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$\boldsymbol{\alpha_s(M_Z^2)}$ from hadronic $\boldsymbol{\tau}$ decays

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We perform an extraction of α_s based on sum rules involving isovector hadronic τ decay data. The particular sum rules employed are constructed specifically to suppress contributions associated with poorly known higher dimension condensates, and hence reduce theoretical systematic uncertainties associated with the treatment of such contributions which are shown to be present in earlier related analyses. Running our results from the $n_f = 3$ to $n_f = 5$ regime we find $\alpha_s(M_Z^2) = 0.1187 \pm 0.0016$, in excellent agreement with the recently updated global fit to electroweak data at the Z scale and other high-scale direct determinations.

I. INTRODUCTION

The value of the running strong coupling, $\alpha_s(\mu^2)$, at some conventionally chosen reference scale is one of the fundamental parameters of the standard model (SM). In what follows, we adhere to standard convention and quote results at the scale $\mu = M_Z$, for $n_f = 5$, in the $\overline{MS}$ scheme, and denote this quantity by $\alpha_s(M_Z^2)$.

The running coupling $\alpha_s(\mu^2)$ has been determined experimentally in a large number of independent processes, over a wide range of scales [1]. The observed variation, by a factor of ~ 3, over the range from $\mu \sim 2$ GeV to $\mu = M_Z$ is in excellent agreement with QCD expectations, and represents a highly nontrivial test of the theory. If, however, one looks in more detail, one finds that the two highest-precision low-energy determinations, that coming from a lattice perturbation theory analysis of UV-sensitive lattice observables [2], and that coming from finite energy sum rule (FESR) analyses of hadronic τ decay data [3–5], are not in good agreement within their mutual errors, the most recent determinations yielding

$$\alpha_s(M_Z^2) = 0.1170 \pm 0.0012 \quad \text{(lattice)} \tag{1}$$

$$\alpha_s(M_Z^2) = 0.1212 \pm 0.0011 \quad (\tau \text{ decay}) \tag{2}$$

for the lattice [2] and τ decay [5] determinations, respectively.

In this paper we revisit the hadronic τ decay extraction, focusing on alternate FESR choices designed specifically to reduce theoretical systematic uncertainties *not included in the error assessment of* Eq. (2) and associated with possible small higher dimension $(D > 8)$ Operator Product Expansion (OPE) contributions assumed negligible in the analyses reported in Refs. [3,5]. We find a shift

in the results for $\alpha_s(M_Z^2)$ in excess of the previously quoted error, and obtain also an improvement in the agreement (i) between the τ decay and direct high-scale determinations and (ii) amongst the separate τ decay extractions obtained from the vector (V), axial vector (A), and vector-plus-axial-vector ($V + A$) channel analyses.

The rest of the paper is organized as follows. In Sec. II we (i) outline the general FESR approach to extracting α_s from hadronic τ decay data, (ii) discuss the relevant features of existing analyses, (iii) point out potential additional theoretical uncertainties in those analyses, associated with the neglect of $D > 8$ OPE contributions, (iv) establish explicitly the presence of such contributions at a level *not* negligible on the scale of the previously quoted errors, and (v) discuss alternate sum rule choices which significantly reduce these uncertainties. In Sec. III we use these alternate sum rules to perform separate V, A and $V + A$ analyses, employing either the ALEPH [3,5–7] or OPAL [8] isovector hadronic τ decay data sets. Our final results for $\alpha_s(M_Z^2)$, together with a discussion of these results, are given in Sec. IV.

II. HADRONIC τ DECAY EXTRACTIONS OF α_s

A. The finite energy sum rule framework

The kinematics of τ decay in the SM allows the inclusive rate for hadronic τ decays mediated by the flavor $ij = ud$, us, V or A currents to be written as a sum of kinematically weighted integrals over the spectral functions $\rho_{V/A;ij}^{(J)}(s)$, associated with the spin $J = 0, 1$ components of the relevant current-current two-point functions [9]. Defining $R_{V/A;ij} \equiv \Gamma[\tau^- \to \nu_\tau \text{hadrons}_{V/A;ij}(\gamma)]/\Gamma[\tau^- \to \nu_\tau e^- \bar{\nu}_e(\gamma)]$ and $y_\tau \equiv s/m_\tau^2$, one has

$$R_{V/A;ij} = 12\pi^2 |V_{ij}|^2 S_{\text{EW}} \int_0^1 dy_\tau (1 - y_\tau)^2$$
$$\times [(1 + 2y_\tau)\rho_{V/A;ij}^{(0+1)}(s) - 2y_\tau \rho_{V/A;ij}^{(0)}(s)] \tag{3}$$

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with V_{ij} the flavor ij Cabibbo-Kobayashi-Maskawa matrix element, S_{EW} a short-distance electroweak correction [10–12], and $\rho_{V/A;ij}^{(0+1)}(s) \equiv \rho_{V/A;ij}^{(1)}(s) + \rho_{V/A;ij}^{(0)}(s)$. We concentrate here on the isovector ($ij = ud$) case.

For $ij = ud$, apart from the π pole contribution to $\rho_{A;ud}^{(0)}$, all contributions to $\rho_{V;ud}^{(0)}(s)$, $\rho_{A;ud}^{(0)}(s)$, are of $O([m_d \mp m_u]^2)$, and hence numerically negligible, allowing the sum of the flavor ud V and A spectral functions $\rho_{V+A;ud}^{(0+1)}(s)$ to be determined directly from experimental results for $dR_{V+A;ud}/ds$. Further separation into V and A components is unambiguous for $n\pi$ states, but requires additional input for $K\bar{K}n\pi$ ($n > 0$) states. Errors on the experimental distribution are thus reduced by working with the $V + A$ sum.

The spectral functions, $\rho_{V/A;ij}^{(0+1)}(s)$, correspond to scalar correlator combinations, $\Pi_{V/A;ij}^{(0+1)}(s) \equiv \Pi_{V/A;ij}^{(1)}(s) + \Pi_{V/A;ij}^{(0)}(s)$, having no kinematic singularities. For any such correlator, $\Pi(s)$, with spectral function $\rho(s)$, and any $w(s)$ analytic in $|s| < M$ with $M > s_0$, analyticity implies the finite energy sum rule relation

$$\int_0^{s_0} w(s)\rho(s)ds = -\frac{1}{2\pi i} \oint_{|s|=s_0} w(s)\Pi(s)ds. \quad (4)$$

For sufficiently large s_0, the OPE representation can be employed on the right-hand side of Eq. (4). The region of applicability of the OPE is extended to lower s_0 by working with "pinched" weights (those satisfying $w(s = s_0) = 0$), which suppress contributions on the right-hand side from the region of the contour near the timelike real axis [13,14].

For FESRs employed hadronic τ decay data, s_0 up to m_τ^2 are kinematically allowed on the right-hand side of Eq. (4). Since $m_\tau = 1.77684(17)$ GeV [15] is $\gg \Lambda_{QCD}$, one expects the integrated OPE to provide a reliable representation over a significant portion of the kinematically allowed s_0 range.

In previous extractions of α_s, FESRs involving $\Pi_T(s) \equiv \Pi_{T;ud}^{(0+1)}(s)$ (with $T = V$, A or $V + A$), pinched polynomial weights, and $s_0 = m_\tau^2$ were employed. Our analysis will employ a range of s_0 and an alternate set of such weights having the generic form $w(y) = \sum_m b_m y^m$, with $y = s/s_0$ [16].

B. Experimental input for the weighted spectral integrals

Data and covariance matrices for the spectral distributions $dR_{T;ud}/ds$, again with $T = V$, A and $V + A$, have been provided by both the ALEPH [3,6,7] and OPAL [8] collaborations. The ALEPH covariances lead to weighted spectral integrals with non-normalization-induced errors a factor of ~ 2 smaller than those obtained using the OPAL results.

In addition, ALEPH has recently provided previously unavailable information on the $V + A$ $K\bar{K}\pi$ distribution [5], a mode for which separate information is not available from OPAL. This is of relevance to performing the separate V and A analyses since recent *BABAR* determinations of the isovector $K\bar{K}\pi$ electroproduction cross sections [17], combined with the conserved vector current hypothesis, allow for a significant improvement in the treatment of the V/A separation in the $K\bar{K}\pi$ channel [5], which channel dominates the uncertainty in the V/A separation for non-strange hadronic τ decays. In view of these advantages, we will focus our discussion on the ALEPH data [18], though we will also perform alternate independent analyses using the OPAL data as input, as a further consistency check.

C. The OPE representation of $\Pi_{V/A;ud}$

1. The $D = 0$ contribution

On the OPE side of Eq. (4), for most weights $w(s)$, and for scales above $s_0 \sim 2$ GeV^2, far and away the dominant contribution comes from the $D = 0$ term, which is conveniently written in terms of the Adler function, $D_T(Q^2) \equiv -Q^2 d\Pi_T(Q^2)/dQ^2$,

$$\oint_{|s|=s_0} dsw(s)[\Pi_T(s)]_{D=0} = \oint_{|s|=s_0} ds \frac{v(s)}{s}[D_T(Q^2)]_{D=0},$$

$$(5)$$

where $Q^2 = -s$ and $v(s) = \int dsw(s)$, with $v(s_0) = 0$. In this form, potentially large logarithms can be summed up point-by-point along the contour through the scale choice $\mu^2 = Q^2$. The resulting "contour-improved" (CIPT) evaluation improves the convergence behavior of the known terms of the integrated $D = 0$ series [19]. An alternate evaluation, referred to as "fixed order perturbation theory" (FOPT), involves choosing a common fixed scale (such as $\mu^2 = s_0$) for all points on the contour. Large logarithms are then unavoidable over at least some portion of the contour. Detailed arguments in favor of the CIPT prescription have been presented in Ref. [5]. We find optimal consistency of our results when employing the CIPT implementation, and thus take the CIPT evaluation as our central one. However, the difference between the CIPT and FOPT evaluations, both truncated at the same given order, lies entirely in contributions of yet higher order. The CIPT-FOPT difference thus serves as one possible measure of the $D = 0$ series truncation uncertainty. It turns out that this difference is, in most cases, significantly larger than other possible estimates of the same uncertainty. We will thus adopt a conservative view and include the full CIPT-FOPT difference as one component of our truncation uncertainty estimate.

The $D = 0$ contribution to $D_{V/A;ij}$ is known to $O(\alpha_s^4)$, and given by

$$[D_{V/A;ij}(Q^2)]_{D=0} = \frac{1}{4\pi^2} \sum_{k\geq 0} d_k^{(0)} \bar{a}^k, \qquad (6)$$

where $\bar{a} = a(Q^2) = \alpha_s(Q^2)/\pi$, with $\alpha_s(\mu^2)$ the running coupling at scale μ^2 in the $\overline{MS}$ scheme, and, for $n_f = 3$, $d_0^{(0)} = d_1^{(0)} = 1$, $d_2^{(0)} = 1.63982$, $d_3^{(0)} = 6.37101$ and $d_4^{(0)} = 49.07570$ [4,20]. The next coefficient, $d_5^{(0)}$, has been estimated to be ~ 275 [4] using methods known to have (i) worked well semiquantitatively for the coefficients of the $D = 0$ series [21] and (ii) produced, in advance of the actual calculation, an accurate prediction for the recently computed $O(a^3)$ $D = 2$ coefficient of the $(J) = (0 + 1)$ $V + A$ correlator sum [22].

2. $D > 0$ OPE contributions

It is the strong numerical dominance of typical OPE integrals by $D = 0$ contributions at scales above $s_0 \sim 2$ GeV^2 that allows the corresponding weighted spectral integrals to be used in making a precision determination of α_s. The impact of uncertainties in the small residual higher D OPE terms can be understood by noting that, for all $w(s)$, the $D = 0$ contribution to the $w(s)$-weighted OPE integral, expanded as a series in $a_0 \equiv a(s_0)$, has the form $C_w[1 + a_0 + O(a_0^2)]$, where both C_w and the coefficients occurring in the $O(a_0^2)$ contribution depend on $w(s)$. Since $a(m_\tau^2) \sim 0.1$, we see that a higher D contribution with a fractional uncertainty r relative to the dominant $D = 0$ term will produce a corresponding fractional uncertainty $\sim 10r$ on $\alpha_s(m_\tau^2)$. (The factor of 10 is reduced somewhat (to ~ 5–6) when one includes the effect of higher order terms.) Thus, e.g., to achieve a determination of $\alpha_s(M_Z^2)$ accurate to $\sim 1\%$ (which corresponds to a determination of $\alpha_s(m_\tau^2)$ accurate to $\sim 3\%$) one needs to reduce the uncertainties in the determination of the higher D contributions, relative to the OPE total, to the sub-0.5% level. How easy it is to satisfy this requirement depends strongly on the choice of weight $w(s)$. We will return to this point below.

Among the $D > 0$ OPE contributions, those with $D = 2$ are either $O(m_{u,d}^2)$ or $O(\alpha_s^2 m_s^2)$ [23] and numerically negligible at the scales we consider. The $D = 4$ OPE terms are, up to numerically tiny $O(m_q^4)$ corrections, determined by the RG invariant light quark, strange quark and gluon condensates, $\langle m_\ell \bar{\ell}\ell \rangle_{\text{RGI}}$, $\langle m_s \bar{s}s \rangle_{\text{RGI}}$ and $\langle aG^2 \rangle_{\text{RGI}}$. Explicit expressions for $[\Pi_{V/A}(Q^2)]_{D=4}^{\text{OPE}}$ may be found in Refs. [23,24].

$D \geq 6$ OPE contributions are potentially more problematic since the relevant condensates are either poorly known or phenomenologically undetermined. Defining effective condensate combinations $C_6, C_8, \cdots$ such that

$$[\Pi(Q^2)]_{D>4}^{\text{OPE}} \equiv \sum_{D=6,8,\cdots} C_D/Q^D \qquad (7)$$

up to logarithmic corrections, proportional to $\alpha_s log(Q^2/\mu^2)$, the $D \geq 6$ contributions to the right-hand side of Eq. (4), for polynomial weights, $w(s) = \sum_{m=0} b_m y^m$, are given by

$$b_2 \frac{C_6}{s_0^2} - b_3 \frac{C_8}{s_0^3} + b_4 \frac{C_{10}}{s_0^4} - b_5 \frac{C_{12}}{s_0^5} + \cdots, \qquad (8)$$

again up to logarithmic corrections, proportional to α_s [25]. Integrated OPE contributions of $D = 2k + 2$ thus scale as $1/s_0^k$ (up to logarithms [26]), and hence as $1/s_0^{k+1}$ relative to the leading $D = 0$ contribution. For pinched weights, the integrals of the logarithmic corrections to Eq. (7) are suppressed, not just by the additional factors of α_s, but also by small numerical factors which result from the structure of the logarithmic integrals, $\oint_{|s|=s_0} ds y^k \ell n(Q^2/\mu^2)/Q^D$, and cancellations inherent in the pinching condition $\sum_m b_m = 0$.

D. The "(km) spectral weight" analyses

Since the kinematic weight, $(1 - y_\tau)^2(1 + 2y_\tau)$, multiplying the $(0 + 1)$ spectral contribution to $R_{T;ud}$ in Eq. (3) has degree 3, the OPE representations of the $R_{T;ud}$ all contain contributions up to $D = 8$, and hence involve three unknowns, α_s, C_6^T and C_8^T, which the single piece of information provided by the corresponding total hadronic τ decay widths (or, equivalently, $R_{T;ud}$) is insufficient to determine.

ALEPH [3,5–7] and OPAL [8] dealt with this problem by constructing additional rescaled spectral integrals, analogous to $R_{T;ud}$, corresponding to a range of alternate weight choices $w(s)$. Explicitly, α_s, $\langle aG^2 \rangle_{\text{RGI}}$, $\delta_{V,A}^{(6)} = -24\pi^2 C_6^{V,A}/m_\tau^6$ and $\delta_{V,A}^{(8)} = -16\pi^2 C_8^{V,A}/m_\tau^8$ (or $\delta_{V+A}^{(D)} = (\delta_V^{(D)} + \delta_A^{(D)})/2$, with $D = 6, 8$) were determined as part of a combined fit to the $s_0 = m_\tau^2$ versions of the $(km) = (00)$, (10), (11), (12), (13) "spectral weight sum rules," FESRs based on the weights, $w^{(km)}(y) = (1 - y)^k y^m w^{(00)}(y)$, where $w^{(00)}(y) = (1 - y)^2(1 + 2y)$ is the kinematic weight occurring on the right-hand side of Eq. (3). ALEPH [3,5–7] performed independent versions of this fit for each of the V, A and $V + A$ channels, while OPAL [8] performed independent fits for the $V + A$ and combined V, A channels.

A crucial input to these analyses was the assumption that $D > 8$ contributions could be safely neglected *for all weights considered in the fit*. In fact, since the polynomial coefficients relevant to $D > 4$ contributions are $(b_2^{(km)}, \cdots, b_7^{(km)}) = (-3, 2, 0, 0, 0, 0)$, $(-3, 5, -2, 0, 0, 0)$, $(-1, -3, 5, -2, 0, 0)$, $(1, -1, -3, 5, -2, 0)$ and $(0, 1, -1, -3, 5, -2)$ for $(km) = (00)$, (10), (11), (12), (13), respectively, we see, from Eq. (8), that all six of the quantities, $C_6, \cdots, C_{16}$, would in principle contribute to at least one of sum rules employed, making a combined fit impossible without this additional assumption.

The neglect of C_{10} through C_{16} in the ALEPH and OPAL analyses creates a theoretical systematic uncertainty not

included in the error assessments of Refs. [3,5–8]. Since the fits are performed with a single s_0 ($s_0 = m_\tau^2$), the differing s_0-dependences of integrated contributions of different D are not operative, and hence neglect of non-negligible $D > 8$ contributions can be compensated for by shifts in the values of fitted parameters relevant to lower D contributions [27]. Indications that such a compensation may, indeed, be at work are provided by (i) the lack of agreement between the values for $\langle aG^2 \rangle_{\mathrm{RGI}}$ obtained from the separate ALEPH V and A analyses [3,5], (ii) the fact that the central fitted values of $\langle aG^2 \rangle_{\mathrm{RGI}}$ obtained in the V, A and $V + A$ CIPT analyses of both groups are uniformly lower than of the updated charmonium sum rule analysis of Ref. [28], and (iii) the poor quality of the 2005 ALEPH A and $V + A$ fits ($\chi^2/\mathrm{dof} = 4.97/1$ and $3.66/1$, respectively) and 2008 ALEPH A fit ($\chi^2/\mathrm{dof} = 3.57/1$).

A further indication that the neglect of $D > 8$ contributions (which are in principle present in the $(km) = (10)$, (11), (12) and (13) spectral weight FESRs) is potentially dangerous is provided by a consideration of the relative sizes of the $D = 6, 8$ and $D = 0$ terms corresponding to the results of the earlier ALEPH and OPAL fits. One should bear in mind that the additional factors of y in the weights $w^{(1m)}(y)$, $m \geq 1$, strongly suppress the correspondingly weighted $D = 0$ integrals, but produce no such suppressions of the integrated higher D contributions, causing the $D > 4$ contributions to play a much larger relative role for these weights than they do for the (00) and (10) weight cases. Taking the 2005 ALEPH V fit as an example, we find that

(i) for the (11) spectral weight FESR, the $D = 6$ and $D = 8$ contributions (which include, as per Eq. (8), the polynomial coefficient factors -1 and -3, respectively) represent, respectively, 5.2% and 7.4% of the leading $D = 0$ contribution, while $D = 10$ and 12 contributions (which would be weighted by the coefficients 5 and -2 from $w^{(11)}$) are assumed negligible;

(ii) for the (12) spectral weight FESR, the $D = 6$ and $D = 8$ contributions (weighted by polynomial coefficients 1 and -1, respectively) represent, respectively, -13.7% and 6.5% of the $D = 0$ contribution, while $D = 10$, 12 and 14 contributions (which would be accompanied by the $w^{(12)}$ polynomial coefficients -3, 5 and -2) are again assumed negligible; and

(iii) for the (13) spectral weight FESR, the $D = 8$ contribution (weighted by polynomial coefficient 1) represents -14.3% of the $D = 0$ contribution, while $D = 10$, 12, 14 and 16 contributions (which would be accompanied by the $w^{(13)}$ polynomial coefficients -1, -3, 5 and -2, respectively) are once more assumed negligible.

Given the $<0.5\%$ tolerance in the sum of $D > 4$ relative to $D = 0$ contributions required for a $\sim 1\%$ determination of $\alpha_s(M_Z^2)$, the neglect of $D > 8$ contributions appears to us to represent a rather strong assumption.

A quantitative test of whether or not such contributions can, in fact, be safely neglected *for all of the weights employed in the ALEPH and OPAL analyses* can be obtained by studying the quality of the fitted OPE representations of the $w^{(km)}(y)$-weighted spectral integrals as a function of s_0. The utility of this test follows from the fact, already noted above, that integrated contributions of different D scale differently with s_0. Thus, if the fitted values of α_s, $\langle aG^2 \rangle_{\mathrm{RGI}}$, C_6 and C_8 are unphysical as a result of shifts induced by the need to compensate for missing $D > 8$ contributions in one or more of the FESRs employed, the fact that this compensation occurs in lower dimension contributions, which scale more slowly with s_0 than do the contributions they are replacing, will show up as a deterioration of the fit quality as s_0 is decreased below the single value $s_0 = m_\tau^2$ used in the ALEPH and OPAL analyses. In contrast, were the fit quality to be maintained at lower s_0, this would provide significant evidence in support of the prescription of neglecting $D > 8$ contributions in the set of FESRs employed in those analyses. We thus define the s_0-dependent fit-qualities,

$$F_T^w(s_0) \equiv \frac{I_{\mathrm{spec}}^w(s_0) - I_{\mathrm{OPE}}^w(s_0)}{\delta I_{\mathrm{spec}}^w(s_0)} \qquad (9)$$

where, as usual, $T = V$, A or $V + A$,

$$\begin{aligned} I_{\mathrm{spec}}^w(s_0) &= \int_0^{s_0} ds\, w(s) \rho_{T;ud}^{(0+1)}(s) \\ I_{\mathrm{OPE}}^w(s_0) &= \frac{-1}{2\pi i} \oint_{|s|=s_0} ds\, w(s) [\Pi_{T;ud}^{(0+1)}(s)]_{\mathrm{OPE}} \end{aligned} \qquad (10)$$

and $\delta I_{\mathrm{spec}}^w(s_0)$ is the error on $I_{\mathrm{spec}}^w(s_0)$, determined using the experimental covariance matrix for $dR_{T;ud}/ds$. One should bear in mind that strong correlations exist between the $I_{\mathrm{spec}}^w(s_0)$ for fixed $w(s)$ but different s_0, and similarly between the $I_{\mathrm{OPE}}^w(s_0)$ for fixed $w(s)$ but different s_0. Because of these correlations, the assumption that $D > 8$ OPE contributions are safely negligible corresponds to the expectation that $|F_T^w(s_0)|$ should remain less than ~ 1 for a range of s_0 below m_τ^2, *and for all of the $w(s)$ employed in the analysis in question*. It turns out that neither the ALEPH nor the OPAL fits satisfy this expectation.

To illustrate this point, we show, in Fig. 1, the fit qualities, $F_V^w(s_0)$, corresponding to the 2005 ALEPH data and fit [3], for a selection of the (km) spectral weights. In the figure, the solid horizontal lines indicate the boundaries $F_V(s_0) = \pm 1$ within which we would expect curves corresponding to a physically meaningful fit to lie. We remind the reader that, although the original 2005 ALEPH $s_0 = m_\tau^2$ A and $V + A$ fits had χ^2/dof significantly >1, the χ^2/dof for the V channel fit was $0.52/1$. The test is thus being applied to the most successful of the previous fits.
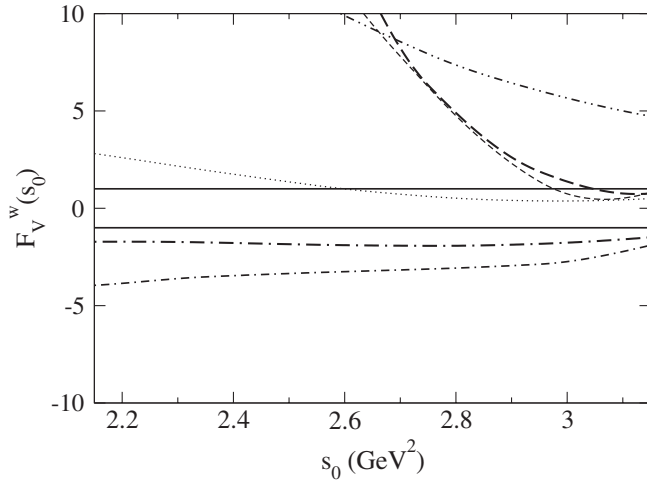
FIG. 1.   Fit qualities, as a function of s_0, for the 2005 ALEPH V fit and the weights $w^{(00)}$, $w^{(12)}$, $w^{(13)}$, w_2, w_3 and $w(y) = y(1-y)^2$. The results for $w^{(00)}$, $w^{(12)}$, $w^{(13)}$, w_2, w_3 and $y(1-y)^2$ are shown by the dotted, medium-dashed, long-dashed, short dot-dashed, long dot-dashed and double-dot-dashed lines, respectively. The right boundary corresponds to the kinematic endpoint, $s_0 = m_\tau^2 \simeq 3.16$ GeV^2.

Also shown in the figure are the V channel fit qualities, $F_V^w(s_0)$, for three additional weights, $w_2(y) = (1-y)^2$, $w_3(y) = 1 - \frac{3}{2}y + \frac{y^3}{2}$ and $w(y) = y(1-y)^2$, all having degree ≤ 3. The weights w_2 and w_3 are the first two members of a series,

$$w_N(y) = 1 - \frac{N}{N-1}y + \frac{1}{N-1}y^N \qquad (11)$$

to which we will return in our own analysis below. From Eq. (8), we see that the only $D > 4$ contribution to the w_2 (respectively, w_3) FESR is $\frac{C_6}{s_0^2}$ (respectively, $-\frac{C_8}{2s_0^3}$). The w_2 (respectively w_3) FESR thus provides a useful independent test of the value of C_6 (respectively C_8) obtained in the earlier fits. The $w(y) = y(1-y)^2$ FESR, with $D > 4$ OPE contribution $-\frac{2C_6}{s_0^2} - \frac{C_8}{s_0^3}$, provides another such test since this linear combination is independent of that appearing in the (00) spectral weight FESR. The strength of the test is enhanced in this case because the factor y in the weight leads to a significant suppression of the $D = 0$ integral, making the $y(1-y)^2$ FESR relatively more sensitive to $D > 4$ contributions. If the neglect of $D > 8$ contributions in the earlier analyses was actually justified, the $s_0 < m_\tau^2$ FESRs corresponding not only to the spectral weights employed in those fits, but also to w_2, w_3, and $y(1-y)^2$ should all be well satisfied using the fitted values of the input $D \le 8$ OPE parameters. It is evident from the figure that this is far from being the case. The poor quality of the ALEPH fit when applied to the w_2, w_3 and $y(1-y)^2$ FESRs, even at $s_0 = m_\tau^2$, and the fact that the nominally good quality of the original fit to the $s_0 = m_\tau^2$ spectral weight FESRs does not persist to lower s_0, clearly establish

the presence of $D > 8$ contamination in at least some of the original fitted FESRs. The deterioration in the fit quality as s_0 is decreased below m_τ^2 seen for all cases shown in the figure is in fact a general feature, one found for all of the weights discussed and all three of the channels investigated in this paper.

One could, of course, attempt to use the s_0 dependence of the $w^{(km)}$-weighted spectral integrals to aid in achieving an improved fit for the $D > 4$ C_D. It is important to bear in mind, however, that the range of s_0 that can be employed in such a fit is limited: to $s_0 < m_\tau^2$ by kinematics, and to s_0 greater than ~ 2 GeV^2, if one wishes to avoid non-negligible "duality violation" (OPE breakdown) [29–31]. In such a relatively restricted window, the number of independent parameters that can be successfully fitted is limited. The (km) spectral weight FESRs thus represent nonoptimal choices for an analysis of this type since their OPE sides typically involve, in addition to the parameter $\alpha_s(m_\tau^2)$ we are primarily interested in determining, a combination of several of the unknown $D > 4$ C_D. It is also worth stressing that the (11), (12) and (13) spectral weight FESRs used in the previous analyses have another feature which makes them nonoptimal for an analysis whose main goal is the determination of α_s. Optimization of such a determination is achieved by using sum rules which enhance, as much as possible, the relative contribution of the integrated $D = 0$ series, since it is in this contribution that the dominant dependence on α_s lies. The $(1m)$, $m \ge 1$, spectral weights, however, do exactly the opposite, the additional factors of y producing rather strong suppressions of the leading $D = 0$ OPE integrals (by factors of ~ 6.5, 17, and 37 relative to the corresponding (00) integral for the (11), (12) and (13) cases, respectively) without any accompanying suppression of higher D contributions (beyond that which may (or may not) be present in the correlator itself).

E. An alternate analysis strategy

In view of the problems displayed by the (km) spectral weight FESR analyses, we turn to FESRs based on the weights, $w_N(y)$ introduced already in Eq. (11) above. The w_N are constructed to share with the (00) spectral weight the presence of a double zero at $s = s_0$ and the resulting suppression of OPE-violating contributions near the timelike point on the OPE contour. For our problem they have, in addition, the following positive features, not shared by the set of (km) spectral weights employed in the ALEPH and OPAL analyses:

(i) the $D = 0$ integrals grow moderately with N rather than decreasing strongly as was the case when one went from the lower to the higher spectral weights;

(ii) at the same time, the coefficient governing the only unsuppressed $D > 4$ contribution (that with $D = 2N + 2$) decreases with N, further enhancing $D = 0$ relative to $D > 4$ contributions;

(iii) because each w_N FESR involves only a single unsuppressed $D > 4$ contribution, the collection of w_N FESRs is well adapted to most efficiently implementing the constraints associated with the s_0 dependence of the correspondingly weighted spectral integrals in the fitting of the unknown $D > 4$ OPE parameters; and

(iv) as N is increased, the $1/s_0^{N+1}$ scaling of the single unsuppressed $D = 2N + 2$ contribution relative to the leading $D = 0$ contribution varies more and more strongly with s_0, increasing the leverage for fitting C_{2N+2} (though the effect is of course offset to some extent by the decrease with N of the polynomial coefficient, $1/(N - 1)$, present in the integrated form of the $D = 2N + 2$ contribution).

To quantify the extent to which the level of $D = 0$ dominance of the w_N FESRs represents an improvement over that of the (km) spectral weight FESRs, we introduce the double ratio, $R^D[w_N, w^{(km)}, s_0]$, defined by

$$R^D[w_N, w^{(km)}, s_0] = \frac{r_{w_N}^D(s_0)}{r_{w^{(km)}}^D(s_0)} \tag{12}$$

where

$$r_w^M(s_0) \equiv \frac{[I_{\mathrm{OPE}}^w(s_0)]_{D=M}}{[I_{\mathrm{OPE}}^w(s_0)]_{D=0}}. \tag{13}$$

$R^D[w_N, w_{km}, s_0]$ represents the suppression of the fractional contribution of dimension D in the w_N FESR relative to that in the $w^{(km)}$ FESR and, by construction, is independent of C_D. Taking $s_0 = m_\tau^2$ to be specific, we find that

(i) $R^6[w_2, w^{(km)}, m_\tau^2] = -1/2.1, -1/2.9, -1/4.4$, and $-1/12$ for $(km) = (00), (10), (11)$ and (12), respectively;

(ii) $R^8[w_3, w^{(km)}, m_\tau^2] = 1/3.1, 1/11, -1/25, -1/26$ and $-1/58$ for $(km) = (00), (10), (11), (12)$ and (13), respectively;

(iii) $R^{10}[w_4, w^{(km)}, m_\tau^2] = -1/6.8, 1/79, -1/126$, and $-1/91$ for $(km) = (10), (11), (12)$ and (13), respectively;

(iv) $R^{12}[w_5, w^{(km)}, m_\tau^2] = -1/44, 1/288$ and $-1/379$ for $(km) = (11), (12)$ and (13), respectively; and

(v) $R^{14}[w_6, w^{(km)}, m_\tau^2] = -1/149$ and $1/814$ for $(km) = (12)$ and (13), respectively.

Neglect of $D > 8$ contributions would thus be between ~ 1 and 3 *orders of magnitude* safer for the w_4, w_5 and w_6 FESRs than it would for the (10), (11), (12) and (13) spectral weight sum rules. Had it been safe for the latter, then it would certainly also be safe for the former. From our fits below, however, we find small, but not entirely negligible, $D = 10, 12, 14$ contributions to the w_4, w_5 and w_6 FESRs, respectively. The analogous contributions, which play a much larger relative role in the higher spectral weight FESRs, account for the problems of the ALEPH

and OPAL spectral weight FESR fits seen in the fit quality plot above.

III. THE w_N FESR ANALYSES

As N gets large, the different $w_N(y)$ become less and less independent, approaching $1 - y$ in the limit that $N \to \infty$. The approach to $1 - y$ also weakens the level of the desired suppression of contributions from the vicinity of the timelike point on the OPE contour. In addition, the reduction of the unsuppressed integrated $D = 2N + 2$ contribution by the factor $1/(N - 1)$ means that these contributions will eventually be driven down to the level of the other, numerically and α_s-suppressed, contributions of $D > 4$ having $D \neq 2N + 2$ [32]. For these reasons we focus, in what follows, on those FESRs corresponding to the limited set of weights $w_2, \cdots, w_6$. A clear demonstration of the independence of the results associated with the different w_N in this set will be given in Sec. IV.

The values of any input parameters, together with details of our treatment of the spectral and OPE integral sides of the w_N FESRs, are given in Secs. III A and III B, respectively. Results for the ALEPH-based V, A and $V + A$ and OPAL-based $V + A$ fits, as well as a breakdown of the contributions to the theoretical errors on the fitted parameters, $\alpha_s(m_\tau^2)$ and C_D, $D = 6, 8 \cdots 14$, are given in Sec. III C. A final assessment and discussion of the results is deferred to Sec. IV.

A. The w_N-weighted spectral integrals

On the spectral integral side of the w_N FESRs, we employ for our main analysis the publicly available 2005 ALEPH V, A and $V + A$ spectral data and covariance matrices [3,6]. Our central results will also follow Ref. [5] in incorporating, in the V and A channels, the improved s-dependent V/A separation of the contribution from the $K\bar{K}\pi$ mode made possible by the recent *BABAR* isovector electroproduction cross-section measurements [17] and the details on the $V + A$ $K\bar{K}\pi$ distribution presented in Ref. [5]. Independent analyses using the 1999 OPAL V, A and $V + A$ data and covariance matrices have also been performed, though in this case we do not have the information on the $K\bar{K}\pi$ distribution needed to make the improved V/A separation for that mode and so will report results below only for the $V + A$ analysis.

We employ as input to the determination of the isovector spectral function from the ALEPH or OPAL distributions the values

$$S_{\mathrm{EW}} = 1.0201(3) \tag{14}$$

$$B_e = 0.17818(32) \tag{15}$$

$$|V_{ud}| = 0.97408(26) \tag{16}$$

where S_{EW} is taken from Ref. [11], the lepton-universality-constrained result for B_e from Ref. [33], and the result for

$|V_{ud}|$ from the most recent update of the $0^+ \to 0^+$ super-allowed nuclear β decay analysis [34]. The π pole contribution to the A and $V + A$ spectral integrals is evaluated using the very accurate determination of $f_\pi |V_{ud}|$ from the $\pi_{\mu 2}$ width [35]. A small global renormalization must also be applied to the ALEPH and OPAL data as a result of small changes to B_e, S_{EW}, $|V_{ud}|$ and the total τ strange branching fraction, B_s, (which enters the most precise determination of the overall $V + A$ normalization, $R_{ud;V+A}$) since the original publications. With the full set of recent *BABAR* and Belle updates to the branching fractions of various strange modes [36], we obtain $R_{ud;V+A} = 3.478(11)$. It is assumed that the continuum parts of the V, A and $V + A$ distributions are all to be rescaled by the same common factor. The uncertainty in $R_{ud;V+A}$ strongly dominates the overall normalization uncertainty on the spectral integrals.

B. The w_N-weighted OPE integrals

For the $D = 0$ contribution we employ the CIPT evaluation as our central determination. We truncate the $D = 0$ Adler function series at $O(\bar{a}^5)$, using the known coefficients for terms up to $O(\bar{a}^4)$ and the estimate $d_5^{(0)} = 275 \pm 275$ of Ref. [4] for the coefficient of the last term. An independent evaluation using the alternate FOPT evaluation is also performed and the variation induced by the uncertainty in $d_5^{(0)}$ and the CIPT-FOPT difference added in quadrature to produce the full truncation uncertainty estimate. An analogous procedure, using however the average of the CIPT and FOPT determinations as central value, and half the difference as the corresponding component of the truncation uncertainty estimate (added linearly to the uncertainty generated by that on $d_5^{(0)}$), was employed in Ref. [4]. Our estimate yields a $D = 0$ truncation uncertainty assessment similar to that of Ref. [4], but significantly more conservative than the alternate estimates based on a combination of the $d_5^{(0)}$ uncertainty and residual scale dependence which have also been employed elsewhere in the literature.

In evaluating the running coupling over the OPE contour we employ the exact analytic solution associated with the 4-loop-truncated β function [37]. The reference scale input needed to specify this solution, taken here to be $\alpha_s(m_\tau^2)$, is to be determined as part of the fitting procedure.

The $D = 2$ contributions, as already noted, are either $O([m_d \pm m_u]^2)$ or $O(\alpha_s^2 m_s^2)$, and hence expected to be numerically negligible. Our central values correspond to neglecting them entirely. The $O([m_d \pm m_u]^2)$ contributions should, in fact, be neglected in any case, as a matter of consistency. The reason is that, even at the highest scale, $s_0 = m_\tau^2$, allowed by kinematics, the OPE representation of the "longitudinal" ($J = 0$) contribution to the experimental spectral distribution (in the $(J) = (0 + 1)/(0)$ decomposition of Eq. (3)) is completely out of control. Not only

do the variously weighted integrated $D = 2$ OPE series display extremely bad convergence, but all truncation schemes for these badly behaved series employed in the literature badly violate constraints associated with spectral positivity [38]. It is thus impossible to use the longitudinal OPE to estimate the $O([m_d \pm m_u]^2)$ longitudinal contributions to the spectral distribution, which means that the spectral functions $\rho_{ud;V/A}^{(0+1)}(s)$ can be determined only up to uncertainties of $O([m_d \mp m_u]^2)$, respectively. It would thus be inconsistent to explicitly include contributions of this same order on the OPE side of the $0 + 1$ FESRs. We have, in any case, verified, by direct computation, that including the integrated $J = 0 + 1$, $D = 2$ OPE contributions would have a negligible impact on our analysis, in agreement with the results for these contributions quoted in the earlier analyses. The $J = 0 + 1$, $D = 2$ computation employed the exact solution for the running masses corresponding to the 4-loop truncated β [37] and γ [39] functions, with PDG06 values for the $\overline{MS}$ scheme light and strange quark masses at scale 2 GeV [35] as input. It is also possible to estimate the contributions from the non-π-pole part of the $J = 0$ spectral distributions and verify that they are safely negligible. For the A channel this estimate employs the spectral model of Ref. [40] for the isovector pseudoscalar channel, a model generated using a combined Borel and finite energy sum rule analysis of the relevant pseudoscalar correlator [40]. The isovector V channel $J = 0$ contributions, being suppressed by a further factor of $[(m_d - m_u)/(m_d + m_u)]^2 \sim 1/10$, are even more negligible.

We employ as basic $D = 4$ input

$$\langle 2m_\ell \bar{\ell}\ell \rangle_{RGI} = -m_\pi^2 f_\pi^2 \quad \text{and} \quad (17)$$

$$\langle aG^2 \rangle_{RGI} = (0.009 \pm 0.007) \text{ GeV}^4 \quad (18)$$

the first result being the GMOR relation [41] and the second the result of Ref. [28]. The remaining $D = 4$ combination, $\langle m_s \bar{s}s \rangle_{RGI}$, then follows from conventional ChPT quark mass ratios [42] and the value,

$$r_c = \frac{\langle \bar{s}s \rangle_{RGI}}{\langle \bar{\ell}\ell \rangle_{RGI}} = 1.1 \pm 0.6, \quad (19)$$

obtained by updating the analysis of Ref. [43], using the range of recent $n_f = 2 + 1$ lattice results for f_{B_s}/f_B as input [44]. Although this value of r_c is nearly twice that employed in the earlier ALEPH and OPAL analyses (whose values, however, are based on somewhat out-of-date input), the difference between the two has negligible impact on the final analysis since the integrated $D = 4$ contributions are both small at the scales employed and, in any case, dominated by the gluon condensate contribution. The sizable uncertainty we quote on r_c, for the same reason, plays a negligible role in our final theoretical error estimate.

$D > 4$ contributions are handled by treating the various C_{2N+2} as fit parameters. C_{2N+2} is fitted, together with $\alpha_s(m_\tau^2)$, to the set of $I_{w_N}(s_0)$ corresponding to a range of s_0. The requirement that the values of $\alpha_s(m_\tau^2)$ obtained in this manner from the different w_N FESRs should be consistent provides a nontrivial check on the reliability of the analysis. We discuss this issue further in Sec. IV.

For the ALEPH-based fits, we work with an equally spaced set of s_0 values, $s_0 = (2.15 + 0.2k)$ GeV^2, $k = 1, \cdots, 6$, adapted to the ALEPH experimental bins. We also study the stability of our fits by either removing the 2.15 GeV^2 point or adding, in addition, $s_0 = 1.95$ GeV^2. For the OPAL-based fits, the analogous s_0 set is $s_0 = (2.176 + 0.192k)$ GeV^2, $k = 1, \cdots, 6$, with stability studied by either removing the lowest point, or adding an additional point with $s_0 = 1.984$ GeV^2.

C. Results

Results for the V, A and $V + A$ fits based on the ALEPH data are presented in the upper portion of Table I. In the table, we display, for each of the w_N, $N = 2, \cdots, 6$, FESRs, the fitted values of $\alpha_s(m_\tau^2)$ and the relevant $D > 4$ coefficient, C_{2N+2}, the latter quoted in the dimensionless form, C_{2N+2}/m_τ^{2N+2}. We remind the reader that, in arriving at these values, we have implemented the improved V/A separation for the $K\bar{K}\pi$ mode, discussed already above. This improvement produces an upward (downward) shift of 0.0013 in the central value of the A (V) determinations of $\alpha_s(m_\tau^2)$, improving further the consistency between the

results of the separate V, A and $V + A$ analyses. The level of consistency, even before this improvement, is significantly better than that displayed by the (km) spectral weight analysis results reported in Ref. [5].

The lower portion of Table I contains the corresponding results for the OPAL-based $V + A$ fits. The results for the separate V and A fits are not displayed in this case, since we lack the information on the $K\bar{K}\pi$ contribution to the inclusive distribution required to perform the improved V/A separation. For completeness, however, we mention that the central values of $\alpha_s(m_\tau^2)$ obtained without this correction lie 0.003 lower (higher) for the V (A) fits. The improved V/A separation, of course, plays no role in the $V + A$ fit. The ALEPH- and OPAL-based results are seen to be in very good agreement within errors.

The experimental errors quoted in the table contain a component associated with the 0.32% normalization uncertainty, which is 100% correlated for all of the separate analyses. The theory error is obtained by adding in quadrature uncertainties associated with (i) the truncation of the $D = 0$ series (itself the quadrature sum of the difference of the CIPT and FOPT fit results and the uncertainty produced by taking $d_5^{(0)} = 275 \pm 275$), (ii) the uncertainties on the $D = 4$ input condensates and (iii) the "stability" uncertainty, generated by varying the lower edge of the fit window employed, as described above.

Individual contributions to the theoretical errors on the fitted parameters, $\alpha_s(m_\tau^2)$ and C_{2N+2}/m_τ^{2N+2}, obtained from the w_N-weighted, ALEPH-based $V + A$ FESRs, are

TABLE I. Results of the w_N FESR fits for $\alpha_s(m_\tau^2)$ and C_{2N+2}/m_τ^{2N+2} obtained using either the ALEPH or OPAL data and covariances. In all entries, the first error is experimental and the second theoretical.

Data set	Channel	Weight w_N	$\alpha_s(m_\tau^2)$	C_{2N+2}/m_τ^{2N+2}
ALEPH	V	w_2	0.321(7)(8)	$-0.000187(29)(56)$
		w_3	0.321(7)(10)	0.000060(36)(60)
		w_4	0.321(7)(11)	0.000015(36)(53)
		w_5	0.321(7)(12)	$-0.000043(33)(44)$
		w_6	0.321(7)(12)	0.000046(27)(35)
	A	w_2	0.319(6)(9)	$-0.000072(24)(60)$
		w_3	0.319(6)(10)	0.000182(28)(71)
		w_4	0.319(6)(11)	$-0.000216(27)(70)$
		w_5	0.319(6)(12)	0.000201(23)(66)
		w_6	0.319(6)(12)	$-0.000166(19)(59)$
	$V + A$	w_2	0.320(5)(8)	$-0.000261(35)(114)$
		w_3	0.320(5)(9)	0.000247(45)(125)
		w_4	0.320(5)(10)	$-0.000208(44)(111)$
		w_5	0.320(5)(11)	0.000166(39)(97)
		w_6	0.320(5)(12)	$-0.000126(34)(88)$
OPAL	$V + A$	w_2	0.322(7)(8)	$-0.000233(59)(114)$
		w_3	0.322(7)(10)	0.000205(74)(120)
		w_4	0.322(7)(11)	$-0.000162(76)(105)$
		w_5	0.322(7)(12)	0.000122(70)(86)
		w_6	0.322(8)(12)	$-0.000091(60)(67)$

TABLE II.   Contributions to the theoretical uncertainties on $\alpha_s(m_\tau^2)$ and C_{2N+2}/m_τ^{2N+2} obtained in the fits to w_N $V + A$ FESRs based on the ALEPH data and covariances.

Observable	Weight w_N	FOPT	$\delta d_5^{(0)}$	$\delta\langle aG^2\rangle$	Stability
$\alpha_s(m_\tau^2)$	w_2	0.0004	0.0056	0.0059	0.0014
	w_3	0.0049	0.0056	0.0059	0.0014
	w_4	0.0068	0.0056	0.0059	0.0013
	w_5	0.0079	0.0055	0.0059	0.0013
	w_6	0.0084	0.0056	0.0059	0.0015
C_{2N+2}/m_τ^{2N+2}	w_2	0.000069	0.000019	0.000084	0.000027
	w_3	0.000090	0.000016	0.000072	0.000044
	w_4	0.000078	0.000013	0.000058	0.000053
	w_5	0.000063	0.000012	0.000045	0.000058
	w_6	0.000051	0.000008	0.000035	0.000062

shown, in the upper and lower halves of Table II, respectively. Results for the OPAL-based $V + A$ and ALEPH-based V and A fits are not quoted separately, the decompositions being similar, with the exception of the stability contributions for the OPAL-based $V + A$ fits, which are a factor of ~ 2 smaller than those for the corresponding ALEPH-based $V + A$ fits. The differences between the results produced by the CIPT and FOPT evaluations of the $D = 0$ OPE contributions are given in the *FOPT* column of the table, while the uncertainties associated with those on $d_5^{(0)}$, $\langle aG^2\rangle_{\rm RGI}$, and the variation of the lower edge of the s_0 fit window appear in the columns headed by $\delta d_5^{(0)}$, $\delta\langle aG^2\rangle$, and *stability*, respectively. The very small uncertainties generated by those on the light and strange condensates (which, for example, produce uncertainties of ± 0.0002 on $\alpha_s(m_\tau^2)$) can be neglected without changing the total theoretical error, and hence are not quoted explicitly in the table. In all cases we symmetrize the quoted errors, taking the larger of the two possibilities in the event that the original error is asymmetric.

We see from the table that the contributions to the theoretical error on $\alpha_s(m_\tau^2)$ are very similar for the various w_N, with the exception of the FOPT-CIPT difference, which is small for w_2 and grows with increasing N. One should bear in mind, however, that, for the kinematic weight, $w^{(00)}$, the FOPT expansion, truncated at a given order, was shown to oscillate about the correspondingly truncated CIPT expansion with a period of about 6 perturbative orders [45]. Studying the FOPT-CIPT difference as a function of truncation order for the various w_N we find evidence for a similar oscillatory pattern, but with the truncation order at which the crossover between the two truncated sums occurs dependent on N. We thus consider the small FOPT-CIPT difference for w_2 an artifact of the particular truncation order of our central results, and expect the difference to grow for the next few truncation orders. For this reason, to be conservative, we take the largest of the FOPT-CIPT differences (that for w_6) as our estimate of the FOPT vs CIPT component of the truncation uncertainty for $\alpha_s(m_\tau^2)$ for all of the w_N FESRs studied. This prescription leads to a common theoretical error of ± 0.012 for all of our determinations of $\alpha_s(m_\tau^2)$.

The results quoted so far take into account short-distance electroweak corrections but do not include long-distance electromagnetic (LDEM) effects. Such LDEM corrections, though believed to be small, have been investigated in detail only for the $\pi\pi$ final hadronic state [46,47]. We study the impact of the $\pi\pi$ LDEM corrections on the V and $V + A$ channel analyses using the form of these corrections given in Ref. [46] (which implementation incorporates a resonance contribution not included in the earlier studies of Refs. [47]). We find that the correction raises $\alpha_s(m_\tau^2)$ by 0.0002–0.0003 (0.0001–0.0002) for the various V ($V + A$) channel w_N FESR analyses. In arriving at our final assessment, reported in the next section, we have included the $\pi\pi$ LDEM correction, assigning it an uncertainty of 100%, in view of the as-yet-undetermined corrections associated with higher multiplicity modes. Even were one to expand this uncertainty several-fold, the impact on our final error would remain entirely negligible.

IV. DISCUSSION AND FINAL RESULTS

A. Discussion

In this subsection we discuss further the reliability and consistency of our extraction of α_s, compare our results for the C_D with those of other analyses, and comment on a number of other relevant points.

1. Impact of the new Belle $\pi\pi$ data

We begin by discussing what impact the recently released Belle $\tau \to \pi\pi\nu_\tau$ data [48] might have on our conclusions. Note that the $\pi\pi$ branching fraction, $B_{\pi\pi}$, measured by Belle is in good agreement with the previous τ measurements reported by ALEPH [6], OPAL [8], CLEO [49], L3 [50] and DELPHI [51]. The unit-normalized number distribution, however, differs slightly in shape from that obtained by ALEPH, being somewhat higher (lower) than ALEPH below (above) the ρ peak. Such a

difference will lead to normalization and s_0-dependence shifts in the weighted V and $V + A$ spectral integrals, causing, in general, shifts in the fitted values of both $\alpha_s(m_\tau^2)$ and the C_{2N+2}. To investigate the size of these effects, we use the new world average for $B_{\pi\pi}$ (including the Belle result) to fix the overall normalization of the Belle $\pi\pi$ distribution and, after adding the difference of the weighted BELLE and ALEPH $\pi\pi$ spectral integral components to the ALEPH spectral integrals, perform a series of "Belle-$\pi\pi$-modified" w_N FESR fits. Since we lack the covariance information needed to fully replace the ALEPH $\pi\pi$ with Belle $\pi\pi$ data, we employ the ALEPH covariance matrix, without change, in the fit. The results thus represent only an exploration of the magnitude of the shift in α_s likely to be associated with such a shift in the shape of the $\pi\pi$ distribution. We find that the Belle-$\pi\pi$-modified V channel (respectively, $V + A$ channel) fits yield $\alpha_s(M_Z^2)$ values lower than those obtained using the ALEPH data alone by ~ 0.00007 (respectively, 0.00013), showing that the impact on our central result (obtained from the $V + A$ channel fits) is negligible on the scale of our other uncertainties. It would nonetheless be extremely interesting to have measured versions of the full nonstrange spectral distribution, including the improved V/A separation made possible by the much higher statistics, from the B factory experiments.

2. Consistency and reliability of the analysis

With regard to the reliability and consistency of our results, we note first that, for each of the V, A and $V + A$ analyses, the same quantity, $\alpha_s(m_\tau^2)$, is obtained from five independent FESR fits. In each of the V, A and $V + A$ channels, we find that the results from the different w_N analyses are in exceedingly good agreement, the variation across the different weight choices being at the ± 0.0001 level, and hence invisible at the precision displayed in Table I. The fitting of the $D > 4$ OPE coefficients, C_D, and concomitant identification of the small $D > 4$ OPE contributions is crucial to achieving this level of agreement, as can be seen from Table III, which shows the ALEPH $V + A$ fit values for $\alpha_s(m_\tau^2)$ already quoted above, together with the corresponding results obtained by ignoring the relevant $D > 4$ contribution, and working at the highest available scale, $s_0 = m_\tau^2$. In assessing the improvement in consistency produced by including the C_D in the fits, one should bear in mind that the non-normalization component of the experimental uncertainty (which is still correlated but, unlike the normalization and theoretical uncertainties, not 100% correlated amongst the different weight cases) is 0.003. The impact of including the $D > 4$ contributions is, not surprisingly, greatest for the w_2 FESR, where the suppression of the $D = 6$ contribution by the polynomial coefficient factor $1/(N - 1)$ ($= 1$ in this case) is the least strong of all the cases studied. The results of the table also show that use of the w_N FESRs has (as intended)

TABLE III. Impact of the inclusion of $D > 4$ OPE contributions on the fitted values for $\alpha_s(m_\tau^2)$ for the ALEPH-based analyses. The column headed *full fit* repeats the values quoted above for the various w_N-weighted $V + A$ FESRs, while that headed *no $D > 4$* contains the corresponding values obtained by working at the maximum scale $s_0 = m_\tau^2$ and neglecting the contribution of dimension $D = 2N + 2$ on the OPE side.

Channel	Weight	*Full fit*	*no $D > 4$*
V	w_2	0.321	0.305
	w_3	0.321	0.320
	w_4	0.321	0.323
	w_5	0.321	0.325
	w_6	0.321	0.325
A	w_2	0.319	0.314
	w_3	0.319	0.312
	w_4	0.319	0.314
	w_5	0.319	0.316
	w_6	0.319	0.318
$V + A$	w_2	0.320	0.310
	w_3	0.320	0.316
	w_4	0.320	0.319
	w_5	0.320	0.321
	w_6	0.320	0.322

been successful in suppressing $D > 4$ relative to $D = 0$ OPE contributions, an effect desirable for optimizing the accuracy of our α_s determination. The table in fact shows that the impact of the full $D > 4$ contribution, in all but the w_2 case, is at a level less than $\sim 50\%$ of the dominant theoretical component of the overall uncertainty, making the impact of higher order corrections to the treatment of the integrated $D > 4$ contributions safely negligible [26].

While the lack of consistency of the results for α_s in the limit that all the C_D are set to zero establishes the independence of the different w_N-weighted FESRs, and hence the nontrivial nature of the consistency observed once the C_D are included in the fits, an even more compelling case for the degree of independence of the different FESRs is provided by the results obtained by fitting the w_N-weighted OPE integrals to the set of w_M-weighted spectral integrals, with $N \neq M$. The results for $\alpha_s(m_\tau^2)$ obtained from this exercise, using the ALEPH data in the $V + A$ channel, are shown in Table IV, whose row (respectively, column) headings give the weight employed for the spectral (respectively, OPE) integrals. Blank entries in the table denote cases where no minimum could be found for the χ^2 function having positive $\alpha_s(m_\tau^2)$. It is evident from the table that the constraints on α_s associated with the set of w_N employed in our analysis enjoy a high degree of independence.

Further evidence for the reliability of our fits for α_s and the C_D is provided by the fact that, unlike the fit qualities associated with the ALEPH fit parameter sets, those associated with our fits remain between -1 and 1 for all three channels, all five w_N, and all s_0 in our fit window. This is

TABLE IV. The fitted values for $\alpha_s(m_\tau^2)$ obtained from an ALEPH-based $V + A$ analysis employing one w_N for the spectral integrals (identified by the row label) but a different w_N for the OPE integrals (identified by the column heading).

	w_2	w_3	w_4	w_5	w_6
w_2	0.320	0.175	$\cdots$	$\cdots$	$\cdots$
w_3	0.435	0.320	0.249	0.194	0.149
w_4	0.499	0.384	0.320	0.277	0.243
w_5	0.541	0.423	0.361	0.320	0.291
w_6	$\cdots$	0.450	0.388	0.349	0.320

illustrated for the V channel in Fig. 2, which shows the $F_V^w(s_0)$ corresponding to our fits (denoted by the heavy lines) for the four weights discussed above ($w^{(00)}$, w_2, w_3 and $w(y) = y(1 - y)^2$) whose OPE integrals do not depend on any of the $C_{D>8}$. Also shown for comparison are the corresponding ALEPH fit results (denoted by the light lines) for this same set of weights and same channel, shown previously in Fig. 1. The comparison makes evident the major improvement represented by our fit results. One might argue that the much improved fit quality in the w_2 and w_3 cases is a result of the fact that our parameters were obtained by fitting to the corresponding spectral integrals. The excellent quality of the fit to the $w^{(00)}$- and $y(1 - y)^2$-weighted spectral integrals, however, is a strong test of the implicit assumption that the form assumed on the OPE side of our FESRs in fact correctly incorporates all relevant OPE contributions, an assumption already shown
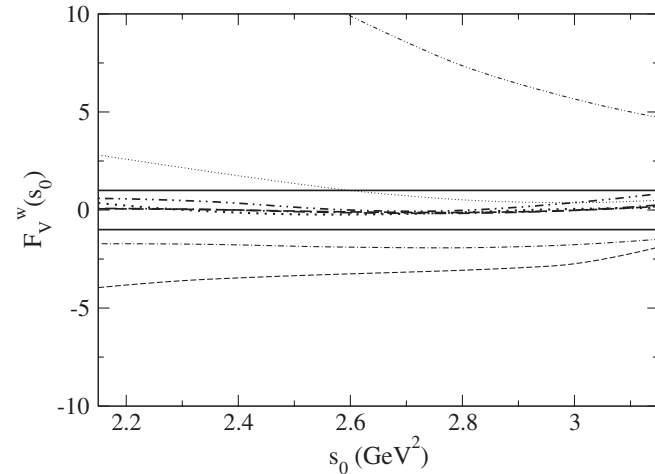

FIG. 2. Comparison of the fit qualities corresponding to (i) our fits and (ii) the 2005 ALEPH fit, as a function of s_0, for the V channel and the weights $w^{(00)}$, w_2, w_3 and $w(y) = y(1 - y)^2$. The light (heavy) dotted line corresponds to the ALEPH fit (our fit) for the weight $w^{(00)}$, the light (heavy) dashed line to the ALEPH fit (our fit) for the weight w_2, the light (heavy) dot-dashed line to the ALEPH fit (our fit) for the weight w_3, and the light (heavy) double-dot-dashed line to the ALEPH fit (our fit) for the weight $y(1 - y)^2$. The right boundary corresponds to the kinematic endpoint, $s_0 = m_\tau^2 \simeq 3.16$ GeV^2.

to fail for the more restrictive forms assumed in the earlier combined spectral weight analyses. We remind the reader that the suppression of the $D = 0$ contribution for the $w(y) = y(1 - y)^2$ case makes the agreement in that case an even more significant test of the reliability of the C_6 and C_8 values obtained using the w_2 and w_3 FESRs.

The situation in the $V + A$ channel, which is the source of our central α_s determination, is similar to that found in the V channel. Specifically, we find

(i) $|F_{V+A}^{w_N}(s_0)|$ for our optimized fits even smaller than those found in the V channel (and hence also uniformly <1 over the whole of the fit window employed);

(ii) $|F_{V+A}^{w_{(km)}}(s_0)|$ corresponding to the 2005 ALEPH optimized fit typically $\gg 1$ away from $s_0 = m_\tau^2$;

(iii) $|F_{V+A}^w(s_0)|$ results produced by applying the optimized 2005 ALEPH values for the $D \le 8$ OPE fit parameters to the degree ≤ 3 weights $w_2(y)$, $w_3(y)$ and $y(1 - y)^2$ not employed in the ALEPH fit significantly >1, even for $s_0 = m_\tau^2$; and

(iv) in contrast, $|F_{V+A}^w(s_0)|$ results produced by applying our optimized $D \le 8$ OPE fit parameters to the degree ≤ 3 weights $w_{(00)}$ and $y(1 - y)^2$ not employed in our fits uniformly <1 through the region of the fit window employed.

In view of the similarity to the V channel results, we do not provide explicit analogues of Figs. 1 and 2 for the $V + A$ channel.

3. $D = 0$ convergence

The next point for discussion is the pattern of convergence of the results for α_s with increasing truncation order. This is relevant to the question of the extent to which our estimate for the $D = 0$ truncation uncertainty is a conservative one. In Table V, we display the results for $\alpha_s(m_\tau^2)$ obtained from full fits to the ALEPH-based $V + A$ w_N FESRs as a function of the truncation order, M, in α_s, employed for the $D = 0$ series. The extremely good consistency (to within ± 0.0001 across the set of w_N employed) allows us to quote a single common value for each truncation order. The behavior of the extracted values of $\alpha_s(m_\tau^2)$ with increasing M appears reasonable and, we would claim, supports the interpretation of our truncation uncertainty estimate of ± 0.010 on $\alpha_s(m_\tau^2)$ as a sensibly

TABLE V. The fitted values for $\alpha_s(m_\tau^2)$ obtained from the ALEPH-based w_N-weighted $V + A$ analyses as a function of the $D = 0$ truncation order, M, where M here specifies that the last term kept in the $D = 0$ series for the Adler function is that proportional to $d_M[\alpha_s(Q^2)]^M$. Our central analyses above correspond to $M = 5$.

M	2	3	4	5
$\alpha_s(m_\tau^2)$	0.375	0.338	0.326	0.320

conservative one. For comparison, the scheme for estimating the truncation uncertainty employed in Ref. [5] produces the less conservative assessment $^{+0.0062}_{-0.0074}$.

4. Comparisons to other determinations of the $D > 4$ parameters, C_D

We turn now to the issue of the extracted values of the $D > 4$ condensate combinations, making comparisons to other determinations of these same combinations appearing in the literature. The analysis above is, of course, designed specifically to reduce $D > 4$ OPE contributions and, as such, is far from optimal for the determination of the C_D. As a result, the precision in our determinations of most of the C_D is not high. In Table VI we compare our results (with the experimental and theoretical errors now combined in quadrature) with those of ALEPH, OPAL and two other recent condensate studies [52,53], focussing on the quantities $C_{6,8}$ obtained in those earlier studies. In the ALEPH and OPAL cases, the errors shown are the nominal ones quoted in the original publications, and do not include the sizeable additional uncertainty associated with the neglect of $D > 8$ contributions discussed already above. In the case of Ref. [52], which employs fits using the weights $w(y) = 1 - y^N$ (which have a zero of order 1 at $y = 1$), we quote only the values considered reliable by the authors themselves, and of these, only the ones corresponding to $\Lambda = 350$ MeV, since it is this value which lies closest to that (346 MeV) associated with our central fit result above. In the case of Ref. [53] we quote only the A channel C_6 result, since this was the only one to display demonstrable stability, within errors, in going from the 2-parameter fit (including contributions up to $D = 6$) to the 3-parameter fit (including contributions up to $D = 8$) [54].

We note that, for the V channel, where the ALEPH fit quality was better, our C_8 values actually agree well with those of ALEPH and OPAL, while our C_6 central values are somewhat larger, but of the same general size. For the A channel, where the ALEPH fit quality was poorer, we have, instead, significant disagreement for C_6, not just in magnitude, but also in the sign of the central value. The significant differences for the A channel are also seen in the $V + A$ channel, as one would expect. Since our values lead to extremely good OPE representations for the $w^{(00)}$, w_2, w_3 and $w(y) = y(1 - y)^2$ spectral integrals in all three

channels, while the ALEPH and OPAL fits do not, it is no surprise that significant differences between our fits and theirs should be found. We note that the disagreement in sign for C_6^A confirms the result found in Refs. [52,53]. As pointed out in those references, the fit results imply a significant breakdown of the vacuum saturation approximation (VSA) for the four-quark $D = 6$ condensates, since VSA values for the V and A channel are in the ratio $-7{:}11$. While it is true that, given the size of the errors, the sign of C_6^A is not firmly established by either our fits or those of Refs. [52,53], nonetheless the relative magnitudes of the V and A results are far from satisfying the VSA relation. To improve on the accuracy of the determinations of the C_D, and investigate such issues further, would require working with a different set of weight functions, chosen in such a way as to suppress $D = 0$ and emphasize higher D contributions.

B. Final results

In order to avoid the additional uncertainties associated with the separation of the observed $V + A$ spectral distribution into its V and A components, we base our final results for α_s on the $V + A$ w_N FESR analyses. As seen above, the agreement of the ALEPH- and OPAL-based $V + A$ results is excellent. The individual ALEPH V and A fits are, in addition, in extremely good agreement with the corresponding $V + A$ results, though, of course, with larger experimental errors. The agreement of the ALEPH V, A and $V + A$ central values is considerably closer than that obtained from the spectral weight analysis of Ref. [5]. It should be stressed that the agreement in the present case is obtained using the value of $\langle aG^2 \rangle_{\mathrm{RGI}}$ determined independently in Ref. [28], in sharp contrast to the A and $V + A$ fits of Ref. [5], which require incompatible, and unambiguously negative, values.

Averaging the $V + A$ results, using the non-normalization component of the experimental errors, we obtain

$$\alpha_s(m_\tau^2) = 0.3209(46)(118) \tag{20}$$

where the first error is experimental (now including the normalization uncertainty) and the second theoretical. The experimental error is identical to that obtained in the spectral weight analysis of Ref. [5], while our theoretical

TABLE VI. Comparison of our results for C_6 and C_8 with those of Refs. [5] (ALEPH), [8] (OPAL), [52] (DS) and [53] (AAS). C_6 is given in units of 10^{-3} GeV^6 and C_8 in units of 10^{-3} GeV^8. The errors quoted are as described in the text.

Reference	C_6^V	C_8^V	C_6^A	C_8^A	C_6^{V+A}	C_8^{V+A}
ALEPH	$-3.6(3)$	$5.0(3)$	$4.6(3)$	$-6.0(3)$	$1.0(5)$	$-1.0(5)$
OPAL	$-3.4(5)$	$5.0(8)$	$2.6(5)$	$-2.6(1.3)$	$-0.3(1.5)$	$1.3(4.2)$
DS	$-8.9(3.0)$	$\cdots$	$-4.3(3.0)$	$\cdots$	$\cdots$	$\cdots$
AAS	$\cdots$	$\cdots$	$-2.4(2.0)$	$\cdots$	$\cdots$	$\cdots$
Our fit	$-5.9(2.0)$	$6.0(7.0)$	$-2.3(2.0)$	$18.1(7.6)$	$-8.4(3.8)$	$25.1(13.2)$

error is larger as a result of the more conservative treatment of the $D = 0$ truncation uncertainty. The theoretical error of the earlier analyses, of course, does not include the additional contribution identified above, associated with the neglect of $D > 8$ OPE contributions.

The $n_f = 5$ result, $\alpha_s(M_Z^2)$, is obtained from the $n_f = 3$ result given in Eq. (20) using the standard self-consistent combination of 4-loop running with 3-loop matching at the flavor thresholds [55]. As shown in Ref. [4], taking $m_c(m_c) = 1.286(13)$ GeV and $m_b(m_b) = 4.164(25)$ GeV [56], the matching thresholds to be $rm_{c,b}(m_{c,b})$ with r varying between 0.7 and 3, and incorporating uncertainties associated with the truncated running and matching, produces a combined evolution uncertainty of 0.0003 on $\alpha_s(M_Z^2)$. Our final result is then

$$\alpha_s(M_Z^2) = 0.1187(3)(6)(15) \qquad (21)$$

where the first uncertainty is due to evolution, the second is experimental and the third theoretical. The difference between this value and that obtained in the earlier spectral weight analysis, 0.1212(11), serves to quantify the impact of the $D > 8$ contributions neglected in the previous analysis.

The result, Eq. (21), is in good agreement with a number of recent independent experimental determinations, specifically,

  (i) the 2008 updates of the global fit to electroweak observables at the Z scale, quoted in Refs. [4,5], which yield $\alpha_s(M_Z^2) = 0.1190(26)$ and $0.1191(27)_{exp}(1)_{th}$, respectively;

  (ii) the combined NLO fit to the inclusive jet cross sections measured by H1 and ZEUS [57], which yields $\alpha_s(M_Z^2) = 0.1198(19)_{exp}(26)_{th}$;

  (iii) the NLO fit to high-Q^2 1-, 2- and 3-jet cross sections measured by H1 (presented at DIS 2008 and the 2008 HERA-LHC workshop [58]) which yields $\alpha_s(M_Z^2) = 0.1182(8)_{exp}(^{+41}_{-31})_{scales}(18)_{pdf}$;

  (iv) the NNLO fit to event shape observables in $e^+e^- \rightarrow$ hadrons at LEP [59], which yields $\alpha_s(M_Z^2) = 0.1240(33)$;

  (v) the SCET analysis, including resummation of next-to-next-to-next-to leading logarithms, of ALEPH and OPAL thrust distributions in $e^+e^- \rightarrow$ hadrons [60], which yields $\alpha_s(M_Z^2) = 0.1172(13)_{exp}(17)_{th}$; and

  (vi) the fit to $e^+e^- \rightarrow$ hadrons cross sections between 2 GeV and 10.6 GeV CM energy [61], which yields $\alpha_s(M_Z^2) = 0.119(^{+9}_{-11})$.

The agreement with the recent updated analysis of $\Gamma[\Upsilon(1s) \rightarrow \gamma X]/\Gamma[\Upsilon(1s) \rightarrow X]$ [62], which replaces the older analysis usually cited in the PDG QCD review section, and yields $\alpha_s(M_Z^2) = 0.119(^{+6}_{-5})$, is also good. Note that the τ decay extraction is considerably more precise than any of the other experimental determinations. In addition, the τ decay and lattice results, whose discrepancy

was noted at the outset, are now seen to be compatible within errors. This compatibility is, in fact, further improved by the increase in $\alpha_s(M_Z)$ found in two recent studies [63,64] which revisit the earlier lattice determination, incorporating lattice data at a wider range of scales than that employed in Ref. [2].

C. Some comments on the recent Beneke-Jamin study and its relation to the present work

After the completion of the work described in this paper, a new exploration of the extraction of α_s from hadronic τ decay data was posted [65]. This study employs a 5-parameter model for the Borel transform of the $D = 0$ component of the Adler function, one whose structure incorporates the form of the known leading UV renormalon and two leading IR renormalon singularities. The parameters of the model are fixed using the known coefficients, $d_{,}^{(0)} \cdots, d_4^{(0)}$, of the $D = 0$ Adler function series expansion, together with the estimated value $d_5^{(0)} = 283$. The study makes the working assumption that the true all-orders result will be well approximated by the Borel sum of the corresponding model Adler function series. The results generated using the model are then argued to favor the use of FOPT over CIPT for the $D = 0$ OPE contribution. It is not clear to us whether extended ansatze for the Borel transform, involving additional parameters, would lead to the same or different conclusions. We do comment, however, that the results for $\alpha_s(m_\tau^2)$ obtained from our FOPT fits, though yielding representations of the spectral integral data which are of nearly as good quality as those produced by the corresponding CIPT fits, are significantly less consistent than those obtained using the CIPT prescription, the results for the $V + A$ channel ranging from 0.320 for w_2 to 0.312 for w_6. Whether one views this as an empirical argument in favor of softening the conclusions of Ref. [65] or not, the arguments of that reference clearly support taking a conservative approach to assessing the $D = 0$ truncation uncertainty.

For readers inclined to adopt the FOPT determination as the central one (in spite of the reduced consistency of its output), we comment that the $\alpha_s(m_\tau^2)$ obtained from the w_2 through w_6 $V + A$ fits correspond to values of $\alpha_s(M_Z^2)$ lying between 0.1186 and 0.1176. The CIPT result, as it turns out, not only displays better consistency, but is also in better agreement with the results reported in Refs. [63,64], which update the original lattice analysis of Ref. [2].

Regarding the values for $\alpha_s(m_\tau^2)$ and $\alpha_s(M_Z^2)$ quoted in Ref. [65], the reader should bear in mind that these result from a $w^{(00)}$-weighted $V + A$ FESR analysis restricted to the single value $s_0 = m_\tau^2$. With only a single s_0, it is not possible to fit C_6^{V+A} and C_8^{V+A}, and central values (and errors) must therefore be assumed for these quantities. The authors of Ref. [65] take the central value for C_6^{V+A} to be given by twice the VSA result and that for C_8^{V+A} to be 0.

Our fifth order FOPT fits in fact return significantly different values.

It is possible to test the consistency of the assumed values for C_6^{V+A} and C_8^{V+A} with the resulting extracted value of $\alpha_s(m_\tau^2)$, as above, by studying the s_0-dependence of the match between the OPE and spectral integral sides of the w_2, w_3, $w^{(00)}$ and $w(y) = y(1-y)^2$ FESRs, whose OPE sides do not depend on any of the $C_{D>8}$. The reader, here, should bear in mind that, in Ref. [65], slightly different values of $d_5^{(0)}$ and $\langle aG^2 \rangle_{\rm RGI}$ were employed than those used above. Using the $d_5^{(0)}$, $\langle aG^2 \rangle_{\rm RGI}$, C_6^{V+A} and C_8^{V+A} values of Ref. [65], together with the resulting $O(\bar{a}^5)$-truncated FOPT fit value for $\alpha_s(m_\tau^2)$, we find the fit qualities, $F_{V+A}^w(s_0)$, displayed in Fig. 3. $F_{V+A}^{w^{(00)}}(s_0)$ is, of course, small near $s_0 = m_\tau^2$ since the value of $\alpha_s(m_\tau^2)$ employed in the calculations was fixed using the $s_0 = m_\tau^2$ version of the $w^{(00)}$ FESR. The deterioration in the fit quality for $w^{(00)}$ as s_0 is decreased, as well as the very poor fit qualities for the other three weights, clearly demonstrates that the values assumed for C_6^{V+A} and C_8^{V+A} are problematic. The value obtained for $\alpha_s(m_\tau^2)$ using these values as input should thus also be treated with caution. We have already noted the results of our own FOPT fits above. Since the $\alpha_s(m_\tau^2)$ values obtained from the w_2 and w_3 FESRs do not show the same degree of consistency as was observed in the CIPT-based fit, it would be necessary to perform a combined fit, using a number of the degree ≤ 3 weights, to improve further on the FOPT determination.
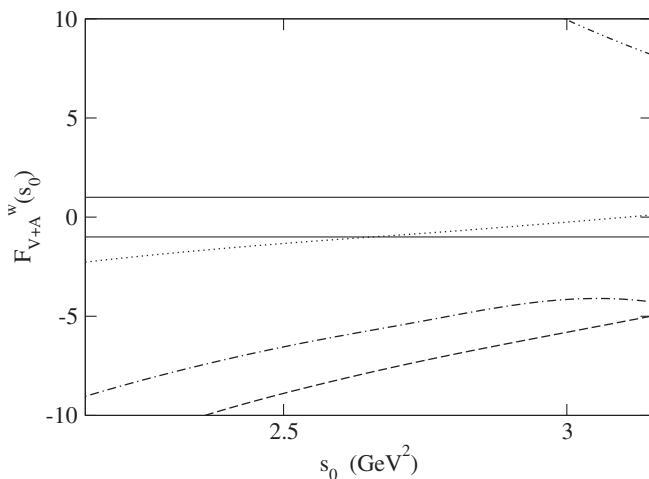

FIG. 3. The fit qualities $F_{V+A}^w(s_0)$ corresponding to the ALEPH data, the OPE parameters of Ref. [65], and the FOPT evaluation of the $D = 0$ OPE contributions, for the $w^{(00)}$, w_2, w_3 and $w(y) = y(1-y)^2$ FESRs. The dotted, dashed, dot-dashed and double-dot-dashed lines correspond to $w^{(00)}$, w_2, w_3 and $y(1 - y)^2$, respectively. The right boundary corresponds to the kinematic endpoint, $s_0 = m_\tau^2 \simeq 3.16$ GeV^2.

D. Final summary and comments

To summarize, we have performed a number of related FESR analyses designed specifically to reduce the impact of poorly known $D > 4$ OPE contributions on the extraction of α_s using hadronic τ decay data. Our results show a high degree of consistency and satisfy constraints not satisfied by other τ decay determinations. Our final result is

$$\alpha_s(M_Z^2) = 0.1187 \pm 0.0016 \qquad (22)$$

where the evolution, experimental and theoretical errors have now been combined in quadrature. The result is in excellent agreement with (and more precise than) alternate independent high-scale experimental determinations. It is, however, significantly lower than the values obtained in the earlier ALEPH and OPAL hadronic τ decay analyses. We have provided clear evidence that the source of this discrepancy lies in the contamination of these earlier combined spectral weight analyses by neglected, but non-negligible, $D > 8$ OPE contributions.

A technical point worth emphasizing from the discussion above is the importance of working with a range of s_0 rather than just the single value $s_0 = m_\tau^2$, and the utility, in this context, of using weights defined in terms of the dimensionless variable $y = s/s_0$. For such weights, the s_0-dependence of the resulting weighted spectral integrals allows one to straightforwardly test any assumptions made about the values of $D > 4$ OPE coefficients, or, better yet, to attempt actual fits to obtain these values using data. Such s_0-dependence studies seem to us unavoidable if one wishes to demonstrate that $D > 4$ OPE contributions have indeed been brought under control at the level ($\sim 0.5\%$ of the full spectral integrals) required for a $\sim 1\%$ precision determination of $\alpha_s(M_Z^2)$. Fortunately, as we have shown, such control is not difficult to achieve, and we have displayed a number of weights which are useful for this purpose. The weights, $w_N(y)$, which isolate individual integrated $D = 2N + 2$ contributions, are related to the kinematic weight, $w^{(00)}(y)$, by slowly varying multiplicative factors [66], and hence produce errors on the spectral integrals that are comparable to, or better than, those for $w^{(00)}$.

We stress that theoretical errors now dominate the uncertainty in the hadronic τ decay determination of $\alpha_s(M_Z^2)$, the $D = 0$ OPE truncation error being the largest among these. Further reduction in experimental errors, and, in particular, improvements in the V/A separation, are likely to be possible using data from the B factories, and such improvements would be useful for further testing the consistency of the V, A and $V + A$ determinations. Given the current situation, however, reduced experimental errors would have little impact on the total error on $\alpha_s(M_Z^2)$.

ACKNOWLEDGMENTS

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[26] Note that higher order corrections to the leading order $D > 4$ OPE contributions to the correlator begin with terms of the form $C_D^{(1)}\alpha_s(Q^2)/Q^D$. These can be reexpressed by expanding $\alpha_s(Q^2)$ as a power series, e.g., in $\alpha_s^0 \equiv \alpha_s(m_\tau^2)$: $\alpha_s(Q^2) = \alpha_s^0 + \sum_{k=2} c_k(t)[\alpha_s^0]^k$, where the $c_k(t)$ are polynomials in $t = \log(Q^2/m_\tau^2)$ whose coefficients are determined by those of the β function. The first order term in the resulting expansion of the α_s-suppressed correlator contribution, $C_D^{(1)}\alpha_s^0/Q^D$, can thus be absorbed into the leading order term C_D/Q^D through a redefinition of the coefficient C_D. This part of the higher order $D > 4$ contribution is thus automatically included whenever C_D is fitted to data. Higher order integrated $D > 4$ "logarithmic corrections" incorporated only in an average sense as part of the fitting procedure thus begin with a term proportional to $[\alpha_s^0]^2 \log(s_0/m_\tau^2)$.

[27] It should be noted that the analysis of Ref. [4], which employs the $w^{(00)}$ spectral weight, uses the 2005 ALEPH fits for the coefficients $\delta^{(6)}$ and $\delta^{(8)}$ as input. Problems with these fit values thus also impact the reliability of that analysis.

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logarithmic integrals, combined with the alternation in signs of the polynomial coefficients associated with the pinching condition $w(y = 1) = 0$. This strong suppression of $D \neq 2N + 2$ contributions is of course also shared by the weights $w(y) = 1 - y^N$, a feature of these weights emphasized in Ref. [52].

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