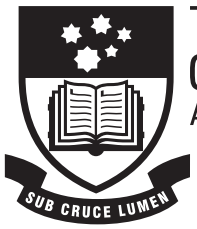

THE UNIVERSITY
OF ADELAIDE
AUSTRALIA

School of Mathematical Sciences
Discipline of Pure Mathematics

Quadrals and their Associated Subspaces

PhD Thesis

Submitted May 2008

David Keith Butler

Contents

Abstract **5**

Signed Statement **6**

Acknowledgements **7**

Introduction **8**

0 Preliminaries **11**

0.1 Algebra 11

0.2 Projective Spaces 18

0.3 Collineations and Polarities 24

0.4 Special sets of lines 29

1 Quadrals **36**

1.1 Ovals 36

1.2 Ovoids 41

1.3 Cones 48

1.4 Oval and Ovoid Cones 58

1.5 Quadrics 63

1.6 Quadrals 83

2 Characterisations of the set of points of a quadral **87**

2.1 Tallini Sets 87

2.2 Quadratic Sets 92

2.3 Characterisation of quadrals by their plane sections 97

3 Existing characterisations of the subspaces associated with quadrals 100

3.1 Lines associated with quadrals in PG$(2, q)$ 100

3.2 Lines associated with Quadrals in PG(n, q) 103

4 Characterisation of the external lines of an oval cone in PG$(3, q)$ **107**

4.1 The external lines of an oval cone in PG$(3, q)$, q even 107

4.2 The external lines of an oval cone in PG$(3, q)$, q odd 114

5 Characterisation of subspaces associated with a parabolic quadric in PG$(4, q)$ **124**

5.1 The external lines of a parabolic quadric in PG$(4, q)$, q even 125

5.2 The secant planes of a parabolic quadric of PG$(4, q)$ 136

5.3 The tangents and generator lines of the non-singular quadric in PG$(4, q)$, q odd 150

6 Characterisation of the external lines of quadrals of parabolic type in PG(n, q)**,** q **even** **152**

6.1 The external lines of a hyperoval cone in PG(n, q), q even 153

6.2 The external lines of a quadral of parabolic type. 160

7 The intersection of ovals and ovoids **166**

7.1 The intersection of two ovals 167

7.2 Relationships Between Shared Points, Shared Tangents and Shared Tangent Planes 168

7.3 Intersections of elliptic quadrics. 171

7.4 Bounds on the intersection of ovoids. 172

7.5 Ovoidal fibrations 175

7.6 The intersection of ovoids sharing all tangents 179

7.7 The external lines or secants that two ovoids may share 184

Conclusion **188**

Bibliography **190**

Abstract

This thesis concerns sets of points in the finite projective space $\mathrm{PG}(n, q)$ that are combinatorially identical to quadrics. A *quadric* is the set of points of $\mathrm{PG}(n, q)$ whose coordinates satisfy a quadradic equation, and the term *quadral* is used in this thesis to mean a set of points with all the combinatorial properties of a quadric.

Most of the thesis concerns the characterisation of certain sets of subspaces associated with quadrals. Characterisations are proved for the external lines of an oval cone in $\mathrm{PG}(3, q)$, of a non-singular quadric in $\mathrm{PG}(4, q)$, q even, and of a large class of cones in $\mathrm{PG}(n, q)$, q even. Characterisations are also proved for the planes meeting the non-singular quadric of $\mathrm{PG}(4, q)$ in a non-singular conic, and for the tangents and generator lines of this quadric for q odd.

The second part of the thesis is concerned with the intersection of ovoids of $\mathrm{PG}(3, q)$. A new bound is proved on the number of points two ovoids can share, and configurations of secants and external lines that two ovoids can share are determined. The structure of ovoidal fibrations is discussed, and this is used to prove new results on the intersection of two ovoids sharing all of their tangents.

Signed Statement

This work contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text.

I give consent to this copy of my thesis, when deposited in the University Library, being made available for loan and photocopying, subject to the provisions of the Copyright Act 1968.

Signed: _____

Date: _____

Acknowledgements

I would like to acknowledge the reciept of a Faculty of Engineering, Computer and Mathematical Sciences Divisional Scholarship, without which it would not have been possible to undertake this research degree.

I must thank my supervisors Susan Barwick and Rey Casse for their support, guidance, criticism, encouragement and chocolate. Sincerest thanks also go to Stephanie Lord, Jan Sutton and Di Parrish for answering all sorts of questions and solving all sorts of problems with a smile.

I would also like to thank the other postgraduate and honours students, who have made the experience not only bearable but actually enjoyable. Thank you to everyone for playing croquet, testing new games, listening to my latest proof and just being there with me on the journey. I particularly want to thank my good friends Ric Green, David Roberts, Ray Vozzo and Jono Tuke for being my good friends.

Finally, I need to thank my precious wife Catherine. How could I possibly live without you my darling, letalone finish a PhD thesis? You supported me right from the first idea that a PhD might be a possibility, all the way along the winding road to this the last long wait. These three years have really been worth it. Not because of the reward I'll get from all this hard work, but because I know even more surely that you'll always be with me, no matter what happens.