Invariant Bilinear Differential Pairings on Parabolic Geometries

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0.1 Abstract

This thesis is concerned with the theory of invariant bilinear differential pairings on parabolic geometries. It introduces the concept formally with the help of the jet bundle formalism and provides a detailed analysis. More precisely, after introducing the most important notations and definitions, we first of all give an algebraic description for pairings on homogeneous spaces and obtain a first existence theorem. Next, a classification of first order invariant bilinear differential pairings is given under exclusion of certain degenerate cases that are related to the existence of invariant linear differential operators. Furthermore, a concrete formula for a large class of invariant bilinear differential pairings of arbitrary order is given and many examples are computed. The general theory of higher order invariant bilinear differential pairings turns out to be much more intricate and a general construction is only possible under exclusion of finitely many degenerate cases whose significance in general remains elusive (although a result for projective geometry is included). The construction relies on so-called splitting operators examples of which are described for projective geometry, conformal geometry and CR geometry in the last chapter.

0.2 Thesis declaration

This work contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by any other person, except where due reference has been made in the text.

I give consent to this copy of my thesis, when deposited in the University library, being made available for loan and photocopying, subject to the provisions of the Copyright Act 1968.

0.3 Acknowledgment

It is my pleasure to express my deepest gratitude towards my supervisor Prof. Michael Eastwood who has skillfully guided me through this thesis and without whom none of this would have been possible. He has not only provided me with invaluable help but also kindled my love for this particular field of mathematics. My time in Adelaide has been thoroughly enjoyable.

This thesis is dedicated to my beautiful daughter Alexis.

Before you can fly, you have to learn how to walk[1]

0.4 Constants and notation used throughout

1. $\mathcal{M}$: a manifold of dimension n (for CR geometry $\mathcal{M}$ will have real dimension $2n+1$).
2. $\mathfrak{g}$: a semisimple Lie algebra of rank l.
3. $[.,.]$: the bracket in $\mathfrak{g}$.
4. G: the simply connected Lie group with Lie algebra $\mathfrak{g}$.
5. $\mathfrak{h}$: a fixed Cartan subalgebra of $\mathfrak{g}$.
6. $\mathfrak{p}$: a parabolic subalgebra of $\mathfrak{g}$.
7. k_0: length of the grading of $\mathfrak{g}$.
8. l_0: number of simple roots in $\mathcal{S}\backslash\mathcal{S}_{\mathfrak{p}}$.
9. α_i, $i=1,...,l$: simple roots of $\mathfrak{g}$.
10. ω_i, $i=1,...,l$: fundamental weights corresponding to the simple roots.
11. $I\subset\{1,...,l\}$: indices that correspond to simple roots in $\mathcal{S}\backslash\mathcal{S}_{\mathfrak{p}}$, i.e. to crossed through nodes.
12. $J=\{1,...,l\}\backslash I$: indices that correspond to simple roots in $\mathcal{S}_{\mathfrak{p}}$.
13. $\mathcal{W}$: the Weyl group of $\mathfrak{g}$.
14. $\mathcal{W}^{\mathfrak{p}}$: the Hasse diagram of G/P.
15. $\rho=\sum_{i=1}^{l}\omega_i$: integral weight in the dominant Weyl chamber which lies closest to the origin.
16. $B(.,.)$: the Killing form of $\mathfrak{g}$.
17. $T(\mathfrak{a})$: the tensor algebra of a Lie algebra $\mathfrak{a}$.
18. $\mathfrak{U}(\mathfrak{a})$: the universal enveloping algebra of a Lie algebra $\mathfrak{a}$.
19. $\mathcal{Z}(\mathfrak{a})$: the center of the universal enveloping algebra $\mathfrak{U}(\mathfrak{a})$.
20. $\mathfrak{gl}_l\mathbb{C}$: the endomorphisms of $\mathbb{C}^l$. These can be identified with $\mathbb{C}^{l\times l}$.

[1]Nolan Wallach, Brisbane winter school in mathematics.

21. $\otimes$: the tensor product symbol.
22. $\odot$: the symbol for the symmetric tensor product.
23. Λ: the symbol for the skew symmetric tensor product.
24. $\circledcirc$: the Cartan product of representations.
25. $\mathcal{G}$: the total space of the principal bundle defining a parabolic geometry.
26. ω: the Cartan connection of $\mathcal{G}$ (the Maurer Cartan form of G is denoted by ω_{MC}). Note that ω is also used as a symbol for the geometric weight of a representation, but the context should make the meaning clear.
27. $\mathcal{A}$: the adjoint tractor bundle.
28. $\mathbb{V}_\lambda$: the irreducible finite dimensional representation (of $\mathfrak{g}$ or $\mathfrak{p}$) that is dual to the irreducible finite dimensional representation of highest weight $\lambda \in \mathfrak{h}^*$.
29. $J^k V$: k-th order jet bundle of a vector bundle V.
30. $\mathcal{J}^k V$: k-th order weighted jet bundle.
31. $\bar{J}^k V$: k-th order semi-holonomic jet bundle.
32. $\bar{\mathcal{J}}^k V$: k-th order restricted semi-holonomic jet bundle.
33. $\mathcal{E}$ or $\mathcal{O}$: the bundle of smooth (or holomorphic) functions of $\mathcal{M}$.
34. $\mathcal{E}^a$: the bundle of tangent vectors.
35. $\mathcal{E}_a$: the bundle of one-forms.
36. V^*: the dual of the vector space (representation, bundle, etc.) V.
37. $M_\mathfrak{p}(\mathbb{V})$: the generalized Verma module associated to a representation $\mathbb{V}$.
38. ∇: a connection.