

# Copulas for Credit Derivative Pricing and other Applications

Glenis Jayne Crane B.Sc., B.A. (Hons), M.Sc., Ph.D.

*Thesis submitted for the degree of*

*Doctor of Philosophy*

*in*

*Applied Mathematics*

*at*

*The University of Adelaide*

*(Faculty of Engineering, Computer and Mathematical Sciences)*

School of Mathematical Sciences



April 5, 2009

March 23, 2009

# Contents

<b>Signed Statement</b>	<b>ix</b>
<b>Acknowledgements</b>	<b>x</b>
<b>Publications Arising from this Study</b>	<b>xi</b>
<b>Abstract</b>	<b>xii</b>
<b>Introduction and Overview</b>	<b>1</b>
<b>1 Literature Review</b>	<b>5</b>
1.1 Copula Functions and Their Applications . . . . .	5
1.2 Definition and Basic Properties . . . . .	7
1.2.1 Copula Functions . . . . .	7
1.2.2 Quasi-Copula Functions . . . . .	9
1.2.3 Tail Dependence Formulae . . . . .	9
1.3 Classes of Copulas . . . . .	10
1.3.1 Elliptical Copulas . . . . .	10
1.3.2 Bertino Copulas . . . . .	11
1.3.3 Copulas with Quadratic Section . . . . .	11
1.3.4 Marshall-Olkin Copulas . . . . .	13
1.3.5 Archimedean Copulas . . . . .	13
1.3.6 Periodic Copulas . . . . .	15

1.3.7	Generalized Diagonal Band Copulas . . . . .	16
1.3.8	Copulas with Fractal Supports . . . . .	17
1.3.9	Empirical Copulas . . . . .	17
1.3.10	Bernstein Copulas . . . . .	18
1.4	Methods of Constructing new 2-copulas . . . . .	19
1.4.1	Convex Combinations of Copulas . . . . .	19
1.4.2	Bivariate Iterated Copulas . . . . .	20
1.4.3	Transformation or Distortion of a 2-copula . . . . .	20
1.4.4	$n$ -copulas . . . . .	22
1.4.5	Methods of Constructing $n$ -copulas . . . . .	23
1.5	Quantiles for Copulas . . . . .	25
1.6	Dynamic Copula Models . . . . .	26
1.7	Practical and Theoretical Applications . . . . .	27
1.7.1	Theoretical Applications . . . . .	31
1.7.2	Estimating Copula Parameters . . . . .	32
1.7.3	Goodness of Fit Tests . . . . .	34
1.8	Collateralized Debt Obligations (CDOs) . . . . .	34
1.8.1	Calibrating Marginal Default Probability . . . . .	38
1.8.2	Correlated Default and Asset Correlation Coefficients . . . . .	44
1.8.3	Correlation factors and Log-Asset Returns . . . . .	47
1.8.4	Survival copulas . . . . .	48
1.8.5	Independent Defaults in a Binomial Framework . . . . .	49
1.8.6	Gaussian One Factor Model (GOFM) . . . . .	50
1.8.7	The Structure of a CDO . . . . .	53
1.8.8	The Structure of a Synthetic CDO . . . . .	55
1.8.9	The Pricing of CDO Tranches . . . . .	57
1.9	Inadequacies of the Gaussian Model . . . . .	61
1.9.1	Base Correlation Model . . . . .	62
1.9.2	Value At Risk and Shortfall . . . . .	62

1.10	Alternatives to the GOFM . . . . .	63
1.10.1	Other Factor Copula Models . . . . .	64
1.10.2	Alternatives to Factor Models . . . . .	66
<b>2</b>	<b>Copula-based Regression Formulae</b>	<b>67</b>
2.1	Introduction . . . . .	67
2.1.1	Background . . . . .	68
2.1.2	Farlie-Gumbel-Morgenstern Copulas . . . . .	71
2.1.3	Iterated FGM Distributions . . . . .	75
2.1.4	Gaussian Copula . . . . .	78
2.1.5	Archimedean Copulas . . . . .	79
2.1.6	Other simple copulas . . . . .	83
2.2	Conclusion . . . . .	84
2.3	Appendix 2.A . . . . .	85
2.4	Appendix 2.B . . . . .	86
<b>3</b>	<b>Pricing Synthetic CDOs</b>	<b>88</b>
3.1	Introduction . . . . .	88
3.2	Distortions of Copulas . . . . .	89
3.2.1	Distortions Described by Durrleman, Nikeghbali and Roncalli. . . . .	91
3.2.2	Distortions Described by Morillas. . . . .	92
3.2.3	New Distortions . . . . .	93
3.2.4	Composition of Distortions . . . . .	95
3.2.5	Conditional Distributions Expressed as Copulas . . . . .	96
3.3	Distorted Gaussian Copula Model . . . . .	96
3.3.1	Model 1: JPMorgan CDO Pricing Model . . . . .	98
3.3.2	Model 2: Gibson Algorithm with Recursion . . . . .	105
3.4	Conclusion . . . . .	112

<b>4</b>	<b>Constructing <math>n</math>-Copulas</b>	<b>114</b>
4.1	Background . . . . .	115
4.2	Method 1 for $n$ -copula Construction . . . . .	116
4.3	Bivariate Copulas Containing Distortions . . . . .	117
4.3.1	Other Subclasses of Copulas . . . . .	121
4.4	Method 2 for $n$ -copula Construction . . . . .	122
4.5	Summary and Suggestions for Future Work . . . . .	125
<b>5</b>	<b>Time and Space Dependent Copulas</b>	<b>126</b>
5.1	Introduction and motivation . . . . .	126
5.1.1	Notation and Definitions . . . . .	127
5.1.2	Method of Darsow et al . . . . .	128
5.1.3	Conditional Copula of Patton . . . . .	130
5.1.4	Pseudo-copulas of Fermanian and Wegkamp . . . . .	132
5.1.5	Galichon model . . . . .	133
5.2	$n$ -dimensional Galichon Model for CDOs . . . . .	136
5.2.1	Generalized $n$ -dimensional model with uncorrelated Brownian Motions. . . . .	145
5.3	Geometric Brownian Motion Model . . . . .	154
5.4	Appendix 5.A . . . . .	159
5.5	Appendix 5.B . . . . .	161
<b>6</b>	<b>Concluding Remarks</b>	<b>164</b>
	<b>Bibliography</b>	<b>166</b>

# List of Tables

1.4.1 Distortions of Durreleman, Nikeghbali and Roncalli. . . . .	22
1.8.1 Standard and Poor's rating matrix (%) . . . . .	40
1.8.2 One year transition probabilities for a <i>BB</i> rated obligor (%) . . .	44
1.8.3 FEB 2006 iTraxx Europe Series 4 quotes . . . . .	56
3.2.1 Distortions described by Durreleman, Nikeghbali and Roncalli. . .	91
3.2.2 New distortions from combinations of known distortions. . . . .	92
3.2.3 Distortions described by Morillas. . . . .	92
3.3.1 iTraxx Europe series 3 and 4 MID quotes. . . . .	101
3.3.2 Comparison 1 of simulated fair prices, 5yrs. . . . .	102
3.3.3 Comparison 2 of simulated fair prices, 5yrs. . . . .	102
3.3.4 Comparison 3 of simulated fair prices, 5yrs. . . . .	102
3.3.5 Comparison 4 of simulated fair prices, 5yrs. . . . .	103
3.3.6 Comparison 5 of simulated fair prices, 1yr. . . . .	103
3.3.7 Comparison 6 of simulated fair prices, 1yr. . . . .	104
3.3.8 Comparison 7 of simulated fair prices, 1yr. . . . .	104
3.3.9 Comparison 8 of simulated fair prices, 1yr. . . . .	104
3.3.10 Comparison 1 of simulated fair prices using recursion, 5yrs. . . .	108
3.3.11 Comparison 2 of simulated fair prices using recursion, 5yrs. . . .	109
3.3.12 Comparison 3 of simulated fair prices using recursion, 5yrs. . . .	109
3.3.13 Comparison 4 of simulated fair prices using recursion, 5yrs. . . .	110
3.3.14 Comparison 5 of simulated fair prices using recursion, 5yrs. . . .	110

3.3.15	Comparison 6 of simulated fair prices using recursion, 5yrs. . . .	110
3.3.16	Comparison 7 of simulated fair prices using recursion, 1yr. . . .	110
3.3.17	Comparison 8 of simulated fair prices using recursion, 1yr. . . .	111
3.3.18	Comparison 9 of simulated fair prices using recursion, 1yr. . . .	111
3.3.19	Comparison 10 of simulated fair prices using recursion, 1yr. . . .	111
3.3.20	Comparison 11 of simulated fair prices using recursion, 1yr. . . .	112
3.3.21	Comparison 12 of simulated fair prices using recursion, 1yr. . . .	112



# List of Figures

1.3.1 Density for FGM Copula distribution in (1.3.4) . . . . .	12
1.3.2 Sampled points from Clayton Copula . . . . .	14
1.3.3 Density for copula distribution in (1.3.12) . . . . .	15
1.3.4 Density for copula distribution in (1.3.13) . . . . .	16
1.4.1 Contours of Gaussian and transformed density . . . . .	22
2.1.1 Scatter plot and conditional mean of Y, given X, for FGM model 1 (simulation with normal marginal distributions). . . . .	73
2.1.2 Scatter plot and conditional mean of Y, given X, for FGM model 4 (simulation with exponential marginal distributions). . . . .	74
2.1.3 Conditional mean and scatter plot of annual Discharge (days), given Peak Discharge ( $cm^3/s$ ). . . . .	76
2.1.4 Scatter plot and conditional mean of JPY/AUD, given the CHF/AUD.	79
2.1.5 Scatter plot and conditional mean of HKD/AUD, given KRW/AUD.	81
2.1.6 Scatter plot and conditional mean of waist size of male subjects, given their forearm size. . . . .	81
2.1.7 Scatter plot and conditional mean of chest size of male subjects, given their waist size. . . . .	82
3.2.1 Comparison of distortions. . . . .	93
3.2.2 Piecewise linear distortion. . . . .	95
3.3.1 Comparison of Expected tranche loss. . . . .	100

4.3.1 Copula density of distribution in (4.3.8) . . . . .	120
4.3.2 Copula density for distribution in (4.3.9) . . . . .	122

# Signed Statement

## Declaration

This work contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text.

I give consent to this copy of my thesis, when deposited in the University Library, being made available for loan and photocopying, subject to the provisions of the Copyright Act 1968.

SIGNED: ..... DATE: .....

# Acknowledgements

There are a number of people I would like to acknowledge for their help during this study. I would like to thank Associate Professor John van der Hoek for his support on this project. John was my principal supervisor until he accepted a teaching position at the University of South Australia. Similarly, I would like to thank Dr Alexei Filinkov, who took on the role of principal supervisor later in the project, for his ongoing support.

I would also like to thank Dr Andrew Metcalfe and Professor Robert Elliott for their advice and for reading parts of my thesis, Dr Liz Cousins and Associate Professor Jim Denier for dealing with many student matters, and also Diane Parish, Dr Peter Gill and Professor Charles Pearce for their kind words and ongoing support. I would also like to thank the School of Mathematics administration staff for their help. Lastly, I want to thank my friends Cheryl Pointon, Ingrid Ahmer, my mum, brother Jeffrey and sister Deborah for their enduring friendship.

# Publications Arising from this Study

Crane G.J. and van der Hoek, J., (2008) Conditional Expectation and Copulas. Australian and New Zealand Journal of Statistics **50**, 1-15.

Crane G.J. and van der Hoek, J., (2008) Using distortions of copulas to price synthetic CDOs. Insurance: Mathematics and Economics **42**, 903-908.

# Abstract

Copulas are multivariate probability distributions, as well as functions which link marginal distributions to their joint distribution. These functions have been used extensively in finance and more recently in other disciplines, for example hydrology and genetics. This study has two components, (a) the development of copula-based mathematical tools for use in all industries, and (b) the application of distorted copulas in structured finance. In the first part of this study, copula-based conditional expectation formulae are described and are applied to small data sets from medicine and hydrology. In the second part of this study we develop a method of improving the estimation of default risk in the context of collateralized debt obligations. Credit risk is a particularly important application of copulas, and given the current global financial crisis, there is great motivation to improve the way these functions are applied. We compose distortion functions with copula functions in order to obtain greater flexibility and accuracy in existing pricing algorithms. We also describe an  $n$ -dimensional dynamic copula, which takes into account temporal and spatial changes.