Copulas for Credit Derivative Pricing and other Applications

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Signed Statement

Declaration

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Publications Arising from this Study

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Crane G.J. and van der Hoek, J., (2008) Using distortions of copulas to price synthetic CDOs. Insurance: Mathematics and Economics **42**, 903-908.

Abstract

Copulas are multivariate probability distributions, as well as functions which link marginal distributions to their joint distribution. These functions have been used extensively in finance and more recently in other disciplines, for example hydrology and genetics. This study has two components, (a) the development of copula-based mathematical tools for use in all industries, and (b) the application of distorted copulas in structured finance. In the first part of this study, copula-based conditional expectation formulae are described and are applied to small data sets from medicine and hydrology. In the second part of this study we develop a method of improving the estimation of default risk in the context of collateralized debt obligations. Credit risk is a particularly important application of copulas, and given the current global financial crisis, there is great motivation to improve the way these functions are applied. We compose distortion functions with copula functions in order to obtain greater flexibility and accuracy in existing pricing algorithms. We also describe an n-dimensional dynamic copula, which takes into account temporal and spatial changes.

Introduction and Overview

Copulas are functions which join multivariate distributions to their marginal distributions. The ability to compose any copula with any choice of margin, means we are able to create a great variety of distributions. Copula functions may also be viewed as multivariate distributions with uniform marginal distributions. The nature of copula functions allows us move away from traditional dependence measures such as linear correlation and toward more general measures of dependence in the form of one or more copula parameters.

While many families of bivariate copulas have been proposed and their properties have been described, progress has been slower in relation to the development of copula-based mathematical tools and the construction of higher dimensional copulas. The reason for the slow progress is that expressions representing higher dimensional copulas and their density functions are often quite lengthy and nontrivial. Generalizations from two dimensions to higher dimensions are sometimes not possible and implementation may require considerable computational effort.

The main aim of this thesis is to create new multidimensional copula-based mathematical tools for use in finance and other areas such as genetics and hydrology. We start by reviewing the theory and application of copula functions in Chapter 1. Given that practitioners are often required to carry out calculations repeatedly, and at the same time achieve fast and accurate results, the copula-based regression formulae of Chapter 2 are proposed. These regression formulae are designed to be simple to implement, and enable the users to make relatively accurate predictions of the conditional mean of a random variable of interest.

The new models in Chapter 2 allow us to move away from traditional expectation formulae, which are models of linear regression, and toward a more general form of regression. More specifically, suppose we have two random variables X and Y. If we assume that X and Y are jointly normal, then the conditional expectation of Y, given X, is

$$\mathbb{E}[Y \mid X] = a + bX,$$

where the constants $a = \mu_y - b\mu_x$ and $b = \rho\sigma_y/\sigma_x$ are such that $\mathbb{E}[Y - a - bX]^2$ is minimized. In the new models, the expectation formula is represented by a copula-based function h. The conditional expectation of Y, given X, in this case is

$$\mathbb{E}[Y \mid X] = h(X),$$

and X and Y do not have to be normal or jointly normal. The new expectation formulae are applied to simulated data, interest rate data, hydrology data and body measurement data.

During the last two decades there has been an expansion in the credit derivatives market, along with a rapid rise in securitization by large financial institutions. Given such changes, one of the challenges facing the finance industry now is the pricing of the latest products in the market. Collateralized debt obligations (CDOs) are an example of a class of financial products which are used in the United States of America (USA) and Europe. Like any other credit derivative, a CDO requires an efficient and accurate method for its pricing. The current market standard for pricing CDOs involves a function called the *Gaussian Copula*.

The Gaussian copula does not have a heavy tail and so its use results in the underpricing of risk. The use of one factor copula models has contributed to the current subprime mortgage crisis in the USA, since investors overestimated their return on CDO deals and underestimated their risk in the current global circumstances. This has motivated researchers including myself to investigate alternative methods and modifications which may overcome the problems arising from the use of the Gaussian copula. The main aim of Chapter 3 is to improve on the copula-based tools for the pricing synthetic CDOs. In particular, two new static models for pricing synthetic CDOs are proposed. These models have four new characteristics,

1. they involve the use of multiplicative generator functions and well known copula functions,
2. they are designed to be less complicated than the present system in that only one dependence parameter is used for the entire structure,
3. the models are easy to implement, and
4. the combination of functions allows us to approximate CDO default distributions with greater accuracy.

New developments in the theory of copula functions are still emerging, although many publications focus only on their application. Few attempts have been made to generate new constructions and new examples of copulas. Thus, we focus on the construction of n-copulas in Chapter 4. Three mathematical methods are combined to form new copulas,

1. the inclusion of distortion functions,
2. the mixing of particular classes of functions, and
3. the use of integral equations involving lower order copulas.

Previously, these methods were applied to copulas separately in order to obtain higher order or new copulas. Many of the examples described in Chapter 4 are novel in that they have been created by a combination of the three methods.

Copulas may be

1. static, so that one can only obtain a snapshot of the dependence between variables of interest,
2. varying with respect to time, or
3. vary with respect to time and space.

Most research papers in finance focus on static copulas or use discrete time rather than continuous time, when modelling dynamic copulas. An option transaction and other financial derivatives may be modelled using copulas which vary in time and space. There are, however, very few descriptions of such dynamic copulas. Thus, the motivation for the work on time and space varying copulas in Chapter 5. Our method makes use of stochastic partial differential equations. In particular, we are interested in modelling the dependence between n Markov diffusions, using a copula-based approach. In order to achieve this goal, we require the forward Kolmogorov equations and an n-copula. The result is a dynamic n-copula which describes the evolution of dependence between the n diffusions. The dynamic n-copulas may be applied to portfolios of risky assets in order to obtain a measure of time-varying aggregated risk.

Ideas for future research in this area are discussed in Chapter 6.

Chapter 1

Literature Review

This chapter consists of

1. a general overview of copula functions and their applications and
2. an overview of the role of copula functions in the pricing of Synthetic CDOs.

1.1 Copula Functions and Their Applications

Although Gumbel and Fréchet were working on very specific examples of copulas, the concept of a general copula function first appeared in 1959 as a solution to the problem of constructing multivariate probability distributions with given marginal distributions [154]. It was not until the 1980s and 1990s that books on the construction of copulas were published, for example [149], [115], [83] and [78]. The purpose of this section is to review recent theoretical and applied research in this area, and in particular to discuss some of the more novel families of copulas.

Before providing a definition of the copula functions, we discuss the reasons why one might use copulas in conjunction with other statistical techniques.

(a) *An alternative to linear correlation between random variables.* Suppose we want to capture the dependence between two random variables. Linear correlation is a sufficient method for describing the association between pairs of normally distributed random variables, otherwise it is not appropriate. Moreover, linear correlation is only a measure of the overall strength of the association between the variables, not a measure of changes across the distribution [86]. Thus, if two distributions are strongly correlated at one extreme, but not otherwise, one would not be able to capture that information if we were using a linear correlation coefficient. Another limitation of linear correlation is that it is not invariant under transformations of the underlying distributions. Copulas and their dependence parameters overcome both of the limitations just described. A more extensive discussion on the properties of dependence parameters and the advantage of using copulas is given in [28]. Copula dependence parameters can also be expressed in terms of population versions of Kendall's tau and Spearman's rho, see [116], which are useful when fitting any particular copula to data.

(b) *A complete representation of multivariate distributions.* Another advantage of the copula functions is that they provide us with a complete representation of all multivariate distributions. The reason we can obtain a complete representation is because we can separate out the marginal distributions from the overall dependence structure using Sklar's Theorem. Sklar's Theorem (see next section) enables us to construct a joint probability distribution by housing given marginal distributions within a copula function. A wide range of copulas and marginal distributions can be defined, and so it is possible to obtain many kinds of multivariate distributions. Sklar's theorem also says that a copula function exists for every multivariate distribution. Thus, we can be sure that we will be able to find a copula, if we are given the multivariate and marginal distributions.

This method for constructing joint distributions is very appealing since one often knows much about the marginal distributions, but not very much about their relationship. Deciding how to choose a suitable copula for a given set of data is the topic of ongoing research. Some copulas will naturally be more suitable for data correlated at one or both of the extremes than others, however, the choice of copula is still somewhat arbitrary. Moreover, researchers are still in the process of developing goodness of fit tests for these functions.

(c) *Copulas may be used in established frameworks.* As will be shown later in this thesis, copulas may be combined with well established mathematical frameworks, since they are distributions in their own right. Conditional distributions can also be expressed in terms of partial derivatives of copulas, so they can be used in Bayesian analysis, sampling algorithms, Markov chain models and other computationally intensive techniques.

1.2 Definition and Basic Properties

1.2.1 Copula Functions

Definition 1. *2-copula.* A function $C : [0,1]^2 \to [0,1]$ is a 2-copula if it satisfies the following properties,

1. $C(u_1, 0) = 0$, $C(0, u_2) = 0, \quad u_1, u_2 \in [0,1]$,
2. $C(u_1, 1) = u_1$, $C(1, u_2) = u_2, \quad u_1, u_2 \in [0,1]$, and
3. For every u_a, u_b, v_a, $v_b \in$ [0,1], such that $u_a \leq u_b$, $v_a \leq v_b$, the volume of C, $V_C([u_a, u_b] \times [v_a, v_b]) \geq 0$, that is

$$C(u_b, v_b) - C(u_b, v_a) - C(u_a, v_b) + C(u_a, v_a) \geq 0.$$

Equivalently, a copula is the restriction on $[0,1]^2$ of a bivariate distribution function with standard uniform marginal distributions. Condition 3 above is usually referred to as the "2-increasing" condition. A theorem by Sklar in 1959 [154] shows how a copula function is able to link a joint distribution function to its marginal distributions.

Sklar's Theorem. Suppose H is a bivariate joint distribution with marginal distributions F and G, then there exists a 2-copula C, such that

$$H(x,y) = C(F(x), G(y)), \quad \text{for all} \quad x, y \in \overline{\mathbb{R}}. \tag{1.2.1}$$

If F and G are continuous distributions then C is unique, otherwise C is uniquely determined on $\text{Ran}F \times \text{Ran}G$ [115]. If a copula C which absolutely continuous, then it has density function

$$c(u_1, u_2) = \nabla_{u_1,u_2} C(u_1, u_2), \tag{1.2.2}$$

where

$$\nabla_{x,y} C(x,y) = \frac{\partial^2 C(x,y)}{\partial x \partial y},$$

that is the mixed partial derivative of C. Moreover

$$C(u_1, u_2) = \int_0^{u_1} \int_0^{u_2} c(s,t) ds dt.$$

It follows that if the density function of H is h, and F and G have associated density functions f and g, then

$$h(x,y) = c(F(x), G(y)) f(x) g(y). \tag{1.2.3}$$

The Fréchet-Hoeffding bounds for a copula C are

$$W(u_1, u_2) \leq C(u_1, u_2) \leq M(u_1, u_2).$$

where

$$\begin{aligned} W(u_1, u_2) &= \max\{u_1 + u_2 - 1, 0\} \\ M(u_1, u_2) &= \min\{u_1, u_2\}. \end{aligned}$$

$W(u_1, u_2)$ and $M(u_1, u_2)$ are also 2-copulas.

1.2.2 Quasi-Copula Functions

Definition 2. *2-Quasi-copula.* A function $Q : [0,1]^2 \to [0,1]$ is a 2-quasi-copula if it satisfies the following properties,

1. $Q(u_1, 0) = 0$, $Q(0, u_2) = 0$ for all $u_1, u_2 \in [0,1]$,
2. $Q(1, u_2) = u_2$, $Q(u_1, 1) = u_1$ for all $u_1, u_2 \in [0,1]$, and
3. Q is non-decreasing in each of its arguments, and

$$\mid Q(u_b, v_b) - Q(u_a, v_a) \mid \leq \mid u_b - u_a \mid + \mid v_b - v_a \mid, \quad \text{for all} \quad u_a, u_b, v_a, v_b \in [0,1].$$

A quasi-copula is a generalization of a copula and may not satisfy the 2-increasing condition, [3],[56]. This class of functions has potential application in the area of fuzzy logic [68]. An equivalent set of conditions to those in Definition 2 is;

1. $Q(u_1, 0) = 0, Q(0, u_2) = 0,$ for all $u_1, u_2 \in [0,1]$,
2. $Q(1, u_2) = u_2, Q(u_1, 1) = u_1$ for all $u_1, u_2 \in [0,1]$, and
3. Q satisfies $Q(u_b, v_b) + Q(u_a, v_a) \geq Q(u_b, v_a) + Q(u_a, v_b)$, for all $0 \leq u_a \leq u_b \leq 1$, $0 \leq v_a \leq v_b \leq 1$.

see [56] and [131].

1.2.3 Tail Dependence Formulae

Tail dependence or extreme measures of dependence in relation to copulas, give an indication of how data pairs are related at the extremes of the distribution. Given random variables U_1 and U_2, the upper λ_U and lower λ_L tail dependence are defined by

$$\begin{aligned} \lambda_U = \lim_{u_1 \to 1} \Pr\{U_1 > u_1 \mid U_2 > u_1\} &= 2 - \lim_{u_1 \to 1} \frac{1 - C(u_1, u_1)}{1 - u_1} \\ &\sim 2 - \lim_{u_1 \to 1} \frac{\ln(C(u_1, u_1))}{\ln(u_1)} \end{aligned}$$

and

$$\lambda_L = \lim_{u_1 \to 0} \Pr\{U_1 < u_1 \mid U_2 < u_1\} \quad = \quad \lim_{u_1 \to 0} \frac{C(u_1, u_1)}{u_1}.$$

If the random variables are independent, then $\lambda_U = 0$. Conversely, if there is perfect dependence between the random variables, then $\lambda_U = 1$. Many copulas have no tail dependence, however it is possible to induce tail dependence by combining copulas in particular ways. Two interesting copulas which have upper tail dependence, $\lambda_U = 2 - 2^{1/\alpha}$, are shown in the Archimedean section below. A good overview of dependence measures is given in [22], and also given in the monographs on copulas mentioned earlier.

1.3 Classes of Copulas

1.3.1 Elliptical Copulas

Gaussian Copula

$$C(u_1, u_2) = \Phi_2(\Phi^{-1}(u_1), \Phi^{-1}(u_2); \rho), \quad \rho \in [-1, 1], \tag{1.3.1}$$

where Φ_2 is the bivariate normal distribution, ρ is the Pearson product-moment correlation coefficient and $\Phi^{-1}(\cdot)$ is the inverse of the standard univariate normal distribution. Although the Gaussian copula is often the first to be chosen when fitting copulas to real data, it may not provide very accurate results, for example in quanto FX pricing, see [8]. A copula transform was applied in [8] in order to provide a better fit to the data. The Student's t copula has more Kurtosis than the Gaussian copula, and so it is often the preferred alternative, see [20], [42] and [19].

Student's t Copula

$$C(u_1, u_2) = T_{\nu,\rho}(t_\nu^{-1}(u_1), t_\nu^{-1}(u_2); \rho), \quad \rho \in [-1, 1], \tag{1.3.2}$$

where $T_{\nu,\rho}$ is the bivariate Student's t distribution, ρ is the correlation coefficient (as in the Gaussian copula), $\nu > 0$ is the number of degrees of freedom, and t_ν^{-1} is the inverse of the univariate Student's t distribution. The upper and lower tail dependence parameters for the Student's t copula are equal in this case, see [44]:

$$\lambda_U = \lambda_L = 2 - 2t_{\nu+1}\left(\sqrt{\nu + 1}\frac{\sqrt{1-\rho}}{\sqrt{1+\rho}}\right).$$

The advantages and disadvantages of using Elliptical copulas are discussed in detail in [45].

1.3.2 Bertino Copulas

Bertino copulas have the form

$$C_\delta(u_1, u_2) = \min\{u_1, u_2\} - \min_{t \in [\{u_1, u_2\}]}\{t - \delta(t)\}, \quad \text{for all} \quad t \in [0, 1], \tag{1.3.3}$$

where $\delta(t)$ is the diagonal section of the copula, that is $\delta_C(t) = C(t, t)$, and $[\{u_1, u_2\}]$ is the closed interval with endpoints u_1 and u_2 [48].

1.3.3 Copulas with Quadratic Section

Farlie-Gumbel-Morgenstern Copula

The Farlie-Gumbel-Morgenstern (**FGM**) copulas are a simple one-parameter family of distributions. Given parameter $\theta \in [-1, 1]$, the bivariate FGM copula is

$$C_\theta(u_1, u_2) = u_1 u_2 + \theta u_1(1 - u_1)u_2(1 - u_2). \tag{1.3.4}$$

The FGM density c_θ, is

$$c_\theta(u_1, u_2) = 1 + \theta(1 - 2u_1)(1 - 2u_2)$$

Figure 1.3.1 shows an example of the FGM density with $\theta = -0.5$.

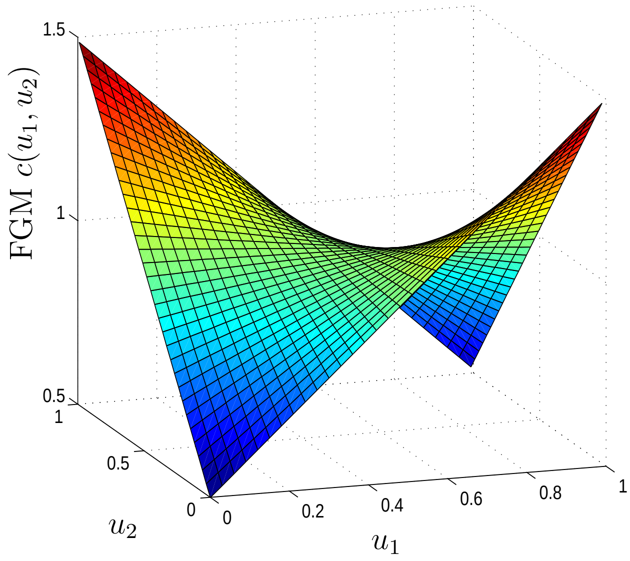

Figure 1.3.1: Density for FGM Copula distribution in (1.3.4)

It is possible to associate Kendall's τ and other scale-free measures of dependence with the FGM dependence parameter θ. In particular, the relationship between τ and θ is $\tau = 2\theta/9$. Given that θ can only take values in $[-1, 1]$, we can only use the relationship above when the dependence between the variables is weak, since that is when we will obtain a relatively small value of τ.

Iterated Farlie-Gumbel-Morgenstern Copula

Authors in [70] describe a method for generalizing the family of FGM copulas and that family is referred to as *iterated FGM copulas*. The first iteration, with parameters α and β, gives us the copula

$$C(u_1, u_2) = u_1 u_2 + \alpha u_1 u_2 (1 - u_1)(1 - u_2) + \beta {u_1}^2 {u_2}^2 (1 - u_1)(1 - u_2), \quad (1.3.5)$$

where

$$|\alpha| \leq 1, \quad \alpha + \beta \geq -1, \quad \text{and} \quad \beta \leq 2^{-1}(3 - \alpha + (9 - 6\alpha - 3\alpha^2)^{1/2}).$$

1.3.4 Marshall-Olkin Copulas

Marshall-Olkin, see [115], have the form

$$C_{\alpha,\beta}(u_1, u_2) = \min\{{u_1}^{1-\alpha}u_2, u_1{u_2}^{1-\beta}\}, \quad \alpha, \beta \in [0, 1]. \tag{1.3.6}$$

1.3.5 Archimedean Copulas

Archimedean 2-copulas have general form

$$C(u_1, u_2) = \varphi^{[-1]}\{\varphi(u_1) + \varphi(u_2)\}. \tag{1.3.7}$$

The function $\varphi : [0, 1] \to [0, \infty]$ is convex and is referred to as the *generator function*. Here, the pseudo-inverse $\varphi^{[-1]} : [0, \infty] \to [0, 1]$ is defined by

$$\varphi^{[-1]}(t) = \begin{cases} \varphi^{-1}(t) & \text{for } 0 \leq t \leq \varphi(0) \\ 0 & \text{for } \varphi(0) \leq t \leq \infty. \end{cases}$$

Gumbel Copula

The Gumbel copula has generator $\varphi(t) = (-\ln(t))^{\alpha}$, $\alpha \in [1, \infty)$, and has the following form

$$C_{\alpha}(u_1, u_2) = \exp\left(-[(-\ln(u_1))^{\alpha} + (-\ln(u_2))^{\alpha}]^{1/\alpha}\right). \tag{1.3.8}$$

The Gumbel copula has more probability mass concentrated in the upper extremes than some of the other copulas, therefore it is thought to be useful in models of severe financial loss, see [86] and [20].

Joe Copula

This example is attributed to Joe [82] and has a heavier right tail than left tail:

$$C(u_1, u_2) = 1 - [(1 - u_1)^{\alpha} + (1 - u_2)^{\alpha} - (1 - u_1)^{\alpha}(1 - u_2)^{\alpha}]^{1/\alpha}, \tag{1.3.9}$$

where $\alpha \in [1, \infty)$. Both, Gumbel and Joe copulas, have upper tail dependence $\lambda_U = 2 - 2^{1/\alpha}$. On the other hand, the Clayton Copula below has a heavy lower tail, so it might be useful for modeling dependence between small losses.

Clayton Copula

The Clayton copula has the form

$$C(u_1, u_2) = ({u_1}^{-1/\theta} + {u_2}^{-1/\theta} - 1)^{-\theta}, \quad \theta > 0. \tag{1.3.10}$$

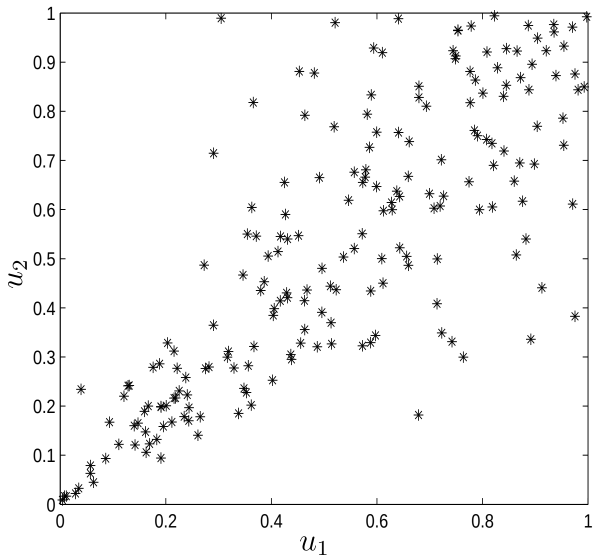

Figure 1.3.2: Sampled points from Clayton Copula

Sampled points from the Clayton copula with $\theta = 4.0$ are shown in Figure 1.3.2.

An extensive list of Archimedean generators and their copulas is given in [115] and S_Plus code for fitting many of these copulas to data is provided in [28]. On the other hand, [161] is a detailed article on sampling from Archimedean copulas. There are further developments in terms of multivariate Archimedean quasi-copulas, [118], simulating from exchangeable Archimedean copulas [163] and generalizations of this class of copulas [33].

1.3.6 Periodic Copulas

Periodic copulas are families of copulas based on Periodic functions, see [2]. Suppose c is the density of a copula C. A density which satisfies the properties of this class of copula, will have the form $c(u_1, u_2) = \tilde{c}(u_1 \pm u_2)$ for a non-negative function $\tilde{c} : \mathbb{R} \to \mathbb{R}$ and it must satisfy

$$\begin{aligned}\int_0^{u_1}\int_0^1 \tilde{c}(x \pm y)dxdy = u_1, \quad \text{for all} \quad u_1 \in [0,1], \\ \int_0^1\int_0^{u_2} \tilde{c}(x \pm y)dxdy = u_2, \quad \text{for all} \quad u_2 \in [0,1].\end{aligned} \tag{1.3.11}$$

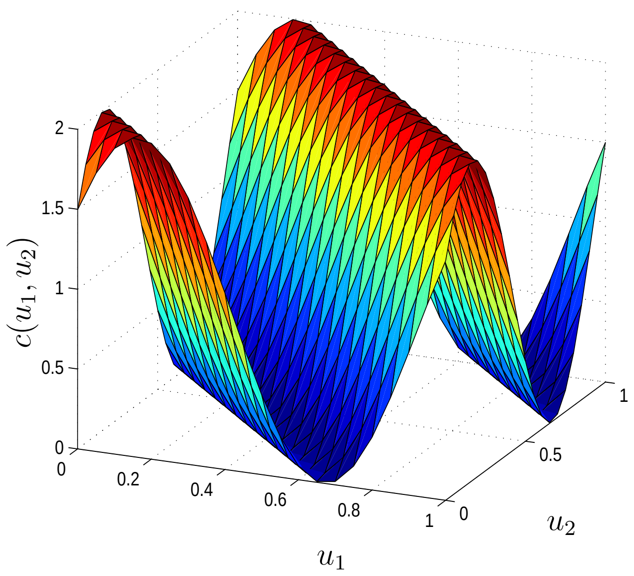

Figure 1.3.3: Density for copula distribution in (1.3.12)

For example, let $\tilde{c}(t) = 1 + \sin(2\pi t + \alpha)$, for $\alpha \in [0, 2\pi)$, then the corresponding copula is

$$\begin{aligned}&C(u_1, u_2) \\ &= (\sin(2\pi u_1 + \alpha) - \sin(\alpha) - \sin(2\pi(u_1 + u_2) + \alpha) + \sin(2\pi u_2 + \alpha))/(2\pi)^2 \\ &+ u_1 u_2\end{aligned} \tag{1.3.12}$$

Figure 1.3.3 shows the density of the Periodic copula described above, with $\alpha = \pi/6$.

1.3.7 Generalized Diagonal Band Copulas

Another special class of copulas is the diagonal band copulas, see [99]. A copula C is a generalized diagonal band copula if it has a density of the form

$$c(u_1, u_2) = \frac{g(\mid u_1 - u_2 \mid) + g(1 - \mid 1 - u_1 - u_2 \mid)}{2}, \tag{1.3.13}$$

where g is the probability density function of a continuous random variable, say Z, on the interval $[0, 1]$. A new class of band copulas is described in [14]. Figure 1.3.4 shows the density function of a diagonal band copula with

$$g(x) = \frac{e}{e-1} e^{-x}.$$

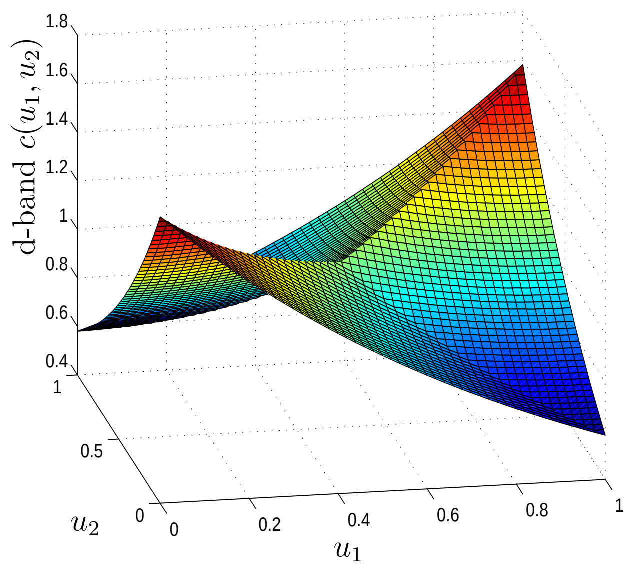

Figure 1.3.4: Density for copula distribution in (1.3.13)

1.3.8 Copulas with Fractal Supports

Copulas with fractal supports are described in [47]. Transformation matrices T are used to describe the partitioning of probability mass on $\mathbf{I}^2 = [0,1] \times [0,1]$, and construct other 2-copulas.

Definition 3. *Transformation matrix for copulas.* A transformation matrix

$$T = \begin{bmatrix} t_{12} & t_{22} \\ t_{11} & t_{21} \end{bmatrix}$$

is defined as a matrix T with nonnegative entries, for which the sum of the entries is 1 and no row or column has every entry zero [47].

The transformation of a copula C is defined as

$$\begin{aligned} T(C)(u,v) &= \sum_{i'<i,j'<j} t_{i'j'} + \frac{u-p_{i-1}}{p_i-p_{i-1}} \sum_{j'<j} t_{ij'} + \frac{v-q_{j-1}}{q_j-q_{j-1}} \sum_{i'<i} t_{i'j} \\ &+ t_{ij} C\left(\frac{u-p_{i-1}}{p_i-p_{i-1}}, \frac{v-q_{j-1}}{q_j-q_{j-1}}\right) \end{aligned}$$

and the volume of $T(C)$ is

$$V_{T(C)}([p_{i-1},u],[q_{j-1},v]) = t_{ij} C\left(\frac{u-p_{i-1}}{p_i-p_{i-1}}, \frac{v-q_{j-1}}{q_j-q_{j-1}}\right).$$

For example, if

$$T = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix},$$

then $T(\min\{u,v\}) = \min\{u,v\}$. A copula C is invariant under this particular transformation.

1.3.9 Empirical Copulas

The Bivariate Empirical Copula

The 2-dimensional Empirical copula is defined as follows; Given data pairs (x_k, y_k), $k = 1, 2, \ldots, n$, from a continuous bivariate distribution, C_E is

$$C_E\left(\frac{i}{n}, \frac{j}{n}\right) = \frac{M \quad \text{such that} \quad x \leq x_i \quad \text{and} \quad y \leq y_j}{n}, \tag{1.3.14}$$

where, M is the number of pairs in the sample, x_i and y_j, $i, j = 1, 2, \ldots, n$ are the order statistics from the sample, see [115]. An algorithm for calculating C_E is shown in [143]. This can be useful when fitting data to a particular copula, since we can find copula parameters such that the least squares fit between C_E and the copula to be fitted to the data is minimized. A theoretical generalization of the Empirical copula is the discrete quasi-copula [131]. Another theoretical development in this area is the use of wavelet expansions and the Empirical copula to estimate other copulas, see [111].

1.3.10 Bernstein Copulas

Given a copula C, its Bernstein approximation of order N is given by

$$B_N(u_1, u_2) = \sum_{i=0}^{N} \sum_{j=0}^{N} C\left(\frac{i}{N}, \frac{j}{N}\right) p_{i,N}(u_1) p_{j,N}(u_2), \tag{1.3.15}$$

where

$$p_{k,N}(x) = \binom{N}{k} x^k (1-x)^{N-k}.$$

Bernstein copulas and their application are described in [137] and [138]. They can be very useful for smoothing data sets.

1.4 Methods of Constructing new 2-copulas

1.4.1 Convex Combinations of Copulas

There are various methods for constructing new 2-copulas from given ones. One method involves constructing a convex combination of two or more known copulas:

$$C(u_1, u_2) = (1-\eta)C_a(u_1, u_2) + \eta C_b(u_1, u_2), \tag{1.4.1}$$

where C_a and C_b are known copulas and $\eta \in [0, 1]$. For example, the two classes of copulas below are constructed in this way.

Positive Linear Spearman Copula

$$C(u_1, u_2) = \begin{cases} \{u_1 + \rho_s(1-u_1)\}u_2, & \text{if } u_2 \leq u_1 \\ \{u_2 + \rho_s(1-u_2)\}u_1, & \text{if } u_2 > u_1, \end{cases} \tag{1.4.2}$$

where $\rho_s \in [0, 1]$ is Spearman's rank correlation coefficient [1].

Mixture Copula

$$\begin{aligned} C_m(u_1, u_2, \theta, \mathbf{w}) &= w_1 C_{Gauss}(u_1, u_2; \rho) + w_2 C_{Gumb}(u_1, u_2; \alpha) \\ &+ (1 - w_1 - w_2)C_{GS}(u_1, u_2; \beta), \end{aligned} \tag{1.4.3}$$

is referred to as a Mixture copula with parameters $\theta = (\rho, \alpha, \beta)$, $\mathbf{w} = (w_1, w_2)^T$ and $u_1, u_2, w_1, w_2 \in [0, 1]$ such that $w_1 + w_2 \leq 1$, see [69]. The copulas involved in the mixture are the bivariate Gaussian copula, the Gumbel Copula and the Gumbel survival copula, C_{GS} with parameter $\beta \in (0, 1]$,

$$C_{GS}(u_1, u_2; \beta) = u_1 + u_2 - 1 + \exp\left\{- \left[(-\log(1-u_1))^{1/\beta} + (-\log(1-u_2))^{1/\beta}\right]^{\beta}\right\}. \tag{1.4.4}$$

Authors in [136] used a Bayesian approach to obtain new Mixture copulas, instead of the method above.

1.4.2 Bivariate Iterated Copulas

The bivariate iterated copulas in [90] provide another approach to building new 2-copulas. A new copula C can be constructed from a known copula $C^*(u_1, u_2)$ and univariate functions f and g as follows

$$C(u_1, u_2) = C^*(u_1, u_2) + f(u_1)g(u_2), \quad (u_1, u_2) \in [0, 1] \times [0, 1]. \qquad (1.4.5)$$

Functions f and g are given and have the following properties

1. f, g are absolutely continuous and $f, g \not\equiv 0$,

2. $f(0) = f(1) = g(0) = g(1) = 0$ and

3. $\min\{\alpha\delta, \beta\gamma\} \geq -\frac{\Delta^*}{(u_b - u_a)(v_a - v_b)}$, where

   $\Delta^* = C^*(u_b, v_b) - C^*(u_b, v_a) - C^*(u_a, v_b) + C^*(u_a, v_a)$,

   with $u_a < u_b, v_a < v_b$, for all $u_a, u_b, v_a, v_b \in [0, 1]$,

   $\alpha = \inf\{f'(u_1) : u_1 \in A\} < 0, \quad \beta = \sup\{f'(u_1) : u_1 \in A\} > 0$,

   $\gamma = \inf\{g'(u_2) : u_2 \in B\} < 0, \quad \delta = \sup\{g'(u_2) : u_2 \in B\} > 0$.

An example of f is

$$f(u_1) = \frac{1 - e^{-\lambda u_1}}{1 - e^{-\lambda}} - u_1, \quad \lambda > 0. \qquad (1.4.6)$$

This construction can be generalized further by introducing a parameter θ, for example,

$$C(u_1, u_2) = C^*(u_1, u_2) + \theta f(u_1)g(u_2), \qquad (1.4.7)$$

see [90] for more detail.

1.4.3 Transformation or Distortion of a 2-copula

A distortion function can be applied to a known 2-copula in order to produce a new 2-copula. It can be useful for inducing Kurtosis in the copula. The general framework for applying distortions to copulas was introduced in [55], and then formally described as *transformations* in [35]. They are also also referred to as *multiplicative generators* in [115]. Distortions are bijective maps from existing copulas to new copulas.

Let C be a bivariate copula and $\psi : [0,1] \to [0,1]$ be a bijective map, then

$$C^{\psi}(u_1, u_2) = \psi^{[-1]}\left(C(\psi(u_1), \psi(u_2))\right), \quad \text{for all} \quad u_1, u_2 \in [0,1] \tag{1.4.8}$$

is a copula, if on the interval $[0,1]$

1. ψ is concave;
2. ψ is strictly increasing;
3. ψ is continuous, and
4. $\psi(0) = 0$ and $\psi(1) = 1$.

Furthermore, the inverse of a multiplicative generator has properties

$$\begin{aligned}
\psi^{[-1]}(t) &= \begin{cases} \psi^{-1}(t) & \text{if} \quad \psi(0) \le t \le 1 \\ 0 \quad \text{if} & 0 \le t \le \psi(0) \end{cases} \\
\psi^{[-1]}(\psi(t)) &= t \quad \text{and} \\
\psi(\psi^{[-1]}(t)) &= \begin{cases} \psi(0) & \text{if} \quad 0 \le t \le \psi(0) \\ t \quad \text{if} & \psi(0) \le t \le 1. \end{cases}
\end{aligned}$$

General distortions induced by functions ψ which are not necessarily bijective, were considered in [34]. The impact of distortions on the properties of copulas is discussed in [35]. Table 1.4.1 shows the distortions considered by Durlemann, Nikeghbali and Roncalli.

Figure 1.4.1 shows how the bivariate Gaussian density with $\rho = 0.5$ is changed when the distortion (3) in Table 1.4.1 is applied with $\beta_1 = 4$ and $\beta_2 = 0.5$.

	$\psi(t)$	$\psi^{-1}(t)$	Restrictions
(1)	$t^{1/\alpha}$	t^{α}	$\alpha \geq 1$
(2)	$\sin(\frac{\pi t}{2})$	$\frac{2}{\pi}\arcsin(t)$	
(3)	$\frac{(\beta_1+\beta_2)t}{\beta_1 t+\beta_2}$	$\frac{\beta_2 t}{\beta_1+\beta_2-\beta_1 t}$	$\beta_1, \beta_2 > 0$
(4)	$\frac{4}{\pi}\arctan(t)$	$\tan(\frac{\pi t}{4})$	
(5)	$\left(\int_0^1 f(t)dt\right)^{-1} \int_0^x f(t)dt$	-	$f \in L^1(]0,1[), f(x) \geq 0, f'(x) \leq 0$

Table 1.4.1: Distortions of Durlemann, Nikeghbali and Roncalli.

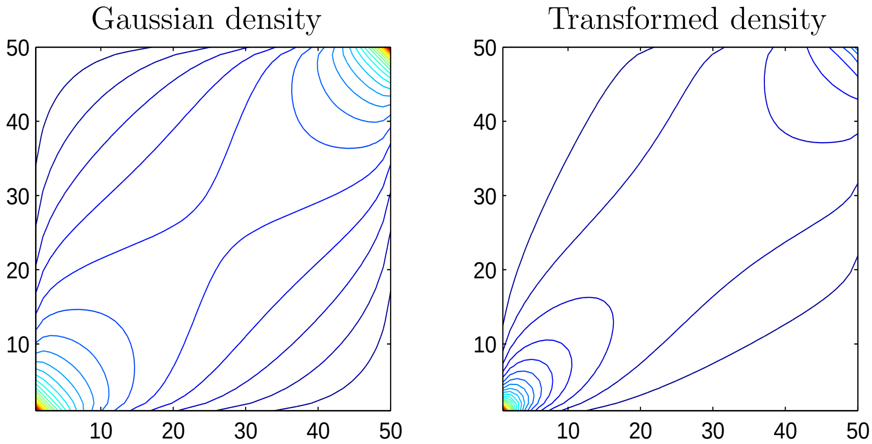

Figure 1.4.1: Contours of Gaussian and transformed density

1.4.4 n-copulas

Definition 2. Let $n \geq 2$ and $\mathbf{u} = (u_1, \ldots, u_n)$. An n dimensional function C is an n-copula if it satisfies

1. $C(\mathbf{u}) = 0$, $\forall \mathbf{u} \in [0,1]^n$, if $u_j = 0$ for at least one j, $1 \leq j \leq n$.
2. $C(\mathbf{u}) = u_j$, if $u_k = 1$ for all $k \neq j$ and $1 \leq j, k \leq n$.
3. Let $[\mathbf{a}, \mathbf{b}]$ denote the n-box $B = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n]$. For every $\mathbf{a}, \mathbf{b} \in [0,1]^n$, such that $\mathbf{a} \leq \mathbf{b}$, the C-volume of B, $V_C([\mathbf{a}, \mathbf{b}]) \geq 0$.

In terms of the copulas, the C-volume of B is $V_C(B) = \Delta_{a_n}^{b_n} \dots \Delta_{a_2}^{b_2} \Delta_{a_1}^{b_1} C(\mathbf{u})$, where

$$\Delta_{a_k}^{b_k} C(\mathbf{u}) = C(u_1, \dots, u_{k-1}, b_k, \dots, u_n) - C(u_1, \dots, u_{k-1}, a_k, \dots, u_n).$$

Elliptical n-copulas exist, for example the Gaussian copula:

$$C_\Sigma(\mathbf{u}) = \Phi_n(\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_n); \Sigma) \quad \mathbf{u} \in [0,1]^n, \tag{1.4.9}$$

where $\Phi_n(\cdot)$ is the multivariate standard normal distribution with correlation matrix $\Sigma = (\varrho_{ij})_{1 \le i,j \le n}$, and $\Phi^{-1}(\cdot)$ is the inverse of the univariate normal distribution.

Archimedean copulas in n dimensions have the form

$$C(u_1, \dots, u_n) = \varphi^{[-1]}\{\varphi(u_1) + \dots + \varphi(u_n)\}, \tag{1.4.10}$$

under some suitable assumptions on φ [108].

1.4.5 Methods of Constructing n-copulas

Method 1

In most cases other than the Gaussian copula, it is not trivial to obtain a higher dimensional version of the 2-copula. For example, Archimedean copulas may be extended to n dimensions, as in (1.4.10), provided that additional conditions are placed on φ [115], [108]. The higher dimensional copulas which are obtained, however, are quite restrictive in terms of their dependence structure. An alternative method for generalizing Archimedean copulas is described in [140] and is designed to overcome the restrictions of the Archimedean n-copulas in (1.4.10) by incorporating more parameters and therefore inducing more flexibility. This construction of n-copulas results in multivariate distributions referred to as *Hierarchical Archimedean copulas*. An example of the construction of a Hierarchical

Archimedean 4-copula is

$$\begin{aligned} C(u_1,u_2,u_3,u_4) &= C\{C_a(u_1,u_2),C_b(u_3,u_4)\} \\ &= \varphi^{-1}(\varphi\circ\varphi_a^{-1}[\varphi_a(u_1)+\varphi_a(u_2)]+\varphi\circ\varphi_b^{-1}[\varphi_b(u_3)+\varphi_b(u_4)]). \end{aligned}$$

Method 2

A method of constructing n-copulas is described in [94]. A multivariate copula C is constructed using the following

$$C(u_1,u_2,\ldots,u_n)=\int_0^1\left(\prod_{i=1}^{n}\nabla_t C_i(u_i,t)\right)dt, \quad u_i\in[0,1], \quad i=1,\ldots,n, \tag{1.4.11}$$

where C_i, $i=1,\ldots,n$ are bivariate copulas. This is a generalization of the copula product discussed in Chapter 5.

Generalizations of this method for constructing 3-copulas are described in [129] and [32]. For example, assume that we have two bivariate copulas

$$C(u_1,u_2,1)=C_{12}(u_1,u_2) \quad \text{and} \quad C(u_1,1,u_3)=C_{13}(u_1,u_3),$$

and a third simple copula, such as the product copula $C_{23}(x,y)=xy$. Now, let ∇_{u_1} be the partial derivative with respect to the first argument, then one can use the formula

$$C(u_1,u_2,u_3) = \int_0^{u_1} C_{23}(\nabla_{u_1}C_{12}(t,u_2),\nabla_{u_1}C_{13}(t,u_3))dt$$

to construct the required 3-copula. For example, let

$$\begin{aligned} C_{12}(u_1,u_2) &= u_1u_2+\theta_1u_1(1-u_1)u_2(1-u_2), && \theta_1\in[-1,1], \\ C_{13}(u_1,u_3) &= u_1u_3+\left\{\frac{\ln(\gamma u_1+1)}{\ln(\gamma+1)}-u_1\right\}\left\{\frac{\ln(\gamma u_3+1)}{\ln(\gamma+1)}-u_3\right\}, && \gamma\in(0,5.2] \end{aligned}$$

and $C_{23}(x,y)=xy$. The range of values for γ is capped at 5.2, since the copula density is not positive for values outside this range. After integration and simplifications, we obtain

$$\begin{aligned} C(u_1, u_2, u_3) &= u_1 u_2 u_3 + \theta_1 u_3 u_1 (1 - u_1) u_2 (1 - u_2) + u_2 \left[\frac{\ln(\gamma u_1 + 1)}{\ln(\gamma + 1)} - u_1 \right] A \\ &+ \theta_1 u_2 (1 - u_2) A \left[\frac{-2\gamma u_1 + (2 + \gamma) \ln(\gamma u_1 + 1)}{\ln(\gamma + 1)} + u_1 (u_1 - 1) \right], \end{aligned}$$

where

$$A = \left[\frac{\ln(\gamma u_3 + 1)}{\ln(\gamma + 1)} - u_3 \right].$$

Other examples are described in Chapter 3.

1.5 Quantiles for Copulas

As an alternative to calculating expectation formulae in models of financial risk, it is suggested that one calculates p-th quantiles [49]. The p-th quantile is defined as the solution x_p of

$$p = H(x_p \mid x_1, \ldots, x_{k-1}), \tag{1.5.1}$$

where H is a k-dimensional distribution. Thus, in terms of a bivariate copula function $C(u_1, u_2)$, for $u_1, u_2 \in [0, 1]$, we have

$$p = H(x_p \mid X_1 = x_1) = \nabla_{u_1} C(F_1(x_1), F_2(x_p)),$$

where $\nabla_{u_1} C$ is the partial derivative of C with respect to the first argument of the copula. Here F_1 and F_2 are given univariate cumulative probability distributions. Therefore, for a specified proportion p and given that $X_1 = x_1$, we can calculate the value of x_p.

The following is an example using the bivariate Gaussian copula, with parameter ρ, and exponential marginal distributions. Suppose that F_1 and F_2 are defined by

$$F_1(x_1) = 1 - e^{-\lambda_1 x_1}, \quad F_2(x_p) = 1 - e^{-\lambda_2 x_p}$$

for constants $\lambda_1, \lambda_2 > 0$. We solve for x_p in

$$p = \Phi\left(\frac{F_2(x_p) - \rho F_1(x_1)}{\sqrt{1-\rho^2}}\right)$$

to obtain

$$x_p = \frac{-\ln\{1 - \rho\Phi^{-1}(1 - e^{-\lambda_1 x_1}) - \sqrt{1-\rho^2}\Phi^{-1}(p)\}}{\lambda_2}, \tag{1.5.2}$$

where $\Phi(\cdot)$ and $\Phi^{-1}(\cdot)$ is the standard univariate normal distribution and its inverse, respectively.

1.6 Dynamic Copula Models

One of the criticisms of copula models is that they are static. The dependence between distributions, such as the movement of stocks or the risk distributions of firms, may change over time. This makes it inappropriate to hold parameters of a copula constant over time. In order to overcome this problem, researchers make the copula dynamic by incorporating time varying distributions and dependence parameters, for example, the bivariate option pricing model described in [156]. Another method of incorporating time into the copula is to associate it with a Markov process. Several models have been constructed in this way and are extensions of the formulae which was initially described in [24]. In particular, such models assume X_s is the initial state of a random variable X at time s. Similarly, X_t is the final state of X at time t. Let F_s and F_t be the initial and final probability distributions associated with X_s and X_t, then their joint distribution can be represented by the copula C_{st}:

$$H_{st}(x_s, x_t) = C_{st}(F_s(x_s), F_t(x_t)) = C_{st}(u_s, u_t) \tag{1.6.1}$$

and

$$\Pr(X_s < x_s \mid X_t = x_t) = \nabla_{u_t} C_{st}(F_s(x_s), F_t(x_t)). \tag{1.6.2}$$

This is used in [165], where the distributions are those of credit grades at two different times. The aim is to model the dependence between initial and final rating grade for a firm. One univariate distribution in the bivariate copula $C(u_s, u_t)$ such that $u_s, u_t \in [0, 1]$, is designated the initial distribution, $u_s = F_s$, of a given credit rating class. The other is designated the final distribution of the credit rating class, $u_t = F_t$. For example, let $X_t = BB$, then we interpret the firm's rating as BB at time t. A transitional distribution or conditional distribution is formed by taking the partial derivative of C_{st}. Also, the cumulative probabilities of the ratings are obtained from the proportion of the total of exposures in the rating class at that time.

Another approach to building time into a copula has been formulated in [125], and applied to Value-at-Risk in [120] and [60]. More details on this approach are given in Chapter 5.

An expression for a 2-copula between two correlated continuous Markov diffusion processes is described in [53]. Very little theoretical work has been carried out in higher dimensions, so there is scope for more research in this area.

1.7 Practical and Theoretical Applications

Applications of copula functions are becoming more prevalent in the literature as researchers realize how applicable they are. Applications include

1. **Finance**

    - *Credit Risk.* The Gaussian copula was used in credit swap valuation, see [100]. In particular, let the time until default of a risky asset be τ, then the associated cumulative distribution of τ up to time t,

$F(t) = \Pr\{\tau < t\}$, may be calculated using

$$F(t) = 1 - \exp\left(-\int_0^t h(s)ds\right),$$

where $h(s)$ is interpreted as the instantaneous default probability obtained from a credit curve. More information about this formula is mentioned in Section 1.8.1. Suppose that we have a pair of risky assets, a and b, then their joint probability of default is given by the Gaussian copula

$$\Pr\{\tau_a < t, \tau_b < t\} = \Phi_2(\Phi^{-1}(F_a(t)), \Phi^{-1}(F_b(t)); \rho). \tag{1.7.1}$$

In Example 3 of [100], the n-dimensional Gaussian copula is used to value a portfolio of n risky assets in a first-to-default contract. The CreditMetrics asset correlation is used as the correlation parameter.

Other applications include asset return dependency [42], pricing portfolios of Credit Derivatives and CDOs [146],[97],[93] and [139]. One factor Gaussian and Clayton copulas are used for the individual conditional marginal default probabilities in [139], and a fast Fourier transform method was applied in order to extract the total number of defaults in a portfolio at time t.

In [77], an alternative to the Gaussian copula for pricing CDOs is presented. The distribution of hazard rate paths, such as $h(t)$ in [100], was specified instead of specifying a particular copula.

- *Auction pricing.* A copula model was applied to Auction pricing in [65]. Marginal distributions in this model are associated with the valuations of private bidders. The Gumbel copula was used to obtain the joint distribution of the valuations.

- *Stock and option pricing.* A bivariate option pricing model appeared in [156]. There are also a number of applications of copulas, including stock and option pricing in [21]. In one example, the marginal distributions of the stock prices are determined by a Generalized Autoregressive Conditional Heteroskedasticity (GARCH) process. A dynamic copula with time-varying correlation parameter is then used to represent the joint distribution between the stocks.

2. **Environmental Science**

   In hydrology, flood data typically consists of flood peak, volume and duration. The joint distribution between these variables (peak, volume and duration) is typically non-Gaussian, so their dependence is not accurately modelled by traditional methods. This makes flood data very suitable for copula models.

   - *Hydrological frequency analysis.* Bivariate hydrological frequency analysis of peak flow and volume of the Rimouski River [38]. In [38], independence, Clayton and Frank copulas were fitted to the data. In the case of the Clayton copula, the relationship between the dependence parameter, α, and Kendall's tau, τ, which is $\alpha = 2\tau/(1-\tau)$, was used to fit the copula to the data. In the last case, the Frank copula, inference for margins (Maximum Likelihood Method) was used to fit the copula to the data.

   - *Flood frequency analysis.* Trivariate flood frequency analysis of the Amite River [164] and Tiber River [150]. Data fitting was carried out in two stages in the these studies. Firstly, the three marginal distributions were fitted to the flood variables, and then log likelihood was used to fit the three dimensional Gumbel-Hougaard copula to the data.

   - *Water cycle estimation.* More recently a 3-copula was used in regional terrestrial water cycle estimation [121]. Microwave temperature is taken to be an indicator of soil moisture and therefore the rainfall on a given region of land. In this water cycle study, the Gumbel copula was chosen to model the joint distribution between benchmark temperature, satellite-measured temperature and latent heat. This model was useful for detecting small rainfall events that were missed by other methods.

3. **Genetics**.

   Traditional methods for analyzing genetic data may be limited in their usefulness, since genetic data is typically non-normal [101]. Copulas were shown to be useful in modelling dependence in two areas of genetics,

   - *Trait linkage analysis.* Computer simulations in [101] suggested that the Gaussian copula is better for testing linkage between non-Gaussian traits than regression and variance-components methods.
   - *Sequence alignment.* DNA and RNA sequences are lined up in order to find similarities. Aligned segments are represented in the form of scoring matrices. In [51], scoring matrices were generated from the sequences and then a copula was used to model the dependence between selected scores from those matrices.
   - *Dependence between genes.* A copula of the form

     $$C(u, v) = uv + \theta uv(1-u)^{\alpha}(1-v)^{\beta},$$

     where $\alpha, \beta \geq 1$, is used to examine directional dependence between genes in [91].

4. **Aggregation and Decision Theory**

   When a firm's directors and policy makers want to make decisions, it is often important to gather information from multiple sources. Quantifying the incoming information is problematic, for example, calculating the aggregated risk of a particular event occurring, given the risks posed from a number individual agencies. The information from individuals is sometimes referred to as expert opinions [85]. Assuming that such opinions can be reformulated into distributions or quantified in some other way, copulas may be applied to the data to provide an aggregated opinion. The use of copulas for aggregating expert opinions was demonstrated in [85].

5. **Signal Processing**

   - *Signal restoration.* Gaussian copulas are combined with Markov chains to find the link between stochastic processes which model hidden and observed signals in [16].

   - *Signal time-frequency distributions.* Copulas can be used to build time-frequency distributions for signals [27]. Let $T(t)$ and $F(f)$ be the cumulative marginal distributions for the time and frequency components of the signal, respectively. Also suppose that C is a copula, then the time-frequency distribution will be

$$P(t, f) = C(T(t), F(f)).$$

An example is

$$\begin{aligned} C(u,v) &= uv + \frac{\epsilon}{4\pi^2 nm}\big(\cos(2\pi[mv-\Delta]) + \cos(2\pi[nu-\Delta]) \\ &- \cos(2\pi[nu+mv-\Delta]) - \cos(2\pi\Delta)\big), \end{aligned} \tag{1.7.2}$$

where $\epsilon \in (-1, 1)$, $\Delta \in [0, 2\pi]$ and $(m, n) \in \mathbb{Z}^2$.

1.7.1 Theoretical Applications

Some of the most recent articles on the theory of copulas are [117], which contains a discussion of the construction of copulas and quasi-copulas with specific type of diagonal section, [153] in which a scalar product for copulas is developed, [31] on threshold copulas, and [19] on tail dependence for t-copulas. Several Bell-type inequalities were also shown to hold for copulas [79] and quasi-copulas [80].

1.7.2 Estimating Copula Parameters

Researchers have taken traditional methods for fitting probability distributions to data and modified them so that copulas can also be fitted to data. Common methods for estimating copula parameters are

- *Exact Maximum Likelihood Method.* The parameters of the marginal distributions and the chosen copula are estimated in one step using this method. We give an example from [52]. Suppose we have N underlying assets and market data associated with the assets is observed at M times. A random sample can be represented by

$$X = \begin{bmatrix} X_{11} & X_{12} & \dots & X_{1N} \\ X_{21} & X_{22} & \dots & X_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ X_{M1} & X_{M2} & \dots & X_{MN} \end{bmatrix}$$

  Let the vector of parameters to be estimated be $\theta = (\theta_1, \theta_2, \dots, \theta_k)$, then the log-likelihood function $l(\theta)$ can be expressed in terms of the copula density c:

$$l(\theta) = \sum_{t=1}^{M} \ln\big(c\big(F_1(x_{1t}), F_2(x_{2t}), \dots, F_N(x_{Nt})\big)\big) + \sum_{t=1}^{M}\sum_{i=1}^{N} \ln\big(f_i(x_{it})\big). \tag{1.7.3}$$

  The maximum likelihood estimator $\hat{\theta}$ of θ is the vector of estimates for θ, which maximizes the log-likelihood function. This method is also explained in [21], and [133].

- *IFM (Inference for margins) Maximum Likelihood Method.* Marginal distribution parameters are estimated first and then those of the copula. For example, let $\breve{\theta} = (\theta_1, \theta_2, \ldots, \theta_N)$ be the set of parameters related only to the univariate marginal distributions, and α be set of dependence parameters of the copula, then the log-likelihood function in this case is

$$l(\breve{\theta}) = \sum_{t=1}^{M} \ln\big(c(F_1(x_{1t};\theta_1),\ldots,F_N(x_{Nt};\theta_N);\alpha)\big) + \sum_{t=1}^{M}\sum_{i=1}^{N} \ln\big(f_i(x_{it};\theta_i)\big). \tag{1.7.4}$$

Estimates if $\breve{\theta}$ are obtained first and then the maximum likelihood method is used to obtain α, see also [21], [133] and [49], in which they fit marginal data to Pareto distributions.

- *Canonical Maximum Likelihood.* Marginal distributions are obtained in empirical form and then are used in the maximum likelihood method. See [21] and [28], in which the authors obtain marginal distributions empirically using a method based on midpoints. Also see [87], [122] and [133] for descriptions of this method.

- *Multi-stage Maximum Likelihood Method.* This is a generalization of a two stage maximum likelihood method, see Patton [126] for details.

- *Non-parametric method.* This method involves calculating non-parametric statistics such as Kendall's τ and Spearman's ρ, then using simple formulae to obtain copula parameters, see in [21], [28] and [58]. For example, $\tau = \alpha/(\alpha+2)$, where α is the dependence parameter of the Clayton copula. Alternatively, $\alpha = 2\tau/(1-\tau)$, therefore, having obtained τ, we can obtain α. In [160], the copula has many locally defined parameters which may be obtained in a similar way.

- *Empirical and Bernstein copulas.* An empirical copula may be generated from any data set and can be combined with the Bernstein copula in order to obtain a multivariate distribution, see [21], [28], [133] and [58]. A substantial amount of the research involving Bernstein copulas is documented in [137].

- *Bayesian approach to fitting copulas.* In this method, Bayesian estimates for the marginal survival distributions are obtained initially, and then a likelihood function is used to obtain a joint posterior distribution, see [72], and for survival models [67] and [134].

1.7.3 Goodness of Fit Tests

There is still considerable research to be done on the goodness of fit of any chosen copula. Both [21] and [28] explain how to use a sampling approach to resolve this problem. Other approaches are described in [142], [122], [57] and [39]. For example, for integrable functions f_1 and f_2, define

$$\langle f_1 \mid \kappa_N \mid f_2 \rangle = \int\int \kappa_N(x_1, x_2) f_1(x_1) f_2(x_2) dx_1 dx_2,$$

where $x_1, x_2 \in \mathbb{R}^N$ and κ_N is a positive definite symmetric kernel. If X_1 and X_2 are random vectors, and f_1 and f_2 are the associated density functions, then

$$\mathbb{E}[\kappa_N(X_1, X_2)] = \int\int \kappa_N(x_1, x_2) f_1(x_1) f_2(x_2) dx_1 dx_2.$$

The squared distance between f_1 and f_2 is

$$\begin{aligned} \Lambda &= \langle f_1 - f_2 \mid \kappa_N \mid f_1 - f_2 \rangle \\ &= \langle f_1 \mid \kappa_N \mid f_1 \rangle - 2\langle f_1 \mid \kappa_N \mid f_2 \rangle + \langle f_2 \mid \kappa_N \mid f_2 \rangle. \end{aligned} \tag{1.7.5}$$

Λ is used as a measure of the goodness of fit between points sampled from a specified copula and real bivariate data [122].

1.8 Collateralized Debt Obligations (CDOs)

During the last two decades there has been an expansion in the credit derivatives market, along with a rapid rise in securitization by large financial institutions.

Given such changes, one of the challenges facing the finance industry now is the pricing of the latest credit derivatives. Collateralized debt obligations (CDOs) are an example of a class of financial products which require efficient methods for their pricing. One reason for the popularity of CDOs is that they enable financial institutions to transfer risk as well as freeing up resources and cash flow for other purposes [127].

To be more precise, CDOs are a class of asset backed securities with underlying collateral made up of loans, mortgages, bonds, credit default swaps (CDS), equity swaps or even other CDOs [114]. All of the underlying are defaultable, hence they are usually referred to as "defaultable instruments" [37]. The general impact of CDOs on a bank's default risk is discussed in [46].

The process involved in CDO pricing is complicated and can be broken down into three parts,

(a) calculating the probability of default of each individual firm,

(b) calculating the dependence between defaulting firms and

(c) calculating the expected loss and fair price of CDO tranches, given the total default.

On step (b) one must calculate a joint default distribution which represents the probability of losses in relation to the entire CDO. A copula can be used in various ways to represent the required joint distribution. A binomial distribution may also be involved in order to obtain the probability that a proportion, say k out of N defaults have occurred by a given time t.

The Gaussian copula is the current market standard in relation to CDO pricing, and some algorithms use it in its conditional bivariate form, while others use the multivariate form mentioned in the n-copula section:

$$C_\Sigma(\mathbf{u}) = \Phi_n(\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_n); \Sigma) \quad \mathbf{u} \in [0,1]^n, \tag{1.8.1}$$

where $\Phi_n(\cdot)$ is the multivariate standard normal distribution with correlation matrix $\Sigma = (\varrho_{ij})_{1\leq i,j\leq n}$, and $\Phi^{-1}(\cdot)$ is the inverse of the univariate standard normal distribution. We note that there are various ways of interpreting the Gaussian correlation parameters ϱ_{ij}, each interpretation depends on the assumptions underlying the model and information available. In a Merton-style model [141] or *asset based* model, ϱ_{ij} is the correlation coefficient between asset values of the i-th and j-th obligor, since one assumes a firm will default when the value of its assets falls below a given threshold. However, one does not need to obtain asset values directly, instead one can formulate the problem in terms of the correlation between default times of each firm. For example, suppose we have a portfolio of n obligors and that random variables $\tau_i, i = 1, \ldots, n$, represent the default times of the n obligors. Then each τ_i will have a corresponding cumulative default probability distribution $F_i(t) = \Pr\{\tau_i \leq t\}, \quad 1 \leq i \leq n$, representing the probability of default before time $t \in \mathbb{R}$. Furthermore, by applying Sklar's theorem, with copula function C, we obtain the joint default distribution H of the n default times:

$$H(t, \ldots, t) = C(F_1(t), \ldots, F_n(t)).$$

For the Gaussian copula (1.8.1) we have

$$H(t, \ldots, t) = \Phi_n(\Phi^{-1}(F_1(t)), \ldots, \Phi^{-1}(F_n(t)); \Sigma). \tag{1.8.2}$$

The following example [43] outlines the steps involved in a Monte Carlo pricing algorithm based on the multivariate Gaussian copula.

1. obtain a sample $u_1, \ldots, u_n$ from the standard Gaussian copula:

   (A) decompose the correlation matrix Σ into $\Sigma = AA^T$,

   (B) obtain $v_1, \ldots, v_n$ from the univariate standard normal distribution, eg. a $\mu_i \sim U_{0,1}$ and set $v_i = \Phi^{-1}(\mu_i), \quad i = 1, \ldots, n$,

(C) set $\mathbf{w} = A\mathbf{v}$ and

(D) set $u_i = \Phi(w_i), \quad i = 1, \ldots, n.$

2. let $\tau_i = F_i^{-1}(u_i)$,

3. compute cash flows implied from the τ_i values,

4. obtain present cash flows with a discounted payoff curve,

5. repeat steps 1 to 4 a sufficient number of times, as required for different situations, and

6. take the average of the present values as the price of the financial product.

Here, $F^{-1}(\cdot)$ is the generalized or pseudo-inverse of $F(\cdot)$ and is defined below.

Definition 2. Let $F(\cdot)$ be a distribution function. Its *quasi-inverse*, $F^{-1}(\cdot)$ with domain $[0, 1]$, is any function satisfying

(α) if $u \in RanF$, then $F^{-1}(u) = x$ for any $x \in \overline{\mathbb{R}}$ such that $F(x) = u$, so $F(F^{-1}(u)) = u$,

(β) if $u \notin RanF$, then $F^{-1}(u) = \inf\{x : F(x) \geq u\} = \sup\{x : F(x) \leq u\}$ and

(γ) $F^{-1}(0) = -\infty$ and $F^{-1}(1) = \infty$.

Therefore, if H is continuous and the pseudo-inverses of $F_i(\cdot)$ are denoted $F_i^{-1}(\cdot)$, then for any $\mathbf{u} \in [0, 1]^n$ we also have,

$$C(\mathbf{u}) = H(F_1^{-1}(u_1), \ldots, F_n^{-1}(u_n)). \tag{1.8.3}$$

Algorithms which use Monte Carlo methods may be computationally cumbersome. This may also be the case if one requires the use of fast Fourier transforms. In order to simplify and speed up computation, models such as the Gaussian One Factor model were developed (see Section 1.11.5).

1.8.1 Calibrating Marginal Default Probability

For each firm in a CDO model, one requires a choice of marginal distribution, and market or manager information to calibrate the distribution parameters.

There are two main types of models within which to determine the marginal default distributions to be used for pricing multi-name credit derivatives.

1. *Structural models.* Structural models were brought into existence by Black and Scholes, broadly speaking. These models require information on the value of a firm's assets and liabilities in order to form default distributions, since they assume that a company will default if the value of it's assets falls below a certain threshold, see [74], [50] and [147].

2. *Reduced form or intensity based models.* Jarrow and Turnbull were the first to produce a reduced form model [81]. These models take the market price of financial products as an indicator of the probability of default. Reduced form models do not require as much detail as structural models [141]. Instead, a reduced form model considers the time of default to be the first jump in an externally given jump diffusion process, since it is not directly observable. Under those assumptions, the formula for default probability before time t is
$$F_i(t) = 1 - \exp\left(-\int_0^t h_i(x)dx\right), \tag{1.8.4}$$
where $t > 0$ and h_i is the i-th intensity [43]. The associated survival probability is
$$S_i(t) = e^{-h_i t}$$
and is calibrated to observed market information, for example bond prices. The calibration allows us to extract an implied h_i from the survival probability formula, see [13]. Several authors have attempted to improve reduced form models. For example, in [26], a skewed double-exponential distribution was used as the marginal distribution of a firm's hazard rate in order

to obtain better results in simulations of correlated default.

An intensity based model was used in the first article on the copula approach to pricing portfolio credit derivatives, see [100]. More recently, in [62] a global as well as an idiosyncratic intensity was incorporated a model of correlated default. More specifically, the default time of the i-th obligor was defined as $\tau_i = inf\{t \geq 0 : N_i(t) + N(t) > 0\}$, where N_i and N was assumed to be Poisson processes with intensities h_i and h, respectively. From these intensities, one can obtain the survival probability

$$S_i(t) = e^{-(h_i+h)t}. \tag{1.8.5}$$

Rather than housing S_i within the Gaussian copula, it may be housed within the survival form of the Gaussian copula (see Section 1.11.3) in order to get the joint survival probability for a number of firms.

Having decided on a copula model and formulae for the marginal default probabilities, one still has to obtain the specific market information for calibration. The four most popular sources of information for calibrating the distribution parameters are

(a) ratings transition matrices constructed by Moody's Investors Service, Inc. or Standard and Poor's service,

(b) asset price and volatility equations,

(c) bond yield spreads or

(d) credit default swap spreads.

An example [73] of part of a rating transition matrix is shown below.

Table 1.8.1 above shows the average cumulative default rates (%). Therefore, the probability of a BBB rated bond defaulting in the second year is $0.55 - 0.24 = 0.31\%$ (from the last row of the table).

	TERM (years)			
RATING	1	2	3	4
AAA	0.00	0.00	0.04	0.07
AA	0.01	0.04	0.10	0.18
A	0.04	0.12	0.21	0.36
BBB	0.24	0.55	0.89	0.55

Table 1.8.1: Standard and Poor's rating matrix (%)

A method of predicting future default rates, rather than just calculating default rate from the history of a firm was described in [89]. The trailing 12-month default rate for month t and rating sample or subgroup of rating k, (referred to as "rating universe k") is

$$\frac{D_{k,t} = \sum_{t-11}^{t} Y_{k,t}}{I_{k,t-11}},$$

where $I_{k,t}$ is the number of firms left in the rating sample k at time t, and is generally known a year ahead. $Y_{k,t}$ is the number of defaulters in month t, that were in the rating sample k as of time $t-11$. Parameter I is relatively stable and a simple autoregressive model is required to obtain the percentage adjustment due to non-credit related withdrawals. On the other hand, the number of bond defaulters Y_i is a source of much variation and so a Poisson distribution is used to calculate it:

$$\Pr\{Y_i = n\} = \frac{e^{-\lambda_i}\lambda_i^n}{n!},$$

where i is the month, and the Poisson parameter, λ_i, must be estimated for each month. A regression estimator can be used for that purpose, for example

$$\ln \lambda_i = \beta_1 G + \beta_2 P, \tag{1.8.6}$$

where G is % of firms in a given rating and P is the industrial production. It depends on the variables found to be most useful in predicting change in credit quality.

Implying default probabilities from asset price and volatility requires the "distance-to-default" formula [23]. Conceptually, distance-to-default is a function of the market value of assets and the associated volatility. However, equity and risk free interest rate r need to be calculated first, since asset value and volatility are not directly observable. More specifically, one of the most common methods of obtaining the asset related parameters is that of Black and Scholes. It is assumed in this method that asset value V_A can be modeled via geometric Brownian motion:

$$dV_A = \mu V_A dt + \sigma_A V_A dB,$$

where μ is the drift rate, σ_A is the asset volatility, and B a Wiener process (Brownian motion). Given X, the book value of the debt at time t, market value of equity V_E and asset value are related by

$$V_E = V_A \Phi(d_1) - e^{-rT} X \Phi(d_2), \tag{1.8.7}$$

where

$$d_1 = \frac{\ln(V_A/X) + (r + \frac{\sigma_A^2}{2})T}{\sigma_A \sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma_A \sqrt{T}.$$

We also have

$$\sigma_E = \frac{V_A}{V_E} \Delta \sigma_A. \tag{1.8.8}$$

The two equations are solved for V_A and σ_A. An example of the values obtained are $V_A = \$12.5\text{bn}$ and $\sigma_A = 9.6\%$. Moody's also have a more sophisticated iterative method for solving the asset volatility, but do not provide the details. Nevertheless, having obtained V_A and σ_A, as well as μ = average return on a firm's assets and X_t = book value of liabilities at time t, one can use the following formula to get the probability of default $p(t)$ at time t:

$$DD = \frac{\ln\left(\frac{V_A}{X_t}\right) + \left(\mu - \frac{\sigma_A^2}{2}\right)}{\sigma_A \sqrt{t}}, \quad p(t) = \Phi(-DD) \tag{1.8.9}$$

Suppose that $\mu = 7\%$, $V_A = \$12.5\text{bn}$, $X_t = \$10\text{bn}$ and $\sigma_A = 9.6\%$, then

$$DD = \frac{\ln\left(\frac{12.5116}{10}\right) + \left(0.07 - \frac{0.0092}{2}\right)}{0.0961} = 3.012,$$

and the one year default probability is $p(t) = \Phi(-3.012) \simeq 13$bp. That is, 13/10000, given that a basis points (bp) is 1/10000. This type of method, see [11], is used to indicate the probability that a firm's assets have fallen below some threshold.

Method (c), calibrating default probability to bond prices, can be implemented by the simple formula

$$Q(T) = \frac{1 - e^{-[y(T) - y^*(T)]T}}{1 - R},$$

where R is recovery rate, $y(T)$ is the yield on a T year corporate zero coupon bond, $y^*(T)$ is the yield on a T year risk free zero coupon bond and $Q(T)$ is the probability of default between $[0, T]$, see [73].
For example, suppose the spreads $[y(T) - y^*(T)]$ for a 5 year and 10 year BBB rated zero coupon bond are 130 and 170 bp, respectively. Default probabilities are $Q(5) = 1 - e^{-0.013*5} = 0.0629$ and $Q(10) = 1 - e^{-0.017*10} = 0.1563$. The probability of default between 5 and 10 years is then $0.1563 - 0.0629 = 0.0934$.

This method was modified so that defaults can happen at any time not just bond maturity dates [75],
Let q_i, the default probability density, be constant in each interval $[t_{i-1}, t_i]$, and

$$\beta_{ij} = \int_{t_{i-1}}^{t_i} v(t)[FW_j(t) - \hat{R}C_j(t)]dt,$$

where $C_j(t)$ is claim made by the bond holders if there is a default at time t, $v(t)$ is the present value of \$1 received at time t, $FW_j(t)$ is the forward price of the j-th bond, given that the forward contract matures at time t and $\hat{R}$ is the expected recovery rate. The probability of default of the j-th bond is

$$q_j = \frac{G_j - B_j - \sum_{i=1}^{j-1} q_i \beta_{ij}}{\beta_{ij}},$$

where G_j and B_j are today's price of a risk free bond and a corporate bond, respectively.

If we were using a reduced form model as mentioned above, we could obtain a constant default intensity h via the formula $h = -\ln(1-p)$, where p is Moody's one-year default probability for the bond of interest [25].

The method of implying default probability from credit default swap (CDS) data (d) is similar to method (c) and one example is given in [73]. Another simple method for approximating default probability $F(t)$ at time t, from CDS spreads is

$$F(t) = 1 - \frac{1}{\left(1 + \frac{S_{CDS}}{1-R/100}\right)^t}$$

where S_{CDS} is the market spread of the CDS, t is the time in years during the life of the CDS and R is the recovery rate, see [112]. This is similar to the risk-neutral probability given in [63]

$$F(T) \approx 1 - \exp\left(-\frac{S_{CDS}T}{1-R}\right).$$

where S_{CDS} is the credit default swap spread and T is the time to maturity. Note that risk neutral probabilities correspond to expected monetary values of 0, whereas real world probabilities correspond to expected utilities of 0 and generally allow for risk aversion.

A hybrid approach which assumes that a firm will default if its asset value falls below a certain threshold and also makes use of historical data, involves setting $P(t)$ to a function of the barrier. By setting the default threshold $z_i = \Phi^{-1}(Q(t))$ or $\Phi^{-1}(F(t))$ as required. An example, of such an approach can be found in the CreditMetrics Technical document [66]. Table 1.8.2 summarizes this method. In column 2 we have transition probability and in column 3, the probability under assumptions of the asset value model. The variables, z_{AAA}, z_{AA}, z_A, z_{BBB}, z_{Def}, etc., are asset return (AR) thresholds and σ_{AR} is standard deviation of the asset returns. An asset return is the percentage change in asset value and, in this model, is assumed to be normally distributed. For example, if $z_A < AR < z_{AA}$ then an obligor is downgraded from rating AAA to AA and $\Pr(AA) = \Phi(z_{AA}/\sigma_{AR}) -$

RATING	Probability from transition matrix (%)	Probability as specified by asset value model
AAA	0.03	$1 - \Phi(z_{AA}/\sigma_{AR})$
AA	0.14	$\Phi(z_{AA}/\sigma_{AR}) - \Phi(z_A/\sigma_{AR})$
A	0.67	$\Phi(z_A/\sigma_{AR}) - \Phi(z_{BBB}/\sigma_{AR})$
BBB	7.73	$\Phi(z_{BBB}/\sigma_{AR}) - \Phi(z_{BB}/\sigma_{AR})$
BB	80.53	$\Phi(z_{BB}/\sigma_{AR}) - \Phi(z_B/\sigma_{AR})$
B	8.84	$\Phi(z_B/\sigma_{AR}) - \Phi(z_{CCC}/\sigma_{AR})$
CCC	1.00	$\Phi(z_{CCC}/\sigma_{AR}) - \Phi(z_{Def}/\sigma_{AR})$
default	1.06	$\Phi(z_{Def}/\sigma_{AR})$

Table 1.8.2: One year transition probabilities for a BB rated obligor (%)

$\Phi(z_A/\sigma_{AR})$.

We solve for z_{Def} and work back up the ratings to obtain the rest of the thresholds. All the thresholds are found in terms of a multiple of σ_{AR}.

1.8.2 Correlated Default and Asset Correlation Coefficients

There are several ways to model correlated default. The simplest models use historical data on credit ratings or default volatility to obtain the correlation coefficients. The problem with simple models is that the resulting correlation coefficients are usually very small and are not good indicators of the influence of correlation on credit risk. The amount of historical data available may also be limited, therefore more complex models of correlated default are required when pricing credit derivatives such as CDOs. For example, copula factor models. The definition of linear correlation is

$$\rho_{E_1E_2} = \frac{\Pr\{E_1 \cap E_2\} - \Pr\{E_1\}\Pr\{E_2\}}{\sqrt{\Pr\{E_1\}(1 - \Pr\{E_1\})\Pr\{E_2\}(1 - \Pr\{E_2\})}}, \tag{1.8.10}$$

so one direct method of obtaining $\rho_{E_1E_2}$ is to set $\Pr\{E_1\}$ and $\Pr\{E_2\}$ to the historic average one-year default rate for a particular rating, see [102]. For example, let $\Pr\{E_1\}$ and $\Pr\{E_2\}$ be the historic average one-year default rate for two B rated companies. The joint default rate is then obtained by calculating

1. the number of B rated firms that defaulted during the year of interest and
2. all possible pairs of such defaulting B rated firms.

Let N_B be the number of defaulted firms in a year, then the number of possible pairs is

$$\frac{N_B(N_B - 1)}{2}.$$

Similarly if N_{Tot} is the total number of B rated firms for that year, the total number of pairs defaulted or not, would be

$$\frac{N_{Tot}(N_{Tot} - 1)}{2}$$

and the joint default $\Pr\{E_1 \cap E_2\}$ in a particular year is the quotient

$$\frac{N_B(N_B - 1)}{N_{Tot}(N_{Tot} - 1)}.$$

It is possible to apply this simple method to data taken over several years, calculate the average over those years, and then calculate the correlation coefficient $\rho_{E_1E_2}$. A problem with this method is that it assumes that the default probability is constant for each rating level [102]. Therefore, when considering yearly joint default rates over a long time interval, one cannot distinguish between the effect of varying default probability and varying correlation. Furthermore, default and correlation may be functions of time, which requires more complex modeling.

Another simple method involves the calculation of an average correlation, see [66]. For a large number, N, of firms in a given credit rating grade, average correlation

ρ is defined by

$$\rho = \frac{N\left(\frac{\sigma^2}{\mu-\mu^2}\right) - 1}{N-1} \approx \frac{\sigma^2}{\mu - \mu^2}, \tag{1.8.11}$$

where σ is the standard deviation of default rates observed from year to year, and μ is average default rate over the years considered. A problem with this approach is that it assumes that average default is constant across all firms in a rating grade and constant with time, which is generally not the case. It is assumed in this method that the default distributions are Gaussian, which is generally not true either.

The joint distribution between asset returns is often assumed to be Gaussian with a correlation coefficient ϱ. A similar idea is to assume that the distributions at time t of asset returns i and j, $F_i(t)$ and $F_j(t)$ are linked by a bivariate Gaussian copula with the dependence parameter θ. If the probability of default or rating migration were represented by the probability that an asset return fell below a certain value, the framework above results in the following pairwise default formula

$$\rho_{ij}(t) = \frac{\Phi_2(\Phi^{-1}[F_i(t)], \Phi^{-1}[F_j(t)]; \theta) - F_i(t)F_j(t)}{\sqrt{F_j(t)(1 - F_i(t))F_j(t)(1 - F_j(t))}}. \tag{1.8.12}$$

Typically, the value of the asset correlation parameter, ϱ, or θ will be much larger than that of pairwise default, ρ_{ij}. For example, CreditMetrics [66] suggest that 20% to 35% is a typical range for average asset correlation across a portfolio. This range results in a default correlation estimate of approximately 1%.

A similar model of a portfolio of bonds in a CDO is described in [13]. The probability of default of each bond is defined by p, the pairwise default correlation coefficient ρ_{ij} is given, and then one determines the uniform asset correlation coefficient ϱ by solving

$$\rho_{ij} = \frac{\Phi_2(\Phi^{-1}(p), \Phi^{-1}(p); \varrho) - p^2}{p(1-p)}. \tag{1.8.13}$$

For example, see [13], suppose that we have m bonds in a portfolio and they all have the same default probability $p = 0.01$. The pairwise default probability ρ_{ij} is the same for the entire portfolio and is set to a value, for example 0.03. An implied $\varrho = 0.2307$ is obtained using a zero finding function such as *fzero* in MATLAB. The equation needs to be rearranged as follows,

$$\Phi_2(\Phi^{-1}(p), \Phi^{-1}(p); \varrho) - p(1-p)\rho_{ij} - p^2 = 0. \tag{1.8.14}$$

Lastly, a very different method of incorporating asset return into a model of correlated default is to assume that it is represented by the function $A_i(t) = k_i e^{b_i t}$ or similar function of time, rather than assuming it is a Gaussian random variable, see [11].

1.8.3 Correlation factors and Log-Asset Returns

It is common practice to correlate defaults by correlating their Log-Asset Returns. The correlation coefficients then appear as entries in the correlation matrix Σ of the multi-dimensional normal distribution [104]. A less cumbersome approach for achieving the same outcomes is to express log-asset returns in factor form and use a special representation of a Gaussian copula called a Gaussian one factor model (GOFM). In this construction we assume

1. there are n obligors in a portfolio,
2. the probability of default of the i-th obligor is F_i,
3. the logarithm of each asset return A_i is normally distributed and
4. the i-th obligor defaults when its asset return falls below a given threshold z_i, then we write
$$F_i(z_i) = \Pr\{A_i \le z_i\} = \Phi(z_i),$$
where $\Phi(\cdot)$ is the standard Gaussian distribution.

We also suppose there exists normally distributed random variables or *factors*, V and ν_i representing the state of the world and specific effects for each firm, respectively [104]. Each market factor ν_i is uncorrelated to the others, and we may express each log-asset return A_i in factor form:

$$\begin{aligned} A_1 &= \rho_1 V + \sqrt{1-\rho_1^2}\nu_1 \\ A_2 &= \rho_2 V + \sqrt{1-\rho_2^2}\nu_2 \\ \dots & \quad \dots \quad \dots \quad \dots . \\ A_n &= \rho_n V + \sqrt{1-\rho_n^2}\nu_n . \end{aligned}$$

where ρ_i is the correlation factor. Then the conditional probability of default for firm i, given each realization of V, is

$$\Pr\{A_i \leq z_i \mid V\} = \Phi\left(\frac{z_i - \rho_i V}{\sqrt{1-\rho_i^2}}\right).$$

This is the one factor Gaussian copula, which is discussed more thoroughly from Section 1.8.6 onward. The correlation between log-asset returns A_i and A_j is obtained from

$$\varrho_{ij} = \rho_i \rho_j .$$

1.8.4 Survival copulas

Some models require the joint probability that n firms have not defaulted by some time t. A survival copula rather than the general copula is usually used in this case. If the survival distribution of the i-th obligor beyond t is

$$S_i(t) = \Pr\{\tau_i > t\} = 1 - F_i(t).$$

and the joint survival distribution is

$$S_n(t, \dots, t) = \Pr\{\tau_1 > t, \dots, \tau_n > t\},$$

then

$$S_n(t, \ldots, t) = \hat{C}(S_1(t), \ldots, S_n(t)), \tag{1.8.15}$$

where $\hat{C}$ is the survival copula. The formula for survival copulas may be derived from the volume formula

$$V_C([F_1(t), 1] \times \ldots \times [F_n(t), 1]).$$

The marginal distributions have to be expressed in terms of survival functions S_i as follows

$$F_i(t) = 1 - S_i(t) \quad \text{and} \quad u_i = S_i(t), \quad i = 1, \ldots, n.$$

For example,

$$\begin{aligned} V_C([F_1(t), 1] \times [F_2(t), 1]) &= 1 - F_1(t) - F_2(t) + C(F_1(t), F_2(t)) \\ &= S_1(t) + S_2(t) - 1 + C\,(1 - S_1(t), 1 - S_2(t)) \\ &= u_1 + u_2 - 1 + C(1 - u_1, 1 - u_2) \\ &= \hat{C}(u_1, u_2), \end{aligned} \tag{1.8.16}$$

for $n = 2$.

1.8.5 Independent Defaults in a Binomial Framework

A good description of the factor model approach in conjunction with the binomial framework can be found in [146]. The main assumptions in this framework are

- default occurs at a given time T,
- interest rates are set to zero,
- there are n obligors with identical exposures of amount L and each has a recovery rate R,
- individual default probabilities p occurring during $[0, T]$ are independent of other individual default probabilities occurring in that time, and

- if X is the number of defaults occurring within the time interval $[0, T]$, the loss given default would be

$$X(1-R)L.$$

In practice, we need to have some type of probability distribution for X to obtain the loss, given default. The Binomial distribution is a common choice for those formulating credit derivative models. In such examples, the probability of $X = k$ defaults until time T is

$$\Pr\{X = k\} = \binom{n}{k} p^k (1-p)^{n-k}$$

and the cumulative probability is

$$\Pr\{X \leq k\} = \sum_{m=0}^{k} \binom{n}{m} p^m (1-p)^{n-m}. \tag{1.8.17}$$

When the number of obligors gets very large, the distribution approaches the normal distribution (from the Central Limit Theorem). This property provides analysts with one method to generate simple explicit CDO pricing formulae, as will be seen in later sections of this chapter. Due to market conventions, the most common copula to be used within the binomial framework (or any other) is the Gaussian one factor copula.

1.8.6 Gaussian One Factor Model (GOFM)

The GOFM was introduced in [100] and applied by many authors, see [37], [17], [43], [97], [13] and [96]. The model can be thought of as a version of Merton's (1974) "firm value model" [63]. To be more specific, the probability of default of each obligor depends on a normally distributed random variable A_i, for example log-asset return of a firm, which comprises two risk factors,

(a) a single external risk factor V which influences all firms in the same way and

(b) an idiosyncratic risk factor ν_i related only to the i-th firm, that is ν_i is independent across firms.

Factors V and ν_i are assumed to be independent standard normal variables and ρ_i is the asset correlation factor. The probability that the i-th obligor defaults before the contract maturity date, conditional on the realization of V, is

$$\begin{aligned} p_i^{(V)} &= \Pr\{A_i \leq z_i \mid V\} \\ &= \Pr\left(\nu_i \leq \frac{z_i - \rho_i V}{\sqrt{1-\rho_i^2}} \;\middle|\; V\right) = \Phi\left(\frac{z_i - \rho_i V}{\sqrt{1-\rho_i^2}}\right). \end{aligned}$$

If the thresholds, $z_i = z$, and correlations $\rho_i = \rho$ are the same throughout the portfolio, we obtain

$$p^{(V)} = \Phi\left(\frac{z - \rho V}{\sqrt{1-\rho^2}}\right). \tag{1.8.18}$$

Combining the general conditional probability of default shown in equation (1.8.18), with binomial and expectation formulae,

$$\Pr\{X = k\} = \int_{-\infty}^{\infty} \Pr\{X = k \mid V = v\}\phi(v)dv,$$

we obtain the cumulative probability of defaults

$$\Pr\{X \leq k\} \;=\; \sum_{m=0}^{k} \binom{n}{m} \int_{-\infty}^{\infty} \Phi\left(\frac{z-\rho V}{\sqrt{1-\rho^2}}\right)^m \left(1 - \Phi\left(\frac{z-\rho V}{\sqrt{1-\rho^2}}\right)\right)^{n-m} \phi(v)dv. \tag{1.8.19}$$

Another very convenient way of calculating the probability that there are exactly k defaults out of n is by recursion, see [61] and [144]. More specifically, suppose that for each $m \in \{0, 1, 2, \ldots, n\}$, $L^{(m)}$ is the conditional portfolio loss, given m obligors have been added to the portfolio. Also assume we know the conditional probability of exactly k defaults out of m, $P_k^{(m|V)} = \Pr\{L^{(m)} = k \mid V\}$, then the recursion for the probability of $k+1$ defaults is

$$P_{k+1}^{(m|V)} = P_{k+1}^{(m-1|V)}(1 - p_m^{(V)}) + P_k^{(m-1|V)} p_m^{(V)}, \tag{1.8.20}$$

where $p_m^{(V)}$ is the conditional default probability of obligor m, given V. Initially one sets

$$\begin{aligned} P_0^{(0|V)} &= 1, \\ P_k^{(0|V)} &= 0, \quad k > 0, \\ P_0^{(m|V)} &= P_0^{(m-1|V)}(1 - p_m^{(V)}). \end{aligned} \tag{1.8.21}$$

This algorithm is explained in more detail in Chapter 3 and is used in the simulation of CDO tranche pricing.

Large Portfolio Formula. Assume χ is the fraction of defaulted obligors and that the asset to market correlations and default thresholds are the same for each asset return and denoted ρ and z, respectively. As the portfolio size approaches ∞, the central limit theorem leads us to the large portfolio formula

$$F_\infty(x) = \Pr\{\chi \leq x\} = \Phi\left(\frac{\sqrt{1-\rho^2}\Phi^{-1}(x) - z}{\rho}\right). \tag{1.8.22}$$

A generalization of the GOFM involving multiple factors and rating transitions also appears in [146]. When all the asset parameters are equal $\varrho = \rho^2$, so some variations of this model, see [13], use the $\sqrt{\varrho}$ in the copula. Sometimes ρ_i is called a "loading factor" [43].

A variation of this model in terms of default time is shown in [97] and is as follows; Assume that $\mathbf{Y} = (Y_1, \ldots, Y_n)$ is a Gaussian vector of latent variables such that $Y_i = \rho_i V + \sqrt{1-\rho_i^2}\nu_i$, where $0 \leq \rho_i \leq 1$ are the loading (correlation) factors, so the covariance of Y_i and Y_j in the portfolio is $Cov\{X_i, X_j\} = \rho_i\rho_j$. We let $\tau_i = F_i^{-1}(\Phi(Y_i))$ and then the conditional probability that the i-th obligor will default before time t, given the realization of V, is

$$\begin{aligned} p_i^{(V)}(t) &= \Pr\{\tau_i \leq t \mid V\} \\ &= \Pr\left(\nu_i \leq \frac{\Phi^{-1}(F_i(t)) - \rho_i V}{\sqrt{1-\rho_i^2}} \,\middle|\, V\right) = \Phi\left(\frac{\Phi^{-1}(F_i(t)) - \rho_i V}{\sqrt{1-\rho_i^2}}\right). \end{aligned}$$

The formulation in (1.8.23) makes use of the general property

$$\Pr\{z \mid V\} \approx \Phi(V\rho, 1-\rho^2),$$

and assumes that z and V are Gaussian. The joint distribution of default times H is

$$H_n(t,\dots,t) = \mathbb{E}\left[\prod_{i=1}^{n} \Pr\{\tau_i \le t \mid V\}\right] = \int_{\mathbb{R}} f(v) \prod_{i=1}^{n} p_i^{(v)}(t)dv, \qquad (1.8.23)$$

where $f(v) = \frac{1}{\sqrt{2\pi}}e^{-v^2/2}$ is the standard normal density function. The conditional survival probability is

$$q_i^{(V)}(t) = 1 - p_i^{(V)}(t),$$

therefore the survival distribution required for pricing basket default swaps [127] as well as CDOs is

$$S_n(t,\dots,t) = \int_{\mathbb{R}} f(v) \prod_{i=1}^{n} q_i^{(v)}(t)dv. \qquad (1.8.24)$$

Translating asset return information into a default times can be achieved by setting $\tau_i = F_i^{-1}(\Phi(A_i))$. It is suggested in [95] that we should remain cautious when making the switch from asset return to default time, since the relationship between correlation ϱ_i and the final correlated default times τ_i may become unclear. This indicates that the interpretation of copula parameters needs care, and perhaps more research should be done in this area.

1.8.7 The Structure of a CDO

Conventional CDOs consist of a portfolio of risky assets which are sold to an entity referred to as a *Special Purpose Vehicle* (SPV) [128]. In order to pay for those assets, the SPV issues bonds, etc., to investors. In this scheme, the investors become *protection sellers* and are provided with periodic premiums depending on how much they invest, whereas the original owner of the portfolio of assets may be thought of as a *protection buyer* and it has to make periodic payments to maintain that protection. A newer type of CDO is referred to as a *synthetic* CDO, since it allows one to transfer credit risk without transferring the ownership of the underlying assets. More detail on synthetic CDO structures will be explained later in this chapter.

Returning briefly to conventional CDOs, we find that they can be divided into two broad subclasses,

(a) *Arbitrage CDOs.* These CDOs are designed merely to make a profit from the difference between the acquisition of the porfolio by the SPV and money arriving from investors. For example, it would be hoped that the cost of acquiring underlying collateral, such as corporate loans, would be much less than the profit from the issuance of the notes (bonds) to investors.

(b) *Balance sheet CDOs.* These CDOs are designed to release regulatory capital and resources associated with the collateral itself [127].

Another difference is that arbitrage CDO structures usually involve new assets especially bought for the deal, see [29]. As well as freeing up regulatory capital, balance sheet CDOs can reduce the use of credit lines to given borrowers.

The subclass of arbitrage CDOs can be divided further still, resulting in

(a_1) cash flow CDOs and

(a_2) market value CDOs,

depending on whether or not the assets are subject to trading. The SPV can trade underlying assets. The value of a tranche depends on current market prices in the case of market value CDOs, since the collateral in the portfolio can be traded. The main source of risk in the case of cash flow CDOs is the loss due to default of assets, since no trading of the underlying occurs [128]. All of the different types of CDOs are reviewed in [128].

1.8.8 The Structure of a Synthetic CDO

In the case of synthetic CDO structures, the SPV does not have to purchase the loans, etc., instead it provides protection against default via credit default swaps (CDSs) with the original bank. Therefore, the credit risk is synthetically transferred off of the balance sheet of the bank. The administration costs involved in actually selling/buying the underlying collateral are no longer a consideration in a synthetic CDO setup. A CDS is a contract which "references" [162] a pool of risky instruments like bonds or mortgages. In this sort of arrangement, the bank makes periodic payments to the SPV, which can be used to pay investors, and in the event of loss on the reference portfolio, the SPV compensates the bank according to the CDS specifications. A synthetic CDO more like insurance than an investment. The SPV still issues bonds to investors and but also invests the proceeds in Government bonds, covered bonds and other collateral [29]. As a consequence of the dealings of the SPV, investors are not only exposed to risk due to loss on the reference items, but also risk associated with the quality of the collateral in which the SPV has invested.

Any CDO structure allows for the transferring of credit risk and re-allocation of resources. An important issue to arise in the meantime is the question of how much the premiums should be, or equivalently how to value the CDO. At present, the protection is divided into *tranches*, which have attachment K_L and detachment (upper attachment) points K_U specifying how much protection the investor has agreed on. For example, if the tranche interval was $[K_L, K_U] = [.03, .10]$, an investor would have to absorb any losses between 3% to 10% of the portfolio Notional, [37].

TRANCHE	**RATING**	**BID**	**ASK**	**IMPLIED CORR**
%	**Moody's**	bps	bps	%
0-3	Caa3	29.5%	31.5%	12
3-6	Baa1	179	199	3
6-9	Aaa	78	84	14
9-12	Aaa	45	50	21
12-22	Aaa	21	25	30

Table 1.8.3: FEB 2006 iTraxx Europe Series 4 quotes

Each tranche also has a name associated with it, such as *senior*, *mezzanine* or *equity* [135]. Tranches also have an external rating such as those shown above, that is AAA, Aaa, AA, Baa, Bbb, CC, and the like. The ratings reflect the quality of the underlying assets and also relate to the level of protection an investor has taken on [135]. Equity tranches often do not get a rating and the originating bank may even absorb the losses associated with them, rather than investors [127]. In Table 1.8.3, the rating of the three highest tranches is the same since they are all assumed to have a low risk of default.

In the event of a default, investors of lower tranches will absorb losses before those holding higher ones. For example if losses are between $3\% - 14\%$, they may be absorbed by investors in a mezzanine tranche, whereas its only when losses exceed that amount that investors of a senior tranche also have to absorb some loss [127]. Consequently, premiums may be less for senior tranche holders than others because of the lower risk. Table 1.8.3 shows typical synthetic CDO quotes for the iTraxx series.

1.8.9 The Pricing of CDO Tranches

Pricing a CDO is similar to pricing a bond or credit default swap, and amounts to calculating quantities such as present value, fair spread and others of interest for each tranche and time step up to maturity. These quantities depend on the portfolio loss distribution, which involves the joint probability of default. There are many different accounts of how the loss distribution fits into CDO tranche pricing schemes, for example [52], [98], [9] and [147]. A standard approach to pricing a synthetic CDO tranche is given in [88]:

Firstly, assume that the reference portfolio is made up of credit default swaps and protection sellers have invested in given tranches. If there is no default, the protection seller receives quarterly payments until the maturity of the deal. If there is a default, and the loss has not exceeded the notional on the tranches lower than that of the protection seller, he/she will also continue to receive payments. However, if the losses cut into the notional of the investor's tranche, then compensation will have to be paid out by the seller to the protection buyer. The formula for the "premium leg" of the $[K_L, K_U]$ tranche describes the amount paid out to the tranche investor and is

$$\text{PremLeg} = \sum_{i=1}^{n} \text{Spd}\Delta t_i(1 - \mathbb{E}[L_{(K_L,K_U)}(t_{i-1})])B(t_0, t_{i-1}), \tag{1.8.25}$$

where

1. $\Delta t_i = t_i - t_{i-1}$ is the time step,
2. B is the discount factor,
3. Spd is the spread and
4. $\mathbb{E}[L_{(K_L,K_U)}]$ is expected loss (as a percentage) of the $[K_L, K_U]$ CDO tranche.

The attachment points K_L and K_U are also expressed as a percentage of the total portfolio notional (Ntl).

The formula for "protection leg" represents the payout to the buyer in the event of default, and is

$$\begin{aligned} \text{ProtLeg} &= \int_{t_0}^{t_n} B(t_0, s) d\mathbb{E}[L_{(K_L,K_U)}(s)] \\ &\approx \sum_{i=1}^{n} \left\{ \mathbb{E}[L_{(K_L,K_U)}(t_i)] - \mathbb{E}[L_{(K_L,K_U)}(t_{i-1})] \right\} B(t_0, t_i). \end{aligned} \quad (1.8.26)$$

Given a continuous loss distribution like the Gaussian copula large portfolio approximation, $F_\infty(x)$ in equation (1.8.22), the expected loss on a single tranche is

$$\mathbb{E}[L_{(K_L,K_U)}] = \frac{1}{K_U - K_L} \left(\int_{K_L}^{1} (x - K_L) dF(x) - \int_{K_U}^{1} (x - K_U) dF(x) \right).$$

All assets have the same properties in a large homogeneous portfolio, so the integral can be solved analytically, see [119]. The result is

$$\mathbb{E}[L_{(K_L,K_U)}] = \frac{\Phi_2(-\Phi^{-1}(K_L), z; -\sqrt{1-\rho^2}) - \Phi_2(-\Phi^{-1}(K_U), z; -\sqrt{1-\rho^2})}{K_U - K_L}. \quad (1.8.27)$$

Similar descriptions of premium and protection leg equations appear in more complicated formulations synthetic CDO tranche pricing, see [17], [98], [18] and [37]. Most of these need Fourier transforms and numerical integration to get the loss distribution, which may be too slow in practice. In contrast to all of these, a much simpler formulae for the premium and protection legs, in terms of hazard rate and survival probabilities, can be found in [144].

Two other methods for calculating tranche loss distribution which warrant discussion, are found in [76], and [63]. The last method is applied to synthetic tranche pricing, however it has the potential to be modified in such a way that it could price other types of CDOs. The authors in [63] also separates the portfolio into two separate parts

(1) the most important individual asset and

(2) group of identical assets.

The motivation for the division is that it allows one to calculate the sensitivity to risk of the most important firm in the portfolio, while treating the rest of the portfolio the same.

Suppose that the distribution of each asset return in the homogenous part of the portfolio is $A_i = \rho_i V + \sqrt{1-\rho_i^2}\nu_i$, where V and ν_i are standard normal as before. In this model, V is interpreted as the market factor return, and $\rho_i = \rho$ is the correlation of V with the homogeneous part of the portfolio. These authors restrict $0 < \rho < 1$. It is also assumed that

- p is the average marginal probability of default for the homogeneous part of the portfolio,
- Ntl is the total notional amount of the portfolio,
- R is the recovery rate and
- each asset in the portfolio defaults if its return falls below a threshold z, so $z = \Phi^{-1}(p_i)$, and the conditional default probability is

$$p_i^{(V)} = \Phi\left(\frac{z - \rho V}{\sqrt{1-\rho^2}}\right) \tag{1.8.28}$$

Loss on the homogeneous part of the portfolio is L^h is

$$L^h = (1-R)\text{Ntl}p_i^{(V)}. \tag{1.8.29}$$

Suppose there exists a single asset in the portfolio with the property that it defaults if A_0 falls below z_0 and the other parameters for this unit of the portfolio are Ntl_0, p_0, R_0, ρ_0, ν_0 and $z_0 = \Phi^{-1}(p_0)$. The probability of default for the single asset, given V, is

$$p_0 = p_{0,i}^{(V)} = \Phi\left(\frac{z_0 - \rho_0 V}{\sqrt{1-\rho_0^2}}\right). \tag{1.8.30}$$

Total portfolio loss is

$$L = \begin{cases} (1 - R_0)Ntl_0 + L^h & \text{with probability } p_0 \\ L^h & \text{with probability } 1 - p_0 \end{cases}.$$

The loss distribution is the probability of a loss greater than a threshold K. For a given value of K, one obtains values B_L and B_U, such that $B_L < B_U$ and

$$B_L = \frac{1}{\rho}\left\{z - \sqrt{1-\rho^2}\Phi^{-1}\left(\frac{K}{(1-R)\mathrm{Ntl}}\right)\right\}$$

and

$$B_U = \frac{1}{\rho}\left\{z - \sqrt{1-\rho^2}\Phi^{-1}\left(\frac{K - (1-R_0)Ntl_0}{(1-R)\mathrm{Ntl}}\right)\right\}.$$

Equivalently

$$\Pr\{L \geq K \mid V\} = \mathbf{1}_{\{V \leq B_L\}} + p_0 \mathbf{1}_{\{B_L < V \leq B_U\}}. \tag{1.8.31}$$

Integrating over V gives rise to $\Pr\{L \geq K\}$ in terms of bivariate and univariate normal distributions:

$$\Pr\{L \geq K\} = \Phi(B_L) + \Phi_2(z_0, B_U; \rho_0) - \Phi_2(z_0, B_L; \rho_0). \tag{1.8.32}$$

The next step involves finding a formula for synthetic CDO tranche default probability and loss. Tranche loss is defined as

$$L_{tr} = \max\{L - K_L, 0\} - \max\{L - K_U, 0\}$$

and tranche probability (expected percentage loss) is

$$P_{tr} = \frac{\mathbb{E}[L_{tr}]}{K_U - K_L} = \frac{\mathbb{E}[\max\{L - K_L, 0\}] - \mathbb{E}[\max\{L - K_U, 0\}]}{K_U - K_L}, \tag{1.8.33}$$

where

$$\begin{aligned} &\mathbb{E}[\max\{L - K, 0\}] \\ &= K[\Phi_2(z_0, B_L; \rho_0) - \Phi(B_L)] + [(1 - R_0)Ntl_0 - K]\Phi_2(z_0, B_U; \rho_0) \\ &+ (1 - R)\mathrm{Ntl}[\Phi_2(z, B_L; \rho) + \Phi_3(z_0, z, B_U; \Sigma) - \Phi_3(z_0, z, B_L; \Sigma)] \end{aligned}$$

and the covariance matrix Σ is

$$\Sigma = \begin{bmatrix} 1 & zz_0 & z_0 \\ zz_0 & 1 & z \\ z_0 & z & 1 \end{bmatrix}.$$

Premium and protection leg equations are essentially the same as described in the previous general model. Tranche present value is the difference between Premleg and ProtLeg. The fair value, sometimes also called tranche premium or break even spread, S, is

$$S = \frac{\text{ProtLeg}}{\text{PremLeg}}. \tag{1.8.34}$$

One of the pitfalls of the model is that the relative error may get large when the number of obligors in the portfolio is less than 100. There may also be issues when choosing ρ and ρ_0. The trivariate normal distribution features in the formulae for tranche losses in this Model. An elegant form of the trivariate normal distribution was formulated in terms of the bivariate distribution and was implemented in MATLAB and other programming languages, see [59].

1.9 Inadequacies of the Gaussian Model

The Gaussian copula in its purist form is limited because it obviously assumes most of the distributions in financial markets are standard normal, when, in fact, they often have fat tails (Kurtosis), [64]. The GOFM is also not very accurate for small diverse portfolios [63].

Even though the Gaussian copula is the market standard, it is not accurate when pricing credit default swap index tranches [64], [159], and does not correspond well with observed iTraxx CDO tranche prices [17]. Furthermore, when the Gaussian copula is calibrated to the lowest tranche of a CDO, it produces implied or base correlations out of step with the market value of the higher tranches [17].

1.9.1 Base Correlation Model

Suppose that the GOFM correlation parameter is calibrated to the market data of a single CDO. A plot of uniform or compound correlation ρ versus tranche attachment point K_L should be linear, however, it resembles a smile. The reason for the discrepancy is that correlation values are a function of both attachment and detachment points. A simple ad hoc method of fixing the problem is called the "Base Correlation method" was introduced in [106] and also discussed in [159], [162]. A plot of base correlation versus detachment point produces a straight line which can be used in determining tranche prices. This method involves "calculating expected loss on a tranche as the difference between the expected loss of two tranches with zero attachment" see [106] and [159]. This method allows us to separate the losses for each level and then get a more appropriate correlation parameter (the base correlation) using the GOFM. The base correlation model is implemented for spreadsheet users in [105].

The base correlation method was criticized in [162]. The authors found that base correlations were not uniform across US or European markets because the structure of traded tranches and tranche points varied. Therefore a more radical alternative may be needed.

1.9.2 Value At Risk and Shortfall

Value at risk (VaR) is typically the 95th or 99th percentile of a portfolio loss distribution. Given that the tail of the normal distribution declines rapidly, the GOFM may underestimate VaR in some situations and overestimate it in others. An alternative which may be used instead of VaR is *Shortfall*. Shortfall is defined as the expected loss size given VaR has been exceeded, see [107].

1.10 Alternatives to the GOFM

Researchers have started creating multiple factor models and using alternative copula factor models in the pricing of CDOs, in order to overcome many of the problems encountered when using the Gaussian copula. Furthermore, a method of testing whether or not it is appropriate to model dependence between pairs of financial products using the Gaussian Copula was formulated in [103]. They found that it was sufficient for modeling dependence between stocks and currencies, but not good for commodities. Several alternatives to the Gaussian copula are now discussed.

A moderate extension of the GOFM was proposed in [4]. These authors introduce random recovery rates and factor loadings to match the fat tails of the CDO loss distribution. Another modification of the GOFM involved the introduction of intra- and inter- group correlations, see [98]. Correlations were also treated in a new way in [5].

An extension in [159] is aimed at overcoming another of the inadequacies of the GOFM, which is that the default of a particular firm can have a large influence on the default probabilities of all the correlated surviving firms. Such a problem is thought worse when one firm defaults within a short time [159], as surviving firms then have a much greater chance of default. The author claims that all models which combine the global risk factor V and idiosyncratic factors ν_i in a linear way, will have that problem with the defaults. An unexpected fast default of a firm is more likely to be caused by the state of a purely external factor than by anything internal. A second factor was introduced in the model to compensate for the effect of V.

1.10.1 Other Factor Copula Models

Student's t Copula Model. Another factor model commonly used in the literature is the Student's t copula and its variations, since it can induce skew and kurtosis in the portfolio default distribution. In this model, one assumes that there exists a vector $(V_1, \ldots, V_n)$ which follows a Student's t distribution with υ degrees of freedom, $V_i = \sqrt{W} X_i$, where $X_i = \rho V + \sqrt{1-\rho^2} G_i$, and $\tau_i = F_i^{-1}(t_\upsilon(V_i))$. The variable V and each G_i are independent Gaussian, but υ/W follows a χ^2_υ distribution, or equivalently, W follows an inverse Gamma distribution with parameters $\upsilon/2$. The default time distributions, conditional on V and W are

$$p_i^{(V,W)}(t) = \Phi\left(\frac{W^{-1/2} t_\upsilon^{-1}(F_i(t)) - \rho V}{\sqrt{1-\rho^2}}\right). \tag{1.10.1}$$

Formula (1.10.1) is an example of the conditional default probability of a two factor model, since it has factors V and W.

Archimedean Copula Models. As mentioned previously, the Clayton copula belongs to the class of Archimedean copulas. In the Clayton copula factor model we let

$$V_i = \varphi\left(-\frac{\ln(u_i)}{V}\right), \quad \tau_i = F_i^{-1}(V_i) \quad \text{and} \quad \varphi(s) = (1+s)^{-1/\theta},$$

where V has a Gamma distribution and u_i are independent uniform random variables. The conditional default probability is

$$p_i^{(V)}(t) = \exp[V(1 - F_i(t)^{-\theta})]. \tag{1.10.2}$$

Another Archimedean copula, which forms the joint default distribution for a portfolio, is the Gumbel copula, which has generating function $\varphi(t) = (-\ln(t))^\theta$. This class of copulas were used to calculate loan loss distribution in [147], and the authors make use of the fact that some copula functions can be represented by Laplace transforms. In this example we are assuming that the model is a structured one, and therefore, we assume that the i-th obligor defaults if X_i falls below the threshold z_i. The probability of default, conditional on factor V, is

$$p_i^{(V)} = \Pr\{X_i \leq z_i \mid V = v\} = \exp\left(-v\varphi(z_i)\right). \tag{1.10.3}$$

The large portfolio approximation associated with the loss distribution is also described in [147].

Double t Model. Recently, it was demonstrated that the double t copula was more accurate for pricing CDO tranches than Gaussian and Clayton copulas, see [17]. In this model we let

$$X_i = \rho\left(\frac{\upsilon - 2}{\upsilon}\right)^{1/2} V + \sqrt{1-\rho^2}\left(\frac{\bar{\upsilon} - 2}{\bar{\upsilon}}\right)^{1/2} \bar{V}_i,$$

where V and each $\bar{V}_i$ are independent random variables having Student t distribution with υ and $\bar{\upsilon}$ degrees of freedom, and $0 \leq \rho \leq 1$. $\tau_i = F_i^{-1}(H(X_i))$ and X_i does not follow a t distribution. $H(.)$ is calculated numerically and depends on ρ. The probability of default, conditional on factor V, is

$$p_i^{(V)} = t_{\bar{\upsilon}}\left\{\left(\frac{\bar{\upsilon}}{\bar{\upsilon} - 2}\right)^{1/2} \frac{H^{-1}(F_i(t)) - \rho\left(\frac{\upsilon-2}{\upsilon}\right)^{1/2} V}{\sqrt{1-\rho^2}}\right\}. \tag{1.10.4}$$

The large portfolio loss distribution for this factor model is

$$F_\infty(x) = t_{\bar{\upsilon}}\left(\frac{\sqrt{1-\rho^2}\,t_{\bar{\upsilon}}^{-1}(x) - z}{\rho}\right). \tag{1.10.5}$$

One problem with this model is that it has the tendency to overprice senior tranches, which suggests this model builds in too much Kurtosis.

NIG Model. The NIG distribution is a mixture of the Gaussian and inverse Gaussian distribution, see [64]:

$$F_{NIG}(x) = \int_0^\infty \Phi\left(\frac{x - (\mu + \beta y)}{\sqrt{y}}\right) f_{IG}(y, \delta\gamma, \gamma^2) dy \tag{1.10.6}$$

and f_{IG} is the density function of the inverse Gaussian distribution for random variable Y, and having parameters $\alpha > 0$ and $\beta > 0$. That is

$$f_{IG}(y;\alpha,\beta) = \begin{cases} \frac{\alpha}{\sqrt{2\pi\beta}} y^{-3/2} exp\big(-\frac{(\alpha-\beta y)^2}{2\beta y}\big), & \text{if } y \geq 0 \\ 0, & \text{if } y \leq 0. \end{cases} \tag{1.10.7}$$

The loss distribution for a large homogeneous portfolio associated with this example (the NIG copula) is described in [88]. Two other factor models are described in [110] and [97]. These models appear to be relatively easy to implement.

1.10.2 Alternatives to Factor Models

A very simple alternative to the GOFM is the positive linear Spearman copula [1]:

$$C(u_1, u_2) = \begin{cases} [u_1 + \rho_s(1-u_1)]u_2, & \text{if } u_2 \leq u_1 \\ [u_2 + \rho_s(1-u_2)]u_1, & \text{if } u_2 > u_1 \end{cases}, \tag{1.10.8}$$

where $\rho_s \in [0,1]$ is Spearman's rank correlation coefficient. Another alternative copula used in [69] is the mixture copula C_m shown in equation (1.4.3).

A very different method for pricing CDO tranches was described in [77]. These authors imply a distribution of hazard rate paths from market prices instead of implying a correlation coefficient. They have eliminated the need to specify a particular copula in their method. Other contrasting models can be found in [84], which is based on a number of gamma processes, and [155], which uses one variable for loss given default instead of a random variable for default and another for loss.

Chapter 2

Copula-based Regression Formulae

2.1 Introduction

As mentioned previously, the original idea of copula functions is attributed to Sklar [154]. Initially, such functions were used in the theory of probabilistic metric spaces and have only been used as practical statistical tools in the last three decades, see [130] and references therein. The use of copula functions is now quite popular in areas such as credit risk, genetics, hydrology, image analysis, etc. Many of the properties of bivariate copulas and their construction have been described in the literature; however there is enormous scope for the development of simple and practical tools in this area. In particular, copula-based quantile regression models, which have been neglected to some extent in the literature. For example, the earliest papers on the application of copula-based quantile regression to data are [15] and [49]. More recently, it has been demonstrated that computer algebra can be used as a tool for generating and transforming copula based quantile functions, [151]. The problem with some of these approaches is that they rely on specific software and involve a considerable amount of computation. Models which make use of simple formulae and are computationally inexpensive are more practically useful. A simple copula-based regression formulae is provided in this chapter, with a view to broadening the range of statistical tools available for analysis of non-linear dependence between random variables. The copula families we focuss on in this chapter are the Farlie-Gumbel-Morgenstern, Gaussian and Archimedean copula families.

2.1.1 Background

Before describing our approach, we return to the idea of linear correlation and regression in order to explain why one might prefer to use copula functions instead of other methods. Suppose we want to capture the dependence between two random variables, X and Y, with distributions G and F respectively. Assume that both X and Y are jointly normal, and G and F are univariate normal distributions. That is, $F = G = \Phi$, where

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \exp\left(-\frac{1}{2}w^2\right) dw.$$

Then the conditional expectation of Y, given X, is

$$\mathbb{E}[Y \mid X] = a + bX,$$

where the constants a and b are such that $\mathbb{E}[Y - a - bX]^2$ is minimized. In this case, the expectation formula represents a simple model of linear regression, and the slope of the regression line, b, can be related back to the linear correlation coefficient between the variables. Linear correlation, however, is only a measure of the overall strength of the association between the variables, not a measure of changes across the distribution [86]. Therefore, if two distributions were strongly dependent at one extreme and marginally dependent elsewhere, a linear correlation coefficient would not be able to capture that information adequately. Consequently, linear models may produce poor estimates the conditional mean at various sites in such data sets.

Another limitation of linear correlation is that it is not invariant under transformations of the underlying distributions, whereas copula-based dependence is. Therefore, copulas and their dependence parameters overcome many of the limitations inherent in existing methods. The dependence parameters may also be expressed in terms of population versions of Kendall's τ and Spearman's ρ, which is useful when fitting a particular copula to data.

The intention of this chapter is to provide flexible regression models represented by a copula-based function h, as an alternative to what is currently available. The conditional expectation of Y, given X, is

$$\mathbb{E}[Y \mid X] = h(X),$$

and X and Y do not have to be normal or jointly normal in terms of their distribution. Such models will most often be nonlinear, however they will not be much more complicated than the original linear regression formulae. We now describe the theory of conditional expectation in terms of copula functions. Most of the examples in this chapter will be bivariate examples. Some generalizations in higher dimensions are also provided. We make use of Farlie-Gumbel-Morgenstern, Gumbel, Clayton and Gaussian copulas, since these are relatively easy to use.

Suppose Y and $X_1, X_2, \ldots, X_n$ are real valued random variables defined on the same probability space and let $\Pr\{Y \leq y \mid X_1, X_2, \ldots, X_n\}$ be the conditional probability that $Y \leq y$ given $X_i, i = 1, 2, \ldots, n$. The corresponding conditional expectation is given by

$$\mathbb{E}[Y|X_1, X_2, \ldots, X_n] = \int_{\mathbb{R}} y \nabla_y \Pr\{Y \leq y \mid X_1, X_2, \ldots, X_n\} dy, \tag{2.1.1}$$

where ∇_y is the partial derivative with respect to y. Before providing examples of the formula in (2.1.1), we need to describe the conditional probability, housed within the integral, in terms of copulas.

The simplest case, $\Pr\{Y \le y \mid X = x\}$, was established in [24]. Suppose that continuous random variables Y and X have joint distribution H. If the marginal distributions of Y and X are F and G, respectively, then the joint distribution can be given by the copula

$$C(u_1, u_2) = H(F^{(-1)}(u_1), G^{(-1)}(u_2)), \quad u_1, u_2 \in [0, 1].$$

Typically this copula will belong to a parameterized family; so we write

$$C(u_1, u_2; \theta), \quad \theta \in \Theta$$

for such a copula. Assume that C is continuous and twice differentiable in $[0, 1]$, then the conditional probability of Y given X is

$$\Pr\{Y \le y \mid X = x\} = \nabla_{u_2} C(F(y), G(x); \theta),$$

where ∇_{u_2} is the partial derivative with respect to the second argument of the copula. Examples of this conditional probability formula are

$$\Pr\{Y \le y \mid X = x\} = \frac{\big(F(y)^{-1/\theta} + G(x)^{-1/\theta} - 1\big)^{-\theta-1}}{G(x)^{-1-1/\theta}}, \quad \theta > 0,$$

for the Clayton copula and

$$\begin{aligned}\Pr\{Y \le y \mid X = x\} &= \exp\Big(-\big[(-\ln(F(y)))^\alpha + (-\ln(G(x)))^\alpha\big]^{1/\alpha}\Big) \times \\ &\quad \big[(-\ln(F(y)))^\alpha + (-\ln(G(x)))^\alpha\big]^{-1-1/\alpha}\big[-\ln(G(x))\big]^{\alpha-1},\end{aligned}$$

$\alpha \in [1, \infty)$, for the Gumbel copula.

For a 3-*copula*, $C(u_1, u_2, u_3)$, the conditional probability of Y given X_1 and X_2 is

$$\Pr\{Y \le y \mid X_1 = x_1, X_2 = x_2\} = \frac{\nabla_{u_2,u_3} C(F(y), G_1(x_1), G_2(x_2); \theta)}{\nabla_{u_2,u_3} C(1, G_1(x_1), G_2(x_2); \theta)}. \quad (2.1.2)$$

The derivation of (2.1.2) is given in Appendix 2.A.

At this point we omit the θ in the notation until specific examples of the parameters are required, such as the α in the Clayton copula. For completeness, however, we provide the formula for a continuous, smooth *n-copula*: Suppose that $F, G_1, G_2, \ldots, G_n$ are the marginal distributions of $Y, X_1, X_2, \ldots, X_n$ respectively and that the joint distribution of the random variables is H. Then

$$C(u_1, u_2, \ldots, u_n) = H(F^{(-1)}(u_1), G_1^{(-1)}(u_2), \ldots, G_n^{(-1)}(u_n)),$$

and the conditional probability of Y given all $X_i, \quad i = 1, 2, \ldots, n$, is

$$\Pr(Y \leq y \mid X_1 = x_1, \ldots, X_n = x_n) = \frac{\nabla_{u_2,\ldots,u_n} C(F(y), G_1(x_1), \ldots, G_n(x_n))}{\nabla_{u_2,\ldots,u_n} C(1, G_1(x_1), \ldots, G_n(x_n))}.$$

A similar formula for the conditional density of a multivariate copula is provided in [6] and [49]. Now, from (2.1.1), we have

$$\begin{aligned}
&\mathbb{E}[Y|X_1 = x_1, \ldots, X_n = x_n] \\
&\quad = \int_{\mathbb{R}} y \frac{\nabla_y \nabla_{u_2,\ldots,u_n} C(F(y), G_1(x_1), \ldots, G_n(x_n))}{\nabla_{u_2,\ldots,u_n} C(1, G_1(x_1), \ldots, G_n(x_n))} dy \\
&\quad = \int_{\mathbb{R}} y \frac{\nabla_{u_1,u_2,\ldots,u_n} C(F(y), G_1(x_1), \ldots, G_n(x_n))}{\nabla_{u_2,\ldots,u_n} C(1, G_1(x_1), \ldots, G_n(x_n))} F'(y) dy.
\end{aligned}$$

2.1.2 Farlie-Gumbel-Morgenstern Copulas

The Farlie-Gumbel-Morgenstern (FGM) copulas are a commonly used one parameter family of distributions. The simplest bivariate case with parameter $\theta \in [-1, 1]$ is defined as

$$C_\theta(u_1, u_2) = u_1 u_2 + \theta u_1(1 - u_1) u_2(1 - u_2), \quad u_1, u_2 \in [0, 1].$$

In this case, the conditional expectation is

$$\mathbb{E}[Y \mid X = x] = \mathbb{E}[Y] + \theta(1 - 2G(x)) \int_{\mathbb{R}} y(1 - 2F(y)) F'(y) dy.$$

This equation was established in relation to the FGM distribution, rather than to the FGM Copula, in [148]. It can also be expressed as

$$\mathbb{E}[Y \mid X = x] = \mathbb{E}[Y] - \theta(1 - 2G(x)) \int_{\mathbb{R}} F(y)(1 - F(y))dy. \tag{2.1.3}$$

We now provide particular examples of this formula, which represent the conditional mean, Y, given realizations of X. In these examples, it was possible to work out the integrals explicitly (see Appendix 2.B). In cases where numerical integration is required, for example if we chose Pareto distributions in the copula, simple techniques such as the Trapezoidal method or quadrature formulae could be used.

Example 2.1. Suppose that the marginal distributions are both $N(0, 1)$. Then (2.1.3) reduces to

$$\mathbb{E}[Y \mid X = x] = -\frac{\theta}{\sqrt{\pi}}\big(1 - 2\Phi(x)\big), \quad \theta \in [-1, 1]. \tag{2.1.4}$$

The solution of the integral component in (2.1.3), under the assumptions set in Example 2.1, is given in Appendix 2.B.

Figure 2.1.1 illustrates the use of the expectation formula (2.1.4) with parameter $\theta = 0.8418$. The nonlinear expectation (conditional mean) curve is superimposed on a scatter plot of the data. The data (100 pairs (x,y) of points) was simulated using a copula-based Gibbs sampling algorithm. Data were simulated in MATLAB version 7 in order to illustrate the nature of the FGM copula-based expectation curve. A simple linear fit to the data may have overestimated the upper tail of the data in this case.

In later sections of this chapter we use real data, which presents us with the problem of estimating the copula parameters, for example θ. In all cases we use the non-parametric technique described in [58] to obtain the required parameters. This method involves matching observed scale-free parameters such as Kendall's τ and Spearman's ρ to θ, given simple equations based on the distributional

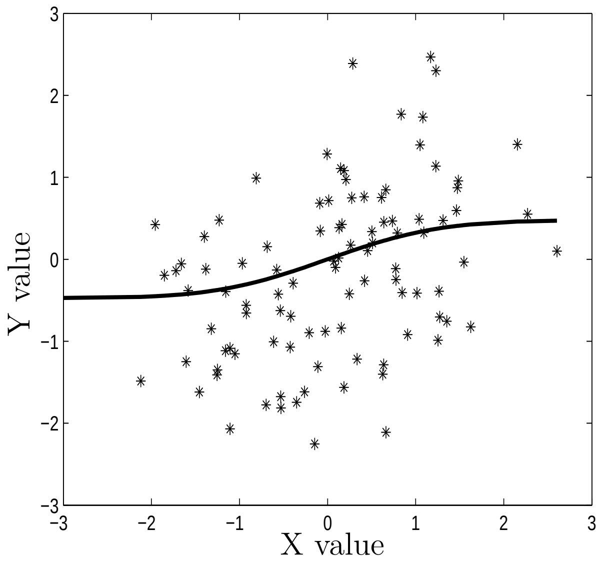

Figure 2.1.1: Scatter plot and conditional mean of Y, given X, for FGM model 1 (simulation with normal marginal distributions).

version of the parameters. In the case of the FGM copula, it has a limitation in that the relationship between τ and θ is $\tau = 2\theta/9$. That is, τ must not exceed the given range, since in that case the original FGM function involved would not be a copula function and would not represent a joint probability distribution. Therefore, if we wish to use this technique to find θ, we are limited to using data pairs which give rise to $\tau \in [-2/9, 2/9]$.

In cases in which τ and ρ are closer to ± 1, it is possible to use the method in [58], provided one chooses another copula function, and have a formula linking its parameter with those mentioned. For example, the Clayton copula parameter α is linked to τ by $\tau = \alpha/(\alpha + 2)$. The relationship of τ to the Gaussian correlation coefficient and the Gumbel copula parameter are shown later sections of this chapter.

Example 2.2. Suppose $F(y)$ is $N(0, 1)$ and $G(x)$ is Gaussian with mean μ and variance σ^2. We have

$$\mathbb{E}[Y \mid X = x] = -\frac{\theta}{\sqrt{\pi}}\left(1 - 2\Phi\left(\frac{x-\mu}{\sigma}\right)\right).$$

Example 2.3. Suppose $F(y)$ is $N(0,1)$, and $G(x) = 1 - \exp(-\exp(\frac{x+\xi}{\alpha}))$, that is negative Gumbel, with $\alpha = 1$ and $\xi = 0$. Then we have

$$\mathbb{E}[Y \mid X = x] = -\frac{\theta}{\sqrt{\pi}}\big(2\exp(-\exp(x)) - 1\big).$$

Example 2.4. Suppose that the marginal distributions are negative exponential,

$$F(y) = 1 - \exp(-\lambda_Y y), \quad G(x) = 1 - \exp(-\lambda_X x),$$

where $\lambda_X, \lambda_Y > 0$. The expectation formula reduces to

$$\mathbb{E}[Y \mid X = x] = \frac{1}{\lambda_Y} + \frac{\theta}{2\lambda_Y}\big(1 - 2\exp(-\lambda_X x)\big).$$

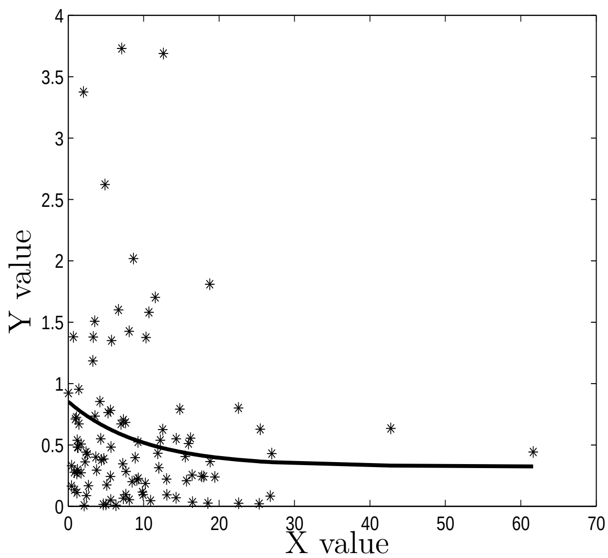

Figure 2.1.2: Scatter plot and conditional mean of Y, given X, for FGM model 4 (simulation with exponential marginal distributions).

Figure 2.1.2 illustrates the use of the expectation formula in Example 2.4 with parameter $\theta = -0.9$. The nonlinear expectation curve is superimposed on a scatter plot of the data (100 pairs (x,y) of points). The data were simulated in MATLAB, as in the previous example. It is clear that the expected value of Y increases as the given value X gets closer to zero, as in the basic trend of the data.

The general formula for an FGM 3-*copula* is

$$\begin{aligned} C(u_1, u_2, u_3) &= u_1 u_2 u_3 \big[1 + \theta_{12}(1-u_1)(1-u_2) + \theta_{13}(1-u_1)(1-u_3) \\ &+ \theta_{23}(1-u_2)(1-u_3) + \theta_{123}(1-u_1)(1-u_2)(1-u_3)\big], \end{aligned}$$

such that the parameters $\theta_{12}, \theta_{13}, \theta_{23}$ and $\theta_{123} \in [-1, 1]$. In terms of an FGM 3-*copula* and random variables Y, X_1 and X_2, the formula for the conditional expectation is

$$\mathbb{E}[Y \mid X_1 = x_1, X_2 = x_2] = \mathbb{E}[Y] - \eta(x_1, x_2) \int_{\mathbb{R}} F(y)(1 - F(y))dy,$$

where

$$\eta(x_1, x_2) = \frac{\theta_{12}(1-2G_1(x_1)) + \theta_{13}(1-2G_2(x_2)) + \theta_{123}(1-2G_1(x_1))(1-2G_2(x_2))}{1 + \theta_{23}(1-2G_1(x_1))(1-2G_2(x_2))},$$

and θ_{12}, θ_{13}, θ_{23} and θ_{123} correspond to those parameters in the general FGM 3-copula formula shown above.

2.1.3 Iterated FGM Distributions

The first iteration in the family of iterated Farlie-Gumbel-Morgenstern distributions, with parameters α and β, is

$$H(u_1, u_2) = u_1 u_2 + \alpha u_1 u_2 (1-u_1)(1-u_2) + \beta {u_1}^2 {u_2}^2 (1-u_1)(1-u_2), \quad u_1, u_2 \in [0, 1],$$

and the second mixed derivative is

$$H_{12}(u_1, u_2) = 1 + \alpha(1 - 2u_1)(1 - 2u_2) + \beta u_1 u_2 (2 - 3u_1)(2 - 3u_2).$$

Conditions under which $H_{12}(u_1, u_2)$ is a density are given in [70] as

$$|\alpha| \leq 1, \quad \alpha + \beta \geq -1, \quad \text{and} \quad \beta \leq 2^{-1}(3 - \alpha + (9 - 6\alpha - 3\alpha^2)^{1/2}).$$

A more tractable way of expressing the permitted range of values is to set $\beta = \alpha\gamma$ and require that both α and $\gamma \in [-1, 1]$.

We obtain

$$\begin{aligned}\mathbb{E}[Y \mid X = x] = \mathbb{E}[Y] + \alpha(1 - 2G(x)) \int_{\mathbb{R}} y(1 - 2F(y))F'(y)dy \\ + \alpha\gamma G(x)(2 - 3G(x)) \int_{\mathbb{R}} yF(y)(2 - 3F(y))F'(y)dy.\end{aligned}$$

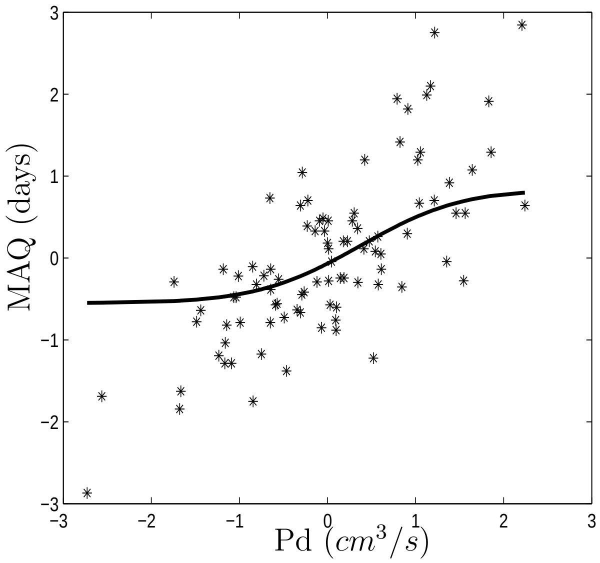

Figure 2.1.3: Conditional mean and scatter plot of annual Discharge (days), given Peak Discharge (cm^3/s).

Example 2.5. Suppose that the marginal distributions are $N(0, 1)$. Then the conditional expectation formula reduces to

$$\mathbb{E}[Y \mid X = x] = -\frac{\alpha}{\sqrt{\pi}}(1 - 2\Phi(x)) - \frac{\beta}{2\sqrt{\pi}}\Phi(x)(2 - 3\Phi(x)). \qquad (2.1.5)$$

The expectation formula given in (2.1.5) is applied to groundwater data which were obtained with permission from the Mekong River Commission (Personal communication from Dr A. Metcalfe), see Figure 2.1.3. Mean annual discharge (MAQ, days), which is the length of the flood season in days, represents one distribution, and Peak discharge (Pd, cm^3/s) of the Mekong River at Vientiane represents the other. It is assumed that both distributions are Gaussian and they have been standardized so that they are $N(0,1)$. The copula parameters are $\alpha = \beta(\alpha\gamma) = 0.9746$. Using the method in [58] involved obtaining Spearman's ρ and using the following formula to estimate the copula parameters, $\rho = \alpha/3 + \alpha\gamma/12$, see also [71]. For ease of computation, we set $\gamma = 1$ and then having calculated ρ in MATLAB, we obtain $\alpha = 12\rho/5$.

If one wishes to use another method to find parameters, such as a more general minimization technique, one must still ensure the copula parameters satisfy the conditions above ($\alpha \in [-1,1]$ and $\gamma \in [-1,1]$). Neither of the methods above are flawless, and the main aim here is to obtain a nonlinear curve which follows the basic trend of the data set. The expectation curve in Figure 2.1.3 overestimates the mean in the lower tail, but otherwise follows the trend in the data quite well. Enabling us to predict the conditional mean MAQ reasonably well, given Peak discharge.

Example 2.6. Suppose that the marginal distributions are negative exponential, as in Example 2.4. Then the conditional expectation formula reduces to

$$\begin{aligned} \mathbb{E}[Y \mid X = x] &= \frac{1}{\lambda_Y} - \frac{\alpha}{2\lambda_Y}[2\exp(-\lambda_X x) - 1] \\ &- \frac{5\beta}{6\lambda_Y}[1 - \exp(-\lambda_X x)][3\exp(-\lambda_X x) - 1]. \end{aligned} \tag{2.1.6}$$

2.1.4 Gaussian Copula

If we have a bivariate Gaussian copula, with dependence parameter ρ, we obtain

$$\mathbb{E}[Y \mid X = x] = \int_{\mathbb{R}} y \nabla_y \Bigg(\Phi \Bigg(\frac{\Phi^{-1}(F(y)) - \rho \Phi^{-1}(G(x))}{\sqrt{1-\rho^2}} \Bigg) \Bigg) dy, \tag{2.1.7}$$

where ∇_y is the partial derivative with respect to y. If both marginal distributions F and G were $N(0,1)$, the copula would revert back to the bivariate normal distribution. The Gaussian copula, however, gives us more flexibility, since it can house any types of univariate distributions, F and G. In (2.1.7), we choose one marginal distribution to be Gaussian to simplify part of the formula, and leave the other one open to choice; $F(y) \sim N(0,1)$, and $G(x)$ is assumed to be a given non-Gaussian distribution. Equation (2.1.7) reduces to

$$\begin{aligned} \mathbb{E}[Y \mid X = x] &= \frac{1}{\sqrt{1-\rho^2}} \int_{\mathbb{R}} y \phi \left(\frac{y - \rho \Phi^{-1}(G(x))}{\sqrt{1-\rho^2}} \right) dy \\ &= \frac{1}{\sqrt{2\pi(1-\rho^2)}} \exp\left[\frac{-\rho^2 [\Phi^{-1}(G(x))]^2}{2(1-\rho^2)} \right] \int_{\mathbb{R}} y \exp\left[\frac{-y^2 + 2y\rho\Phi^{-1}(G(x))}{2(1-\rho^2)} \right] dy. \end{aligned} \tag{2.1.8}$$

Exchange rate data for the Swiss Franc [CHF] and Japenese Yen [JPY] relative to the Australian dollar[AUD] for 2006, were taken from the Reserve Bank of Australia website. Figure 2.1.4 illustrates the application of (2.1.8) to the bank data above. The conditional mean of Y, given X, is superimposed on a scatter plot of the data. Values of the JPY/AUD from 2006 (N =115) were assumed to be Gaussian and were standardized so that they were $N(0,1)$. Therefore, the Gaussian marginal distribution $F(y)$ is associated with the Japanese data (JPY/AUD), and $G(x) = 1 - \exp(-\exp(\frac{x+\xi}{\alpha}))$, with parameters $\xi = -0.9544$ and $\alpha = 0.015409$ is associated with the Swiss data (CHF/AUD). The dependence parameter $\rho = 0.7750$ was calculated by first obtaining Kendall's τ from the data and using the relationship

$$\rho = \sin(\frac{\pi\tau}{2}).$$

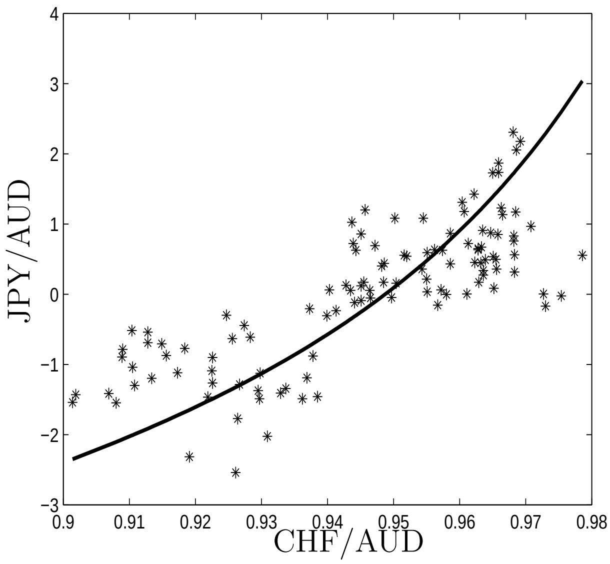

Figure 2.1.4: Scatter plot and conditional mean of JPY/AUD, given the CHF/AUD.

In this example, the numerical integration was carried out using the Trapezoidal method. The overshoot at the upper end of the data suggests that perhaps that the Gaussian copula-based expectation formula is not the best choice in this case. Choosing the best copula for any data set is still the subject of debate, and this point is discussed in the conclusion of this chapter.

2.1.5 Archimedean Copulas

Archimedean copulas are a large family defined by

$$\varphi(C(u_1, u_2, \ldots, u_n)) = \varphi(u_1) + \varphi(u_2) + \ldots + \varphi(u_n),$$

where φ is the generating function (see [115], Chapter 4). The simplest bivariate case of the conditional expectation is given by

$$\mathbb{E}[Y \mid X = x] = -\varphi'(G(x)) \int_{\mathbb{R}} y \frac{\varphi''(C(F(y), G(x)))\varphi'(F(y))}{[\varphi'(C(F(y), G(x)))]^3} F'(y) dy. \qquad (2.1.9)$$

Recall that the Gumbel copula formula is

$$C_\theta(u_1, u_2) = \exp\left[-\{(-\ln(u_1))^\theta + (-\ln(u_2))^\theta\}^{1/\theta}\right], \quad \theta \geq 1,$$

and the Clayton copula formula is

$$C_\alpha(u_1, u_2) = \max\{({u_1}^{-\alpha} + {u_2}^{-\alpha} - 1)^{-1/\alpha}, 0\}, \quad \alpha \in [-1, \infty) \setminus \{0\}.$$

The relationship between Kendall's τ and the Archimedean copula parameters in these examples is $\theta = 1/(1-\tau)$ and $\tau = \alpha/(\alpha+2)$, respectively; see [109].

Figure 2.1.5 demonstrates the application of (2.1.9) with the Clayton copula to exchange rate data. The data come from the Reserve Bank as above. In this example, we use the Hong Kong dollar [HKD] and Korean Won [KRW], relative to the Australian Dollar [AUD]. We want to predict the HKD/AUD, given the KRW/AUD. The estimate for the Clayton parameter was $\alpha = 1.2987$. As both marginal distributions were standardized to $N(0,1)$, we were able to use a form of Gauss-Hermite Quadrature to perform the numerical integration in MATLAB. Test data for Figures 2.1.6 and 2.1.7 were obtained from the *Mass and Physical Measurements for Male Subjects* study on the StatSci.org website. Body measurements were taken from 22 males between the ages of 16 and 30 years. All measurements are in centimetres, and the marginal distributions are assumed to be Gaussian. The data were standardized. In the first example, Figure 2.1.6, we are given maximum circumference of the Forearm and want to predict the expected Waist size. The Gumbel copula with $\theta = 2.2647$ was used. The value of θ was estimated from Kendall's τ as described above. In Figure 2.1.7, we want to predict the chest size from the waist size. In this case, we used the Clayton copula with $\alpha = 2.3582$, estimated from Kendall's τ. A small amount of numerical instability became evident when calculating the expectation curves in Figures 2.1.6 and 2.1.7, due to the sparse amount of data. The curves are still able to provide a good indication of the basic trend in the data.

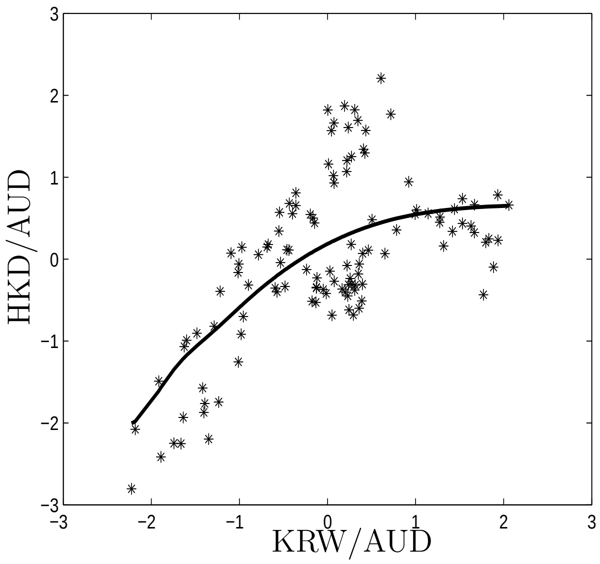

Figure 2.1.5: Scatter plot and conditional mean of HKD/AUD, given KRW/AUD.

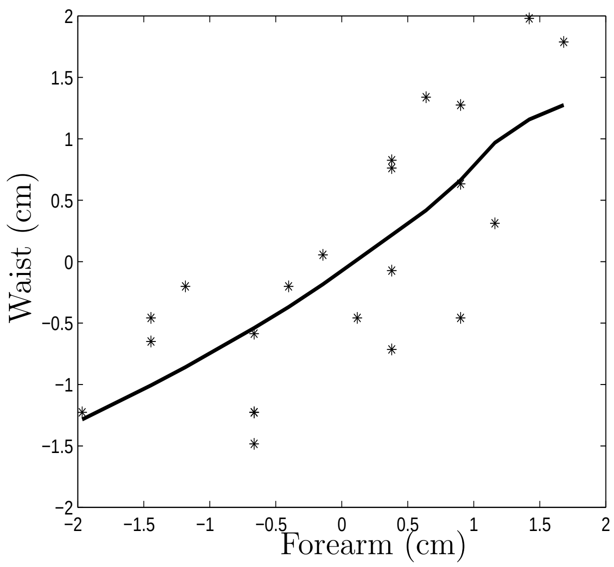

Figure 2.1.6: Scatter plot and conditional mean of waist size of male subjects, given their forearm size.

A general pattern for the n-copula case appears to be elusive, so we provide formulae for $n = 3, 4, 5$. In the trivariate case, assume φ is three times differentiable,

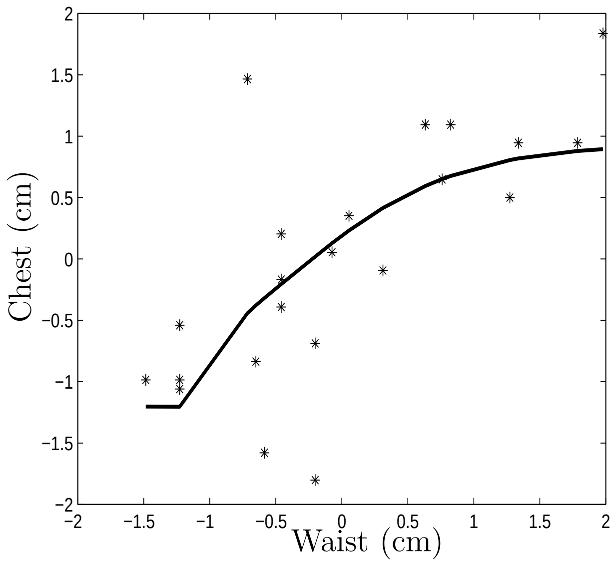

Figure 2.1.7: Scatter plot and conditional mean of chest size of male subjects, given their waist size.

C is a copula and the marginal distributions are F, G_1 and G_2. Then

$$\mathbb{E}[Y \mid X_1 = x_1, X_2 = x_2] = \gamma(x_1, x_2)\int_{\mathbb{R}} y\varphi'(F(y))(\beta_1(y, x_1, x_2) - \beta_2(y, x_1, x_2))F'(y)dy,$$

where

$$\gamma(x_1, x_2) = \frac{\left[\varphi'(C(1, G_1(x_1), G_2(x_2)))\right]^3}{\varphi''(C(1, G_1(x_1), G_2(x_2)))},$$

$$\beta_1(y, x_1, x_2) = \frac{\varphi^{(3)}(C(F(y), G_1(x_1), G_2(x_2)))}{\left[\varphi'(C(F(y), G_1(x_1), G_2(x_2)))\right]^4},$$

$$\beta_2(y, x_1, x_2) = \frac{3\left[\varphi''(C(F(y), G_1(x_1), G_2(x_2)))\right]^2}{\left[\varphi'(C(F(y), G_1(x_1), G_2(x_2)))\right]^5},$$

and $\varphi^{(3)}$ is the third derivative of φ.

Conditional copulas required for conditional expectation formulae in cases $n = 4$ and $n = 5$ follow. Suppose $\mathbf{u} = (u_1, u_2, u_3, u_4)$, the first four derivatives of φ exist and C is a smooth copula, then

$$\nabla_{u_1,u_2,u_3,u_4} C(\mathbf{u}) = -\prod_{j=1}^{4} \varphi'(u_j)\left(\omega_1 - \omega_2 + \omega_3\right),$$

where

$$\omega_1 = \frac{\varphi^{(4)}(C(\mathbf{u}))}{\left[\varphi'(C(\mathbf{u}))\right]^5}, \quad \omega_2 = \frac{10\varphi^{(3)}(C(\mathbf{u}))\varphi''(C(\mathbf{u}))}{\left[\varphi'(C(\mathbf{u}))\right]^6} \quad \text{and} \quad \omega_3 = \frac{15\left[\varphi''(C(\mathbf{u}))\right]^3}{\left[\varphi'(C(\mathbf{u}))\right]^7}.$$

Similarly, suppose $\mathbf{u} = (u_1, \ldots, u_5)$, the first five derivatives of φ exist and C is a smooth copula on $[0, 1]$, then

$$\nabla_{u_1,\ldots,u_5} C(\mathbf{u}) = -\prod_{j=1}^{5} \varphi'(u_j)\left(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\right),$$

where

$$\alpha_1 = \frac{\varphi^{(5)}(C(\mathbf{u}))}{\left[\varphi'(C(\mathbf{u}))\right]^6}, \quad \alpha_2 = -\left(\frac{15\varphi^{(4)}(C(\mathbf{u}))\varphi''(C(\mathbf{u})) + 10\left[\varphi^{(3)}(C(\mathbf{u}))\right]^2}{\left[\varphi'(C(\mathbf{u}))\right]^7}\right),$$

$$\alpha_3 = \frac{105\varphi^{(3)}(C(\mathbf{u}))[\varphi''(C(\mathbf{u}))]^2}{\left[\varphi'(C(\mathbf{u}))\right]^8} \quad \text{and} \quad \alpha_4 = -105\frac{\left[\varphi''(C(\mathbf{u}))\right]^4}{\left[\varphi'(C(\mathbf{u}))\right]^9}.$$

2.1.6 Other simple copulas

Another simple copula established in [36] is

$$C_\eta(u_1, u_2) = u_1 u_2 + \frac{2\eta - 1}{\pi^2}\sin(\pi u_1)\sin(\pi u_2) \quad u_1, u_2 \in [0, 1]$$

In this case, we obtain

$$\mathbb{E}[Y \mid X = x] = \mathbb{E}[Y] + (2\eta - 1)\cos(\pi G(x)) \int_{\mathbb{R}} y \cos(\pi F(y)) F'(y) dy.$$

2.2 Conclusion

In this chapter, we have provided several simple models such as that of (2.1.4), which for the most part require only that one can calculate the combinations of the cumulative distributions in the examples. General formulae such as in (2.1.8) and (2.1.9), which can easily be computed using either Gauss-Hermite Quadrature or the Trapezoidal method, have also been described. The computer code required to implement the mathematics is not very complicated. The expectation curves are easily generated in MATLAB and other languages such as R (S-Plus). The resulting nonlinear copula-based regression curves are meant to enable one to make reasonable predictions of one random variable given another. This method may not be as accurate as spline fitting or alternatives, but it is also not as complicated and is designed for those having to make many similar calculations very quickly. We have used the parameter-fitting method suggested in [58] for ease of computation, but other methods may be used.

Deciding which marginal distributions fit the data is much easier than deciding which copula to choose. It is our view that, out of the three copula families we have chosen to try, the Archimedean copulas have the most potential as non-linear regressors. There are a great variety of Archimedean copulas, and the numerical integration involved in the calculation of the required curves is rapid and uncomplicated. Overall, the choice of copula and the goodness of fit is a problem of current research. We make no apology for this, since techniques are still being developed for deciding which copula is the best choice for a given data set; see [10], [142], [122] and [39]. Fortunately there are many copulas from which to choose and as the area progresses, methods for copula calibration and specification will become more handable. Given that copulas have recently become so popular, it is hoped that these formulae will be of use to actuaries and statisticians.

2.3 Appendix 2.A

Derivation of equation(2.1.2)**.** Let Y, X_1 and X_2 be continuous random variables with distribution functions F, G_1 and G_2, respectively.

$$
\begin{aligned}
& \Pr\{Y \le y \mid X_1 = x_1, X_2 = x_2\} \\
= & \lim_{\substack{\epsilon \to 0^+ \\ \eta \to 0^+}} \Pr\{Y \le y \mid X_1 \in (x_1 - \epsilon, x_1], X_2 \in (x_2 - \eta, x_2]\} \\
= & \lim_{\substack{\epsilon \to 0^+ \\ \eta \to 0^+}} \frac{\Pr\{Y \le y, X_1 \in (x_1 - \epsilon, x_1], X_2 \in (x_2 - \eta, x_2]\}}{\Pr\{X_1 \in (x_1 - \epsilon, x_1], X_2 \in (x_2 - \eta, x_2]\}} \\
= & \lim_{\substack{\epsilon \to 0^+ \\ \eta \to 0^+}} \frac{P_1 - P_2 - P_3 + P_4}{P_5 - P_6 - P_7 + P_8} \\
= & \lim_{\substack{\epsilon \to 0^+ \\ \eta \to 0^+}} \frac{C_1 - C_2 - C_3 + C_4}{C_5 - C_6 - C_7 + C_8} \\
= & \lim_{\substack{\epsilon \to 0^+ \\ \eta \to 0^+}} \frac{C_1 - C_2 - C_3 + C_4}{\{G_1(x_1) - G_1(x_1 - \epsilon)\}\{G_2(x_2) - G_2(x_2 - \eta)\}} \div \\
& \frac{C_5 - C_6 - C_7 + C_8}{\{G_1(x_1) - G_1(x_1 - \epsilon)\}\{G_2(x_2) - G_2(x_2 - \eta)\}} \\
= & \frac{\nabla_{u_2,u_3} C(F(y), G_1(x_1), G_2(x_2))}{\nabla_{u_2,u_3} C(1, G_1(x_1), G_2(x_2))},
\end{aligned}
$$

where

$$
\begin{aligned}
P_1 &= \Pr\{Y \le y, X_1 \le x_1, X_2 \le x_2\}, \\
P_2 &= \Pr\{Y \le y, X_1 \le x_1, X_2 \le x_2 - \eta\}, \\
P_3 &= \Pr\{Y \le y, X_1 \le x_1 - \epsilon, X_2 \le x_2\}, \\
P_4 &= \Pr\{Y \le y, X_1 \le x_1 - \epsilon, X_2 \le x_2 - \eta\}, \\
P_5 &= \Pr\{X_1 \le x_1, X_2 \le x_2\}, \\
P_6 &= \Pr\{X_1 \le x_1, X_2 \le x_2 - \eta\}, \\
P_7 &= \Pr\{X_1 \le x_1 - \epsilon, X_2 \le x_2\}, \\
P_8 &= \Pr\{X_1 \le x_1 - \epsilon, X_2 \le x_2 - \eta\}
\end{aligned}
$$

and

$$
\begin{aligned}
C_1 &= C(F(y), G_1(x_1), G_2(x_2)), \\
C_2 &= C(F(y), G_1(x_1), G_2(x_2 - \eta)), \\
C_3 &= C(F(y), G_1(x_1 - \epsilon), G_2(x_2)), \\
C_4 &= C(F(y), G_1(x_1 - \epsilon), G_2(x_2 - \eta)), \\
C_5 &= C(1, G_1(x_1), G_2(x_2)), \\
C_6 &= C(1, G_1(x_1), G_2(x_2 - \eta)), \\
C_7 &= C(1, G_1(x_1 - \epsilon), G_2(x_2)), \\
C_8 &= C(1, G_1(x_1 - \epsilon), G_2(x_2 - \eta)).
\end{aligned}
$$

2.4 Appendix 2.B

Solution of the integral component in equation(2.1.3), assuming $F(y)$ is $N(0,1)$.

Suppose that

$$F(y) = \Phi(y) = \frac{1}{2} + \frac{1}{2}\mathrm{erf}(y/\sqrt{2}).$$

Then

$$
\begin{aligned}
& -\int_{\mathbb{R}} y(2F(y)-1)F'(y)dy \\
= & -\int_{\mathbb{R}} y[2(\frac{1}{2}+\frac{1}{2}\text{erf}(y/\sqrt{2}))-1]\frac{1}{\sqrt{2\pi}}\exp(-y^2/2)dy \\
= & \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \frac{d}{dy}\left(\exp(-y^2/2)\right)\text{erf}(y/\sqrt{2})dy \\
= & \lim_{b\to\infty}\frac{1}{\sqrt{2\pi}}\exp(-b^2/2)\text{erf}(b/\sqrt{2}) - \lim_{a\to-\infty}\frac{1}{\sqrt{2\pi}}\exp(-a^2/2)\text{erf}(a/\sqrt{2}) \\
- & \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\exp(-y^2/2)\frac{\sqrt{2}}{\sqrt{\pi}}\exp(-y^2/2)dy \\
= & -\frac{1}{\pi}\int_{-\infty}^{+\infty}\exp(-y^2)dy \\
= & -\frac{1}{\pi}\sqrt{\pi} \\
= & -\frac{1}{\sqrt{\pi}}.
\end{aligned}
$$

Chapter 3

Pricing Synthetic CDOs

3.1 Introduction

As mentioned in Chapter 1, the current standard for pricing Collateralized Debt Obligations (CDOs) is the Gaussian one factor model (GOFM). One of the fundamental problems with the GOFM is that once it has been calibrated to a single tranche of a CDO, it does not price any of the other tranches very accurately. To be more precise, one of the assumptions of the GOFM is that all asset values in a CDO portfolio are dependent on a common external factor. That dependence is modeled with a dependence or *correlation* parameter. Calibrating the model to real data (usually related the lowest tranche of the CDO), gives rise to an implied dependence value. Using that implied dependence value to price the rest of the tranches will result in the underpricing the rest of the CDO. The primary reason for the underpricing is that CDO tranche loss distributions have fat tails, whereas the standard Gaussian copula function does not. In order to overcome this problem, practitioners fit a Gaussian copula to each tranche separately and thus obtain a different dependence parameter each time. Having a separate parameter for each tranche provides some additional information regarding the riskiness of that tranche, however, it is also cumbersome and complicated. It would be preferable to be able to fit a suitable copula function to the CDO and only use a single dependence parameter for all tranches.

Underlying a CDO, a transaction which transfers credit risk [61], is a reference portfolio of risky assets. In the case of a Synthetic CDO, the reference portfolio contains credit default swaps. In this chapter distortion functions are applied to the Gaussian copula in order to produce a fat tailed portfolio loss distribution for a Synthetic Collateralized Debt Obligation (sCDO). The process of distorting the copula function results in more realistic CDO tranche prices than those produced via the Gaussian copula alone. Accurate pricing is the key to understanding the real risk of credit derivatives such as CDOs. Distortion functions have not previously been used in this area. In fact, there are very few applications of distortion functions in the literature at all, and this is one of the motivations behind the present chapter. Two different models will be used to generate the tranche prices from the loss distribution,

1. a mixture model combined with the JPMorgan algorithm and
2. the model described in [61], which uses a recursion method which was first described in [5].

The new models only require a single dependence parameter for the entire portfolio rather than one parameter per tranche. The intention behind the reduction of parameters in the models is to provide practitioners with a simpler and more flexible alternative to current CDO pricing methods.

3.2 Distortions of Copulas

The first formal description of distortions of copulas appeared in [35]. More recently distortions have been discussed in [113], [92] and [123]. These functions are also known as *transformations* [35] and *multiplicative generators* in [115]. Regardless of the name, these functions are bijective maps which take existing copulas to new copulas.

Let C be a bivariate copula and $\psi : [0,1] \to [0,1]$ be a distortion function, then

$$C^{\psi}(u_1, u_2) = \psi^{[-1]}(C(\psi(u_1), \psi(u_2))), \quad u_1, u_2 \in [0,1]. \tag{3.2.1}$$

is also a copula function if

(c1) ψ is concave on [0,1]

(c2) ψ is strictly increasing on [0,1]

(c3) ψ is continuous and twice differentiable on [0,1], and

(c4) $\psi(0) = 0$ and $\psi(1) = 1$,

see [35]. Conditions (c1) to (c4) are the strongest assumptions for ensuring that $C^{\psi}(u_1, u_2)$ is a copula. Condition (c3) can be replaced by the weaker assumption that ψ be continuous and piecewise linear, since functions can be uniformly approximated by a sequence of functions satisfying (c1) to (c4). Another weak condition for $C^{\psi}(u_1, u_2)$ to be a copula, which does not require (c1), however, it requires that one can prove that the transformed density function of $C^{\psi}(u_1, u_2)$ is positive on the open interval $(0,1) \times (0,1)$ is given in [35]. It may also be noted that a distortion is called *strict* if $\psi^{[-1]}(t) = \psi^{-1}(\max\{t, \psi(0)\})$ then $\psi^{[-1]} \equiv \psi^{-1}$. That is, ψ has an inverse ψ^{-1}, as opposed to the weaker condition of having a psuedo-inverse $\psi^{[-1]}$.

Assuming C^{ψ} is smooth, then the distorted copula density c^{ψ} is given by

$$\begin{aligned} c^{\psi}(u_1, u_2) &= \nabla_{u_1, u_2} C^{\psi}(u_1, u_2) \\ &= \eta \Big\{ [\psi'(C^{\psi})]^2 \nabla_{12} C(\psi(u_1), \psi(u_2)) \\ &- \psi''(C^{\psi}) \nabla_1 C(\psi(u_1), \psi(u_2)) \nabla_2 C(\psi(u_1), \psi(u_2)) \Big\}, \end{aligned} \tag{3.2.2}$$

where

$$\eta = \frac{\psi'(u_1)\psi'(u_2)}{[\psi'(C^\psi)]^3}$$

and ∇_i is the partial derivative with respect to the i-th argument in the copula.

In the following sections we describe known and new distortion functions. The performance of several distortion functions in the context of CDO pricing is tested via simulation later in this chapter.

3.2.1 Distortions Described by Durrleman, Nikeghbali and Roncalli.

Authors described how the properties of copulas are changed after distortions have been applied in [35]. The specific distortions described are shown in Table 3.2.1.

	$\psi(t)$	$\psi^{-1}(t)$	Restrictions
(D1)	$t^{1/\alpha}$	t^α	$\alpha \geq 1$
(D2)	$\sin(\frac{\pi t}{2})$	$\frac{2}{\pi}\arcsin(t)$	
(D3)	$\frac{(\beta_1+\beta_2)t}{\beta_1 t+\beta_2}$	$\frac{\beta_2 t}{\beta_1+\beta_2-\beta_1 t}$	$\beta_1, \beta_2 > 0$
(D4)	$\frac{4}{\pi}\arctan(t)$	$\tan(\frac{\pi t}{4})$	
(D5)	$\left(\int_0^1 f(t)dt\right)^{-1} \int_0^x f(t)dt$	-	$f \in L^1(]0,1[), f(x) \geq 0,$ $f'(x) \leq 0$

Table 3.2.1: Distortions described by Durrleman, Nikeghbali and Roncalli.

The second conjecture in [35], allows us to form new distortions from convex combinations of existing distortions. If $\psi_1(t)$ is a continuous, twice differentiable distortion with inverse $\psi_1^{-1}(t)$ for $t \in [0,1]$, then $\psi(t) = 1 - \psi_1^{-1}(1-t)$ is also a distortion. Therefore, the functions in Table 3.2.2 are examples of distortions.

$\psi(t)$	$\psi^{-1}(t)$	Restrictions
$\frac{(\gamma+1)-\exp[\ln(\gamma+1)(1-t)]}{\gamma}$	$1-\frac{\ln(\gamma(1-t)+1)}{\ln(\gamma+1)}$	$\gamma>0$
$\frac{1+\ln[t+(1-t)\exp(-\alpha)]}{\alpha}$	$\frac{\exp[-\alpha(1-t)]-\exp(-\alpha)}{1-\exp(-\alpha)}$	$\alpha>0$
$1-\left(\frac{2-2t}{2-t}\right)^{1/\alpha}$	$\frac{2-2(1-t)^{\alpha}}{(1-t)^{\alpha}-2}$	$\alpha>1/3$
$1-\sqrt{2}\mathrm{erf}^{-1}[\mathrm{erf}(1/\sqrt{2})(1-t)]$	$1-\frac{\mathrm{erf}((1-t)/\sqrt{2})}{\mathrm{erf}(1/\sqrt{2})}$	

Table 3.2.2: New distortions from combinations of known distortions.

3.2.2 Distortions Described by Morillas.

A total of twenty four distortions are shown in [113]. Three of those distortions $\psi(\cdot)$ and their inverses $\psi^{-1}(\cdot)$, are shown in Table 3.2.3.

	$\psi(t)$	$\psi^{-1}(t)$	Restrictions
(M1)	$\frac{\ln(\gamma t+1)}{\ln(\gamma+1)}$	$\big(\exp[t\ln(\gamma+1)]-1\big)/\gamma$	$\gamma>0$
(M2)	$\frac{1-\exp(-\alpha t)}{1-\exp(-\alpha)}$	$-\big(\ln[1-(1-\exp(-\alpha))t]\big)/\alpha$	$\alpha>0$
(M3)	$\frac{t^{\alpha}}{2-t^{\alpha}}$	$\left(\frac{2t}{1+t}\right)^{1/\alpha}$	$\alpha\in(0,1/3]$.

Table 3.2.3: Distortions described by Morillas.

The distortions $\psi(t)$ used in Figure 3.2.1 are (D3) with $\beta_1=1$ and $\beta_2=1/4$, indicated by [- -] , (D2), indicated by [-.-], (M1) with $\gamma=3$, indicated by [—], and the identity t, indicated by [....].

Several other ways of combining known distortions in order to create new distortions were described in [113].

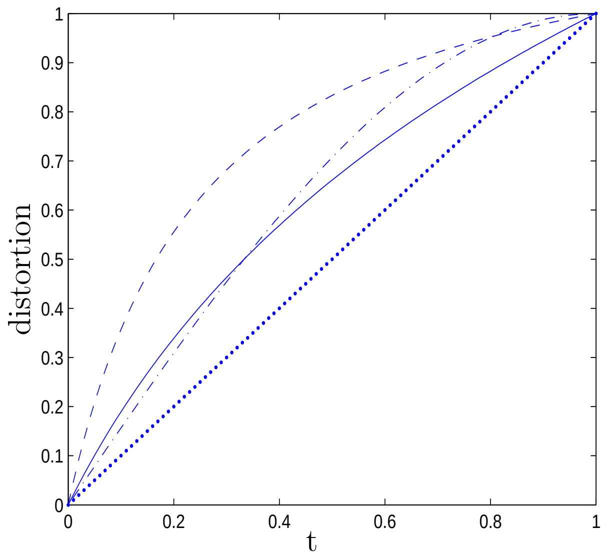

Figure 3.2.1: Comparison of distortions.

3.2.3 New Distortions

Distortion using Erf Function. We obtained the following distortion.

$$\psi(t) = \frac{\operatorname{erf}(t/\sqrt{2})}{\operatorname{erf}(1/\sqrt{2})}. \tag{3.2.3}$$

The inverse of the erf distortion is

$$\psi^{-1}(t) = \sqrt{2}\operatorname{erf}^{-1}[\operatorname{erf}(1/\sqrt{2})t].$$

The second derivative of (3.2.3) is

$$\psi''(t) = \frac{-t\sqrt{2}\exp(-t^2/2)}{\sqrt{\pi}\operatorname{erf}(1/\sqrt{2})}$$

It is clear that ψ is strictly increasing since

$$\operatorname{erf}(a/\sqrt{2}) < \operatorname{erf}(b/\sqrt{2})$$

is true for $a, b \in [0, 1]$ such that $a < b$. It is easy to verify that the other conditions required for ψ to be a distortion function are also satisfied.

Given the relationship between the erf function, standard normal distribution function, Φ, then

$$\psi(t) = \frac{2\Phi(t) - 1}{2\Phi(1) - 1} \tag{3.2.4}$$

is also a distortion.

Piecewise Linear Distortion. Piecewise linear distortions were first presented in [157]. More recently, related work was presented by these authors in [158]. An N knot piecewise linear distortion is defined as

$$\psi(t) = \sum_{i=0}^{N} y_i(t)\mathbf{1}_{\{\beta_i < t \leq \beta_{i+1}\}}, \tag{3.2.5}$$

where

$$\begin{array}{lll}
y_0(t) = \frac{\eta_1}{\beta_1} t, & \text{if} \quad 0 \leq t \leq \beta_1 & \\
y_i(t) = \eta_i + \left(\frac{\eta_{i+1} - \eta_i}{\beta_{i+1} - \beta_i}\right)(t - \beta_i), & \text{if} \quad \beta_i < t \leq \beta_{i+1}, & i = 1, \ldots, N-1 \\
y_N(t) = \eta_N + \left(\frac{1 - \eta_N}{1 - \beta_N}\right)(t - \beta_N), & \text{if} \quad \beta_N < t \leq 1 &
\end{array}$$

and for concavity, we require that

$$\frac{\eta_i - \eta_{i-1}}{\beta_i - \beta_{i-1}} > \frac{\eta_{i+1} - \eta_i}{\beta_{i+1} - \beta_i}.$$

The inverse of the N knot distortion is

$$\psi^{-1}(t) = \sum_{i=0}^{N} g_i(t)\mathbf{1}_{\{\eta_i < t \leq \eta_{i+1}\}}, \tag{3.2.6}$$

where

$$\begin{array}{lll}
g_0(t) = \frac{\beta_1}{\eta_1} t, & \text{if} \quad 0 \leq t \leq \eta_1 & \\
g_i(t) = \beta_i + \left(\frac{\beta_{i+1} - \beta_i}{\eta_{i+1} - \eta_i}\right)(t - \eta_i), & \text{if} \quad \eta_i < t \leq \eta_{i+1}, & i = 1, \ldots, N-1 \\
g_N(t) = \beta_N + \left(\frac{1 - \beta_N}{1 - \eta_N}\right)(t - \eta_N), & \text{if} \quad \eta_N < t \leq 1 &
\end{array}$$

An example of a four knot piecewise linear distortion is shown in Figure 3.2.2. In this example the parameters are $\beta = [0.1\ 0.3\ 0.5\ 0.7]$ and $\eta = [0.4\ 0.7\ 0.85\ 0.95]$.

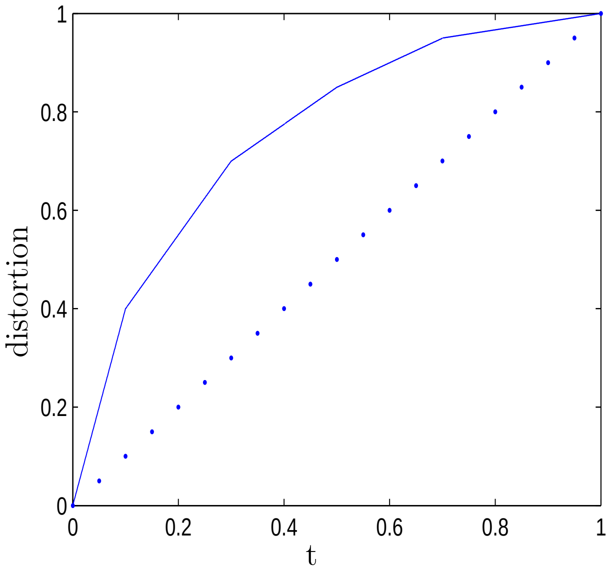

Figure 3.2.2: Piecewise linear distortion.

3.2.4 Composition of Distortions

Creating new distortions from the composition of existing distortions gives us greater flexibility. Suppose $\psi_1(t)$ and $\psi_2(t)$ are continuous, twice differentiable distortions for $t \in [0,1]$, then so is their composition $\psi(t) = \psi_1(\psi_2(t))$. For example,

$$\frac{\ln(\gamma t^{1/\alpha}+1)}{\ln(\gamma+1)}, \quad \alpha \geq 1, \gamma > 0,$$

$$\frac{2\Phi\left[\frac{(\beta_1+\beta_2)t}{\beta_1 t+\beta_2}\right]-1}{2\Phi(1)-1}, \quad \gamma, \beta_1, \beta_2 > 0$$

and

$$\frac{\ln\left(\gamma\frac{2\Phi(t^{1/\alpha})-1}{2\Phi(1)-1}+1\right)}{\ln(\gamma+1)}, \quad \alpha \geq 1, \gamma > 0.$$

A large number of alternative compositions are possible and lead to new families of copulas when applied to existing copulas.

3.2.5 Conditional Distributions Expressed as Copulas

Suppose that we have uniformly distributed random variables U_1, U_2 on $[0, 1]$. Given that

$$\Pr\{U_1 \leq u_1, U_2 \leq u_2\} = C^{\psi}(u_1, u_2),$$

the conditional distribution

$$\Pr\{U_1 \leq u_1 \mid U_2 = u_2\} = \nabla_{u_2} C^{\psi}(u_1, u_2).$$

In particular

$$\nabla_{u_2} C^{\psi}(u_1, u_2) = \frac{\nabla_{u_2} C(\psi(u_1), \psi(u_2))}{\psi'[\psi^{-1}\{C(\psi(u_1), \psi(u_2))\}]} \psi'(u_2). \tag{3.2.7}$$

Substituting for u_1 and u_2 with known marginal distributions $F_1(x_1)$, $F_2(x_2)$, one obtains

$$\Pr\{X_1 \leq x_1 \mid X_2 = x_2\} = \frac{\nabla_{u_2} C(\psi(F_1(x_1)), \psi(F_2(x_2)))}{\psi'[\psi^{-1}\{C(\psi(F_1(x_1)), \psi(F_2(x_2)))\}]} \psi'(F_2(x_2)), \tag{3.2.8}$$

where ψ' is the first derivative of the distortion function. Thus, the partial derivative of the copula now represents the conditional distribution $\Pr\{X_1 \leq x_1 \mid X_2 = x_2\}$, which often plays an important role in credit derivative pricing. The effectiveness of various conditional distributions will be compared the next section of this chapter.

3.3 Distorted Gaussian Copula Model

We have chosen two CDO pricing algorithms, that of

(a) JPMorgan and

(b) Gibson, in which to incorporate the distorted copula and simulate synthetic CDO tranche prices.

In all cases, the results are compared to those of the original Gaussian one factor model (GOFM). In framework 1, we followed the pricing algorithm set out in [105]. In framework 2, we used the pricing algorithm described in [61].

We assume that the underlying portfolio has a mixture of risky assets. Default occurs when the asset falls below a particular threshold. In Model 1 the default probability is tied to credit spread and horizon time, whereas in Model 2 it is tied to credit spread and default time.

Recall that for random variables, asset value (A) with cumulative distribution $F_1(a)$, and global factor (V), with cumulative distribution $F_2(v)$, the Gaussian copula represents the joint distribution

$$\Pr\{A \leq a, V \leq v\} = \Phi_2(\Phi^{-1}(F_1(a)), \Phi^{-1}(F_2(v)); \rho), \tag{3.3.1}$$

where ρ is the correlation parameter. In the new framework we want the joint distribution to be represented by the distorted copula,

$$\Pr\{A \leq a, V \leq v\} = \psi^{(-1)}[\Phi_2(\Phi^{-1}(\psi[F_1(a)]), \Phi^{-1}(\psi[F_2(v)]; \rho)]. \tag{3.3.2}$$

In the original Gaussian copula model [100], the marginal distributions for log asset return and global variables are considered to be standard normal random variables. The Gaussian copula reverts back to the bivariate Gaussian distribution in this case, since $\Phi^{-1}(\Phi(a)) = a$ and $\Phi^{-1}(\Phi(v)) = v$. The parameter ρ is a factor loading linked to pairwise asset correlation, which is ρ^2 if we assume that the dependence is the same for all assets.

In the models of this chapter, we also assume that ρ is the Gaussian copula correlation coefficient and captures the dependence between asset value and global factor. The conditional distribution for firm/credit default, given the realization of a global factor, is arrived at by differentiating the copula. This method for deriving the conditional distribution is simpler than that of the original GOFM. The distribution F_2 is assumed to be standard normal. From equation (3.2.8), the conditional probability that a firm/credit defaults given the realization of global factor V, $p^{(V)}$ is

$$p^{(V)} = \frac{\Phi\left(\frac{\Phi^{-1}(\psi[F_1(a)])-\rho\Phi^{-1}(\psi[F_2(v)])}{\sqrt{1-\rho^2}}\right)\psi'(F_2(v))}{\psi'(\psi^{(-1)}[\Phi_2(\Phi^{-1}(\psi[F_1(a)]),\Phi^{-1}(\psi[F_2(v)]);\rho)])}. \tag{3.3.3}$$

3.3.1 Model 1: JPMorgan CDO Pricing Model

We now describe the most important parameters in the JPMorgan framework, and show how the distortion can be incorporated into the existing model. The horizon date, h, is defined as

$$\mathrm{h} = (D_M - D_V)/360,$$

where D_M is maturity date and D_V is the valuation date, measured in days. The *Cleanspread* (Cs) of the portfolio is

$$Cs = \frac{\mathrm{spd}}{1-R},$$

where R is the recovery rate and *spd* is the average credit spread of the entire portfolio in basis points (bps). For example, if we have 125 credit default swaps in the portfolio then we would have to calculate the mean of the 125 associated credit spreads.

In order to obtain a single value for individual credit default probability, we take the mean of all individual default probabilities in the portfolio:

$$\begin{aligned}\Pr\{A_i \le x*\} &= \mathrm{PD(h)} \\ &= 1 - e^{-(\mathrm{Cs}\ \times\ \mathrm{h}/10000)}. \end{aligned} \tag{3.3.4}$$

Thus, all firms/credits are assumed to have the same default distribution PD and this is substituted for F_1 in the copula. Therefor, the conditional distribution represented by the original GOFM takes the form

$$p^{(V)} = \Phi\left(\frac{\Phi^{-1}(PD(h)) - \sqrt{\varrho}V}{\sqrt{1-\varrho}}\right). \tag{3.3.5}$$

From equation (3.3.3) we obtain the conditional default distribution in terms of distorted copula

$$p^{(V)} = \frac{\Phi\left(\frac{\Phi^{-1}(\psi[PD(h)]) - \rho\Phi^{-1}(\psi[F_2(v)])}{\sqrt{1-\rho^2}}\right)\psi'(F_2(v))}{\psi'(\psi^{(-1)}[\Phi_2(\Phi^{-1}(\psi[PD(h)]), \Phi^{-1}(\psi[F_2(v)]); \rho)])} \tag{3.3.6}$$

and substitute the result into the equation for portfolio loss PFL, given any particular realization of V,

$$\mathrm{PFL}(v) = p^{(V=v)}(1-R). \tag{3.3.7}$$

Let Ntl be the notional of the entire portfolio, then tranche loss TL, given any particular realization of V is

$$\mathrm{TL}(v) = \min\{\max\{\mathrm{PFL}(v)\mathrm{Ntl}, 0\}, K_U\mathrm{Ntl}\}/\mathrm{Ntl}, \tag{3.3.8}$$

where K_U is the tranche upper attachment point. The total expected tranche loss is calculated numerically by approximating the standard formula

$$\mathbb{E}[\mathrm{TL}] = \int_{\mathbb{R}} \mathrm{TL}(v) f_2(v) dv. \tag{3.3.9}$$

We integrate across all values of V using a simple method such as the Trapezoidal method. If f_2 is standard normal then a modified Gaussian quadrature formula can be used. Figure 3.3.1 compares the expected tranche loss using

- the original Gaussian copula, indicated by [*],
- the distortion (D3) applied to the Gaussian copula, indicated by [+] with $\beta_1 = 1$ and $\beta_2 = 0.5$, and
- the piecewise multinode distortion also applied to the Gaussian copula, indicated by [o] with $\beta = [\ 0.3\ 0.4\ 0.5\ 0.7]$ and $\eta = [0.65\ 0.85\ 0.93\ 0.97]$.

F_2 is assumed to be standard normal in these simulations. It is evident that both distortions lift the tail of the loss distribution markedly.

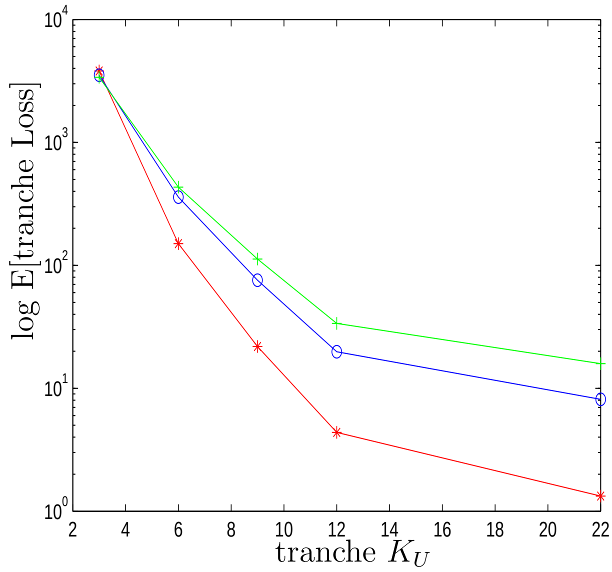

Figure 3.3.1: Comparison of Expected tranche loss.

Results of Model 1

Tables 3.3.3 to 3.3.6 show simulated tranche prices, whereas Table 3.3.1 shows real MID quotes (mean of BID and ASK prices) of the iTraxx Series 3 and Series 4 CDO tranches on dates specified. Except for the $0-3\%$ tranche, quotes are in basis points (bps). The partial set of data were kindly provided by CreditFlux Newsletter. Given that a complete set of data was not accessible, it was not possible to make a direct comparison between the real data and simulations, so feasible parameter values were chosen and fair tranche prices were simulated from those. A variety of distorted Gaussian copulas were used (see paragraph with each Table for details) and a maturity date (horizon time, h) of one year or five years.

The parameter settings were

- Portfolio Notional = \$1 Million
- Recovery rate = 40%
- Discount rate = 2%

TRANCHE	QUOTE (S4)	IMPLIED	QUOTE (S3)	BASE
	21 FEB 06	CORR	12 APRIL 05	CORR
%	bps	%	bps	%
0-3	26.3%	12%	23.8%	20
3-6	78.0	3	151.0	29
6-9	25.0	12	46.0	37
9-12	11.5	17	21.0	43
12-22	5.3	24	13.5	59

Table 3.3.1: iTraxx Europe series 3 and 4 MID quotes.

In comparison 1, Table 3.3.2 and comparison 3, Table 3.3.4, CDO tranche prices were simulated using

1. the original Gaussian copula (GOFM),
2. distortion (D3) from Table 3.2.1 with $\beta_1 = 1$ and $\beta_2 = 0.5$, applied to the Gaussian copula (DURL) and
3. the piecewise linear multiknot distortion with β = [0.3 0.4 0.5 0.7] and η = [0.65 0.85 0.93 0.97] applied to the Gaussian copula (PWLD).

Average credit spread $spd = 40$ bps, $h = 5$ years and the dependence parameter is as indicated in the last column of each table.

Two different logarithmic distortions of the Gaussian copula are compared to that of the original Gaussian copula (GOFM) in comparison 2, Table 3.3.3, comparison 4, Table 3.3.5, comparison 7, Table 3.3.8 and comparison 8, Table 3.3.9. The first distortion is (M1) from Table 3.2.3 with $\gamma = 5$ (LOG), and the second is the composition of (M1) and $\sqrt{t}$, with $\gamma = 5$ (LOG SQ).

The Durrleman (D3) and piecewise linear distortions lead to much larger increases in tranche values compared to the logarithmic distortions. All distortions produced far more realistic values in relation to the highest tranche. Lowering the

TRANCHE	GOFM	DURL	PWLD	SIM CORR
%	bps	bps	bps	%
0-3	14.3%	9.9%	9.1%	45
3-6	262.2	300.2	376.4	45
6-9	79.8	152.5	170.4	45
9-12	28.0	83.9	80.9	45
12-22	4.9	26.7	20.6	45

Table 3.3.2: Comparison 1 of simulated fair prices, 5yrs.

TRANCHE	GOFM	LOG	LOG SQ	SIM CORR
%	bps	bps	bps	%
0-3	14.3%	10.8%	13.4%	45
3-6	262.2	286.9	242.7	45
6-9	79.8	138.9	96.1	45
9-12	28.0	74.2	45.7	45
12-22	4.9	22.7	12.9	45

Table 3.3.3: Comparison 2 of simulated fair prices, 5yrs.

TRANCHE	GOFM	DURL	PWLD	SIM CORR
%	bps	bps	bps	%
0-3	18.9%	12.7%	11.2%	30
3-6	159.7	296.6	423.8	30
6-9	16.9	115.2	116.1	30
9-12	1.9	43.6	29.0	30
12-22	0.1	6.0	2.57	30

Table 3.3.4: Comparison 3 of simulated fair prices, 5yrs.

dependence parameter tended to deflate the highest tranche prices and cause a mixture of changes in the lower tranches.

TRANCHE	GOFM	LOG	LOG SQ	SIM CORR
%	bps	bps	bps	%
0-3	18.9%	13.8%	16.3%	30
3-6	159.7	268.8	199.5	30
6-9	16.9	96.2	57.7	30
9-12	1.9	34.2	20.0	30
12-22	0.1	4.3	3.1	30

Table 3.3.5: Comparison 4 of simulated fair prices, 5yrs.

The horizon time in comparison 5, Table 3.3.6, comparison 6, Table 3.3.7 and comparison 7, Table 3.3.8 is one year.

TRANCHE	GOFM	DURL	PWLD	SIM CORR
%	bps	bps	bps	%
0-3	18.76%	16.51%	17.22%	45
3-6	67.13	194.75	160.99	45
6-9	9.75	50.25	33.73	45
9-12	1.95	15.03	8.81	45
12-22	0.18	2.12	1.08	45

Table 3.3.6: Comparison 5 of simulated fair prices, 1yr.

In comparison 6, the average credit spread $spd = 35$ bps and the dependence parameter $\rho = 0.45$. The parameters in the Durrleman (D3) distortion were set the same as in the previous example, whereas in the multiknot distortion, $\beta = [0.17\ 0.23\ 0.35\ 0.6]$ and $\eta = [0.55\ 0.7\ 0.85\ 0.9]$.

The same log distortions with $\gamma = 5$ are used in comparison 8, Table 3.3.9, however the $spd = 35$ bps.

In comparisons 1 and 2, the distortions increase all prices except for the lowest tranche which is decreased. The combination of log and square root distortions

TRANCHE	GOFM	DURL	PWLD	SIM CORR
%	bps	bps	bps	%
0-3	16.34%	14.60%	14.18%	45
3-6	49.60	153.39	181.52	45
6-9	6.78	36.93	43.38	45
9-12	1.29	10.48	12.49	45
12-22	0.11	1.40	1.71	45

Table 3.3.7: Comparison 6 of simulated fair prices, 1yr.

TRANCHE	GOFM	LOG	LOG SQ	SIM CORR
%	bps	bps	bps	%
0-3	18.76%	16.80%	17.45%	45
3-6	67.13	178.66	133.33	45
6-9	9.75	44.88	36.40	45
9-12	1.95	13.099	11.83	45
12-22	0.18	1.79	1.78	45

Table 3.3.8: Comparison 7 of simulated fair prices, 1yr.

TRANCHE	GOFM	LOG	LOG SQ	SIM CORR
%	bps	bps	bps	%
0-3	16.34%	14.83%	15.28%	45
3-6	49.60	140.59	107.10	45
6-9	6.78	32.93	27.9	45
9-12	1.29	9.12	8.66	45
12-22	0.11	1.18	1.22	45

Table 3.3.9: Comparison 8 of simulated fair prices, 1yr.

was able to decrease the value of both of the lowest CDO tranches. This was not a typical result. In most cases, the distortions increased all tranche values except for the equity tranche, relative to the tranche values of the GOFM.

Many other combinations of distortions could be applied to the copulas and may give quite promising results. The piecewise linear distortion has an extra advantage in that it can be fitted to any other distortion. However, some of the distortions may be faster to compute or easier to fit to data. The marginal distributions may also be changed, for example we could assume that F_2 is a Skew t distribution or exponential distribution.

3.3.2 Model 2: Gibson Algorithm with Recursion

One of the assumptions of the previous model is that as the number of risky assets in the portfolio gets large, the total portfolio loss converges to the conditional Gaussian copula formula originally used. The distortion is then used to induce a skew in the portfolio loss distribution. Some may find it undesirable to use the distortion globally after the asymptotics have been finalized in a theoretical setting. Therefore, we present another method, that in [61] combined with the recursion formula described in [5], since no such assumptions are made. This method has capacity to handle a variety of individual conditional default distributions if required. The original recursion formula taken from [5] was described briefly in Chapter 1. Since the author uses vastly different notation in [61] to that used in [5], the formulae is redescribed below.

Let the cumulative distribution F_2 be associated with global factor be V and the default time probability distribution $Q_i(t)$, $i = 1, 2, \ldots, N$ be associated with the default time τ of each credit default swap. The distorted copula represents the joint probability that the credit default swap defaults before time t and the global factor is less than some value v,

$$\Pr\{\tau \leq t, V \leq v\} = C^{\psi}(Q_i(t), F_2(v)).$$

The default probability for each credit default swap, conditional on V, be $q_{i,t}^V$, $i = 1, \ldots, N$. The In the original GOFM, $Q_i(t \mid V)$ is set to the conditional Gaussian copula. In this model we apply a distortion function to the conditional Gaussian copula in the same way as Model 1, so

$$q_{i,t}^V = \frac{\Phi\left(\frac{\Phi^{-1}(\psi[Q_i(t)])-\rho\Phi^{-1}(\psi[F_2(v)])}{\sqrt{1-\rho^2}}\right)\psi'(F_2(v))}{\psi'(\psi^{(-1)}[\Phi_2(\Phi^{-1}(\psi[Q_i(t)]), \Phi^{-1}(\psi[F_2(v)]); \rho)])}, \tag{3.3.10}$$

Note that the distribution $F_1(a)$, individual credit default swap default probability of Model 1 is replaced by the distribution, $Q_i(t)$ for all items in the portfolio, $i = 1, 2, \ldots, N$. Both distributions are exponential in nature, in particular, given the hazard rate h, $Q_i(t) = 1 - \exp(-ht)$.

In order to keep the model simple, one distortion will be applied per portfolio simulation, so that all distributions $q_{i,t}^V$ for $i = 1, \ldots, N$, will be equal. In a portfolio of size k, let $p^k(j, t \mid V)$ be the probability that exactly j defaults occur by time t, conditional on V. The idea is to assume one knows the default distribution of k credit default swaps and denote this

$$p^k(j, t \mid V), \quad j = 0, \ldots, k. \tag{3.3.11}$$

Adding one more credit default swap, gives us the default distribution for $k + 1$ swaps.

$$\begin{aligned} p^{k+1}(0, t \mid V) &= p^k(0, t \mid V)[1 - q_{k+1,t}^V] \\ p^{k+1}(j, t \mid V) &= p^k(j, t \mid V)[1 - q_{k+1,t}^V] + p^k(j-1, t \mid V)q_{k+1,t}^V \\ &\quad j = 1, \ldots, k \\ p^{k+1}(k+1, t \mid V) &= p^k(k, t \mid V)q_{k+1,t}^V. \end{aligned} \tag{3.3.12}$$

Initially, we choose $p^0(0, t \mid V) = 1$ for $k = 0$, then solve $p^N(j, t \mid V)$ for $j = 0, \ldots, N$ credit default swaps. Suppose that f_2 is the probability density associated with V, then the unconditional default distribution (that j defaults occur by time t) is

$$p(j,t) = \int_{-\infty}^{\infty} p^N(j,t \mid V) f_2(V) dV. \tag{3.3.13}$$

The integral above has to be solved numerically. Given that $f_2(V)$ was assumed to be standard normal, we chose to use a Gaussian Quadrature formula to obtain a solution.

Payments are calculated quarterly. Expected tranche loss up to payment time T_i is

$$\mathbb{E}[L_i] = \sum_{j=0}^{N} p(j,T_i) \max\{\min\{jA(1-R), K_U\} - K_L, 0\}, \tag{3.3.14}$$

where R is the recovery rate, $N = 100$ is the number of credit default swaps, A_i is notional amount of credit default swap i. Therefore, the loss from the i-th default is $A_i(1-R)$. The upper and lower tranche attachment points are K_U and K_L, as in the previous model. The Protection or *Contingent* Leg, Ct, which is the expected discounted payment the tranche investor must make when defaults impact on the tranche, is

$$\mathrm{Ct} = \sum_{i=1}^{N} D_i(\mathbb{E}[L_i] - \mathbb{E}[L_{i-1}]), \tag{3.3.15}$$

where D_i is the risk-free discount factor for payment date i.

The Default Leg, $DefL$, (the fee) the tranche investor receives for providing a type of insurance is

$$\mathrm{DefL} = s \sum_{i=1}^{n} D_i \Delta_i((K_U - K_L) - \mathbb{E}[L_i]), \tag{3.3.16}$$

where $\Delta_i \approx T_i - T_{i-1}$ and s is the spread per annum paid to the investor. Given that MTM = Default Leg - Contingent, and setting $MTM = 0$ results in

$$s_{par} = \frac{\mathrm{Ct}}{\sum_{i=1}^{n} D_i \Delta_i((K_U - K_L) - \mathbb{E}[L_i])} \tag{3.3.17}$$

Results of Model 2

Parameters were initially identical to those in [61], that is

- Single-name spread = 60 bps
- Notional amount per credit = \$10 Million
- Recovery Rate = 40%
- Default hazard rate = 1%, so $q_i(t) = 1 - \exp(-0.01t)$ and
- Asset dependence (correlation) parameter = 30%
- Maturity date = 5 years
- Total number of credit default swaps, N = 100
- Constant interest rate = 5% (continuously compounded)

It was possible to replicate the tranche fair prices (par spread) in [61] with good accuracy, so that a comparison could be made between prices from the Gaussian Copula and those from the distorted Gaussian Copula. To be specific, the original tranche fair prices in the Gibson paper were 1507 (tranche 0-3), 315 (tranche 3-10)and 7(tranche 10-100) bps, which are close to the values in column 1 (GOFM) of Table 3.3.10. Results for three tranche levels and GOFM, Durrleman (D3), Table 3.2.1 and Log distortion (M1), Table 3.2.3 models are shown in Table 3.3.10. Parameter values in the Durrleman distortion (D3) of the Gaussian copula

TRANCHE	GOFM	DURL	LOG	SIM CORR
bps	bps	bps	bps	%
0-3	1512.3	1061.4	1223.5	30
3-10	314.9	330.6	322.5	30
10-100	7.4	14.5	11.9	30

Table 3.3.10: Comparison 1 of simulated fair prices using recursion, 5yrs.

(DURL) were $\beta_1 = 1.0$ and $\beta_2 = 0.5$, and that in the log distortion (M1) was $\gamma = 3.0$. These distortions cause changes in the weighting of the tranche prices in the CDO. More specifically, the highest tranche is given much greater value. In Table 3.3.11, we compare GOFM and two different piecewise linear distortion models. Parameters for PWLD1 were β =[0.25 0.3 0.5 0.6] and η = [0.35 0.4 0.6 0.7], and for PWLD2 the values were β =[0.2 0.3 0.7 0.8] and η = [0.61 0.65 0.8 0.9]. The second piecewise linear distortion adds considerable value to the higher tranche prices, while taking considerable weight out of the lowest tranche. The highest tranche increases in value in this case, while the lowest tranche decreases.

TRANCHE	GOFM	PWLD1	PWLD2	SIM CORR
bps	bps	bps	bps	%
0-3	1512.3	1227.4	730.9	30
3-10	314.9	332.8	392.4	30
10-100	7.4	10.0	19.2	30

Table 3.3.11: Comparison 2 of simulated fair prices using recursion, 5yrs.

TRANCHE	GOFM	DURL	LOG	SIM CORR
bps	bps	bps	bps	%
0-3	1105.2	793.9	903.6	45
3-10	326.3	325.0	324.0	45
10-100	13.8	20.4	18.0	45

Table 3.3.12: Comparison 3 of simulated fair prices using recursion, 5yrs.

The effect of the distortions in comparison 5 and 6 are similar to that of the previous tables.

In the following tables the maturity to one year, so that more comparisons could be made. In most cases the highest two tranches increase in value, while the

TRANCHE	GOFM	PWLD1	PWLD2	SIM CORR
bps	bps	bps	bps	%
0-3	1105.2	904.4	602.3	45
3-10	326.3	340.8	369.9	45
10-100	13.8	16.4	24.0	45

Table 3.3.13: Comparison 4 of simulated fair prices using recursion, 5yrs.

TRANCHE	GOFM	DURL	LOG	SIM CORR
bps	bps	bps	bps	%
0-3	2078.5	1438.5	1668.3	15
3-10	272.5	323.6	305.6	15
10-100	1.9	8.1	5.5	15

Table 3.3.14: Comparison 5 of simulated fair prices using recursion, 5yrs.

TRANCHE	GOFM	PWLD1	PWLD2	SIM CORR
bps	bps	bps	bps	%
0-3	2078.5	1680.9	932.4	15
3-10	272.5	302.0	396.6	15
10-100	1.9	3.6	13.1	15

Table 3.3.15: Comparison 6 of simulated fair prices using recursion, 5yrs.

TRANCHE	GOFM	DURL	LOG	SIM CORR
bps	bps	bps	bps	%
0-3	1802.2	1438.3	1563.8	30
3-10	152.1	244.7	213.0	30
10-100	1.4	4.2	3.2	30

Table 3.3.16: Comparison 7 of simulated fair prices using recursion, 1yr.

lowest tranche decreases. If the original bank were to retain the equity tranche instead of trading it, then adjustments could be made for the decrease in tranche

TRANCHE	GOFM	PWLD1	PWLD2	SIM CORR
bps	bps	bps	bps	%
0-3	1802.2	1662.5	1408.9	30
3-10	152.1	184.2	270.2	30
10-100	1.4	2.1	4.5	30

Table 3.3.17: Comparison 8 of simulated fair prices using recursion, 1yr.

TRANCHE	GOFM	DURL	LOG	SIM CORR
bps	bps	bps	bps	%
0-3	1268.2	1089.9	1198.6	45
3-10	273.3	310.1	286.4	45
10-100	6.8	9.1	7.7	45

Table 3.3.18: Comparison 9 of simulated fair prices using recursion, 1yr.

price. At the same time, the higher tranches which would be traded, would not be underpriced like they are when the Gaussian copula is used by itself. It should be acknowledged that the results would differ slightly if the model were calibrated to real data.

TRANCHE	GOFM	PWLD1	PWLD2	SIM CORR
bps	bps	bps	bps	%
0-3	1268.2	1304.1	1047.9	45
3-10	273.3	264.0	329.2	45
10-100	6.8	6.4	9.6	45

Table 3.3.19: Comparison 10 of simulated fair prices using recursion, 1yr.

TRANCHE	GOFM	DURL	LOG	SIM CORR
bps	bps	bps	bps	%
0-3	2144.5	1830.8	1952.9	15
3-10	53.4	152.7	115.0	15
10-100	0.0	0.6	0.3	15

Table 3.3.20: Comparison 11 of simulated fair prices using recursion, 1yr.

TRANCHE	GOFM	PWLD1	PWLD2	SIM CORR
bps	bps	bps	bps	%
0-3	2144.5	1999.0	1763.3	15
3-10	53.4	81.7	187.4	15
10-100	0.0	0.1	0.7	15

Table 3.3.21: Comparison 12 of simulated fair prices using recursion, 1yr.

3.4 Conclusion

We have been able to incorporate several distortion functions into existing synthetic CDO tranche pricing systems. In all cases, the distortion functions prove to be a good tool for inducing a fat tail in the portfolio loss distribution. In framework 1 (JP Morgan), we simulated tranche prices similar to those of iTraxx series 3 and 4 in the case of $h = 1$ year. In framework 2 (recursion method) it was possible to apply the distortions to individual credit default swap conditional default probabilities and induce a fat tail in the final CDO tranche prices. The second of the piecewise linear distortions used in framework 2 was one of the best for shifting the fair value of the tranches. Framework 2 is preferable to framework 1, since it is more flexible and makes less assumptions.

There are many other distortions which could be tried in the future and may prove useful in this field. These functions have the potential to enable analysts to price credit risk with greater accuracy. Furthermore, the combination of copulas and distortion functions could be used to price a variety of credit derivatives, or even to create new financial products.

Chapter 4

Constructing n-Copulas

We have seen many classes of bivariate copulas in Chapter 1, however, there are situations in which we may want a joint distribution between three or more variables.

When one has several random variables of interest, it is generally easy to fit each variable to a univariate distribution. Choosing a joint distribution or dependence structure for all the variables, however, is not so easy and the choices are not so obvious. If each variable has a standard normal distribution and the correlation matrix is known, then their joint distribution may be assumed to be multivariate normal. In many physical situations, however, variables may follow extreme value distributions or a mixture of distributions such that no obvious joint distribution fits the information available. One choice of dependence structure in these situations is a n-copula. The construction of new classes of n-copulas is a relatively new area, so that there are not many methods or examples described in the literature. The purpose of this chapter is to combine more than one method of constructing n-copulas in order to produce new examples. We also build on the results described in [90].

4.1 Background

The following new class of bivariate copulas is described in [132]:

$$C_\theta(u_1, u_2) = u_1 u_2 + \theta f(u_1) g(u_2), \quad \text{for all} \quad u_1, u_2 \in [0, 1], \tag{4.1.1}$$

where f and g are non-zero real functions with the following properties

1. f, g are absolutely continuous,
2. $f(0) = f(1) = g(0) = g(1) = 0$,
3. $\theta \in [-1/\max\{\alpha\gamma, \beta\delta\}, -1/\min\{\alpha\delta, \beta\gamma\}]$

and

$$\begin{aligned} & \min\{\alpha\delta, \beta\gamma\} \geq -1 \\ \alpha &= \inf\{f'(u_1) : u_1 \in A\} < 0, \\ \beta &= \sup\{f'(u_1) : u_1 \in A\} > 0, \\ \gamma &= \inf\{g'(u_2) : u_2 \in B\} < 0, \\ \delta &= \sup\{g'(u_2) : u_2 \in B\} > 0, \\ A &= \{u_1 \in [0, 1] : f'(u_1) \quad \text{exists}\}, \\ B &= \{u_2 \in [0, 1] : f'(u_2) \quad \text{exists}\}. \end{aligned}$$

A typical example of f is $f(u) = u(1 - u)$ which is an inverted parabola. The Farlie-Gumbel-Morgenstern family of copulas fits into this class of copulas. The bivariate copulas are generalized further in [90], so that one may incorporate copulas which are vastly different from the Farlie-Gumbel- Morgenstern family. The first generalization of bivariate copulas is

$$C_\theta(u_1, u_2) = C^*(u_1, u_2) + f(u_1) g(u_2), \quad \text{for all} \quad u_1, u_2 \in [0, 1], \tag{4.1.2}$$

where f and g are absolutely continuous and

$$\min\{\alpha\delta, \beta\gamma\} \geq -\frac{\Delta^*}{(u_b - u_a)(v_b - v_a)} \tag{4.1.3}$$

such that $u_a, u_b, v_a, v_b \in [0,1]$. Δ^* is the volume of $C^*(u_1, u_2)$, that is

$$\Delta^* = C^*(u_b, v_b) - C^*(u_b, v_a) - C^*(u_a, v_b) + C^*(u_a, v_a). \tag{4.1.4}$$

The second generalization is

$$C_\theta(u_1, u_2) = C^*(u_1, u_2) + \theta f(u_1) g(u_2), \quad \text{for all} \quad u_1, u_2 \in [0,1], \tag{4.1.5}$$

where $C^*(u_1, u_2)$ is any known bivariate copula and f and g have the same properties as those listed previously.

Note 1: The following condition on θ ensures that C_θ is a copula

$$\theta \geq \frac{-\Delta^*}{f'(u_1) g'(u_2)}, \quad f'(u_1) g'(u_2) \geq 0. \tag{4.1.6}$$

4.2 Method 1 for n-copula Construction

Authors in [30] have recently extended the class of copulas shown in equation (4.1.1) to n dimensions:

$$C_\theta(\mathbf{u}) = \prod_{i=1}^{n} u_i + \theta \prod_{i=1}^{n} f_i(u_i), \quad \text{for all} \quad \mathbf{u} \in [0,1]^n, \tag{4.2.1}$$

where $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and the component functions f have the same properties as those in the bivariate case, and the admissible range of θ is

$$-1/\sup_{u \in D^+} \left(\prod_{i=1}^{n} f_i'(u_i) \right) \leq \theta \leq -1/\inf_{u \in D^-} \left(\prod_{i=1}^{n} f_i'(u_i) \right), \tag{4.2.2}$$

where

$$D^- = \{\mathbf{u} \in [0,1]^n : \prod_{i=1}^{n} f_i'(u_i) < 0\}$$
$$D^+ = \{\mathbf{u} \in [0,1]^n : \prod_{i=1}^{n} f_i'(u_i) > 0\}.$$

4.3 Bivariate Copulas Containing Distortions

In Chapter 1 and Chapter 3 we described distortions which can be applied to known copulas in order to obtain a new ones:

$$C^{\psi}(u_1, u_2) = \psi^{[-1]}\left(C(\psi(u_1), \psi(u_2))\right), \quad \text{for all} \quad u_1, u_2 \in [0,1],$$

provided that

1. ψ is concave;
2. ψ is strictly increasing;
3. ψ is continuous and twice differentiable, and
4. $\psi(0) = 0$ and $\psi(1) = 1$

on the interval $[0,1]$. In this section we suggest an alternative method of obtaining new copulas by combining distortions with known copulas. We obtained the idea from [90], in which the authors suggest that a copula may be constructed from univariate probability distributions in the following way; Suppose that F_1, F_2, G_1 and G_2 are univariate distribution functions. If

$$f(u_1) = \frac{F_1(u_1)}{F_1(1)} - \frac{F_2(u_1)}{F_2(1)} \quad \text{and} \quad g(u_2) = \frac{G_1(u_2)}{G_1(1)} - \frac{G_2(u_2)}{G_2(1)}, \tag{4.3.1}$$

then

$$C(u_1, u_2) = C^*(u_1, u_2) + \left\{\frac{F_1(u_1)}{F_1(1)} - \frac{F_2(u_1)}{F_2(1)}\right\}\left\{\frac{G_1(u_2)}{G_1(1)} - \frac{G_2(u_2)}{G_2(1)}\right\} \tag{4.3.2}$$

is a copula. We noticed that non-negative univariate probability distribution functions have similar properties to distortion functions. Therefore we propose

Theorem 4.1. Given a known absolutely continuous copula $C^*(u_1, u_2)$ and distinct distortion functions ψ_1 and ψ_2, then

$$C(u_1, u_2) = C^*(u_1, u_2) + [\psi_1(u_1) - \psi_2(u_1)][\psi_1(u_2) - \psi_2(u_2)] \quad \text{for all} \quad u_1, u_2 \in [0, 1] \tag{4.3.3}$$

is also a copula, given the conditions below, and assuming that

$$[\psi_1(u_1) - \psi_2(u_1)][\psi_1(u_2) - \psi_2(u_2)] \leq \min\{u_1, u_2\} - C^*(u_1, u_2). \tag{4.3.4}$$

Assume also that the known copula is continuous, twice differentiable and the distortions are at least once differentiable on $[0, 1]$, then the density of the new copula is

$$\nabla_{u_1,u_2} C(u_1, u_2) = \nabla_{u_1,u_2} C^*(u_1, u_2) + [\psi_1'(u_1) - \psi_2'(u_1)][\psi_1'(u_2) - \psi_2'(u_2)], \tag{4.3.5}$$

which is non-negative if

$$[\psi_1'(u_1) - \psi_2'(u_1)][\psi_1'(u_2) - \psi_2'(u_2)] \geq -\nabla_{u_1,u_2} C^*(u_1, u_2). \tag{4.3.6}$$

Proof of general copula properties:

$$\begin{aligned}
C(0, u_2) &= C^*(0, u_2) + [\psi_1(0) - \psi_2(0)][\psi_1(u_2) - \psi_2(u_2)] \\
&= 0 + (0 - 0)[\psi_1(u_2) - \psi_2(u_2)] \\
&= 0, \\
C(u_1, 0) &= C^*(u_1, 0) + [\psi_1(u_1) - \psi_2(u_1)][\psi_1(0) - \psi_2(0)] \\
&= 0 + [\psi_1(u_1) - \psi_2(u_1)](0 - 0) \\
&= 0
\end{aligned}$$

and

$$\begin{aligned} C(1,u_2) &= C^*(1,u_2) + [\psi_1(1) - \psi_2(1)][\psi_1(u_2) - \psi_2(u_2)] \\ &= u_2 + (1-1)[\psi_1(u_2) - \psi_2(u_2)] \\ &= u_2, \\ C(u_1,1) &= C^*(u_1,1) + [\psi_1(u_1) - \psi_2(u_1)][\psi_1(1) - \psi_2(1)] \\ &= u_1 + [\psi_1(u_1) - \psi_2(u_1)](1-1) \\ &= u_1. \end{aligned}$$

The simplest generalization of this new subclass of copulas is

$$C(u_1,u_2) = C^*(u_1,u_2) + \theta[\psi_1(u_1) - \psi_2(u_1)][\psi_1(u_2) - \psi_2(u_2)] \tag{4.3.7}$$

for some parameter θ such that

$$\theta \geq \frac{-\nabla_{u_1,u_2} C^*(u_1,u_2)}{[\psi_1'(u_1) - \psi_2'(u_1)][\psi_1'(u_2) - \psi_2'(u_2)]} \quad \text{and} \quad [\psi_1'(u_1) - \psi_2'(u_1)][\psi_1'(u_2) - \psi_2'(u_2)] \geq 0.$$

In this context we only require that ψ is once differentiable and may be able to relax the concavity condition as long as the derivatives of the distortions satisfy the condition shown above. In this case the relationship between Spearman's ρ of the new copula and that of the given copula $\tilde{\rho}_s$ is

$$\rho_s = \tilde{\rho}_s + 12 \int_0^1 \{\psi_1(u_1) - \psi_2(u_1)\} du_1 \int_0^1 \{\psi_1(u_2) - \psi_2(u_2)\} du_2.$$

Example 4.1.

$$C(u_1,u_2) = u_1 u_2 + \left\{ \frac{\ln(\gamma u_1 + 1)}{\ln(\gamma + 1)} - u_1 \right\} \left\{ \frac{\ln(\gamma u_2 + 1)}{\ln(\gamma + 1)} - u_2 \right\}, \quad \gamma \in (0, 5.2]$$

and

$$\rho_s = 12 \left[\frac{1}{2} + \frac{\ln(\gamma+1) - \gamma}{\gamma \ln(\gamma+1)} \right]^2.$$

Example 4.2.

$$C(u_1, u_2) = \Phi_2(\Phi^{-1}(u_1), \Phi^{-1}(u_2); \rho) + [\sin(\frac{\pi}{2}u_1) - u_1][\sin(\frac{\pi}{2}u_2) - u_2],$$
$$\rho \in [-0.6, 0], \tag{4.3.8}$$

where $\Phi_2(\cdot)$ is the bivariate normal, and $\Phi^{-1}(\cdot)$ inverse univariate normal distribution, and

$$\rho_s = \frac{6}{\pi} \arcsin(\frac{\rho}{2}) + 12\left(\frac{2}{\pi} - \frac{1}{2}\right)^2.$$

Figure 4.3.1 shows the density of the copula in Example 4.2, with $\rho = -0.1$.

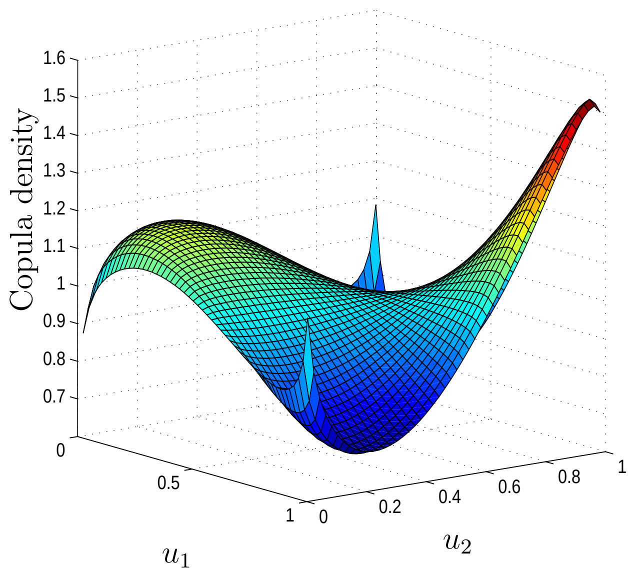

Figure 4.3.1: Copula density of distribution in (4.3.8)

Example 4.3.

$$C(u_1, u_2) = u_1u_2 + \left\{\frac{1 - e^{-\lambda u_1}}{1 - e^{-\lambda}} - u_1\right\}\left\{\frac{1 - e^{-\lambda u_2}}{1 - e^{-\lambda}} - u_2\right\}, \quad \lambda \in (0, 2.1]$$

and

$$\rho_s = 12\left[\frac{2 - e^{-\lambda}}{-\lambda(1 - e^{-\lambda})} - \frac{1}{2}\right]^2.$$

An example in which the distortion is convex is

Example 4.4.

$$C(u_1, u_2) = u_1u_2 + \theta \left\{u_1 - \frac{u_1}{2-u_1}\right\}\left\{u_2 - \frac{u_2}{2-u_2}\right\}, \quad \theta \in [-1, 2] \qquad (4.3.9)$$

and

$$\rho_s = 12\theta\left(\frac{3}{2} - 2\ln(2)\right)^2.$$

Example 4.5.

$$\begin{aligned} C(u_1, u_2) &= \Phi_2(\Phi^{-1}(u_1), \Phi^{-1}(u_2); \rho) + \left\{u_1 - \frac{u_1}{2-u_1}\right\}\left\{u_2 - \frac{u_2}{2-u_2}\right\} \\ & \quad \rho \in [-0.6, 0.1] \end{aligned}$$

and

$$\rho_s = \frac{6}{\pi}\arcsin(\frac{\rho}{2}) + 12\left(\frac{3}{2} - 2\ln(2)\right)^2.$$

Example 4.6.

$$\begin{aligned} C(u_1, u_2) &= \left\{\frac{\ln(\gamma_1 u_1 + 1)}{\ln(\gamma_1 + 1)} - \frac{\ln(\gamma_2 u_1 + 1)}{\ln(\gamma_2 + 1)}\right\}\left\{\frac{\ln(\gamma_1 u_2 + 1)}{\ln(\gamma_1 + 1)} - \frac{\ln(\gamma_2 u_2 + 1)}{\ln(\gamma_2 + 1)}\right\} \\ &+ \Phi_2(\Phi^{-1}(u_1), \Phi^{-1}(u_2); \rho). \end{aligned}$$

The range of suitable values for ρ will depend on the ranges for γ_1 and γ_2. Both γ_1 and γ_2 should be at least $n > 0$. The range for parameters such as ρ was calculated numerically in these examples. Figure 4.3.2 shows the density of the copula in Example 4.4, with $\theta = 0.5$.

4.3.1 Other Subclasses of Copulas

Another method of obtaining a new copula is

$$C(u_1, u_2) = \psi_1^{[-1]}[C^*(\psi_1(u_1), \psi_1(u_2))] + [\psi_2(u_1) - \psi_3(u_1)][\psi_2(u_2) - \psi_3(u_2)],$$

for all $u_1, u_2 \in [0, 1]$. In this case, at least ψ_1 must be concave.

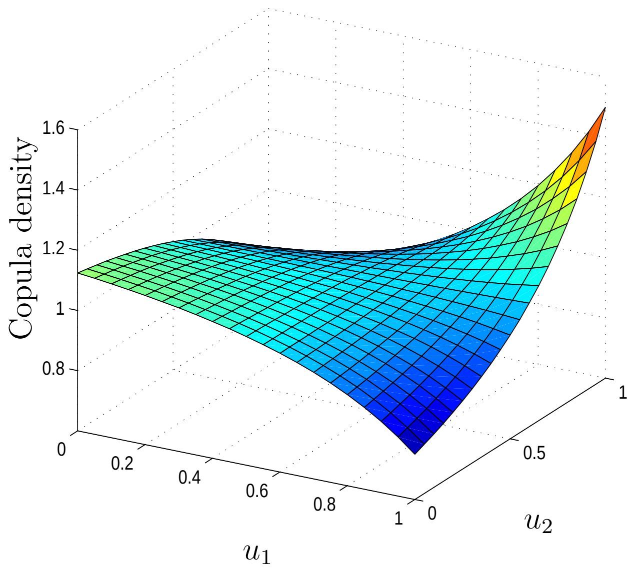

Figure 4.3.2: Copula density for distribution in (4.3.9)

4.4 Method 2 for n-copula Construction

A method by which lower dimensional copulas can be used to construct higher dimensional ones was described in [129] and [24]. We now apply that method to a combination of bivariate copulas in order to obtain 3-copulas.

Example 4.7. Suppose we have two specified bivariate copulas such that

$$\begin{aligned} C(u_1,1,u_3) &= C_{13}(u_1,u_3), \quad C(u_1,u_2,1) = C_{12}(u_1,u_2), \\ C_{12}(u_1,u_2) &= u_1u_2 + \theta_1 u_1(1-u_1)u_2(1-u_2), \quad \theta_1 \in [-1,1] \quad \text{and} \\ C_{13}(u_1,u_3) &= u_1u_3 + \theta_2 \left[\sin(\frac{\pi}{2}u_1) - u_1\right]\left[\sin(\frac{\pi}{2}u_3) - u_3\right], \quad \theta_2 \in [-1,1.7]. \end{aligned}$$

Now, if we let $C_{23}(x,y) = xy$, for all $x,y \in [0,1]$, then we may use

$$C(u_1,u_2,u_3) = \int_0^{u_1} C_{23}(\nabla_{u_1}C_{12}(t,u_2), \nabla_{u_1}C_{13}(t,u_3))dt$$

to construct the 3-copula

$$\begin{aligned} C(u_1,u_2,u_3) = &\theta_1 u_1(1-u_1)u_2(1-u_2)u_3 + \theta_1\theta_2 u_2(1-u_2)\left[\sin(\frac{\pi}{2}u_3) - u_3\right] \\ + &\ u_1u_2u_3 + \theta_2 u_2 \left[\sin(\frac{\pi}{2}u_1) - u\right]\left[\sin(\frac{\pi}{2}u_3) - u_3\right] \times \vartheta, \end{aligned}$$

where

$$\vartheta = \left[\sin(\frac{\pi}{2}u_1)(1-2u_1) - \frac{4}{\pi}\{\cos(\frac{\pi}{2}u_1)-1\} - u_1(1-u_1)\right].$$

Example 4.8. Let $C_{23}(x,y) = xy$,

$$C_{12}(u_1,u_2) = u_1u_2 + \left\{\frac{\ln(\gamma u_1+1)}{\ln(\gamma+1)} - u_1\right\}\left\{\frac{\ln(\gamma u_2+1)}{\ln(\gamma+1)} - u_2\right\}, \quad \gamma \in (0, 5.2]$$

and

$$C_{13}(u_1,u_3) = u_1u_3 + \left\{\frac{\ln(\gamma u_1+1)}{\ln(\gamma+1)} - u_1\right\}\left\{\frac{\ln(\gamma u_3+1)}{\ln(\gamma+1)} - u_3\right\}, \quad \gamma \in (0, 5.2].$$

then

$$C(u_1,u_2,u_3) = u_1u_2u_3 + \left(\frac{\ln(\gamma u_2+1)}{\ln(\gamma+1)} - u_2\right)\left(\frac{\ln(\gamma u_3+1)}{\ln(\gamma+1)} - u_3\right) \times Z$$
$$+ \left[\frac{\ln(\gamma u_1+1)}{\ln(\gamma+1)} - u_1\right]\left[u_2\left(\frac{\ln(\gamma u_3+1)}{\ln(\gamma+1)} - u_3\right) + u_3\left(\frac{\ln(\gamma u_2+1)}{\ln(\gamma+1)} - u_2\right)\right],$$

where

$$Z = \left\{\frac{\gamma^2 u_1}{[\ln(\gamma+1)]^2(\gamma u_1+1)} - 2\frac{\ln(\gamma u_1+1)}{\ln(\gamma+1)} + u_1\right\}.$$

Example 4.9. Similarly, let $C_{12}(u_1,u_2) = u_1u_2 + \theta_1 u_1(1-u_1)u_2(1-u_2)$, with $\theta_1 \in [-1,1]$,

$$C_{23}(x,y) = xy \text{ and } C_{13}(u_1,u_3) = u_1u_3 + \theta_2\left\{\frac{1-e^{-\lambda u_1}}{1-e^{-\lambda}} - u_1\right\}\left\{\frac{1-e^{-\lambda u_3}}{1-e^{-\lambda}} - u_3\right\},$$

where

$$\theta_2 \geq \frac{-(1-e^{-\lambda})^2}{(\lambda e^{-\lambda u_1} - 1 + e^{-\lambda})\lambda e^{-\lambda u_2} - 1 + e^{-\lambda})},$$

then

$$\begin{aligned} C(u_1,u_2,u_3) &= u_1u_2u_3 + \theta_1 u_3 u_1(1-u_1)u_2(1-u_2) + \theta_1\theta_2 u_2(1-u_2)Y \\ &+ \theta_2 u_2\left[\frac{1-e^{-\lambda u_1}}{1-e^{-\lambda}} - u_1\right]\left[\frac{1-e^{-\lambda u_3}}{1-e^{-\lambda}} - u_3\right], \end{aligned}$$

where

$$Y = \left(\frac{1-e^{-\lambda u_3}}{1-e^{-\lambda}} - u_3\right)\left[u_1(u_1-1) + \frac{(2u_1-1)e^{-\lambda u_1}+1}{1-e^{-\lambda}} - \frac{2(1-e^{-\lambda u_1})}{\lambda(1-e^{-\lambda})}\right].$$

Example 4.10. Let $C_{12}(u_1,u_2) = u_1u_2 + \theta_1 u_1(1-u_1)u_2(1-u_2), \quad \theta_1 \in [-1,1]$,

$$\begin{aligned} C_{23}(x,y) &= xy \quad \text{and} \\ C_{13}(u_1,u_3) &= u_1u_3 + \left\{\frac{\ln(\gamma u_1+1)}{\ln(\gamma+1)} - u_1\right\}\left\{\frac{\ln(\gamma u_3+1)}{\ln(\gamma+1)} - u_3\right\}, \quad \gamma \in (0, 5.2]. \end{aligned}$$

then

$$\begin{aligned} C(u_1,u_2,u_3) = u_1u_2u_3 + \theta_1 u_3 u_1(1-u_1)u_2(1-u_2) + \theta_1 u_2(1-u_2)\Upsilon \\ + \quad u_2\left[\frac{\ln(\gamma u_1+1)}{\ln(\gamma+1)} - u_1\right]\left[\frac{\ln(\gamma u_3+1)}{\ln(\gamma+1)} - u_3\right], \end{aligned}$$

where

$$\Upsilon = \left[\frac{\ln(\gamma u_3+1)}{\ln(\gamma+1)} - u_3\right]\left\{\frac{-2\gamma u_1 + (2+\gamma)\ln(\gamma u_1+1)}{\ln(\gamma+1)} + u_1(u_1-1)\right\}. \quad (4.4.1)$$

Another interesting example based on the original class of copulas suggested in [132] is

Example 4.11. Given $n, m \in \mathbb{N}^*$, let $C_{23}(x,y) = xy$,

$$C_{12}(u_1,u_2) = u_1u_2 + \theta_1 u_1({u_1}^n - 1)u_2({u_2}^n - 1), \quad \theta_1 \in [-\frac{1}{n^2}, 1]$$

and

$$C_{13}(u_1,u_3) = u_1u_3 + \theta_2 u_1({u_1}^m - 1)u_3({u_3}^m - 1), \quad \theta_2 \in [-\frac{1}{m^2}, 1],$$

then

$$\begin{aligned} C(u_1,u_2,u_3) = u_1u_2u_3\left\{1 + \theta_1({u_2}^n-1)({u_1}^n-1) + \theta_2({u_3}^m-1)({u_1}^m-1)\right\} \\ + \quad u_1u_2u_3\left(\theta_1\theta_2({u_2}^n-1)({u_3}^m-1)\left[\frac{(n+1)(m+1)}{n+m+1}{u_1}^{n+m} - {u_1}^n - {u_1}^m + 1\right]\right), \end{aligned}$$

with

$$\rho_s = 12\theta_1\left(\frac{1}{n+2} - \frac{1}{2}\right)^2$$

and

$$\rho_s = 12\theta_2\left(\frac{1}{m+2} - \frac{1}{2}\right)^2.$$

4.5 Summary and Suggestions for Future Work

In this chapter we have extended the class of copula functions described in [90]. The new class of bivariate copulas includes a variety of distortion functions, including those which are non-negative probability distributions. We have also used a construction method to build 3-copulas from specified 2-copulas. The construction may be extended to n-dimensions. This method allows us to provide explicit examples of unique higher dimensional copulas. Given the results in [30], functions such as

Conjecture 1. Given a known copula C^* and distortions ψ_1 and ψ_2 then

$$C(u_1, u_2) = C^*(u_1, u_2) + \theta[\psi_1(u_1) - \psi_2(u_1)]^n[\psi_1(u_2) - \psi_2(u_2)]^n$$

is also a copula.

Conjecture 2. Let $\mathbf{u} = (u_1, u_2, \ldots, u_n)$. Given a known copula C^* and distortions ψ_1 and ψ_2, then

$$C(\mathbf{u}) = C^*(\mathbf{u}) + \theta \prod_{i=1}^{n} [\psi_1(u_i) - \psi_2(u_i)]$$

is also a copula.

It is also possible to derive 3-copulas which are a combination of explicit expressions in u_1, u_2, u_3, but also contain an integral component, which can be solved numerically.

Chapter 5

Time and Space Dependent Copulas

5.1 Introduction and motivation

Mapping joint probability distribution functions to copula functions is straight forward when they are static, due to Sklar's Theorem. On the other hand, mapping time dependent probability functions to copula functions is more problematic. In this chapter we

(a) review the techniques for creating time dependent copulas and

(b) extend the method described in [53], [54], since it incorporates both time and space. These equations are the first of their kind in higher dimensions, since only 2-dimensional examples have previously been described.

There are at least two areas in which the time dependent copulas of this chapter are applicable,

- *Credit derivatives.* We would assume in this application that we have a portfolio of n firms, and the stochastic process $X_i(t)$ is the value of a i-th firm's assets at time t. Each marginal distribution associated with $X_i(t)$ would represent the probability of the firm's value falling below some threshold, given certain information at time zero. The time varying copula would represent the evolution of the joint distribution or state of the entire portfolio.

- *Genetic drift.* For example, each $X_i(t)$ may represent the frequency of a particular gene at time t. Each marginal distribution would represent the probability that the frequency of a particular gene had fallen below some threshold. The copula would relate to the evolution of a group of genes of interest.

5.1.1 Notation and Definitions

In order to understand some of the issues surrounding the mapping of copulae to distributions it is necessary to go back to some of the basic definitions and some notation in relation to the probability distributions of interest. We will assume throughout the chapter that we have an underlying probability space $(\Omega, \mathcal{F}, \Pr)$, where Ω is a set of points ω, $\mathcal{F}$ is a σ-algebra of subsets of Ω, and Pr is a probability on $\mathcal{F}$.

Let $X_k(t_k) : T \times \Omega \to \mathbb{R}$ be the k–th stochastic process for $t_k \in T$, where $T \subset \mathbb{R}$ is an interval of time. The notation used for a transitional probability function in this chapter is

$$\Pr\{X_k(t_i) \leq x_i \mid X_k(t_j) = x_j\} = F(t_i, x_i \mid t_j, x_j), \quad t_i > t_j. \tag{5.1.1}$$

If $t_j = 0$ then it is quite common to suppress the zero and so the notation the distribution in this case would be $F(t_i, x_i \mid x_0^j)$.

Definition 5.0. *Filtration (adapted process).* A filtration on $(\Omega, \mathcal{F})$ is a family $\mathbb{F} = (\mathcal{F}_t)_{t \in T}$ of σ-algebras $\mathcal{F}_t \subset \mathcal{F}$ such that for $s < t$, $\mathcal{F}_s \subset \mathcal{F}_t$, [145].

A common filtration is the Brownian Filtration, which will be assumed to be the default filtration in Subsection 5.1.5 and Section 5.2.

5.1.2 Method of Darsow et al

Authors in [24] were the first to attempt to map a transitional probability function to a copula. To understand this mapping we introduce the Markov Property and Process.

Definition 5.1. *Markov Property.* A stochastic process $X_i(t)$ with $a \leq t \leq b$ is said to satisfy the Markov property if for any $a \leq t_1 \leq t_2 \ldots, \leq t_n \leq t$, the equality

$$\Pr\{X_i(t) \leq x_i \mid X_i(t_1), X_i(t_2), \ldots, X_i(t_n)\} = \Pr\{X_i(t) \leq x_i \mid X_i(t_n)\}$$

holds for any $x_i \in \mathbb{R}$. A stochastic process is called a continuous-time *Markov Process* if it satisfies the Markov property described in Definition 5.1. The following notation will be used for an unconditional cumulative probability function at time $t_i \geq 0$

$$\Pr\{X_i(t_i) \leq x_i\} = F_{t_i}(x_i) \tag{5.1.2}$$

for a stochastic process $X_i(t_i)$ and $x_i \in \mathbb{R}$, and also let

$$\nabla_{x_i} F = \frac{\partial F}{\partial x_i},$$

then the corresponding density function f in this case is such that

$$f(x_i) = \nabla_{x_i} F(x_i).$$

Provided F_{t_j} is continuous and C is at least once differentiable, then a univariate transitional probability function F (Markov process) can be mapped to a bivariate copula C by setting

$$F(t_i, x_i \mid t_j, x_j) = \nabla_{u_2} C\left(F_{t_i}(x_i), F_{t_j}(x_j)\right), \tag{5.1.3}$$

where ∇_{u_2} is the partial derivative with respect to the second argument in C [24]. Note that $u_2 = F_{t_j}(x_j)$. Thus, the partial derivative of the copula now represents a transitional probability since the first marginal distribution is associated with time t_i and the second with time t_j. In this chapter we will assume that the random variables considered are continuous, however several results can also be formulated if we suppose a standard way for associating a unique copula to a set of discontinuous random variables [24], [145].

This method is particularly useful for building Markov chains, since this is the case in which the random variables are discontinuous. One of the most important innovations which enabled the authors [24] to link copulas to Markov processes was to introduce the idea of a copula product;

Definition 5.2. *Copula product.* Let C_a and C_b be bivariate copulas, then the product of C_a and C_b is the function $C_a * C_b : [0,1]^2 \to [0,1]$, such that

$$(C_a * C_b)(x,y) = \int_0^1 \nabla_z C_a(x,z) \nabla_z C_b(z,y) dz. \tag{5.1.4}$$

This product is essentially the copula equivalent of the Chapman-Kolmogorov equation, as stated in Theorem 3.2 of [24]. We restate that theorem here (with modified notation).

Theorem 3.2. Let $X_i(t)$, $t \in T$ be a real stochastic process, and for each $s, t \in T$ let C_{st} denote the copula of the random variables $X_i(s)$ and $X_i(t)$. The following are equivalent:

1. The transition probabilities $F(t, \mathbf{A} \mid s, x_s) = \Pr\{X_i(t) \in \mathbf{A} \mid X_i(s) = x_s\}$ of the process satisfy the Chapman-Kolmogorov equations

$$F(t, \mathbf{A} \mid s, x_s) = \int_{\mathbb{R}} F(t, \mathbf{A} \mid u, \xi) F(u, d\xi \mid s, x_s) \tag{5.1.5}$$

for all Borel sets $\mathbf{A}$, for all $s < t \in T$, for all $u \in (s,t) \cap T$ and for almost all $x_s \in \mathbb{R}$.

2. For all $s, u, t \in T$ satisfying $s < u < t$,

$$C_{st} = C_{su} * C_{ut}. \tag{5.1.6}$$

This paper has advanced both the theory of copulas and techniques for building Markov processes. This method was used in [165] to formulate a Markov chain model of the dependence in credit risk. The discrete stochastic variable $X_i(t)$ is interpreted as the rating grade of a firm at a particular point in time. A variety of copulas were fitted to the data and gave mixed results. Therefore, no copula was the best for all data sets. This type of mapping of the transition distribution to the copula is very simple, however, one consequence is that an n-dimensional transition function requires a $2n$-dimensional copula. In other words, as the dimension of the copula increases, the calculation of the transition function becomes more and more computationally cumbersome.

The method in [24] has also been extended in [145], so that an n-dimensional Markov process can be represented by a combination of bivariate copulas and margins. Hence,

$$\begin{aligned}
&\Pr\{X_i(t_1) \leq x_1, \ldots, X_i(t_n) \leq x_n\} \\
&\quad = \prod_{i=2}^{n} \Pr\{X_i(t_i) \leq x_i \mid X_i(t_1) = x_1, \ldots, X_i(t_{i-1}) = x_{i-1}\} \Pr\{X_i(t_1) \leq x_1\} \\
&\quad = \prod_{i=2}^{n} \Pr\{X_i(t_i) \leq x_i \mid X_i(t_{i-1}) = x_{i-1}\} \Pr\{X_i(t_1) \leq x_1\} \\
&\quad = \frac{\prod_{i=2}^{n} C_{t_{i-1},t_i}(F_{t_{i-1}}(x_{i-1}), F_{t_i}(x_i))}{\prod_{i=2}^{n-1} F_{t_i}(x_i)}.
\end{aligned} \tag{5.1.7}$$

5.1.3 Conditional Copula of Patton

Another approach to building time into a copula was formulated in [125]. In order to explain this approach, we need to recall more definitions and set up notation.

Firstly, let $\mathcal{F}$ be a σ-algebra or *conditioning set*, then

$$\Pr\{X_i \leq x_i \mid \mathcal{F}\} = F_i(x_i \mid \mathcal{F}) \tag{5.1.8}$$

The multivariate analogue of equation (5.1.8) is

$$\Pr\{X \leq x \mid \mathcal{F}\} = H(x \mid \mathcal{F}) \tag{5.1.9}$$

for $x = (x_1, x_, \ldots, x_n)^T$ such that the volume of H, $V_H(R) \geq 0$, for all rectangles $R \in \mathbb{R}^n$ with their vertices in the domain of H, [145],

$$\begin{aligned} &H(+\infty, x_i, +\infty, \ldots, +\infty \mid \mathcal{F}) = F_i(x_i \mid \mathcal{F}), \quad \text{and} \\ &H(-\infty, x_i, \ldots, x_n \mid \mathcal{F}) = 0 \quad \text{for all} \quad x_1, \ldots, x_n \in \mathbb{R}. \end{aligned}$$

Here F_i is the i-th univariate marginal distribution of H. See [125] for a bivariate version of H. As expected, the density of the conditional H is

$$h(x \mid \mathcal{F}) = \nabla_{x_1,\ldots,x_n} H(x \mid \mathcal{F}). \tag{5.1.10}$$

In equation (5.1.8), the distribution is atypical since it may be conditional on a vector of variables, not just one, as opposed to a typical univariate transition distribution.

The author in [125] mapped the conditional distribution $H(x \mid \mathcal{F})$, defined above, to a copula of the same order. That is, for all $x_i \in \mathbb{R}$ and $i = 1, 2, \ldots, n$,

$$H(x_1, \ldots, x_n \mid \mathcal{F}) = C(F_1(x_1 \mid \mathcal{F}), F_2(x_2 \mid \mathcal{F}), \ldots, F_n(x_n \mid \mathcal{F}) \mid \mathcal{F}). \tag{5.1.11}$$

$\mathcal{F}$ is a sub-algebra or in other words a conditioning set. Such conditioning is necessary for C to satisfy all the conditions of a conventional copula. The relationship between the conditional density h and copula density c is

$$h(x_1, x_2, \ldots, x_n \mid \mathcal{F}) = c(u_1, u_2, \ldots, u_n \mid \mathcal{F}) \prod_{i=1}^{n} f_i(x_i \mid \mathcal{F}), \tag{5.1.12}$$

where $u_i \equiv F_i(x_i \mid \mathcal{F})$, $i = 1, 2, \ldots, n$ and f_i, $i = 1, 2, \ldots, n$ are univariate conditional densities.

In terms of time varying distributions, we can think of the conditioning set as the history of all the variables in the distribution. In the case of the Markov processes, it is only the last time point which is of importance. The implication of this type of conditioning is that the marginal distributions in the copula can no longer be typical transition probabilities, but are atypical conditional probabilities. Hence, if each X_i represented the value of an asset at time t, the associated distribution F_i would represent the distribution of X_i, given that we knew the value of all the assets in the model, $X_1, X_2, \ldots, X_n$, at some previous time, for example $t-1$. In other words, we can rewrite the time-varying version of the distribution and copula above as

$$\begin{aligned} & H_t(x_t^1, x_t^2, \ldots, x_t^n \mid \mathcal{F}_{t-1}) \\ &= C_t(F_t^1(x_t^1 \mid \mathcal{F}_{t-1}), F_t^2(x_t^2 \mid \mathcal{F}_{t-1}), \ldots, F_t^n(x_t^n \mid \mathcal{F}_{t-1}) \mid \mathcal{F}_{t-1}), \end{aligned} \quad (5.1.13)$$

where

$$\mathcal{F}_{t-1} = \sigma(x_{t-1}^1, x_{t-1}^2, \ldots, x_{t-1}^n, x_{t-2}^1, x_{t-2}^2, \ldots, x_{t-2}^n, \ldots, x_1^1, x_1^2, \ldots, x_1^n).$$

In [125], the marginal distributions are characterized by Autoregressive (AR) and generalized autoregressive conditional heteroskedasticity (GARCH) processes. Ultimately, they are handled in the same way as other time series processes. More recent work of Patton, which also employs time varying copulas, appears in [124].

5.1.4 Pseudo-copulas of Fermanian and Wegkamp

As we have seen above, Markov processes are only defined with respect to their own history, not the history of other processes. Therefore, the method in [125] is good for some applications but not practical for others. If we want marginal distributions of processes, conditional on their own history, for example Markov processes, and want to use a mapping similar to that shown in [125], then it is possible via a conditional pseudo-copula. Authors in [41] introduced the notion of conditional pseudo-copula in order to cover a wider range of applications than the conditional copula in [125]. The definition of a pseudo-copula is

Definition 5.3. *Pseudo-copula.* A function $C : [0,1]^n \rightarrow [0,1]$ is called an n-dimensional pseudo-copula if

1. for every $\mathbf{u} \in [0,1]^n$, C($\mathbf{u}$) = 0 when at least one coordinate of $\mathbf{u}$ is zero,
2. $C(1,1,\ldots,1) = 1$, and
3. for every $\mathbf{u}, \mathbf{v} \in [0,1]^n$ such that $\mathbf{u} \leq \mathbf{v}$, the volume of C, $V_C \geq 0$.

The pseudo-copula satisfies most of the conditions of a conventional copula except for $C(1,1,u_k,1,\ldots,1) = u_k$, so the marginal distributions of a pseudo-copula may not be uniform. The definition of a conditional pseudo-copula is

Definition 5.4. *Conditional pseudo-copula.* Given a joint distribution H associated with $X_1, X_2, \ldots, X_n$, an n-dimensional conditional pseudo-copula with respect to sub-algebras
$\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n)$ and $\mathcal{G}$ is a random function $C(\cdot \mid \mathcal{F}, \mathcal{G}) : [0,1]^n \rightarrow [0,1]$ such that

$$H(x_1, x_2, \ldots, x_n) = C(F_1(x_1 \mid \mathcal{F}_1), F_2(x_2 \mid \mathcal{F}_2), \ldots, F_n(x_n \mid \mathcal{F}_n) \mid \mathcal{F}, \mathcal{G}) \quad (5.1.14)$$

almost everywhere, for every $(x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$, see [40].

5.1.5 Galichon model

More recently, a dynamic bivariate copula was used to correlate Markov diffusion processes, see [53], [54]. Unlike the previous models of time dependent copulas, this model addresses the issue of spacial as well as time dependence. The model uses a partial differential approach to obtain a representation of the time dependent copula. An outline of the main result follows. Consider two Markov diffusion processes $X_1(t)$ and $X_2(t)$, $t \in [0, T]$, which represent two risky financial assets, for example options with a maturity date T. The diffusions are such that

$$\begin{aligned} dX_1(t) &= \mu_1(X(t))dt + \tilde{\sigma}_1(X(t))dB_1(t) \\ dX_2(t) &= \mu_2(X(t))dt + \tilde{\sigma}_2(X(t))dB_2(t) \\ dB_1(t)dB_2(t) &= \rho_{12}(X_1(t), X_2(t))dt, \end{aligned} \tag{5.1.15}$$

where $X(t) = (X_1(t), X_2(t))^T$, μ_i, $\tilde{\sigma}_i$, for $i = 1, 2$, are the drift and diffusion coefficients, respectively. The Brownian motion terms are correlated with coefficient $\rho_{12} \in [-1, 1]$. One would like an expression for the evolution of a copula between the distributions F_1, F_2 of $X_1(t)$ and $X_2(t)$, conditional on information at time $t = 0$, $\mathcal{F}_{t_0}$. Firstly a joint bivariate distribution H is mapped to a copula C, by

$$H(t, x_1, x_2 \mid \mathcal{F}_{t_0}) = C(t, F_1(t, x_1 \mid \mathcal{F}_{t_0}), F_2(t, x_2 \mid \mathcal{F}_{t_0}) \mid \mathcal{F}_{t_0}) \tag{5.1.16}$$

then the Kolmogorov forward equation is used to obtain an expression for $\nabla_t C$. Letting

$$u_1 = F_1(t, x_1 \mid \mathcal{F}_{t_0}) \quad \text{and} \quad u_2 = F_2(t, x_2 \mid \mathcal{F}_{t_0}), \quad u_1, u_2 \in [0, 1],$$

$x = (x_1, x_2)^T$ and shortening the notation for the copula to $C(t, u_1, u_2)$, then the time dependent copula in [53] is

$$
\begin{aligned}
&\nabla_t C(t, u_1, u_2) \\
= \;& \frac{1}{2}\tilde{\sigma}_1^2(x) f_1^2(t, x_1 \mid \mathcal{F}_{t_0}) \nabla_{u_1}^2 C(t, u_1, u_2) \\
+ \;& \frac{1}{2}\tilde{\sigma}_2^2(x) f_2^2(t, x_2 \mid \mathcal{F}_{t_0}) \nabla_{u_2}^2 C(t, u_1, u_2) \\
- \;& \nabla_{u_1} C(t, u_1, u_2) \mathcal{B}_1 F_1(t, x_1 \mid \mathcal{F}_{t_0}) \\
+ \;& \int_{(-\infty, x_2]} \nabla_{u_1, u_2} C(t, u_1, u_2) f_2(t, z_2 \mid \mathcal{F}_{t_0}) \mathcal{B}_1 F_1(t, z_1 \mid \mathcal{F}_{t_0}) dz_2 \\
- \;& \nabla_{u_2} C(t, u_1, u_2) \mathcal{B}_2 F_2(t, x_2 \mid \mathcal{F}_{t_0}) \\
+ \;& \int_{(-\infty, x_1]} \nabla_{u_1, u_2} C(t, u_1, u_2) f_1(t, z_1 \mid \mathcal{F}_{t_0}) \mathcal{B}_2 F_2(t, z_2 \mid \mathcal{F}_{t_0}) dz_1 \\
+ \;& \tilde{\sigma}_1(x)\tilde{\sigma}_2(x)\rho_{12}(x_1, x_2) f_1(t, x_1 \mid \mathcal{F}_{t_0}) f_2(t, x_2 \mid \mathcal{F}_{t_0}) \nabla_{u_1, u_2} C(t, u_1, u_2),
\end{aligned}
\tag{5.1.17}
$$

where $\mathcal{B}_1$ and $\mathcal{B}_2$ are the following operators, given any function $g \in C^2(\mathbb{R})$,

$$
\begin{aligned}
\mathcal{B}_1 g &= \left\{ \nabla_{x_1} \left(\frac{1}{2}\tilde{\sigma}_1^2(x) \right) - \mu_1(x) \right\} \nabla_{x_1} g + \left(\frac{1}{2}\tilde{\sigma}_1^2(x) \right) \nabla_{x_1}^2 g \\
\mathcal{B}_2 g &= \left\{ \nabla_{x_2} \left(\frac{1}{2}\tilde{\sigma}_2^2(x) \right) - \mu_2(x) \right\} \nabla_{x_2} g + \left(\frac{1}{2}\tilde{\sigma}_2^2(x) \right) \nabla_{x_2}^2 g
\end{aligned}
$$

and

$$\nabla_{x_i} g = \frac{\partial g}{\partial x_i}, \quad \nabla_{x_i}^2 g = \frac{\partial^2 g}{\partial x_i^2}.$$

For the greatest flexibility we would choose

$$inf\{x_i : F_i(t, x_i \mid \mathcal{F}_{t_0}) \geq u_i\} = F_i^{-1}(t, u_i \mid \mathcal{F}_{t_0}), \quad u_i \in [0, 1].$$

That is, F_i^{-1} is the pseudo-inverse. If $X_1(t)$ and $X_2(t)$ are individually Markov, that is, $\tilde{\sigma}_i$ and μ_i depend only on x_i, for $i = 1, 2$, then the formula for the time dependent copula simplifies to

$$
\begin{aligned}
\nabla_t C(t, u_1, u_2) &= \tilde{\sigma}_1(x)\tilde{\sigma}_2(x)\rho_{12}(x_1, x_2) f_1(t, x_1 \mid \mathcal{F}_{t_0}) f_2(t, x_2 \mid \mathcal{F}_{t_0}) \nabla_{u_1, u_2} C(t, u_1, u_2) \\
&+ \frac{1}{2}\tilde{\sigma}_1^2(x) f_1^2(t, x_1 \mid \mathcal{F}_{t_0}) \nabla_{u_1}^2 C(t, u_1, u_2) + \frac{1}{2}\tilde{\sigma}_2^2(x) f_2^2(t, x_2 \mid \mathcal{F}_{t_0}) \nabla_{u_2}^2 C(t, u_1, u_2).
\end{aligned}
\tag{5.1.18}
$$

Another sophisticated dynamic copula model based on Markov chains, referred to as the *Markov Copula*, appears in [12]. The Markov Copula is a tool for pricing and hedging credit index derivatives and ratings-triggered corporate step-up bonds.

The main aim of this chapter is to extend the two dimensional dynamic copula model of Galichon. We derive an n-dimensional version of the model in [53]. A reformulation is also given, in which linear combinations of independent Brownian motion terms are used.

5.2 n-dimensional Galichon Model for CDOs

Suppose we have an $n \times n$ system of stochastic differential equations, such that $X(t) \in \mathbb{R}^n$ and $B(t)$ is an n-dimensional Brownian motion. The vector $X(t)$ could represent a portfolio of risky assets, as in a CDO. We want to find a partial differential equation with respect to a time dependent n-copula, which gives us information on the riskiness of the package of assets. As in the 2-dimensional model, t is a scalar such that $t \in (0, T]$. Throughout this section the default filtration will be the Brownian Filtration at time t_0, so that we may say each stochastic process $X_i(t)$, $i = 1, 2, \ldots, n$ is conditional on $\mathcal{F}_{t_0}$. In this case the diffusions are such that

$$dX(t) = \mu(X(t))dt + \tilde{A}dB(t) \tag{5.2.1}$$

$$dB_i(t)dB_j(t) = \rho_{ij}(X_i(t), X_j(t))dt, \tag{5.2.2}$$

where

$$dX(t) = \begin{pmatrix} dX_1(t) \\ dX_2(t) \\ \vdots \\ dX_n(t) \end{pmatrix}, \quad dB(t) = \begin{pmatrix} dB_1(t) \\ dB_2(t) \\ \vdots \\ dB_n(t) \end{pmatrix}$$

$$\mu(X(t)) = \begin{pmatrix} \mu_1(X(t)) \\ \mu_2(X(t)) \\ \vdots \\ \mu_n(X(t)) \end{pmatrix}, \quad \tilde{\sigma}(X(t)) = \begin{pmatrix} \tilde{\sigma}_1(X(t)) \\ \tilde{\sigma}_2(X(t)) \\ \vdots \\ \tilde{\sigma}_n(X(t)) \end{pmatrix}.$$

Note that in this case μ and $\tilde{\sigma}$ are n-vector functions which represent the drift and diffusion coefficients of the process, respectively. Let $\tilde{A}$ be

$$\tilde{A} = diag(\tilde{\sigma}(X(t))) = \begin{pmatrix} \tilde{\sigma}_1(X(t)) & 0 & \dots & \dots & 0 \\ 0 & \tilde{\sigma}_2(X(t)) & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & \dots & \dots & 0 & \tilde{\sigma}_n(X(t)). \end{pmatrix}$$

The correlation coefficients $\rho_{ij} \in [-1, 1]$ and let ρ be

$$\rho = \begin{pmatrix} 1 & \rho_{12}(X_1(t), X_2(t)) & \dots & \rho_{1n}(X_1(t), X_n(t)) \\ \rho_{21}(X_2(t), X_1(t)) & 1 & \dots & \rho_{2n}(X_2(t), X_n(t)) \\ \vdots & & & \vdots \\ \rho_{n1}(X_n(t), X_1(t)) & \dots & \dots & 1 \end{pmatrix}.$$

Three conditions are required for the existence and uniqueness of a solution to equation (5.2.1):

1. Coefficients $\mu(x)$ and $\tilde{\sigma}(x)$ must be defined for $x \in \mathbb{R}^n$ and measurable with respect to x.

2. For $x, y \in \mathbb{R}^n$, there exists a constant K such that

$$\begin{aligned} \| \mu(x) - \mu(y) \| &\leq K \| x - y \|, \\ \| \tilde{\sigma}(x) - \tilde{\sigma}(y) \| &\leq K \| x - y \|, \\ \| \mu(x) \|^2 + \| \tilde{\sigma}(x) \|^2 &\leq K^2(1 + \| x \|^2) \end{aligned}$$

   and

3. $X(0)$ does not depend on $B(t)$ and $\mathbb{E}[X(0)^2] < \infty$.

Theorem 5.1. The time dependent n-copula $\nabla_t C(t,u)$ between a vector of distributions $u_i = F_i(t, x_i \mid x_0)$, $i = 1, \ldots, n$, associated with the Markov diffusions $X(t) = [X_1(t), \ldots, X_n(t)]^T$, conditional on information at time $t = 0$, $\mathcal{F}_{t_0} = x_0$ is

$$\nabla_t C(t,u) = \frac{1}{2}\sum_{i=1}^{n}\int_{(-\infty,\bar{x}]}\tilde{\sigma}_i(z)^2 f_i^2(t,x_i \mid x_0)\nabla_{z_1,.,\hat{z}_i,.,z_n}\nabla^2_{u_i}C(t,u)d\bar{z}$$
$$+\sum_{i=1}^{n}\left(-\nabla_{u_i}C(t,u)\mathcal{B}_t^i F_i(t,x_i \mid x_0) + \int_{(-\infty,\bar{x}]}\nabla_{z_1,.,\hat{z}_i,.,z_n}\nabla_{u_i}C(t,u)\mathcal{B}_t^i F_i(t,z_i \mid x_0)d\bar{z}\right)$$
$$+\frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\int_{(-\infty,\check{x}]}\rho_{ij}(x_i,x_j)\tilde{\sigma}_i(z)\tilde{\sigma}_j(z)f_i(t,x_i|x_0)f_j(t,x_j|x_0)\nabla_{z_1,.,\hat{z}_i,\hat{z}_j,.,z_n}\nabla_{u_i,u_j}C(t,u)d\check{z},$$
$$(5.2.3)$$

where $x_i = F_i^{-1}(t, u_i \mid \mathcal{F}_{t_0})$, $i = 1, \ldots, n$. The intervals for the integration are

$$(-\infty,\bar{x}] = (-\infty,x_1]\times\ldots\times(-\infty,x_{i-1}]\times(-\infty,x_{i+1}]\times\ldots\times(-\infty,x_n] \quad \text{and}$$
$$(-\infty,\check{x}] = (-\infty,x_1]\times\ldots\times(-\infty,x_{i-1}]\times(-\infty,x_{i+1}]\times\ldots\times(-\infty,x_{j-1}]$$
$$\times(-\infty,x_{j+1}]\ldots\times(-\infty,x_n].$$

Also note that

$$d\bar{z} = dz_1 dz_2 \ldots dz_{i-1}dz_{i+1}\ldots dz_{n-1}dz_n \quad \text{and}$$
$$d\check{z} = dz_1 dz_2 \ldots dz_{i-1}dz_{i+1}\ldots dz_{j-1}dz_{j+1}\ldots dz_{n-1}dz_n.$$

Thus, the i-th term is excluded in the first two integrals on the right hand side of Theorem 5.1. Similarly, in the last integral the i-th and j-th terms are excluded. Furthermore, for any smooth function g

$$\nabla_{z_1,.,\hat{z}_i,.,z_n} g = \frac{\partial^{n-1}g}{\partial z_1\ldots\partial z_{i-1}\partial z_{i+1}\ldots\partial z_n}$$
$$\text{and}\quad \nabla_{z_1,.,\hat{z}_i,\hat{z}_j,.,z_n} g = \frac{\partial^{n-2}g}{\partial z_1\ldots\partial z_{i-1}\partial z_{i+1}\ldots z_{j-1}\partial z_{j+1}\ldots\partial z_n}.$$

The operators $\mathcal{B}_t^i$, $i = 1, \ldots, n$ are the same as in the two dimensional model. If the diffusions are individually Markov, that is, each σ_k and μ_k depends only on

x_k then the expression for $\nabla_t C$ simplifies to

$$\nabla_t C(t,u) = \frac{1}{2}\sum_{i=1}^{n} \tilde{\sigma}_i(x_i)^2 f_i^2(t, x_i \mid x_0)\nabla^2_{u_i} C(t,u)$$

$$+\ \frac{1}{2}\mathrm{Tr}\big\{\big[\mathcal{H}_u^C(t,u) - \mathrm{diag}\{\nabla^2_{u_1}C(t,u), \nabla^2_{u_2}C(t,u), \ldots, \nabla^2_{u_n}C(t,u)\}\big]D\tilde{A}\rho\tilde{A}^T D^T\big\},$$

where

$$D = \begin{pmatrix} f_1 & 0 & \ldots & 0 \\ 0 & f_2 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & f_n \end{pmatrix}.$$

Proof. In this case 1-dimensional Ito formula for each component of $X(t)$ is

$$\begin{aligned} dg(X_i(t)) &= \big\{\nabla_{x_i} g(X_i(t))\mu_i(X(t)) + \frac{1}{2}\nabla^2_{x_i} g(X_i(t))\tilde{\sigma}_i^2(X(t))\big\}dt \\ &+ \nabla_{x_i} g(X_i(t))\tilde{\sigma}_i(X(t))dB_i(t). \end{aligned}$$

Define the vector ∇_x of partial derivatives with respect to components of x, as

$$\nabla_x g(X(t)) = \begin{pmatrix} \nabla_{x_1} g(X(t)) \\ \nabla_{x_2} g(X(t)) \\ \vdots \\ \nabla_{x_n} g(X(t)) \end{pmatrix}$$

and the Hessian matrix of $g(X(t))$

$$\mathcal{H}_x^g(X(t)) \equiv \Big(\Big(\nabla_{x_i x_j} g(X(t))\Big)\Big)_{1\le i,j\le n}. \tag{5.2.4}$$

In this case, assume $g \in C^2(\mathbb{R}^n)$, then the n-dimensional Ito formula for $g(X(t))$ is

$$\begin{aligned} &dg(X(t)) \\ &= \big\{\langle \nabla_x g(X(t)), \mu(X(t))\rangle + \frac{1}{2}\mathrm{Tr}\big(\mathcal{H}_x^g(X(t))\tilde{A}\rho\tilde{A}^T\big)\big\}dt + \nabla_x g(X(t))^T \tilde{A}dB(t), \end{aligned}$$

where $\langle a, b\rangle = a^T b$ for any vectors a and b. Let the operators $\mathcal{A}$ on distributions (Kolmogorov backward equations), analogous to those in [53], [54], be called $\mathcal{A}_t^i$

and $\mathcal{A}_t^n$ for the 1- an n-dimensional case, respectively. With respect to typical distributions $F_i(t, x_i \mid \tau, \xi_i)$ and $H(t, x \mid \tau, \xi)$, the operators are

$$\mathcal{A}_t^i F_i(t, x_i \mid \tau, \xi_i) = \mu_i(x) \nabla_{\xi_i} F_i(t, x_i \mid \tau, \xi_i) + \frac{1}{2} \tilde{\sigma}_i^2 \nabla_{\xi_i}^2 F_i(t, x_i \mid \tau, \xi_i) \tag{5.2.5}$$

and

$$\mathcal{A}_t^n H(t, x \mid \tau, \xi) = \langle \nabla_\xi H(t, x \mid \tau, \xi), \mu(x) \rangle + \frac{1}{2} \mathrm{Tr} \left(\mathcal{H}_\xi^H (t, x \mid \tau, \xi) \tilde{A} \rho \tilde{A}^T \right). \tag{5.2.6}$$

The operators $\mathcal{A}_t^i$, $i = 1, \ldots, n$ and $\mathcal{A}_t^n$ are not used in the rest of the formulation, but are mentioned briefly, in view of the fact the Kolmogorov forward equations, which are required, are the associated adjoint operators of these. Assuming the density functions of H and F are h and f, respectively, then the adjoint operators $\mathcal{A}_t^{i*}$, $i = 1, \ldots, n$ and $\mathcal{A}_t^{n*}$ have the form

$$\mathcal{A}_t^{i*} f_i(t, x_i \mid \tau, \xi_i) = -\nabla_{x_i} \big[\mu_i(x) f_i(t, x_i \mid \tau, \xi_i) \big] + \nabla_{x_i}^2 \big[\frac{1}{2} \tilde{\sigma}_i^2 f_i(t, x_i \mid \tau, \xi_i) \big] \tag{5.2.7}$$

and

$$\begin{aligned} &\mathcal{A}_t^{n*} h(t, x \mid \tau, \xi) \\ &\quad = -\sum_{i=1}^{n} \nabla_{x_i} \big[\mu_i(x) h(t, x \mid \tau, \xi) \big] + \frac{1}{2} \sum_{i,j=1}^{n} \nabla_{x_i, x_j} \big[\rho_{ij}(x_i, x_j) \tilde{\sigma}_i(x) \tilde{\sigma}_j(x) h(t, x \mid \tau, \xi) \big]. \end{aligned} \tag{5.2.8}$$

The marginal density functions f_i and joint density h, are such that $f_i(t, x_i \mid \mathcal{F}_{t_0}) = f_i(t, x_i \mid x_0)$, and $H(t, x \mid \mathcal{F}_{t_0}) = H(t, x \mid x_0)$, where $x_0 = (x_1 = X_1(0), x_2 = X_2(0), \ldots, x_n = X_n(0))$, see Appendix 5.A. In other words, the assumption made here is that all the distributions are conditional on the entire vector of realizations of x at time zero. As in the 2-dimensional case, it is possible to express the operator $\mathcal{A}_t^{n*}$ in terms of the operators $\mathcal{A}_t^{i*}$ associated with the univariate distributions;

$$\mathcal{A}_t^{n*} g = \sum_{i=1}^{n} \mathcal{A}_t^{i*} g + \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \nabla_{x_i, x_j} \big[\rho_{ij}(x_i, x_j) \tilde{\sigma}_i(x) \tilde{\sigma}_j(x) g \big]. \tag{5.2.9}$$

Given that

$$\nabla_t f_i(t, x_i \mid x_0) = \mathcal{A}_t^{i*} f_i(t, x_i \mid x_0), \tag{5.2.10}$$

we can integrate the left hand side of (5.2.10) with respect to x_i, call it $\mathcal{B}_t^i$, and we obtain

$$\begin{aligned}\mathcal{B}_t^i F_i(t, x_i \mid x_0) &= \int_{(-\infty, x_i]} \nabla_t f_i(t, z_i \mid x_0) dz_i \\ &= \int_{(-\infty, x_i]} \nabla_t \nabla_{z_i} F_i(t, z_i \mid x_0) dz_i \\ &= \nabla_t F_i(t, x_i \mid x_0).\end{aligned} \tag{5.2.11}$$

Integrating the right hand side of (5.2.10) with respect to x_i gives us

$$\begin{aligned}\int_{(-\infty, x_i]} \mathcal{A}_t^{i*} f_i(t, z_i \mid x_0) dz_i &= -\mu_i(x) f_i(t, x_i \mid x_0) + \nabla_{x_i} \big[\frac{1}{2}\tilde{\sigma}_i^2(x) f_i(t, x_i \mid x_0)\big] \\ &= \big[\nabla_{x_i}\{\frac{1}{2}\tilde{\sigma}_i^2(x)\} - \mu_i(x)\big] \nabla_{x_i} F_i(t, x_i \mid x_0) + \frac{1}{2}\tilde{\sigma}_i^2(x) \nabla_{x_i}^2 F_i(t, x_i \mid x_0),\end{aligned}$$

so

$$\mathcal{B}_t^i F_i(t, x_i \mid x_0) = \big[\nabla_{x_i}\{\frac{1}{2}\tilde{\sigma}_i^2(x)\} - \mu_i(x)\big] \nabla_{x_i} F_i(t, x_i \mid x_0) + \frac{1}{2}\tilde{\sigma}_i^2(x) \nabla_{x_i}^2 F_i(t, x_i \mid x_0). \tag{5.2.12}$$

Similarly, integrating over $\mathcal{A}_t^{n*}$ will give us the analogous operator $\mathcal{B}_n^i$ for the multivariate distribution H. Now, since

$$\begin{aligned}&\int_{(-\infty, x]} \mathcal{A}_t^{n*} h(t, z \mid x_0) dz \\ &= \sum_{i=1}^{n} \int_{(-\infty, x]} \mathcal{A}_t^{i*} h(t, z \mid x_0) dz + \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \int_{(-\infty, x]} \nabla_{z_i, z_j} \big[\rho_{ij}(z_i, z_j) \tilde{\sigma}_i(z) \tilde{\sigma}_j(z) h(t, z \mid x_0)\big] dz,\end{aligned} \tag{5.2.13}$$

where $(-\infty, x] = (-\infty, x_1] \times \ldots \times (-\infty, x_n]$, it is possible to get an expression for $\mathcal{B}_t^n$ in terms of $\mathcal{B}_t^i$. That is, let $\mathcal{B}_t^n H(t, x \mid x_0) = \nabla_t H(t, x \mid x_0)$ and given that

$h(t, x \mid x_0) = \nabla_{x_1,.,x_n} H(t, x \mid x_0)$, we have

$$\begin{aligned}\mathcal{B}_t^n H(t,x \mid x_0) &= \frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{n} \int_{(-\infty,x]} \nabla_{z_i,z_j}\left[\rho_{ij}(z_i,z_j)\tilde{\sigma}_i(z)\tilde{\sigma}_j(z)\nabla_{z_1,\ldots,z_n} H(t,z \mid x_0)\right]dz \\ &+ \sum_{i=1}^{n}\int_{(-\infty,x]} \mathcal{A}_t^{i*}\nabla_{z_1,\ldots,z_n} H(t,z \mid x_0)dz. \qquad (5.2.14)\end{aligned}$$

The right hand side of equation (5.2.14) can be expressed in terms in terms of the univariate operators $\mathcal{B}_t^i$, $i = 1, 2 \ldots, n$.

$$\begin{aligned}\mathcal{B}_t^n H(t,x \mid x_0) &= \frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{n} \int_{(-\infty,\check{x}]} \rho_{ij}(x_i,x_j)\tilde{\sigma}_i(z)\tilde{\sigma}_j(z)\nabla_{z_1,.,\hat{z}_i,\hat{z}_j,.,z_n}\nabla_{z_i,z_j} H(t,z \mid x_0)d\check{z} \\ &+ \sum_{i=1}^{n}\int_{(-\infty,\bar{x}]} \mathcal{B}_t^{i}\nabla_{z_1,.,\hat{z}_i,.,z_n} H(t,z \mid x_0)d\bar{z}. \qquad (5.2.15)\end{aligned}$$

Let

$$H(t, x \mid x_0) = C(t, F_1(t, x_1 \mid x_0), F_2(t, x_2 \mid x_0), \ldots, F_n(t, x_n \mid x_0) \mid x_0) \qquad (5.2.16)$$

where C is an n-copula defined on $[0, T] \times [0, 1]^n$. At this point we shorten the notation so that $C(t, F(t, x \mid x_0))$ is the same copula as above. We now seek an expression for $\mathcal{B}_t^n C(t, F(t, x \mid x_0))$ by substituting for H with C in equation (5.2.15). Letting $F_i(t, x_i \mid x_0) = u_i$, $i = 1, 2, \ldots, n$, and $u = (u_1, \ldots, u_n)^T$, then from the first term in equation (5.2.15) we obtain

$$\begin{aligned}
&\sum_{i=1}^{n}\int_{(-\infty,\bar{x}]}\mathcal{B}_t^i\nabla_{z_1,.,\hat{z}_i,.,z_n}H(t,z\mid x_0)d\bar{z}\\
&= \sum_{i=1}^{n}\int_{(-\infty,\bar{x}]}\mathcal{B}_t^i\nabla_{z_1,.,\hat{z}_i,.,z_n}C(t,F(t,z\mid x_0))d\bar{z}\\
&= \sum_{i=1}^{n}\Big(\int_{(-\infty,\bar{x}]}\{\nabla_{z_i}\frac{\tilde{\sigma}_i^2(z)}{2}-\mu_i(z)\}\nabla_{z_i}\nabla_{z_1,.,\hat{z}_i,.,z_n}C(t,F(t,z\mid x_0))d\bar{z}\\
&+ \int_{(-\infty,\bar{x}]}\frac{\tilde{\sigma}_i^2(z)}{2}\nabla_{z_i}^2\nabla_{z_1,.,\hat{z}_i,.,z_n}C(t,F(t,z\mid x_0))d\bar{z}\Big)\\
&= \sum_{i=1}^{n}\Big(\int_{(-\infty,\bar{x}]}\{\nabla_{z_i}\frac{\tilde{\sigma}_i^2(z)}{2}-\mu_i(z)\}f_i(t,z_i\mid x_0)\nabla_{u_i}\nabla_{z_1,.,\hat{z}_i,.,z_n}C(t,u)d\bar{z}\\
&+ \int_{(-\infty,\bar{x}]}\frac{\tilde{\sigma}_i^2(z)}{2}f_i^2(t,z_i\mid x_0)\nabla_{u_i}^2\nabla_{z_1,.,\hat{z}_i,.,z_n}C(t,u)d\bar{z}\\
&+ \int_{(-\infty,\bar{x}]}\frac{\tilde{\sigma}_i^2(z)}{2}\nabla_{z_i}f_i(t,z_i\mid x_0)\nabla_{u_i}\nabla_{z_1,.,\hat{z}_i,.,z_n}C(t,u)d\bar{z}\Big)\\
&= \sum_{i=1}^{n}\Big(\int_{(-\infty,\bar{x}]}\{\nabla_{z_i}\frac{\tilde{\sigma}_i^2(z)}{2}-\mu_i(z)\}\nabla_{z_i}F_i(t,z_i\mid x_0)\nabla_{u_i}\nabla_{z_1,.,\hat{z}_i,.,z_n}C(t,u)d\bar{z}\\
&+ \int_{(-\infty,\bar{x}]}\frac{\tilde{\sigma}_i^2(z)}{2}\nabla_{z_i}^2F_i(t,z_i\mid x_0)\nabla_{u_i}\nabla_{z_1,.,\hat{z}_i,.,z_n}C(t,u)d\bar{z}\\
&+ \int_{(-\infty,\bar{x}]}\frac{\tilde{\sigma}_i^2(z)}{2}f_i^2(t,z_i\mid x_0)\nabla_{u_i}^2\nabla_{z_1,.,\hat{z}_i,.,z_n}C(t,u)d\bar{z}\Big)\\
&= \sum_{i=1}^{n}\int_{(-\infty,\bar{x}]}\nabla_{z_1,.,\hat{z}_i,.,z_n}\nabla_{u_i}C(t,u)\mathcal{B}_t^iF_i(t,z_i\mid x_0)d\bar{z}\\
&+ \frac{1}{2}\sum_{i=1}^{n}\int_{(-\infty,\bar{x}]}\tilde{\sigma}_i^2(z)f_i^2(t,z_i\mid x_0)\nabla_{z_1,.,\hat{z}_i,.,z_n}\nabla_{u_i}^2C(t,u)d\bar{z}.
\end{aligned}$$

Since z is a dummy variable and the multiple integrals exclude that over $(-\infty,x_i]$, we can write

$$\begin{aligned}
&\sum_{i=1}^{n}\int_{(-\infty,\bar{x}]}\mathcal{B}_t^i\nabla_{z_1,.,\hat{z}_i,.,z_n}H(t,z\mid x_0)d\bar{z}\\
&= \sum_{i=1}^{n}\int_{(-\infty,\bar{x}]}\nabla_{z_1,.,\hat{z}_i,.,z_n}\nabla_{u_i}C(t,u)\mathcal{B}_t^iF_i(t,z_i\mid x_0)d\bar{z}\\
&+ \frac{1}{2}\sum_{i=1}^{n}\int_{(-\infty,\bar{x}]}\tilde{\sigma}_i^2(z)f_i^2(t,x_i\mid x_0)\nabla_{z_1,.,\hat{z}_i,.,z_n}\nabla_{u_i}^2C(t,u)d\bar{z}. \qquad (5.2.17)
\end{aligned}$$

From the second term in equation (5.2.15) we have

$$
\begin{aligned}
&\frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\int_{(-\infty,\check{x}]}\rho_{ij}(x_i,x_j)\tilde{\sigma}_i(z)\tilde{\sigma}_j(z)\nabla_{z_1,.,\hat{z}_i,\hat{z}_j,.,z_n}\nabla_{z_i,z_j}H(t,z\mid x_0)d\check{z}\\
&= \frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\int_{(-\infty,\check{x}]}\rho_{ij}(x_i,x_j)\tilde{\sigma}_i(z)\tilde{\sigma}_j(z)\nabla_{z_1,.,\hat{z}_i,\hat{z}_j,.,z_n}\nabla_{z_i,z_j}C(t,F(t,z\mid x_0))d\check{z}\\
&= \frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\int_{(-\infty,\check{x}]}\rho_{ij}(x_i,x_j)\tilde{\sigma}_i(z)\tilde{\sigma}_j(z)f_i(t,x_i\mid x_0)f_j(t,x_j\mid x_0)\nabla_{z_1,.,\hat{z}_i,\hat{z}_j,.,z_n}\nabla_{u_i,u_j}C(t,u)d\check{z}
\end{aligned}
\tag{5.2.18}
$$

so

$$
\begin{aligned}
\mathcal{B}_t^nC(t,u) &= \sum_{i=1}^{n}\int_{(-\infty,\bar{x}]}\nabla_{z_1,.,\hat{z}_i,.,z_n}\nabla_{u_i}C(t,u)\mathcal{B}_t^iF_i(t,z_i\mid x_0)d\bar{z}\\
&+ \frac{1}{2}\sum_{i=1}^{n}\int_{(-\infty,\bar{x}]}\tilde{\sigma}_i^2(z)f_i^2(t,x_i\mid x_0)\nabla_{z_1,.,\hat{z}_i,.,z_n}\nabla_{u_i}^2C(t,u)d\bar{z}\\
&+ \frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\int_{(-\infty,\check{x}]}\rho_{ij}(x_i,x_j)\tilde{\sigma}_i(z)\tilde{\sigma}_j(z)f_i(t,x_i\mid x_0)f_j(t,x_j\mid x_0)\nabla_{z_1,.,\hat{z}_i,\hat{z}_j,.,z_n}\nabla_{u_i,u_j}C(t,u)d\check{z}.
\end{aligned}
\tag{5.2.19}
$$

Now, we also have

$$
\begin{aligned}
\nabla_tH(t,x\mid x_0) &= \mathcal{B}_t^nH(t,x\mid x_0)\\
&= \nabla_tC(t,F(t,x\mid x_0))+\sum_{i=1}^{n}\nabla_{u_i}C(t,F(t,x\mid x_0))\nabla_tF_i(t,x_i\mid x_0)\\
&= \nabla_tC(t,u)+\sum_{i=1}^{n}\nabla_{u_i}C(t,u)\mathcal{B}_t^iF_i(t,x_i\mid x_0).
\end{aligned}
\tag{5.2.20}
$$

Matching equation (5.2.19) and (5.2.20) and rearranging, we obtain

$$
\begin{aligned}
&\nabla_t C(t,u)\\
&= \sum_{i=1}^{n}\left(-\nabla_{u_i}C(t,u)\mathcal{B}_t^i F_i(t,x_i \mid x_0) + \int_{(-\infty,\bar{x}]}\nabla_{z_1,.,\hat{z}_i,.,z_n}\nabla_{u_i}C(t,u)\mathcal{B}_t^i F_i(t,z_i \mid x_0)d\bar{z}\right)\\
&+ \frac{1}{2}\sum_{i=1}^{n}\int_{(-\infty,\bar{x}]}\tilde{\sigma}_i^2(z)f_i^2(t,x_i \mid x_0)\nabla_{z_1,.,\hat{z}_i,.,z_n}\nabla_{u_i}^2 C(t,u)d\bar{z}\\
&+\frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\int_{(-\infty,\check{x}]}\rho_{ij}(x_i,x_j)\tilde{\sigma}_i(z)\tilde{\sigma}_j(z)f_i(t,x_i|x_0)f_j(t,x_j|x_0)\nabla_{z_1,.,\hat{z}_i,\hat{z}_j,.,z_n}\nabla_{u_i,u_j}C(t,u)d\check{z}.
\end{aligned}
$$

If the equations are individually Markov, so that each σ_k and μ_k depends only on x_k, then

$$
\sum_{i=1}^{n}\left(-\nabla_{u_i}C(t,u)\mathcal{B}_t^i F_i(t,x_i \mid x_0)+\int_{(-\infty,\bar{x}]}\nabla_{z_1,.,\hat{z}_i,.,z_n}\nabla_{u_i}C(t,u)\mathcal{B}_t^i F_i(t,z_i \mid x_0)d\bar{z}\right) = 0,
$$

so the expression for $\nabla_t C$ simplifies to

$$
\begin{aligned}
\nabla_t C(t,u) &= \frac{1}{2}\sum_{i=1}^{n}\tilde{\sigma}_i(x_i)^2 f_i^2(t,x_i \mid x_0)\nabla_{u_i}^2 C(t,u)\\
&+\frac{1}{2}\mathrm{Tr}\big\{\big[\mathcal{H}_u^C(t,u) - \mathrm{diag}\{\nabla_{u_1}^2 C(t,u), \nabla_{u_2}^2 C(t,u), \ldots, \nabla_{u_n}^2 C(t,u)\}\big]D\tilde{A}\rho\tilde{A}^T D^T\big\}.
\end{aligned}
$$

□

5.2.1 Generalized n-dimensional model with uncorrelated Brownian Motions.

In this case, we start with an $n \times n$ system of stochastic differential equations such that $X(t) \in \mathbb{R}^n$ and $B(t)$ is an n-dimensional Brownian motion. We want to find a partial differential equation with respect to a time dependent n-copula. As in the previous formulation, t a scalar such that $t \in (0,T]$. In this case the diffusions are such that

$$dX(t) = \mu(X(t))dt + \sigma(X(t))^T dB(t), \tag{5.2.21}$$

where

$$dX(t) = \begin{pmatrix} dX_1(t) \\ dX_2(t) \\ \vdots \\ dX_n(t) \end{pmatrix}, \quad \mu(X(t)) = \begin{pmatrix} \mu_1(X(t)) \\ \mu_2(X(t)) \\ \vdots \\ \mu_n(X(t)) \end{pmatrix}, \quad dB(t) = \begin{pmatrix} dB_1(t) \\ dB_2(t) \\ \vdots \\ dB_n(t) \end{pmatrix}$$

and

$$\begin{aligned} \sigma(X(t)) &= \begin{pmatrix} \sigma_{11}(X(t)) & \sigma_{21}(X(t)) & \dots & \sigma_{n1}(X(t)) \\ \dots & \dots & \dots & \dots \\ \sigma_{1n}(X(t)) & \sigma_{2n}(X(t)) & \dots & \sigma_{nn}(X(t)) \end{pmatrix} \\ &= \left[\sigma_1(X(t)), \sigma_2(X(t)), \dots, \sigma_n(X(t))\right]. \end{aligned}$$

In this model, σ is a matrix, rather than a vector. The coefficients $\mu(X(t))$ and $\sigma(X(t))$ represent the drift and diffusion of the process, respectively. Three conditions are required for the existence and uniqueness of a solution to equation (5.2.21):

1. Coefficients $\mu(x)$ and $\sigma(x)$ must be defined for $x \in \mathbb{R}^n$ and are measurable with respect to x.

2. For $x, y \in \mathbb{R}^n$, there exists a constant K such that

$$\begin{aligned} \| \mu(x) - \mu(y) \| &\leq K \| x - y \|, \\ \| \sigma(x) - \sigma(y) \| &\leq K \| x - y \|, \\ \| \mu(x) \|^2 + \| \sigma(x) \|^2 &\leq K^2(1 + \| x \|^2) \end{aligned}$$

   and

3. $X(0)$ does not depend on $B(t)$ and $\mathbb{E}[X(0)^2] < \infty$.

In this case the components of $B(t)$ are independent Brownian motions, so that $dB_i(t)dB_j(t) = 0, \quad i \neq j$.

Theorem 5.2. The time dependent n-copula $\nabla_t C(t,u)$ between a vector of distributions $F_i(t, x_i \mid x_0)$, $i = 1, \ldots, n$, associated with the Markov diffusions $X(t) = [X_1(t), \ldots, X_n(t)]^T$, conditional on information at time $t = 0$, $\mathcal{F}_{t_0} = x_0$ is

$$\begin{aligned}\nabla_t C(t,u) &= \frac{1}{2}\sum_{i=1}^{n}\int_{(-\infty,\bar{x}]}\|\sigma_i(z)\|^2 f_i^2(t,x_i \mid x_0)\nabla_{z_1,.,\hat{z}_i,.,z_n}\nabla^2_{u_i}C(t,u)d\bar{z}\\ &+\sum_{i=1}^{n}\Big(-\nabla_{u_i}C(t,u)\mathcal{G}^i_t F_i(t,x_i \mid x_0)+\int_{(-\infty,\bar{x}]}\nabla_{z_1,.,\hat{z}_i,.,z_n}\nabla_{u_i}C(t,u)\mathcal{G}^i_t F_i(t,z_i \mid x_0)d\bar{z}\Big)\\ &+\frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\int_{(-\infty,\check{x}]}\big(\sigma(z)^T\sigma(z)\big)_{ij} f_i(t,x_i \mid x_0)f_j(t,x_j \mid x_0)\nabla_{z_1,.,\hat{z}_i,\hat{z}_j,.,z_n}\nabla_{u_i,u_j}C(t,u)d\check{z}.\end{aligned} \tag{5.2.22}$$

If the equations are individually Markov, that is each σ_k and μ_k depends only on x_k then the expression for $\nabla_t C$ simplifies to

$$\begin{aligned}\nabla_t C(t,u) &= \frac{1}{2}\sum_{i=1}^{n}\|\sigma_i(x_i)\|^2 f_i^2(t,x_i \mid x_0)\nabla^2_{u_i}C(t,u)\\ &+\frac{1}{2}\mathrm{Tr}\Big\{\bar{\sigma}(x)^T\bar{\sigma}(x)D\big[\mathcal{H}^C_u(t,u)-\mathrm{diag}\{\nabla^2_{u_1}C(t,u),\ldots,\nabla^2_{u_n}C(t,u)\}\big]D^T\Big\},\end{aligned} \tag{5.2.23}$$

where $D = diag(f)$ as in the previous model.

Proof. The 1-dimensional Ito formula for each component of $X(t)$ is

$$\begin{aligned} dg(X_i(t)) &= \nabla_{x_i} g(X_i(t))dX_i(t) + \frac{1}{2}\nabla^2_{x_i} g(X_i(t))(dX_i(t))^2 \\ &= \big\{\nabla_{x_i} g(X_i(t))\mu_i(X(t)) + \frac{1}{2}\nabla^2_{x_i} g(X_i(t))\|\sigma_i(X(t))\|^2\big\}dt \\ &+ \nabla_{x_i} g(X_i(t))\langle\sigma_i(X(t)), dB(t)\rangle, \end{aligned} \tag{5.2.24}$$

for $i = 1, 2, \dots, n$. Now define the vector ∇_x of partial derivatives with respect to components of x, as

$$\nabla_x g(X(t)) = \begin{pmatrix} \nabla_{x_1} g(X(t)) \\ \nabla_{x_2} g(X(t)) \\ \vdots \\ \nabla_{x_n} g(X(t)) \end{pmatrix}$$

and the Hessian matrix of $g(X(t))$

$$\mathcal{H}^g_x(X(t)) \equiv \Big(\Big(\nabla_{x_i,x_j} g(X(t))\Big)\Big)_{1\le i,j\le n}. \tag{5.2.25}$$

In this example, assume $g \in C^2(\mathbb{R}^n)$, then the n-dimensional Ito formula for $g(X(t))$ is

$$\begin{aligned} dg(X(t)) &= \langle\nabla_x g(X(t)), dX(t)\rangle + \frac{1}{2}dX(t)^T\mathcal{H}^g_x(X(t))dX(t) \\ &= \big\{\langle\nabla_x g(X(t)), \mu(X(t))\rangle + \frac{1}{2}Tr\big[\sigma(X(t))^T\sigma(X(t))\mathcal{H}^g_x(X(t))\big]\big\}dt \\ &+ \nabla_x g(X(t))^T\sigma(X(t))^T dB(t), \end{aligned} \tag{5.2.26}$$

where Tr is the trace of the matrix; $Tr(A) = \sum_i a_{ii}$.

Suppose τ and t are scalars such that $\tau \in [0, t)$ and $t \in (0, T]$, $x_i, \xi_i \in \mathbb{R}$ and $x, \xi \in \mathbb{R}^n$. The Kolmogorov backward equations restated in terms of the operators $\mathcal{L}^i_t$, $\mathcal{L}^n_t$ for $X_i(t)$ and $X(t)$, respectively, are

$$\mathcal{L}^i_t F_i(t, x_i \mid \tau, \xi_i) = \mu_i(\xi_i)\nabla_{\xi_i} F_i(t, x_i \mid \tau, \xi_i) + \frac{1}{2}\|\sigma_i(\xi_i)\|^2\nabla^2_{\xi_i} F_i(t, x_i \mid \tau, \xi_i) \tag{5.2.27}$$

for $i = 1, 2, \ldots, n$, where $\|a\|^2 = \sum_{k=1}^n a_k^2$ and

$$\mathcal{L}_t^n H(t, x \mid \tau, \xi) = \langle \nabla_\xi H(t, x \mid \tau, \xi), \mu(\xi) \rangle + \frac{1}{2}\mathrm{Tr}\{\sigma(\xi)^T \sigma(\xi) \mathcal{H}_\xi^H(t, x \mid \tau, \xi)\}. \tag{5.2.28}$$

The general form of the adjoint operators $\mathcal{L}_t^{i*}$, $\mathcal{L}_t^{n*}$, which are required in this model are

$$\mathcal{L}_t^{i*} f_i(t, x_i \mid \tau, \xi_i) = -\nabla_{x_i}[\mu_i(x) f_i(t, x_i \mid \tau, \xi_i)] + \frac{1}{2}\nabla_{x_i}^2 \big[\|\sigma_i(x)\|^2 f_i(t, x_i \mid \tau, \xi_i)\big], \tag{5.2.29}$$

assuming $\tau \in [0, T)$ and $t \in (\tau, T]$, and

$$\begin{aligned} &\mathcal{L}_t^{n*} h(t, x \mid \tau, \xi) \\ &\quad = -\sum_{i=1}^n [\mu_i(x) \nabla_{x_i} h(t, x \mid \tau, \xi)] + \frac{1}{2}\sum_{i,j=1}^n \nabla_{x_i, x_j}\big[\big(\sigma(x)^T \sigma(x)\big)_{ij} h(t, x \mid \tau, \xi)\big]. \end{aligned} \tag{5.2.30}$$

The marginal density functions which we will assume to have in this example are $f_i(t, x_i \mid x_0)$, where $x_0 = (x_1 = X_1(0), x_2 = X_2(0), \ldots, x_n = X_n(0))$, see appendix A. Thus, $f_i = f_i(t, x_i \mid x_0)$, $h = h(t, x \mid x_0)$, $F_i = F_i(t, x_i \mid x_0)$ and $H = H(t, x \mid x_0)$. Thus, the assumption here is that all distributions are conditional on the entire vector of realizations of x at time zero. It is also possible to have variations with regard to the distributions and this will be discussed at the end of the derivation. As in the two dimensional case, it is possible to express the operator $\mathcal{L}_t^{n*}$ in terms of the operators associated with the univariate distributions $\mathcal{L}_t^{i*}$;

$$\mathcal{L}_t^{n*} g = \sum_{i=1}^n \mathcal{L}_t^{i*} g + \frac{1}{2}\sum_{\substack{i,j=1 \\ i \neq j}}^n \nabla_{x_i, x_j}\big[\big(\sigma(x)^T \sigma(x)\big)_{ij} g\big]. \tag{5.2.31}$$

for $g \in C^2(\mathbb{R}^n)$. Integrating $\mathcal{L}_t^{i*} f_i$ as in equation (5.2.29) with respect to x_i gives us

$$\int_{(-\infty, x_i]} \mathcal{L}_t^{i*} f_i(t, z_i \mid x_0) dz_i = -\mu_i(x) f_i + \frac{1}{2}\nabla_{x_i}\big[\|\sigma_i(x)\|^2 f_i\big]. \tag{5.2.32}$$

for $i = 1, 2, \ldots, n$. Given that $\nabla_{x_i} F_i(t, x_i \mid x_0) = f_i(t, x_i \mid x_0)$, then

$$\begin{aligned}
&\int_{(-\infty,x_i]} \mathcal{L}_t^{i*} f_i(t, z_i \mid x_0) dz_i \\
&= -\mu_i(x)\nabla_{x_i} F_i(t, x_i \mid x_0) + \frac{1}{2}\nabla_{x_i}\left[\|\sigma_i(x)\|^2 \nabla_{x_i} F_i(t, x_i \mid x_0)\right] \\
&= \quad -\mu_i(x)\nabla_{x_i} F_i(t, x_i \mid x_0) + \frac{1}{2}\nabla_{x_i}\|\sigma_i(x)\|^2 \nabla_{x_i} F_i(t, x_i \mid x_0) \\
&+ \quad \frac{1}{2}\|\sigma_i(x)\|^2 \nabla^2_{x_i} F_i(t, x_i \mid x_0) \\
&= \quad \left(\frac{1}{2}\nabla_{x_i}\|\sigma_i(x)\|^2 - \mu_i(x)\right) \nabla_{x_i} F_i(t, x_i \mid x_0) \\
&+ \quad \frac{1}{2}\|\sigma_i(x)\|^2 \nabla^2_{x_i} F_i(t, x_i \mid x_0). \qquad (5.2.33)
\end{aligned}$$

If we let $\nabla_t F_i(t, x_i \mid x_0) = \mathcal{G}_t^i F_i(t, x_i \mid x_0)$, then the operator $\mathcal{G}_t^i$ on F_i is

$$\begin{aligned}
\mathcal{G}_t^i F_i(t, x_i \mid x_0) \quad &= \quad \left[\frac{1}{2}\nabla_{x_i}\|\sigma_i(x)\|^2 - \mu_i(x)\right] \nabla_{x_i} F_i(t, x_i \mid x_0) \\
&+ \quad \frac{1}{2}\|\sigma_i(x)\|^2 \nabla^2_{x_i} F_i(t, x_i \mid x_0). \qquad (5.2.34)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_{(-\infty,x]} \mathcal{L}_t^{n*} h(t, z \mid x_0) dz \\
&= \quad \sum_{i=1}^{n} \int_{(-\infty,x]} \mathcal{L}_t^{i*} h(t, z \mid x_0) dz + \frac{1}{2}\sum_{\substack{i,j=1 \\ i \neq j}}^{n} \int_{(-\infty,x]} \nabla_{z_i,z_j}\left[\left(\sigma(z)^T\sigma(z)\right)_{ij} h(t, z \mid x_0)\right] dz,
\end{aligned} \qquad (5.2.35)$$

where $(-\infty, x] = (-\infty, x_1] \times \ldots \times (-\infty, x_n]$. Suppose that

$$\mathcal{G}_t^n H(t, x \mid x_0) = \nabla_t H(t, x \mid x_0) = \int_{(-\infty,x]} \mathcal{L}_t^{n*} h(t, z \mid x_0) dz.$$

We find that $\mathcal{G}_t^n$ can be expressed in terms of $\mathcal{G}_t^i$;

$$\begin{aligned}
\mathcal{G}_t^n H(t, x \mid x_0) \quad &= \quad \sum_{i=1}^{n} \int_{(-\infty,\bar{x}]} \mathcal{G}_t^i \nabla_{z_1,.,\hat{z}_i,.,z_n} H(t, z \mid x_0) d\bar{z} \\
&\quad + \frac{1}{2}\sum_{\substack{i,j=1 \\ i \neq j}}^{n} \int_{(-\infty,\check{x}]} \left(\sigma(z)^T\sigma(z)\right)_{ij} \nabla_{z_1,.,\hat{z}_i,\hat{z}_j,.,z_n} \nabla_{z_i,z_j} H(t, z \mid x_0) d\check{z}.
\end{aligned} \qquad (5.2.36)$$

When each σ_k only depends on x_k, then equation (5.2.36) simplifies further to

$$\mathcal{G}_t^n H(t,x \mid x_0) = \sum_{i=1}^{n} \int_{(-\infty,\bar{x}]} \mathcal{G}_t^i \nabla_{z_1,.,\hat{z}_i,.,z_n} H(t,z \mid x_0) d\bar{z}$$
$$+\frac{1}{2}\mathrm{Tr}\big\{\bar{\sigma}(x)^T\bar{\sigma}(x)\left(\mathcal{H}_x^H(t,x\,|x_0) - \mathrm{diag}\{\nabla^2_{x_1}H(t,x\,|x_0),\ldots,\nabla^2_{x_n}H(t,x\,|x_0)\}\right)\big\}, \quad (5.2.37)$$

where

$$\bar{\sigma}(x) = \begin{pmatrix} \sigma_{11}(x_1) & \sigma_{21}(x_2) & \ldots & \sigma_{n1}(x_n) \\ \ldots & \ldots & \ldots & \ldots \\ \sigma_{1n}(x_1) & \sigma_{2n}(x_2) & \ldots & \sigma_{nn}(x_n) \end{pmatrix}.$$

Let

$$H(t,x \mid x_0) = C(t, F_1(t,x_1 \mid x_0), F_2(t,x_2 \mid x_0), \ldots, F_n(t,x_n \mid x_0) \mid x_0) \quad (5.2.38)$$

be an n-copula defined on $[0,T] \times [0,1]^n$. At this point we shorten the notation so that $C(t, F(t,x \mid x_0))$ is consistent with the notation used in Theorem 5.1. Applying the most general form of $\mathcal{G}_t^n$ in equation (5.2.36) to $C(t, F(x \mid x_0))$, we obtain

$$\mathcal{G}_t^n C(t,F(t,x \mid x_0)) = \sum_{i=1}^{n} \int_{(-\infty,\bar{x}]} \mathcal{G}_t^i \nabla_{z_1,.,\hat{z}_i,.,z_n} C(t,F(t,z \mid x_0)) d\bar{z}$$
$$+\frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{n} \int_{(-\infty,\check{x}]} \left(\sigma(z)^T\sigma(z)\right)_{ij} \nabla_{z_1,.,\hat{z}_i,\hat{z}_j,.,z_n} \nabla_{x_i,x_j} C(t,F(t,z \mid x_0)) d\check{z}. \quad (5.2.39)$$

Let $F_i(t,x_i \mid x_0) = u_i$, for all $i = 1,2,\ldots,n$, and $u = (u_1,\ldots,u_n)^T$ then first term on the right hand side becomes

$$\sum_{i=1}^{n} \int_{(-\infty,\bar{x}]} \mathcal{G}_t^i \nabla_{z_1,.,\hat{z}_i,.,z_n} C(t,F(t,z \mid x_0)) d\bar{z}$$
$$= \sum_{i=1}^{n} \int_{(-\infty,\bar{x}]} \nabla_{z_1,.,\hat{z}_i,.,z_n} \nabla_{u_i} C(t,u) \mathcal{G}_t^i F_i(t,z_i \mid x_0) d\bar{z}$$
$$+ \frac{1}{2}\sum_{i=1}^{n} \int_{(-\infty,\bar{x}]} \|\sigma_i(z)\|^2 f_i^2(t,x_i \mid x_0) \nabla_{z_1,.,\hat{z}_i,.,z_n} \nabla^2_{u_i} C(t,u) d\bar{z}, \quad (5.2.40)$$

since

$$\sum_{i=1}^{n}\int_{(-\infty,\bar{x}]}\mathcal{G}_t^i\nabla_{z_1,.,\hat{z}_i,.,z_n}C(t,F(t,z\mid x_0))d\bar{z}$$
$$= \sum_{i=1}^{n}\int_{(-\infty,\bar{x}]}\big[\frac{1}{2}\nabla_{z_i}\|\sigma_i(z)\|^2-\mu_i(z)\big]\nabla_{z_i}\nabla_{z_1,.,\hat{z}_i,.,z_n}C(t,F(t,z\mid x_0))d\bar{z}$$
$$+ \sum_{i=1}^{n}\int_{(-\infty,\bar{x}]}\|\sigma_i(z)\|^2\ \nabla_{z_i}^2\nabla_{z_1,.,\hat{z}_i,.,z_n}C(t,F(t,z\mid x_0))d\bar{z}$$
$$= \sum_{i=1}^{n}\int_{(-\infty,\bar{x}]}\big[\frac{1}{2}\nabla_{z_i}\|\sigma_i(z)\|^2-\mu_i(z)\big]f_i(t,z_i\mid x_0)\nabla_{u_i}\nabla_{z_1,.,\hat{z}_i,.,z_n}C(t,u)d\bar{z}$$
$$+ \frac{1}{2}\sum_{i=1}^{n}\int_{(-\infty,\bar{x}]}\|\sigma_i(z)\|^2 f_i^2(t,z_i\mid x_0)\nabla_{u_i}^2\nabla_{z_1,.,\hat{z}_i,.,z_n}C(t,u)d\bar{z}$$
$$+ \frac{1}{2}\sum_{i=1}^{n}\int_{(-\infty,\bar{x}]}\|\sigma_i(z)\|^2\nabla_{z_i}f_i(t,z_i\mid x_0)\nabla_{u_i}\nabla_{z_1,.,\hat{z}_i,.,z_n}C(t,u)d\bar{z}$$
$$= \sum_{i=1}^{n}\int_{(-\infty,\bar{x}]}\big[\frac{1}{2}\nabla_{z_i}\|\sigma_i(z)\|^2-\mu_i(z)\big]\nabla_{z_i}F_i(t,z_i\mid x_0)\nabla_{u_i}\nabla_{z_1,.,\hat{z}_i,.,z_n}C(t,u)d\bar{z}$$
$$+ \frac{1}{2}\sum_{i=1}^{n}\int_{(-\infty,\bar{x}]}\|\sigma_i(z)\|^2 f_i^2(t,z_i\mid x_0)\nabla_{u_i}^2\nabla_{z_1,.,\hat{z}_i,.,z_n}C(t,u)d\bar{z}$$
$$+ \frac{1}{2}\sum_{i=1}^{n}\int_{(-\infty,\bar{x}]}\|\sigma_i(z)\|^2\nabla_{z_i}^2F_i(t,z_i\mid x_0)\nabla_{u_i}\nabla_{z_1,.,\hat{z}_i,.,z_n}C(t,u)d\bar{z}$$
$$= \sum_{i=1}^{n}\int_{(-\infty,\bar{x}]}\nabla_{z_1,.,\hat{z}_i,.,z_n}\nabla_{u_i}C(t,u)\mathcal{G}_t^iF_i(t,z_i\mid x_0)d\bar{z}$$
$$+ \frac{1}{2}\sum_{i=1}^{n}\int_{(-\infty,\bar{x}]}\|\sigma_i(z)\|^2 f_i^2(t,z_i\mid x_0)\nabla_{z_1,.,\hat{z}_i,.,z_n}\nabla_{u_i}^2C(t,u)d\bar{z}. \tag{5.2.41}$$

since z is a dummy variable and the multiple integrals exclude that over $(-\infty,x_i]$, we can write

$$\sum_{i=1}^{n}\int_{(-\infty,\bar{x}]}\mathcal{G}_t^i\nabla_{z_1,.,\hat{z}_i,.,z_n}C(t,F(t,z\mid x_0))d\bar{z}$$
$$= \sum_{i=1}^{n}\int_{(-\infty,\bar{x}]}\nabla_{z_1,.,\hat{z}_i,.,z_n}\nabla_{u_i}C(t,u)\mathcal{G}_t^iF_i(t,z_i\mid x_0)d\bar{z}$$
$$+ \frac{1}{2}\sum_{i=1}^{n}\int_{(-\infty,\bar{x}]}\|\sigma_i(z)\|^2 f_i^2(t,x_i\mid x_0)\nabla_{z_1,.,\hat{z}_i,.,z_n}\nabla_{u_i}^2C(t,u)d\bar{z}. \tag{5.2.42}$$

With regard to the second term, we obtain

$$\frac{1}{2}\sum_{\substack{i,j=1\\i\neq j}}^{n}\int_{(-\infty,\check{x}]}\big(\sigma(z)^T\sigma(z)\big)_{ij}\nabla_{z_1,.,\hat{z}_i,\hat{z}_j,.,z_n}\nabla_{x_i,x_j}C(t,F(t,z\mid x_0))d\check{z}$$
$$=\frac{1}{2}\sum_{\substack{i,j=1\\i\neq j}}^{n}\int_{(-\infty,\check{x}]}\big(\sigma(z)^T\sigma(z)\big)_{ij}f_i(t,x_i\mid x_0)f_j(t,x_j\mid x_0)\nabla_{z_1,.,\hat{z}_i,\hat{z}_j,.,z_n}\nabla_{u_i,u_j}C(t,u)d\check{z}. \tag{5.2.43}$$

Setting $\mathcal{G}_t^n H(t,x\mid x_0)=\mathcal{G}_t^n C(t,F(t,x\mid x_0))$, we obtain

$$\mathcal{G}_t^n H(t,x\mid x_0)=\sum_{i=1}^{n}\int_{(-\infty,\bar{x}]}\nabla_{z_1,.,\hat{z}_i,.,z_n}\nabla_{u_i}C(t,u)\mathcal{G}_t^i F_i(t,z_i\mid x_0)d\bar{z}$$
$$+\ \frac{1}{2}\sum_{i=1}^{n}\int_{(-\infty,\bar{x}]}\|\sigma_i(z)\|^2 f_i^2(t,x_i\mid x_0)\nabla_{z_1,.,\hat{z}_i,.,z_n}\nabla_{u_i}^2 C(t,u)d\bar{z}$$
$$+\frac{1}{2}\sum_{\substack{i,j=1\\i\neq j}}^{n}\int_{(-\infty,\check{x}]}\big(\sigma(z)^T\sigma(z)\big)_{ij}f_i(t,x_i\mid x_0)f_j(t,x_j\mid x_0)\nabla_{z_1,.,\hat{z}_i,\hat{z}_j,.,z_n}\nabla_{u_i,u_j}C(t,u)d\check{z}. \tag{5.2.44}$$

We also have

$$\begin{aligned}\nabla_t H(t,x\mid x_0) &= \mathcal{G}_t^n H(t,x\mid x_0)\\ &= \nabla_t C(t,F(t,x\mid x_0))+\sum_{i=1}^{n}\nabla_{u_i}C(t,F(t,x\mid x_0))\nabla_t F_i(t,x_i\mid x_0)\\ &= \nabla_t C(t,u)+\sum_{i=1}^{n}\nabla_{u_i}C(t,u)\mathcal{G}_t^i F_i(t,x_i\mid x_0). \end{aligned} \tag{5.2.45}$$

Matching equation (5.2.44) and (5.2.45) and rearranging, we obtain

$$\nabla_t C(t,u)=\frac{1}{2}\sum_{i=1}^{n}\int_{(-\infty,\bar{x}]}\|\sigma_i(z)\|^2 f_i^2(t,x_i\mid x_0)\nabla_{z_1,.,\hat{z}_i,.,z_n}\nabla_{u_i}^2 C(t,u)d\bar{z}$$
$$+\sum_{i=1}^{n}\Big(-\nabla_{u_i}C(t,u)\mathcal{G}_t^i F_i(t,x_i\mid x_0)+\int_{(-\infty,\bar{x}]}\nabla_{z_1,.,\hat{z}_i,.,z_n}\nabla_{u_i}C(t,u)\mathcal{G}_t^i F_i(t,z_i\mid x_0)d\bar{z}\Big)$$
$$+\ \frac{1}{2}\sum_{\substack{i,j=1\\i\neq j}}^{n}\int_{(-\infty,\check{x}]}\big(\sigma(z)^T\sigma(z)\big)_{ij}f_i(t,x_i\mid x_0)f_j(t,x_j\mid x_0)\nabla_{z_1,.,\hat{z}_i,\hat{z}_j,.,z_n}\nabla_{u_i,u_j}C(t,u)d\check{z}. \tag{5.2.46}$$

If the equations are individually Markov, that is each σ_k and μ_k depends only on x_k then the expression for $\nabla_t C$ simplifies to

$$\nabla_t C(t,u) = \frac{1}{2}\sum_{i=1}^{n} \|\sigma_i(x_i)\|^2 f_i^2(t, x_i \mid x_0)\nabla^2_{u_i} C(t,u) + \frac{1}{2}\mathrm{Tr}\Big\{\bar{\sigma}(x)^T\bar{\sigma}(x)D\big[\mathcal{H}^C_u(t,u) - \mathrm{diag}\{\nabla^2_{u_1}C(t,u),\dots,\nabla^2_{u_n}C(t,u)\}\big]D^T\Big\}, \tag{5.2.47}$$

where $D = diag(f)$ as in the previous model. □

5.3 Geometric Brownian Motion Model

Suppose we have an $n \times n$ system of stochastic differential equations, such that $X(t) \in \mathbb{R}^n$ and $B(t)$ is an n-dimensional Brownian motion.

$$dX(t) = UX(t)dt + AYdB(t) \tag{5.3.1}$$

$$dB_i(t)dB_j(t) = \rho_{ij}(X_i(t), X_j(t))dt, \tag{5.3.2}$$

where

$$dX(t) = \begin{pmatrix} dX_1(t) \\ dX_2(t) \\ \vdots \\ dX_n(t) \end{pmatrix}, \quad dB(t) = \begin{pmatrix} dB_1(t) \\ dB_2(t) \\ \vdots \\ dB_n(t) \end{pmatrix}$$

In this example, the drift and volatility (diffusion) coefficients, $\acute{\mu}_i$, $\acute{\sigma}_i$, respectively, are constants in $\mathbb{R}$ and let

$$U = \begin{pmatrix} \acute{\mu}_1 & 0 & \dots & \dots & 0 \\ 0 & \acute{\mu}_2 & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & \dots & \dots & 0 & \acute{\mu}_n \end{pmatrix}, \quad A = \begin{pmatrix} \acute{\sigma}_1 & 0 & \dots & \dots & 0 \\ 0 & \acute{\sigma}_2 & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & \dots & \dots & 0 & \acute{\sigma}_n \end{pmatrix}$$

and

$$Y = \begin{pmatrix} X_1(t) & 0 & \dots & 0 \\ 0 & X_2(t) & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & X_n(t) \end{pmatrix}.$$

The correlation coefficients $\rho_{ij} \in [-1, 1]$ and let ρ be

$$\rho = \begin{pmatrix} 1 & \rho_{12}(X_1(t), X_2(t)) & \dots & \rho_{1n}(X_1(t), X_n(t)) \\ \rho_{21}(X_2(t), X_1(t)) & 1 & \dots & \rho_{2n}(X_2(t), X_n(t)) \\ \vdots & & & \vdots \\ \rho_{n1}(X_n(t), X_1(t)) & \dots & \dots & 1 \end{pmatrix}.$$

Theorem 5.3. The time dependent n-copula $\nabla_t C(t, u)$ between a vector of distributions $F_i(t, x_i \mid x_0)$, $i = 1, \dots, n$, associated with the geometric Brownian motions $X(t) = [X_1(t), \dots, X_n(t)]^T$, conditional on information at time $t = 0$, $\mathcal{F}_{t_0} = x_0$ is

$$\begin{aligned} \nabla_t C(t, u) &= \frac{1}{2} \sum_{i=1}^{n} x_i^2 \acute{\sigma}_i^2 f_i^2(t, x_i \mid x_0) \nabla_{u_i}^2 C(t, u) \\ &+ \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \acute{\sigma}_i \acute{\sigma}_j x_i x_j \rho_{ij}(x_i, x_j) f_i(t, x_i \mid x_0) f_j(t, x_j \mid x_0) \nabla_{u_i u_j} C(t, u). \end{aligned} \tag{5.3.3}$$

Proof. In this case 1-dimensional Ito formula, given a function $g \in C^2(\mathbb{R})$ for each component of $X(t)$ is

$$\begin{aligned} & dg(X_i(t)) \\ &= X_i(t)\big\{\big(\nabla_{x_i} g(X_i(t))\acute{\mu}_i + \frac{1}{2}\nabla_{x_i}^2 g(X_i(t))\acute{\sigma}_i^2 X_i(t)\big)dt + \nabla_{x_i} g(X_i(t))\acute{\sigma}_i dB_i(t)\big\}. \end{aligned} \tag{5.3.4}$$

Similarly, assume $g \in C^2(\mathbb{R}^n)$, and define the Hessian $\mathcal{H}_x^g(X(t))$, and vectors $\nabla_x g(X(t))$, $dB(t)$ and $dX(t)$ as in the previous models. Letting

$$Y = \begin{pmatrix} X_1(t) & 0 & \dots & 0 \\ 0 & X_2(t) & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & X_n(t) \end{pmatrix},$$

then the n-dimensional Ito formula for $g(X(t))$ is

$$\begin{aligned} & dg(X(t)) \\ = \; & \big\{\nabla_x g(X(t))^T U X(t) + \frac{1}{2}\mathrm{Tr}\big(\mathcal{H}_x^g(X(t)) AY\rho YA\big)\big\} dt + \nabla_x g(X(t))^T AY dB(t). \end{aligned}$$

Let the operators $\mathcal{J}$ on distributions of $X_i(t)$, $i = 1, \dots, n$ and $X(t)$ be called $\mathcal{J}_t^i$ and $\mathcal{J}_t^n$ for the 1- an n-dimensional case, respectively. With respect to typical distributions $F_i(t, x_i \mid \tau, \xi_i)$ and $H(t, x \mid \tau, \xi)$, the operators (Kolmogorov backward equations) are

$$\mathcal{J}_t^i F_i(t, x_i \mid \tau, \xi_i) = \acute{\mu}_i x_i \nabla_{\xi_i} F_i(t, x_i \mid \tau, \xi_i) + \frac{1}{2}\acute{\sigma}_i^2 x_i^2 \nabla_{\xi_i}^2 F_i(t, x_i \mid \tau, \xi_i) \qquad (5.3.5)$$

and

$$\mathcal{J}_t^i H(t, x \mid \tau, \xi) = \nabla_\xi H(t, x \mid \tau, \xi)^T U \xi + \frac{1}{2}\mathrm{Tr}\big(\mathcal{H}_\xi^H(t, x \mid \tau, \xi) AY\rho YA\big), \qquad (5.3.6)$$

where

$$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix}, \qquad \nabla_\xi H(t, x \mid \tau, \xi) = \begin{pmatrix} \nabla_{\xi_1} H(t, x \mid \tau, \xi) \\ \nabla_{\xi_2} H(t, x \mid \tau, \xi) \\ \vdots \\ \nabla_{\xi_n} H(t, x \mid \tau, \xi) \end{pmatrix}.$$

The adjoint operators $\mathcal{J}_t^{i*}$ and $\mathcal{J}_t^{n*}$, that is the Kolmogorov forward equations, required for this model are

$$\mathcal{J}_t^{i*} F_i(t, x_i \mid \tau, \xi_i) = -\nabla_{x_i}[\mu_i x_i F_i(t, x_i \mid \tau, \xi_i)] + \frac{1}{2}\nabla_{x_i}^2[\acute{\sigma}_i x_i F_i(t, x_i \mid \tau, \xi_i)] \quad (5.3.7)$$

and

$$\mathcal{J}_t^{n*} = -\sum_{i=1}^{n} \nabla_{x_i}[\mu_i x_i H(t,x \mid \tau,\xi)] + \frac{1}{2}\sum_{i,j=1}^{n} \nabla_{x_i,x_j}[\rho_{ij}(x_i,x_j)\acute{\sigma}_i\acute{\sigma}_j x_i x_j H(t,x \mid \tau,\xi)]. \tag{5.3.8}$$

The derivation of the 1-dimensional case for the distribution we require, $p(t,x_i \mid x_0)$, is given in Appendix 5.B. As in the previous case, it is possible to express the operator $\mathcal{J}_t^{n*}$ in terms of the operators $\mathcal{J}_t^{i*}$,$i=1,\ldots,n$, associated with the univariate distributions;

$$\mathcal{J}_t^{n*} g = \sum_{i=1}^{n} \mathcal{J}_t^{i*} g + \frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{n} \nabla_{x_i,x_j}\left[\rho_{ij}(x_i,x_j)\acute{\sigma}_i\acute{\sigma}_j x_i x_j g\right]. \tag{5.3.9}$$

Integrating over these operators in a similar way to those of the previous models, we get new operators $\mathcal{K}_t^i$ and $\mathcal{K}_t^n$

$$\mathcal{K}_t^i F_i(t,x_i \mid x_0) = \left[\acute{\sigma}_i^2 - \acute{\mu}_i + \frac{1}{2}x_i\nabla_{x_i}\acute{\sigma}_i^2\right] x_i \nabla_{x_i} F_i(t,x_i \mid x_0) + \frac{1}{2}x_i^2\acute{\sigma}_i^2\nabla_{x_i}^2 F_i(t,x_i \mid x_0), \tag{5.3.10}$$

$$\begin{aligned}\mathcal{K}_t^n H(t,x \mid x_0) &= \sum_{i=1}^{n}\int_{(-\infty,\bar{x}]} \mathcal{K}_t^i \nabla_{z_1,.,\hat{z}_i,.,z_n} H(t,z \mid x_0) d\bar{z} \\ &+ \frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\int_{(-\infty,\check{x}]} \rho_{ij}(x_i,x_j)\acute{\sigma}_i\acute{\sigma}_j x_i x_j \nabla_{z_1,.,\hat{z}_i,\hat{z}_j,.,z_n}\nabla_{z_i,z_j} H(t,z \mid x_0) d\check{z}.\end{aligned} \tag{5.3.11}$$

We map the distribution H to a copula C in the usual way and seek an expression for $\mathcal{K}_t^n C(t,F(t,x \mid x_0))$. In this example the dynamic copula is

$$\begin{aligned}&\mathcal{K}_t^n C(t,F(t,x \mid x_0)) \\ &= \sum_{i=1}^{n} \nabla_{u_i} C(t,u)\mathcal{K}_t^i F_i(t,x_i \mid x_0) + \frac{1}{2}\sum_{i=1}^{n} x_i^2\acute{\sigma}_i^2 f_i^2(t,x_i \mid x_0)\nabla_{u_i}^2 C(t,u) \\ &+ \frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{n} \rho_{ij}(x_i,x_j)\acute{\sigma}_i\acute{\sigma}_j x_i x_j f_i(t,x_i \mid x_0) f_j(t,x_j \mid x_0)\nabla_{u_i,u_j} C(t,u),\end{aligned}$$

since the coefficients are constant and the ρ_{ij} depends only on x_i and x_j. As before, we also have

$$\begin{aligned}\nabla_t H(t, x \mid x_0) &= \mathcal{K}_t^n H(t, x \mid x_0) \\ &= \nabla_t C(t, F(t, x \mid x_0)) + \sum_{i=1}^{n} \nabla_{u_i} C(t, F(t, x \mid x_0)) \nabla_t F_i(t, x_i \mid x_0) \\ &= \nabla_t C(t, u) + \sum_{i=1}^{n} \nabla_{u_i} C(t, u) \mathcal{K}_t^i F_i(t, x_i \mid x_0). \end{aligned} \tag{5.3.12}$$

Matching equation (5.3.12) and (5.3.12) and canceling terms, we obtain

$$\begin{aligned}\nabla_t C(t, u) &= \frac{1}{2} \sum_{i=1}^{n} x_i^2 \acute{\sigma}_i^2 f_i^2(t, x_i \mid x_0) \nabla_{u_i}^2 C(t, u) \\ &+ \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \acute{\sigma}_i \acute{\sigma}_j x_i x_j \rho_{ij}(x_i, x_j) f_i(t, x_i \mid x_0) f_j(t, x_j \mid x_0) \nabla_{u_i u_j} C(t, u). \end{aligned} \tag{5.3.13}$$

□

5.4 Appendix 5.A

For each component $X_i(t)$ of $X(t)$, we have a one dimensional diffusion process

$$dX_i(t) = \mu_i(X(t))dt + \sigma_i(X(t))dB_i(t), \tag{5.4.1}$$

with $a \leq t \leq b$ and $x_i \in \mathbb{R}$. Three conditions are required for the existence and uniqueness of a solution to equation (5.4.1);

1. Coefficients $\mu_i(x)$ and $\sigma_i(x)$ must be defined for $x \in \mathbb{R}^n$ and are measurable with respect to x.

2. For $x, y \in \mathbb{R}^n$, there exists a constant K such that

$$\begin{aligned} \| \mu_i(x) - \mu_i(y) \| &\leq K \| x - y \|, \\ \| \sigma_i(x) - \sigma_i(y) \| &\leq K \| x - y \|, \\ \| \mu_i(x) \|^2 + \| \sigma_i(x) \|^2 &\leq K^2(1 + \| x \|^2) \end{aligned}$$

   and

3. $X_i(0)$ does not depend on $B(t)$ and $\mathbb{E}[X_i(0)^2] < \infty$.

We now require the adjoint operator on the density functions rather than the distribution functions, which can be arrived at from the Kolmogorov forward equations. The derivation of the 1-dimensional case is as follows, see [152]. For any function g and random process $X_i(t)$ with density $p(t, y_i \mid x_0)$, we have

$$\mathbb{E}[g(X_i(t)) \mid X(0) = x_0] = \int_0^\infty p(t, y_i \mid x_0) g(y_i) dy_i. \tag{5.4.2}$$

Suppose we also have the diffusion process shown in equation (5.4.1), and

$x_0 = (x_1 = X_1(0), x_2 = X_2(0), \ldots, x_n = X_n(0))$, then

$$\begin{aligned} &\mathbb{E}[g(X_i(t)) \mid X(0) = x_0] = g(X_i(0)) \\ &+ \mathbb{E}[\int_{t_0}^t \big(\nabla_{x_i} g(X_i(s))\mu_i(X(s)) + \frac{1}{2}\nabla^2_{x_i} g(X_i(s))\sigma_i^2(X(s))\big)ds \mid X(0) = x_0]. \end{aligned} \tag{5.4.3}$$

From the definition of expectation

$$\mathbb{E}[\int_{t_0}^{t} \nabla_{x_i} g(X_i(s))\mu_i(X(s))ds \mid X(0) = x_0] = \int_{t_0}^{t}\int_{0}^{\infty} \nabla_{y_i} g(y_i)\mu_i(y)p(s, y_i \mid x_0)dy_i ds$$

and

$$\begin{aligned}\mathbb{E}[\int_{t_0}^{t} \frac{1}{2}\nabla^2_{x_i} g(X_i(s))\sigma_i^2(X(s))ds \mid X(0) = x_0] \\ = \int_{t_0}^{t}\int_{0}^{\infty} \frac{1}{2}\nabla^2_{y_i} g(y_i)\sigma_i^2(y)p(s, y_i \mid x_0)dy_i ds,\end{aligned}$$

so

$$\begin{aligned}\int_{0}^{\infty} p(t, y_i \mid x_0)g(y_i)dy_i &= g(X_i(0)) + \int_{t_0}^{t}\int_{0}^{\infty} \nabla_{y_i} g(y_i)\mu_i(y)p(s, y_i \mid x_0)dy_i ds \\ &+ \int_{t_0}^{t}\int_{0}^{\infty} \frac{1}{2}\nabla^2_{y_i} g(y_i)\sigma_i^2(y)p(s, y_i \mid x_0)dy_i ds. \qquad (5.4.4)\end{aligned}$$

Differentiating both sides with respect to t, we have

$$\begin{aligned}\int_{0}^{\infty} \nabla_t p(t, y_i \mid x_0)g(y_i)dy_i &= \int_{0}^{\infty} \nabla_{y_i} g(y_i)\mu_i(y)p(t, y_i \mid x_0)dy_i \\ &+ \frac{1}{2}\nabla^2_{y_i} g(y_i)\sigma_i^2(y)p(t, y_i \mid x_0)dy_i. \qquad (5.4.5)\end{aligned}$$

Integrating the first term of the right hand side of equation 5.4.5 by parts, we obtain

$$\begin{aligned}&\int_{0}^{\infty} \nabla_{y_i} g(y_i)\mu_i(y)p(t, y_i \mid x_0)dy_i \\ &= g(y_i)\mu_i(y)p(t, y_i \mid x_0)\Big|_{0}^{\infty} - \int_{0}^{\infty} g(y_i)\nabla_{y_i}[\mu_i(y)p(t, y_i \mid x_0)]dy_i \\ &= -\int_{0}^{\infty} g(y_i)\nabla_{y_i}[\mu_i(y)p(t, y_i \mid x_0)]dy_i\end{aligned}$$

since the first component is zero. Similarly, integrating the second term on the right hand side of equation (5.4.5) by parts, we obtain

$$\begin{aligned}
&\int_0^\infty \frac{1}{2}\nabla^2_{y_i} g(y_i)\sigma_i^2(y)p(t,y_i \mid x_0)dy_i \\
&= \frac{1}{2}\nabla_{y_i} g(y_i)\sigma_i^2(y)p(t,y_i \mid x_0)\Big|_0^\infty - \int_0^\infty \frac{1}{2}\nabla_{y_i} g(y_i)\nabla_{y_i}[\sigma_i^2(y)p(t,y_i \mid x_0)]dy_i \\
&= -\int_0^\infty \frac{1}{2}\nabla_{y_i} g(y_i)\nabla_{y_i}[\sigma_i^2(y)p(t,y_i \mid x_0)]dy_i \\
&= -\frac{1}{2}g(y_i)\nabla_{y_i}[\sigma_i^2(y)p(t,y_i \mid x_0)]\Big|_0^\infty + \frac{1}{2}\int_0^\infty g(y_i)\nabla^2_{y_i}[\sigma_i^2(y)p(t,y_i \mid x_0)]dy_i \\
&= \frac{1}{2}\int_0^\infty g(y_i)\nabla^2_{y_i}[\sigma_i^2(y)p(t,y_i \mid x_0)]dy_i,
\end{aligned}$$

so

$$\begin{aligned}
&\int_0^\infty g(y_i)\nabla_t p(t,y_i \mid x_0)dy_i \\
&\qquad = \int_0^\infty g(y_i)\Big\{-\nabla_{y_i}[\mu_i(y)p(t,y_i \mid x_0)] + \frac{1}{2}\nabla^2_{y_i}[\sigma_i^2(y)p(t,y_i \mid x_0)]\Big\}dy_i
\end{aligned}$$

and we have the implied relationship

$$\nabla_t p(t,y_i \mid x_0) = -\nabla_{y_i}[\mu_i(y)p(t,y_i \mid x_0)] + \frac{1}{2}\nabla^2_{y_i}[\sigma_i^2(y)p(t,y_i \mid x_0)] \tag{5.4.6}$$

with the initial condition

$$\lim_{t\to t_0} p(t,y_i \mid x_0) = \delta_{x_i(0)}(y_i),$$

where $\delta_{x_i(0)}(y_i)$ is the Dirac delta function. The Kolmogorov forward equation can be written in terms of the operator $\mathcal{A}^*$, which is the adjoint of $\mathcal{A}$ (from the Kolmogorov backward equation),

$$\nabla_t p(t,y_i \mid x_0) = \mathcal{A}_t^{i*} p(t,y_i \mid x_0).$$

5.5 Appendix 5.B

For each component $X_i(t)$ of $X(t)$, we have a one dimensional diffusion process

$$dX_i(t) = \acute{\mu}_i X_i(t)dt + \acute{\sigma}_i(X(t))X_i(t)dB_i(t). \tag{5.5.1}$$

The derivation of the Kolmogorov forward equation for the 1-dimensional case is as follows. Suppose we have the diffusion process shown in equation (5.5.1), and $x_0 = (x_1 = X_1(0), x_2 = X_2(0), \ldots, x_n = X_n(0))$, then

$$\begin{aligned} &\mathbb{E}[g(X_i(t)) \mid X(0) = x_0] = g(X_i(0)) \\ &+ \ \mathbb{E}[\int_{t_0}^{t} \{X_i(s)\nabla_{x_i} g(X_i(s))\acute{\mu}_i + \frac{1}{2}X_i^2(s)\nabla_{x_i}^2 g(X_i(s))\acute{\sigma}_i^2\} ds \mid X(0) = x_0]. \end{aligned} \tag{5.5.2}$$

From the definition of expectation in equation (5.4.2) and the equation above

$$\mathbb{E}[\int_{t_0}^{t} X_i(s)\nabla_{x_i} g(X_i(s))\acute{\mu}_i ds \mid X(0) = x_0] = \int_{t_0}^{t}\int_{0}^{\infty} y_i \nabla_{y_i} g(y_i)\acute{\mu}_i p(s, y_i \mid x_0) dy_i ds$$

and

$$\mathbb{E}[\int_{t_0}^{t} \frac{1}{2}X_i^2(s)\nabla_{x_i}^2 g(X_i(s))\acute{\sigma}_i^2 ds \mid X(0) = x_0] = \int_{t_0}^{t}\int_{0}^{\infty} \frac{1}{2}y_i^2 \nabla_{y_i}^2 g(y_i)\acute{\sigma}_i^2 p(s, y_i \mid x_0) dy_i ds,$$

so

$$\begin{aligned} \int_0^\infty p(t, y_i \mid x_0) g(y_i) dy_i &= g(X_i(0)) + \int_{t_0}^{t}\int_{0}^{\infty} y_i \nabla_{y_i} g(y_i)\acute{\mu}_i p(s, y_i \mid x_0) dy_i ds \\ &+ \int_{t_0}^{t}\int_{0}^{\infty} \frac{1}{2}y_i^2 \nabla_{y_i}^2 g(y_i)\acute{\sigma}_i^2 p(s, y_i \mid x_0) dy_i ds. \end{aligned} \tag{5.5.3}$$

Differentiating both sides with respect to t, we have

$$\begin{aligned} \int_0^\infty \nabla_t p(t, y_i \mid x_0) g(y_i) dy_i &= \int_0^\infty \{y_i \nabla_{y_i} g(y_i)\acute{\mu}_i p(t, y_i \mid x_0) \\ &+ \frac{1}{2}y_i^2 \nabla_{y_i}^2 g(y_i)\acute{\sigma}_i^2 p(t, y_i \mid x_0)\} dy_i. \end{aligned}$$

Integrating the first term of the right hand side of equation (5.5.4) by parts, we obtain

$$\begin{aligned} &\int_0^\infty y_i \nabla_{y_i} g(y_i)\acute{\mu}_i p(t, y_i \mid x_0) dy_i \\ &= y_i g(y_i)\mu_i(y) p(t, y_i \mid x_0)\Big|_0^\infty - \int_0^\infty g(y_i)\nabla_{y_i}[\mu_i(y) p(t, y_i \mid x_0)] dy_i \\ &= -\int_0^\infty g(y_i)\nabla_{y_i}[y_i\mu_i(y) p(t, y_i \mid x_0)] dy_i \end{aligned}$$

since the first component is zero. Similarly, integrating the second term on the right hand side of equation (5.5.4) by parts, we obtain

$$\begin{aligned}
&\int_0^\infty \frac{1}{2}y_i^2\nabla_{y_i}^2 g(y_i)\acute{\sigma}_i^2 p(t,y_i \mid x_0)dy_i \\
&= \frac{1}{2}y_i^2\nabla_{y_i} g(y_i)\acute{\sigma}_i^2 p(t,y_i \mid x_0)\Big|_0^\infty - \int_0^\infty \frac{1}{2}y_i^2\nabla_{y_i} g(y_i)\nabla_{y_i}[y_i^2\acute{\sigma}_i^2 p(t,y_i \mid x_0)]dy_i \\
&= -\int_0^\infty \frac{1}{2}y_i^2\nabla_{y_i} g(y_i)\nabla_{y_i}[y_i^2\acute{\sigma}_i^2 p(t,y_i \mid x_0)]dy_i \\
&= -\frac{1}{2}g(y_i)\nabla_{y_i}[y_i^2\acute{\sigma}_i^2 p(t,y_i \mid x_0)]\Big|_0^\infty + \frac{1}{2}\int_0^\infty g(y_i)\nabla_{y_i}^2[y_i^2\acute{\sigma}_i^2 p(t,y_i \mid x_0)]dy_i \\
&= \frac{1}{2}\int_0^\infty g(y_i)\nabla_{y_i}^2[y_i^2\acute{\sigma}_i^2 p(t,y_i \mid x_0)]dy_i,
\end{aligned}$$

so

$$\begin{aligned}
&\int_0^\infty g(y_i)\nabla_t p(t,y_i \mid x_0)dy_i \\
&= \int_0^\infty g(y_i)\big\{-\nabla_{y_i}[y_i\acute{\mu}_i p(t,y_i \mid x_0)] + \frac{1}{2}\nabla_{y_i}^2[y_i^2\acute{\sigma}_i^2 p(t,y_i \mid x_0)]\big\}dy_i
\end{aligned}$$

and we have the implied relationship

$$\nabla_t p(t,y_i \mid x_0) = -\nabla_{y_i}[y_i\acute{\mu}_i p(t,y_i \mid x_0)] + \frac{1}{2}\nabla_{y_i}^2[y_i^2\acute{\sigma}_i^2 p(t,y_i \mid x_0)]$$

with the initial condition

$$\lim_{t\to t_0} p(t,y_i \mid x_0) = \delta_{x_i(0)}(y_i),$$

where $\delta_{x_i(0)}(y_i)$ is the Dirac delta function. The Kolmogorov forward equation can be written in terms of the operator $\mathcal{J}^*$;

$$\nabla_t p(t,y_i \mid x_0) = \mathcal{J}_t^{i*} p(t,y_i \mid x_0).$$

Chapter 6

Concluding Remarks

Refinements which could be carried out as part of a post-doctoral project are as follows,

(a) Hedging and Sensitivity Calculations. Hedging and sensitivity calculations for CDOs is a under-explored area. No research has been done on hedging CDOs which have been priced using distorted copulas. There are very few results of sensitivity tests which show how variations in individual CDS prices influence the final tranche prices. Similarly, tests on the effect of changing individual default distributions have also not been carried out, so research could be carried out in that area.

(b) Extending Specialized CDO Models. It is possible to extend the CDO models of Greenberg et al. [63] and Schönbucher [147]. For example, in the case of [63], one could extend the work to four dimensions or more providing one has an algorithm for calculating the multivariate normal distribution. It may be possible to form a similar type of implementation using other types of factor copulas, approximations, etc.

(c) Making Links to Individual Default. Asset values are usually not observable on the market. Instead of asset values, analysts have to rely on equity, volatility and other parameters [13]. Therefore, there may be room for research into which variables can be used as the best indicators of a firm's probability of default.

(d) Quantile Research. There is still much to be explored in relation to the use of pth quantiles, which involves the generalization of work done in Chapter 2 on the conditional mean. Numerical experiments could be carried out to find properties of the conditional median of copulas and distorted copulas.

(e) Estimating Parameters. Parameters in copulas can be estimated in several ways, maximum likelihood method, method of inference functions for margins, canonical maximum likelihood method and using dependence measures such as Kendall's tau, the empirical copula and other techniques related to moment generating functions. Some of these methods could be developed further, in order to make parameter estimation easier.

(f) The Squaring of CDOs. Another under-explored area in the world of CDOs is the CDO^2. This squaring of CDOs involves taking a portfolio of synthetic CDO tranches and tranching those, and thereby providing an extra layer to the so-called capital structure [7]. The authors in [7] have started modelling CDO^2s. The super tranches created in that model need to be priced. Thus we have yet another area where work could be done in the future.

(g) Distorting n-copulas. The theory of distorted copulas may be extended to higher dimensions, so that sampling experiments and data fitting in n dimensions can be carried out. This would include generating formulae for n-copula densities.

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