: There are B black points and q^4 planes of $\mathscr{S}$ through each, so $|X_2| = Bq^4$.
Count π, then P. There are S planes of $\mathscr{S}$ and B_S black points in each, so $|X_2| = SB_S$.

$$\text{So,} \qquad q^4 B = SB_S. \tag{5.6}$$

Substituting Equation 5.5 into Equation 5.6 gives

$$\begin{aligned} q^4 B &= (q^6 + q^4 + q^3 + q^2 - q^2 B_S)B_S \\ q^2 B &= (q^4 + q^2 + q + 1)B_S - B_S^2. \end{aligned} \tag{5.7}$$

Let $X_3 = \{(\ell, \pi) \mid \; \ell \text{ is a light line}, \pi \text{ is a plane of } \mathscr{S} \text{ through } \ell\;\}$.
Count ℓ, then π: There are B dark lines by Corollary 5.2.5 and so there are $q^6 + q^5 + 2q^4 + 2q^3 + 2q^2 + q + 1 - B$ light lines. Each of these lines has q^2 planes of $\mathscr{S}$ through it, so

$$\begin{aligned} |X_3| &= q^2(q^6 + q^5 + 2q^4 + 2q^3 + 2q^2 + q + 1 - B) \\ &= q^8 + q^7 + 2q^6 + 2q^5 + 2q^4 + q^3 + q^2 - q^2 B. \end{aligned}$$

Count π, then ℓ: There are $S = q^6 + q^4 + q^3 + q^2 - q^2 B_S$ planes of $\mathscr{S}$ (from Equation 5.5) and each contains $q^2 + q + 1$ light lines, so

$$\begin{aligned} |X_3| &= (q^6 + q^4 + q^3 + q^2 - q^2 B_S)(q^2 + q + 1) \\ &= q^8 + q^7 + 2q^6 + 2q^5 + 3q^4 + 2q^3 + q^2 - (q^4 + q^3 + q^2)B_S. \end{aligned}$$

Equating the above two expressions for $|X_3|$ and simplifying gives:

$$q^2 B = (q^4 + q^3 + q^2)B_S - q^4 - q^3 \tag{5.8}$$

Substituting Equation 5.8 into Equation 5.7 gives

$$\begin{aligned} (q^4 + q^3 + q^2)B_S - q^4 - q^3 &= (q^4 + q^2 + q + 1)B_S - B_S^2 \\ B_S^2 + (q^3 - (q+1))B_S - q^3(q+1) &= 0 \\ (B_S + q^3)(B_S - (q+1)) &= 0 \end{aligned}$$

The number B_S is non-negative, since it is a number of points, so the above equation implies that $B_S = q + 1$. □

Using the above lemma and its proof, the number of each type of point, line and plane can now be determined.

Corollary 5.2.7 *There are q^3+q^2+q+1 black points, q^4 white points, q^3+q^2+q+1 dark lines, $q^6+q^5+2q^4+q^3+q^2$ light lines and q^6+q^4 planes of $\mathscr{S}$.*

Proof From Equation 5.5, and using the fact that $B_S = q + 1$ (Lemma 5.2.6), $S = q^6 + q^4 + q^3 + q^2 - q^2(q + 1) = q^6 + q^4$.

From Equation 5.6 and using the above value of S, $q^4 B = (q^6 + q^4)(q + 1)$, so $B = (q^2 + 1)(q + 1) = q^3 + q^2 + q + 1$. That is, there are $q^3 + q^2 + q + 1$ black points. All other points of $\mathrm{PG}(4, q)$ are white points, so there are q^4 white points. By Corollary 5.2.5, the number of dark lines is $q^3 + q^2 + q + 1$. All other lines of $\mathrm{PG}(4, q)$ are light lines, so there are $q^6 + q^5 + 2q^4 + q^3 + q^2$ light lines. □

The next lemma gives a useful relationship between the number of black points and the number of dark lines in a plane not in $\mathscr{S}$. In order to state the lemma, the following notation is introduced. If π is a plane, then the number of black points in π is denoted by B_π and the number of dark lines in π is denoted by D_π.

Lemma 5.2.8 *Suppose π is a plane not in $\mathscr{S}$. Then $B_\pi = qD_\pi + 1$.*

Proof Every plane of $\mathscr{S}$ meets π in a line or a single point. So the number of planes in $\mathscr{S}$ may be counted by considering the planes of $\mathscr{S}$ through each point of π. Each of the D_π dark lines in π lies on no planes of $\mathscr{S}$ and each of the $(q^2+q+1-D_\pi)$ light lines in π lies on q^2 planes of $\mathscr{S}$. Thus the number of planes of $\mathscr{S}$ meeting π in a line is $q^2(q^2 + q + 1 - D_\pi) = q^4 + q^3 + q^2 - q^2 D_\pi$. The remaining planes of $\mathscr{S}$ meet π in a single point. So the number of planes of $\mathscr{S}$ meeting π in a point is $q^6 + q^4 - (q^4 + q^3 + q^2 - q^2 D_\pi) = q^6 - q^3 - q^2 + q^2 D_\pi$.

Let $X = \{(P, \sigma) \mid P$ is a point of π, σ is a plane of $\mathscr{S}$ through $P\}$. The size of X is counted in two ways.

Count P, then σ: There are B_π black points in π, each lying on q^4 planes of $\mathscr{S}$. There are $(q^2+q+1-B_\pi)$ white points in π, each lying on q^4+q^2 planes of $\mathscr{S}$. So

$$\begin{aligned}|X| &= q^4 B_\pi + (q^2+q+1-B_\pi)(q^4+q^2)\\ &= q^6+q^5+2q^4+q^3+q^2-q^2B_\pi.\end{aligned}$$

Count σ, then P: There are $q^4+q^3+q^2-q^2D_\pi$ planes of $\mathscr{S}$ containing $q+1$ points of π. There are $q^6-q^3-q^2+q^2D_\pi$ planes of $\mathscr{S}$ containing one point of π. So

$$\begin{aligned}|X| &= (q^4+q^3+q^2-q^2D_\pi)(q+1)+q^6-q^3-q^2+q^2D_\pi\\ &= q^6+q^5+2q^4+q^3-q^3D_\pi.\end{aligned}$$

Equating the above two expressions for $|X|$ gives

$$\begin{aligned}q^6+q^5+2q^4+q^3+q^2-q^2B_\pi &= q^6+q^5+2q^4+q^3-q^3D_\pi\\ B_\pi &= qD_\pi+1.\end{aligned}$$

□

There are two useful corollaries of the above result:

Corollary 5.2.9 *Suppose π is a plane not in $\mathscr{S}$ containing no dark lines. Then π contains exactly one black point.*

Proof By Lemma 5.2.8, $B_\pi = qD_\pi + 1 = q\cdot 0+1 = 1$. □

Corollary 5.2.10 *Suppose ℓ is a line containing no black points. Then through ℓ there are q^2 planes of $\mathscr{S}$, each containing exactly $q+1$ black points, and $q+1$ planes not in $\mathscr{S}$, each containing exactly one black point.*

Proof Since ℓ contains no black points, it is a light line by Corollary 5.2.4. That is, ℓ lies on q^2 planes of $\mathscr{S}$ and $q+1$ planes not in $\mathscr{S}$. Each plane of $\mathscr{S}$ contains exactly $q+1$ black points by Lemma 5.2.6. It remains to show that a plane through ℓ that is not in $\mathscr{S}$ contains exactly one black point.

Let π be a plane through ℓ that is not in $\mathscr{S}$. Now ℓ contains only white points, and white points lie on no dark lines by Lemma 5.2.3. Thus there are no dark lines meeting ℓ. Since any line in π meets ℓ, this implies that there are no dark lines in π. Hence, π contains exactly one black point by Corollary 5.2.9. □

Note that there is no guarantee at this stage that there exist planes with no dark lines, or lines with no black points. The next lemma gives a relationship between the number of black points, dark lines and planes of $\mathscr{S}$ in a 3-space. In order to do this, the following notation is introduced. If Σ is a 3-space, then the number of planes of $\mathscr{S}$ in Σ is denoted by S_Σ, the number of black points in Σ is denoted by B_Σ and the number of dark lines in Σ is denoted by D_Σ.

Lemma 5.2.11 *Let Σ be a 3-space. Then there exists a non-negative integer k such that $D_\Sigma = k(q+1)$, $B_\Sigma = q^2+1+qk$ and $S_\Sigma = q^3+q-qk$.*

Proof By counting sets of pairs, equations will be formed relating the three numbers S_Σ, B_Σ and D_Σ.

Let $X_1 = \{(P,\pi) \mid P \text{ is a point of } \Sigma, \pi \text{ is a plane of } \mathscr{S} \text{ through } P\}$.
Count P, then π: There are B_Σ black points in Σ, each lying on q^4 planes of $\mathscr{S}$. There are $q^3+q^2+q+1-B_\Sigma$ white points in Σ, each lying on q^4+q^2 planes of $\mathscr{S}$. So

$$\begin{aligned}|X_1| &= q^4 B_\Sigma + (q^3+q^2+q+1-B_\Sigma)(q^4+q^2)\\ &= q^7+q^6+2q^5+2q^4+q^3+q^2-q^2B_\Sigma.\end{aligned}$$

Count π, then P: There are S_Σ planes of $\mathscr{S}$ in Σ, each containing q^2+q+1 points of Σ. There are $q^6+q^4-S_\Sigma$ planes of $\mathscr{S}$ not in Σ, which each meet Σ in a line and so contain $q+1$ points of Σ. So

$$\begin{aligned}|X_1| &= (q^2+q+1)S_\Sigma + (q^6+q^4-S_\Sigma)(q+1)\\ &= q^2S_\Sigma + q^7+q^6+q^5+q^4.\end{aligned}$$

Equating the above two expressions for $|X_1|$ gives:

$$\begin{aligned}q^7+q^6+2q^5+2q^4+q^3+q^2-q^2B_\Sigma &= q^2S_\Sigma+q^7+q^6+q^5+q^4\\ B_\Sigma &= q^3+q^2+q+1-S_\Sigma \qquad (5.9)\end{aligned}$$

Let $X_2 = \{(\ell, \pi) \mid \ \ell \text{ is a line in } \Sigma, \pi \text{ is a plane of } \mathscr{S} \text{ through } \ell\ \}$.
Count ℓ, then π: There are D_Σ dark lines in Σ, each lying on no planes of $\mathscr{S}$. There are $q^4 + q^3 + 2q^2 + q + 1 - D_\Sigma$ light lines in Σ, each on q^2 planes of $\mathscr{S}$. So,

$$\begin{aligned} |X_2| &= q^2(q^4 + q^3 + 2q^2 + q + 1 - D_\Sigma) \\ &= q^6 + q^5 + 2q^4 + q^3 + q^2 - q^2 D_\Sigma \end{aligned}$$

Count π, then ℓ: There are S_Σ planes of $\mathscr{S}$ contained in Σ, each containing q^2+q+1 lines of Σ. There are $(q^6 + q^4 - S_\Sigma)$ planes of $\mathscr{S}$ not in Σ, each meeting Σ in one line. So,

$$\begin{aligned} |X_2| &= (q^2 + q + 1)S_\Sigma + q^6 + q^4 - S_\Sigma \\ &= (q^2 + q)S_\Sigma + q^6 + q^4 \end{aligned}$$

Equating the above two expressions for $|X_2|$ gives:

$$\begin{aligned} q^6 + q^5 + 2q^4 + q^3 + q^2 - q^2 D_\Sigma &= (q^2 + q)S_\Sigma + q^6 + q^4 \\ qD_\Sigma &= (q + 1)(q^3 + q - S_\Sigma) \end{aligned} \tag{5.10}$$

From this equation it follows that $(q + 1)$ divides qD_Σ. Since q and $(q + 1)$ are coprime, this implies that $(q + 1)$ divides D_Σ.

Let $D_\Sigma = k(q + 1)$, for some integer $k \geq 0$. Substituting this into Equation 5.10 gives:

$$\begin{aligned} qk(q + 1) &= (q + 1)(q^3 + q - S_\Sigma) \\ S_\Sigma &= q^3 + q - qk \end{aligned}$$

Finally, substituting this value into Equation 5.9 gives:

$$B_\Sigma = q^3 + q^2 + q + 1 - (q^3 + q - qk) = q^2 + 1 + qk$$

□

The above lemma has these two useful corollaries.

Corollary 5.2.12 *Suppose Σ is a 3-space containing exactly q^2+1 black points. Then these points form an ovoid of Σ.*

Proof For the 3-space Σ, the number of black points is $B_\Sigma = q^2+1$. By Lemma 5.2.11, it follows that $D_\Sigma = 0$. In particular, no plane of Σ contains a dark line. Corollary 5.2.9 implies that a plane of Σ not in $\mathscr{S}$ contains exactly one black point. By Lemma 5.2.6, a secant plane contains $q+1$ black points, so every plane of Σ contains 1 or $q+1$ black points. Theorem 2.3.3 implies that the set of black points in Σ is either a line or an ovoid. Since there are q^2+1 black points in Σ, the set of black points in Σ is an ovoid. □

Corollary 5.2.13 *Suppose ℓ is a line containing no black points, then every 3-space through ℓ contains q^2+1, q^2+q+1 or q^2+2q+1 black points.*

Proof By Corollary 5.2.10, ℓ has through it q^2 planes of $\mathscr{S}$ with $q+1$ black points each, and $q+1$ planes not in $\mathscr{S}$ with one black point each.

Let Σ be a 3-space through ℓ. By Lemma 5.2.11, there exists a non-negative integer k such that $B_\Sigma = q^2+1+qk$. Denote by a the number of planes of Σ through ℓ containing exactly one black point. Consider the black points in the planes of Σ through ℓ. There are a planes of Σ through ℓ with one black point, and $q+1-a$ planes of Σ through ℓ with $q+1$ black points, so

$$a \cdot 1 + (q+1-a)(q+1) = q^2+1+qk$$
$$a+k = 2$$

Since both a and k are non-negative, it follows that $k \leq 2$. Thus the number of black points in Σ is q^2+1, q^2+q+1 or q^2+2q+1. □

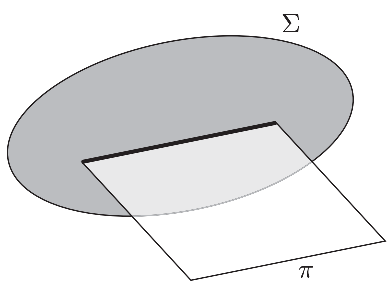
Note that there is no guarantee at this stage that there exists a line with no black points, or a 3-space with q^2+1 black points.

The two possibilities stated in Theorem 5.2.1 may now be isolated. Firstly, the case where there exists a 3-space containing no planes of $\mathscr{S}$ is considered.

Lemma 5.2.14 *Suppose there exists a 3-space Σ containing no planes of $\mathscr{S}$. Then the set of dark lines $\mathscr{M}$ is a $(q+1)$-cover of Σ and $\mathscr{S}$ is the set of planes of* $\mathrm{PG}(4,q)$ *not containing a line of $\mathscr{M}$.*

Proof By Lemma 5.2.11, there is a non-negative integer k such $D_\Sigma = k(q+1)$, $B_\Sigma = q^2+1+qk$ and $S_\Sigma = q^3+q-qk$. Since Σ contains no planes of $\mathscr{S}$, it follows that $k = q^2+1$ and so $D_\Sigma = (q^2+1)(q+1) = q^3+q^2+q+1$ and $B_\Sigma = q^2+1+q(q^2+1) = q^3+q^2+q+1$. By Corollary 5.2.7, these are all the black points and dark lines, so Σ is the set of black points, and all the dark lines are contained in Σ.

By Lemma 5.2.3, there are $q+1$ dark lines through any black point. Thus the set of dark lines forms a $(q+1)$-cover of Σ (see Definition 0.4.3). It remains to show that a plane not in $\mathscr{S}$ contains a dark line, as then $\mathscr{S}$ is the set of planes which contain no dark lines.



Consider a plane π not in $\mathscr{S}$. If π is contained in Σ, then by Lemma 0.4.4 it has $q+1$ dark lines, as the dark lines form a $(q+1)$-cover. If π is not contained in Σ, then it meets Σ in a line and so has $q+1$ black points. Lemma 5.2.8 then implies that π has one dark line. (Note that this dark line is $\pi \cap \Sigma$, since all the dark lines are in Σ.) Thus a plane not in $\mathscr{S}$ has at least one dark line and so $\mathscr{S}$ is the set of planes which contain no dark lines. □

In light of the above lemma, from this point the following assumption is made. This assumption is Condition (III) in Theorem 5.2.2:

Assume from now on that every 3-space contains at least one plane of $\mathscr{S}$.

This assumption makes it possible to use Corollaries 5.2.9, 5.2.10, 5.2.12 and 5.2.13, which did not apply in the situation of Lemma 5.2.14.

Lemma 5.2.15 *There exists a line containing no black points, and a 3-space containing q^2+1 black points.*

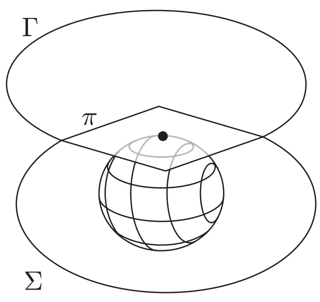
Proof Suppose every line contains a black point. Theorem 0.2.18 for $\mathrm{PG}(4,q)$ states that if $\mathscr{K}$ is a set of points in $\mathrm{PG}(4,q)$ such that every line meets $\mathscr{K}$, then $\mathscr{K}$ has at least q^3+q^2+q+1 points. Moreover, if $\mathscr{K}$ has exactly q^3+q^2+q+1 points, then it is a 3-space. Since the set of black points has size q^3+q^2+q+1 points, it is a 3-space.

However, by Lemma 5.2.11, a 3-space with q^3+q^2+q+1 black points contains no planes of $\mathscr{S}$ (see the proof of Lemma 5.2.14). This is a contradiction to the assumption that every 3-space contains a plane of $\mathscr{S}$, so there exists a line with no black points. Let ℓ be such a line.

By Corollary 5.2.10, there exist $q+1$ planes through ℓ containing exactly one black point. Let Γ be the 3-space spanned by two of these planes. Using the proof of Corollary 5.2.13 (with $a=2$), it follows that there are q^2+1 black points in Γ. □

Lemma 5.2.16 *Any 3-space contains q^2+1, q^2+q+1 or q^2+2q+1 black points.*

Proof Let Σ be a 3-space of $\mathrm{PG}(4,q)$ containing exactly q^2+1 black points, which exists by Lemma 5.2.15. Let Γ be any 3-space of $\mathrm{PG}(4,q)$. Then either $\Gamma=\Sigma$ in which case Γ contains exactly q^2+1 black points, or Γ meets Σ in a plane $\pi=\Gamma\cap\Sigma$.



The black points in Σ form an ovoid by Corollary 5.2.12. The planes in Σ meet this ovoid in one point or the $q+1$ points of an oval. In either case, there exists a line in π containing no black points. Thus there is a line in Γ containing no black points. By Corollary 5.2.13, Γ contains q^2+1, q^2+q+1 or q^2+2q+1 black points. □

Lemma 5.2.17 *The set of black points in a plane not in $\mathscr{S}$ is either a single point, the $q+1$ points on a dark line, or the set of $2q+1$ black points on a pair of dark lines.*

Proof Let π be a plane not in $\mathscr{S}$. An upper bound for B_π may be found by considering the black points in the 3-spaces through π. Let $X = \{(\Sigma, P) \mid \Sigma$ is a 3-space through π, P is a black point in $\Sigma\}$. The size of X is counted in two ways. Count Σ, then π: By Lemma 5.2.16, the 3-spaces through π contain q^2+1, q^2+q+1 or q^2+2q+1 black points. Denote by $A_{2\pi}$ the number of 3-spaces through π with q^2+2q+1 black points, by $A_{1\pi}$ the number with q^2+q+1 black points and by $A_{0\pi}$ the number with q^2+1 black points. Then

$$\begin{aligned}|X| &= (q^2+2q+1)A_{2\pi} + (q^2+q+1)A_{1\pi} + (q^2+1)A_{0\pi} \\ &= (q^2+q+1)(A_{2\pi}+A_{1\pi}+A_{0\pi}) + qA_{2\pi} - qA_{0\pi} \\ &= (q^2+q+1)(q+1) + qA_{2\pi} - qA_{0\pi} \\ &= q^3+2q^2+2q+1+qA_{2\pi}-qA_{0\pi}.\end{aligned}$$

Count P, then Σ: There are B_π black points in π which are contained in each of the $q+1$ 3-spaces through π. There are $(q^3+q^2+q+1-B_\pi)$ black points not in π which are contained in one 3-space through π. So

$$\begin{aligned}|X| &= (q+1)B_\pi + q^3+q^2+q+1-B_\pi \\ &= qB_\pi + q^3+q^2+q+1.\end{aligned}$$

Equating these two expressions for $|X|$ gives

$$\begin{aligned}qB_\pi + q^3+q^2+q+1 &= q^3+2q^2+2q+1+qA_{2\pi}-qA_{0\pi} \\ B_\pi &= q+1+A_{2\pi}-A_{0\pi} \\ B_\pi &\le q+1+A_{2\pi} \\ B_\pi &\le q+1+q+1 \\ B_\pi &\le 2q+2.\end{aligned}$$

By Lemma 5.2.8, $B_\pi = qD_\pi + 1$. So,

$$\begin{aligned}qD_\pi + 1 &\le 2q+2 \\ D_\pi &\le 2+\frac{1}{q}.\end{aligned}$$

So $D_\pi \le 2$ as D_π is an integer.

If $D_\pi = 0$, then there is one black point in π by Corollary 5.2.9. If $D_\pi = 1$, then there is one dark line in π, which contains $q+1$ black points. However, $B_\pi = qD_\pi + 1 = q + 1$, so the black points in π are those on the one dark line. Finally, if $D_\pi = 2$, then there are two dark lines in π, which together contain $2q+1$ black points. However, $B_\pi = qD_\pi + 1 = 2q + 1$, so the black points in π are those on the two dark lines. □

Corollary 5.2.18 *The black points in a plane of $\mathscr{S}$ form an oval.*

Proof Let α be a plane of $\mathscr{S}$ and let ℓ be a line in α. Note that a plane of $\mathscr{S}$ contains no dark lines, so ℓ is a light line. Thus ℓ lies on q^2 planes of $\mathscr{S}$ and $q+1$ planes not in $\mathscr{S}$. Let π be a plane through ℓ that is not in $\mathscr{S}$. From Lemma 5.2.17, the only lines in π containing $q+1$ black points are dark lines, and all other lines contain 0, 1 or 2 black points. Since ℓ is a light line, it follows that ℓ contains 0, 1 or 2 black points. Hence, every line in σ contains 0, 1 or 2 black points, and so the set of black points in α is an arc. By Lemma 5.2.6, there are $q+1$ black points in α, so the set of black points in α is an oval. □

Lemma 5.2.19 *The set of black points is a parabolic quadric and $\mathscr{S}$ is the set of its secant planes.*

Proof Denote the set of black points by $\mathscr{B}$. By Lemma 5.2.17 and Corollary 5.2.18, every plane meets $\mathscr{B}$ in a point, a line, a pair of lines, or an oval. So $\mathscr{B}$ is a quadral by Theorem 2.3.1. Also, every point of $\mathscr{B}$ lies in q^4 planes of $\mathscr{S}$, each meeting $\mathscr{B}$ in an oval. Thus $\mathscr{B}$ is a non-singular quadric in $\mathrm{PG}(4, q)$ by Corollary 2.3.2. Finally, the planes of $\mathscr{S}$ are the planes that meet $\mathscr{B}$ in an oval. Since $\mathscr{B}$ is a quadric, the planes of $\mathscr{S}$ are the planes that meet $\mathscr{B}$ in a non-singular conic. That is, $\mathscr{S}$ is the set of secant planes of $\mathscr{B}$. □

This completes the proof of Theorems 5.2.1 and 5.2.2.

5.3 The tangents and generator lines of the non-singular quadric in $\mathrm{PG}(4, q)$, q odd

Theorem 5.2.2 can be used to provide a new characterisation of the tangent and generator lines of the non-singular quadric in $\mathrm{PG}(4, q)$ when q is odd. In order to do this, the dual nature of the quadric in $\mathrm{PG}(4, q)$ is used.

Theorem 5.3.1 *Let $\mathscr{L}$ be a set of lines in* $\mathrm{PG}(4, q)$*, q odd, such that*

(I) Every point of $\mathrm{PG}(4, q)$ *lies on at least one line not in $\mathscr{L}$,*

(II) Every plane of $\mathrm{PG}(4, q)$ *contains $q + 1$ or $q^2 + q + 1$ lines of $\mathscr{L}$,*

(III) Every 3-space of $\mathrm{PG}(4, q)$ *contains $q^3 + q^2 + q + 1$ or $q^3 + 2q^2 + q + 1$ lines of $\mathscr{L}$.*

Then $\mathscr{L}$ is the set of tangents and generator lines of a parabolic quadric of $\mathrm{PG}(4, q)$*.*

Proof Denote by $\mathscr{L}^c$ the set of lines of $\mathrm{PG}(4, q)$ not in $\mathscr{L}$. Then $\mathscr{L}^c$ has the following properties:

(I^c) Every point of $\mathrm{PG}(4, q)$ lies on at least one line of $\mathscr{L}^c$,

(II^c) Every plane of $\mathrm{PG}(4, q)$ contains q^2 or 0 lines of $\mathscr{L}^c$,

(III^c) Every 3-space of $\mathrm{PG}(4, q)$ contains $q^4 + q^2$ or q^4 lines of $\mathscr{L}^c$.

The set of lines $\mathscr{L}^c$ may be considered as a set of planes of the dual space $\mathrm{PG}(4, q)^*$. That is, $\mathscr{L}^c$ is a set of planes of $\mathrm{PG}(4, q)^*$ with the following properties:

(I^{c*}) Every 3-space of $\mathrm{PG}(4, q)^*$ contains at least one plane of $\mathscr{L}^c$,

(II^{c*}) Every line of $\mathrm{PG}(4, q)^*$ lies on q^2 or 0 planes of $\mathscr{L}^c$,

(III^{c*}) Every point of $\mathrm{PG}(4, q)^*$ lies on $q^4 + q^2$ or q^4 planes of $\mathscr{L}^c$.

By Theorem 5.2.2, $\mathscr{L}^c$ is the set of secant planes of a parabolic quadric of $\mathrm{PG}(4,q)^*$. Denote this quadric by $\mathscr{P}^*$ and the set of its tangent hyperplanes by $\mathscr{P}$. Then $\mathscr{P}$ is a set of points in $\mathrm{PG}(4,q)$, which is a parabolic quadric by Lemma 1.5.22. The set of secant planes of $\mathscr{P}^*$ is the set of external lines and secants of $\mathscr{P}$ by Lemma 1.5.44. Thus $\mathscr{L}^c$ is the set of external lines and secants of the parabolic quadric $\mathscr{P}$ in $\mathrm{PG}(4,q)$. Hence, $\mathscr{L}$ is the set of tangents and generator lines of the parabolic quadric $\mathscr{P}$ in $\mathrm{PG}(4,q)$. $\square$

Chapter 6

Characterisation of the external lines of quadrals of parabolic type in $\mathrm{PG}(n, q)$, q even

In the previous two chapters, the external lines of an oval cone in $\mathrm{PG}(3, q)$ and of a parabolic quadric in $\mathrm{PG}(4, q)$ (q even) were characterised. The quadrals involved are examples of quadrals of parabolic type. Since it is possible to characterise their external lines, it is reasonable to expect that it is possible to characterise the external lines of other quadrals of parabolic type. This chapter presents new characterisation results for the external lines of the quadrals of parabolic type in $\mathrm{PG}(n, q)$, q even. That is, Theorems 4.1.1 and 5.1.1 are generalised to $\mathrm{PG}(n, q)$.

First, a characterisation is proved for the external lines of an oval cone in $\mathrm{PG}(n, q)$, q even, $n \geq 3$. As for $n = 3$, this set of external lines is also the set of external lines of a hyperoval cone, and so the set of external lines of $q + 1$ other oval cones.

Next, a characterisation is proved for the external lines of any quadral of parabolic type in $\mathrm{PG}(n, q)$, q even. The proof of this result combines the common elements of the previous characterisations, and makes use of Tallini's characterisation of quadrals given in Theorems 2.1.10 and 2.1.11.

6.1 The external lines of a hyperoval cone in $\mathrm{PG}(n,q)$, q even

In Chapter 4, the set of external lines of an oval cone in $\mathrm{PG}(3,q)$, q even, was characterised by the number of lines through a point and in a plane. The aim of this section is to generalise this result to the external lines of an oval cone in $\mathrm{PG}(n,q)$, $n \geq 4$.

Let $\mathscr{C}$ be an oval cone of $\mathrm{PG}(n,q)$, $n \geq 4$, q even, and denote by $\mathscr{L}$ the set of external lines of $\mathscr{C}$. By Lemmas 1.4.9 and 1.4.4, $\mathscr{C}$ has the following properties:

(I) The number of lines of $\mathscr{L}$ through a point of $\mathrm{PG}(n,q)$ is 0 or $\frac{1}{2}q^{n-1}$,

(II) The number of lines of $\mathscr{L}$ in a plane of $\mathrm{PG}(n,q)$ is 0, q^2 or $\frac{1}{2}q(q-1)$.

However, the set of external lines of any quadral of parabolic type also has the above properties, by Theorem 1.6.7. Thus the above properties are not enough to characterise $\mathscr{L}$, and a third property must be used.

A natural third property is the number of lines of $\mathscr{L}$ in a 3-space. That is, an option is to include the following property, which follows from Lemma 1.4.5.

(III) The number of lines of $\mathscr{L}$ in a 3-space of $\mathrm{PG}(4,q)$ is 0, q^4 or $\frac{1}{2}q^3(q-1)$.

Including this property does make it possible to complete the characterisation. However, further investigation reveals that it is not actually necessary to know the exact number of lines of $\mathscr{L}$ in each 3-space. It is only necessary to assume that if a 3-space contains a line of $\mathscr{L}$, then the number of lines of $\mathscr{L}$ it contains is congruent to 1 (mod $q+1$). Thus the following theorem is proved.

Theorem 6.1.1 *Let $\mathscr{L}$ be a non-empty set of lines in $\mathrm{PG}(n,q)$, $n \geq 4$, q even, such that*

(I) The number of lines of $\mathscr{L}$ through a point of $\mathrm{PG}(n,q)$ is 0 or $\frac{1}{2}q^{n-1}$,

(II) The number of lines of $\mathscr{L}$ in a plane of $\mathrm{PG}(n,q)$ is 0, q^2 or $\frac{1}{2}q(q-1)$,

(III) The number of lines of $\mathscr{L}$ in a 3-space of $\mathrm{PG}(n,q)$ *is* 0, *or congruent to* 1 $\pmod{q+1}$.

Then $\mathscr{L}$ is the set of external lines of a hyperoval cone $\bar{\mathscr{C}}$ of $\mathrm{PG}(n,q)$. *Further, $\mathscr{L}$ is the set of external lines of each of the $q+2$ oval cones contained in $\bar{\mathscr{C}}$.*

The proof of Theorem 6.1.1

Let $\mathscr{L}$ be a set of lines of $\mathrm{PG}(n,q)$, $n \geq 4$, q even, with the properties described in Theorem 6.1.1. By a series of lemmas, it is proved that $\mathscr{L}$ is the set of external lines of a hyperoval cone of $\mathrm{PG}(n,q)$. In order to make the argument clearer, some terminology is now introduced.

- A point on 0 lines of $\mathscr{L}$ is called a *black* point. A point on $\frac{1}{2}q^{n-1}$ lines of $\mathscr{L}$ is called a *white* point.
- A plane containing 0 lines of $\mathscr{L}$ is called an *null plane*. A plane containing q^2 lines of $\mathscr{L}$ is called a *tangent plane*. A plane containing $\frac{1}{2}q(q-1)$ lines of $\mathscr{L}$ is called a *secant plane*.

The first lemma can now be stated.

Lemma 6.1.2 *Let ℓ be a line of $\mathscr{L}$. Then through ℓ there are θ_{n-3} tangent planes and q^{n-2} secant planes.*

Proof Denote by T_ℓ the number of tangent planes through ℓ. Since every plane through ℓ contains a line of $\mathscr{L}$, the remaining $\theta_{n-2} - T_\ell$ planes through ℓ are secant planes. Denote by L_ℓ the number of lines of $\mathscr{L}$ meeting ℓ in one point. The number L_ℓ is counted in two ways.

Each point on ℓ is a white point and lies on $\frac{1}{2}q^{n-1}$ lines of $\mathscr{L}$. Counting this way, the line ℓ itself has been included $q+1$ times, so

$$\begin{aligned} L_\ell &= \tfrac{1}{2}q^{n-1}(q+1) - (q+1) \\ &= \tfrac{1}{2}(q+1)(q^{n-1}-2) \\ &= \tfrac{1}{2}(q+1)\big[(q-1)\theta_{n-2} - 1\big]. \end{aligned}$$

On the other hand, each line of $\mathscr{L}$ meeting ℓ is contained in a plane through ℓ. Each tangent plane through ℓ contains q^2 lines of $\mathscr{L}$ and each secant plane through ℓ contains $\frac{1}{2}q(q-1)$ lines of $\mathscr{L}$. Counting this way, the line ℓ itself has been included θ_{n-2} times, so

$$\begin{aligned} L_\ell &= q^2T_\ell + (\theta_{n-2} - T_\ell)\tfrac{1}{2}q(q-1) - \theta_{n-2} \\ &= \tfrac{1}{2}q(q+1)T_\ell + (\tfrac{1}{2}q^2 - \tfrac{1}{2}q - 1)\theta_{n-2} \\ &= \tfrac{1}{2}(q+1)\big[qT_\ell + (q-2)\theta_{n-2}\big]. \end{aligned}$$

Equating these two expressions for L_ℓ and dividing by $\frac{1}{2}(q+1)$ gives

$$\begin{aligned} qT_\ell + (q-2)\theta_{n-2} &= (q-1)\theta_{n-2} - 1 \\ qT_\ell &= \theta_{n-2} - 1 = q\theta_{n-3} \\ T_\ell &= \theta_{n-3}. \end{aligned}$$

So there are θ_{n-3} tangent planes through ℓ, which implies there are q^{n-2} secant planes through ℓ. □

Note that this lemma ensures the existence of both tangent planes and secant planes, since $\mathscr{L}$ is non-empty. The following lemma will make it possible to determine the structure of the black points in the tangent and secant planes. In order to make the discussion easier, the following notation is introduced. If π is a plane, then denote by W_π the number of white points in π and by $\mathscr{B}_\pi$ the set of black points in π.

Lemma 6.1.3 *Let π be a plane containing at least one line of $\mathscr{L}$. Then W_π is a multiple of $q+1$.*

Proof Denote by L_π the number of lines of $\mathscr{L}$ in π, and by $L_{1\pi}$ the number of lines of $\mathscr{L}$ meeting π in one point. The number $L_{1\pi}$ is counted in two ways.

Consider the lines of $\mathscr{L}$ through the points of π. Each black point in π lies on no lines of $\mathscr{L}$, and each white point in π lies on $\frac{1}{2}q^{n-1}$ lines of $\mathscr{L}$. Counting this way, each line of $\mathscr{L}$ in π has been included $(q+1)$ times – once for each point on it. So $L_{1\pi} = \frac{1}{2}q^{n-1}W_\pi - (q+1)L_\pi$.

On the other hand, every line of $\mathscr{L}$ meeting π is contained in a 3-space through π, so consider the lines of $\mathscr{L}$ in the 3-spaces through π. The plane π contains a line of $\mathscr{L}$, so every 3-space through π contains a line of $\mathscr{L}$. By Condition (III) of Theorem 6.1.1, the number of lines of $\mathscr{L}$ in every 3-space through π is congruent to 1 (mod $q+1$). Denote by A_k the number of 3-spaces through π containing $k(q+1)+1$ lines of $\mathscr{L}$, for $k \geq 0$. Counting this way, every line of $\mathscr{L}$ in π has been included θ_{n-3} times – once for each 3-space through π. So $L_{1\pi} = \sum_k \big[k(q+1)+1\big]A_k - \theta_{n-3}L_\pi$.

Equating these two expressions for $L_{1\pi}$ gives

$$\begin{aligned} \tfrac{1}{2}q^{n-1}W_\pi - (q+1)L_\pi &= \sum_k \big[k(q+1)+1\big]A_k - \theta_{n-3}L_\pi \\ q^{n-1}W_\pi - 2(q+1)L_\pi &= 2\sum_k \big[k(q+1)+1\big]A_k - 2\theta_{n-3}L_\pi \end{aligned} \tag{6.1}$$

Since π contains lines of $\mathscr{L}$, it follows that $L_\pi = q^2$ or $\frac{1}{2}q(q-1)$. In either case, $2L_\pi \equiv 2 \pmod{q+1}$, so reducing Equation 6.1 modulo $(q+1)$ gives

$$\begin{aligned} (-1)^{n-1}W_\pi &\equiv 2\sum_k A_k - 2\theta_{n-3} \pmod{q+1} \\ &\equiv 2\theta_{n-3} - 2\theta_{n-3} \pmod{q+1} \\ &\equiv 0 \pmod{q+1} \\ W_\pi &\equiv 0 \pmod{q+1}. \end{aligned}$$

That is, W_π is a multiple of $q+1$. □

Using the above lemma, it is now possible to find the number of black points in a tangent plane.

Corollary 6.1.4 *Each tangent plane has exactly one black point and the lines of $\mathscr{L}$ in the plane are the lines not through this black point.*

Proof Let π be a tangent plane. Then π has q^2 lines of $\mathscr{L}$, which do not pass through any black point in π. Since the number of lines not through a point in a plane is q^2, the plane π has at most one black point. Thus $W_\pi = q^2+q$ or q^2+q+1. But $W_\pi \equiv 0 \pmod{q+1}$, so $W_\pi = q^2+q$. That is, π contains exactly one black point, and the q^2 lines of $\mathscr{L}$ in π are the lines not through this point. $\square$

In order to find the structure of the black points in a secant plane, first the number of black points on a line not in $\mathscr{L}$ is calculated.

Lemma 6.1.5 *A line not in $\mathscr{L}$ contains 1, 2 or $q+1$ black points.*

Proof Let m be a line not in $\mathscr{L}$. Denote by W_m the number of white points on m, by S_m the number of secant planes through m and by T_m the number of tangent planes through m. Finally, denote by L_m the number of lines of $\mathscr{L}$ meeting m in one point. The number L_m is counted in two ways.

Each white point on m lies on $\frac{1}{2}q^{n-1}$ lines of $\mathscr{L}$, and each black point on m lies on no lines of $\mathscr{L}$. So $L_m = \frac{1}{2}q^{n-1}W_m$. On the other hand, each secant plane through m contains $\frac{1}{2}q(q-1)$ lines of $\mathscr{L}$, and each tangent plane through m contains q^2 lines of $\mathscr{L}$. The remaining planes through m contain no lines of $\mathscr{L}$, so $L_m = \frac{1}{2}q(q-1)S_m + q^2T_m$. Thus

$$\frac{1}{2}q^{n-1}W_m = \frac{1}{2}q(q-1)S_m + q^2T_m$$
$$q^{n-2}W_m = (q-1)S_m + 2qT_m.$$

Suppose m is contained in a tangent plane π. By Corollary 6.1.4, the lines of π not in $\mathscr{L}$ all pass through the unique black point in π. Thus m contains exactly one black point.

Suppose m is not contained in a tangent plane. That is, suppose $T_m = 0$. The above equation then becomes $q^{n-2}W_m = (q-1)S_m$. Since q and $q-1$ are coprime, it follows that q^{n-2} divides S_m. Now $S_m \le \theta_{n-2} < 2q^{n-2}$. Thus $S_m = 0$ or q^{n-2} and so $W_m = 0$ or $q-1$. Hence m contains $q+1$ or 2 black points. $\square$

Denote by $\mathscr{B}$ the set of black points. Then the above lemma implies that $\mathscr{B}$ is a Tallini set and that $\mathscr{L}$ is the set of external lines of $\mathscr{B}$. Before $\mathscr{B}$ can be proved to be a hyperoval cone, more must be proved about the structure of the black points in a secant plane and a null plane.

Lemma 6.1.6 *The black points in a secant plane form a hyperoval.*

Proof Let π be a secant plane and consider a line ℓ of $\mathscr{L}$ in π. Every line of π meets ℓ, and so every line of π contains a white point. By Lemma 6.1.5, this implies that every line of π contains 0, 1 or 2 black points. That is, the set $\mathscr{B}_\pi$ of black points in π is an arc. By Lemma 6.1.5, any line not in $\mathscr{L}$ contains a black point, so the set of lines of $\mathscr{L}$ in π is the set of external lines of $\mathscr{B}_\pi$. Thus $\mathscr{B}_\pi$ has exactly $\frac{1}{2}q(q-1)$ external lines.

By Corollary 1.1.7, it follows that $\mathscr{B}_\pi$ is either an oval or a hyperoval. Thus $W_\pi = q^2$ or $q^2 - 1$. By Lemma 6.1.3, the number of white points in π is a multiple of $q+1$, so $W_\pi = q^2 - 1$ and $\mathscr{B}_\pi$ is a hyperoval. □

Lemma 6.1.7 *The set of black points in a null plane contains a line.*

Proof Let π be a null plane and suppose $\mathscr{B}_\pi$ contains no lines. Then every line of π contains 0, 1 or 2 black points. That is, the set $\mathscr{B}_\pi$ is an arc. Every arc has external lines, so there exists a line in π with no black points. By Lemma 6.1.5, the lines with no black points are the lines of $\mathscr{L}$, so π contains a line of $\mathscr{L}$. This is a contradiction, so $\mathscr{B}_\pi$ contains a line. □

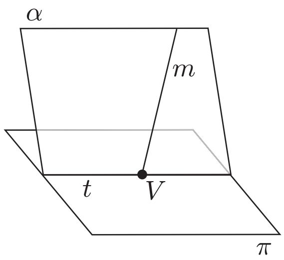
The set of black points $\mathscr{B}$ is a Tallini set by Lemma 6.1.5. In order to show that $\mathscr{B}$ is a hyperoval cone, it is necessary to show that the singular space of $\mathscr{B}$ is an $(n-3)$-space. The following lemma makes this possible.

Lemma 6.1.8 *A black point contained in a secant plane is a non-singular point of $\mathscr{B}$. A black point contained in a tangent plane is a singular point of $\mathscr{B}$.*

Proof Let π be a secant plane and let P be a black point in π. The set of black points in π is a hyperoval, and every point on a hyperoval lies on $q+1$ secants. Thus P lies on a secant of $\mathscr{B}$ and so P is a non-singular point of $\mathscr{B}$.

Let π be a tangent plane and let V be the unique black point in π. To show that V is a singular point of $\mathscr{B}$, it must be shown that the lines through V are either tangents or generator lines of $\mathscr{B}$.

Consider a line m through V. If m is a line of π, then it is a tangent of $\mathscr{B}$. Suppose m is not a line of π.



Let t be a line of π through V and let $\alpha = t \oplus m$. By Lemma 6.1.6, a secant plane meets $\mathscr{B}$ in a hyperoval and so contains no tangents of $\mathscr{B}$. Since t is a tangent of $\mathscr{B}$ in α, the plane α is not a secant plane.

If α is a tangent plane, then P is its only black point, so the line m is a tangent of $\mathscr{B}$. If α is a null plane, then the set of black points in α is a Tallini set containing a line, which also has a tangent t. Using the list of plane Tallini sets in Lemma 2.1.4, this implies that the set of black points in α is either a line through V or a pair of lines through V. In both cases the line m is either a tangent or a generator line of $\mathscr{B}$. (Note that this also applies in the case $q = 2$.)

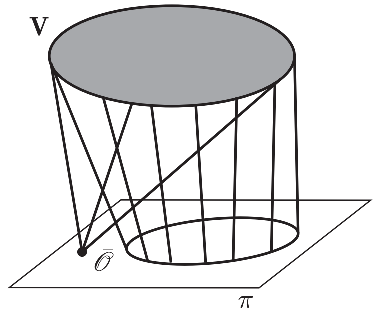
Thus every line through V is a tangent or generator line of $\mathscr{B}$. That is, V is a singular point of $\mathscr{B}$. □

The proof of Theorem 6.1.1 can now be completed.

Lemma 6.1.9 *The set of black points $\mathscr{B}$ is a hyperoval cone of* PG(n, q) *and $\mathscr{L}$ is its set of external lines.*

Proof The set of black points $\mathscr{B}$ is a Tallini set by Lemma 6.1.5. Denote by $\mathbf{V}$ the singular space of $\mathscr{B}$.

Let ℓ be a line of $\mathscr{L}$. Then every black point lies in a unique plane through ℓ. By Lemma 6.1.8, the black points in the tangent planes through ℓ are singular points and the black points in the secant planes through ℓ are non-singular points. Thus the singular points of $\mathscr{B}$ are the black points in the tangent planes through ℓ. By Lemma 6.1.2, the line ℓ lies on θ_{n-3} tangent planes. By Corollary 6.1.4, each of these planes contains one black point, so there are θ_{n-3} singular points of $\mathscr{B}$. Thus the singular space $\mathbf{V}$ of $\mathscr{B}$ is an $(n-3)$-space.



Let π be a secant plane containing the hyperoval $\bar{\mathscr{O}}$ of black points. Since a secant plane contains no singular points, the plane π is skew to $\mathbf{V}$. Thus $\mathscr{B}$ is the hyperoval cone $\mathbf{V}\bar{\mathscr{O}}$ by Lemma 2.1.6. Since a line not in $\mathscr{L}$ contains a black point, it follows that $\mathscr{L}$ is the set of external lines of $\mathbf{V}\bar{\mathscr{O}}$. $\square$

In the above lemma, if $\mathscr{O}$ is an oval obtained by removing one point of the hyperoval $\bar{\mathscr{O}}$, then the oval cone $\mathbf{V}\mathscr{O}$ has the same set of external lines as the hyperoval cone $\mathbf{V}\bar{\mathscr{O}}$ by Lemma 1.4.11. Thus $\mathscr{L}$ is the set of external lines of $q+2$ oval cones. This completes the proof of Theorem 6.1.1.

6.2 The external lines of a quadral of parabolic type.

In the previous section, the set of external lines of an oval cone of $\mathrm{PG}(n,q)$, q even, was characterised by some of its combinatorial properties. In this section, this result is generalised to the set of external lines of any quadral of parabolic type.

Let $\mathscr{Q}$ be a quadral of parabolic type in $\mathrm{PG}(n,q)$, q even, $n \geq 2$ (see Definition 1.6.2) and let $\mathscr{L}$ be the set of external lines of $\mathscr{Q}$. Then $\mathscr{L}$ has the following two properties:

(I) The number of lines of $\mathscr{L}$ through any point is 0 or $\frac{1}{2}q^{n-1}$,

(II) The number of lines of $\mathscr{L}$ in any plane is 0, q^2 or $\frac{1}{2}q(q-1)$.

Property (I) is given in Theorem 1.6.7. Property (II) follows from the fact that every plane meets $\mathscr{Q}$ in a point, a line, a pair of lines, an oval or the whole plane.

By Lemma 3.1.4, only the first property is needed to show that a set of lines in $\mathrm{PG}(2,q)$ is the external lines of an oval. In the proof of Theorem 4.1.1, both were used to show that a set of lines in $\mathrm{PG}(3,q)$ is the external lines of an oval cone. However, the two conditions do not seem to be sufficient to prove that a set of lines in $\mathrm{PG}(n,q)$, $n \geq 4$, is the external lines of a quadral of parabolic type.

In the proof of Theorem 6.1.1, many of the crucial arguments relied on the fact that a plane containing q^2 lines of "$\mathscr{L}$" also contains a "black" point. For this reason, the condition that every plane contains a "black" point has been chosen in order to complete the characterisation. That is, the following theorem is proved.

Theorem 6.2.1 *Let $\mathscr{L}$ be a set of lines in* $\mathrm{PG}(n, q)$*,* $n \geq 4$*,* q *even, such that*

(I) The number of lines of $\mathscr{L}$ through a point of $\mathrm{PG}(n, q)$ *is* 0 *or* $\frac{1}{2}q^{n-1}$*,*

(II) The number of lines of $\mathscr{L}$ in a plane of $\mathrm{PG}(n, q)$ *is* 0*,* q^2 *or* $\frac{1}{2}q(q-1)$*,*

(III) Every plane of $\mathrm{PG}(n, q)$ *contains at least one point which lies on no lines of $\mathscr{L}$.*

Then $\mathscr{L}$ is the set of external lines of a quadral of parabolic type.

The proof of Theorem 6.2.1

Let $\mathscr{L}$ be a set of lines of $\mathrm{PG}(n, q)$, $n \geq 4$, q even, with the properties described in Theorem 6.2.1. By a series of lemmas, it is proved that $\mathscr{L}$ is the set of external lines of a quadral of parabolic type. In order to make the argument clearer, some terminology is introduced.

- A point on 0 lines of $\mathscr{L}$ is called a *black* point. A point on $\frac{1}{2}q^{n-1}$ lines of $\mathscr{L}$ is called a *white* point.
- A plane containing 0 lines of $\mathscr{L}$ is called an *null plane.* A plane containing q^2 lines of $\mathscr{L}$ is called a *tangent plane.* A plane containing $\frac{1}{2}q(q-1)$ lines of $\mathscr{L}$ is called a *secant plane.*

Firstly, note that a hyperplane is a quadral of parabolic type which has no external lines. Thus if $\mathscr{L}$ is empty, it is the set of external lines of a quadral of parabolic type. Assume from now on that $\mathscr{L}$ is non-empty.

Lemma 6.2.2 *Let ℓ be a line of $\mathscr{L}$. Then through ℓ there are q^{n-2} secant planes and θ_{n-3} tangent planes.*

Proof Let T_ℓ be the number of tangent planes through ℓ. Then the number of secant planes through ℓ is $\theta_{n-2} - T_\ell$, since there are no lines of $\mathscr{L}$ in a null plane. Denote by L_ℓ the number of lines of $\mathscr{L}$ meeting ℓ in one point. The number L_ℓ is counted in two ways.

Each point on ℓ lies on $\frac{1}{2}q^{n-1}$ lines of $\mathscr{L}$ including ℓ. Counting this way, ℓ itself has been included $q+1$ times, so $L_\ell = \frac{1}{2}q^{n-1}(q+1) - (q+1)$.

On the other hand, each tangent plane through ℓ contains q^2 lines of $\mathscr{L}$, and each secant plane through ℓ contains $\frac{1}{2}q(q-1)$ lines of $\mathscr{L}$. Counting this way, ℓ itself has been included θ_{n-2} times, so $L_\ell = q^2 T_\ell + \frac{1}{2}q(q-1)(\theta_{n-2} - T_\ell) - \theta_{n-2}$.

Equating these two expressions for L_ℓ and solving for T_ℓ gives $T_\ell = \theta_{n-3}$. That is, the number of tangent planes through ℓ is θ_{n-3}. Thus the number of secant planes through ℓ is $\theta_{n-2} - \theta_{n-3} = q^{n-2}$. □

Note that since $\mathscr{L}$ is non-empty, the above lemma implies the existence of both tangent planes and secant planes. The next lemma determines the number of black points in a tangent plane.

Lemma 6.2.3 *Let π be a tangent plane. Then there is exactly one black point in π and the lines of $\mathscr{L}$ in π are the lines not through this point.*

Proof There is at least one black point in π by Condition III of Theorem 6.2.1. Let V be a black point in π. There are q^2 lines of π not through V. On the other hand, there are q^2 lines of $\mathscr{L}$ in π, none of which contain a black point. Thus V is the only black point in π and the lines of $\mathscr{L}$ in π are the lines not through V. □

Using the above lemma, the number of black points on a line not in $\mathscr{L}$ can now be counted.

Lemma 6.2.4 *Let m be a line not in $\mathscr{L}$. Then m has 1, 2 or $q+1$ black points.*

Proof Suppose the tangent plane π passes through ℓ and denote the unique black point in π by V. By Lemma 6.2.3, the lines of π not through V are lines of $\mathscr{L}$, so m passes through V, since it is not a line of $\mathscr{L}$. Thus m contains exactly one black point.

Suppose m is contained in no tangent planes. Denote by S_m the number of secant planes through m and by W_m the number of white points on m. Denote by L_m the number of lines of ℓ meeting m in one point. The number L_m is counted in two ways.

Each white point on m lies on $\frac{1}{2}q^{n-1}$ lines of $\mathscr{L}$ and the remaining points lie on no lines of $\mathscr{L}$, so $L_m = \frac{1}{2}q^{n-1}W_m$. Each secant plane through m contains $\frac{1}{2}q(q-1)$ lines of $\mathscr{L}$ and the remaining planes contain no lines of $\mathscr{L}$. So $L_m = \frac{1}{2}q(q-1)S_m$.

Equating these two expressions gives

$$\frac{1}{2}q^{n-1}W_m = \frac{1}{2}q(q-1)S_m$$
$$q^{n-2}W_m = (q-1)S_m.$$

Since $(q-1)$ and q are coprime, it follows that q^{n-2} divides S_m. Since S_m is a number of planes through a line, $S_m \leq \theta_{n-2} < 2q^{n-2}$. Thus $S_m = 0$ or q^{n-2}, and so $W_m = 0$ or $q-1$. Hence the number of black points on m is either $q+1$ or 2. $\square$

Denote by $\mathscr{B}$ the set of black points. The above lemma implies that $\mathscr{B}$ is a Tallini set and that the lines of $\mathscr{L}$ are the external lines of $\mathscr{B}$. In order to show that $\mathscr{B}$ is a quadral of parabolic type, the structure of the black points in a secant plane needs to be determined.

Lemma 6.2.5 *Let π be a secant plane. Then the black points in π form a hyperoval or an oval.*

Proof Denote the set of black points in π by $\mathscr{B}_\pi$ and let ℓ be a line of $\mathscr{L}$ in π. Every line of π meets ℓ in a white point, so by Lemma 6.2.4, every line of π contains 0, 1 or 2 black points. Thus $\mathscr{B}_\pi$ is an arc. The lines of $\mathscr{L}$ in π are the lines with no black points, so $\mathscr{B}_\pi$ has exactly $\frac{1}{2}q(q-1)$ external lines. By Corollary 1.1.7, it follows that $\mathscr{B}_\pi$ is either an oval or a hyperoval. □

Now the size of the Tallini set $\mathscr{B}$ can be calculated.

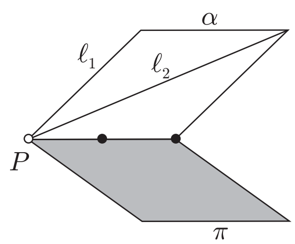
Lemma 6.2.6 *There are at least θ_{n-1} black points.*

Proof Let ℓ be a line of $\mathscr{L}$. By Lemma 6.2.5, each of the q^{n-2} secant planes through ℓ contains at least $q+1$ black points, and by Lemma 6.2.3, each of the θ_{n-3} tangent planes through ℓ contains exactly one black point. Thus $|\mathscr{B}| \geq (q+1)q^{n-2} + \theta_{n-3} = \theta_{n-1}$. □

The above lemma makes it possible to use the characterisation of large Tallini sets for $q > 2$ given in Theorem 2.1.10. For $q = 2$, the following lemma is required.

Lemma 6.2.7 *Suppose $q = 2$. Then no plane has exactly 2 or 6 black points.*

Proof Suppose π is a plane containing exactly 2 black points. Then there exists a line ℓ in π containing no black points. By Lemma 6.2.4, the line ℓ is a line of $\mathscr{L}$. Thus π is a tangent plane or a secant plane. By Lemmas 6.2.3 and 6.2.5, the number of black points in π is 1, 3 or 4. This is a contradiction, so no plane has exactly 2 black points.



Suppose π is a plane containing exactly 6 black points and let P be the unique white point in π. Suppose Π is a 3-space through π containing two lines ℓ_1 and ℓ_2 of $\mathscr{L}$ through P. Consider the plane $\alpha = \ell_1 \oplus \ell_2$. The plane α meets π in a line. Since all the points of π except P are black, this line contains two black points. By Lemma 6.2.3, a tangent plane contains exactly one black point. Thus α is a secant plane. However, the number of lines of $\mathscr{L}$ in a secant plane is $\frac{1}{2}q(q-1) = 1$. This is a contradiction, since π contains two lines of $\mathscr{L}$. So every 3-space through Π contains at most one line of $\mathscr{L}$ through P.

The number of 3-spaces through π is $\theta_{n-3} = 2^{n-2}-1$. Each of these 3-spaces contains at most one line of $\mathscr{L}$ through P. Since each line of $\mathscr{L}$ through P is contained in a 3-space through π, this implies that there are at most $2^{n-2}-1$ lines of $\mathscr{L}$ through P. But the number of lines of $\mathscr{L}$ through P is $\frac{1}{2}q^{n-1} = 2^{n-2}$. This is a contradiction, so no plane contains exactly 6 black points. □

The proof of Theorem 6.2.1 can now be completed.

Lemma 6.2.8 *The set of lines $\mathscr{L}$ is the set of external lines of a quadral of parabolic type.*

Proof The set of black points $\mathscr{B}$ is a Tallini set with at least θ_{n-1} points by Lemmas 6.2.4 and 6.2.6. If $q = 2$, then no plane meets $\mathscr{B}$ in 0, 2 or 6 points by Lemma 6.2.7. By Theorems 2.1.10 and 2.1.11, the Tallini set $\mathscr{B}$ is either a hyperplane plus a subspace, a quadric of hyperbolic type or a quadric of parabolic type plus a subspace contained in its nuclear space.

If $\mathscr{B}$ contains a hyperplane, then it has no external lines and $\mathscr{L}$ is empty, which is a contradiction. If $\mathscr{B}$ is a quadral of hyperbolic type containing no hyperplanes, then Theorem 1.6.7 implies that a point not in $\mathscr{B}$ lies on greater than $\frac{1}{2}q^{n-1}$ external lines. This is also a contradiction. Thus, $\mathscr{B}$ is a quadral $\mathscr{Q}$ of parabolic type plus a subspace contained in its nuclear space $\mathbf{N}$.

If P is a point in the nuclear space $\mathbf{N}$ but not in $\mathscr{Q}$, then every line through P meets $\mathscr{Q}$ in a unique (black) point. Thus a line through P is not a line of $\mathscr{L}$. Since $\mathscr{L}$ is the set of external lines of $\mathscr{B}$, this implies that $\mathscr{L}$ is also the set of external lines of the quadral $\mathscr{Q}$ of parabolic type. □

Chapter 7

The intersection of ovals and ovoids

In Chapter 3, Theorem 3.2.6 of Durante and Olanda [29] was presented, which characterises the set of external lines of an ovoid by its combinatorial properties. In the proof of this result, the set of points lying on no external lines is shown to be an ovoid. Thus, an ovoid Ω is uniquely determined by its set of external lines. It is reasonable to ask how many of the external lines are needed to uniquely determine Ω. An alternative way to ask this question is to ask how many external lines two distinct ovoids can share. This naturally leads to the question of how many points two distinct ovoids can share, and in what configuration. These questions are the subject of this chapter.

The chapter begins with an investigation of the intersection of ovals, and then moves on to prove several existing and new results concerning the relationships between the number of shared points, lines and planes of ovoids. Using this information, a new bound is proved for the number of points two distinct ovoids can share. This new bound makes it possible to investigate configurations of shared external lines, secants and points. Finally, the action of certain collineations on ovoids is used to prove new results on the intersection of two ovoids sharing all of their tangents.

7.1 The intersection of two ovals

Let $\mathscr{O}_A$ and $\mathscr{O}_B$ be two ovals in $\mathrm{PG}(2, q)$. A point in both Ω_A and Ω_B is called a *shared point*. Similarly, a line is called a *shared tangent* when it is a tangent of both ovals. Similar definitions apply for shared secants and external lines. Note that it is possible for a shared tangent to meet each oval in a different point. In this section, several results are collected about the number of shared points or lines that $\mathscr{O}_A$ and $\mathscr{O}_B$ may have.

First it is appropriate to describe how two non-singular conics meet. This is a corollary of Lemma 1.5.25.

Theorem 7.1.1 [36, Chapter 7] *Two distinct conics share at most* 4 *points.*

When q is odd, all ovals are conics, so the above theorem describes the situation for all ovals. When q is even, there exist ovals that are not conics. Moreover, every oval is contained in a hyperoval which contains $q + 1$ other ovals, and two ovals contained in the same hyperoval share exactly q points. Thus it is the intersection of hyperovals that is of interest for q even.

Theorem 7.1.2 [36, Chapter 8] *Two distinct hyperovals of* $\mathrm{PG}(2, q)$*, q even, share at most* $\frac{1}{2}q + 1$ *points. Hence, two distinct ovals of* $\mathrm{PG}(2, q)$*, q even, sharing a nucleus, share at most* $\frac{1}{2}q$ *points.*

It is also possible to consider the number of shared tangents that two ovals have. For q even, two distinct ovals may share a nucleus and so share all of their tangents. Otherwise two distinct ovals share exactly one tangent. For q odd, the set of tangents of a non-singular conic $\mathscr{C}$ is a non-singular conic of the dual plane $\mathrm{PG}(2, q)^*$ by Lemma 1.5.22. Also, $\mathscr{C}$ is determined by its set of tangents. Thus two non-singular conics in $\mathrm{PG}(2, q)$, q odd, share at most 4 tangents.

The following result shows that two ovals cannot share all of their secants.

Corollary 7.1.3 *Let $\mathscr{O}$ be an oval of* $\mathrm{PG}(2, q)$ *and let $\mathscr{S}$ be the set of secants of $\mathscr{O}$. Then $\mathscr{O}$ is the only oval with $\mathscr{S}$ as its set of secants.*

Proof The points on $\mathscr{O}$ lie on q secants of $\mathscr{O}$. If q is even, the points not on $\mathscr{O}$ lie on 0 or $\frac{1}{2}q$ secants by Lemma 1.1.14. If q is odd, the points not on $\mathscr{O}$ lie on $\frac{1}{2}(q+1)$ or $\frac{1}{2}(q-1)$ secants by Lemma 1.1.8. Thus the points of $\mathrm{PG}(2,q)$ which lie on q secants are the points of $\mathscr{O}$. Hence, $\mathscr{O}$ is the only oval with $\mathscr{S}$ as its set of secants.

□

This result will be useful when the shared secants and external lines of ovoids are considered in Section 7.7.

7.2 Relationships Between Shared Points, Shared Tangents and Shared Tangent Planes

Let Ω_A and Ω_B be two ovoids. A point in both Ω_A and Ω_B is called a *shared point*. Similarly, a line is called a *shared tangent* when it is a tangent of both ovoids, and a plane is called a *shared tangent plane* when it is a tangent plane of both ovoids. Similar definitions apply for shared external lines, shared secants and shared secant planes. Note that a shared tangent may meet the two ovoids in different points, and a shared tangent plane may meet the two ovoids in different points.

In this section, several relationships between the numbers of shared points, shared tangents and shared tangent planes are proved. The first result provides a relationship for all q between the number of shared points and the number of shared tangent planes. This result was proved by Bruen and Hirschfeld [14] for elliptic quadrics, but the proof can be applied to any ovoid.

Lemma 7.2.1 *Let Ω_A and Ω_B be two ovoids of $\mathrm{PG}(3,q)$. Then the number of shared tangent planes is the same as the number of shared points of Ω_A and Ω_B.*

Proof Let x be the number of shared points of Ω_A and Ω_B and z the number of shared tangent planes. Consider the set $X = \{(P, \pi) \mid P \text{ is a point of } \Omega_A, \pi \text{ is a tangent plane of } \Omega_B, P \in \pi \}$. The size of X is counted in two ways.
Count P first, then π: If $P \in \Omega_B$, then it lies on 1 tangent plane of Ω_B. If $P \notin \Omega_B$, then it lies on $q+1$ tangent planes of Ω_B.

Thus $|X| = x \cdot 1 + (q^2 + 1 - x)(q + 1) = (q + 1)(q^2 + 1) - qx$.
Count π first, then P: If π is a tangent plane of Ω_A, then it has 1 point of Ω_A. Otherwise, π has $q + 1$ points of Ω_A.
Thus $|X| = z \cdot 1 + (q^2 + 1 - z)(q + 1) = (q + 1)(q^2 + 1) - qz$.
Hence, $(q+1)(q^2+1) - qx = (q+1)(q^2+1) - qz$ and so $x = z$. That is, the number of shared points is the same as the number of shared tangent planes. □

The above result implies that an ovoid is uniquely determined by its set of tangent planes. This can also be seen using duality. The tangent planes of an ovoid Ω form an ovoid Ω^* of the dual space $\mathrm{PG}(3, q)^*$, and the tangent planes of Ω^* in the dual space are the points of Ω. Thus Ω is determined by Ω^*.

When q is odd, the tangents of an elliptic quadric $\mathscr{E}$ through a point P are only coplanar when P lies on $\mathscr{E}$. Thus the points of $\mathscr{E}$ are determined by the tangents of $\mathscr{E}$. So, two distinct elliptic quadrics in $\mathrm{PG}(3, q)$, q odd, cannot share all of their tangents.

This is not the case when q is even. When q is even, the tangents of an ovoid form a general linear complex, and there are many examples of ovoids sharing the same general linear complex of tangents. However, the structure of the tangents of an ovoid when q is even does make it possible to fully describe the shared tangents of two ovoids.

Lemma 7.2.2 *Let Ω_A and Ω_B be two ovoids of $\mathrm{PG}(3, q)$, q even. If Ω_A and Ω_B do not share all of their tangents, then the shared tangents form a regular spread, the set of transversals of two skew lines, or $q + 1$ pencils sharing a line.*

Proof Denote by $\mathscr{T}_A$ the set of tangents of Ω_A and by $\mathscr{T}_B$ the set of tangents of Ω_B. By Theorem 1.2.18, the sets of lines $\mathscr{T}_A$ and $\mathscr{T}_B$ are general linear complexes, so the set of shared tangents is a linear congruence. A linear congruence contained in a general linear complex is one of the possibilities listed (see Section 0.4). □

Note that by Lemma 1.2.19, the tangents meeting a secant ℓ of an ovoid Ω in $\mathrm{PG}(3, q)$, q even, are the transversals of ℓ and its polar line. So, if the polar of a shared secant ℓ is the same with respect to both ovoids, then the two ovoids either share of all their tangents, or they share the tangents meeting ℓ. This information can be used to give some conditions for when two ovoids share all of their tangents.

Lemma 7.2.3 *Let* Ω_A *and* Ω_B *be two ovoids of* $\mathrm{PG}(3, q)$, *q even. Suppose there exist two shared points P and Q, each lying in a shared tangent plane. Then all tangents of* Ω_A *meeting the shared secant PQ are shared tangents.*

Proof Denote the polarity of Ω_A by σ_A, the polarity of Ω_B by σ_B, and the line PQ by ℓ. Since the shared point P lies on a shared tangent plane, it follows that $\sigma_A(P) = \sigma_B(P)$. Similarly $\sigma_A(Q) = \sigma_B(Q)$.
Thus $\sigma_A(\ell) = \sigma_A(PQ) = \sigma_A(P) \cap \sigma_A(Q) = \sigma_B(P) \cap \sigma_B(Q) = \sigma_B(PQ) = \sigma_B(\ell)$.

By Lemma 1.2.19, the tangents of Ω_A meeting ℓ are the transversals of ℓ and $\sigma_A(\ell)$, and the tangents of Ω_B meeting ℓ are the transversals of ℓ and $\sigma_B(\ell)$. But $\sigma_B(\ell) = \sigma_A(\ell)$, so the tangents of Ω_A meeting ℓ are also tangents of Ω_B. That is, they are shared tangents. □

Lemma 7.2.4 *Let* Ω_A *and* Ω_B *be two ovoids of* $\mathrm{PG}(3, q)$, *q even. Suppose that* Ω_A *and* Ω_B *have at least three shared points, each lying in a shared tangent plane. Then* Ω_A *and* Ω_B *share all of their tangents.*

Proof Let P, Q and R be three shared points, each lying in a shared tangent plane. By Lemma 7.2.3, the ovoids Ω_A and Ω_B share all tangents meeting the line PQ. By Lemma 1.2.19, these shared tangents are the transversals of two skew lines. Lemma 7.2.2 implies that if Ω_A and Ω_B share any further tangents then they must share *all* of their tangents. By Lemma 7.2.3, the ovoids Ω_A and Ω_B also share all tangents meeting QR and RP. Thus Ω_A and Ω_B share all of their tangents. □

Using the above relationships, it is possible to give bounds on the intersection of ovoids. First, the intersection of elliptic quadrics is described.

7.3 Intersections of elliptic quadrics.

In this section, some results are stated concerning the intersection of elliptic quadrics in $\mathrm{PG}(3,q)$.

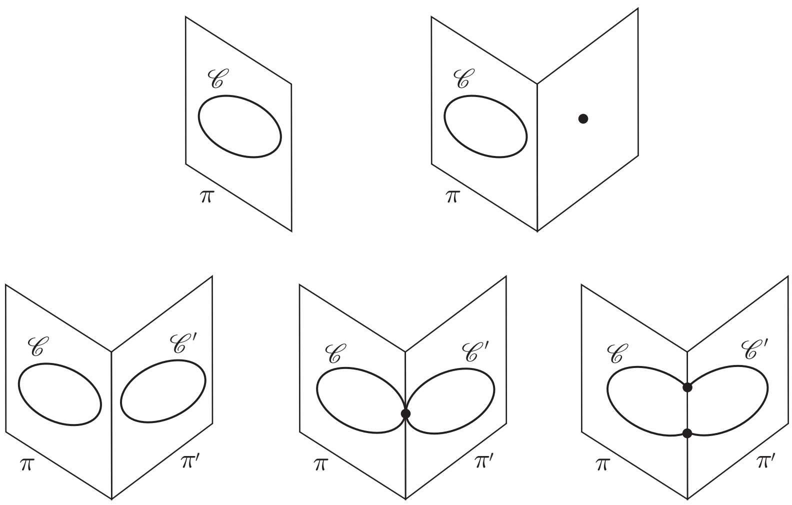
Let $\mathscr{E}_1$ and $\mathscr{E}_2$ be distinct elliptic quadrics of $\mathrm{PG}(3,q)$. The intersection of $\mathscr{E}_1$ and $\mathscr{E}_2$ is the set of points in $\mathrm{PG}(3,q)$ satisfying the equations of both $\mathscr{E}_1$ and $\mathscr{E}_2$. This set of points is known as a *quartic curve*. It is known that a quartic curve has at most $2(q+1)$ points (see [15]). Thus the following theorem holds

Lemma 7.3.1 [15] *Two distinct elliptic quadrics in* $\mathrm{PG}(3,q)$ *intersect in at most* $2(q+1)$ *points.*

It is important to note that the above bound is too large for the smallest values of q. If $q=2$, an elliptic quadric has 5 points and two distinct elliptic quadrics share at most 3 points. If $q=3$, an elliptic quadric has 10 points, and two distinct elliptic quadrics share at most 5 points. If $q=4$, an elliptic quadric has 17 points, and two distinct elliptic quadrics share at most 9 points.

If the intersection of the two quadrics is known to contain the points of a non-singular conic in a plane, then more can be said. The following lemma is taken from the results in [15].

Lemma 7.3.2 [15] *Let* $\mathscr{E}_1$ *and* $\mathscr{E}_2$ *be distinct elliptic quadrics of* $\mathrm{PG}(3,q)$. *Let* π *be a plane and let* $\mathscr{C}$ *be a non-singular conic in* π. *Suppose that* $\mathscr{E}_1 \cap \mathscr{E}_2$ *contains* $\mathscr{C}$. *Then* $\mathscr{E}_1 \cap \mathscr{E}_2$ *is one of the following sets of points.*



- *The non-singular conic* $\mathscr{C}$*,*
- *The non-singular conic* $\mathscr{C}$*, plus one point lying in a shared tangent plane,*
- *The non-singular conic* $\mathscr{C}$*, plus a second non-singular conic* $\mathscr{C}'$ *in a second plane* π'*.*

Note that the maximum possible intersection of two elliptic quadrics is provided by two non-singular conics $\mathscr{C}_1$ and $\mathscr{C}_2$ in two planes such that $\mathscr{C}_1$ and $\mathscr{C}_2$ are disjoint. It is important to note that if the points in $\mathrm{PG}(3, q^2)$ satisfying the equations of $\mathscr{E}_1$ and $\mathscr{E}_2$ are also considered, then the two conics $\mathscr{C}_1$ and $\mathscr{C}_2$ are no longer disjoint but share exaclty two points on the line $\pi \cap \pi'$.

The above statements apply when q is even or odd. When q is even, the two elliptic quadrics may share all of their tangents, which further restricts the possible intersections. The following theorem was proved by Glynn [34].

Theorem 7.3.3 *Let* $\mathscr{E}_1$ *and* $\mathscr{E}_2$ *be distinct elliptic quadrics of* $\mathrm{PG}(3, q)$*,* q *even. If* $\mathscr{E}_1$ *and* $\mathscr{E}_2$ *share all of their tangents, then they either meet in one point, or they meet in the* $q + 1$ *points of a non-singular conic section.*

7.4 Bounds on the intersection of ovoids.

In 1959, Segre [51] proved the following result about the size of the intersection of ovoids.

Theorem 7.4.1 [51] *Let* Ω_A *and* Ω_B *be distinct ovoids of* $\mathrm{PG}(3, q)$*. Then* Ω_A *and* Ω_B *share at most* $\frac{1}{2}q(q + 1) + 1$ *points.*

Proof Let Ω be an ovoid and let $\mathscr{K}$ be a set of $\frac{1}{2}q(q+1)+2$ points contained in Ω. It will be shown that if $\mathscr{K}$ is contained in a larger cap, then this cap is also contained in Ω. This will be done by showing that a point not on Ω lies on a secant of $\mathscr{K}$.

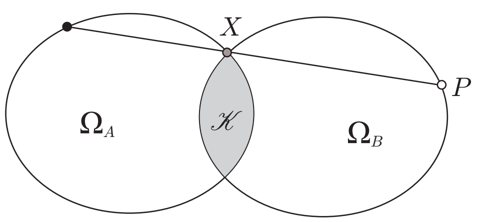
Let $P \notin \Omega$. There are $\frac{1}{2}q(q-1)$ secants of Ω and $q+1$ tangents of Ω through P. Thus there are $\frac{1}{2}q(q+1)+1$ lines through P meeting Ω. However, $\mathscr{K}$ contains $\frac{1}{2}q(q+1)+2$ points, and is contained in Ω. Hence, at least one of the lines through P has two points of $\mathscr{K}$.

Thus any cap containing $\mathscr{K}$ is contained in Ω. In particular, any ovoid containing $\mathscr{K}$ coincides with Ω. Hence, two distinct ovoids cannot share $\frac{1}{2}q(q+1)+2$ points.
$\square$

In the cases of q odd, $q = 2$ and $q = 4$, every ovoid is an elliptic quadric and the intersections of elliptic quadrics were described in Section 7.3. In all of these cases, the maximum number of points that two elliptic quadrics can share is smaller than the bound given in Theorem 7.4.1. For q even, $q > 4$, the bound given in Theorem 7.4.1 is the best existing bound for general ovoids.

If two ovoids share all of their tangents, then there is a smaller bound, proved by Glynn [34]. Glynn's proof used special representations of ovoids, but a more direct proof is given here.

Theorem 7.4.2 [34] *Let Ω_A and Ω_B be distinct ovoids of* $\mathrm{PG}(3,q)$, *q even, sharing all their tangents. Then Ω_A and Ω_B share at most $\frac{1}{2}q(q-1)$ points.*



Proof Let $\mathscr{K} = \Omega_A \cap \Omega_B$ and let P be a point of Ω_B not in Ω_A. Let X be a point of $\mathscr{K}$ and consider the line PX.

The line PX is a secant of Ω_B, so it contains no further point of $\mathscr{K}$. Since all tangents of Ω_A are tangents of Ω_B and PX is a secant of Ω_B, the line PX is not a tangent of Ω_A. Since PX contains the point X of Ω_A, the line PX is a secant of Ω_A. Hence, each point of $\mathscr{K}$ lies on a secant of Ω_A through P, and each such secant contains at most one point of $\mathscr{K}$. There are $\frac{1}{2}q(q-1)$ secants of Ω_A through P, so there are at most $\frac{1}{2}q(q-1)$ points of $\mathscr{K}$. □

Note that in the above proof, it is not actually necessary for Ω_A and Ω_B to share all of their tangents. It is only necessary for them to share the tangents through a point in one but not the other.

Theorems 7.4.1 and 7.4.2 are the best existing bounds for the size of the intersection of two ovoids. However, using Theorem 7.4.2 and the structure of the shared tangents of two ovoids, it is possible to improve the bound given in Theorem 7.4.1.

Theorem 7.4.3 *Let Ω_A and Ω_B be distinct ovoids of* $\mathrm{PG}(3,q)$*, q even. Then Ω_A and Ω_B share at most $\frac{1}{2}q^2+1$ points.*

Proof Suppose Ω_A and Ω_B share $\frac{1}{2}q^2+1+a$ points for $a \geq 1$. Then they also share $\frac{1}{2}q^2+1+a$ tangent planes by Lemma 7.2.1. Each of these tangent planes contains one point of Ω_A. There are $\frac{1}{2}q^2-a$ points of $\Omega_A \setminus \Omega_B$, so there are at most $\frac{1}{2}q^2-a$ shared tangent planes containing a point of $\Omega_A \setminus \Omega_B$. It follows that the number of shared tangent planes containing a point of $\Omega_A \cap \Omega_B$ is at least $2a+1 \geq 3$. Thus there exist three shared points each lying on a shared tangent plane. Lemma 7.2.4 then implies that Ω_A and Ω_B share all of their tangents.

Two distinct ovoids sharing all of their tangents share at most $\frac{1}{2}q(q-1)$ points by Theorem 7.4.2. But Ω_A and Ω_B share $\frac{1}{2}q^2+1+a$ points, which is greater than $\frac{1}{2}q(q-1)$. This is a contradiction, so Ω_A and Ω_B share at most $\frac{1}{2}q^2+1$ points. □

Note that T. Penttila (personal communication) has found examples of Tits ovoids in $\mathrm{PG}(3,8)$ sharing exactly $\frac{1}{2}\times 8^2+1=33$ points, so the above bound is sharp. That is, it is the best possible bound for at least the cases $q=2$, 4 and 8.

The bound given in Theorem 7.4.3 will be used later in this chapter to investigate the structure of the external lines and secants that two ovoids share.

7.5 Ovoidal fibrations

Thus far, upper bounds on the number of points that two distinct ovoids share have been considered. In Section 7.6, further results will be proved on ovoids sharing all their tangents. In order to do this, certain sets of mutually disjoint ovoids are used.

Definition 7.5.1 *A set of $q+1$ mutually disjoint ovoids of* $\mathrm{PG}(3,q)$ *is called an* ovoidal fibration *of* $\mathrm{PG}(3,q)$.

Note that since the number of points in $\mathrm{PG}(3,q)$ is $(q+1)(q^2+1)$, the ovoids of an ovoidal fibration partition the points of $\mathrm{PG}(3,q)$. Ebert [30] first constructed an ovoidal fibration using the orbits of a subgroup of a cyclic Singer group. In [22], these ovoids were shown to be elliptic quadrics. Cossidente and Veerecke [23] used group theory to construct an ovoidal fibration of $\mathrm{PG}(3,8)$ using both elliptic quadrics and Tits ovoids.

The first result in this section shows that when q is even, it is possible to construct an ovoidal fibration from any ovoid. This result was mentioned without proof in [28].

Theorem 7.5.2 *Let Ω be an ovoid of* $\mathrm{PG}(3,q)$, *q even, and let $\mathscr{S}$ be a regular spread contained in the set of tangents of Ω. Let G be the kernel of $\mathscr{S}$, and let $G(\Omega) = \{g(\Omega) \mid g \in G\}$ be the set of images of Ω under G. Then $G(\Omega)$ is an ovoidal fibration.*

Proof Firstly note that the set of tangents of Ω is a general linear complex by Lemma 1.2.18. This general linear complex contains many elliptic linear congruences. An elliptic linear congruence is a regular spread, so there do exist regular spreads contained in the set of tangents of Ω.

The group G is the set of collineations fixing each line of the regular spread $\mathscr{S}$, so by Lemma 0.4.10, $|G| = q+1$. Thus $G(\Omega)$ is a set of $q+1$ ovoids. It is required to show that any two ovoids of $G(\Omega)$ are disjoint.

Let g_1 and g_2 be collineations in G and suppose P is a point of $g_1(\Omega) \cap g_2(\Omega)$. Then $P = g_1(Q_1)$ for some $Q_1 \in \Omega$ and also $P = g_2(Q_2)$ for some $Q_2 \in \Omega$. Thus $Q_1 = g_1^{-1}(P)$ and $Q_2 = g_2^{-1}(P)$.

Let ℓ be the unique line of $\mathscr{S}$ through P. Then ℓ is fixed by both g_1^{-1} and g_2^{-1}, since these are collineations in G. Thus $g_1^{-1}(P) = Q_1 \in \ell$ and $g_2^{-1}(P) = Q_2 \in \ell$. However, ℓ is a tangent of Ω and so contains exactly one point of Ω. Since Q_1 and Q_2 are both points of Ω on ℓ, it follows that $Q_1 = Q_2$.

Now $g_2(Q_2) = P$ and $g_1(Q_2) = g_1(Q_1) = P$. By Lemma 0.4.7, no two collineations in G send Q_2 to the same point. Thus $g_1 = g_2$ and $g_1(\Omega) = g_2(\Omega)$.

Hence, two distinct ovoids of $G(\Omega)$ are disjoint, and so $G(\Omega)$ is an ovoidal fibration.

□

If the ovoid Ω is an elliptic quadric and the regular spread $\mathscr{S}$ is chosen correctly, then the above construction produces the ovoidal fibration originally constructed by Ebert in [30]. There are some very special properties of this fibration, which will be described now.

Lemma 7.5.3 *Let $\mathscr{E}$ be an elliptic quadric of* PG$(3, q)$*, q even. Then there exists a cyclic Singer group S of* PG$(3, q)$ *with the following properties:*

- *S contains a subgroup G of order $q+1$, which is the kernel of a regular spread $\mathscr{S}$ contained in the set of tangents of $\mathscr{E}$;*
- *S contains a subgroup H of order $q^2 + 1$ such that the orbits of H are the elliptic quadrics in the ovoidal fibration $G(\mathscr{E})$, including $\mathscr{E}$ itself;*
- *Each line of* PG$(3, q)$ *not in $\mathscr{S}$ is a tangent of exactly one of the elliptic quadrics of $G(\mathscr{E})$.*

Proof Let S be a cyclic Singer group of $\mathrm{PG}(3,q)$. Since S is cyclic, there is a unique subgroup of S of order d for each number d dividing the order of S (Lemma 0.1.6). Now $|S| = q^3+q^2+q+1 = (q+1)(q^2+1)$, so S has unique subgroups of size $q+1$ and q^2+1. Denote these subgroups by G and H respectively.

In [33], Glynn showed that the orbits of G form a regular spread (Lemma 0.4.11). That is, each orbit forms a line of $\mathrm{PG}(3,q)$ and the set of these lines is a regular spread. In [30], Ebert showed that the orbits of the group H form an ovoidal fibration. That is, the orbits of H are disjoint ovoids partitioning the points of $\mathrm{PG}(3,q)$. In [22], it was shown that these ovoids are in fact elliptic quadrics.

Let $\mathscr{S}$ be the regular spread formed by the orbits of G and let $\mathscr{E}'$ be one of the orbits of H. Note that since G is of order $q+1$, it is the kernel of $\mathscr{S}$ by Corollary 0.4.8.

Let $P \in \mathscr{E}'$ and let ℓ be the unique line of the spread $\mathscr{S}$ through P. Suppose Q is a point of $\mathscr{E}'$ on ℓ. Since $\mathscr{E}'$ is an orbit of H, $Q = h(P)$ for some $h \in H$. On the other hand, since ℓ is an orbit of G, $Q = g(P)$ for some $g \in G$. Then $g(P) = h(P)$. From the definition of a cyclic Singer group (Definition 0.3.6), it follows that $g = h$. The element g is contained in both G and H, so its order divides both $q+1$ and q^2+1 by Corollary 0.1.4. However, $q+1$ and q^2+1 are coprime when q is even, so the order of g is 1. That is, $g = \iota$ and so $P = Q$. Hence, the line ℓ contains exactly one point of $\mathscr{E}'$. Since there are q^2+1 lines of $\mathscr{S}$ and q^2+1 points of $\mathscr{E}'$, this implies that the regular spread $\mathscr{S}$ is contained in the set of tangents of $\mathscr{E}'$.

Let $g \in G$ and $h \in H$. Then $hg = gh$, since g and h are elements of the abelian group S. Hence $h(g(\mathscr{E}')) = g(h(\mathscr{E}')) = g(\mathscr{E}')$, since $\mathscr{E}'$ is an orbit of H. Thus $g(\mathscr{E}')$ is fixed by every element of H. Since no element of H fixes any point and $|H| = |g(\mathscr{E}')| = q^2+1$, it follows that $g(\mathscr{E}')$ is an orbit of H. That is, the orbits of H are the elliptic quadrics in the fibration $G(\mathscr{E}')$.

In [30], it was shown that the number of lines that are tangents of all of the elliptic quadrics in the fibration $G(\mathscr{E}')$ is q^2+1, and that every other line of $\mathrm{PG}(3,q)$ is a tangent of exactly one of the elliptic quadrics. That is, any line not in $\mathscr{S}$ is a tangent of exactly one elliptic quadric of the fibration $G(\mathscr{E}')$.

Finally, every elliptic quadric of $\mathrm{PG}(3,q)$ is projectively equivalent to $\mathscr{E}'$, so the result follows. □

This section is completed with a result on general ovoidal fibrations. In the above lemma, the lines of the regular spread $\mathscr{S}$ are the shared tangents of the elliptic quadrics in the ovoidal fibration. Also, every line not in $\mathscr{S}$ is a tangent of one of the elliptic quadrics in the fibration. This property is also true of the other existing examples of ovoidal fibrations, even those containing two different types of ovoids. The result below shows that it is true for *any* ovoidal fibration.

Theorem 7.5.4 *Let $\mathscr{F}$ be an ovoidal fibration of* $\mathrm{PG}(3, q)$*, q even. Then there exists a regular spread $\mathscr{S}$ such that $\mathscr{S}$ is the set of shared tangents of every pair of ovoids of $\mathscr{F}$. Moreover, every line of* $\mathrm{PG}(3, q)$ *not in $\mathscr{S}$ is a tangent of exactly one of the ovoids in $\mathscr{F}$.*

Proof Denote the $q+1$ ovoids of the fibration $\mathscr{F}$ by Ω_0, Ω_1, …, Ω_q. Let π be a plane and let s be the number of ovoids Ω_i from $\mathscr{F}$ such that π is a secant plane of Ω_i. The remaining $q+1-s$ ovoids of $\mathscr{F}$ have π as a tangent plane. The ovoids of $\mathscr{F}$ partition the points of π, so $s(q+1)+(q+1-s)\cdot 1 = q^2+q+1$. It follows that $s = q$. Thus every plane is a tangent plane of exactly one ovoid of $\mathscr{F}$.

Let ℓ be a line of $\mathrm{PG}(3, q)$ and suppose ℓ is not a tangent of any ovoid of $\mathscr{F}$. Then ℓ contains 0 or 2 points of each ovoid of $\mathscr{F}$. This implies there are an even number of points of ℓ lying on the ovoids of $\mathscr{F}$. Since the ovoids of $\mathscr{F}$ partition all points of $\mathrm{PG}(3, q)$, this implies there are an even number of points on ℓ. However, ℓ contains $q+1$ points, which is odd. This is a contradiction, so ℓ is a tangent of at least one ovoid of $\mathscr{F}$.

Suppose π is a tangent plane of Ω_j and denote the point $\pi\cap\Omega_j$ by N_j. The plane π is a secant plane of Ω_i for $i \neq j$. For each $i \neq j$, denote the oval where π meets Ω_i by $\mathscr{O}_i$ and denote the nucleus of $\mathscr{O}_i$ by N_i. Since every line of $\mathrm{PG}(3, q)$ is a tangent of at least one ovoid of $\mathscr{F}$, it follows that every line of π passes through at least one of the points N_0, …, N_q. Denote by $\mathscr{N}$ the set of points $\{N_0, \ldots, N_q\}$.

Let P be a point of π not in the set $\mathscr{N}$. Every line through P contains at least one point of $\mathscr{N}$. There are $q+1$ lines through P, so every line through P contains exactly one point of $\mathscr{N}$. This implies that the $q+1$ points of $\mathscr{N}$ are distinct. It also implies that if a line contains a point not in $\mathscr{N}$, then it contains exactly one point of $\mathscr{N}$. The line joining N_0 and N_q contains two points of $\mathscr{N}$, and so every point on this line is a point of $\mathscr{N}$. Thus the set $\mathscr{N}$ is a line. The line $\mathscr{N}$ is necessarily a tangent of each ovoid of $\mathscr{F}$. Each of the remaining lines of π is a tangent of exactly one ovoid of $\mathscr{F}$.

Thus, every plane contains one line which is a tangent of every ovoid of $\mathscr{F}$. This implies that the set of lines that are tangents of every ovoid of Ω forms a spread $\mathscr{S}$. Every line not in this spread is a tangent of exactly one ovoid of $\mathscr{F}$. Thus $\mathscr{S}$ is the set of shared tangents of any two ovoids of $\mathscr{F}$. Since the set of shared tangents of two ovoids forms a linear congruence, the spread $\mathscr{S}$ is a *regular* spread. □

The above result will be useful in future research on the possible fibrations that can exist. The fibrations themselves are useful for investigating the intersection of ovoids, which is the subject of the following section.

7.6 The intersection of ovoids sharing all tangents

The question of how two ovoids may meet is a very difficult one, and is not even fully solved for elliptic quadrics. Many authors have considered the question under the assumption that the two ovoids share all of their tangents. In this section, some existing results about this question will be generalised.

In his PhD thesis, Glynn [34] showed that two distinct elliptic quadrics sharing all of their tangents meet in 1 or $q+1$ points. Bagchi and Sastry [1] showed that an elliptic quadric and a Tits ovoid sharing all of their tangents meet in $q \pm \sqrt{2q}+1$ points. If it is only assumed that *one* of the ovoids is of a known type, then the exact size of the intersection has not yet been isolated. However, Bagchi and Sastry [2] and Ball [3] have showed the following result, each using different methods.

Theorem 7.6.1 [2, 3] *Let Ω_A and Ω_B be ovoids of* $\mathrm{PG}(3,q)$*, q even, sharing all of their tangents, and suppose that Ω_A is either an elliptic quadric or a Tits ovoid. Then Ω_A and Ω_B share an odd number of points. In particular, they share at least one point.*

Here, Theorem 7.6.1 is generalised to *any* pair of ovoids sharing all of their tangents. That is, it is not necessary to assume one is an elliptic quadric or a Tits ovoid.

Theorem 7.6.2 *Let Ω_A and Ω_B be ovoids of* PG$(3, q)$*, q even, sharing all of their tangents. Then Ω_A and Ω_B share an odd number of points. In particular, they share at least one point.*

Proof Let Ω_A and Ω_B be two ovoids sharing all of their tangents. Let ρ be the null polarity defined by their shared general linear complex of tangents, and let $\mathscr{S}$ be a regular spread contained in this linear complex. Let G be the kernel of $\mathscr{S}$ and let $G(\Omega_B) = \{g(\Omega_B) \mid g \in G\}$ be the set of images of Ω_B under G. Then the $q+1$ ovoids of $G(\Omega_B)$ form an ovoidal fibration by Theorem 7.5.2. Thus the ovoids partition all of the points of PG$(3, q)$ and the points of Ω_A in particular.

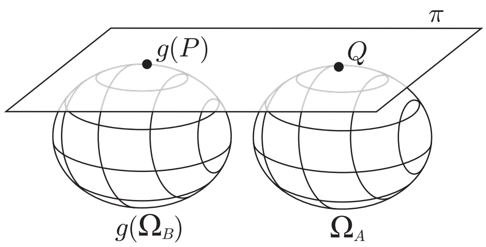
For each $g \in G$ define $x_g = |g(\Omega_B) \cap \Omega_A|$. In particular, $x_\iota = |\Omega_B \cap \Omega_A|$, where ι is the identity collineation. Since the ovoids of $G(\Omega_B)$ partition the points of Ω_A,

$$\sum_{g \in G} x_g = |\Omega_A|$$

$$x_\iota + \sum_{g \in G \setminus \{\iota\}} x_g = q^2 + 1$$

$$|\Omega_B \cap \Omega_A| + \sum_{g \in G \setminus \{\iota\}} x_g = q^2 + 1. \tag{7.1}$$

It will be shown that the sum $\sum_{g \in G \setminus \{\iota\}} x_g$ on the left-hand side of Equation 7.1 is an even number. In order to show this, it will be proved that $x_g = x_{g^{-1}}$ for any $g \in G$. Let $g \in G$ and consider the ovoids $g(\Omega_B)$ and Ω_A. These ovoids share x_g points, and Lemma 7.2.1 implies that they also share x_g tangent planes.



Let π be one of these shared tangent planes. Then π contains one point of $g(\Omega_B)$ and one point of Ω_A. Let $g(P)$ be the point of $g(\Omega_B)$ in π (for some $P \in \Omega_B$), and let Q be the point of Ω_A in π.

The tangent plane of Ω_A at Q is the image of Q under the polarity of Ω_A. That is, $\pi = \rho(Q)$. Similarly, the tangent plane of Ω_B at P is $\rho(P)$, so the tangent plane of $g(\Omega_B)$ at $g(P)$ is $g\rho(P)$. That is, $\pi = g\rho(P)$. Hence $\rho(Q) = g\rho(P) = \rho g^{-1}(P)$ by Corollary 0.4.12. Thus $Q = g^{-1}(P)$ and Q is a point of $g^{-1}(\Omega_B) \cap \Omega_A$.

Each tangent plane of Ω_A contains a different point of Ω_A, so the x_g shared tangent planes of $g(\Omega_B)$ and Ω_A define x_g points of $g^{-1}(\Omega_B) \cap \Omega_A$. Thus $x_{g^{-1}} \geq x_g$. However, the same argument beginning with g^{-1} shows that $x_g \geq x_{g^{-1}}$, so $x_g = x_{g^{-1}}$ for all $g \in G$.

Suppose $g = g^{-1}$. Then $g^2 = \iota$, so if $g \neq \iota$, the collineation g has order 2. However, the order of g divides the order of G by Corollary 0.1.4. The order of G is $q + 1$, which is an odd number, so g has odd order. Hence, $g = \iota$. That is, $g = g^{-1}$ if and only if $g = \iota$.

Hence, every summand in $\sum_{g \in G \setminus \{\iota\}} x_g$ occurs twice, and so the sum is an even number. Since q is even, the number $q^2 + 1$ is odd, so Equation 7.1 implies that $|\Omega_B \cap \Omega_A|$ is odd. That is, Ω_A and Ω_B share an odd number of points. In particular, Ω_A and Ω_B share at least one point. □

The above result applies to any pair of ovoids sharing all tangents. If one of the ovoids is assumed to be an elliptic quadric, then it is possible to strengthen the statement slightly. In [4], the following result was shown.

Theorem 7.6.3 [4] *Let Ω be an ovoid of* $\mathrm{PG}(3, q)$*, q even. Then either* $|\Omega \cap \mathscr{E}| \equiv 1 \pmod 4$ *for all elliptic quadrics $\mathscr{E}$ sharing all tangents with Ω, or* $|\Omega \cap \mathscr{E}| \equiv 3 \pmod 4$ *for all elliptic quadrics $\mathscr{E}$ sharing all tangents with Ω.*

Using the ovoidal fibration as before, and the cyclic Singer group associated with the elliptic quadric as described in Lemma 7.5.3, it will be shown that only one of these cases actually occurs. That is, the following result is proved.

Theorem 7.6.4 *Let Ω be an ovoid of* $\mathrm{PG}(3, q)$*, q even, and let $\mathscr{E}$ be an elliptic quadric sharing all tangents with Ω. Then* $|\Omega \cap \mathscr{E}| \equiv 1 \pmod 4$.

Proof By Lemma 7.5.3, there exists a cyclic Singer group S of $\mathrm{PG}(3, q)$ with the following properties:

- S contains a subgroup G of order $q + 1$ which is the kernel of a regular spread $\mathscr{S}$ contained in the set of tangents of $\mathscr{E}$,

- S contains a subgroup H of order q^2+1 such that the orbits of H are the elliptic quadrics in the ovoidal fibration $G(\mathscr{E})$, including $\mathscr{E}$ itself,
- Each line of $\mathrm{PG}(3,q)$ not in $\mathscr{S}$ is a tangent of exactly one of the elliptic quadrics of $G(\mathscr{E})$.

The two subgroups G and H will be used to prove the result. For each $g \in G$, define $x_g = |\Omega \cap g(\mathscr{E})|$. In particular, if $g = \iota$, then $x_\iota = |\Omega \cap \mathscr{E}|$. The elliptic quadrics of $G(\mathscr{E})$ partition the points of Ω, so

$$x_\iota + \sum_{g \in G \setminus \{\iota\}} x_g = q^2 + 1. \tag{7.2}$$

From the proof of Theorem 7.6.2, $x_g = x_{g^{-1}}$ for all $g \in G$. Since the order of G is odd, $g = g^{-1}$ if and only if $g = \iota$, so G may be partitioned into ι and $\frac{1}{2}q$ pairs $\{g, g^{-1}\}$. Let A be a subset of G formed by choosing one member from each of these pairs. Then Equation 7.2 becomes

$$x_\iota + 2\sum_{g \in A} x_g = q^2 + 1. \tag{7.3}$$

It will be shown that x_g is even for each $g \neq \iota$.

Let $g \in G$ and let P be a point of $g(\mathscr{E})$. Let $H_P = \{h \in H \mid \ P \in h(\Omega)\ \}$ and denote $|H_P|$ by m. Let Q be another point of $g(\mathscr{E})$. Since $g(\mathscr{E})$ is an orbit of H, there exists a collineation $h' \in H$ such that $Q = h'(P)$. The point P lies on $h(\Omega)$ if and only if $h \in H_P$. So the point $Q = h'(P)$ lies on $h'h(\Omega)$ if and only if $h \in H_P$. Hence, for each point $Q \in g(\mathscr{E})$ there are $|H_P| = m$ collineations $h \in H$ such that $Q \in h(\Omega)$.

Let $h \in H$ and consider the intersection of the ovoid $h(\Omega)$ with $g(\mathscr{E})$:

$$\begin{aligned} |h(\Omega) \cap g(\mathscr{E})| &= |h^{-1}h(\Omega) \cap h^{-1}g(\mathscr{E})| \\ &= |\Omega \cap g(\mathscr{E})| \qquad \text{since } g(\mathscr{E}) \text{ is an orbit of } H. \\ &= x_g \end{aligned}$$

Thus, $|h(\Omega) \cap g(\mathscr{E})| = x_g$ for every $h \in H$. Next is is shown that $m = x_g$.

Consider the set of pairs $X = \{(P,h) \mid \ P \in g(\mathscr{E}),\ h \in H \text{ and } P \in h(\Omega)\ \}$. The size of X will be counted in two ways. Counting P then h, there are q^2+1 points

$P \in g(\mathscr{E})$, and for each of these points there are m collineations $h \in H$ such that $P \in h(\Omega)$. So $|X| = (q^2+1)m$. Counting h then P, there are q^2+1 collineations $h \in H$ and for each of these collineations there are x_g points of $g(\mathscr{E})$ in $h(\Omega)$. So $|X| = (q^2+1)x_g$. Thus $(q^2+1)m = (q^2+1)x_g$, and it follows that $m = x_g$.

Suppose $g \neq \iota$ and let ℓ be a tangent of $g(\mathscr{E})$ not in the regular spread $\mathscr{S}$. The spread $\mathscr{S}$ is the set of shared tangents of any pair of elliptic quadrics of the fibration $G(\mathscr{E})$. So, the line ℓ is not a tangent of any elliptic quadric in $G(\mathscr{E})$ except $g(\mathscr{E})$. That is, ℓ is either a secant or an external line of $g'(\mathscr{E})$ for each $g' \in G \setminus \{g\}$. Let $B = \{g' \in G \mid \ \ell \text{ is a secant of } g'(\mathscr{E})\ \}$.

The ovoids $\mathscr{E}$ and Ω share all of their tangents, so ℓ is not a tangent of Ω, since it is not tangent of $\mathscr{E}$. Also, every collineation $h \in H$ fixes $\mathscr{E}$, and so fixes its set of tangents. Thus any ovoid $h(\Omega)$ for $h \in H$ has the same set of tangents as $\mathscr{E}$. Hence, ℓ is not tangent of any ovoid $h(\Omega)$ for $h \in H$. That is, for each $h \in H$, the line ℓ is either a secant or an external line of $h(\Omega)$. Let s be the number of collineations $h \in H$ such that ℓ is a secant of $h(\Omega)$.

Consider the set $Y = \{(P,h) \mid \ P \in \ell,\, h \in H \text{ and } P \in h(\Omega)\ \}$. The size of Y will be counted in two ways. Counting P then h, there is one point on ℓ in $g(\mathscr{E})$ and there are two points on ℓ in each of the ovoids $g'(\mathscr{E})$ for $g' \in B$. For a point $P \in g'(\mathscr{E})$, there are $x_{g'}$ collineations $h \in H$ such that $P \in h(\Omega)$, so $|Y| = x_g + 2\sum_{g' \in B} x_{g'}$. Counting h then P, there are s collineations $h \in H$ such that $h(\Omega)$ contains two points of ℓ. The remaining collineations $h \in H$ are such that $h(\Omega)$ contains no points of ℓ. So $|Y| = 2s$. Thus $x_g + 2\sum_{g' \in B} x_{g'} = 2s$. From this equation, it follows that x_g is even for $g \neq \iota$. In light of this, write $x_g = 2z_g$ for each $g \in G \setminus \{\iota\}$.

Substituting these values into Equation 7.3 gives:

$$\begin{aligned} x_\iota + 2\sum_{g \in A} x_g &= q^2+1 \\ x_\iota + 2\sum_{g \in A} 2z_g &= q^2+1 \\ x_\iota + 4\sum_{g \in A} z_g &= q^2+1 \end{aligned}$$

Since q is even, the number q^2 is a multiple of 4, so this equation implies that $x_\iota \equiv 1 \pmod 4$. That is, $|\Omega \cap \mathscr{E}| \equiv 1 \pmod 4$. $\square$

7.7 The external lines or secants that two ovoids may share

Let Ω be an ovoid of $\mathrm{PG}(3,q)$ and let $\mathscr{L}$ be the set of external lines of Ω. In Theorem 3.2.6, the set of points lying on no lines of $\mathscr{L}$ was shown to be an ovoid. Thus Ω is the only ovoid with $\mathscr{L}$ as its set of external lines. Another way to say this is to say that two ovoids cannot share all of their external lines. In this section, further restrictions are placed on the set of external lines and secants that two ovoids may share. This is done by finding subsets $\mathscr{L}'$ of $\mathscr{L}$ such that Ω is the only ovoid containing $\mathscr{L}'$ among its external lines. For the first result, $\mathscr{L}'$ is the set of external lines meeting a line.

Theorem 7.7.1 *Let Ω be an ovoid of $\mathrm{PG}(3,q)$, and let m be any line of $\mathrm{PG}(3,q)$. Denote by $\mathscr{L}$ the set of external lines of Ω meeting m, and by $\mathscr{B}$ the set of points lying on no line of $\mathscr{L}$. If q is odd, then $\mathscr{B} = \Omega$. If q is even and m is a tangent of Ω, then $\mathscr{B} = \Omega$. If q is even and m is not a tangent of Ω, then $\mathscr{B}$ is Ω, plus the set of points on the polar of ℓ.*

Proof Let P be a point of $\mathscr{B}$. That is, P is a point lying on no lines of $\mathscr{L}$. Since $\mathscr{L}$ is the set of all external lines meeting m, if $P \in m$, then it lies on no external lines of Ω and so $P \in \Omega$.

Suppose $P \notin m$, and denote the plane $P \oplus m$ by π. The external lines of Ω in π all meet m, and so the lines of $\mathscr{L}$ in π are the external lines of Ω in π.

Suppose π is a tangent plane. The only point of π not lying on any external lines in π is the point where π meets Ω, so $P \in \Omega$.

Suppose π is a secant plane meeting Ω in the oval $\mathscr{O}$. Then the lines of $\mathscr{L}$ in π are the external lines of $\mathscr{O}$ in π.

If q is odd, the points in π lying on no external lines of $\mathscr{O}$ are the points of $\mathscr{O}$. Thus $P \in \Omega$ and it follows that $\mathscr{B} = \Omega$ for q odd.

If q is even, then the points in π lying on no external lines of $\mathscr{O}$ are the points of $\mathscr{O}$, plus the nucleus of $\mathscr{O}$. Denote by σ the polarity associated with Ω. Then the nucleus of $\mathscr{O}$ is the point $\sigma(\pi)$. Thus either $P \in \Omega$, or $P = \sigma(\pi)$. The line $\sigma(m)$ is the union of the points $\sigma(\pi)$ for the planes π through m. So $\mathscr{B} \subseteq \Omega \cup \sigma(m)$.

If m is a tangent, then each point on m not on Ω lies on an external line of Ω. Thus $\mathscr{B} = \Omega$. If m is not a tangent, then $\sigma(m)$ is skew to m, and the transversals of m and $\sigma(m)$ are tangents. Thus $\mathscr{B} = \Omega \cup \sigma(m)$. □

In the above lemma, for q even, the set of points on no lines of $\mathscr{L}$ may be larger than the ovoid Ω. However, Ω is still the only ovoid with $\mathscr{L}$ contained in its set of external lines. Thus the following corollaries are proved:

Corollary 7.7.2 *Let Ω be an ovoid of* $\mathrm{PG}(3, q)$ *and let m be a line of* $\mathrm{PG}(3, q)$*. Denote by $\mathscr{L}$ the set of external lines of Ω meeting m. Then Ω is the only ovoid of* $\mathrm{PG}(3, q)$ *containing $\mathscr{L}$ among its external lines*

Corollary 7.7.3 *Let Ω_A and Ω_B be distinct ovoids of* $\mathrm{PG}(3, q)$ *and let m be a line of* $\mathrm{PG}(3, q)$*. There exists an external line of Ω_A meeting m which is not an external line of Ω_B.*

Using the dual nature of an ovoid, it is possible to prove the following results for secants.

Corollary 7.7.4 *Let Ω be an ovoid of* $\mathrm{PG}(3, q)$ *and let m be a line of* $\mathrm{PG}(3, q)$*. Let $\mathscr{S}$ be the set of secants of Ω meeting m. Then Ω is the only ovoid containing $\mathscr{S}$ among its secants.*

Proof Denote the set of tangent planes of Ω by Ω^*. Then Ω^* is an ovoid of the dual space $\mathrm{PG}(3, q)^*$, and the secants of Ω in $\mathrm{PG}(3, q)$ are the external lines of Ω^* in $\mathrm{PG}(3, q)^*$ (see the discussion after Lemma 1.2.16). Thus, $\mathscr{S}$ is the set of lines of $\mathrm{PG}(3, q)^*$ meeting m that are external lines of Ω^*. By Corollary 7.7.3, Ω^* is the only ovoid of $\mathrm{PG}(3, q)^*$ containing $\mathscr{S}$ among its external lines.

The ovoid Ω of $\mathrm{PG}(3, q)$ is the set of tangent planes of the ovoid Ω^* in $\mathrm{PG}(3, q)^*$, so Ω is uniquely determined by Ω^*. Thus Ω is the only ovoid containing $\mathscr{S}$ among its secants. □

Corollary 7.7.5 *Let Ω_A and Ω_B be distinct ovoids of* $\mathrm{PG}(3,q)$ *and let m be a line of* $\mathrm{PG}(3,q)$*. There exists a secant of Ω_A meeting m which is not a secant of Ω_B.*

It is natural to ask whether similar results may be proved if $\mathscr{L}$ is only a subset of the external lines meeting m. This question is answered by the following theorem for q even.

Theorem 7.7.6 *Let Ω be an ovoid of* $\mathrm{PG}(3,q)$*, q even.*

(i) Let m be a line of $\mathrm{PG}(3,q)$ *and let Y be a set of $\frac{1}{2}q+1$ secant planes of Ω through m. Let $\mathscr{S}$ be the set of secants of Ω in the planes of Y. Then Ω is the only ovoid containing $\mathscr{S}$ among its secants.*

(ii) Let ℓ be a line of $\mathrm{PG}(3,q)$ *and let X be a set of $\frac{1}{2}q+1$ points of ℓ not on Ω. Let $\mathscr{L}$ be the set of external lines of Ω passing through the points of X. Then Ω is the only ovoid containing $\mathscr{L}$ among its external lines.*

Proof (i) Suppose Ω' is an ovoid containing $\mathscr{S}$ among its secants. Let π be a plane of the set Y and denote the oval where π meets Ω by $\mathscr{O}$. By Lemma 7.1.3, the only oval with the same secants as $\mathscr{O}$ is $\mathscr{O}$ itself. Thus the ovoid Ω' also meets π in the oval $\mathscr{O}$.

If m is a secant then each secant plane through m contains $q-1$ points of Ω not on m. Since Ω and Ω' coincide in the planes of Y, this implies that $|\Omega\cap\Omega'|\geq(\frac{1}{2}q+1)(q-1)+2=\frac{1}{2}q^2+\frac{1}{2}q+1>\frac{1}{2}q^2+1$. If m is a tangent, then each secant plane through it contains q points of Ω not on m. Thus $|\Omega\cap\Omega'|\geq(\frac{1}{2}q+1)q+1>\frac{1}{2}q^2+1$. Finally, if m is an external line, then each secant plane through it contains $q+1$ points of Ω not on m. Thus $|\Omega\cap\Omega'|\geq(\frac{1}{2}q+1)(q+1)>\frac{1}{2}q^2+1$. Hence Ω and Ω' coincide by Theorem 7.4.3.

(ii) Denote the set of tangent planes of Ω by Ω^*. Then Ω^* is an ovoid of the dual space $\mathrm{PG}(3,q)^*$, and the external lines of Ω in $\mathrm{PG}(3,q)$ are the secants of Ω^* in $\mathrm{PG}(3,q)^*$. Also, the points not on Ω are the secant planes of Ω^* in $\mathrm{PG}(3,q)^*$. Thus, the set X is a set of $\frac{1}{2}q+1$ secant planes of $\mathrm{PG}(3,q)^*$ through ℓ. The set $\mathscr{L}$ is the set of secants of Ω^* in these planes. Thus Ω^* is the only ovoid of $\mathrm{PG}(3,q)^*$ containing $\mathscr{L}$ among its secants. Since Ω is the set of tangent planes of Ω^* in $\mathrm{PG}(3,q)^*$, it is uniquely determined by Ω^*. Thus Ω is the only ovoid of $\mathrm{PG}(3, q)$ containing $\mathscr{L}$ among its external lines. $\square$

For q odd, all ovoids are elliptic quadrics, so there is a more accurate version of the above result for q odd. It also applies to elliptic quadrics in the q even case.

Theorem 7.7.7 *Let $\mathscr{E}$ be an elliptic quadric of $\mathrm{PG}(3, q)$.*

(i) Let α, β and γ be three distinct secant planes of $\mathscr{E}$ and let $\mathscr{S}$ be the set of secants of $\mathscr{E}$ in α, β and γ. Then $\mathscr{E}$ is the only ovoid of $\mathrm{PG}(3, q)$ containing $\mathscr{S}$ among its secants.

(ii) Let P, Q and R be three distinct points not on $\mathscr{E}$ and let $\mathscr{L}$ be the set of external lines of $\mathscr{E}$ through P, Q and R. Then $\mathscr{E}$ is the only ovoid of $\mathrm{PG}(3, q)$ containing $\mathscr{L}$ among its external lines.

Proof (i) Suppose Ω is an ovoid of $\mathrm{PG}(3, q)$ containing $\mathscr{S}$ among its secants. Consider the non-singular conic $\alpha \cap \mathscr{E}$. By Lemma 7.1.3, the only oval with the same secants as $\alpha \cap \mathscr{E}$ is $\alpha \cap \mathscr{E}$. Thus α meets Ω in the non-singular conic $\alpha \cap \mathscr{E}$. If q is even, then Theorem 2.3.4 implies that Ω is an elliptic quadric.

Since the two elliptic quadrics $\mathscr{E}$ and Ω share the secants in the planes α, β and γ, Lemma 7.1.3 implies that the two elliptic quadrics coincide in these three planes. By Lemma 7.3.2, it follows that $\mathscr{E} = \Omega$.

(ii) Denote the set of tangent planes of $\mathscr{E}$ by $\mathscr{E}^*$. Then $\mathscr{E}^*$ is an elliptic quadric of the dual space $\mathrm{PG}(3, q)^*$. The set of lines $\mathscr{L}$ is the set of secants of $\mathscr{E}^*$ in the three secant planes P, Q and R of $\mathscr{E}^*$ in $\mathrm{PG}(3, q)^*$. Thus $\mathscr{E}^*$ is the only ovoid of $\mathrm{PG}(3, q)^*$ containing $\mathscr{L}$ among its secants. Since $\mathscr{E}$ is uniquely determined by $\mathscr{E}^*$, it follows that $\mathscr{E}$ is the only ovoid of $\mathrm{PG}(3, q)$ containing $\mathscr{L}$ among its external lines. $\square$

Included in the above proof is the remarkable fact that if an ovoid and an elliptic quadric share all the external lines through a point not on either, then the ovoid is also an elliptic quadric. This completes the discussion of the points and lines that two ovoids may share.

Conclusion

In this thesis, several new results have been proved characterising sets of subspaces associated with quadrals. Existing results have characterised the set of tangents and generator lines of a non-singular quadric in $\mathrm{PG}(n,q)$, the set of external lines of a non-singular quadral in $\mathrm{PG}(3,q)$, and the set of secants of an ovoid in $\mathrm{PG}(3,q)$ (See Chapter 3).

The following new characterisations have been presented here. Theorems 4.1.1 and 4.2.1 give characterisations of the external lines of an oval cone in $\mathrm{PG}(3,q)$ for q odd and q even. The oval cone was the only irreducible quadral in $\mathrm{PG}(3,q)$ that did not have a characterisation for its external lines. Theorem 5.1.1 gives a characterisation of the external lines of a parabolic quadric in $\mathrm{PG}(4,q)$. Previously, only tangents and generator lines of quadrics in $\mathrm{PG}(n,q)$, $n \geq 4$ had been characterised. Theorems 5.2.1 and 5.2.2 give characterisations of the secant planes of a parabolic quadric in $\mathrm{PG}(4,q)$. These are the first characterisations of a set of planes associated with a quadral. Finally, Theorems 6.1.1 and 6.2.1 give characterisations of the external lines of quadrals of parabolic type in $\mathrm{PG}(n,q)$.

The above results lead to certain possibilities for future research into characterisations of subspaces.

Let $\mathscr{L}$ be a set of lines of $\mathrm{PG}(3,q)$ with the following properties.

(I) Every point of $\mathrm{PG}(3,q)$ lies on 0, $\frac{1}{2}q(q+1)$ or $\frac{1}{2}q(q-1)$ lines of $\mathscr{L}$,

(II) Every plane of $\mathrm{PG}(3,q)$ contains 0, q^2 or $\frac{1}{2}q(q-1)$ lines of $\mathscr{L}$.

The set of external lines of an ovoid or a hyperbolic quadric have these properties. If q is odd, then the set of external lines of an oval cone also has these properties. A natural extension to the characterisations given in this thesis is to completely classify all sets of lines in $\mathrm{PG}(3,q)$ with Properties (I) and (II).

The set of secant planes of a parabolic quadric in $\mathrm{PG}(n,q)$, n even, is also worth further investigation. If $\mathscr{P}$ is a parabolic quadric in $\mathrm{PG}(n,q)$, n even, then arguments similar to those in the proof of Lemma 1.5.41 can be used to calculate the number of secant planes through each line and each point of $\mathrm{PG}(n,q)$. In fact, there are two possibilities for the number of secant planes through a point, and two possibilities for the number of secant planes through a line. It should be possible to characterise the set of secant planes of $\mathscr{P}$ using arguments analogous to those in the proof of Theorem 5.2.1.

As well as results characterising sets of subspaces associated with quadrals, this thesis also presents several new results about ovoids of $\mathrm{PG}(3,q)$. Theorem 7.4.3 improves the existing bound on the number of points that two distinct ovoids share when q is even. Theorem 7.5.2 describes a method for constructing ovoidal fibrations, and Theorem 7.5.4 proves that all ovoidal fibrations have an associated regular spread. Theorem 7.6.2 proves that any two ovoids sharing all their tangents must share an odd number of points. Until now this was only known to be true if one of the ovoids is a Tits ovoid or an elliptic quadric. Theorem 7.6.4 proves that an elliptic quadric and an ovoid sharing all their tangents must share a number of points congruent to $1 \pmod 4$. Until now, this number was only known to be 1 or $3 \pmod 4$. The methods used to prove these results are new, and very different to the existing methods. Finally, Section 7.7 gives configurations of external lines and secants that determine ovoids.

The above results suggest several problems for future research. The bound on the number of shared points of two ovoids given in Theorem 7.4.3 is the best possible for $q = 2$, 4 or 8. Future research could focus on whether this is true for $q > 8$. The intersection of ovoids sharing all tangents was discussed in Section 7.6, and certain sets of ovoids sharing a regular spread of tangents were discussed in Section 7.5. Investigating the intersection of ovoids sharing other configurations of tangents is a natural extension of this work.

The kernel of a regular spread was used in Theorem 7.5.2 to construct an ovoidal fibration in PG$(3, q)$, q even. By Theorem 7.5.4, there is a regular spread of tangents associated with any ovoidal fibration in PG$(3, q)$, q even. This suggests that it may be possible to construct new ovoidal fibrations using a small selection of ovoids and the kernel of a regular spread. Finally, Chapter 7 gives several conditions to decide if two ovoids coincide, or to decide if an ovoid is an elliptic quadric. These conditions were based on information about the number of points, tangents, secants or external lines that two ovoids share. Further investigations into the possible ways that ovoids intersect will produce other conditions, and may provide the necessary tools to help classify all ovoids of PG$(3, q)$, q even.

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