Decontracted double BRST symmetry on the lattice

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We present the Curci-Ferrari model on the lattice. In the massless case the topological interpretation of this model with its double Becchi-Rouet-Stora-Tyutin (BRST) symmetry relates to the Neuberger $0/0$ problem which we extend to include the ghost/antighost symmetric formulation of the nonlinear-covariant Curci-Ferrari gauges on the lattice. The introduction of a Curci-Ferrari mass term, however, serves to regulate the $0/0$ indeterminate form of physical observables observed by Neuberger. While such a mass m decontracts the double BRST/anti-BRST algebra, which is well known to result in a loss of unitarity, observables can be meaningfully defined in the limit $m \to 0$ via l'Hospital's rule. At finite m, the topological nature of the partition function used as the gauge-fixing device seems lost. We discuss the gauge parameter ξ and mass m dependence of the model and show how both cancel when $m \equiv m(\xi)$ is appropriately adjusted with ξ.

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I. INTRODUCTION

In the covariant continuum formulation of gauge theories, in terms of local field systems, one has to deal with the redundant degrees of freedom due to gauge invariance. Within the language of local quantum field theory, the machinery for that is based on the so-called Becchi-Rouet-Stora-Tyutin (BRST) symmetry which is a global symmetry and can be considered the quantum version of local gauge invariance [1,2]. In short, one starts out from the representations of a BRST algebra on indefinite metric spaces with assuming the existence (and completeness) of a nilpotent BRST charge Q_B. The physical Hilbert space can then be defined as the equivalence classes of BRST closed (which are annihilated by Q_B) modulo exact states (which are BRST variations of others). In QED this machinery reduces to the usual Gupta-Bleuler construction. For the generalization thereof, in non-Abelian gauge theories, all is well in perturbation theory also. Beyond perturbation theory, however, there is a problem with such a construction that has not been fully and comprehensively addressed as yet. It relates to the famous Gribov ambiguity [3], the existence of so-called Gribov copies that satisfy the Lorenz condition [4] (or any other local gauge-fixing condition) but are related by gauge transformations, and are thus physically equivalent. As a result of this ambiguity, the usual definitions of a BRST charge fail to be globally valid.

A rigorous nonperturbative framework is provided by lattice gauge theory. Its strength and beauty derives from the fact that gauge fixing is not required. However, in order to arrive at a nonperturbative definition of non-Abelian gauge theories in the continuum, from a lattice formulation, we need to be able to perform the continuum limit in a formally watertight way. There is the gap in our present understanding. The same problem as described above comes back to haunt us in another dress when attempting to fix a gauge via BRST formulations on the lattice. There it is known as the Neuberger problem which asserts that the expectation value of any gauge invariant (and thus physical) observable in a lattice BRST formulation will always be of the indefinite form $0/0$ [5].

The BRST algebra requires the introduction of further unphysical degrees of freedom. These are the Faddeev-Popov ghosts and antighosts which violate the spin-statistics theorem of local quantum field theory on positive definite metric (Hilbert) spaces. Contrary to what the name antighost might suggest, however, in the usual linear-covariant gauges the treatment of ghosts and antighosts is completely asymmetric. On the other hand, it is also known for many years that it is possible to extend the BRST algebra to be entirely symmetric with respect to (w.r.t.) ghosts and antighosts. This additional symmetry arises naturally in the Landau gauge but can also be extended to more general gauges, the so-called Curci-Ferrari gauges, at the expense of quartic-ghost self-interactions. The most interesting feature of these gauges for our purpose, however, is that they allow the introduction of a mass term for gluons and ghosts [6]. While such a Curci-Ferrari mass m breaks the nilpotency of the BRST and anti-BRST charges, which is known to result in a loss of unitarity [7,8] and which therefore meant that this relatively old model received little attention for many years, it also serves to regulate the Neuberger zeros in a lattice formulation. In [9] this was exemplified in a simple Abelian toy model where the zeros in the numerator and denominator of expectation values become proportional to m^2 and allow to compute a finite value for $m^2 \to 0$ via l'Hospital's rule.

For the $SU(N)$ gauge theory on a finite four-dimensional lattice things are naturally much more complicated than in the toy model. In this paper we develop a full lattice

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formulation of the time-honored model by Curci and Ferrari with its decontracted double BRST/anti-BRST and ghost-mass term, as announced in [10]. After introducing the general setup for double BRST on the lattice in Sec. II, we next review Neuberger's no-go-theorem in a generalized version to include the ghost/antighost symmetric case of the nonlinear-covariant Curci-Ferrari gauges for $m^2 = 0$ in Sec. III, a case originally excluded by Neuberger. At nonvanishing Curci-Ferrari mass the partition function of the model used as the gauge-fixing device is shown to be polynomial in m^2 and to be thus nonvanishing, in a special gauge-parameter limit in Sec. IV. In this way regularizing the Neuberger zeros, the leading power of that polynomial can be extracted from a suitable number of derivatives (w.r.t. m^2) before the limit $m^2 \to 0$ is taken, in the spirit of l'Hospital's rule. This could provide a lattice BRST model without Neuberger problem. The massive Curci-Ferrari model is no longer purely topological in nature, however, and as a result, its gauge-parameter ξ independence requires tuning of the Curci-Ferrari mass with ξ as explained in Sec. V. The gauge-orbit independence of this procedure is discussed in Sec. VI. A short summary is given in Sec. VII, and our conclusions and outlook are provided in Sec. VIII. Several appendices are provided with supplementary derivations.

II. DOUBLE BRST ON THE LATTICE

For the topological lattice formulation of the double BRST symmetry of the ghost/antighost symmetric covariant gauges, we start out from the standard gauge-fixing functional $V_U[g]$ of covariant gauges which here assumes the role of a Morse potential on a gauge orbit,

$$V_U[g] = -\frac{1}{2\rho}\sum_i\sum_{j\sim i}\mathrm{tr}U^g_{ij} = -\frac{1}{\rho}\sum_{x,\mu}\mathrm{Re}\,\mathrm{tr}U^g_{x,\mu}. \quad (1)$$

Here, in the first form, $U_{ij} \in SU(N)$ is the directed link variable connecting nearest neighbor sites i and j. The sum $j \sim i$ denotes summation over all nearest neighbors j of site i. We assume periodic boundary conditions. The double sum thus runs twice over all links $\langle ij\rangle$, and with $U^\dagger_{ij} = U_{ji}$ it is therefore equivalent to the simple sum over links in the second form, where $U_{x,\mu}$ stands for the same link field U at position x in direction μ. The constant ρ is the normalization of the $SU(N)$ generators X. We use anti-Hermitian $[X^a, X^b] = f^{abc}X^c$ with $\mathrm{tr}X^aX^b = -\rho\delta^{ab}$. We explicitly only need the fundamental representation, where $\rho = \rho_{\mathrm{fund}} = 1/2$.

As usual, under gauge transformations the link variables U transform

$$U_{ij} \to U^g_{ij} = g^\dagger_i U_{ij} g_j. \quad (2)$$

BRST transformations s and anti-BRST transformations $\bar{s}$ in the topological setting do not act on the link variables U directly, but on the gauge transformations g_i like infinitesimal right translations in the gauge group with real ghost and antighost Grassmann fields c^a_i, $\bar{c}^a_i$ as parameters, respectively,

$$sg = gX^ac^a = gc, \qquad \bar{s}g = gX^a\bar{c}^a = g\bar{c}, \quad (3)$$

where we introduced Lie-algebra valued, anti-Hermitian ghost fields $c_i \equiv X^a c^a_i$ with $c^\dagger_i = -c_i$, and analogous antighost fields $\bar{c}_i \equiv X^a\bar{c}^a_i$. For consistency, we furthermore require

$$sg^\dagger = (sg)^\dagger = -cg^\dagger, \qquad \bar{s}g^\dagger = (\bar{s}g)^\dagger = -\bar{c}g^\dagger. \quad (4)$$

For the gauge-transformed link variables this then implies

$$sU^g_{ij} = -c_iU^g_{ij} + U^g_{ij}c_j, \qquad \bar{s}U^g_{ij} = -\bar{c}_iU^g_{ij} + U^g_{ij}\bar{c}_j. \quad (5)$$

The BRST transformations for (anti)ghosts and Nakanishi-Lautrup fields b are straightforward lattice analogues (per site) of their continuum counterparts, see, e.g., Refs. [11,12],

$$sc^a = -\tfrac{1}{2}(c\times c)^a, \quad (6)$$

$$s\bar{c}^a = b^a - \tfrac{1}{2}(\bar{c}\times c)^a, \quad (7)$$

$$sb^a = -\tfrac{1}{2}(c\times b)^a - \tfrac{1}{8}((c\times c)\times\bar{c})^a. \quad (8)$$

The relatively obvious notation of using the "cross-product" herein refers to the structure constants for $SU(N)$, for example, $(\bar{c}\times c)^a \equiv f^{abc}\bar{c}^bc^c$.

In the ghost/antighost symmetric gauges as considered here, the anti-BRST variations are obtained by substituting $c \to \bar{c}$ and $\bar{c} \to -c$ according to Faddeev-Popov conjugation. Thus,

$$\bar{s}c^a = -b^a - \tfrac{1}{2}(\bar{c}\times c)^a, \quad (9)$$

$$\bar{s}\bar{c}^a = -\tfrac{1}{2}(\bar{c}\times\bar{c})^a, \quad (10)$$

$$\bar{s}b^a = -\tfrac{1}{2}(\bar{c}\times b)^a + \tfrac{1}{8}((\bar{c}\times\bar{c})\times c)^a. \quad (11)$$

The action of the topological lattice model for gauge fixing *à la* Faddeev-Popov with double BRST invariance can then be written in compact form as

$$S_{\mathrm{GF}} = is\bar{s}\Big(V_U[g] + i\frac{\xi}{2\rho}\sum_i\mathrm{tr}\bar{c}_ic_i\Big). \quad (12)$$

This is the lattice counterpart of the continuum gauge-fixing Lagrangian

$$\mathcal{L}_{\mathrm{GF}} = \frac{i}{2}s\bar{s}(A^a_\mu A^a_\mu - i\xi\bar{c}^ac^a) \quad \text{with } S_{\mathrm{GF}} = \int d^Dx\,\mathcal{L}_{\mathrm{GF}} \quad (13)$$

in D Euclidean dimensions.

For the purpose of a self-contained presentation we work out the double (anti)BRST variation on the right of (12) explicitly in Appendix A. This leads to

$$S_{\rm GF} = \sum_i \Big\{ -ib_i^a F_i^a(U^g) - i\bar{c}_i^a M_{\rm FP\it i}^a[c] + \frac{\xi}{2} b_i^a b_i^a + \frac{\xi}{8}(\bar{c}_i \times c_i)^2 \Big\}, \qquad (14)$$

where

$$F_i^a(U^g) = -\frac{1}{2\rho}\sum_{j\sim i} \mathrm{tr}(X^a(U_{ij}^g - U_{ji}^g)) \qquad (15)$$

defines, of course, the standard gauge-fixing form of covariant gauges with the continuum limit,

$$F_i^a(U^g) \stackrel{a\to 0}{\longrightarrow} a^2 \partial_\mu A_\mu^{ga} + \mathcal{O}(a^4). \qquad (16)$$

The Faddeev-Popov operator $M_{\mathrm{FP}ij}^{ab}$ is obtained from the short-hand notation in (14),

$$\sum_i \bar{c}_i^a M_{\mathrm{FP}i}^a[c] = \sum_{i,j} \bar{c}_i^a M_{\mathrm{FP}ij}^{ab} c_j^b, \qquad (17)$$

and given explicitly for later reference in alternative forms in Eqs. (A11) or (A15). It is symmetric w.r.t. simultaneous interchanges of color and site indices, and identical to the one obtained in [13] as the Hessian of $V_U[g]$ from variations along one-parameter subgroups of the $SU(N)$ gauge group. In the continuum limit it reduces to the symmetrized and thus Hermitian

$$M_{\mathrm{FP}ij}^{ab} \stackrel{a\to 0}{\to} -a^2 \tfrac{1}{2}(\partial D^{ab} + D^{ab}\partial)\delta(x-y) + \mathcal{O}(a^4)$$

of the ghost/antighost symmetric Curci-Ferrari gauges. In contrast, the Faddeev-Popov operator of the linear-covariant gauges for $\xi \neq 0$ is not a Hessian because it is not symmetric. It can be read off as a by-product of our BRST derivation from Eq. (A9). In particular, this non-symmetric Faddeev-Popov operator needs to be used when implementing other linear-covariant gauges such as the Feynman gauge with $\xi = 1$ on the lattice as discussed in [14,15]. In Landau gauge $\xi = 0$ the distinction is an illusion. To keep the symmetric Hessian for $\xi \neq 0$, however, is only possible within the ghost/antighost symmetric framework where it necessarily comes along with the quartic-ghost self-interactions in (14).

The full symmetry of the ghost/antighost symmetric Curci-Ferrari gauges [6,12] is given by a semidirect product of a global $SL(2,\mathbb{R})$, which includes ghost number and Faddeev-Popov conjugation, with the BRST/anti-BRST symmetries as used above [16]. This is the global symmetry of the Landau gauge, and it is sometimes referred to as extended BRST symmetry, see [1].

Among the general class of all covariant gauges [11], with a Lagrangian which is polynomial in the fields, Lorentz, globally gauge and BRST invariant, and renormalizable in $D = 4$, the ghost/antighost symmetric case is special and interesting in that it allows to smoothly connect to the Landau gauge for $\xi \to 0$, without changing the global symmetry properties.

In particular, introducing with [11] a second gauge parameter $\beta \in [0, 1]$, to interpolate between the various generalized covariant gauges, the linear-covariant gauges of standard Faddeev-Popov theory correspond to the line $\beta = 0$ in the two gauge-parameter plane (ξ, β). Along this line, the global symmetry changes abruptly when reaching the Landau-gauge limit; and for $\beta = 1$, one obtains a mirror image of standard Faddeev-Popov theory with the roles of ghosts and antighosts interchanged. The ghost/antighost symmetric gauges discussed here then correspond to the line $\beta = 1/2$. The $\xi = 0$ gauge is β independent. The whole interval for $\beta \in [0, 1]$ at $\xi = 0$ is equivalent and corresponds to the Landau gauge. The important difference is, however, that the $SL(2,\mathbb{R})$ symmetric line at $\beta = 1/2$ provides a unique class of covariant gauges which share the full extended BRST symmetry of the Landau gauge for any value of ξ. The limit $\xi \to 0$ is thus a smooth one, as far as this symmetry is concerned, only along the line of $\beta = 1/2$. The price to pay are the quartic-ghost self-interactions in (14) which again vanish only in the Landau-gauge limit.

For a further discussion of the general ghost creating gauges, and their geometrical interpretation, see [12]. The one-loop renormalization was first discussed in [11], for explicit calculations of renormalization constants and anomalous dimensions of the ghost/antighost symmetric case up to including the three-loop level, see [8,17]. The Dyson-Schwinger equations of these gauges were studied in [18]. A nonrenormalization theorem relating to the Curci-Ferrari mass was recently reported in [19].

III. THE NEUBERGER PROBLEM

Following Neuberger, we introduce an auxiliary parameter t in the Euclidean partition function to be used as the gauge-fixing device via the Faddeev-Popov procedure of inserting unity into the unfixed partition function of $SU(N)$ lattice gauge theory. The gauge-fixing action of the double BRST invariant model given by (12) consists of two terms both of which are separately BRST (and anti-BRST) exact. Multiplying the 1st term in (12) by the real parameter t amounts to a mere redefinition of the Morse potential which should have no further effect. We can therefore write the gauge-fixing partition function with double BRST,

$$Z_{\rm GF}(t) = \int d[g, b, \bar{c}, c] \exp\Big\{ -is\bar{s}\Big(tV_U[g] + i\frac{\xi}{2\rho}\sum_i \mathrm{tr}\bar{c}_i c_i \Big)\Big\}, \qquad (18)$$

which is independent of the set of link variables $\{U\}$ and the gauge parameter ξ because of its topological nature. Moreover, the t independence is really no different from the ξ independence here, and it is thus rather obvious.

Explicitly, the derivative with respect to t (or ξ) produces the expectation value of a BRST exact operator which vanishes, i.e.,

$$Z'_{\mathrm{GF}}(t) = 0. \tag{19}$$

At $t = 0$ on the other hand, we obtain with the BRST variations given in (A4) and (A12) of Appendix A,

$$Z_{\mathrm{GF}}(0) = \mathcal{N} \int d[b, \bar{c}, c] \exp\left\{ -\sum_i \left(\frac{\xi}{2} b_i^a b_i^a + \frac{\xi}{8} (\bar{c}_i \times c_i)^2 \right) \right\}, \tag{20}$$

where the volume of the gauge group on the lattice, from the invariant integrations $\prod_i dg_i$ via the Haar measure over $g_i \in SU(N)$ per site i, is absorbed in the constant $\mathcal{N}$. The Gaussian integrations over the Nakanishi-Lautrup fields b are also well defined and produce a factor $(2\pi/\xi)^{(N^2-1)/2}$ per site.

One might be tempted to conclude at this point that the quartic-ghost self-interactions in (20) might remove the uncompensated Grassmann integrations of the linear-covariant gauges where no such self-interactions occur. The ghost/antighost integrations at $t = 0$ also factorize into independent integrations $d\bar{c}_i^a dc_i^a$ over $2(N^2 - 1)$ Grassmann variables per site. For $N = 3$, for example, the 4th order term of the exponential in (20) produces a monomial in $\bar{c}_i^a$, c_i^a which contains each of these 16 Grassmann variables exactly once, so that their integration might produce a nonvanishing result. This is not the case, however. Working out the prefactor of this monomial, as we will do explicitly in the more general case with including a nonvanishing Curci-Ferrari mass m below, one finds that the prefactor of this term in (20) vanishes in the massless case and thus,

$$Z_{\mathrm{GF}}(0) = 0. \tag{21}$$

Because of the t independence (19), this implies the vanishing of the gauge-fixing partition function (18) of the ghost/antighost or $SL(2, \mathbb{R})$ symmetric formulation with double BRST invariance in the same way as that of standard Faddeev-Popov theory observed in [5]. As for the latter, the sign-weighted sum over all Gribov copies, as originally proposed to generalize the Faddeev-Popov procedure in presence of Gribov copies [20,21], vanishes.

This cancellation of Gribov copies is well known [22]. The fact that it also arises here, in the ghost/antighost symmetric formulation with its quartic self-interactions, directly relates to the topological interpretation [23,24] of the Neuberger zero: Z_{GF} can be viewed as the partition function of a Witten-type topological model to compute the Euler characteristic χ of the gauge group. On the lattice the gauge group is a direct product of $SU(N)$'s per site, and because the Euler characteristic factorizes,

$$Z_{\mathrm{GF}} = \chi(SU(N)^{\#\mathrm{sites}}) = \chi(SU(N))^{\#\mathrm{sites}} = 0^{\#\mathrm{sites}}.$$

For $t = 0$ the action in (18) decouples from the link-field configuration and $Z_{\mathrm{GF}}(0)$, albeit computing the same topological invariant, has of course no effect in terms of fixing a gauge. In the present formulation, with $Z_{\mathrm{GF}}(0)$ in (20), the independent Grassmann integrations per site of the quartic-ghost term which contains the curvature of $SU(N)$ each compute its Euler characteristic via the Gauss-Bonnet theorem [25]. This explicitly produces one factor of zero per site on the lattice. It provides the topological explanation for the vanishing of the prefactor of the corresponding monomial of degree $2(N^2 - 1)$ in the Grassmann variables $\bar{c}$, c, which could otherwise exist in the expansion of the exponential in (20) for all odd N. For $N = 3$, for example, the zero in this prefactor arises, upon normal ordering, from a cancellation of 368 nonvanishing individual terms when expanding the square of the square of the quartic-ghost self-interaction. This cancellation would be rather unnatural to arise accidentally, without such explanation.

The vanishing of the gauge-fixing partition function at the $t = 0$ part in Neuberger's argument, in the ghost/antighost symmetric gauges with their $SL(2, \mathbb{R})\rtimes$ double BRST symmetry, therefore most directly reflects the topological origin of the Neuberger zero. Equation (20) precisely represents a product of one Gauss-Bonnet integral expression for $\chi(SU(N))$ per site of the lattice.

Note that the gauge parameter ξ can be removed completely from the expression for $Z_{\mathrm{GF}}(0)$ in Eq. (20) by a rescaling $\sqrt{\xi} b \to b$ and $\sqrt[4]{\xi}\bar{c} \to \bar{c}$, $\sqrt[4]{\xi} c \to c$, which leaves the integration measure unchanged. The same rescaling for the full gauge-fixing partition function $Z_{\mathrm{GF}}(t)$ in (18), which amounts to replacing the action in S_{GF} in (14) by

$$S_{\mathrm{GF}}(t) = \sum_i \left\{ -itb_i^a F_i^a(U^g) - it\bar{c}_i^a M_{\mathrm{FP}i}^a[c] + \frac{\xi}{2} b_i^a b_i^a + \frac{\xi}{8} (\bar{c}_i \times c_i)^2 \right\}, \tag{22}$$

furthermore shows that t and ξ really represent a single parameter $t/\sqrt{\xi}$. Setting $t = 0$ in Neuberger's argument is therefore the same as the $\xi \to \infty$ limit which is usually what is considered as the Gauss-Bonnet limit in topological quantum field theory [25]. As mentioned above, there is no gauge fixing in this limit, but it provides a simple way to compute the value (zero here) of the partition function which is independent of $t/\sqrt{\xi}$.

In the opposite limit, that of the Landau gauge $\xi \to 0$ or $t/\sqrt{\xi} \to \infty$, of course, $Z_{\mathrm{GF}}(t)$ still reduces to the sign-weighted sum over all Gribov copies as usual [20,21],

$$Z_{\mathrm{GF}}(t) \to \sum_{\mathrm{copies}\{g^{(i)}\}} \mathrm{sign}(\det M_{\mathrm{FP}}(U^{g^{(i)}})), \tag{23}$$

which because of the t (and ξ) independence (19) thus computes the same topological zero [22–24], in this case via the Poincaré-Hopf theorem [25].

IV. THE MASSIVE CURCI-FERRARI MODEL ON THE LATTICE

In the previous section we have seen that the quartic-ghost self-interactions of the $SL(2,\mathbb{R})\rtimes$ double BRST symmetric Curci-Ferrari gauges have no effect on the disastrous conclusion of the $0/0$ problem in lattice BRST. They rather serve to reveal most clearly the topological origin of this problem.

We will demonstrate explicitly below that this zero can be regularized, however, by introducing a Curci-Ferrari mass m, as proposed in [9,10]. The gauge-fixing action $S_{\rm GF}$ is thereby once more replaced by

$$S_{\rm mGF}(t) = i(s\bar{s} - im^2)\Big(tV_U[g] + i\xi\sum_i \mathrm{tr}\bar{c}_i c_i\Big) \qquad (24)$$

[where we dropped in the 2nd term the factor $1/(2\rho) = 1$, in the fundamental representation]. The BSRT and anti-BRST transformations of U^g, $\bar{c}$, and c in Eqs. (5)–(7), (9), and (10) of Sec. II remain unchanged. Those for the Nakanishi-Lautrup b fields, Eqs. (8) and (11), are replaced by [12]

$$sb^a = im^2c^a - \tfrac{1}{2}(c\times b)^a - \tfrac{1}{8}((c\times c)\times\bar{c})^a, \qquad (25)$$

$$\bar{s}b^a = im^2\bar{c}^a - \tfrac{1}{2}(\bar{c}\times b)^a + \tfrac{1}{8}((\bar{c}\times\bar{c})\times c)^a. \qquad (26)$$

In the derivation of the explicit form for $S_{\rm mGF}(t)$, using these modified (anti)BRST transformations, the only modification in comparison to Sec. II and Appendix A, arises from $s(\bar{c}^a b^a)$ in (A12), which now becomes

$$s(\bar{c}^a b^a) = -im^2\bar{c}^a c^a + b^a b^a + \tfrac{1}{4}(\bar{c}\times c)^2. \qquad (27)$$

The additional first term on the right contributes an additional term $-i(\xi/2)m^2\bar{c}_i^a c_i^a$ to the gauge-fixing Lagrangian, cf. Eq. (A5). Together with the same contribution from the explicit mass term $-i(\xi/2)m^2\bar{c}_i^a c_i^a$ in (24) we therefore obtain *twice that* as the final ghost mass term of the massive Curci-Ferrari model (this subtlety will be worth remembering for later). The action of the massive Curci-Ferrari model therefore becomes, explicitly,

$$S_{\rm mGF}(t) = m^2tV_U[g] + \sum_i\Big\{-itb_i^aF_i^a(U^g) - it\bar{c}_i^aM^a_{{\rm FP}i}[c] + \frac{\xi}{2}b_i^ab_i^a - im^2\xi\bar{c}_i^ac_i^a + \frac{\xi}{8}(\bar{c}_i\times c_i)^2\Big\}. \qquad (28)$$

BRST and anti-BRST transformations are no longer nilpotent at finite m^2, but we have [1,6,12]

$$s^2 = im^2\sigma^+, \qquad \bar{s}^2 = -im^2\sigma^-, \qquad s\bar{s}+\bar{s}s = -im^2\sigma^0, \qquad (29)$$

where $\sigma^\pm$ and σ^0 generate the global $SL(2,\mathbb{R})$ including ghost number and Faddeev-Popov conjugation. The Curci-Ferrari mass decontracts the $sl(2,\mathbb{R})\rtimes$ double BRST algebra of the massless case to the osp(1|2) superalgebra extension of the Lie algebra of the three-dimensional Lorentz group $SL(2,\mathbb{R})$. Conversely, the $m^2\to 0$ limit is interpreted as an Inonu-Wigner contraction of the simple superalgebra osp(1|2) [1,12]. The BRST and anti-BRST invariance of the massive Curci-Ferrari action in (24) itself follows readily from this algebra as given in (29), noting that only $\bar{c}$ and c transform nontrivially under the $SL(2,\mathbb{R})$.

We emphasize that this algebra decontraction has from the very beginning been known to lead to a breakdown of unitarity when attempting a BRST-cohomology construction of a physical Hilbert space in analogy to the massless case [6]. In fact, explicit examples exist for states of negative norm surviving in any such construction [7,8]. They do not belong to BRST quartets and can therefore not be removed by the quartet mechanism [1]. Only through the algebra contraction by $m^2\to 0$ do these states reduce to zero norm components which have no effect on the physical S-matrix elements, and the unitarity of the S matrix is restored in this limit.

Here we deliberately do not want to interpret the mass parameter by Curci and Ferrari as a physical mass. It rather serves to meaningfully define a limit $m^2\to 0$ on the lattice, perhaps in parallel with the continuum limit, to recover nilpotent (anti)BRST transformations.

To study the parameter dependence, we first define the partition function of the massive Curci-Ferrari model, explicitly listing all three parameters (even though these again really only represent two independent ones as we will show below),

$$Z_{\rm mGF}(t,\xi,m^2) = \int d[g,b,\bar{c},c]\exp\{-S_{\rm mGF}(t)\}, \qquad (30)$$

with $S_{\rm mGF}(t)$ from (24) or (28). We note in passing that the terms proportional to m^2 in the massive Curci-Ferrari action (28) are given by

$$\mathcal{O}(t,\xi) \equiv tV_U[g] - i\xi\sum_i\bar{c}_i^ac_i^a, \qquad (31)$$

or, in the continuum,

$$\mathcal{O}(t,\xi) = \int d^Dx\Big(\frac{t}{2}A^a_\mu(x)A^a_\mu(x) - i\xi\bar{c}^a(x)c^a(x)\Big). \qquad (32)$$

For $t=1$ this coincides with the on-shell BRST invariant (at $m^2=0$) operator proposed by Kondo as a possible candidate for a dimension 2 condensate [26]. The doubling of the explicit ghost mass term in (12), by the BRST variation of $\bar{c}b$ in (27) as mentioned above, is crucial here. Without this difference in the relative factor of 2 between the two terms in $\mathcal{O}(t,\xi)$ and the gauge-fixing functional

$$-iW_{\rm GF} = tV_U[g] - i\frac{\xi}{2}\sum_i\bar{c}_i^ac_i^a, \qquad (33)$$

one could not have both, the on-shell BRST invariance of

$\mathcal{O}$ and the gauge-fixing action in (12) from the double BRST variation $S_{\rm GF} = s\bar{s}W_{\rm GF}$, at the same time.

The observation that the mass terms in (28) are given by $m^2\mathcal{O}(t,\xi)$ could in principle be used to obtain the expectation value of Kondo's operator from the derivative

$$\langle \mathcal{O}(t,\xi)\rangle = -\frac{\partial}{\partial m^2}\ \ln Z_{\rm mGF}(t,\xi,m^2)|_{m^2=0}, \tag{34}$$

upon insertion into the unfixed partition function of lattice gauge theory, i.e., with taking the additional expectation value in the gauge-field ensemble. As any other observable at $m^2 = 0$ this expectation value as it stands, unfortunately, of course also suffers from Neuberger's $0/0$ problem of lattice BRST.

In order to demonstrate that the Curci-Ferrari mass regulates the Neuberger zero, for $t = 0$ we will verify by explicit calculation that

$$Z_{\rm mGF}(0,\xi,m^2) \neq 0. \tag{35}$$

In fact, from (28) and (30),

$$Z_{\rm mGF}(0,\xi,m^2) = \mathcal{N}\int d[b,\bar{c},c]\exp\biggl\{-\sum_i\biggl(\frac{\xi}{2}b_i^a b_i^a - im^2\xi\bar{c}_i^a c_i^a + \frac{\xi}{8}(\bar{c}_i\times c_i)^2\biggr)\biggr\}, \tag{36}$$

which again factorizes into independent Grassmann (and b-field) integrations per site on the lattice. Using the same rescaling $\sqrt{\xi}b \to b$ and $\sqrt[4]{\xi}\bar{c} \to \bar{c}$, $\sqrt[4]{\xi}c \to c$ as mentioned in the last section, we obtain

$$Z_{\rm mGF}(0,\xi,m^2) = (V_N(2\pi)^{(N^2-1)/2}I_N(m^2\sqrt{\xi}))^{\#\rm sites}, \tag{37}$$

where V_N is the group volume of $SU(N)$, and

$$I_N(\hat{m}^2) = \int\prod_{a=1}^{N^2-1} d(i\bar{c}^a)dc^a\exp\biggl\{i\hat{m}^2\bar{c}\cdot c - \frac{1}{8}(\bar{c}\times c)^2\biggr\}, \tag{38}$$

where we used the rather obvious abbreviations $\bar{c}\cdot c = \bar{c}^a c^a$, $(\bar{c}\times c)^a = f^{abc}\bar{c}^b c^c$, and $\hat{m}^2 = m^2\sqrt{\xi}$. Note that we define the Grassmann integration measure to include the imaginary unit i with the real antighosts $\bar{c}$ so as to reproduce the result of integrating over complex conjugate Grassmann variables $c^a \pm i\bar{c}^a$. Expanding the exponential and collecting the relevant powers in the ghost/antighost variables, for $SU(2)$ we straightforwardly obtain

$$I_2(\hat{m}^2) = \tfrac{3}{4}\hat{m}^2(1 + \tfrac{4}{3}\hat{m}^4). \tag{39}$$

For $SU(3)$ the computation is a bit more tedious, the result is

$$I_3(\hat{m}^2) = \frac{45}{64}\hat{m}^4\biggl(1 + 4\hat{m}^4 + \frac{64}{15}\hat{m}^8 + \frac{64}{45}\hat{m}^{12}\biggr). \tag{40}$$

In both cases we factorized the leading power for $\hat{m}^2 \to 0$. $I_N(\hat{m}^2)$ is polynomial in $\hat{m}^2 = m^2\sqrt{\xi}$ of degree $N^2 - 1$, for all N. The successively lower powers of $\hat{m}^2$ decrease by 2 in each step in this polynomial, reflecting an increasing power of the quartic-ghost self-interactions contributing to each term. Therefore, the polynomials $I_N(\hat{m}^2)$ are odd/even in $\hat{m}^2$ for N even/odd.

Because the polynomial is odd for all even N, there can thus not be an order-zero term in the first place. The powers of the quartic interactions alone never match the number of independent Grassmann variables, and the Neuberger zero at $\hat{m}^2 = 0$ arises rather trivially for even N, for the same reason that the Euler characteristic of an odd-dimensional manifold, here of dimension $N^2 - 1$, necessarily vanishes.

For N odd, $I_N(\hat{m}^2)$ is an even polynomial which could in principle have an order zero, constant term. The fact that this term is absent, e.g., as explicitly verified for $SU(3)$ in (40), reflects the vanishing of the Euler characteristic of $SU(N)$ also for odd N, as mentioned above. The even dimension $N^2 - 1$ of the algebra is irrelevant in this case, because, for the purpose of cohomology, the parameter space of $SU(N)$ behaves as a product of odd-dimensional spheres $S^3\times S^5\times S^7\times\cdots S^{2N-1}$ [27].

The polynomials $I_N(\hat{m}^2)$ do not have a constant term in either case and therefore vanish with $\hat{m}^2 \to 0$, i.e., $I_N(0) = 0$, as expected. Moreover, the scaling argument used here and in the last section shows that the partition function (30) of the massive Curci-Ferrari model can only depend on two of the three parameters,

$$Z_{\rm mGF}(t,\xi,m^2) = f(t/\sqrt{\xi},\xi m^4). \tag{41}$$

An independent route of deriving this generic form, from the equations of motions, will be presented below. In this section we explicitly obtained $f(0,y)$ with $y = \hat{m}^4$ to constrain this function $f(x,y)$ of two variables along the $x = t/\sqrt{\xi} = 0$ line, and verified that

$$Z_{\rm mGF}(0,\xi,m^2) = f(0,\xi m^4) \propto \begin{cases}(\xi m^4)^{\#\text{sites}/2}, & N=2\\ (\xi m^4)^{\#\text{sites}}, & N=3\end{cases}$$

for $m^2 \to 0$. Because of the topological explanation of the zero obtained in this limit, i.e., $f(0,0) = 0$, as discussed in the last section, this actually constrains f to vanish along the entire $y = 0$ line, $f(x,0) = 0$ for all $x = t/\sqrt{\xi}$.

For $x = 0$ we can in principle therefore define a nonvanishing, finite limit,

$$\lim_{m^2\to 0}(\xi m^4)^{-N_{\rm tot}}Z_{\rm mGF}(0,\xi,m^2) = \text{const} \tag{42}$$

with an appropriate power $N_{\rm tot} = \#$ of sites on a finite lattice for odd N, or half that for even N. This constant could thus be inserted into the unfixed lattice gauge theory measure without harm, i.e., avoiding the zero in (21). Because $x = t/\sqrt{\xi} = 0$, however, this still has no effect in terms of gauge fixing by the Faddeev-Popov procedure either. We need to get away from $x = 0$, at least by a small amount, in order to suppress those parts of the gauge orbits with large violations of the Lorenz condition. At a finite

Curci-Ferrari mass m^2, however, this is aggravated by the fact that the gauge-fixing partition function of the Curci-Ferrari model is no longer that of a topological model, and we thus no longer have the t independence (or x independence) of (19) either. We can therefore not as yet conclude at this point that the constant in (42) will essentially remain unchanged when going to some finite $x = t/\sqrt{\xi} \neq 0$ as we must.

We are not quite there yet, and we will therefore have to have a closer look at the parameter dependence of the massive Curci-Ferrari model in the next section.

V. PARAMETER DEPENDENCES

From Eqs. (24) and (30) or (28) we immediately obtain the following (logarithmic) derivatives:

$$\begin{aligned} t\frac{\partial}{\partial t}\ln Z_{\mathrm{mGF}}(t,\xi,m^2) &= -i\langle(s\bar{s}-im^2)tV_U[g]\rangle_{m^2},\\ 2\xi\frac{\partial}{\partial \xi}\ln Z_{\mathrm{mGF}}(t,\xi,m^2) &= -i\left\langle(s\bar{s}-im^2)\left(-i\xi\sum_i \bar{c}_i^a c_i^a\right)\right\rangle_{m^2},\\ m^2\frac{\partial}{\partial m^2}\ln Z_{\mathrm{mGF}}(t,\xi,m^2) &= -\langle m^2\mathcal{O}(t,\xi)\rangle_{m^2}, \end{aligned} \tag{43}$$

where the subscripts m^2 on the right denote expectation values within the Curci-Ferrari model at finite mass. In particular, the derivative w.r.t. m^2 in the last line differs from (34) only in that m^2 has not been set to zero here yet. All these expectation values can, in general, depend on the link-field configuration $\{U\}$ which acts as a background field to the model. Independence of $\{U\}$ is only guaranteed to hold in the topological limit $m^2 \to 0$.

From the definition of $\mathcal{O}$ in (31), we thus find that

$$\begin{aligned} &\left(t\frac{\partial}{\partial t} + 2\xi\frac{\partial}{\partial \xi} - m^2\frac{\partial}{\partial m^2}\right)\ln Z_{\mathrm{mGF}}(t,\xi,m^2)\\ &\quad = -i\langle s\bar{s}\mathcal{O}(t,\xi)\rangle_{m^2}. \end{aligned} \tag{44}$$

The standard argument that the expectation value of an (anti)BRST exact operator vanishes does not hold at finite m^2. Neither are BRST and anti-BRST variations nilpotent, nor is $\mathcal{O}$ invariant under the BRST or anti-BRST transformations. However, the equations of motion for (anti) ghost and Nakanishi-Lautrup fields on the lattice, i.e., their lattice Dyson-Schwinger equations, can be used to show that, indeed,

$$\langle s\bar{s}\mathcal{O}(t,\xi)\rangle_{m^2} = 0, \tag{45}$$

even at finite m^2. This is shown explicitly in Appendix B. Therefore,

$$\left(t\frac{\partial}{\partial t} + 2\xi\frac{\partial}{\partial \xi} - m^2\frac{\partial}{\partial m^2}\right)Z_{\mathrm{mGF}}(t,\xi,m^2) = 0. \tag{46}$$

This differential equation entails that we can write the partition function of the model in the generic form (41).

As we already did in the previous sections, we therefore continue to use the new parameters $x = t/\sqrt{\xi}$ and $\hat{m}^2 = m^2\sqrt{\xi}$ from now on, writing

$$Z_{\mathrm{mGF}} \equiv Z_{\mathrm{mGF}}(x, \hat{m}^2). \tag{47}$$

Again using rescaled fields $\sqrt{\xi}b \to b$, $\sqrt[4]{\xi}\bar{c} \to \bar{c}$, $\sqrt[4]{\xi}c \to c$ and with $\sqrt[4]{\xi}\bar{s} \to \bar{s}$, $\sqrt[4]{\xi}s \to s$, so that the (anti)BRST transformations of Eqs. (6)–(11) remain formally unchanged, the only modification is the replacement of m^2 by $\hat{m}^2$ in those of the massive model in Eqs. (25) and (26). Correspondingly, all other relations above are then converted by the formal replacements $\xi \to 1$, $t \to x$, and $m^2 \to \hat{m}^2$. In particular,

$$\begin{aligned} S_{\mathrm{mGF}}(x) &= i(s\bar{s} - i\hat{m}^2)\left(xV_U[g] - \frac{i}{2}\sum_i \bar{c}_i^a c_i^a\right)\\ &= \sum_i\left\{-ixb_i^aF_i^a(U^g) - ix\bar{c}_i^aM_{\mathrm{FP}i}^a[c] + \frac{1}{2}b_i^ab_i^a\right.\\ &\qquad \left. + \frac{1}{8}(\bar{c}_i \times c_i)^2\right\} + \hat{m}^2\mathcal{O}(x), \end{aligned} \tag{48}$$

with

$$\mathcal{O}(x) = xV_U[g] - i\sum_i \bar{c}_i^a c_i^a. \tag{49}$$

The two independent derivatives remaining are readily read off in an analogous way to give

$$\begin{aligned} \frac{\partial}{\partial x}\ln Z_{\mathrm{mGF}}(x,\hat{m}^2) &= -i\langle(s\bar{s} - i\hat{m}^2)V_U[g]\rangle_{\hat{m}^2},\\ \frac{\partial}{\partial \hat{m}^2}\ln Z_{\mathrm{mGF}}(x,\hat{m}^2) &= -\langle\mathcal{O}(x)\rangle_{\hat{m}^2}. \end{aligned} \tag{50}$$

In the absence of a topological argument for the gauge-parameter independence at finite Curci-Ferrari mass, the best we can do to achieve independence of $x = t/\sqrt{\xi}$ is to allow an x dependent mass parameter $\hat{m}^2 \equiv \hat{m}^2(x)$. In particular, the $x = 0$ results of the previous section are then to be interpreted as being expressed in terms of $\hat{m}^2(0)$. These results will remain unchanged for $x \neq 0$, if we adjust the mass function $\hat{m}^2(x)$ with x in the partition function Z_{mGF}, accordingly; that is, if

$$\begin{aligned} 0 &= \frac{d}{dx}Z_{\mathrm{mGF}}(x,\hat{m}^2(x))\\ &= \left(\frac{\partial}{\partial x} + \frac{d\hat{m}^2}{dx}\frac{\partial}{\partial \hat{m}^2}\right)Z_{\mathrm{mGF}}(x,\hat{m}^2(x)). \end{aligned} \tag{51}$$

From Eq. (50) we see that this requires that

$$\frac{d\hat{m}^2}{dx} = -i\frac{\langle(s\bar{s} - i\hat{m}^2)V_U[g]\rangle_{\hat{m}^2}}{\langle\mathcal{O}(x)\rangle_{\hat{m}^2}}. \tag{52}$$

This might not appear to be a very profound insight, because we simply arranged matters by hand to achieve gauge-parameter independence in this way. The crucial question at this point is whether the tuning of the Curci-Ferrari mass parameter with x is possible independent of the link configuration $\{U\}$ which is far from obvious here.

Otherwise we would have to choose a different trajectory in the parameter space $(x, \hat{m}^2)$ for different gauge orbits which would be of little use then, as far as the Faddeev-Popov gauge-fixing procedure is concerned. If it is possible, on the other hand, we can then use the value of the mass $\hat{m}_0^2 = \hat{m}^2(0)$ at $x = 0$ to regulate the Neuberger zero and use the x and $\{U\}$ independent, nonvanishing and finite constant

$$\lim_{\hat{m}_0^2 \to 0} (\hat{m}_0^4)^{-N_{\mathrm{tot}}} Z_{\mathrm{mGF}}(x, \hat{m}^2(x)) = \mathrm{const} \tag{53}$$

as the starting definition of Faddeev-Popov gauge fixing on the lattice. Then, of course, we would also expect that there should be a topological meaning to this constant which is so far, however, unfortunately unknown to us.

VI. ORBIT INDEPENDENT GAUGE- PARAMETER EXPANSION OF THE CURCI-FERRARI MASS

As we have seen in the previous section, the gauge-parameter independence of the gauge-fixing partition function Z_{mGF} of the massive Curci-Ferrari model will in general require the rescaled Curci-Ferrari mass parameter $\hat{m}^2 = m^2\sqrt{\xi}$ to depend on the gauge parameter in a nontrivial way, i.e., $\hat{m}^2 \equiv \hat{m}^2(x)$, via $x = t/\sqrt{\xi}$.

Gauge-parameter independence requires the derivative of $\hat{m}^2(x)$ to be given by Eq. (52). Together with the condition that $\hat{m}^2(0) = \hat{m}_0^2$, we can use this equation to obtain the coefficients of $\hat{m}^2(x)$ in a Taylor series expansion around $x = 0$, where we can do explicit calculations. Importantly, we can then verify that these coefficients will not depend on the gauge orbit, i.e., on (the class of gauge-equivalent) link configurations $\{U\}$. Being based on the tensor method of invariant integrations over the gauge-group elements per site i of the lattice, this perturbative expansion of the Curci-Ferrari mass parameter at small x will in fact be gauge-orbit independent at every order. As always, of course, nothing can be learned in such an expansion about possible nonanalytic contributions. We therefore assume the analyticity of the massive Curci-Ferrari model in the "would-be-Gauss-Bonnet" limit $x \to 0$. This should surely be valid in the massless limit, but we need to assume here that the limits $x \to 0$ and $\hat{m}_0^2 \to 0$ can be interchanged, in addition. While this is all well on a finite lattice, it certainly needs to be kept in mind when studying the model in the infinite-volume and continuum limits.

On a finite lattice, it is relatively straightforward to show that $\hat{m}^2$ is in fact independent of the gauge parameter x at 1st order in this expansion, i.e., that

$$\frac{d\hat{m}^2}{dx}\bigg|_{x=0} = 0. \tag{54}$$

This is simply because the numerator in Eq. (52) vanishes at $x = 0$ while the denominator is a finite number. To see this explicitly, first consider at $x = 0$ Eq. (36) with our new variables and rescaled fields, before the gauge-group integrations,

$$Z_{\mathrm{mGF}}(0, \hat{m}_0^2) = \int d[g, b, \bar{c}, c] \exp\bigg\{-\sum_i \bigg(\frac{1}{2} b_i^a b_i^a + \frac{1}{8}(\bar{c}_i \times c_i)^2 - i\hat{m}_0^2 \bar{c}_i \cdot c_i\bigg)\bigg\}. \tag{55}$$

As mentioned above, it decouples from the link-field configuration and factorizes. Relative to this partition function, we obtain

$$\langle i\bar{c}_i^a c_j^b \rangle_{\hat{m}^2}|_{x=0} = \frac{\delta^{ab}\delta_{ij}}{N^2 - 1} \frac{I_N'(\hat{m}_0^2)}{I_N(\hat{m}_0^2)}, \tag{56}$$

which is easily verified from the rules of Grassmann integration and Eqs. (36)–(38). Because $\mathcal{O}(0) = -i\sum_i \bar{c}_i^a c_i^a$, the denominator in (52) at $x = 0$ is obtained from the trace in (56),

$$\langle \mathcal{O}(x) \rangle_{\hat{m}^2}|_{x=0} = \bigg\langle -i\sum_i \bar{c}_i^a c_i^a \bigg\rangle_{\hat{m}^2}\bigg|_{x=0} = -(\#\mathrm{sites}) \times \frac{I_N'(\hat{m}_0^2)}{I_N(\hat{m}_0^2)}, \tag{57}$$

which on a finite lattice with nonvanishing $\hat{m}_0^2$ is finite.

For the numerator in (52) we have to compute

$$\frac{\partial}{\partial x} \ln Z_{\mathrm{mGF}}(x, \hat{m}^2) = -i\langle (s\bar{s} - i\hat{m}^2) V_U[g] \rangle_{\hat{m}^2} = \langle i(b, F) + i(\bar{c}, M_{\mathrm{FP}} c) - \hat{m}^2 V_U[g] \rangle_{\hat{m}^2}, \tag{58}$$

where the brackets are introduced for summation over site and color indices. At $x = 0$, the b-field integration is Gaussian and the first term in (58), linear in b, therefore vanishes. Because the gauge fields decouple from the measure in (55), Eq. (56) produces the trace of the Faddeev-Popov matrix M_{FP} in the second term on the right of (58) for $x = 0$. With

$$\mathrm{tr}\, M_{\mathrm{FP}}(U^g) = -2C_2^\rho V_U[g], \tag{59}$$

where C_2^ρ is the quadratic Casimir invariant in representation ρ, $C_2^f = (N - 1/N)/2$ in the fundamental one, we see that the second becomes proportional to the third, and both proportional to the expectation value of the Morse potential $V_U[g]$. This is linear in U^g and contains exactly one element g_i (or $g_i^\dagger$) in each and every term for which the invariant integration over the gauge group at the particular site i produces a zero. Therefore,

$$\langle V_U[g] \rangle_{\hat{m}^2}|_{x=0} = 0, \tag{60}$$

which establishes (54). It means that the derivative of $\hat{m}(x)$ is of the order x near $x = 0$, or

$$\hat{m}^2(x) = \hat{m}_0^2 + \mathcal{O}(x^2). \tag{61}$$

In order to compute the constant in the second order term, and verify that it is nonvanishing and independent of $\{U\}$, we can consider the derivative w.r.t. x of the numerator in (52), or

$$\frac{\partial^2}{\partial x^2}\ln Z_{\mathrm{mGF}}(x,\hat{m}^2)=\langle(i(b,F)+i(\bar{c},M_{\mathrm{FP}}c)-\hat{m}^2V_U[g])^2\rangle_{\hat{m}^2}-\langle i(b,F)+i(\bar{c},M_{\mathrm{FP}}c)-\hat{m}^2V_U[g]\rangle^2_{\hat{m}^2}. \quad (62)$$

The second (disconnected) term again vanishes at $x=0$. Expanding the square in the first, we once more use that every term linear in the b field will also vanish at $x=0$, and therefore

$$\frac{\partial^2}{\partial x^2}\ln Z_{\mathrm{mGF}}(x,\hat{m}^2)|_{x=0}=\langle(ib,F)^2\rangle_{\hat{m}^2,0}+\langle(i\bar{c},M_{\mathrm{FP}}c)^2\rangle_{\hat{m}^2,0}+\hat{m}_0^4\langle V_U^2[g]\rangle_{\hat{m}^2,0}-2\hat{m}_0^2\langle(i\bar{c},M_{\mathrm{FP}}c)V_U[g]\rangle_{\hat{m}^2,0}, \quad (63)$$

where we used short-hand $\langle\cdots\rangle_{\hat{m}^2,0}\equiv\langle\cdots\rangle_{\hat{m}^2}|_{x=0}$.

We calculate and discuss each of the individual terms on the right of (63) separately in Appendix C (with the exception that, by the same argument that led to (59), the last two are essentially the same again, i.e., both $\propto\langle V_U^2\rangle_{\hat{m}^2,0}$). In particular, we show there that they are indeed all independent of $\{U\}$. The results for $SU(N)$ gauge groups in D Euclidean dimensions are summarized in Table I, where I_N is defined in Eq. (38) and J_N analogously by

$$J_N(\hat{m}^2)=\int\prod_{a=1}^{N^2-1}d(i\bar{c}^a)dc^a\Big(-\frac{1}{8}(\bar{c}\times c)^2\Big)\times\exp\Big\{i\hat{m}^2\bar{c}\cdot c-\frac{1}{8}(\bar{c}\times c)^2\Big\}. \quad (64)$$

A comparison of the integral expression for J_N with the analogous one for I_N in Eq. (38) shows that, with an explicit interaction inserted in the integral, $J_N(\hat{m}^2)$ is a polynomial in $\hat{m}^2$ of two orders less than $I_N(\hat{m}^2)$. Just as $I_N(\hat{m}^2)$, this polynomial has no zero-order, constant term because the Euler characteristic of $SU(N)$ vanishes and we essentially obtain a Gauss-Bonnet integral for $\hat{m}^2\to 0$ again. Therefore,

$$J_N(\hat{m}^2)\sim I_N(\hat{m}^2)\to 0, \qquad \text{for } \hat{m}^2\to 0. \quad (65)$$

Explicitly, for $SU(2)$, see Appendix C,

$$J_2(\hat{m}^2)=\tfrac{1}{8}I_2''(\hat{m}^2)=\tfrac{3}{4}\hat{m}^2, \quad (66)$$

while for $SU(3)$ a straightforward but a bit more tedious computation analogous to that used for Eq. (40) yields

$$J_3(\hat{m}^2)=\frac{135}{64}\hat{m}^4+\frac{45}{8}\hat{m}^8+3\hat{m}^{12}. \quad (67)$$

With the results in Table I we can go back to the derivative of the Curci-Ferrari mass parameter $\hat{m}^2(x)$ w.r.t. x. Recall that the independence of the massive Curci-Ferrari model on the gauge parameter $x=t/\sqrt{\xi}$ requires this derivative to be given by the ratio, cf. Eqs. (50) and (52),

$$\frac{d\hat{m}^2}{dx}=\frac{\frac{\partial}{\partial x}\ln Z_{\mathrm{mGF}}(x,\hat{m}^2)}{\langle\mathcal{O}(x)\rangle_{\hat{m}^2}}. \quad (68)$$

With (63) and the results from the table, in the limit $x\to 0$ we therefore find

$$\frac{d\hat{m}^2}{dx}\xrightarrow{x\to 0}x2D\Big\{\frac{N^2-1}{N}\frac{I_N(\hat{m}_0^2)}{I_N'(\hat{m}_0^2)}-\frac{2}{N}\frac{J_N(\hat{m}_0^2)}{I_N'(\hat{m}_0^2)}-\hat{m}_0^4\frac{I_N(\hat{m}_0^2)}{I_N'(\hat{m}_0^2)}-2\hat{m}_0^2\frac{1}{N}\Big\}, \quad (69)$$

for $N\geq 3$, and

$$\frac{d\hat{m}^2}{dx}\xrightarrow{x\to 0}x2D\Big\{3\frac{I_2(\hat{m}_0^2)}{I_2'(\hat{m}_0^2)}-\frac{1}{4}\frac{I_2''(\hat{m}_0^2)}{I_2'(\hat{m}_0^2)}-2\hat{m}_0^4\frac{I_2(\hat{m}_0^2)}{I_2'(\hat{m}_0^2)}-2\hat{m}_0^2\Big\}=-x2D\hat{m}_0^2\gamma_2(\hat{m}_0^2) \quad (70)$$

for $SU(2)$, where, using Eq. (39) for I_2,

$$\gamma_2(\hat{m}_0^2)=\frac{3+18\hat{m}_0^4+8\hat{m}_0^8}{3+12\hat{m}_0^4}. \quad (71)$$

In either case, the expansion of the Curci-Ferrari mass parameter around $x=0$ for $SU(N)$ in D dimensions can finally be written in the general form

$$\hat{m}^2(x)=\hat{m}_0^2-x^2D\hat{m}_0^2\gamma_N(\hat{m}_0^2)+\mathcal{O}(x^3), \quad (72)$$

where $\gamma_N(\hat{m}_0^2)$ is a ratio of polynomials in $\hat{m}_0^2$ obtained via (69), for $N\geq 3$, from

$$\gamma_N(\hat{m}_0^2)=\frac{2}{N}\Big(1+\frac{J_N(\hat{m}_0^2)}{\hat{m}_0^2I_N'(\hat{m}_0^2)}\Big)-\Big(\frac{N^2-1}{N}-\hat{m}_0^4\Big)\times\frac{I_N(\hat{m}_0^2)}{\hat{m}_0^2I_N'(\hat{m}_0^2)}. \quad (73)$$

TABLE I. The ratios of expectation values needed to compute the leading contribution to the right-hand side of Eq. (68) from the individual terms in Eq. (63) for the numerator in the limit $x\to 0$.

Ratio of expectation values	$N=2$	$N\geq 3$
$\langle(ib,F)^2\rangle_{\hat{m}^2,0}/\langle\mathcal{O}(x)\rangle_{\hat{m}^2,0}$	$6D\frac{3\hat{m}_0^2+4\hat{m}_0^6}{3+12\hat{m}_0^4}$	$2D(N-\frac{1}{N})\frac{I_N(\hat{m}_0^2)}{I_N'(\hat{m}_0^2)}$
$\langle V_U^2[g]\rangle_{\hat{m}^2,0}/\langle\mathcal{O}(x)\rangle_{\hat{m}^2,0}$	$-4D\frac{3\hat{m}_0^2+4\hat{m}_0^6}{3+12\hat{m}_0^4}$	$-2D\frac{I_N(\hat{m}_0^2)}{I_N'(\hat{m}_0^2)}$
$\langle(i\bar{c},M_{\mathrm{FP}}c)V_U[g]\rangle_{\hat{m}^2,0}/\langle\mathcal{O}(x)\rangle_{\hat{m}^2,0}$	$2D$	$2D/N$
$\langle(i\bar{c},M_{\mathrm{FP}}c)^2\rangle_{\hat{m}^2,0}/\langle\mathcal{O}(x)\rangle_{\hat{m}^2,0}$	$-2D\frac{6\hat{m}_0^2}{3+12\hat{m}_0^4}$	$-2D\frac{2}{N}\frac{J_N(\hat{m}_0^2)}{I_N'(\hat{m}_0^2)}$

For $N = 3$, using (40) and (67), we find explicitly

$$\gamma_3(\hat{m}_0^2) = \frac{1}{3}\frac{90 + 855\hat{m}_0^4 + 1692\hat{m}_0^8 + 1088\hat{m}_0^{12} + 192\hat{m}_0^{16}}{90 + 720\hat{m}_0^4 + 1152\hat{m}_0^8 + 512\hat{m}_0^{12}}. \tag{74}$$

To work out γ_N for general N in the limit $\hat{m}_0^2 \to 0$, remember that $I_N(\hat{m}_0^2)$ is an odd/even polynomial in $\hat{m}_0^2$ for N even/odd. In either case there is no constant term $I_N(\hat{m}_0^2) \to 0$ for $\hat{m}_0^2 \to 0$. Therefore

$$\frac{I_N(\hat{m}_0^2)}{\hat{m}_0^2 I_N'(\hat{m}_0^2)} \xrightarrow{\hat{m}_0^2 \to 0} \begin{cases} 1, & N \text{ even}, \\ \frac{1}{2}, & N \text{ odd}. \end{cases} \tag{75}$$

For the leading power n of $\hat{m}_0^2$ in $I_N(\hat{m}_0^2)$ near $\hat{m}_0^2 = 0$, which is $n = 1$ when N is even and $n = 2$ when N is odd, we need to expand the exponential of the ghost self-interactions in its integral representation (38) to a power p such that

$$n + 2p = N^2 - 1.$$

Otherwise the Grassmann integrations over $N^2 - 1$ ghost (and antighost) variables will produce zero. Therefore,

$$p = \begin{cases} \frac{N^2-2}{2}, & N \text{ even}, \\ \frac{N^2-3}{2}, & N \text{ odd}. \end{cases} \tag{76}$$

Comparing the definition of J_N in Eq. (C37) to that of I_N in (38), we see that the exponential of the ghost self-interactions in the integral representation of J_N needs to be expanded to one power less, i.e., to the power $p - 1$ for the leading term. Comparing the prefactors of these terms in each case we therefore find

$$\frac{J_N(\hat{m}_0^2)}{I_N(\hat{m}_0^2)} \xrightarrow{\hat{m}_0^2 \to 0} \frac{p!}{(p-1)!} = p. \tag{77}$$

With (75) and (76) this then implies

$$\frac{J_N(\hat{m}_0^2)}{\hat{m}_0^2 I_N'(\hat{m}_0^2)} \xrightarrow{\hat{m}_0^2 \to 0} \begin{cases} \frac{N^2-2}{2}, & N \text{ even}, \\ \frac{N^2-3}{4}, & N \text{ odd}. \end{cases} \tag{78}$$

For even $N \geq 4$ we thus obtain from Eq. (73)

$$\gamma_N \to \frac{2}{N}\left(1 + \frac{N^2 - 2}{2}\right) - \frac{N^2 - 1}{N} = \frac{1}{N}, \tag{79}$$

and twice that for $SU(2)$, cf. Eq. (71), where $\gamma_2 \to 1 = 2/N$ for $N = 2$, in the limit $\hat{m}_0^2 \to 0$. This doubling in the $SU(2)$ case essentially comes about because for the expectation values containing terms which are at most quadratic in the link variables U at this order, only for $N = 2$ we obtain contributions from two types of invariant integrals, Eqs. (C3) and (C6), while only the gauge-group integrations of the form in (C3) contribute for $N \geq 3$ at this order [additional contributions similar to those for $SU(2)$ here, will arise, e.g. for $SU(3)$ at the next order etc.].

For all odd $N \geq 3$, at the present order, (75) and (76) therefore give

$$\gamma_N \to \frac{2}{N}\left(1 + \frac{N^2 - 3}{4}\right) - \frac{N^2 - 1}{2N} = \frac{1}{N}, \tag{80}$$

which again leads to the same result as obtained for the even $N \geq 4$ above, i.e.,

$$\gamma_N(\hat{m}_0^2) \to \frac{1}{N}, \qquad \text{for } \hat{m}_0^2 \to 0, \tag{81}$$

and all $N \geq 3$.

All these results are gauge-orbit independent as they must. While this is merely necessary, but not sufficient, it demonstrates that we can get away from $x = 0$, at least perturbatively in a small x expansion. This is of qualitative importance as a nonzero value of $x = t/\sqrt{\xi}$, no matter how small, corresponds to a large but finite ξ at $t = 1$ and thus eliminates the gauge freedom.

In summary, the gauge-orbit independence of the constant in the second order term of the small x expansion of $\hat{m}^2(x)$ is a direct consequence of the invariant integrations over the gauge-group elements at each site. The gauge-group integrations can in fact be performed at any order in this Taylor expansion around $x = 0$ because the action is independent of the gauge group there [cf. Eq. (55)]. Moreover, the invariant tensor method can be used to demonstrate how these integrations will ensure that the coefficients in this Taylor expansion are indeed independent of $\{U\}$ at any order in x. This gauge-orbit independence of the mass expansion is verified explicitly for the constant in the second order term of (61) from the results in Appendix C. In particular, as we have shown above, in D Euclidean dimensions the gauge-parameter expansion of the Curci-Ferrari mass in Eq. (61) becomes

$$\hat{m}^2(x) = \hat{m}_0^2(1 - D\gamma_N(\hat{m}_0^2)x^2 + \mathcal{O}(x^3)), \tag{82}$$

where γ_2 and γ_3 for $SU(2)$ and $SU(3)$ are explicitly given by Eqs. (71) and (74). Moreover, at leading order in the Curci-Ferrari mass parameter, $\gamma_N(\hat{m}_0^2)$ is finite and of the order $1/N$ in the limit $\hat{m}_0^2 \to 0$. For general $SU(N)$ gauge groups we found

$$\gamma_N(\hat{m}_0^2) = \begin{cases} 1 + \mathcal{O}(\hat{m}_0^4), & N = 2, \\ \frac{1}{N} + \mathcal{O}(\hat{m}_0^4), & N \geq 3. \end{cases} \tag{83}$$

The leading x dependence of the Curci-Ferrari mass is therefore order N suppressed. Without need to adjust the Curci-Ferrari mass parameter $\hat{m}^2$ with x, the gauge-fixing partition function Z_{mGF} of the massive Curci-Ferrari model therefore becomes gauge parameter independent at least up to the order x^3 in the large N limit.

VII. SUMMARY

We have formulated the Curci-Ferrari model on the lattice. In the massless case this model provides an explicit demonstration of the topological origin of the Neuberger 0/0 problem of lattice BRST. The starting point of Neuberger's original argument was the observation of uncompensated Grassmann integrations producing a zero result in a certain limit. This same limit in the massless Curci-Ferrari model with its double BRST symmetry and quartic-ghost self-interactions corresponds to the Gauss-Bonnet limit, $\xi \to \infty$, of a topological model that computes the Euler characteristic of the lattice gauge group which vanishes for compact Lie groups. The fact that the Neuberger zero is independent of this special limit then follows directly from the gauge-parameter ξ independence of the topological model.

Introducing a Curci-Ferrari mass term regularizes the Neuberger zero. The analogue of the Gauss-Bonnet-Neuberger limit here corresponds to the gauge parameter $\xi \to \infty$ limit together with the Curci-Ferrari mass $m^2 \to 0$ such that the product $m^2\sqrt{\xi}$ remains finite, i.e., $m^2\sqrt{\xi} \to \hat{m}^2$ for some finite $\hat{m}^2$. In this limit, the partition function of the massive Curci-Ferrari model on a finite lattice is obtained as a polynomial in the new mass parameter $\hat{m}^2$ and is hence nonvanishing. The $0/0$ problem is thus avoided. However, the massive Curci-Ferrari model is no longer a purely topological model. BRST and anti-BRST are explicitly broken by the Curci-Ferrari mass. The $sl(2,\mathbb{R})\rtimes$ double BRST algebra of the massless Curci-Ferrari model is decontracted into a simple superalgebra for $m^2 \neq 0$. As a result of this BRST breaking, meaning that neither BRST nor anti-BRST transformations are nilpotent anymore, the gauge-fixing partition function of the massive Curci-Ferrari model is *a priori* not independent of the gauge parameter ξ. This implies that the Curci-Ferrari mass has to depend on ξ so as to restore total ξ independence, a requirement which in turn allows one to determine this ξ dependence of m^2. A gauge-orbit independent Taylor series expansion of $m^2(\xi)$, in D dimensions of the form, cf. Eq. (82),

$$m^2(\xi) = \frac{\hat{m}_0^2}{\sqrt{\xi}}\left(1 - \frac{D\gamma_N}{\xi} + \cdots\right), \tag{84}$$

around the Gauss-Bonnet limit of $1/\xi \to 0$ is possible with ξ-independent mass parameter $\hat{m}_0^2$, to show that one can meaningfully define a limit $\hat{m}_0^2 \to 0$ in the spirit of l'Hospital's rule. In this limit, $\gamma_N \to 1/N$, for $SU(N)$ gauge theory with $N \geq 3$ [and $\gamma_2 \to 1$ for $SU(2)$].

VIII. CONCLUSIONS AND OUTLOOK

The explicit BRST breaking by the Curci-Ferrari mass term is well known to lead to unitarity violations in the continuum quantum field theory. The BRST-cohomology construction of a positive physical Hilbert space breaks down together with this BRST breaking. There are explicit examples of negative norm states mixing into what would otherwise be defined as the physical subspace [7,8]. For that reason, the parameter m^2 should not be interpreted as a physical mass but rather only as a regulator for the Neuberger $0/0$ problem of lattice BRST.

In the Landau-gauge limit with $\xi = 0$, for example, it has the effect of reweighting different Gribov copies depending on their value for the Morse potential on the gauge orbit, $V_U[g]$, which avoids their perfect cancellation. The analogue of the Poincaré-Hopf theorem for the Euler characteristic of the lattice gauge group, Eq. (23) for finite m^2, becomes

$$Z_{\mathrm{mGF}}(t, 0, m^2) = \sum_{\mathrm{copies}\{g^{(i)}\}} \mathrm{sign}(\det M_{\mathrm{FP}}(U^{g^{(i)}})) \times \exp\{-m^2 t V_U[g^{(i)}]\}. \tag{85}$$

At finite m^2 this leads to a suppression of Gribov copies outside the fundamental modular region, i.e., a suppression of all copies relative to the absolute minima of $V_U[g]$.

In particular, because we furthermore have $V_U[g] \propto -\mathrm{tr}M_{\mathrm{FP}}$, each positive (negative) eigenvalue will increase (suppress) the weight of a given copy. The situation is thus similar to that in the Gribov-Zwanziger approach [3,28] which includes a horizon functional to suppress Gribov copies with negative eigenvalues, outside the first Gribov region. As in the massive Curci-Ferrari model, this leads to a certain BRST breaking in the Gribov-Zwanziger framework also. While the renormalizability of this framework is maintained [28], the unitarity violations when attempting a BRST-cohomology construction of a physical Hilbert space remain to be a problem there as well.

The so-called soft BRST breaking of the Gribov-Zwanziger framework, which really means that it is a nonperturbative BRST breaking, should not be a problem, if it is effective in restricting the Landau gauge to the fundamental modular region, the set of absolute minima along the gauge orbits [29]. This should be intuitively expected, of course, for a perfect gauge fixing, but because of the existence of many degenerate absolute minima it is not mathematically obvious.

Note that we could at least formally achieve the same restriction here, if we define the Landau-gauge limit of the massive Curci-Ferrari model as the limit of vanishing gauge-parameter, $\xi \to 0$, at finite mass parameter $\hat{m}^2 = m^2\sqrt{\xi}$ (analogous to the opposite limit $\xi \to \infty$ at fixed $\hat{m}^2$ considered above). In fact, the nonrenormalization theorems in [19] can be used to show that $\hat{m}^2$ is a renormalization group invariant mass parameter of the continuum Curci-Ferrari model in the Landau gauge which can be verified explicitly at three loops in the $\overline{\mathrm{MS}}$ scheme from the results in [17]. It seems therefore not unreasonable to define a Landau-gauge limit as $\xi \to 0$ at fixed $\hat{m}^2$ also on the lattice, which would then imply $m^2 = \hat{m}^2/\sqrt{\xi} \to \infty$ in (85), so as to suppress all copies but the absolute minima of $V_U[g]$. In principle, this limit could thus be realized by the procedure of simulated annealing with a control "temperature" $\sqrt{\xi}/\hat{m}^2$ and a sufficiently slow annealing schedule (more and more slowly on larger lattices which poses the practical limitation of this procedure).

The arguments for the Gribov-Zwanziger framework to achieve this further restriction to the absolute minima in lattice formulations [30], furthermore involve the thermodynamic or infinite-volume limit, in which the common boundary of the fundamental modular region and the first Gribov region dominate the minimal Landau-gauge configuration space [31]. To find absolute minima in lattice simulations is not feasible for large lattices as this is a nonpolynomially hard computational problem. When sampling the local minima of the first Gribov region on a finite lattice, as is usually done in minimal lattice Landau-gauge implementations numerically, BRST breaking and unitarity violations could therefore be a potential problem there also.

In fact, related to this potential problem of minimal Landau-gauge implementations on finite lattices might be the question of a gluon mass and the infrared behavior of the gluon and ghost propagators in Landau gauge. While early lattice studies of those propagators [32,33] supported the predicted infrared behavior based on their Dyson-Schwinger equations qualitatively well [34,35], small but significant differences are increasingly being observed nowadays in detailed comparisons and studies of finite-volume effects [36–40]. In particular, as a result of these differences, the Kugo-Ojima confinement criterion of local quantum field theory is now frequently stated to be violated in the infinite-volume limit:

Based on the assumption that BRST invariance is unbroken, and that a nilpotent BRST charge can be defined nonperturbatively, the continuum prediction is that the infrared dominant correlations are mediated by the Faddeev-Popov ghosts, whose propagator is then found to be infrared enhanced in the Landau gauge, while the gluon propagator is found to be infrared suppressed. This infrared behavior can be completely understood in terms of confinement in QCD [2,41]; it is a consequence of the celebrated Kugo-Ojima confinement criterion. The subsequent verification of this infrared behavior with a variety of different functional methods in the continuum meant a remarkable success [42–44]. In fact, it is directly tied to the validity and applicability of the framework of local quantum field theory for non-Abelian gauge theories beyond perturbation theory. Consistent with the conditions for confinement in local quantum field theory, these predictions all lead to a conformal infrared behavior for gluonic Green's functions which is yet to be observed in lattice simulations. Because simulations must necessarily be done in a finite volume, where such a behavior can strictly never be observed, finite-volume effects have long been blamed for the remaining mismatch with the continuum studies. Recently, these effects have been analyzed carefully in the Dyson-Schwinger equations [36,37]. These results together with latest lattice data on huge lattices [38–40], corresponding to physical lengths of more than 20 fm in each direction, clearly rule out finite-volume effects as the sole origin for the observed discrepancies.

Rather, the most likely origin for these discrepancies potentially hints at a much more profound problem: a BRST breaking that a sampling of minima of the gauge-fixing potential in lattice simulations might bring about much like the restriction to the first Gribov region does in the Gribov-Zwanziger framework. The observed evidence of a massive infrared behavior of the gluon propagator together with a massless free ghost propagator without infrared enhancement in the infinite-volume limit in current lattice Landau-gauge implementations in fact suggests that this is the case. While this infrared behavior can be explained quite naturally as a dynamical effect related to the formation of a dimension 2 condensate in the Gribov-Zwanziger framework [45,46], the question of BRST breaking remains to be addressed. Any reweighting of Gribov copies, inside or outside the first Gribov region, appears to correspond to a BRST breaking procedure similar in effect to the introduction of an explicit Curci-Ferrari mass. Only if the numerical lattice procedure converges towards a correct sampling of the fundamental modular region in the infinite volume and continuum limits, can the BRST breaking effects be expected to go away. The most important remaining question then is to see whether the massive infrared behavior of the gluon propagator persists or not. If it does, it will preclude the infrared enhancement of the ghost propagator necessary for confinement in ghost/antighost symmetric gauges such as the Landau gauge. Because this Kugo-Ojima confinement criterion is based on the assumption of the existence of a nilpotent BRST charge, however, its potential violation can certainly not be inferred from the results of any BRST breaking procedure as this would not be within the rules of local quantum field theory for covariant gauge fields on indefinite metric spaces.

An interesting alternative procedure based on stereographic projection to define lattice gauge fields is provided by the modified lattice Landau gauge of Ref. [47]. This gives rise to a manifestly BRST invariant lattice formulation. The Neuberger $0/0$ problem is avoided there because it is not the vanishing Euler characteristic of the lattice gauge group that is calculated by the gauge-fixing partition function in this case, but that of a stereographically projected manifold. In this approach the pure lattice-artifact Gribov problem of compact $U(1)$ is avoided because the Faddeev-Popov operator is positive, and there are thus no cancellations between Gribov copies [47]. Consequently, there are none along the maximal Abelian subgroup $U(1)^{N-1}$ of $SU(N)$ either. This is just enough, however, to remove the complete cancellation of Gribov copies in $SU(N)$ also. The remaining cancellations between copies of either sign in $SU(N)$, which persist in the continuum limit, are necessarily incomplete because the Euler characteristic of the coset manifold is nonzero. It is essentially determined by that of the even-dimensional spheres $S^2 \times S^4 \times \cdots S^{2N-2}$ or, more precisely, of the corresponding even-dimensional, real projective spaces $\mathbb{R}P^{2n}$, of one dimension less than the odd-dimensional spheres of the group manifold. The perhaps most promising feature of the modified lattice Landau gauge is, however, that it provides a way to perform gauge-fixed Monte Carlo simulations sampling all Gribov copies of either sign in the spirit of BRST. Bridging the gap between gauge-fixed lattice and continuum studies this should, in particular, resolve the present discrepancies observed in the QCD Green's functions once and for all, and at the same time put our theoretical knowledge of QCD and all gauge theories of the standard model on solid ground in a completely nonperturbative manner.

Meanwhile, the decontracted double BRST supersymmetry of the massive Curci-Ferrari model on the lattice provides an interesting test bed with a controlled BRST breaking and regularized Neuberger zeros. It might have its own interesting topological features and interpretation as indicated by its potential gauge-orbit and gauge-parameter independence. This certainly deserves further study, especially with regard to its potential to generalize the topological field theory relation between the Gauss-Bonnet and Poincaré-Hopf theorems [25,48,49]. In fact, an extension to Morse theory similar to (85), but corresponding to the introduction of an imaginary Curci-Ferrari mass parameter [i.e., using a real parameter $\phi \equiv im^2 t$ in (85)], and its relation to a generalized Gauss-Bonnet theorem reminiscent of (36) were discussed in Ref. [50]. This might help to understand how it is possible to achieve the gauge parameter and orbit independence in the massive Curci-Ferrari model in general. It might provide an interesting new topological interpretation of the Curci-Ferrari model within the decontracted double BRST osp(1|2) superalgebra framework.

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APPENDIX A: BRST DERIVATION OF FADDEEV-POPOV OPERATOR AND GAUGE-FIXING ACTION

In this Appendix we show explicitly how the double (anti)BRST variation in the gauge-fixing action of the topological model in Eq. (12) leads to the explicit form given in (14). This lattice transcription of a well-known continuum procedure is mainly given for the reader's convenience and ready reference.

Performing the anti-BRST variation on the right-hand side in Eq. (12) first, we obtain

$$\bar{s}V_U[g] = \frac{1}{2\rho}\sum_i\sum_{j\sim i}\mathrm{tr}(\bar{c}_i U^g_{ij} - \bar{c}_j U^g_{ij}) = -\sum_i \bar{c}^a_i F^a_i(U^g), \tag{A1}$$

where

$$F^a_i(U^g) = -\frac{1}{2\rho}\sum_{j\sim i}\mathrm{tr}(X^a(U^g_{ij} - U^g_{ji})) \tag{A2}$$

is used in the standard gauge-fixing condition of covariant gauges. In the continuum limit it reduces to

$$F^a_i(U^g) \overset{a\to 0}{\longrightarrow} a^2\partial_\mu A^{ga}_\mu + \mathcal{O}(a^4). \tag{A3}$$

With Eqs. (9) and (10) we furthermore have

$$\bar{s}(\bar{c}^a c^a) = \bar{c}^a b^a, \tag{A4}$$

and therefore, for the gauge-fixing action, the alternative form

$$S_{\mathrm{GF}} = -i\sum_i s\bigg(\bar{c}^a_i\bigg(F^a_i(U^g) + \frac{i\xi}{2}b^a_i\bigg)\bigg). \tag{A5}$$

As in the continuum formulation, in this form it looks exactly like the gauge-fixing action of standard Faddeev-Popov theory for the linear-covariant gauge. The specific features of the ghost/antighost symmetric framework show when working out the remaining BRST variation. From the first term we have (i),

$$\sum_i (s\bar{c}^a_i)F^a_i = -\frac{1}{2\rho}\sum_i\sum_{j\sim i}\mathrm{tr}(b_i(U^g_{ij} - U^g_{ji})) + \frac{1}{4\rho}\sum_i\sum_{j\sim i}\mathrm{tr}(\{\bar{c}_i, c_i\}(U^g_{ij} - U^g_{ji})). \tag{A6}$$

Herein, the first term on the right implements the gauge-fixing condition as in standard Faddeev-Popov theory. The second term, containing the anticommutator $\{\bar{c}, c\}$, is characteristic of ghost/antighost symmetry because it combines with the remaining quadratic ghost terms to produce a Hermitian Faddeev-Popov operator (for any gauge parameter ξ). To see this explicitly, consider (ii),

$$\sum_i \bar{c}^a_i(sF^a_i) = \frac{1}{2\rho}\sum_i\sum_{j\sim i}\mathrm{tr}(\bar{c}_i c_i U^g_{ij} - \bar{c}_i U^g_{ij} c_j + c_j U^g_{ji}\bar{c}_i - c_i\bar{c}_i U^g_{ji}), \tag{A7}$$

so that the difference $(i) - (ii)$ yields

$$\begin{aligned}\sum_i s(\bar{c}^a_i F^a_i) &= -\frac{1}{2\rho}\sum_i\sum_{j\sim i}\mathrm{tr}(b_i(U^g_{ij} - U^g_{ji})) \\ &\quad + \frac{1}{2\rho}\sum_i\sum_{j\sim i}\mathrm{tr}\bigg(\bar{c}_i U^g_{ij} c_j - c_i U^g_{ij}\bar{c}_j \\ &\quad - [\bar{c}_i, c_i]\frac{1}{2}(U^g_{ij} + U^g_{ji})\bigg) \\ &\equiv \sum_i b^a_i F^a_i + \sum_{i,j}\bar{c}^a_i M^{ab}_{\mathrm{FP}ij} c^b_j,\end{aligned} \tag{A8}$$

which defines the lattice Faddeev-Popov operator M_{FP} of the ghost/antighost symmetric Curci-Ferrari gauges.

Note that the terms in (A7) can be written in the form

$$\sum_i \bar{c}_i^a(sF_i^a) = \frac{1}{4\rho}\sum_{i,j\sim i}\bar{c}_i^a\{\mathrm{tr}([X^a, X^b](U_{ij}^g - U_{ji}^g))(c_i^b + c_j^b) + \mathrm{tr}(\{X^a, X^b\}(U_{ij}^g + U_{ji}^g))(c_i^b - c_j^b)\}. \quad \text{(A9)}$$

This yields the standard Faddeev-Popov operator of the linear-covariant gauges on the lattice. It differs by the quadratic ghost terms in (A6) from the ghost/antighost symmetric one, M_{FP} in (A8). These terms are of the form

$$-\frac{1}{4\rho}\sum_i\sum_{j\sim i}\mathrm{tr}(\{\bar{c}_i, c_i\}(U_{ij}^g - U_{ji}^g)) = \frac{1}{2}\sum_i \bar{c}_i^a f^{abc}F_i^c(U^g)c_i^b. \quad \text{(A10)}$$

In lattice Landau gauge, where $F_i^a(U^g) = 0$, the forms from Eqs. (A7) and (A8) both, of course, lead to the same Faddeev-Popov operator, as they do in the continuum where the standard and symmetric versions differ by an analogous term $\propto f^{abc}\partial_\mu A_\mu^c/2$ which vanishes for $\xi = 0$. For $\xi \neq 0$, on the other hand, the two Faddeev-Popov operators do differ and, in particular, only the ghost/antighost symmetric framework based on (A8) leads to a Hermitian one which can be written in the alternative form,

$$\sum_{i,j}\bar{c}_i^a M_{\mathrm{FP}ij}^{ab}c_j^b = -\frac{1}{4\rho}\sum_{x,\mu}\{\mathrm{tr}(\{X^a, X^b\}(U_{x,\mu}^g + U_{x,\mu}^{g\dagger})) \times (\bar{c}_{x+\hat{\mu}}^a - \bar{c}_x^a)(c_{x+\hat{\mu}}^b - c_x^b) + \mathrm{tr}([X^a, X^b] \times (U_{x,\mu}^g - U_{x,\mu}^{g\dagger}))(\bar{c}_x^a(c_{x+\hat{\mu}}^b - c_x^b) - (\bar{c}_{x+\hat{\mu}}^a - \bar{c}_x^a)c_x^b)\}. \quad \text{(A11)}$$

We have added and subtracted the term proportional to $\bar{c}_x^a c_x^b$ in the last line here to underpin that in the continuum limit the M_{FP} herein reduces to the ghost/antighost symmetric Faddeev-Popov operator,

$$M_{\mathrm{FP}ij}^{ab} \overset{a\to 0}{\longrightarrow} -a^2\tfrac{1}{2}(\partial D^{ab} + D^{ab}\partial)\delta(x-y) + \mathcal{O}(a^4).$$

To complete the derivation of the gauge-fixing action in the ghost/antighost symmetric framework, we furthermore need work out the BRST variation of $s\bar{s}(\bar{c}^a c^a) = s(\bar{c}^a b^a)$ from (6)–(8). This, however, is done in exactly the same way as in the continuum, the result is (*iii*),

$$s(\bar{c}^a b^a) = b^a b^a + \tfrac{1}{4}(\bar{c}\times c)^2. \quad \text{(A12)}$$

Putting together all terms from (*i*) to (*iii*) we obtain the full gauge-fixing action with extended double BRST invariance on the lattice in the form

$$S_{\mathrm{GF}} = \sum_i\left\{-ib_i^a F_i^a(U^g) - i\bar{c}_i^a M_{\mathrm{FP}i}^a[c] + \frac{\xi}{2}b_i^a b_i^a + \frac{\xi}{8}(\bar{c}_i\times c_i)^2\right\}, \quad \text{(A13)}$$

where we introduced the short-hand notation that

$$M_{\mathrm{FP}i}^a[c] \equiv -\frac{1}{4\rho}\sum_{j\sim i}\{\mathrm{tr}([X^a, X^b](U_{ij}^g - U_{ji}^g))c_j^b + \mathrm{tr}(\{X^a, X^b\}(U_{ij}^g + U_{ji}^g))(c_i^b - c_j^b)\},$$

which corresponds to the ghost/antighost symmetric Faddeev-Popov operator in (A11). In particular, we have

$$\sum_i \bar{c}_i^a M_{\mathrm{FP}i}^a[c] = \sum_{i,j}\bar{c}_i^a M_{\mathrm{FP}ij}^{ab}c_j^b, \quad \text{(A14)}$$

with $M_{\mathrm{FP}ij}^{ab}$ in a simplified alternative form given by

$$M_{\mathrm{FP}ij}^{ab} = -\frac{1}{2\rho}\sum_{k\sim i}\{\mathrm{Re}\,\mathrm{tr}(\{X^a, X^b\}U_{ik}^g)\delta_{ij} - 2\mathrm{Re}\,\mathrm{tr}(X^b X^a U_{ik}^g)\delta_{kj}\}. \quad \text{(A15)}$$

The Faddeev-Popov operator of lattice Landau gauge was first derived in [13]. It might be worth recalling that the derivation presented there, based on the differentials of the gauge-fixing potential $V_U[g]$ along one-parameter subgroups of the $SU(N)$ gauge group, by construction leads directly to the Hermitian M_{FP} in (A15), and not to that of standard Faddeev-Popov theory on the lattice following from (A9). They are equivalent in Landau gauge, of course. Their subtle difference needs to be kept in mind, however, when attempting to implement smeared covariant gauges for $\xi \neq 0$ on the lattice as was done, e.g., in Refs. [14,15]. It reflects the different symmetry properties of standard Faddeev-Popov theory and the ghost/antighost symmetric framework for $\xi \neq 0$.

APPENDIX B: PROOF OF EQ. (45)

In this Appendix we give the explicit proof of Eq. (45),

$$\langle s\bar{s}\mathcal{O}(t,\xi)\rangle_{m^2} = 0, \quad \text{(B1)}$$

from the Dyson-Schwinger equations of the lattice model. It would actually not really be necessary because the scaling argument used in Sec. IV, in particular, Eq. (41), implies the differential equation for the gauge-fixing partition function $Z_{\mathrm{mGF}}(t,\xi,m^2)$ of the massive Curci-Ferrari model in (46) which is sufficient to establish (B1).

To verify explicitly that this expectation value vanishes at all values of the Curci-Ferarri mass m, first note that

$$s\bar{s}\mathcal{O}(t,\xi) = ts\bar{s}V_U[g] - i\xi s\bar{s}\sum_i \bar{c}_i^a c_i^a. \quad \text{(B2)}$$

From Eqs. (A1) and (A8) in the previous Appendix,

$$s\bar{s}V_U[g] = -\sum_i b_i^a F_i^a - \sum_{i,j}\bar{c}_i^a M_{\mathrm{FP}ij}^{ab}c_j^b, \quad \text{(B3)}$$

while from Eqs. (27) and (A4),

$$s\bar{s}\sum_i \bar{c}_i^a c_i^a = \sum_i\left\{b_i^a b_i^a - im^2\bar{c}_i^a c_i^a + \frac{1}{4}(\bar{c}_i\times c_i)^2\right\}. \quad \text{(B4)}$$

With our short-hand bracket notation for the summation over color and site indices as in the main text, we can thus write

$$s\bar{s}\mathcal{O}(t,\xi) = -t(b,F) - t(\bar{c}, M_{\rm FP}c) - i\xi(b,b) - m^2\xi(\bar{c},c) - i\frac{\xi}{4}((\bar{c}\times c),(\bar{c}\times c)). \tag{B5}$$

On the other hand, with (28), from

$$\frac{\partial S_{\rm mGF}}{\partial b_i^a} = -itF_i^a + \xi b_i^a,$$

we obtain the equation of motion for the Nakanishi-Lautrup field on the lattice as

$$\left\langle \frac{\partial S_{\rm mGF}}{\partial b_i^a} b_j^b \right\rangle_{m^2} = \langle -itF_i^a b_j^b + \xi b_i^a b_j^b \rangle_{m^2} = \delta^{ab}\delta_{ij}. \tag{B6}$$

Likewise, from the left derivative w.r.t. the antighosts,

$$\frac{\partial S_{\rm mGF}}{\partial_L \bar{c}_i^a} = -itM^a_{{\rm FP}i}[c] - im^2\xi c_i^a - \frac{\xi}{4}((\bar{c}\times c)\times c)_i^a,$$

the Dyson-Schwinger equation for the ghosts in the massive Curci-Ferrari model on the lattice is obtained as

$$\begin{aligned}\left\langle \frac{\partial S_{\rm mGF}}{\partial_L \bar{c}_i^a}\bar{c}_j^b \right\rangle_{m^2} &= -it\langle M^a_{{\rm FP}i}[c]\bar{c}_j^b\rangle_{m^2} - im^2\xi\langle c_i^a\bar{c}_j^b\rangle_{m^2} \\ &\quad - \frac{\xi}{4}\langle(((\bar{c}\times c)\times c)_i^a \bar{c}_j^b\rangle_{m^2} \\ &= \delta^{ab}\delta_{ij}. \end{aligned}\tag{B7}$$

The difference of the two Dyson-Schwinger equations (B6) and (B7) is zero, and upon taking the trace over color and site indices, we therefore find

$$\begin{aligned} 0 &= {\rm Tr}\left\langle \frac{\partial S_{\rm mGF}}{\partial_L \bar{c}_i^a}\bar{c}_j^b \right\rangle_{m^2} - {\rm Tr}\left\langle \frac{\partial S_{\rm mGF}}{\partial b_i^a} b_j^b \right\rangle_{m^2} \\ &= it\langle(\bar{c}, M_{\rm FP}c)\rangle_{m^2} + im^2\xi\langle(\bar{c},c)\rangle_{m^2} - \frac{\xi}{4}\langle((\bar{c}\times c), \\ &\quad (\bar{c}\times c))\rangle_{m^2} + it\langle(b,F)\rangle_{m^2} - \xi\langle(b,b)\rangle_{m^2}. \end{aligned}\tag{B8}$$

Comparing (B5) and (B8) we therefore observe that

$$\langle s\bar{s}\mathcal{O}(t,\xi)\rangle_{m^2} = i{\rm Tr}\left\langle \frac{\partial S_{\rm mGF}}{\partial_L \bar{c}_i^a}\bar{c}_j^b \right\rangle_{m^2} - i{\rm Tr}\left\langle \frac{\partial S_{\rm mGF}}{\partial b_i^a} b_j^b \right\rangle_{m^2} = 0,$$

as promised.

APPENDIX C: EXPECTATION VALUES IN THE "WOULD-BE-GAUSS-BONNET" LIMIT

Here we describe the explicit calculations to derive the results for the expectation values in Eq. (63), as summarized in Table I, which are needed at the 2nd order in the expansion of the Curci-Ferrari mass parameter $\hat{m}^2(x)$ around $x = 0$, the "would-be-Gauss-Bonnet" limit.

For the first term we use $\langle b_i^a b_j^b\rangle_{\hat{m}^2,0} = \delta^{ab}\delta_{ij}$ to obtain

$$\langle(ib,F)^2\rangle_{\hat{m}^2,0} = -\langle(F,F)\rangle_{\hat{m}^2,0}. \tag{C1}$$

Remember the explicit form (15) of the gauge condition,

$$F_i^a(U^g) = -\sum_{j\sim i}{\rm tr}(X^a(g_i^\dagger U_{ij}g_j - g_j^\dagger U_{ji}g_i)). \tag{C2}$$

All terms in the sum of the squares of this condition are quadratic in g and in $g^\dagger$. For $N \geq 3$ the only nonvanishing results of the group integration arise in terms where all the g_i's match up pairwise with $g_i^\dagger$'s at identical sites (the special case of $N = 2$ will be discussed separately below). We can then use for the fundamental g's at that site i,

$$V_N^{-1}\int dg_i (g_i)_{kl}(g_i^\dagger)_{mn} = \frac{1}{N}\delta_{kn}\delta_{lm}. \tag{C3}$$

In one particular term $F_i^aF_i^a$, without summation over i, only the two mixed terms then survive and we have (no sum i)

$$\begin{aligned} V_N^{-1}\int dg_i F_i^aF_i^a &= -2V_N^{-1}\int dg_i\Big(\sum_{j\sim i}{\rm tr}(X^a g_i^\dagger U_{ij}g_j)\Big) \\ &\quad\times\Big(\sum_{k\sim i}{\rm tr}(X^a g_k^\dagger U_{ki}g_i)\Big) \\ &= -\frac{2}{N}\sum_{k,j\sim i}{\rm tr}(X^aX^a g_k^\dagger U_{ki}U_{ij}g_j) \\ &= \Big(1-\frac{1}{N^2}\Big)\sum_{k,j\sim i}{\rm tr}(g_k^\dagger U_{ki}U_{ij}g_j). \end{aligned}\tag{C4}$$

The integrations over the gauge-group elements at all sites $k\sim i$, attached to site i, by the same argument yield a nonvanishing result only if $j = k$ in the double sum over all neighbors of site i in (C4). For those terms the group integration yields ${\rm tr}U_{ji}U_{ij} = N$. Therefore, in D dimensions,

$$\langle(F,F)\rangle_{\hat{m}^2,0} = \mathcal{N}\int\prod_i dg_i(F,F) = (\#{\rm sites})\times 4DC_2^f, \tag{C5}$$

for $SU(N)$ with $N \geq 3$, where $C_2^f = \frac{1}{2}(N - 1/N)$ is the value of the quadratic Casimir operator in the fundamental representation.

For $N = 2$ we obtain an additional contribution to $\langle(F,F)\rangle_{\hat{m}^2,0}$ from the squares of the two terms in the gauge condition (C2). This is because for $g\in SU(2)$,

$$V_2^{-1}\int dg_i (g_i)_{kl}(g_i)_{mn} = \tfrac{1}{2}\epsilon_{km}\epsilon_{ln}. \tag{C6}$$

Again, however, only the squares of the same links contribute in the double sum over the neighbors of site i. For these, e.g.,

$$(\mathrm{tr}X^a g_j^\dagger U_{ji} g_i)^2 V_2^{-2} \xrightarrow{\int dg_i dg_j} \frac{3}{4}\det U_{ji} = C_2^f. \tag{C7}$$

There are two such terms in $F_i^a F_i^a$ for each of the $2D$ links attached to site i. Therefore, the total additional contribution from those terms in $SU(2)$ equals that from the mixed terms calculated for general N above; we have

$$\langle (F, F)\rangle_{\hat{m}^2,0} = (\#\text{sites}) \times 8DC_2^f, \qquad \text{for } SU(2). \tag{C8}$$

Next, to compute the expectation value $\propto \langle V_U^2\rangle_{\hat{m}^2,0}$ of

$$V_U^2[g] = \Big(\sum_{i,j\sim i} \mathrm{tr} g_i^\dagger U_{ij} g_j\Big)\Big(\sum_{k,l\sim k} \mathrm{tr} g_k^\dagger U_{kl} g_l\Big), \tag{C9}$$

we consider one term for fixed (neighbors) i and j in the double sum. Integrating this term over g_j via (C3),

$$V_N^{-1}\int dg_j\, \mathrm{tr} g_i^\dagger U_{ij} g_j \Big(\sum_{k,l\sim k} \mathrm{tr} g_k^\dagger U_{kl} g_l\Big) = \frac{1}{N}\sum_{l\sim j} \mathrm{tr} g_i^\dagger U_{ij} U_{jl} g_l, \tag{C10}$$

for $N \geq 3$, where the only contribution arises when $k = j$. Then, l must be one of the neighbors of j. Successive integration over g_i singles out that neighbor of j with $l = i$,

$$V_N^{-1}\int dg_i \frac{1}{N}\sum_{l\sim j} \mathrm{tr} g_i^\dagger U_{ij} U_{jl} g_l = \frac{1}{N}\mathrm{tr} U_{ij} U_{ji} = 1.$$

We obtain one such contribution for every one of the $2D$ neighbors $j \sim i$ at site i, thus summing $\sum_{i,j\sim i}$ yields

$$\langle V_U^2[g]\rangle_{\hat{m}^2,0} = (\#\text{sites}) \times 2D, \qquad N \geq 3. \tag{C11}$$

Note that the number of sites in all these expectation values cancels with that in (57) when computing the ratio of Eq. (52).

Again, for $N = 2$ in $SU(2)$ we obtain an additional contribution from (C6). Starting again from the contribution to (C9) for fixed neighbors i and j as in (C10), we now obtain for $N = 2$

$$V_2^{-1}\int dg_j\, \mathrm{tr} g_i^\dagger U_{ij} g_j \Big(\sum_{k,l\sim k} \mathrm{tr} g_k^\dagger U_{kl} g_l\Big) = \frac{1}{2}\sum_{l\sim j} \mathrm{tr} g_i^\dagger U_{ij} U_{jl} g_l + \frac{1}{2}\sum_{k\sim j} \epsilon_{su}\epsilon_{rt}(g_i^\dagger U_{ij})_{sr}(g_k^\dagger U_{kj})_{ut}. \tag{C12}$$

The second group integration over g_i then produces, in addition to the above, an according contribution from the second term, which is given by

$$V_2^{-1}\int dg_i \frac{1}{2}\sum_{k\sim j} \epsilon_{su}\epsilon_{rt}(g_i^\dagger U_{ij})_{sr}(g_k^\dagger U_{kj})_{ut} = \frac{1}{4}\epsilon_{su}\epsilon_{rt}\epsilon_{su}\epsilon_{vw}(U_{ij})_{vr}(U_{ij})_{wt} = \frac{1}{4}4\det U_{ij} = 1. \tag{C13}$$

This equals the first term obtained for all N; and together the two again give for $SU(2)$ twice the result for $N \geq 3$ in (C11) above, i.e.,

$$\langle V_U^2[g]\rangle_{\hat{m}^2,0} = (\#\text{sites}) \times 4D, \qquad N = 2. \tag{C14}$$

The hardest task here is to compute the last remaining term in (63), $\langle (i\bar{c}, M_{\mathrm{FP}} c)^2\rangle_{\hat{m}^2,0}$. For a start, we first note that

$$\langle i\bar{c}_i^a c_j^b i\bar{c}_k^c c_l^d\rangle_{\hat{m}^2,0} = (\delta^{ab}\delta_{ij}\delta^{cd}\delta_{kl} - \delta^{ad}\delta_{il}\delta^{bc}\delta_{jk})A_{ik}^{ac},$$

where

$$A_{ij}^{ab} = \langle (i\bar{c}c)_i^a (i\bar{c}c)_j^b\rangle_{\hat{m}^2,0}, \tag{C15}$$

using the notation $(i\bar{c}c)_i^a \equiv i\bar{c}_i^a c_i^a$ without implicit summations over a and i, here. The expectation value in (C15) is of course independent of $\{U\}$. It depends on the site indices only in that we need to distinguish whether $i = j$ or not, i.e.,

$$A_{ij}^{ab} = \begin{cases} P_N^{ab}(\hat{m}_0^2), \quad P_N^{ab} = \begin{cases} P_N^{ba}, & a \neq b \\ 0, & a = b \end{cases}; & i = j. \\ Q_N(\hat{m}_0^2), \quad \text{independent of } \{a, b\}; & i \neq j, \end{cases} \tag{C16}$$

where both P_N^{ab} and Q_N are site independent. We have

$$Q_N(\hat{m}_0^2) = \frac{1}{(N^2-1)^2}\Big(\frac{I_N'(\hat{m}_0^2)}{I_N(\hat{m}_0^2)}\Big)^2, \tag{C17}$$

from (56), while P_N^{ab} has a more complicated structure, in general. Only its sum simplifies,

$$\sum_{a,b} P_N^{ab}(\hat{m}_0^2) = 2\sum_{a,b>a} P_N^{ab}(\hat{m}_0^2) = \frac{I_N''(\hat{m}_0^2)}{I_N(\hat{m}_0^2)}, \tag{C18}$$

which follows immediately from its definition via (C15) and (C16) and with Eqs. (36)–(38). With these results,

$$\begin{aligned}\langle (i\bar{c}, M_{\mathrm{FP}} c)^2\rangle_{\hat{m}^2,0} &= \Big\langle \sum_{i,j,a,b} (M_{ii}^{aa} M_{jj}^{bb} - M_{ij}^{ab} M_{ji}^{ba}) A_{ij}^{ab}\Big\rangle_{\hat{m}^2,0} \\ &= 2Q_N(\hat{m}_0^2)\sum_{i,j>i,a,b} \langle (M_{ii}^{aa} M_{jj}^{bb} \\ &\quad - M_{ij}^{ab} M_{ji}^{ba})\rangle_{\hat{m}^2,0} + 2\sum_{i,a,b>a} P_N^{ab}(\hat{m}_0^2) \\ &\quad \times \langle (M_{ii}^{aa} M_{ii}^{bb} - M_{ii}^{ab} M_{ii}^{ba})\rangle_{\hat{m}^2,0}.\end{aligned} \tag{C19}$$

The first part with the contributions from different sites $j \neq i$ would be a disaster for the intended $\hat{m}_0^2 \to 0$ limit: Because $I_N'(\hat{m}_0^2)/I_N(\hat{m}_0^2)$ always is of the order $1/\hat{m}_0^2$, we have that $Q_N(\hat{m}_0^2)$ is of the order $1/\hat{m}_0^4$. Recalling that we need to divide all terms computed here by the expectation value in (57) which is proportional to $I_N'(\hat{m}_0^2)/I_N(\hat{m}_0^2)$, this then implies that the second derivative w.r.t. x of $\hat{m}^2(x)$ would contain a contribution proportional $1/\hat{m}_0^2$ at $x=0$ and therefore become infinite in the limit $\hat{m}_0^2 \to 0$. Luckily, this contribution turns out to be zero for all N because of a cancellation between the two terms in the expectation value of this part in (C19). To see this, first consider

$$\sum_{a,b} M_{ii}^{aa} M_{jj}^{bb} = 4C_2^{f^2} \sum_{k\sim i}\sum_{l\sim j} \mathrm{Re\,tr}(g_i^\dagger U_{ik} g_k)\mathrm{Re\,tr}(g_j^\dagger U_{jl} g_l)$$
$$= C_2^{f^2} \sum_{k\sim i}\sum_{l\sim j} (\mathrm{tr}U_{ik}^g\,\mathrm{tr}U_{jl}^g + \mathrm{tr}U_{ki}^g\,\mathrm{tr}U_{lj}^g + \mathrm{tr}U_{ik}^g\,\mathrm{tr}U_{lj}^g + \mathrm{tr}U_{ki}^g\,\mathrm{tr}U_{jl}^g). \tag{C20}$$

Because $i \neq j$, only the first two terms in the brackets contribute when integrating g_i via (C3). For those, the different group integrations over g_i and g_j, by the method now familiar, then yield

$$V_N^{-2}\int dg_i dg_j \sum_{a,b} M_{ii}^{aa} M_{jj}^{bb} = 2C_2^{f^2}\sum_{k\sim i}\sum_{l\sim j}\delta_{li}\delta_{kj}. \tag{C21}$$

After summation over j we can replace j by k here. The fact that in this sum we need to restrict $j \neq i$ does not matter because the nonzero contributions arise for $j = k$, where k is a neighbor of i and thus necessarily different from i. Subsequent summation over the neighbors l of k now, then due to the second Kronecker symbol picks up neighbor i of site k. We have

$$\sum_{i,j\neq i} V_N^{-2}\int dg_i dg_j \sum_{a,b} M_{ii}^{aa}M_{jj}^{bb} = 2C_2^{f^2}\sum_{i,k\sim i}\sum_{l\sim k}\delta_{li}$$
$$= (\#\mathrm{sites}) \times D\Big(N - \frac{1}{N}\Big)^2. \tag{C22}$$

As before, there is an additional contribution from (C6) for $SU(2)$ here also. As i and j are different, after integrating g_i this time, this contribution is nonzero only for precisely the other two terms in the brackets above. Using (C6) it is straightforward to verify that the result for those two terms again matches the contribution just calculated from (C3). We can therefore summarize that, for all N including $N=2$,

$$\sum_{i,j\neq i,a,b}\langle M_{ii}^{aa}M_{jj}^{bb}\rangle_{\hat{m}^2,0} = (\#\mathrm{sites})\times 2D2C_2^{f^2}(1+\delta_{N,2}). \tag{C23}$$

For the second term in the first part of (C19) with $i \neq j$, which allowed us to drop the diagonal terms in both Faddeev-Popov operators, we need that in the fundamental representation,

$$(X^bX^a)_{ij}(X^aX^b)_{kl} = \frac{1}{4}\Big(N - \frac{2}{N}\Big)\delta_{il}\delta_{jk} + \frac{1}{4N^2}\delta_{ij}\delta_{kl},$$

and

$$(X^aX^b)_{ij}(X^aX^b)_{kl} = -\frac{1}{2N}\delta_{il}\delta_{jk} + \frac{1}{4}\Big(1+\frac{1}{N^2}\Big)\delta_{ij}\delta_{kl},$$

so that we can write (for $i \neq j$)

$$\sum_{a,b} M_{ij}^{ab}M_{ji}^{ba} = 4\sum_{k\sim i}\sum_{l\sim j}\mathrm{Re\,tr}(X^bX^a g_i^\dagger U_{ik}g_k)\mathrm{Re\,tr}(X^aX^b g_j^\dagger U_{jl} g_l)\delta_{kj}\delta_{li}$$
$$= \sum_{k\sim i}\sum_{l\sim j}(\mathrm{tr}X^bX^aU_{ij}^g + \mathrm{tr}X^aX^bU_{ji}^g)(\mathrm{tr}X^aX^bU_{ji}^g + \mathrm{tr}X^bX^aU_{ij}^g)\delta_{kj}\delta_{li}$$
$$= \sum_{k\sim i}\sum_{l\sim j}\Big\{\frac{1}{2}(N^2-2) + \frac{1}{2N^2}\mathrm{tr}U_{ij}^g\,\mathrm{tr}U_{ji}^g - \frac{1}{2N}(\mathrm{tr}U_{ij}^gU_{ij}^g + \mathrm{tr}U_{ji}^gU_{ji}^g) + \frac{1}{4}\Big(1+\frac{1}{N^2}\Big)((\mathrm{tr}U_{ij}^g)^2 + (\mathrm{tr}U_{ji}^g)^2)\Big\}\delta_{kj}\delta_{li}. \tag{C24}$$

The group integrations via (C3) and, in addition, for the special case of $N=2$ via (C6) proceed as before. The explicit results of the corresponding calculations above can all be reduced to essentially using two relations summarized as follows:

$$\langle \mathrm{tr}(g_i^\dagger U_{ij} g_j)\mathrm{tr}(g_k^\dagger U_{kl} g_l)\rangle_{\hat{m}^2,0} = \delta_{il}\delta_{ki} + \delta_{N,2}\delta_{ik}\delta_{jl},$$
$$\langle \mathrm{tr}(g_i^\dagger U_{ij} g_j g_k^\dagger U_{kl} g_l)\rangle_{\hat{m}^2,0} = N\delta_{il}\delta_{ki} - \delta_{N,2}\delta_{ik}\delta_{jl}. \tag{C25}$$

These are the basic terms that arise at the quadratic order in x, and hence in the gauge-transformed links U^g, of our mass expansion (recall that the terms linear in x vanish in this expansion because the linear order terms in U^g do upon the group integrations). Using the relations (C25) in (C24), we obtain

$$\sum_{a,b}\langle M_{ij}^{ab}M_{ji}^{ba}\rangle_{\hat{m}^2,0} = \Big\{\frac{1}{2}(N^2-2) + \frac{1}{2N^2} + \delta_{N,2}\Big(\frac{1}{N} + \frac{1}{2}\Big(1+\frac{1}{N^2}\Big)\Big)\Big\}\sum_{k\sim i}\sum_{l\sim j}\delta_{kj}\delta_{li}, \tag{C26}$$

and hence, together with (C23),

$$\sum_{i,j\neq i,a,b}\langle (M_{ii}^{aa}M_{jj}^{bb} - M_{ij}^{ab}M_{ji}^{ba})\rangle_{\hat{m}^2,0} = (\#\text{sites})2D\times\begin{cases}\frac{1}{2}\Big(N-\frac{1}{N}\Big)^2 - \frac{1}{2}(N^2-2) - \frac{1}{2N^2} = 0, & N\geq 3,\\ \Big(N-\frac{1}{N}\Big)^2 - \frac{1}{2}(N^2-2) - \frac{1}{2N^2} - \Big(\frac{1}{N}+\frac{1}{2}\Big(1+\frac{1}{N^2}\Big)\Big) = 0, & N=2.\end{cases} \tag{C27}$$

The two terms from (C23) and (C26) therefore cancel and the first part in (C19) thus vanishes in either case, whether $N = 2$ or $N \geq 3$. At the same time this cancels the otherwise quite disastrous singularity of the mass expansion in the $\hat{m}_0^2 \to 0$ limit, as promised.

In the second term in (C19), the one with $i = j$, we need products of diagonal entries of the Faddeev-Popov operator of the form [cf. Eq. (A15); no implicit sum over i here either],

$$\begin{aligned} M_{ii}^{ab} &= -\sum_{k\sim i}\frac{1}{2}(\mathrm{tr}\{X^a,X^b\}U_{ik}^g + \mathrm{tr}\{X^a,X^b\}U_{ki}^g)\\ &= \sum_{k\sim i}\bigg\{\frac{1}{2N^2}\delta^{ab}(\mathrm{tr}U_{ik}^g + \mathrm{tr}U_{ki}^g)\\ &\quad + \frac{1}{2}d^{abc}(\mathrm{tr}iX^cU_{ik}^g + \mathrm{tr}iX^cU_{ki}^g)\bigg\}, \end{aligned} \tag{C28}$$

where we have used the identity

$$\{X^a,X^b\} = -\frac{1}{N}\delta^{ab} - id^{abc}X^c. \tag{C29}$$

For $SU(2)$ we set $d^{abc} = 0$ which then leaves us with only the first term in (C28). Because, e.g.,

$$\begin{aligned}&\langle \mathrm{tr}(g_i^\dagger U_{ik}g_k)\mathrm{tr}(iX^a g_l^\dagger U_{li}g_i)\rangle_{\hat{m}^2,0}\\ &\quad = \frac{1}{N}\langle \mathrm{tr}(iX^a g_l^\dagger U_{li}U_{ik}g_k)\rangle_{\hat{m}^2,0} = 0,\end{aligned} \tag{C30}$$

due to the tracelessness of the generators X^a, there will be no terms linear in the X's in the expectation values of squares of the operators in (C28) for $N \geq 3$ either. The terms quadratic in the X's simplify from relations as follows:

$$\begin{aligned}&\langle \mathrm{tr}(iX^a g_i^\dagger U_{ik}g_k)\mathrm{tr}(iX^b g_l^\dagger U_{li}g_i)\rangle_{\hat{m}^2,0}\\ &\quad = \frac{1}{N}\langle \mathrm{tr}(iX^b g_l^\dagger U_{li}U_{ik}g_k iX^a)\rangle_{\hat{m}^2,0} = \frac{1}{2N}\delta^{ab}\delta_{kl}.\end{aligned} \tag{C31}$$

We do not have to worry about corresponding terms from (C6) for $N = 2$ here because these contain the d symbols which are zero in $SU(2)$. With (C25), (C30), and (C31) we therefore obtain

$$\begin{aligned}\langle M_{ii}^{ab}M_{ii}^{cd}\rangle_{\hat{m}^2,0} &= \sum_{k\sim i}\bigg\{\frac{1}{2N^2}\delta^{ab}\delta^{cd}(1+\delta_{N,2}) + \frac{1}{4N}d^{abe}d^{cde}\bigg\}\\ &\overset{N>2}{=} \frac{2D}{4N}\Big(\frac{2}{N}\delta^{ab}\delta^{cd} + d^{abe}d^{cde}\Big).\end{aligned} \tag{C32}$$

We therefore have, implicitly summing over color indices again but not over sites i (yet),

$$\begin{aligned}\langle (i\bar{c}^aM^{ab}c^b)_i(i\bar{c}^cM^{cd}c^d)_i\rangle_{\hat{m}^2,0} &= \langle (i\bar{c}^ac^b)_i(i\bar{c}^cc^d)_i\rangle_{\hat{m}^2,0}\langle M_{ii}^{ab}M_{ii}^{cd}\rangle_{\hat{m}^2,0}\\ &= -\frac{2D}{4N}\bigg\{\frac{2}{N}\langle (\bar{c}^ac^a)_i^2\rangle_{\hat{m}^2,0} + d^{abe}d^{cde}\langle (\bar{c}^ac^b\bar{c}^cc^d)_i\rangle_{\hat{m}^2,0}\bigg\}\\ &= -\frac{2D}{4N}\langle f^{abe}(\bar{c}^ac^b)_if^{cde}(\bar{c}^cc^d)_i\rangle_{\hat{m}^2,0}\\ &= -\frac{2D}{4N}\langle (\bar{c}\times c)_i^2\rangle_{\hat{m}^2,0},\end{aligned} \tag{C33}$$

for $N \geq 3$, and twice that for $N = 2$ again, where the d's are zero and where the $x = 0$ expectation value of the remaining first term above agrees with that of the quartic-ghost interaction,

$$-\langle (\bar{c}\times c)_i^2\rangle_{\hat{m}^2,0} = \langle (i\bar{c}^ac^a)_i^2\rangle_{\hat{m}^2,0} = \frac{I_2''(\hat{m}_0^2)}{I_2(\hat{m}_0^2)}. \tag{C34}$$

Finally, summing in (C33) over the sites i of the lattice, and from (C19) with (C26), we obtain

$$\begin{aligned}&\langle (i\bar{c}, M_{\mathrm{FP}}c)^2\rangle_{\hat{m}^2,0}\\ &\quad = 2D(\#\text{sites})\begin{cases}-\frac{1}{4N}\langle (\bar{c}\times c)^2\rangle_{\hat{m}^2,0} \equiv \frac{2}{N}\frac{J_N(\hat{m}_0^2)}{I_N(\hat{m}_0^2)}, & N\geq 3,\\ -\frac{1}{4}\langle (\bar{c}\cdot c)^2\rangle_{\hat{m}^2,0} = \frac{1}{4}\frac{I_2''(\hat{m}_0^2)}{I_2(\hat{m}_0^2)}, & N=2.\end{cases}\end{aligned} \tag{C35}$$

The only piece left to compute is the expectation value (at $x = 0$) of the quartic-ghost interaction,

$$-\frac{1}{8}\langle(\bar{c}\times c)^2\rangle_{\hat{m}^2,0} \equiv \frac{J_N(\hat{m}_0^2)}{I_N(\hat{m}_0^2)}. \tag{C36}$$

The integral expression for J_N, analogous to that for I_N, cf. Eq. (38), is given by

$$J_N(\hat{m}^2) = \int \prod_{a=1}^{N^2-1} d(i\bar{c}^a)dc^a\left(-\frac{1}{8}(\bar{c}\times c)^2\right) \times \exp\left\{i\hat{m}^2\bar{c}\cdot c - \frac{1}{8}(\bar{c}\times c)^2\right\}. \tag{C37}$$

For $SU(2)$ and $SU(3)$, respectively, the resulting $J_2(\hat{m}^2)$ and $J_3(\hat{m}^2)$ are given explicitly in Eqs. (66) and (67). This completes the computations of the terms in (63). The results are summarized in Table I.

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