TRANSIENT MODELLING OF ARBITRARY PIPE NETWORKS BY A LAPLACE-DOMAIN ADMITTANCE MATRIX

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Abstract

An alternative to modeling of the transient behavior of pipeline systems in the time-domain is to model these systems in the frequency-domain using Laplace transform techniques. Despite the ability of current methods to deal with many different hydraulic element types, a limitation with almost all frequency-domain methods for pipeline networks is that they are only able to deal with systems of a certain class of configuration, namely, networks not containing second order loops. This paper addresses this limitation by utilizing graph theoretic concepts to derive a Laplace-domain network admittance matrix relating the nodal variables of pressure and demand for a network comprised of pipes, junctions and reservoirs. The adopted framework allows complete flexibility with regard to the topological structure of a network and as such, it provides an extremely useful general basis for modeling the frequency-domain behavior of pipe networks. Numerical examples are given for a 7-pipe and 51-pipe network, demonstrating the utility of the method.

INTRODUCTION

Modeling of the transient behavior of fluid transmission line (pipeline) networks is of interest in many applications including hydraulic and pneumatic control systems [*Boucher and Kitsios*, 1986], biological systems (*e.g.* arterial blood flow) [*John*, 2004], and pipeline distribution systems (*e.g.* gas, petroleum, and water) [*Fox*, 1977; *Chaudhry*, 1987; *Wylie and Streeter*, 1993]. Two approaches for modeling such systems are discretized time-domain methods (*e.g.* the method of characteristics (MOC) [*Wylie and Streeter*, 1993]) or linearized frequency-domain (or Laplace-domain) methods. The focus of this paper is on the latter of the two methods.

A pipeline network's transient behavior can be completely described in the frequency-domain by the frequencies dependent distribution of magnitude and phase of the fluid variables (the variable of interest are typically pressure and flow), as opposed to the time-domain representation of temporal fluctuations in these variables. Frequency-domain models are used to compute the relationship between the frequency distribution of the transient fluid variables at any points of interest within the system.

Frequency-domain models are given by the solution of the Laplace-transform of the linearized underlying fluid equations. An advantage of frequency-domain methods is that the true distributed space/continuous time nature of the system is retained and analytic relationships between system components and the transient behavior of system can be derived. It is this latter point of the amenability of frequency-domain methods to analytic work that has seen its emergence in the field of pipe leak and blockage detection (*e.g.* [*Lee et al.*, 2005; *Mohapatra et al.*, 2006]). The analytic nature of frequency-domain methods is that they are extremely computationally efficient in comparison to their costly numerical time-domain counterparts [*Zecchin et al.*, 2005]. Additionally, the absence of discretization schemes by these methods means that complications with organizing the computational grid to satisfy the Courant condition are avoided *Kim* [2007].

The two main approaches used construct frequency-domain representations of pipeline systems are the transfer matrix method [*Chaudhry*, 1970, 1987] and the impedance method [*Wylie*, 1965; *Wylie and Streeter*, 1993]. The transfer matrix method is extremely versatile as it can be applied to a broad class of systems involving many different hydraulic elements. However, despite this utility, a limitation is that it is only able to deal with networks of a certain class of configuration, namely systems containing only first order loops [*Fox*, 1977] (explained later). The impedance method can, theoretically, be applied to any system (comprised of elements for which impedance relationship exist), but the algebraic nature of the method has seen its application to only relatively simple first order systems. Recently, *Kim* [2007] presented a method for systematically organizing the impedance equations into a matrix form to facilitate the application of the impedance method to systems of an arbitrary configuration called the *impedance matrix method*.

Within this paper, an alternative systematic approach to developing a frequency-domain model of a pipe network of arbitrary configuration is developed. The arbitrary network is posed in a graph-theoretic framework (similar to that used with the treatment of steady state pipe networks [*Collins et al.*, 1978] and transient electrical circuits [*Desoer and Kuh*, 1969; *Chen*, 1983]) from which matrix relationships are derived, relating the unknown nodal pressures and flows to the known nodal pressures and flows. As such, and *admittance matrix* characterization of the network is achieved. This work focuses only on networks comprised of reservoirs, junctions and pipes. The importance of this work is that it provides a systematic, analytic model of pipe networks that is not limited in the class of network configuration that can be addressed.

BACKGROUND

Fluid line network equations

Given a network comprised of a set of nodes $\mathcal{N} = \{1, 2, ..., n_n\}$ and fluid lines $\Lambda = \{1, 2, ..., n_\lambda\}$, the network problem involves the solution of the distributions of pressure p_j and flow q_j along the lines $j \in \Lambda$ subject to the boundary conditions at the n_n nodes. Equations (1)-(7) below outline outline the network equations, and can be divided into the following four groups: (1) and (2) are the fluid dynamic equations of motion and mass continuity for each fluid line; (3) and (4) are the nodal equations of equal pressures in pipe ends connected to the same node for junctions (nodes for which the inline pressure is the free variable) and reservoirs (nodes for which the outflow is the free variable) respectively; (5) and (6) are the nodal equations of mass conservation for junctions and demand nodes; and, (7) is the initial conditions. The network problem can be stated as the solution of the distributions $p_j, q_j, j \in \Lambda$ for time $t \in \mathbb{R}_+$ where

$$\frac{\partial q_j}{\partial t} + \frac{A_j}{\rho}\frac{\partial p_j}{\partial x} + \tau_j\left(q_j\right) = 0, \qquad x \in [0, l_j],\ j \in \Lambda, \tag{1}$$

$$\frac{\partial p_j}{\partial t} + \frac{\rho c_j^2}{A_j}\frac{\partial q_j}{\partial x} = 0, \qquad x \in [0, l_j],\ j \in \Lambda, \tag{2}$$

$$p_j(\varphi_{j,i}, t) = p_k(\varphi_{k,i}, t), \qquad j, k \in \Lambda_i,\ i \in \mathcal{N}/\mathcal{N}_r \tag{3}$$

$$p_j(\varphi_{j,i}, t) = \psi_{r,i}(t), \qquad j \in \Lambda_i,\ i \in \mathcal{N}_r, \tag{4}$$

$$\sum_{j \in \Lambda_{d,i}} q_j(\varphi_{j,i}, t) - \sum_{j \in \Lambda_{u,i}} q_j(\varphi_{j,i}, t) = 0, \qquad i \in \mathcal{N}/\left(\mathcal{N}_d \cup \mathcal{N}_r\right) \tag{5}$$

$$\sum_{j \in \Lambda_{d,i}} q_j(\varphi_{j,i}, t) - \sum_{j \in \Lambda_{u,i}} q_j(\varphi_{j,i}, t) = \theta_{d,i}(t), \quad i \in \mathcal{N}_d \tag{6}$$

$$p_j(x, 0) = p_j^0(x),\ q_j(x, 0) = q_j^0(x), \qquad x \in [0, l_j], j \in \Lambda \tag{7}$$

where the symbols are defined as follows: for the fluid lines x is the axial coordinate, ρ is the fluid density, c_j is the fluid line wavespeed for pipe j, A_j is the cross-sectional area, τ_j is the cross sectional frictional resistance, and l_j is the pipe length; for the nodes, Λ_i is the set of pipes connected to node i, $\Lambda_{u,i}$ ($\Lambda_{d,i}$) is the set of pipes for which node i is upstream (downstream), $\psi_{r,i}$ is the controlled (known) temporally varying reservoir pressure for the reservoir nodes in the reservoir node set $\mathcal{N}_r$, $\theta_{d,i}$ is the controlled (known) temporally varying nodal demand for the demand nodes in the demand node set $\mathcal{N}_d$; p_j^0 and q_j^0 are the initial distribution of pressure and flow in each pipe $j \in \Lambda$; and $\varphi_{j,i}$ is a special function, defined on Λ_i, to indicate the end of pipe j that is incident to node i, that is

$$\varphi_{j,i} = \begin{cases} 0 & \text{if } j \in \Lambda_{u,j} \\ l_j & \text{if } j \in \Lambda_{d,j} \end{cases}.$$

(Note that in (3) and (5), / denotes the minus operation for sets.)

Basic Laplace-Domain Transmission Line Equations

To achieve the requirement of linearity and homogeneous initial conditions, the standard approach for Laplace-domain methods is to linearize the system (1)-(6) about the initial conditions (7) [*Chaudhry*, 1987; *Wylie and Streeter*, 1993] and consider the transient fluctuations in p_j and q_j about these values. The nonlinearities arise in the frictional loss term τ_j in (1) for turbulent flows only.

As with many systems of PDEs that describe wave propagation, the linearized (1) and (2) can be expressed as the transformed telegrapher's equations [*Brown*, 1962; *Stecki and Davis*, 1986]

$$\begin{cases} Z_{s,j}(x,s)\,Q_j(x,s) = -\dfrac{\partial P_j(x,s)}{\partial x} \\ Y_{s,j}(x,s)\,P_j(x,s) = -\dfrac{\partial Q_j(x,s)}{\partial x} \end{cases} \tag{8}$$

on $x \in [0, l_j]$, where $s \in \mathbb{C}$ is the Laplace variable ($\mathbb{C}$ is the set of complex numbers), P_j is the transformed pressure, Q_j is the transformed flow, $Z_{s,j}$ is the series impedance per unit length (describes the effect of mass flow on the pressure gradient) and $Y_{s,j}$ is the shunt admittance per unit length (describes the compressibility effect in the flow driven by the pressure). Despite the simplicity of (8), as $Z_{s,j}$ and $Y_{s,j}$ are transforms of linear operators, (8) can be used to describe a range of fluid line types including unsteady friction and compressible flows [*Stecki and Davis*, 1986]. For a uniform line, an elegantly simple expression for wave propagation results from (8), namely

$$\begin{cases} P_j(x,s) = A_j(s)e^{-\widetilde{\Gamma}_j(s)x} + B_j(s)e^{\widetilde{\Gamma}_j(s)x} \\ Q_j(x,s) = \left(A_j(s)e^{-\widetilde{\Gamma}_j(s)x} - B_j(s)e^{\widetilde{\Gamma}_j(s)x}\right) Z_{c,j}^{-1}(s) \end{cases} \tag{9}$$

on $x \in [0, l_j]$, where $\widetilde{\Gamma}_j(s) = \sqrt{Y_{s,j}(s)Z_{s,j}(s)}$ is the propagation operator [*Brown*, 1962; *Stecki and Davis*, 1986] which essentially describes the frequency dependent attenuation and phase change per unit length that a traveling wave experiences, and $Z_{c,j}(s) = \sqrt{Z_{s,j}(s)/Y_{s,j}(s)}$ is the characteristic impedance of the pipeline, which describes the phase lag and wave magnitude of the flow traveling wave that accompanies a pressure traveling wave, and $A_j(s)$ and $B_j(s)$ are the positive and negative traveling waves forms that are dependent on the boundary conditions to fluid line j.

Within a network setting, explicit boundary conditions A_j and B_j to a pipe cannot be specified, as the boundary conditions are comprised of the interactions of the variables of coincident pipes as governed by the node equations (3)-(6). Therefore, to determine the distributions of P_j and Q_j along each line in a network, methods are required to describe the interaction of the pipes at their endpoints. The existing methods that address these issues are surveyed in the following.

Previous work on frequency-domain methods for networks

Classical Methods for Restricted Types of Pipe Networks

The transfer matrix method [*Chaudhry*, 1970], one of the classical methods for pipeline system modelling, utilizes matrix expressions for each pipe (or hydraulic element) that relate the pressure and flow at the upstream and downstream ends. The resulting end to end transfer matrix of a hydraulic system is achieved by the ordered multiplication of the hydraulic element matrices. An advantage of the transfer matrix method is that it can incorporate a whole range of hydraulic elements (*e.g.* valves, tanks, emitters *etc.*). However, the main limitation, is that it can only be applied to certain network structures, that is, systems with pipes in series, systems with branched pipes, and more generally, systems containing only first order loops [*Fox*, 1977]. First order loops are loops that are either disjoint or nested in only one of the arcs of the outer loop. An example of first and second order looping is given in Figure 1.

The other classical method for the frequency-domain modelling of pipeline systems is the impedance method [*Wylie*, 1965]. This approach adopts a system description in terms of the distribution of hydraulic impedance throughout the system, where the hydraulic impedance at a point is defined as the ratio of transformed pressure to transformed flow. Upstream to downstream impedance functions for each hydraulic element are used to describe the variation in impedance across each element. As with the transfer matrix method, a strength of the impedance method is that it can be generalized to be applied to any system involving arbitrary hydraulic elements. Theoretically, this method can be applied to networks of arbitrary configuration by simultaneously solving the nonlinear end to end impedance functions. However, the large algebraic effort required by the impedance method has traditionally seen its application to only simple first order networks.

Current Methods for Modeling Arbitrary Networks

There has been limited application of Laplace-domain methods for modeling arbitrarily configured pipe networks, and these are briefly surveyed below.

In *Ogawa* [1980]; *Ogawa et al.* [1994], system matrix transfer functions for pressure and velocity sinusoidal amplitude distributions were derived for arbitrary networks. In this work, spatial earthquake vibrations were the transient state driver for the system, and as such, the fluid line equations incorporated axial displacement terms. *Ogawa* [1980]; *Ogawa et al.* [1994] reduce their model to a set of two unknowns for each pipe (one coefficient for each pipe's positive and negative traveling waves).

Muto and Kanei [1980] applied a transfer matrix type approach to a simple second order looping system, however, no general approach for an arbitrary system was outlined in this work. Employing a modal approximation to the transcendental fluid line functions, *Margolis and Yang* [1985] developed a rational transfer function bond graph approximation for a fluid line. This served as the basis for a network model, however, only tree networks were considered.

Recently *John* [2004], applied an impedance based method to a tree network model of the human arterial system.

An alternative methodology of utilizing the frequency-domain pipeline transfer functions within a network setting was adopted by *Reddy et al.* [2006]. In this paper, *Reddy et al.* [2006] analytically invert the rational transfer function approximations proposed *Kralik et al.* [1984] to develop a discrete time-domain network model. Case study specific matrices are constructed to relate the fluid variables at the pipe end points.

Boucher and Kitsios [1986]; *Wang et al.* [2000] employ a transmission line model to describe the pressure wave attenuation within an air pipe network. This work is a simplification of the original work done by *Auslander* [1968], in that the pipes are modelled as pure timedelays, and the resistance effects are lumped at the nodes. The variables within the system are the incident and emergent waves from the pipes to the nodes, for which a scattering matrix equation is set up that describes the relationship between these based on the nodal constraints.

Kim [2007] proposed a model to deal with an arbitrary network structure called the *address oriented impedance matrix*. This method starts from the basis of the set of link and node equations and follows through an algorithm to generate the address matrix that accounts for the network connectivity. All pressure heads are normalized by a reference flow rate, and as such, hydraulic impedance is the fluid variable adopted in this method. This method can be viewed as a systematic generalization of the impedance method to networks of a complicated configuration. Based on an IPREM type approach [*Suo and Wylie*, 1989], the method was successfully used to calibrate the unknown parameters of a hydraulic model to synthetically generated time-domain data *Kim* [2008]. Despite the methods ability to model networks, the algorithm for constructing the address matrix is quite involved and does not fully utilize the structure of the network to reduce the matrix size relating the network variables.

The formulation presented in this paper differs from this past work in that a network admittance matrix is derived. This matrix maps from the network nodal pressures to the nodal outflows. Dealing purely with nodal variables provides a smaller system of equations than that achieved by dealing with wave form coefficients for each pipe. Additionally, graph theoretic concepts implemented in electrical circuit theory were adopted within this formulation. This facilitates a simple and systematic treatment of the network connectivity equations, thus avoiding the need for manual, or algorithm based methods for constructing appropriate network matrices.

NETWORK ADMITTANCE MATRIX FORMULATION

The Laplace-domain admittance matrix equation for the solution of linearized network equations (1)-(6), subject to homogeneous initial conditions (7), is presented in the following. This is the main result of the paper. For convenience the network is treated as a single component graph $\mathcal{G}(\mathcal{N}, \Lambda)$ of arbitrary configuration consisting of the node set $\mathcal{N}$ as defined previously and link set Λ which, in keeping with graph theory notation [*Diestel*, 2000], is redefined as

$$\Lambda = \{\lambda_1, \lambda_2, ..., \lambda_{n_\lambda}\} = \{(i,k) : \exists \text{ a directed link from node } i \text{ to node } k\},$$

where each link describes the connectivity of a pipe and the directed nature of the link describes the sign convention for the positive flow direction.

From (9) it is clear that, for homogeneous initial conditions, the distributions of pressure and flow in a fluid line are uniquely determined by the boundary conditions. In the following it will be shown that the full state of the network (*i.e.* the distributions of pressure and flow along each link) are uniquely determined by the nodal pressures and nodal outflows symbolized by the vectors

$$\boldsymbol{\Psi}(s) = [\Psi_1(s) \cdots \Psi_{n_n}(s)]^T, \quad \boldsymbol{\Theta}(s) = [\Theta_1(s) \cdots \Theta_{n_n}(s)]^T,$$

respectively, where the nodal outflow is a generic term describing the controlled demand for a demand node, the free outflow into (or out of) a reservoir at a reservoir node and zero for a junction. Further, it is shown that these nodal properties are related to each other by the simple equation

$$\boldsymbol{Y}(s)\boldsymbol{\Psi}(s) = \boldsymbol{\Theta}(s) \tag{10}$$

where $\boldsymbol{Y}(s)$ is a $n_n \times n_n$ symmetric matrix function that describes the dynamic *admittance* relationship between all the nodal pressures $\boldsymbol{\Psi}$ and the nodal outflows $\boldsymbol{\Theta}$. That is, the network admittance matrix $\boldsymbol{Y}$ determines the nodal outflows $\boldsymbol{\Theta}$ that are *admitted* from an input of nodal pressures $\boldsymbol{\Psi}$.

Derivation of Network Matrix for an Arbitrary Network Configuration

For each $s \in \mathbb{C}$, the system state is given by the distributions of pressure and flow, $P_j(x_j, s), Q_j(x_j, s)$, on $x_j \in [0, l_j]$ of each line $\lambda_j \in \Lambda$. These states can be represented as the $n_\lambda \times 1$ vectors

$$\boldsymbol{P}(\boldsymbol{x}, s) = [P_1(x_1, s) \cdots P_{n_\lambda}(x_{n_\lambda}, s)]^T,$$
$$\boldsymbol{Q}(\boldsymbol{x}, s) = [Q_1(x_1, s) \cdots Q_{n_\lambda}(x_{n_\lambda}, s)]^T$$

where $\boldsymbol{x} = [x_1, \ldots, x_{n_\lambda}]^T$ is the vector of spatial coordinates for all links. Using this notation, the matrix version of the telegrapher's equations [*Elfadel et al.*, 2002] relating the states $P_j(x_j, s)$ and $Q_j(x_j, s)$ can be formulated. The matrix telegrapher's equations are usually used for parallel multi-transmission lines [*Elfadel et al.*, 2002] or multi-state wave propagation lines [*Brown and Tentarelli*, 2001], where, in such situations the axial coordinate is common to all states.

Here the states represent those from different lines, and as such there is no common axial coordinate, but a vector of coordinates $\boldsymbol{x}$. Therefore, the spatial differential operator takes the form of the diagonal matrix $\mathrm{diag}\, d/d\boldsymbol{x}$ where $d/d\boldsymbol{x} = [d/dx_1 \cdots d/dx_{n_\lambda}]$. The telegrapher's equations for a fluid line network are

$$\boldsymbol{Z_s}(s)\,\boldsymbol{Q}(\boldsymbol{x},s) = -\mathrm{diag}\frac{d}{d\boldsymbol{x}}\,\boldsymbol{P}(\boldsymbol{x},s) \tag{11}$$

$$\boldsymbol{Y_s}(s)\,\boldsymbol{P}(\boldsymbol{x},s) = -\mathrm{diag}\frac{d}{d\boldsymbol{x}}\,\boldsymbol{Q}(\boldsymbol{x},s) \tag{12}$$

where $\boldsymbol{Z_s}$ and $\boldsymbol{Y_s}$ are the diagonal $n_\lambda \times n_\lambda$ series impedance and shunt admittance matrices whose entries correspond to the respective functions for each individual link. Equations (11) and (12) are not simply diagonal for other transmission line types where there is a greater interaction amongst the state variables. For example, for electrical transmission line networks [*Elfadel et al.*, 2002; *Maffucci and Miano*, 1998], the electro-magnetic field associated with the voltage and current on each individual line influences the state distributions on the other lines. Similarly, in the case of vibration analysis tubing systems [*Brown and Tentarelli*, 2001; *Tentarelli and Brown*, 2001], the fluid states and many tube wall states are highly coupled through fluid-structure interactions (*e.g.* Bourdon effect, frequency-dependent wall shear, Poisson coupling). Analogously to (8), (11) and (12) can be solved to yield

$$\boldsymbol{P}(\boldsymbol{x},s) = e^{-\widetilde{\boldsymbol{\Gamma}}(s)\mathrm{diag}\boldsymbol{x}}\boldsymbol{A}(s) + e^{\widetilde{\boldsymbol{\Gamma}}(s)\mathrm{diag}\boldsymbol{x}}\boldsymbol{B}(s), \tag{13}$$

$$\boldsymbol{Q}(\boldsymbol{x},s) = \boldsymbol{Z_c}^{-1}(s)\left[e^{-\widetilde{\boldsymbol{\Gamma}}(s)\mathrm{diag}\boldsymbol{x}}\boldsymbol{A}(s) - e^{\widetilde{\boldsymbol{\Gamma}}(s)\mathrm{diag}\boldsymbol{x}}\boldsymbol{B}(s)\right], \tag{14}$$

where $\boldsymbol{A}, \boldsymbol{B}$ are complex $n_\lambda \times 1$ vector functions whose elements depend on the boundary conditions on $\boldsymbol{P}$ and $\boldsymbol{Q}$, and

$$\begin{aligned} \widetilde{\boldsymbol{\Gamma}}(s) &= [\boldsymbol{Z_s}(s)\boldsymbol{Y_s}(s)]^{\frac{1}{2}} &&= \mathrm{diag}\left\{\widetilde{\Gamma}_1(s), \ldots, \widetilde{\Gamma}_{n_\lambda}(s)\right\}, \\ \boldsymbol{Z_c}(s) &= \left[\boldsymbol{Z_s}(s)\boldsymbol{Y_s}^{-1}(s)\right]^{\frac{1}{2}} &&= \mathrm{diag}\left\{Z_{c,1}(s), \ldots, Z_{c,n_\lambda}(s)\right\}, \end{aligned}$$

are the propagation operator and characteristic impedance matrices respectively.

As expressed in (13) and (14), for each link $\lambda_j \in \Lambda$, the distribution of the state on $x_j \in [0, l_j]$ is entirely dependent on the boundary conditions for the line. As was illustrated in the previous section, the full state of the line can be reconstructed by knowledge of any two of the line's state variables at the line's endpoints. Generalizing this statement to a network, it is seen that the full network state $\boldsymbol{P}(\boldsymbol{x}, s), \boldsymbol{Q}(\boldsymbol{x}, s), \boldsymbol{x} \in [0, l_1] \times \cdots \times [0, l_{n_\lambda}]$ can be constructed from the vector of the state values at the links upstream endpoints ($\boldsymbol{P}(\mathbf{0}, s)$ and $\boldsymbol{Q}(\mathbf{0}, s)$) or the vector of the state values at the links downstream endpoints ($\boldsymbol{P}(\boldsymbol{l}, s)$ and $\boldsymbol{Q}(\boldsymbol{l}, s)$), where, with the adopted notation, the upstream state values for a link occur at $\boldsymbol{x} = \mathbf{0}$ and the link downstream state values occur at $\boldsymbol{x} = \boldsymbol{l} = [l_1, \ldots, l_{n_\lambda}]^T$. In an analogous manner to the single dimensional transfer matrix method [*Chaudhry*, 1987], (13) and (14) can be solved to yield the following $2n_\lambda$ dimensional transfer matrix equations between the upstream variables $\boldsymbol{P}(\mathbf{0}, s), \boldsymbol{Q}(\mathbf{0}, s)$ and the downstream variables $\boldsymbol{P}(\boldsymbol{l}, s), \boldsymbol{Q}(\boldsymbol{l}, s)$. That is

$$\left[\begin{array}{c} \boldsymbol{P}(\boldsymbol{l}, s) \\ \hline \boldsymbol{Q}(\boldsymbol{l}, s) \end{array}\right] = \left[\begin{array}{c|c} \cosh \boldsymbol{\Gamma}(s) & -\boldsymbol{Z_c}(s) \sinh \boldsymbol{\Gamma}(s) \\ \hline -\boldsymbol{Z_c}^{-1}(s) \sinh \boldsymbol{\Gamma}(s) & \cosh \boldsymbol{\Gamma}(s) \end{array}\right] \left[\begin{array}{c} \boldsymbol{P}(\mathbf{0}, s) \\ \hline \boldsymbol{Q}(\mathbf{0}, s) \end{array}\right] \tag{15}$$

where $\boldsymbol{\Gamma}(s) = \widetilde{\boldsymbol{\Gamma}} \operatorname{diag} \boldsymbol{l}$, and the definition of the hyperbolic trigonometric operations on the matrices arises naturally from the definition of the matrix exponential [*Horn and Johnson*, 1991]. Note that (15) is simply a generalization of the standard 2×2 transfer matrix to n_λ independent (unjoined) links.

Equation (15) represents the relationship between the end points of each individual link, but the boundary conditions on each link must be imposed to determine the relationship between the $4\,n_\lambda$ state elements of the link endpoints. As expressed in (3)-(6), the constraints on the link ends incident to common nodes are the continuity of pressure at the link end points attached to each node, and the conservation of mass at each nodal point. Given the vector of nodal pressures $\boldsymbol{\Psi}$, the transform equivalent of (3) and (4) in matrix form is

$$\left[\begin{array}{c}\boldsymbol{P}(\boldsymbol{0},s)\\\hline\boldsymbol{P}(\boldsymbol{l},s)\end{array}\right]=\left[\begin{array}{c|c}\boldsymbol{N_u}&\boldsymbol{N_d}\end{array}\right]^T\boldsymbol{\Psi}(s)\,, \tag{16}$$

where $\boldsymbol{N_u}$ and $\boldsymbol{N_d}$ are $n_n \times n_\lambda$ upstream and downstream topological matrices defined by

$$\{\boldsymbol{N_u}\}_{i,j}=\begin{cases}1 & \text{if} \quad \lambda_j\in\Lambda_{u,i}\\ 0 & \text{otherwise}\end{cases}, \quad \text{and} \quad \{\boldsymbol{N_d}\}_{i,j}=\begin{cases}1 & \text{if} \quad \lambda_j\in\Lambda_{d,i}\\ 0 & \text{otherwise}\end{cases} \tag{17}$$

The sum $\boldsymbol{N_u}+\boldsymbol{N_d}$ is the standard incidence matrix is used to describe the connectivity of undirected graphs and $\boldsymbol{N_u}-\boldsymbol{N_d}$ for directed graphs [*Diestel*, 2000]. It is seen in (16) that the $2n_\lambda$ variables of upstream and downstream pressure are uniquely identified by the n_n variables of nodal pressure. Similarly, given the vector of nodal outflows $\boldsymbol{\Theta}$, the transform of the nodal continuity constraints (5) and (6) can be expressed in the following matrix form

$$\left[\begin{array}{c|c}-\boldsymbol{N_u}&\boldsymbol{N_d}\end{array}\right]\left[\begin{array}{c}\boldsymbol{Q}(\boldsymbol{0},s)\\\hline\boldsymbol{Q}(\boldsymbol{l},s)\end{array}\right]=\boldsymbol{\Theta}(s)\,, \tag{18}$$

which is equivalent to saying that the flow into the node (from the downstream end of the relevant links, *e.g.* $\Lambda_{d,i}$) minus the flow out from the node (into the upstream end of the relevant links, *e.g.* $\Lambda_{u,i}$) is equal to the nodal outflow Θ_i.

By considering (15), (16) and (18), a full set of equations that govern the transient network state is achieved. Keeping in mind that the objective is to determine the admittance relationship between the nodal pressures $\boldsymbol{\Psi}$ and the nodal outflows $\boldsymbol{\Theta}$, it is convenient to express (15) in the form of relating the link end pressures to the link end flows as

$$\left[\begin{array}{c}\boldsymbol{Q}(\boldsymbol{0},s)\\\hline\boldsymbol{Q}(\boldsymbol{l},s)\end{array}\right]=\left[\begin{array}{c|c}\boldsymbol{Z_c}^{-1}(s)\coth\boldsymbol{\Gamma}(s)&-\boldsymbol{Z_c}^{-1}(s)\text{csch}\boldsymbol{\Gamma}(s)\\\hline\boldsymbol{Z_c}^{-1}(s)\text{csch}\boldsymbol{\Gamma}(s)&-\boldsymbol{Z_c}^{-1}(s)\coth\boldsymbol{\Gamma}(s)\end{array}\right]\left[\begin{array}{c}\boldsymbol{P}(\boldsymbol{0},s)\\\hline\boldsymbol{P}(\boldsymbol{l},s)\end{array}\right] \tag{19}$$

where $\coth \boldsymbol{A} = [\tanh \boldsymbol{A}]^{-1}$, and $\mathrm{csch}\boldsymbol{A} = [\sinh \boldsymbol{A}]^{-1}$. Combining (19) with (16) and (18) yields the following relationship between the nodal pressures and outflows

$$\boldsymbol{\Theta}(s) = \left[\begin{array}{c|c} \boldsymbol{N_u} & \boldsymbol{N_d} \end{array}\right] \left[\begin{array}{c|c} -\boldsymbol{Z_c}^{-1}(s)\coth \boldsymbol{\Gamma}(s) & \boldsymbol{Z_c}^{-1}(s)\mathrm{csch}\boldsymbol{\Gamma}(s) \\ \hline \boldsymbol{Z_c}^{-1}(s)\mathrm{csch}\boldsymbol{\Gamma}(s) & -\boldsymbol{Z_c}^{-1}(s)\coth \boldsymbol{\Gamma}(s) \end{array}\right] \left[\begin{array}{c|c} \boldsymbol{N_u} & \boldsymbol{N_d} \end{array}\right]^T \boldsymbol{\Psi}(s) \quad (20)$$

The expression (20) has an elegant structure to it that is worth some discussion. The dynamics of the system (*i.e.* the pressure to flow transfer functions for each link) are contained completely within the inner matrix, as the incidence matrices $\boldsymbol{N_u}$ and $\boldsymbol{N_d}$ are simply constant matrices with elements either 0 or 1. The connectivity constraints of the network are described by the pre- and post-multiplying of the block incidence matrix $[\boldsymbol{N_u}|\boldsymbol{N_d}]$ and its transpose. The action of the post-multiplication by $[\boldsymbol{N_u}|\boldsymbol{N_d}]^T$ can be seen as the mapping from the n_n nodal pressures to the $2n_\lambda$ link end pressures, as in (16). The inner matrix in (20) then maps from the link end pressures to the link end outflows, as in (19). Finally, the pre-multiplication of $[\boldsymbol{N_u}|\boldsymbol{N_d}]$ then maps from the $2n_\lambda$ link end flows to the n_n nodal outflows, as in (18). Equation (20) is also clearly symmetric.

The expression (20) can be reduced to the desired form of $\boldsymbol{Y\Psi} = \boldsymbol{\Theta}$, from (10), where $\boldsymbol{Y}$ is the network matrix and is given by (using the more common functions of tanh and sinh)

$$\{\boldsymbol{Y}(s)\}_{i,k} = \begin{cases} \dfrac{1}{Z_j(s)\sinh \Gamma_j(s)} & \text{if } \lambda_j = \{(i,k),(k,i)\} \cap \Lambda_i \neq \emptyset \\ \displaystyle\sum_{\lambda_j \in \Lambda_i} \dfrac{-1}{Z_j(s)\tanh \Gamma_j(s)} & \text{if } k = i \\ 0 & \text{otherwise} \end{cases} . \quad (21)$$

Details of the reduction of the matrix expression pre-multiplying $\boldsymbol{\Psi}$ in (20) to the form in (21) are given in the appendix. A brief discussion of the form of (21) is in order. The first case in (21) corresponds to all the off diagonal elements $\{\boldsymbol{Y}(s)\}_{i,k}, i \neq k$, for which there exists a link λ_j between nodes i and k regardless of the links direction, (*i.e.* either $\lambda_j = (i,k)$ or $\lambda_j = (k,i)$ for $\lambda_j \in \Lambda_i$). Moreover, when there is a link between nodes i and k, the term $\{\boldsymbol{Y}(s)\}_{i,k} = [Z_j(s)\sinh(\Gamma_j(s))]^{-1}$, corresponds to the transfer function describing the contribution of the pressure at node k to the flow in link λ_j at node i, and hence its contribution to the outflow Θ_i. The second case corresponds to all the diagonal terms in $\boldsymbol{Y}(s)$ where the summation is taken over the set Λ_i, which is the set of all links incident to node i. The terms in the summation $-[Z_j(s)\tanh(\Gamma_j(s))]^{-1}$ correspond to the transfer function for the contribution that the pressure at node i makes to the flow in link λ_j at node i. Consequently, the sum of these individual functions correspond to the transfer function describing the contribution that the nodal pressure Ψ_i makes to the outflow Θ_i.

Connection of Network Matrix with Electrical Circuit Admittance Matrix

The form of (10) mirrors that seen in electrical circuits [*Monticelli*, 1999] where the nodal current injections $\boldsymbol{I}(s)$ are related to the nodal voltages $\boldsymbol{V}(s)$ (with respect to some reference node) via the relationship $\mathcal{Y}(s)\boldsymbol{V}(s) = \boldsymbol{I}(s)$, where $\mathcal{Y}(s)$ is the nodal admittance matrix. This representation of electrical circuits is achieved by the application of Kirchoffs current laws to the circuit nodes in conjunction with the end to end element dynamics. As such the admittance matrix can be expanded as $\mathcal{Y}(s) = \boldsymbol{N}\mathcal{Y}_e(s)\boldsymbol{N}^T$ [*Desoer and Kuh*, 1969], where $\boldsymbol{N} = \boldsymbol{N_u} - \boldsymbol{N_d}$ is the node-link incidence matrix for a directed graph, and $\mathcal{Y}_e$ is a diagonal matrix of the individual element admittance functions. There are clearly links between $\boldsymbol{Y}$ in (20) and $\mathcal{Y}$, however, the fundamental difference is that the links in fluid networks are distributed, and the elements in electrical circuits are lumped. Each lumped electrical element has only two states (current and voltage change) which are related by a single element admittance transfer function, therefore the network representation $\mathcal{Y}$ is much simpler. For the fluid lines, the upstream and downstream states are different and related via transfer matrices, which necessitates separate consideration of the upstream and downstream nodes as displayed in the division of the incidence matrix into $\boldsymbol{N_u}$ and $\boldsymbol{N_d}$.

Derivation of Network Transfer Matrix for a Network Comprised of Reservoirs, Demand Nodes and Junctions

The focus in this section is the derivation of an input-output matrix transfer function relating the unknown nodal heads and outflows to the known nodal heads and outflows. As specified in the network equations (1)-(7), there are three types of nodes, junctions, demand nodes (controlled temporal demand θ_d), and reservoirs (controlled temporal nodal head ψ_r). As junctions are simply a special case of demand nodes (*i.e.* $\theta_d = 0$), the network is assumed to consist entirely of demand nodes and reservoirs, that is $\mathcal{N} = \mathcal{N}_d \cup \mathcal{N}_r$. At these nodes, the non-specified variable is free. That is, at a reservoir, the inflow or outflow is a free variable, and at a demand node, the nodal pressure is a free variable. Given a system with n_r reservoirs, and n_d demand nodes ($n_n = n_r + n_d$), the nodal variables $\boldsymbol{\Psi}$ and $\boldsymbol{\Theta}$ can be partitioned as follows

$$\boldsymbol{\Psi}(s) = \left[\begin{array}{c} \boldsymbol{\Psi}_d(s) \\ \hline \boldsymbol{\Psi}_r(s) \end{array}\right], \qquad \boldsymbol{\Theta}(s) = \left[\begin{array}{c} \boldsymbol{\Theta}_d(s) \\ \hline \boldsymbol{\Theta}_r(s) \end{array}\right]$$

where the nodes are ordered so that the first n_d are the demand nodes and the last n_r are the reservoirs, (*i.e.* $\boldsymbol{\Psi}_d$ and $\boldsymbol{\Theta}_d$ are $n_d \times 1$ vectors that correspond to the demand nodes, and $\boldsymbol{\Psi}$ and $\boldsymbol{\Theta}_r$ are $n_r \times 1$ vectors correspond to the reservoirs). Using these partitioned vectors, the matrix equation (10) can be expressed in the following partitioned form

$$\left[\begin{array}{c|c} \boldsymbol{Y}_d(s) & \boldsymbol{Y}_{d\text{-}r}(s) \\ \hline \boldsymbol{Y}_{r\text{-}d}(s) & \boldsymbol{Y}_r(s) \end{array}\right] \left[\begin{array}{c} \boldsymbol{\Psi}_d(s) \\ \hline \boldsymbol{\Psi}_r(s) \end{array}\right] = \left[\begin{array}{c} \boldsymbol{\Theta}_d(s) \\ \hline \boldsymbol{\Theta}_r(s) \end{array}\right] \tag{22}$$

where $\boldsymbol{Y}_d$ is the $n_d \times n_d$ system matrix for the subsystem comprised of the demand nodes, $\boldsymbol{Y}_r$ is the $n_r \times n_r$ system matrix for the subsystem comprised of the reservoir nodes, and $\boldsymbol{Y}_{d\text{-}r}$ ($\boldsymbol{Y}_{r\text{-}d}$) are the $n_d \times n_r$ ($n_r \times n_d$) partitions of the network matrix that corresponding to the outflow contribution at the demand (reservoir) nodes admitted from the nodal pressures at the reservoir (demand) nodes. Note that $\boldsymbol{Y}_d$ and $\boldsymbol{Y}_r$ are symmetric and $\boldsymbol{Y}_{d\text{-}r} = \boldsymbol{Y}_{r\text{-}d}^T$. From (22), the unknown nodal pressures and outflows can be expressed as a function of the reservoir pressures and demands by reorganising the matrix equation (22) as

$$\left[\begin{array}{c} \boldsymbol{\Psi}_d(s) \\ \hline \boldsymbol{\Theta}_r(s) \end{array}\right] = \left[\begin{array}{c|c} \boldsymbol{Y}_d^{-1}(s) & -\boldsymbol{Y}_d^{-1}(s)\,\boldsymbol{Y}_{d\text{-}r}(s) \\ \hline \boldsymbol{Y}_{r\text{-}d}(s)\,\boldsymbol{Y}_d^{-1}(s) & \boldsymbol{Y}_r(s) - \boldsymbol{Y}_{r\text{-}d}(s)\,\boldsymbol{Y}_d^{-1}(s)\,\boldsymbol{Y}_{d\text{-}r}(s) \end{array}\right] \left[\begin{array}{c} \boldsymbol{\Theta}_d(s) \\ \hline \boldsymbol{\Psi}_r(s) \end{array}\right] \tag{23}$$

for all $s \in \mathbb{C}$ for which $\boldsymbol{Y}_d$ is nonsingular. So from (23) it is seen that there exists an analytic transfer matrix relationship between the unknown nodal pressures and outflows and the known nodal pressures and demands for a fluid line network of an arbitrary configuration. The form of these equations can be explained in an intuitive manner as follows. Concerning the expression for $\boldsymbol{\Psi}_d$ in (23), which can be written as $\boldsymbol{\Psi}_d = \boldsymbol{Y}_d^{-1}[\boldsymbol{\Theta}_d - \boldsymbol{Y}_{d\text{-}r}\boldsymbol{\Psi}_r]$. The term $\boldsymbol{Y}_{d\text{-}r}\boldsymbol{\Psi}_r$ corresponds to the contribution of the outflow admitted from the demand nodes as a result of the pressures at the reservoir nodes. Therefore $\boldsymbol{\Theta}_d - \boldsymbol{Y}_{d\text{-}r}\boldsymbol{\Psi}_r$ is clearly the remaining outflow at the demand nodes resulting from the pressures at the demand nodes. Finally, $\boldsymbol{Y}_d^{-1}$ is the map from this quantity (the remaining outflow) to the pressure at the demand nodes $\boldsymbol{\Psi}_d$. A similar explanation can be given for the block matrix equation for $\boldsymbol{\Theta}_r$.

From a computational perspective, an advantageous attribute about (23) is that the n_d unknowns $\mathbf{\Psi}_d$ are uncoupled from the n_r unknowns $\mathbf{\Theta}_r$. This means that the unknown nodal pressures $\mathbf{\Psi}_r$ can be computed independently from the unknown nodal outflows $\mathbf{\Theta}_r$, thus reducing the order of the linear system to n_d, the number of known nodal outflow nodes. Computing (23) on $s \in \mathbb{I}_+$ (the positive imaginary axis) provides a frequency-domain model for such networks of arbitrary configuration, and as such, it is an important contribution of this paper.

EXAMPLES

In the following, two network case studies are presented: network-1, a 7-pipe/6-node network, and network-2, a 51-pipe/35-node network. The networks frequency response calculated by the network admittance matrix is compared to the frequency response calculated by the method of characteristics (MOC). A turbulent flow state was assumed for both case studies, for which the time-domain frictionloss model τ and the transmission line parameters $\Gamma(s)$ and $Z_c(s)$ are given as

$$\tau(q) = \frac{f_0|q_0|}{2rA}\, q(t) + O\left\{q^2(t)\right\}, \Gamma(s) = \frac{l}{c}\sqrt{s\left(s + \frac{f_0|q_0|}{2rA}\right)}, \; Z_c(s) = \frac{\rho c}{A}\sqrt{\frac{s + \frac{f_0|q_0|}{2rA}}{s}}.$$

As τ_j is nonlinear, these case studies provide an example of the utility of the admittance matrix method to approximate nonlinear systems.

Small Network in Steady-Oscillatory State

Network-1 of Figure 2(a) is possibly the simplest example of a second order system. Given the nodal and link ordering in Figure 2(a), the upstream and downstream incidence matrices for this network are

$$\boldsymbol{N_u} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{N_d} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(recall that the rows correspond to nodes and the columns to links), the state vectors for the network are the pressures $\boldsymbol{\Psi}(s) = [\Psi_1(s) \cdots \Psi_5(s) \,|\, \Psi_6(s)]^T$, and the nodal outflows $\boldsymbol{\Theta}(s) = [\Theta_1(s) \cdots \Theta_5(s) \,|\, \Theta_6(s)]^T$ (the partitions correspond to the outflow control and pressure control nodes as in the previous section), and the network link matrices are $\boldsymbol{\Gamma}(s) = \text{diag}\{\Gamma_1(s), \ldots, \Gamma_7(s)\}$, and $\boldsymbol{Z}_c(s) = \text{diag}\{Z_{c,1}(s), \ldots, Z_{c,6}(s)\}$. The network admittance matrix can be expressed as

$$\boldsymbol{Y}(s) = \left[\begin{array}{c|c} \boldsymbol{Y}_d(s) & \boldsymbol{Y}_{d\text{-}r}(s) \\ \hline \boldsymbol{Y}_{r\text{-}d}(s) & \boldsymbol{Y}_r(s) \end{array}\right] =$$

$$\left[\begin{array}{ccccc|c} -t_1(s) & s_1(s) & 0 & 0 & 0 & 0 \\ s_1(s) & -\sum\limits_{j=1,2,3} t_j(s) & s_2(s) & s_3(s) & 0 & 0 \\ 0 & s_2(s) & -\sum\limits_{j=2,4,5} t_j(s) & s_4(s) & s_5(s) & 0 \\ 0 & s_3(s) & s_4(s) & -\sum\limits_{j=3,4,6} t_j(s) & s_6(s) & 0 \\ 0 & 0 & s_5(s) & s_6(s) & -\sum\limits_{j=5,6,7} t_j(s) & s_7(s) \\ \hline 0 & 0 & 0 & 0 & s_7(s) & -t_7(s) \end{array}\right] \tag{24}$$

where $t_j(s) = [Z_c(s) \tanh \Gamma_j(s)]^{-1}$ and $s_j(s) = [Z_c(s) \sinh \Gamma_j(s)]^{-1}$, (the partitions correspond to the matrix partitioning from (22)). For the outflow control nodes, Node 1 is the only demand node (*i.e.* $\theta_i(s) = 0, i = 2, 3, 4, 5$), and at the only head control node (reservoir) $\Psi_6(s) = 0$. Therefore, from (23), the unknown nodal heads can be expressed as

$$\left[\begin{array}{c}\Psi_1(s)\\ \vdots\\ \Psi_5(s)\\ \hline \Theta_6(s)\end{array}\right] = \left[\begin{array}{c}\left\{\boldsymbol{Y}_d^{-1}(s)\right\}_{1,1}\\ \vdots\\ \left\{\boldsymbol{Y}_d^{-1}(s)\right\}_{5,1}\\ \hline s_7(s)\left\{\boldsymbol{Y}_d^{-1}(s)\right\}_{5,1}\end{array}\right]\Theta_1(s). \tag{25}$$

As seen in (25), the computation of the unknown nodal values involves the inversion of a complex 6×6 matrix, of which only the first column is used.

For the numerical studies of network-1 the parameters were taken as; pipe diameters $= \{60, 50, 35, 50, 35, 50, 60\}$ mm, pipe lengths $= \{31, 52, 34, 41, 26, 57, 28\}$ m, and the wavespeeds and the Darcy-Weisbach friction factors were set to 1000 m/s and 0.02, respectively, for all pipes. The demand at node 1 was taken as a sinusoid of amplitude 0.025 L/s about a base demand level of 10 L/s. For the MOC model, a frequency sweep was performed for frequencies up to 15 Hz. Figure 3 presents the amplitude of the sinusoidal pressure fluctuations observed at node 1 computed by the Laplace-domain admittance matrix, and the discrete Fourier transform (DFT) of the MOC in steady oscillatory state. Extremely good matches between the two methods are observed.

Large Network in Transient State

The original formulation for network-2 was maintained [*Vítkovský*, 2001] with the following exceptions: pipe lengths were rounded to the nearest 5 meters and the wavespeeds were all made to be 1000 m/s to ensure a Courant number of 1; the nodal demands were doubled to increase the flow through the network. For brevity, the network details are not given here, but the range of network parameters are $[450, 895]$ m for pipe lengths, $[304, 1524]$ mm for pipe sizes, and $[80, 280]$ L/s for nodal demands (for case study details, the reader is referred to [*Vítkovský*, 2001]).

In order to avoid burdensome computational requirements, network-2 was analyzed in the transient state as opposed to the steady-oscillatory state used for network-1. This meant that the frequency response was computed from a single MOC simulation of the system excited by a finite energy input. The network was excited into a transient state by a pulse flow perturbation at nodes $\{14, 17, 28\}$ of duration $\{0.055, 0.025, 0.075\}$ s and of magnitude $\{70, 50, 100\}$ L/s.

A Plot of the frequency response at node 25 for network-2 is given in Figure 4 (due to the densely distributed harmonics, only the range 0 - 2 Hz is shown). The DFT of the MOC pressure trace is almost indistinguishable from that of the admittance matrix model. This illustrates that even for a network of a large size, the linear admittance matrix model provides an extremely good approximation of the nonlinear MOC model.

CONCLUSIONS

The majority of existing methods for modeling the frequency-domain behavior of a transient fluid line system have been limited to dealing only with certain classes of network types, namely, those that do not contain second order loops. In this paper, a completely new formulation is derived that is able to deal with networks comprised of pipes, junctions, demand nodes, and reservoirs that are of an arbitrary configuration. The derived representation takes the form of an admittance matrix that maps from the nodal pressures to the nodal demands. The analytic nature of this representation enables significant qualitative insight into the structure of a network, and the dependency of the relationship of the nodal states on the individual pipeline transfer functions. In addition to the qualitative insight, the admittance matrix serves as the basis for an efficient model for computing the frequency response of a network of unknown nodal states subject to known nodal inputs. The numerical examples have demonstrated that the method serves as an excellent linear approximation for a turbulent state pipeline network.

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Nomenclature

l_k, $\boldsymbol{l}$	The length of $k-$th pipe and the vector of the pipe lengths
$\boldsymbol{N_d}$, $\boldsymbol{N_u}$	Node link incidence matrix for downstream and upstream link ends
$P(x,s)$	Transformed pressure distribution along a line
$Q(x,s)$	Transformed flow distribution along a line
s	Laplace variable
$\boldsymbol{Y}(s)$	Network admittance matrix
$\boldsymbol{Y}_d(s)$, $\boldsymbol{Y}_r(s)$	Partition of $\boldsymbol{Y}(s)$ corresponding to the connections between the demand (reservoir) nodes
$\boldsymbol{Y}_{r\text{-}d}(s)$, $\boldsymbol{Y}_{d\text{-}r}(s)$	Partition of $\boldsymbol{Y}(s)$ corresponding to the outflow at the reservoir (demand) nodes driven by the pressures at the demand (reservoir) nodes
$Y_s(s)$, $\boldsymbol{Y_s}(s)$	Shunt admittance, and shunt admittance matrix
$Z_{c,j}(s)$, $\boldsymbol{Z_c}(s)$	Characteristic impedance for link j, and characteristic impedance matrix j
$Z_s(s)$, $\boldsymbol{Z_s}(s)$	Series impedance, and series impedance matrix j
$\mathcal{G}(\mathcal{N},\Lambda)$	Graph on node set $\mathcal{N}$ and link set Λ
$\mathcal{N}$, $\mathcal{N}_d$, $\mathcal{N}_r$	Set of all nodes, demand nodes and reservoir nodes
$\Gamma_j(s)$, $\boldsymbol{\Gamma}(s)$	Propagation operator for link j, and propagation operator matrix
λ_k	The $k-$th link in Λ
$\boldsymbol{\Lambda}$, $\boldsymbol{\Lambda}_i$	Set of all links, and set of links incident to node i
$\boldsymbol{\Lambda}_{d,i}$, $\boldsymbol{\Lambda}_{u,i}$	Set of links whose downstream (upstream) node is node i
$\boldsymbol{\Psi}(s)$, $\boldsymbol{\Psi}_d(s)$, $\boldsymbol{\Psi}_r(s)$	Transformed pressure for all network nodes, demand nodes, and reservoir nodes
$\boldsymbol{\Theta}(s)$, $\boldsymbol{\Theta}_d(s)$, $\boldsymbol{\Theta}_r(s)$	Transformed outflow for all network nodes, demand nodes and reservoir nodes

APPENDIX: REDUCTION OF ADMITTANCE MATRIX $\boldsymbol{Y}(s)$

Multiplying through the block matrices in (20) leads to the following expression for the matrix in (10) that relates the nodal pressures to the nodal flows,

$$\begin{aligned}\boldsymbol{Y}(s) = &- \boldsymbol{N_u}\boldsymbol{Z_c}^{-1}(s)\coth\boldsymbol{\Gamma}(s)\,\boldsymbol{N_u}^T + \boldsymbol{N_d}\boldsymbol{Z_c}^{-1}(s)\text{csch}\boldsymbol{\Gamma}(s)\,\boldsymbol{N_u}^T \\ &+ \boldsymbol{N_u}\boldsymbol{Z_c}^{-1}(s)\text{csch}\boldsymbol{\Gamma}(s)\,\boldsymbol{N_d}^T - \boldsymbol{N_d}\boldsymbol{Z_c}^{-1}(s)\coth\boldsymbol{\Gamma}(s)\,\boldsymbol{N_d}^T \end{aligned} \tag{26}$$

To determine the explicit form of $\boldsymbol{Y}$, each matrix expression is considered separately. Based on a purely algebraic argument exploiting the structure of the incidence matrices $\boldsymbol{N_u}$ and $\boldsymbol{N_d}$, and the diagonal nature of $\boldsymbol{Z}_c$, it can be found that

$$\left\{\boldsymbol{N}_d\boldsymbol{Z_c}^{-1}\text{csch}\boldsymbol{\Gamma}\boldsymbol{N}_u^T\right\}_{i,k} = \begin{cases}\text{csch}\Gamma_j/Z_{c,j} & \text{if } \lambda_j = (k,i) \in \Lambda_{d,i} \\ 0 & \text{otherwise}\end{cases},$$

$$\left\{\boldsymbol{N}_u\boldsymbol{Z_c}^{-1}\text{csch}\boldsymbol{\Gamma}\boldsymbol{N}_d^T\right\}_{i,k} = \begin{cases}\text{csch}\Gamma_j/Z_{c,j} & \text{if } \lambda_j = (i,k) \in \Lambda_{u,i} \\ 0 & \text{otherwise}\end{cases}.$$

$$\left\{\boldsymbol{N}_d\boldsymbol{Z_c}^{-1}\coth\boldsymbol{\Gamma}\boldsymbol{N}_d^T\right\}_{i,k} = \begin{cases}\displaystyle\sum_{\lambda_j\in\Lambda_{d,i}}\frac{\coth(\Gamma_j L_j)}{Z_{c,j}} & \text{if } k = i \\ 0 & \text{otherwise}\end{cases},$$

$$\left\{\boldsymbol{N}_u\boldsymbol{Z_c}^{-1}\coth\boldsymbol{\Gamma}\boldsymbol{N}_u^T\right\}_{i,k} = \begin{cases}\displaystyle\sum_{\lambda_j\in\Lambda_{u,i}}\frac{\coth(\Gamma_j L_j)}{Z_{c,j}} & \text{if } k = i \\ 0 & \text{otherwise}\end{cases}.$$

Finally, gathering all these matrices together, (26) can be re-expressed as (10) and (21).

FIGURES FOR MANUSCRIPT

Figure Captions

Figure 1: Example of a first order looped network without the dashed link, and a second order looped network with the dashed link.

Figure 2: Example Networks: (a) network-1 (6 nodes, 7 pipes), and (b) network-4 (35 nodes, 51 pipes) from *Vítkovský* [2001].

Figure 3: Sinusoidal pressure amplitude response for 7-pipe network at node 6 for the admittance matrix model ($-$) and the method of characteristics in steady oscillatory state ($\circ$).

Figure 4: Pressure frequency response magnitudes for 51-pipe network at node 25 for the admittance matrix model ($-$) and the DFT of the method of characteristics ($\cdot$).

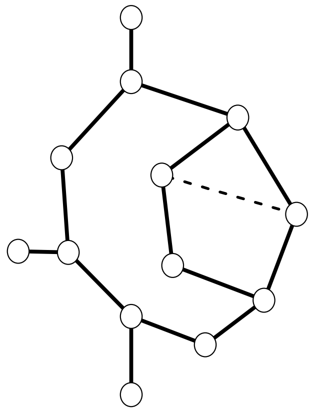

Figure 1: Example of a first order looped network without the dashed link, and a second order looped network with the dashed link.

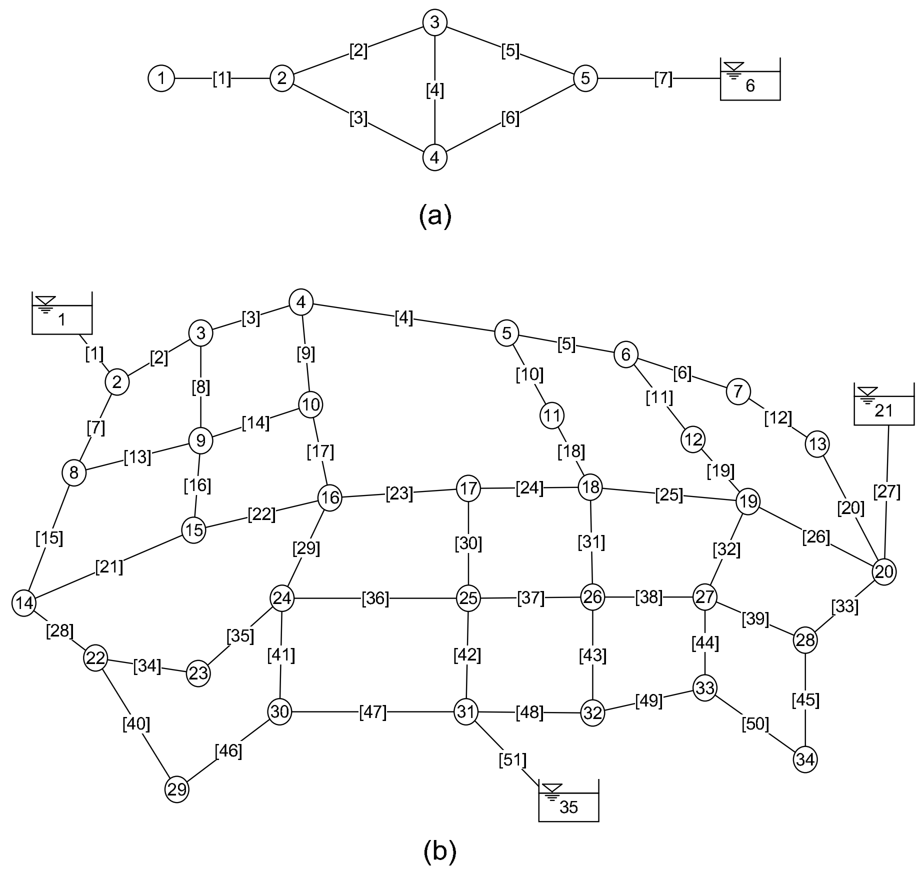

Figure 2: Example Networks: (a) network-1 (6 nodes, 7 pipes), and (b) network-4 (35 nodes, 51 pipes) from *Vítkovský* [2001].

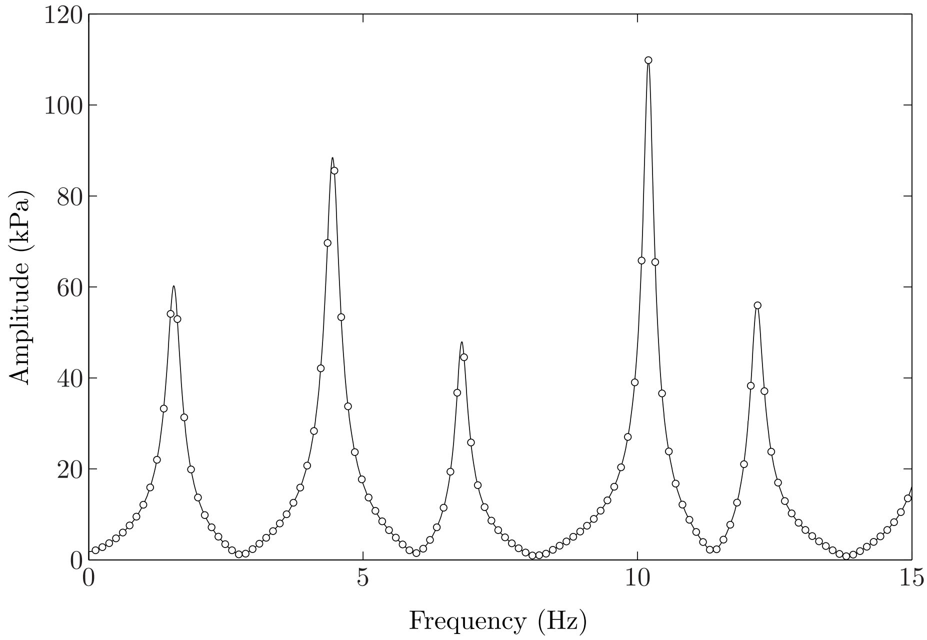

Figure 3: Sinusoidal pressure amplitude response for 7-pipe network at node 6 for the admittance matrix model (−) and the method of characteristics in steady oscillatory state (∘).

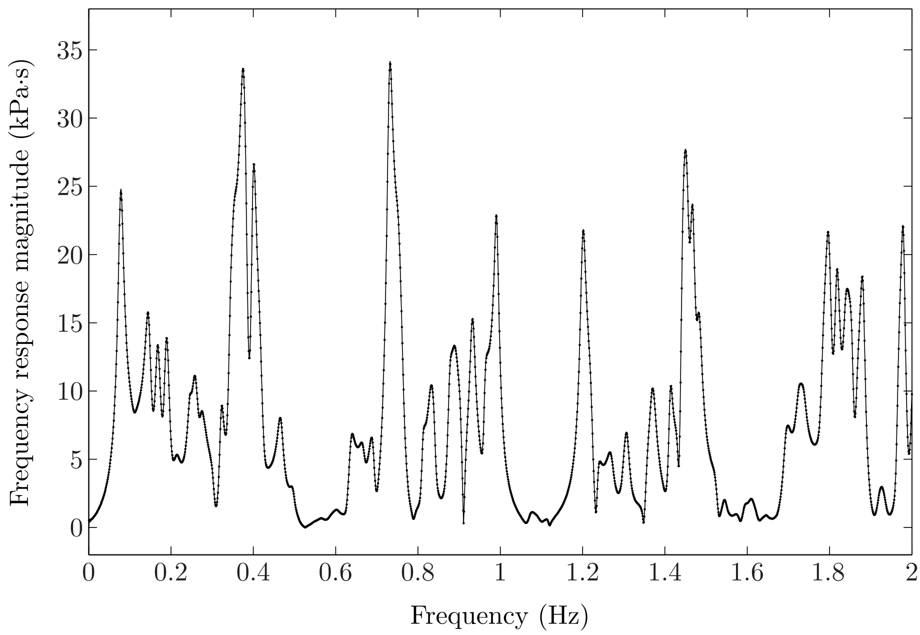

Figure 4: Pressure frequency response magnitudes for 51-pipe network at node 25 for the admittance matrix model (−) and the DFT of the method of characteristics (·).