

Loop Groups, Higgs Fields and Generalised String Classes

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Abstract

We consider various generalisations of the string class of a loop group bundle. The string class is the obstruction to lifting a bundle whose structure group is the loop group LG to one whose structure group is the Kac-Moody central extension of the loop group.

We develop a notion of higher string classes for bundles whose structure group is the group of based loops, ΩG . In particular, we give a formula for characteristic classes in odd dimensions for such bundles which are associated to characteristic classes for G -bundles in the same way that the string class is related to the first Pontrjagyn class of a certain G -bundle associated to the loop group bundle in question. This provides us with a theory of characteristic classes for ΩG -bundles analogous to Chern-Weil theory in finite dimensions. This also gives us a geometric interpretation of the well-known transgression map $H^{2k}(BG) \rightarrow H^{2k-1}(G)$.

We also consider the obstruction to lifting a bundle whose structure group is not the loop group but the semi-direct product of the loop group with the circle, $LG \rtimes S^1$. We review the theory of bundle gerbes and their application to central extensions and lifting problems and use these methods to obtain an explicit expression for the de Rham representative of the obstruction to lifting such a bundle. We also relate this to a generalisation of the so-called ‘caloron correspondence’ (which relates LG -bundles over M to G -bundles over $M \times S^1$) to a correspondence which relates $LG \rtimes S^1$ -bundles over M to G -bundles over S^1 -bundles over M .

Signed Statement

This work contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text.

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