Decision–Feedback Equalisation with Fixed–Lag Smoothing in Nonlinear Channels

by

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Appendix A

Terms in decomposition of (2.16)

This appendix gives the terms involved in the isolation of quantities that involve the time index $t-n$ in equation (2.16), where (2.16) is reproduced below as (A.1):

$$
\begin{aligned}
Y_{t-k} = &\sum_{i_1=0}^{N} h_1(i_1)X_{t-k-i_1} + \sum_{i_1=0}^{N}\sum_{i_2=i_1}^{N}\sum_{j_1=0}^{N} h_3(i_1,i_2,j_1)X_{t-k-i_1}X_{t-k-i_2}X^*_{t-k-j_1} \\
&+ \sum_{i_1=0}^{N}\sum_{i_2=i_1}^{N}\sum_{i_3=i_2}^{N}\sum_{j_1=0}^{N}\sum_{j_2=j_1}^{N} h_5(i_1,i_2,i_3,j_1,j_2) \\
&\times X_{t-k-i_1}X_{t-k-i_2}X_{t-k-i_3}X^*_{t-k-j_1}X^*_{t-k-j_2} + \dots \\
&+ \sum_{i_1=0}^{N}\sum_{i_2=i_1}^{N}\cdots\sum_{i_{p+1}=i_p}^{N}\sum_{j_1=0}^{N}\sum_{j_2=j_1}^{N}\cdots\sum_{j_p=j_{p-1}}^{N} h_{2p+1}(i_1,\dots,i_{p+1},j_1,\dots,j_p) \\
&\times X_{t-k-i_1}\dots X_{t-k-i_{p+1}}X^*_{t-k-j_1}\dots X^*_{t-k-j_p} + V_{t-k}.
\end{aligned} \tag{A.1}
$$

As discussed in section 2.3.2, rewrite (A.1) as follows, keeping terms to third–order only ($p=1$):

$$
Y_{t-k} = Y^{(<)}_{t-k} + Y^{(=)}_{t-k} + Y^{(>)}_{t-k} + Y^{(<,<,<)}_{t-k} + \dots + Y^{(>,>,>)}_{t-k} + V_{t-k}, \tag{A.2}
$$

where the individual terms $Y^{(<)}_{t-k}, \dots, Y^{(>,>,>)}_{t-k}$ are listed in (A.3)–(A.23) overleaf.

A.1 First–Order terms

$$
Y^{(<)}_{t-k} = \sum_{i_1=0}^{n-k-1} h_1(i_1)X_{t-k-i_1}, \tag{A.3}
$$

$$
Y^{(=)}_{t-k} = h_1(n-k)X_{t-n}, \tag{A.4}
$$

and

$$Y_{t-k}^{(>)} = \sum_{i_1=n-k+1}^{N} h_1(i_1) X_{t-k-i_1}. \tag{A.5}$$

A.2 Third–Order Terms

$$Y_{t-k}^{(<,<,<)} = \sum_{i_1=0}^{n-k-1} \sum_{i_2=i_1}^{n-k-1} \sum_{j_1=0}^{n-k-1} h_3(i_1, i_2, j_1) X_{t-k-i_1} X_{t-k-i_2} X^*_{t-k-j_1}, \tag{A.6}$$

$$Y_{t-k}^{(<,<,=)} = X^*_{t-n} \sum_{i_1=0}^{n-k-1} \sum_{i_2=i_1}^{n-k-1} h_3(i_1, i_2, n-k) X_{t-k-i_1} X_{t-k-i_2}, \tag{A.7}$$

$$Y_{t-k}^{(<,<,>)} = \sum_{i_1=0}^{n-k-1} \sum_{i_2=i_1}^{n-k-1} \sum_{j_1=n-k+1}^{N} h_3(i_1, i_2, j_1) X_{t-k-i_1} X_{t-k-i_2} X^*_{t-k-j_1}, \tag{A.8}$$

$$Y_{t-k}^{(<,=,<)} = X_{t-n} \sum_{i_1=0}^{n-k-1} \sum_{j_1=0}^{n-k-1} h_3(i_1, n-k, j_1) X_{t-k-i_1} X^*_{t-k-j_1}, \tag{A.9}$$

$$Y_{t-k}^{(<,=,=)} = |X_{t-n}|^2 \sum_{i_1=0}^{n-k-1} h_3(i_1, n-k, n-k) X_{t-k-i_1}, \tag{A.10}$$

$$Y_{t-k}^{(<,=,>)} = X_{t-n} \sum_{i_1=0}^{n-k-1} \sum_{j_1=n-k+1}^{N} h_3(i_1, n-k, j_1) X_{t-k-i_1} X^*_{t-k-j_1}, \tag{A.11}$$

$$Y_{t-k}^{(<,>,<)} = \sum_{i_1=0}^{n-k-1} \sum_{i_2=n-k+1}^{N} \sum_{j_1=0}^{n-k-1} h_3(i_1, i_2, j_1) X_{t-k-i_1} X_{t-k-i_2} X^*_{t-k-j_1}, \tag{A.12}$$

$$Y_{t-k}^{(<,>,=)} = X^*_{t-n} \sum_{i_1=0}^{n-k-1} \sum_{i_2=n-k+1}^{N} h_3(i_1, i_2, n-k) X_{t-k-i_1} X_{t-k-i_2}, \tag{A.13}$$

$$Y_{t-k}^{(<,>,>)} = \sum_{i_1=0}^{n-k-1} \sum_{i_2=n-k+1}^{N} \sum_{j_1=n-k+1}^{N} h_3(i_1, i_2, j_1) X_{t-k-i_1} X_{t-k-i_2} X^*_{t-k-j_1}, \tag{A.14}$$

$$Y_{t-k}^{(=,=,<)} = X^2_{t-n} \sum_{j_1=0}^{n-k-1} h_3(n-k, n-k, j_1) X^*_{t-k-j_1}, \tag{A.15}$$

$$Y_{t-k}^{(=,=,=)} = h_3(n-k, n-k, n-k) X_{t-n} |X_{t-n}|^2, \tag{A.16}$$

$$Y_{t-k}^{(=,=,>)} = X^2_{t-n} \sum_{j_1=n-k+1}^{N} h_3(n-k, n-k, j_1) X^*_{t-k-j_1}, \tag{A.17}$$

$$Y_{t-k}^{(=,>,<)} = X_{t-n} \sum_{i_2=n-k+1}^{N} \sum_{j_1=0}^{n-k-1} h_3(n-k, i_2, j_1) X_{t-k-i_2} X^*_{t-k-j_1}, \tag{A.18}$$

$$Y_{t-k}^{(=,>,=)} = |X_{t-n}|^2 \sum_{i_2=n-k+1}^{N} h_3(n-k, i_2, n-k) X_{t-k-i_2}, \tag{A.19}$$

$$Y_{t-k}^{(=,>,>)} = X_{t-n} \sum_{i_2=n-k+1}^{N} \sum_{j_1=n-k+1}^{N} h_3(n-k, i_2, j_1) X_{t-k-i_2} X_{t-k-j_1}^{*}, \tag{A.20}$$

$$Y_{t-k}^{(>,>,<)} = \sum_{i_1=n-k+1}^{N} \sum_{i_2=i_1}^{N} \sum_{j_1=0}^{n-k-1} h_3(i_1, i_2, j_1) X_{t-k-i_1} X_{t-k-i_2} X_{t-k-j_1}^{*}, \tag{A.21}$$

$$Y_{t-k}^{(>,>,=)} = X_{t-n}^{*} \sum_{i_1=n-k+1}^{N} \sum_{i_2=i_1}^{N} h_3(i_1, i_2, n-k) X_{t-k-i_1} X_{t-k-i_2}, \tag{A.22}$$

and

$$Y_{t-k}^{(>,>,>)} = \sum_{i_1=n-k+1}^{N} \sum_{i_2=i_1}^{N} \sum_{j_1=n-k+1}^{N} h_3(i_1, i_2, j_1) X_{t-k-i_1} X_{t-k-i_2} X_{t-k-j_1}^{*}. \tag{A.23}$$

Appendix B

Terms in (2.22) to third–order

This appendix gives the terms $A_{t-k}^{(l,m)}$ in (2.22) for a third–order ($p = 1$) Volterra series model, where (2.22) is reproduced below as (B.1):

$$Y_{t-k} = \sum_{l=0}^{p+1} \sum_{m=0}^{p} A_{t-k}^{(l,m)} X_{t-n}^{l} \left(X_{t-n}^{*}\right)^{m} + V_{t-k}. \tag{B.1}$$

For $p = 1$, the terms $A_{t-k}^{(l,m)}$ in (B.1) are as follows:

$$\begin{aligned} A_{t-k}^{(0,0)} = & \sum_{i_1=0}^{n-k-1} h_1(i_1) X_{t-k-i_1} \\ & + \sum_{i_1=n-k+1}^{N} h_1(i_1) X_{t-k-i_1} \\ & + \sum_{i_1=0}^{n-k-1} \sum_{i_2=i_1}^{n-k-1} \sum_{j_1=0}^{n-k-1} h_3(i_1, i_2, j_1) X_{t-k-i_1} X_{t-k-i_2} X_{t-k-j_1}^{*} \\ & + \sum_{i_1=0}^{n-k-1} \sum_{i_2=i_1}^{n-k-1} \sum_{j_1=n-k+1}^{N} h_3(i_1, i_2, j_1) X_{t-k-i_1} X_{t-k-i_2} X_{t-k-j_1}^{*} \\ & + \sum_{i_1=0}^{n-k-1} \sum_{i_2=n-k+1}^{N} \sum_{j_1=0}^{n-k-1} h_3(i_1, i_2, j_1) X_{t-k-i_1} X_{t-k-i_2} X_{t-k-j_1}^{*} \\ & + \sum_{i_1=0}^{n-k-1} \sum_{i_2=n-k+1}^{N} \sum_{j_1=n-k+1}^{N} h_3(i_1, i_2, j_1) X_{t-k-i_1} X_{t-k-i_2} X_{t-k-j_1}^{*} \\ & + \sum_{i_1=n-k+1}^{N} \sum_{i_2=i_1}^{N} \sum_{j_1=0}^{n-k-1} h_3(i_1, i_2, j_1) X_{t-k-i_1} X_{t-k-i_2} X_{t-k-j_1}^{*} \\ & + \sum_{i_1=n-k+1}^{N} \sum_{i_2=i_1}^{N} \sum_{j_1=n-k+1}^{N} h_3(i_1, i_2, j_1) X_{t-k-i_1} X_{t-k-i_2} X_{t-k-j_1}^{*}, \end{aligned} \tag{B.2}$$

$$A_{t-k}^{(0,1)} = \sum_{i_1=0}^{n-k-1} \sum_{i_2=i_1}^{n-k-1} h_3(i_1, i_2, n-k) X_{t-k-i_1} X_{t-k-i_2}$$

$$
\begin{aligned}
&+ \sum_{i_1=0}^{n-k-1} \sum_{i_2=n-k+1}^{N} h_3(i_1, i_2, n-k) X_{t-k-i_1} X_{t-k-i_2} \\
&+ \sum_{i_1=n-k+1}^{N} \sum_{i_2=i_1}^{N} h_3(i_1, i_2, n-k) X_{t-k-i_1} X_{t-k-i_2}, \qquad \text{(B.3)}
\end{aligned}
$$

$$
\begin{aligned}
A_{t-k}^{(1,0)} &= h_1(n-k) \\
&+ \sum_{i_1=0}^{n-k-1} \sum_{j_1=0}^{n-k-1} h_3(i_1, n-k, j_1) X_{t-k-i_1} X_{t-k-j_1}^* \\
&+ \sum_{i_1=0}^{n-k-1} \sum_{j_1=n-k+1}^{N} h_3(i_1, n-k, j_1) X_{t-k-i_1} X_{t-k-j_1}^* \\
&+ \sum_{i_2=n-k+1}^{N} \sum_{j_1=0}^{n-k-1} h_3(n-k, i_2, j_1) X_{t-k-i_2} X_{t-k-j_1}^* \\
&+ \sum_{i_2=n-k+1}^{N} \sum_{j_1=n-k+1}^{N} h_3(n-k, i_2, j_1) X_{t-k-i_2} X_{t-k-j_1}^*, \qquad \text{(B.4)}
\end{aligned}
$$

$$
\begin{aligned}
A_{t-k}^{(1,1)} &= \sum_{i_1=0}^{n-k-1} h_3(i_1, n-k, n-k) X_{t-k-i_1} \\
&+ \sum_{i_2=n-k+1}^{N} h_3(n-k, i_2, n-k) X_{t-k-i_2}, \qquad \text{(B.5)}
\end{aligned}
$$

$$
\begin{aligned}
A_{t-k}^{(2,0)} &= \sum_{j_1=0}^{n-k-1} h_3(n-k, n-k, j_1) X_{t-k-j_1}^* \\
&+ \sum_{j_1=n-k+1}^{N} h_3(n-k, n-k, j_1) X_{t-k-j_1}^*, \qquad \text{(B.6)}
\end{aligned}
$$

and

$$
A_{t-k}^{(2,1)} = h_3(n-k, n-k, n-k). \qquad \text{(B.7)}
$$

Appendix C

Terms in (2.27) to third–order

This appendix gives the terms $\hat{A}^{(l,m)}_{t-k|t-k}$ in (2.27) for a third–order ($p = 1$) Volterra series model, where (2.27) is reproduced below as (C.1):

$$\hat{V}_{t-k} = Y_{t-k} - \sum_{l=0}^{p+1} \sum_{m=0}^{p} \hat{A}^{(l,m)}_{t-k|t-k} X^{l}_{t-n} \left(X^{*}_{t-n}\right)^{m}. \tag{C.1}$$

For $p = 1$, the terms $\hat{A}^{(l,m)}_{t-k|t-k}$ in (C.1) are as follows, obtained from (B.2)–(B.7) by the substitution of filtered FLSDFE estimators $\hat{X}_{t|t}$ for symbols X_t:

$$\begin{aligned}
\hat{A}^{(0,0)}_{t-k|t-k} = & \sum_{i_1=0}^{n-k-1} h_1(i_1) \hat{X}_{t-k-i_1|t-k-i_1} \\
& + \sum_{i_1=n-k+1}^{N} h_1(i_1) \hat{X}_{t-k-i_1|t-k-i_1} \\
& + \sum_{i_1=0}^{n-k-1} \sum_{i_2=i_1}^{n-k-1} \sum_{j_1=0}^{n-k-1} h_3(i_1, i_2, j_1) \\
& \times \hat{X}_{t-k-i_1|t-k-i_1} \hat{X}_{t-k-i_2|t-k-i_2} \hat{X}^{*}_{t-k-j_1|t-k-j_1} \\
& + \sum_{i_1=0}^{n-k-1} \sum_{i_2=i_1}^{n-k-1} \sum_{j_1=n-k+1}^{N} h_3(i_1, i_2, j_1) \\
& \times \hat{X}_{t-k-i_1|t-k-i_1} \hat{X}_{t-k-i_2|t-k-i_2} \hat{X}^{*}_{t-k-j_1|t-k-j_1} \\
& + \sum_{i_1=0}^{n-k-1} \sum_{i_2=n-k+1}^{N} \sum_{j_1=0}^{n-k-1} h_3(i_1, i_2, j_1) \\
& \times \hat{X}_{t-k-i_1|t-k-i_1} \hat{X}_{t-k-i_2|t-k-i_2} \hat{X}^{*}_{t-k-j_1|t-k-j_1} \\
& + \sum_{i_1=0}^{n-k-1} \sum_{i_2=n-k+1}^{N} \sum_{j_1=n-k+1}^{N} h_3(i_1, i_2, j_1)
\end{aligned}$$

$$
\begin{aligned}
&\times \hat{X}_{t-k-i_1|t-k-i_1}\hat{X}_{t-k-i_2|t-k-i_2}\hat{X}^*_{t-k-j_1|t-k-j_1}\\
&+\sum_{i_1=n-k+1}^{N}\sum_{i_2=i_1}^{N}\sum_{j_1=0}^{n-k-1} h_3(i_1,i_2,j_1)\\
&\times \hat{X}_{t-k-i_1|t-k-i_1}\hat{X}_{t-k-i_2|t-k-i_2}\hat{X}^*_{t-k-j_1|t-k-j_1}\\
&+\sum_{i_1=n-k+1}^{N}\sum_{i_2=i_1}^{N}\sum_{j_1=n-k+1}^{N} h_3(i_1,i_2,j_1)\\
&\times \hat{X}_{t-k-i_1|t-k-i_1}\hat{X}_{t-k-i_2|t-k-i_2}\hat{X}^*_{t-k-j_1|t-k-j_1},
\end{aligned}
\tag{C.2}
$$

$$
\begin{aligned}
\hat{A}^{(0,1)}_{t-k|t-k} &= \sum_{i_1=0}^{n-k-1}\sum_{i_2=i_1}^{n-k-1} h_3(i_1,i_2,n-k)\hat{X}_{t-k-i_1|t-k-i_1}\hat{X}_{t-k-i_2|t-k-i_2}\\
&+\sum_{i_1=0}^{n-k-1}\sum_{i_2=n-k+1}^{N} h_3(i_1,i_2,n-k)\hat{X}_{t-k-i_1|t-k-i_1}\hat{X}_{t-k-i_2|t-k-i_2}\\
&+\sum_{i_1=n-k+1}^{N}\sum_{i_2=i_1}^{N} h_3(i_1,i_2,n-k)\hat{X}_{t-k-i_1|t-k-i_1}\hat{X}_{t-k-i_2|t-k-i_2},
\end{aligned}
\tag{C.3}
$$

$$
\begin{aligned}
\hat{A}^{(1,0)}_{t-k|t-k} &= h_1(n-k)\\
&+\sum_{i_1=0}^{n-k-1}\sum_{j_1=0}^{n-k-1} h_3(i_1,n-k,j_1)\hat{X}_{t-k-i_1|t-k-i_1}\hat{X}^*_{t-k-j_1|t-k-j_1}\\
&+\sum_{i_1=0}^{n-k-1}\sum_{j_1=n-k+1}^{N} h_3(i_1,n-k,j_1)\hat{X}_{t-k-i_1|t-k-i_1}\hat{X}^*_{t-k-j_1|t-k-j_1}\\
&+\sum_{i_2=n-k+1}^{N}\sum_{j_1=0}^{n-k-1} h_3(n-k,i_2,j_1)\hat{X}_{t-k-i_2|t-k-i_2}\hat{X}^*_{t-k-j_1|t-k-j_1}\\
&+\sum_{i_2=n-k+1}^{N}\sum_{j_1=n-k+1}^{N} h_3(n-k,i_2,j_1)\hat{X}_{t-k-i_2|t-k-i_2}\hat{X}^*_{t-k-j_1|t-k-j_1},
\end{aligned}
\tag{C.4}
$$

$$
\begin{aligned}
\hat{A}^{(1,1)}_{t-k|t-k} &= \sum_{i_1=0}^{n-k-1} h_3(i_1,n-k,n-k)\hat{X}_{t-k-i_1|t-k-i_1}\\
&+\sum_{i_2=n-k+1}^{N} h_3(n-k,i_2,n-k)\hat{X}_{t-k-i_2|t-k-i_2},
\end{aligned}
\tag{C.5}
$$

$$\begin{aligned}\hat{A}^{(2,0)}_{t-k|t-k} = &\sum_{j_1=0}^{n-k-1} h_3(n-k, n-k, j_1)\hat{X}^{*}_{t-k-j_1|t-k-j_1} \\ &+ \sum_{j_1=n-k+1}^{N} h_3(n-k, n-k, j_1)\hat{X}^{*}_{t-k-j_1|t-k-j_1},\end{aligned} \tag{C.6}$$

and

$$\hat{A}^{(2,1)}_{t-k|t-k} = h_3(n-k, n-k, n-k). \tag{C.7}$$

Appendix D

Terms in (2.30) to third–order

This appendix gives the terms $\tilde{A}^{(l,m)}_{t-k|t-k}$ in (2.30) for a third–order ($p = 1$) Volterra series model, where (2.30) is reproduced below as (D.1):

$$\begin{aligned}\hat{X}_{t-n|t} &= \underset{X_{t-n}\in\mathcal{A}_M}{\arg\min} \sum_{k=0}^{n} \left| \sum_{l=0}^{p+1} \sum_{m=0}^{p} \left(A^{(l,m)}_{t-k} - \hat{A}^{(l,m)}_{t-k|t-k} \right) X^{l}_{t-n} \left(X^{*}_{t-n}\right)^{m} + V_{t-k} \right|^2 \\ &= \underset{X_{t-n}\in\mathcal{A}_M}{\arg\min} \sum_{k=0}^{n} \left| \sum_{l=0}^{p+1} \sum_{m=0}^{p} \tilde{A}^{(l,m)}_{t-k} X^{l}_{t-n} \left(X^{*}_{t-n}\right)^{m} + V_{t-k} \right|^2 . \end{aligned} \tag{D.1}$$

For $p = 1$, the terms $\tilde{A}^{(l,m)}_{t-k|t-k} = A^{(l,m)}_{t-k} - \hat{A}^{(l,m)}_{t-k|t-k}$ in (D.1) are as follows, obtained from (B.2)–(B.7) and (C.2)–(C.7):

$$\begin{aligned}\tilde{A}^{(0,0)}_{t-k|t-k} = &\sum_{i_1=0}^{n-k-1} h_1(i_1)\left(X_{t-k-i_1} - \hat{X}_{t-k-i_1|t-k-i_1}\right) \\ &+ \sum_{i_1=n-k+1}^{N} h_1(i_1)\left(X_{t-k-i_1} - \hat{X}_{t-k-i_1|t-k-i_1}\right) \\ &+ \sum_{i_1=0}^{n-k-1} \sum_{i_2=i_1}^{n-k-1} \sum_{j_1=0}^{n-k-1} h_3(i_1,i_2,j_1) \\ &\times \Big(X_{t-k-i_1} X_{t-k-i_2} X^{*}_{t-k-j_1} \\ &- \hat{X}_{t-k-i_1|t-k-i_1} \hat{X}_{t-k-i_2|t-k-i_2} \hat{X}^{*}_{t-k-j_1|t-k-j_1} \Big) \\ &+ \sum_{i_1=0}^{n-k-1} \sum_{i_2=i_1}^{n-k-1} \sum_{j_1=n-k+1}^{N} h_3(i_1,i_2,j_1) \\ &\times \Big(X_{t-k-i_1} X_{t-k-i_2} X^{*}_{t-k-j_1} \end{aligned}$$

$$
\begin{aligned}
& - \hat{X}_{t-k-i_1|t-k-i_1}\hat{X}_{t-k-i_2|t-k-i_2}\hat{X}^*_{t-k-j_1|t-k-j_1}\Big) \\
& + \sum_{i_1=0}^{n-k-1}\sum_{i_2=n-k+1}^{N}\sum_{j_1=0}^{n-k-1} h_3(i_1,i_2,j_1) \\
& \times \Big(X_{t-k-i_1}X_{t-k-i_2}X^*_{t-k-j_1} \\
& - \hat{X}_{t-k-i_1|t-k-i_1}\hat{X}_{t-k-i_2|t-k-i_2}\hat{X}^*_{t-k-j_1|t-k-j_1}\Big) \\
& + \sum_{i_1=0}^{n-k-1}\sum_{i_2=n-k+1}^{N}\sum_{j_1=n-k+1}^{N} h_3(i_1,i_2,j_1) \\
& \times \Big(X_{t-k-i_1}X_{t-k-i_2}X^*_{t-k-j_1} \\
& - \hat{X}_{t-k-i_1|t-k-i_1}\hat{X}_{t-k-i_2|t-k-i_2}\hat{X}^*_{t-k-j_1|t-k-j_1}\Big) \\
& + \sum_{i_1=n-k+1}^{N}\sum_{i_2=i_1}^{N}\sum_{j_1=0}^{n-k-1} h_3(i_1,i_2,j_1) \\
& \times \Big(X_{t-k-i_1}X_{t-k-i_2}X^*_{t-k-j_1} \\
& - \hat{X}_{t-k-i_1|t-k-i_1}\hat{X}_{t-k-i_2|t-k-i_2}\hat{X}^*_{t-k-j_1|t-k-j_1}\Big) \\
& + \sum_{i_1=n-k+1}^{N}\sum_{i_2=i_1}^{N}\sum_{j_1=n-k+1}^{N} h_3(i_1,i_2,j_1) \\
& \times \Big(X_{t-k-i_1}X_{t-k-i_2}X^*_{t-k-j_1} \\
& - \hat{X}_{t-k-i_1|t-k-i_1}\hat{X}_{t-k-i_2|t-k-i_2}\hat{X}^*_{t-k-j_1|t-k-j_1}\Big),
\end{aligned}
\tag{D.2}
$$

$$
\begin{aligned}
\tilde{A}^{(0,1)}_{t-k|t-k} = & \sum_{i_1=0}^{n-k-1}\sum_{i_2=i_1}^{n-k-1} h_3(i_1,i_2,n-k) \\
& \times \Big(X_{t-k-i_1}X_{t-k-i_2} - \hat{X}_{t-k-i_1|t-k-i_1}\hat{X}_{t-k-i_2|t-k-i_2}\Big) \\
& + \sum_{i_1=0}^{n-k-1}\sum_{i_2=n-k+1}^{N} h_3(i_1,i_2,n-k) \\
& \times \Big(X_{t-k-i_1}X_{t-k-i_2} - \hat{X}_{t-k-i_1|t-k-i_1}\hat{X}_{t-k-i_2|t-k-i_2}\Big) \\
& + \sum_{i_1=n-k+1}^{N}\sum_{i_2=i_1}^{N} h_3(i_1,i_2,n-k) \\
& \times \Big(X_{t-k-i_1}X_{t-k-i_2} - \hat{X}_{t-k-i_1|t-k-i_1}\hat{X}_{t-k-i_2|t-k-i_2}\Big),
\end{aligned}
\tag{D.3}
$$

$$
\begin{aligned}
\tilde{A}^{(1,0)}_{t-k|t-k} = & \sum_{i_1=0}^{n-k-1} \sum_{j_1=0}^{n-k-1} h_3(i_1, n-k, j_1) \\
& \times \left(X_{t-k-i_1} X^*_{t-k-j_1} - \hat{X}_{t-k-i_1|t-k-i_1} \hat{X}^*_{t-k-j_1|t-k-j_1} \right) \\
& + \sum_{i_1=0}^{n-k-1} \sum_{j_1=n-k+1}^{N} h_3(i_1, n-k, j_1) \\
& \times \left(X_{t-k-i_1} X^*_{t-k-j_1} - \hat{X}_{t-k-i_1|t-k-i_1} \hat{X}^*_{t-k-j_1|t-k-j_1} \right) \\
& + \sum_{i_2=n-k+1}^{N} \sum_{j_1=0}^{n-k-1} h_3(n-k, i_2, j_1) \\
& \times \left(X_{t-k-i_2} X^*_{t-k-j_1} - \hat{X}_{t-k-i_2|t-k-i_2} \hat{X}^*_{t-k-j_1|t-k-j_1} \right) \\
& + \sum_{i_2=n-k+1}^{N} \sum_{j_1=n-k+1}^{N} h_3(n-k, i_2, j_1) \\
& \times \left(X_{t-k-i_2} X^*_{t-k-j_1} - \hat{X}_{t-k-i_2|t-k-i_2} \hat{X}^*_{t-k-j_1|t-k-j_1} \right),
\end{aligned} \tag{D.4}
$$

$$
\begin{aligned}
\tilde{A}^{(1,1)}_{t-k|t-k} = & \sum_{i_1=0}^{n-k-1} h_3(i_1, n-k, n-k) \left(X_{t-k-i_1} - \hat{X}_{t-k-i_1|t-k-i_1} \right) \\
& + \sum_{i_2=n-k+1}^{N} h_3(n-k, i_2, n-k) \left(X_{t-k-i_2} - \hat{X}_{t-k-i_2|t-k-i_2} \right),
\end{aligned} \tag{D.5}
$$

$$
\begin{aligned}
\tilde{A}^{(2,0)}_{t-k|t-k} = & \sum_{j_1=0}^{n-k-1} h_3(n-k, n-k, j_1) \left(X^*_{t-k-j_1} - \hat{X}^*_{t-k-j_1|t-k-j_1} \right) \\
& + \sum_{j_1=n-k+1}^{N} h_3(n-k, n-k, j_1) \left(X^*_{t-k-j_1} - \hat{X}^*_{t-k-j_1|t-k-j_1} \right),
\end{aligned} \tag{D.6}
$$

and

$$
\tilde{A}^{(2,1)}_{t-k|t-k} = 0. \tag{D.7}
$$

Appendix E

Terms in decomposition of (2.51)

This appendix gives the terms involved in the isolation of quantities that involve the time index $t-n$ in equation (2.51). Retaining only third–order terms, we rewrite (2.51) as follows,

$$\begin{aligned} Y_{t-k} = {} & Y_{t-k}^{(<)} + Y_{t-k}^{(=)} + Y_{t-k}^{(>)} + Y_{t-k}^{(<,<)} + Y_{t-k}^{(<,=)} + Y_{t-k}^{(<,>)} + Y_{t-k}^{(=,=)} + Y_{t-k}^{(=,>)} \\ & + Y_{t-k}^{(>,>)} + Y_{t-k}^{(<,<,<)} + Y_{t-k}^{(<,<,=)} + Y_{t-k}^{(<,<,>)} + Y_{t-k}^{(<,=,=)} + Y_{t-k}^{(<,=,>)} \\ & + Y_{t-k}^{(<,>,>)} + Y_{t-k}^{(=,=,=)} + Y_{t-k}^{(=,=,>)} + Y_{t-k}^{(=,>,>)} + Y_{t-k}^{(>,>,>)} + V_{t-k}, \end{aligned} \tag{E.1}$$

where, for ease of reference, (2.51) is reproduced below as (E.2):

$$\begin{aligned} Y_{t-k} = {} & \sum_{k_1=0}^{N} h_1(k_1) X_{t-k-k_1} + \sum_{k_1=0}^{N} \sum_{k_2=k_1}^{N} h_2(k_1,k_2) X_{t-k-k_1} X_{t-k-k_2} + \dots \\ & + \sum_{k_1=0}^{N} \sum_{k_2=k_1}^{N} \cdots \sum_{k_p=k_{p-1}}^{N} h_p(k_1,k_2,\dots,k_p) X_{t-k-k_1} \dots X_{t-k-k_p} + V_{t-k}. \end{aligned} \tag{E.2}$$

E.1 First–Order terms

$$Y_{t-k}^{(<)} = \sum_{k_1=0}^{n-k-1} h_1(k_1) X_{t-k-k_1}, \tag{E.3}$$

$$Y_{t-k}^{(=)} = h_1(n-k) X_{t-n}, \tag{E.4}$$

and

$$Y_{t-k}^{(>)} = \sum_{k_1=n-k+1}^{N} h_1(k_1) X_{t-k-k_1}. \tag{E.5}$$

E.2 Second–Order terms

$$Y_{t-k}^{(<,<)} = \sum_{k_1=0}^{n-k-1} \sum_{k_2=k_1}^{n-k-1} h_2(k_1,k_2) X_{t-k-k_1} X_{t-k-k_2}, \tag{E.6}$$

$$Y_{t-k}^{(<,=)} = X_{t-n} \sum_{k_1=0}^{n-k-1} h_2(k_1,n-k) X_{t-k-k_1}, \tag{E.7}$$

$$Y_{t-k}^{(<,>)} = \sum_{k_1=0}^{n-k-1} \sum_{k_2=n-k+1}^{N} h_2(k_1,k_2) X_{t-k-k_1} X_{t-k-k_2}, \tag{E.8}$$

$$Y_{t-k}^{(=,=)} = h_2(n-k,n-k) X_{t-n}^2, \tag{E.9}$$

$$Y_{t-k}^{(=,>)} = X_{t-n} \sum_{k_2=n-k+1}^{N} h_2(n-k,k_2) X_{t-k-k_2}, \tag{E.10}$$

and

$$Y_{t-k}^{(>,>)} = \sum_{k_1=n-k+1}^{N} \sum_{k_2=k_1}^{N} h_2(k_1,k_2) X_{t-k-k_1} X_{t-k-k_2}. \tag{E.11}$$

E.3 Third–Order terms

$$Y_{t-k}^{(<,<,<)} = \sum_{k_1=0}^{n-k-1} \sum_{k_2=k_1}^{n-k-1} \sum_{k_3=k_2}^{n-k-1} h_3(k_1,k_2,k_3) X_{t-k-k_1} X_{t-k-k_2} X_{t-k-k_3}, \tag{E.12}$$

$$Y_{t-k}^{(<,<,=)} = X_{t-n} \sum_{k_1=0}^{n-k-1} \sum_{k_2=k_1}^{n-k-1} h_3(k_1,k_2,n-k) X_{t-k-k_1} X_{t-k-k_2}, \tag{E.13}$$

$$Y_{t-k}^{(<,<,>)} = \sum_{k_1=0}^{n-k-1} \sum_{k_2=k_1}^{n-k-1} \sum_{k_3=n-k+1}^{N} h_3(k_1,k_2,k_3) X_{t-k-k_1} X_{t-k-k_2} X_{t-k-k_3}, \tag{E.14}$$

$$Y_{t-k}^{(<,=,=)} = X_{t-n}^2 \sum_{k_1=0}^{n-k-1} h_3(k_1,n-k,n-k) X_{t-k-k_1}, \tag{E.15}$$

$$Y_{t-k}^{(<,=,>)} = X_{t-n} \sum_{k_1=0}^{n-k-1} \sum_{k_3=n-k+1}^{N} h_3(k_1,n-k,k_3) X_{t-k-k_1} X_{t-k-k_3}, \tag{E.16}$$

$$Y_{t-k}^{(<,>,>)} = \sum_{k_1=0}^{n-k-1} \sum_{k_2=n-k+1}^{N} \sum_{k_3=k_2}^{N} h_3(k_1,k_2,k_3) X_{t-k-k_1} X_{t-k-k_2} X_{t-k-k_3}, \tag{E.17}$$

$$Y_{t-k}^{(=,=,=)} = h_3(n-k,n-k,n-k) X_{t-n}^3, \tag{E.18}$$

$$Y_{t-k}^{(=,=,>)} = X_{t-n}^2 \sum_{k_3=n-k+1}^{N} h_3(n-k, n-k, k_3) X_{t-k-k_3}, \tag{E.19}$$

$$Y_{t-k}^{(=,>,>)} = X_{t-n} \sum_{k_2=n-k+1}^{N} \sum_{k_3=k_2}^{N} h_3(n-k, k_2, k_3) X_{t-k-k_2} X_{t-k-k_3}, \tag{E.20}$$

and

$$Y_{t-k}^{(>,>,>)} = \sum_{k_1=n-k+1}^{N} \sum_{k_2=k_1}^{N} \sum_{k_3=k_2}^{N} h_3(k_1, k_2, k_3) X_{t-k-k_1} X_{t-k-k_2} X_{t-k-k_3}. \tag{E.21}$$

Appendix F

Terms A_{t-k} and B_{t-k} in (2.52)

This appendix gives the terms A_{t-k} and B_{t-k} in (2.52), to third order, *viz.*

$$
\begin{aligned}
A_{t-k} = & \sum_{k_1=0}^{n-k-1} h_1(k_1) X_{t-k-k_1} + \sum_{k_1=n-k+1}^{N} h_1(k_1) X_{t-k-k_1} \\
& + \sum_{k_1=0}^{n-k-1} \sum_{k_2=k_1}^{n-k-1} h_2(k_1,k_2) X_{t-k-k_1} X_{t-k-k_2} \\
& + \sum_{k_1=0}^{n-k-1} \sum_{k_2=n-k+1}^{N} h_2(k_1,k_2) X_{t-k-k_1} X_{t-k-k_2} \\
& + h_2(n-k,n-k) + \sum_{k_1=n-k+1}^{N} \sum_{k_2=k_1}^{N} h_2(k_1,k_2) X_{t-k-k_1} X_{t-k-k_2} \\
& + \sum_{k_1=0}^{n-k-1} \sum_{k_2=k_1}^{n-k-1} \sum_{k_3=k_2}^{n-k-1} h_3(k_1,k_2,k_3) X_{t-k-k_1} X_{t-k-k_2} X_{t-k-k_3} \\
& + \sum_{k_1=0}^{n-k-1} \sum_{k_2=k_1}^{n-k-1} \sum_{k_3=n-k+1}^{N} h_3(k_1,k_2,k_3) X_{t-k-k_1} X_{t-k-k_2} X_{t-k-k_3} \\
& + \sum_{k_1=0}^{n-k-1} h_3(k_1,n-k,n-k) X_{t-k-k_1} \\
& + \sum_{k_1=0}^{n-k-1} \sum_{k_2=n-k+1}^{N} \sum_{k_3=k_2}^{N} h_3(k_1,k_2,k_3) X_{t-k-k_1} X_{t-k-k_2} X_{t-k-k_3} \\
& + \sum_{k_3=n-k+1}^{N} h_3(n-k,n-k,k_3) X_{t-k-k_3} \\
& + \sum_{k_1=n-k+1}^{N} \sum_{k_2=k_1}^{N} \sum_{k_3=k_2}^{N} h_3(k_1,k_2,k_3) X_{t-k-k_1} X_{t-k-k_2} X_{t-k-k_3},
\end{aligned}
\tag{F.1}
$$

and

$$\begin{aligned}
B_{t-k} = h_1(n-k) &+ \sum_{k_1=0}^{n-k-1} h_2(k_1, n-k) X_{t-k-k_1} \\
&+ \sum_{k_2=n-k+1}^{N} h_2(n-k, k_2) X_{t-k-k_2} \\
&+ \sum_{k_1=0}^{n-k-1} \sum_{k_2=k_1}^{n-k-1} h_3(k_1, k_2, n-k) X_{t-k-k_1} X_{t-k-k_2} \\
&+ \sum_{k_1=0}^{n-k-1} \sum_{k_3=n-k+1}^{N} h_3(k_1, n-k, k_3) X_{t-k-k_1} X_{t-k-k_3} \\
&+ h_3(n-k, n-k, n-k) \\
&+ \sum_{k_2=n-k+1}^{N} \sum_{k_3=k_2}^{N} h_3(n-k, k_2, k_3) X_{t-k-k_2} X_{t-k-k_3}. \qquad \text{(F.2)}
\end{aligned}$$

Appendix G

Terms $\hat{A}_{t-k|t-k}$ and $\hat{B}_{t-k|t-k}$ in (2.53)

This appendix gives the terms $\hat{A}_{t-k|t-k}$ and $\hat{B}_{t-k|t-k}$ in (2.53), to third order, *viz.*

$$
\begin{aligned}
\hat{A}_{t-k|t-k} = &\sum_{k_1=0}^{n-k-1} h_1(k_1)\hat{X}_{t-k-k_1|t-k-k_1} + \sum_{k_1=n-k+1}^{N} h_1(k_1)\hat{X}_{t-k-k_1|t-k-k_1} \\
&+ \sum_{k_1=0}^{n-k-1}\sum_{k_2=k_1}^{n-k-1} h_2(k_1,k_2)\hat{X}_{t-k-k_1|t-k-k_1}\hat{X}_{t-k-k_2|t-k-k_2} \\
&+ \sum_{k_1=0}^{n-k-1}\sum_{k_2=n-k+1}^{N} h_2(k_1,k_2)\hat{X}_{t-k-k_1|t-k-k_1}\hat{X}_{t-k-k_2|t-k-k_2} \\
&+ h_2(n-k,n-k) \\
&+ \sum_{k_1=n-k+1}^{N}\sum_{k_2=k_1}^{N} h_2(k_1,k_2)\hat{X}_{t-k-k_1|t-k-k_1}\hat{X}_{t-k-k_2|t-k-k_2} \\
&+ \sum_{k_1=0}^{n-k-1}\sum_{k_2=k_1}^{n-k-1}\sum_{k_3=k_2}^{n-k-1} h_3(k_1,k_2,k_3) \\
&\times \hat{X}_{t-k-k_1|t-k-k_1}\hat{X}_{t-k-k_2|t-k-k_2}\hat{X}_{t-k-k_3|t-k-k_3} \\
&+ \sum_{k_1=0}^{n-k-1}\sum_{k_2=k_1}^{n-k-1}\sum_{k_3=n-k+1}^{N} h_3(k_1,k_2,k_3) \\
&\times \hat{X}_{t-k-k_1|t-k-k_1}\hat{X}_{t-k-k_2|t-k-k_2}\hat{X}_{t-k-k_3|t-k-k_3} \\
&+ \sum_{k_1=0}^{n-k-1} h_3(k_1,n-k,n-k)\hat{X}_{t-k-k_1|t-k-k_1} \\
&+ \sum_{k_1=0}^{n-k-1}\sum_{k_2=n-k+1}^{N}\sum_{k_3=k_2}^{N} h_3(k_1,k_2,k_3) \\
&\times \hat{X}_{t-k-k_1|t-k-k_1}\hat{X}_{t-k-k_2|t-k-k_2}\hat{X}_{t-k-k_3|t-k-k_3}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{k_3=n-k+1}^{N} h_3(n-k,n-k,k_3)\hat{X}_{t-k-k_3|t-k-k_3} \\
&+\sum_{k_1=n-k+1}^{N}\sum_{k_2=k_1}^{N}\sum_{k_3=k_2}^{N} h_3(k_1,k_2,k_3) \\
&\times \hat{X}_{t-k-k_1|t-k-k_1}\hat{X}_{t-k-k_2|t-k-k_2}\hat{X}_{t-k-k_3|t-k-k_3},
\end{aligned} \tag{G.1}
$$

and

$$
\begin{aligned}
\hat{B}_{t-k|t-k} = h_1(n-k) &+ \sum_{k_1=0}^{n-k-1} h_2(k_1,n-k)\hat{X}_{t-k-k_1|t-k-k_1} \\
&+ \sum_{k_2=n-k+1}^{N} h_2(n-k,k_2)\hat{X}_{t-k-k_2|t-k-k_2} \\
&+ \sum_{k_1=0}^{n-k-1}\sum_{k_2=k_1}^{n-k-1} h_3(k_1,k_2,n-k)\hat{X}_{t-k-k_1|t-k-k_1}\hat{X}_{t-k-k_2|t-k-k_2} \\
&+ \sum_{k_1=0}^{n-k-1}\sum_{k_3=n-k+1}^{N} h_3(k_1,n-k,k_3)\hat{X}_{t-k-k_1|t-k-k_1}\hat{X}_{t-k-k_3|t-k-k_3} \\
&+ h_3(n-k,n-k,n-k) \\
&+ \sum_{k_2=n-k+1}^{N}\sum_{k_3=k_2}^{N} h_3(n-k,k_2,k_3)\hat{X}_{t-k-k_2|t-k-k_2}\hat{X}_{t-k-k_3|t-k-k_3}.
\end{aligned} \tag{G.2}
$$

Appendix H

State Transition Probability : $n = 0$

This appendix derives the state transition probability given by (3.2).

The estimator $\hat{X}_{t|t}$ of chapter 2 is a function of the symbols $X_t, \ldots, X_{t-N}$ and the previous estimators $\hat{X}_{t-1|t-1}, \ldots, \hat{X}_{t-N|t-N}$. Pool these variables to form the state vector $\mathbf{S}_{t|t}$, with domain $\mathbf{s}_{t|t} \in \mathcal{A}_M^{2(N+1)}$. Although given earlier in (3.1), the definition for $\mathbf{S}_{t|t}$ is reproduced for convenience below as (H.1):

$$\mathbf{S}_{t|t} = \begin{bmatrix} X_t \\ \vdots \\ X_{t-N} \\ \hat{X}_{t|t} \\ \vdots \\ \hat{X}_{t-N|t-N} \end{bmatrix}. \tag{H.1}$$

For $m \in \mathbb{Z}^+$ we wish to determine the state transition probability

$$\mathbb{P}(\mathbf{S}_{t|t} = \mathbf{s}_{t|t} | \mathbf{S}_{t-1|t-1} = \mathbf{s}_{t-1|t-1} \cap \cdots \cap \mathbf{S}_{t-m|t-m} = \mathbf{s}_{t-m|t-m}), \tag{H.2}$$

and to subsequently show that $\mathbf{S}_{t|t}$ is a first–order Markov process [50], in that

$$\begin{aligned} &\mathbb{P}(\mathbf{S}_{t|t} = \mathbf{s}_{t|t} | \mathbf{S}_{t-1|t-1} = \mathbf{s}_{t-1|t-1} \cap \cdots \cap \mathbf{S}_{t-m|t-m} = \mathbf{s}_{t-m|t-m}) \\ &= \mathbb{P}(\mathbf{S}_{t|t} = \mathbf{s}_{t|t} | \mathbf{S}_{t-1|t-1} = \mathbf{s}_{t-1|t-1}). \end{aligned} \tag{H.3}$$

The operator $\mathbb{P}$ in (H.2) and (H.3) is defined to be a discrete multivariate probability distribution over the $2(N + m + 1)$ random variables X_{t-i} and $\hat{X}_{t-i|t-i}$, for $i \in \{0, \ldots, N + m\}$, where the domain of each X_{t-i} and $\hat{X}_{t-i|t-i}$ is $\mathcal{A}_M$.

When one or more of the variables $X_t, \ldots, X_{t-N-m}, \hat{X}_{t|t}, \ldots, \hat{X}_{t-N-m|t-N-m}$ is missing from the argument of $\mathbb{P}$, we understand this to mean that we have a marginal probability distribution. The state probability $\mathbb{P}(\mathbf{S}_{t|t} = \mathbf{s}_{t|t})$, for instance, is given by

$$\begin{aligned}
\mathbb{P}(\mathbf{S}_{t|t} = \mathbf{s}_{t|t}) &= \mathbb{P}(X_t = x_t \cap \cdots \cap X_{t-N} = x_{t-N} \\
&\quad \cap \hat{X}_{t|t} = \hat{x}_{t|t} \cap \cdots \cap \hat{X}_{t-N|t-N} = \hat{x}_{t-N|t-N}) \\
&= \sum_{x_{t-N-1} \in \mathcal{A}_M} \cdots \sum_{x_{t-N-m} \in \mathcal{A}_M} \\
&\quad \sum_{\hat{x}_{t-N-1|t-N-1} \in \mathcal{A}_M} \cdots \sum_{\hat{x}_{t-N-m|t-N-m} \in \mathcal{A}_M} \\
&\quad \mathbb{P}(X_t = x_t \cap \cdots \cap X_{t-N-m} = x_{t-N-m} \\
&\quad \cap \hat{X}_{t|t} = \hat{x}_{t|t} \cap \cdots \cap \hat{X}_{t-N-m|t-N-m} = \hat{x}_{t-N-m|t-N-m}),
\end{aligned} \tag{H.4}$$

where we have summed over the domain of each of the 'missing' variables X_{t-N-1}, $\ldots, X_{t-N-m}, \hat{X}_{t-N-1|t-N-1}, \ldots, \hat{X}_{t-N-m|t-N-m}$.

For a given probability distribution $\mathbb{P}$, the conditional probability of an event A assuming that B has occurred, denoted $\mathbb{P}(A|B)$, is given by [62]

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}. \tag{H.5}$$

Use (H.5) to write out (H.2) in full, where for brevity we denote the events $X_{t-i} = x_{t-i}$ by $\mathbb{A}_i$, and $\hat{X}_{t-i|t-i} = \hat{x}_{t-i|t-i}$ by $\mathbb{B}_i$, for $i \in \{0, \ldots, N+m\}$:

$$\begin{aligned}
&\mathbb{P}(\mathbf{S}_{t|t} = \mathbf{s}_{t|t} | \mathbf{S}_{t-1|t-1} = \mathbf{s}_{t-1|t-1} \cap \cdots \cap \mathbf{S}_{t-m|t-m} = \mathbf{s}_{t-m|t-m}) \\
&= \frac{\mathbb{P}(\mathbf{S}_{t|t} = \mathbf{s}_{t|t} \cap \cdots \cap \mathbf{S}_{t-m|t-m} = \mathbf{s}_{t-m|t-m})}{\mathbb{P}(\mathbf{S}_{t-1|t-1} = \mathbf{s}_{t-1|t-1} \cap \cdots \cap \mathbf{S}_{t-m|t-m} = \mathbf{s}_{t-m|t-m})} \\
&= \frac{\mathbb{P}(\mathbb{A}_0 \cap \cdots \cap \mathbb{A}_{N+m} \cap \mathbb{B}_0 \cap \cdots \cap \mathbb{B}_{N+m})}{\mathbb{P}(\mathbb{A}_1 \cap \cdots \cap \mathbb{A}_{N+m} \cap \mathbb{B}_1 \cap \cdots \cap \mathbb{B}_{N+m})}.
\end{aligned} \tag{H.6}$$

Using (H.5) again, isolate $\mathbb{B}_0$ in the numerator of (H.6):

$$\begin{aligned}
&\mathbb{P}(\mathbb{A}_0 \cap \cdots \cap \mathbb{A}_{N+m} \cap \mathbb{B}_0 \cap \cdots \cap \mathbb{B}_{N+m}) \\
&= \mathbb{P}(\mathbb{B}_0 | \mathbb{A}_0 \cap \cdots \cap \mathbb{A}_{N+m} \cap \mathbb{B}_1 \cap \cdots \cap \mathbb{B}_{N+m}) \\
&\quad \times \mathbb{P}(\mathbb{A}_0 \cap \cdots \cap \mathbb{A}_{N+m} \cap \mathbb{B}_1 \cap \cdots \cap \mathbb{B}_{N+m}).
\end{aligned} \tag{H.7}$$

Now remove the events $\mathbb{A}_{N+1}, \ldots, \mathbb{A}_{N+m}, \mathbb{B}_{N+1}, \ldots, \mathbb{B}_{N+m}$ from the first term on the right–hand side of (H.7), as $\hat{X}_{t|t}$ only depends on $X_t, \ldots, X_{t-N}$ and $\hat{X}_{t-1|t-1}, \ldots,$

$\hat{X}_{t-N|t-N}$. We thus simplify (H.7) to

$$\begin{aligned}&\mathbb{P}(\mathbb{A}_0 \cap \cdots \cap \mathbb{A}_{N+m} \cap \mathbb{B}_0 \cap \cdots \cap \mathbb{B}_{N+m})\\&= \mathbb{P}(\mathbb{B}_0 | \mathbb{A}_0 \cap \cdots \cap \mathbb{A}_N \cap \mathbb{B}_1 \cap \cdots \cap \mathbb{B}_N)\\&\quad \times \mathbb{P}(\mathbb{A}_0 \cap \cdots \cap \mathbb{A}_{N+m} \cap \mathbb{B}_1 \cap \cdots \cap \mathbb{B}_{N+m}).\end{aligned} \tag{H.8}$$

Isolate $\mathbb{B}_1$ from the second term on the right–hand side of (H.8):

$$\begin{aligned}&\mathbb{P}(\mathbb{A}_0 \cap \cdots \cap \mathbb{A}_{N+m} \cap \mathbb{B}_0 \cap \cdots \cap \mathbb{B}_{N+m})\\&= \mathbb{P}(\mathbb{B}_0 | \mathbb{A}_0 \cap \cdots \cap \mathbb{A}_N \cap \mathbb{B}_1 \cap \cdots \cap \mathbb{B}_N)\\&\quad \times \mathbb{P}(\mathbb{B}_1 | \mathbb{A}_0 \cap \cdots \cap \mathbb{A}_{N+m} \cap \mathbb{B}_2 \cap \cdots \cap \mathbb{B}_{N+m})\\&\quad \times \mathbb{P}(\mathbb{A}_0 \cap \cdots \cap \mathbb{A}_{N+m} \cap \mathbb{B}_2 \cap \cdots \cap \mathbb{B}_{N+m}).\end{aligned} \tag{H.9}$$

Remove the events $\mathbb{A}_0, \mathbb{A}_{N+2}, \ldots, \mathbb{A}_{N+m}, \mathbb{B}_{N+2}, \ldots, \mathbb{B}_{N+m}$ from the second term on the right–hand side of (H.9), as $\hat{X}_{t-1|t-1}$ only depends on $X_{t-1}, \ldots, X_{t-N-1}$ and $\hat{X}_{t-2|t-2}$, $\ldots, \hat{X}_{t-N-1|t-N-1}$. We thus simplify (H.9) to

$$\begin{aligned}&\mathbb{P}(\mathbb{A}_0 \cap \cdots \cap \mathbb{A}_{N+m} \cap \mathbb{B}_0 \cap \cdots \cap \mathbb{B}_{N+m})\\&= \mathbb{P}(\mathbb{B}_0 | \mathbb{A}_0 \cap \cdots \cap \mathbb{A}_N \cap \mathbb{B}_1 \cap \cdots \cap \mathbb{B}_N)\\&\quad \times \mathbb{P}(\mathbb{B}_1 | \mathbb{A}_1 \cap \cdots \cap \mathbb{A}_{N+1} \cap \mathbb{B}_2 \cap \cdots \cap \mathbb{B}_{N+1})\\&\quad \times \mathbb{P}(\mathbb{A}_0 \cap \cdots \cap \mathbb{A}_{N+m} \cap \mathbb{B}_2 \cap \cdots \cap \mathbb{B}_{N+m}).\end{aligned} \tag{H.10}$$

Observe that $\mathbb{A}_0$ is independent of $\mathbb{A}_1 \cap \cdots \cap \mathbb{A}_{N+m} \cap \mathbb{B}_2 \cap \cdots \cap \mathbb{B}_{N+m}$, since X_t is independent of $X_{t-1}, \ldots, X_{t-N-m}$ and $\hat{X}_{t-2|t-2}, \ldots, \hat{X}_{t-N-m|t-N-m}$. Thus (H.10) simplifies further to

$$\begin{aligned}&\mathbb{P}(\mathbb{A}_0 \cap \cdots \cap \mathbb{A}_{N+m} \cap \mathbb{B}_0 \cap \cdots \cap \mathbb{B}_{N+m})\\&= \mathbb{P}(\mathbb{B}_0 | \mathbb{A}_0 \cap \cdots \cap \mathbb{A}_N \cap \mathbb{B}_1 \cap \cdots \cap \mathbb{B}_N)\\&\quad \times \mathbb{P}(\mathbb{B}_1 | \mathbb{A}_1 \cap \cdots \cap \mathbb{A}_{N+1} \cap \mathbb{B}_2 \cap \cdots \cap \mathbb{B}_{N+1})\\&\quad \times \mathbb{P}(\mathbb{A}_0)\mathbb{P}(\mathbb{A}_1 \cap \cdots \cap \mathbb{A}_{N+m} \cap \mathbb{B}_2 \cap \cdots \cap \mathbb{B}_{N+m}).\end{aligned} \tag{H.11}$$

Now apply (H.5) to the denominator of (H.6), and isolate $\mathbb{B}_1$:

$$\begin{aligned}&\mathbb{P}(\mathbb{A}_1 \cap \cdots \cap \mathbb{A}_{N+m} \cap \mathbb{B}_1 \cap \cdots \cap \mathbb{B}_{N+m})\\&= \mathbb{P}(\mathbb{B}_1 | \mathbb{A}_1 \cap \cdots \cap \mathbb{A}_{N+m} \cap \mathbb{B}_2 \cap \cdots \cap \mathbb{B}_{N+m})\\&\quad \times \mathbb{P}(\mathbb{A}_1 \cap \cdots \cap \mathbb{A}_{N+m} \cap \mathbb{B}_2 \cap \cdots \cap \mathbb{B}_{N+m}).\end{aligned} \tag{H.12}$$

Remove the events $\mathbb{A}_{N+2}, \ldots, \mathbb{A}_{N+m}, \mathbb{B}_{N+2}, \ldots, \mathbb{B}_{N+m}$ from the first term on the right–hand side of (H.12), as $\hat{X}_{t-1|t-1}$ depends only on $X_{t-1}, \ldots, X_{t-N-1}$ and $\hat{X}_{t-2|t-2}, \ldots, \hat{X}_{t-N-1|t-N-1}$. We thus simplify (H.12) to

$$\begin{aligned}
&\mathbb{P}(\mathbb{A}_1 \cap \cdots \cap \mathbb{A}_{N+m} \cap \mathbb{B}_1 \cap \cdots \cap \mathbb{B}_{N+m}) \\
&= \mathbb{P}(\mathbb{B}_1 | \mathbb{A}_1 \cap \cdots \cap \mathbb{A}_{N+1} \cap \mathbb{B}_2 \cap \cdots \cap \mathbb{B}_{N+1}) \\
&\quad \times \mathbb{P}(\mathbb{A}_1 \cap \cdots \cap \mathbb{A}_{N+m} \cap \mathbb{B}_2 \cap \cdots \cap \mathbb{B}_{N+m}). \qquad \text{(H.13)}
\end{aligned}$$

Finally, observe that the right–hand side of (H.13) is a factor of the right–hand side of (H.11). From (H.6), (H.11) and (H.13), we thus obtain the result

$$\begin{aligned}
&\mathbb{P}(\mathbf{S}_{t|t} = \mathbf{s}_{t|t} | \mathbf{S}_{t-1|t-1} = \mathbf{s}_{t-1|t-1} \cap \cdots \cap \mathbf{S}_{t-m|t-m} = \mathbf{s}_{t-m|t-m}) \\
&= \mathbb{P}(\mathbb{A}_0)\mathbb{P}(\mathbb{B}_0 | \mathbb{A}_0 \cap \cdots \cap \mathbb{A}_N \cap \mathbb{B}_1 \cap \cdots \cap \mathbb{B}_N) \\
&= \mathbb{P}(X_t = x_t)\mathbb{P}(\hat{X}_{t|t} = \hat{x}_{t|t} | X_t = x_t \cap \cdots \cap X_{t-N} = x_{t-N} \\
&\quad \cap \hat{X}_{t-1|t-1} = \hat{x}_{t-1|t-1} \cap \cdots \cap \hat{X}_{t-N|t-N} = \hat{x}_{t-N|t-N}). \qquad \text{(H.14)}
\end{aligned}$$

Note that the right–hand side of (H.14) is independent of $m \in \mathbb{Z}^+$, and thus (H.14) holds for $m = 1$ as well as for arbitrary $m \in \mathbb{Z}^+$. Hence (H.3) is true for all $m \in \mathbb{Z}^+$ and $\mathbf{S}_{t|t}$ of (H.1) is a first–order Markov process [50].

Setting $m = 1$ in (H.14) we have the state transition probability

$$\begin{aligned}
&\mathbb{P}(\mathbf{S}_{t|t} = \mathbf{s}_{t|t} | \mathbf{S}_{t-1|t-1} = \mathbf{s}_{t-1|t-1}) \\
&= \mathbb{P}(X_t = x_t)\mathbb{P}(\hat{X}_{t|t} = \hat{x}_{t|t} | X_t = x_t \cap \cdots \cap X_{t-N} = x_{t-N} \\
&\quad \cap \hat{X}_{t-1|t-1} = \hat{x}_{t-1|t-1} \cap \cdots \cap \hat{X}_{t-N|t-N} = \hat{x}_{t-N|t-N}). \qquad \text{(H.15)}
\end{aligned}$$

The first term $\mathbb{P}(X_t = x_t)$ in the right–hand side of (H.15) is the *a priori* probability of symbol value $x_t \in \mathcal{A}_M$. $\mathbb{P}(X_t = x_t)$ is the relative frequency at which the baseband symbol x_t is transmitted into the digital communication or data storage channel. The second term gives the conditional probability of the filtered estimate $\hat{x}_{t|t} \in \mathcal{A}_M$, where we have fixed values $x_t, \ldots, x_{t-N}, \hat{x}_{t-1|t-1}, \ldots, \hat{x}_{t-N|t-N}$ for the variables $X_t, \ldots, X_{t-N}, \hat{X}_{t-1|t-1}, \ldots, \hat{X}_{t-N|t-N}$ of which $\hat{X}_{t|t}$ is a function. This functional dependence is illustrated in section 2.3 for the case of MPSK and MQAM signalling, and 2.4 for BPSK signalling. Note that the second term in the right-hand side of (H.15) is a probability distribution function, rather than a deterministic expression, since $\hat{X}_{t|t}$ also depends on the additive noise variate V_t, as shown in the models of chapter 2.

Appendix I

State Transition Probability : $n \in \{1, \ldots, N\}$

This appendix derives the state transition probability given by (3.179).

For $n \in \{1, \ldots, N\}$ the smoothed estimator $\hat{X}_{t-n|t}$ of chapter 2 is a function of the symbols X_t, ..., X_{t-N-n} and the filtered estimators $\hat{X}_{t|t}$, ..., $\hat{X}_{t-n+1|t-n+1}$, $\hat{X}_{t-n-1|t-n-1}$, ..., $\hat{X}_{t-N-n|t-N-n}$. Pool these variables, together with the 'missing' filtered estimator $\hat{X}_{t-n|t-n}$, to form the state vector $\mathbf{S}_{t-n|t}$, with domain $\mathbf{s}_{t|t} \in \mathcal{A}_M^{2N+2n+3}$. Although given earlier in (3.178), the definition for $\mathbf{S}_{t-n|t}$ is reproduced for convenience below as (I.1):

$$\mathbf{S}_{t-n|t} = \begin{bmatrix} X_t \\ \vdots \\ X_{t-N-n} \\ \hat{X}_{t|t} \\ \vdots \\ \hat{X}_{t-N-n|t-N-n} \\ \hat{X}_{t-n|t} \end{bmatrix}. \tag{I.1}$$

For $m \in \mathbb{Z}^+$ we wish to determine the state transition probability

$$\mathbb{P}(\mathbf{S}_{t-n|t} = \mathbf{s}_{t-n|t} | \mathbf{S}_{t-n-1|t-1} = \mathbf{s}_{t-n-1|t-1} \cap \cdots \cap \mathbf{S}_{t-n-m|t-m} = \mathbf{s}_{t-n-m|t-m}), \tag{I.2}$$

and to subsequently show that $\mathbf{S}_{t-n|t}$ is a first–order Markov process [50], in that

$$\begin{aligned} &\mathbb{P}(\mathbf{S}_{t-n|t} = \mathbf{s}_{t-n|t} | \mathbf{S}_{t-n-1|t-1} = \mathbf{s}_{t-n-1|t-1} \cap \cdots \cap \mathbf{S}_{t-n-m|t-m} = \mathbf{s}_{t-n-m|t-m}) \\ &= \mathbb{P}(\mathbf{S}_{t-n|t} = \mathbf{s}_{t-n|t} | \mathbf{S}_{t-n-1|t-1} = \mathbf{s}_{t-n-1|t-1}). \end{aligned} \tag{I.3}$$

The operator $\mathbb{P}$ in (I.2) and (I.3) is defined to be a discrete multivariate probability distribution over the $2N + 2n + 3m + 3$ random variables X_{t-i}, $\hat{X}_{t-i|t-i}$, and $\hat{X}_{t-n-j|t-j}$, where $i \in \{0, \ldots, N + n + m\}$ and $j \in \{0, \ldots, m\}$. The domain of each variate is the baseband alphabet $\mathcal{A}_M$. (See (2.13) for MPSK signalling, for example.)

Where one or more of the variables X_{t-i}, $\hat{X}_{t-i|t-i}$, and $\hat{X}_{t-n-j|t-j}$ is missing from the argument of $\mathbb{P}$, we understand this to mean that we have a marginal probability distribution. The state probability $\mathbb{P}(\mathbf{S}_{t-n|t} = \mathbf{s}_{t-n|t})$, for instance, is given by

$$\begin{aligned}\mathbb{P}(\mathbf{S}_{t-n|t} = \mathbf{s}_{t-n|t}) = \mathbb{P}(&X_t = x_t \cap \cdots \cap X_{t-N-n} = x_{t-N-n} \cap \hat{X}_{t|t} = \hat{x}_{t|t} \cap \ldots \\ &\cap \hat{X}_{t-N-n|t-N-n} = \hat{x}_{t-N-n|t-N-n} \cap \hat{X}_{t-n|t} = \hat{x}_{t-n|t}) \\ = &\sum_{x_{t-N-n-1} \in \mathcal{A}_M} \cdots \sum_{x_{t-N-n-m} \in \mathcal{A}_M} \\ &\sum_{\hat{x}_{t-N-n-1|t-N-n-1} \in \mathcal{A}_M} \cdots \sum_{\hat{x}_{t-N-n-m|t-N-n-m} \in \mathcal{A}_M} \\ &\sum_{\hat{x}_{t-n-1|t-1} \in \mathcal{A}_M} \cdots \sum_{\hat{x}_{t-n-m|t-m} \in \mathcal{A}_M} \\ &\mathbb{P}(X_t = x_t \cap \cdots \cap X_{t-N-n-m} = x_{t-N-n-m} \cap \hat{X}_{t|t} = \hat{x}_{t|t} \cap \ldots \\ &\cap \hat{X}_{t-N-n-m|t-N-n-m} = \hat{x}_{t-N-n-m|t-N-n-m} \\ &\cap \hat{X}_{t-n|t} = \hat{x}_{t-n|t} \cap \cdots \cap \hat{X}_{t-n-m|t-m} = \hat{x}_{t-n-m|t-m}),\end{aligned} \tag{I.4}$$

where the sums are over the set of possible values of the 'missing' discrete random variables X_{t-i}, $\hat{X}_{t-i|t-i}$ and $\hat{X}_{t-n-j|t-j}$, respectively, for $i \in \{N+n+1, \ldots, N+n+m\}$ and $j \in \{1, \ldots, m\}$.

For a given probability distribution $\mathbb{P}$, the conditional probability of an event A assuming that B has occurred, denoted $\mathbb{P}(A|B)$, equals [62]

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}. \tag{I.5}$$

Use (I.5) to write out (I.2) in full, where for brevity we denote the events $X_{t-i} = x_{t-i}$ by $\mathbb{A}_i$, $\hat{X}_{t-i|t-i} = \hat{x}_{t-i|t-i}$ by $\mathbb{B}_i$, and $\hat{X}_{t-n-j|t-j} = \hat{x}_{t-n-j|t-j}$ by $\mathbb{C}_j$, with $i \in \{0, \ldots, N+n+m\}$ and $j \in \{0, \ldots, m\}$:

$$\begin{aligned}&\mathbb{P}(\mathbf{S}_{t-n|t} = \mathbf{s}_{t-n|t} | \mathbf{S}_{t-n-1|t-1} = \mathbf{s}_{t-n-1|t-1} \cap \cdots \cap \mathbf{S}_{t-n-m|t-m} = \mathbf{s}_{t-n-m|t-m}) \\ &= \frac{\mathbb{P}(\mathbf{S}_{t-n|t} = \mathbf{s}_{t-n|t} \cap \cdots \cap \mathbf{S}_{t-n-m|t-m} = \mathbf{s}_{t-n-m|t-m})}{\mathbb{P}(\mathbf{S}_{t-n-1|t-1} = \mathbf{s}_{t-n-1|t-1} \cap \cdots \cap \mathbf{S}_{t-n-m|t-m} = \mathbf{s}_{t-n-m|t-m})} \\ &= \frac{\mathbb{P}(\mathbb{A}_0 \cap \cdots \cap \mathbb{A}_{N+n+m} \cap \mathbb{B}_0 \cap \cdots \cap \mathbb{B}_{N+n+m} \cap \mathbb{C}_0 \cap \cdots \cap \mathbb{C}_m)}{\mathbb{P}(\mathbb{A}_1 \cap \cdots \cap \mathbb{A}_{N+n+m} \cap \mathbb{B}_1 \cap \cdots \cap \mathbb{B}_{N+n+m} \cap \mathbb{C}_1 \cap \cdots \cap \mathbb{C}_m)}.\end{aligned} \tag{I.6}$$

Using (I.5) again, isolate $\mathbb{C}_0$ in the numerator of (I.6):

$$\begin{aligned}&\mathbb{P}(\mathbb{A}_0 \cap \cdots \cap \mathbb{A}_{N+n+m} \cap \mathbb{B}_0 \cap \cdots \cap \mathbb{B}_{N+n+m} \cap \mathbb{C}_0 \cap \cdots \cap \mathbb{C}_m) \\ &= \mathbb{P}(\mathbb{C}_0 | \mathbb{A}_0 \cap \cdots \cap \mathbb{A}_{N+n+m} \cap \mathbb{B}_0 \cap \cdots \cap \mathbb{B}_{N+n+m} \cap \mathbb{C}_1 \cap \cdots \cap \mathbb{C}_m) \\ &\quad \times \mathbb{P}(\mathbb{A}_0 \cap \cdots \cap \mathbb{A}_{N+n+m} \cap \mathbb{B}_0 \cap \cdots \cap \mathbb{B}_{N+n+m} \cap \mathbb{C}_1 \cap \cdots \cap \mathbb{C}_m).\end{aligned} \tag{I.7}$$

Now remove the events $\mathbb{A}_{N+n+1}, \ldots, \mathbb{A}_{N+n+m}, \mathbb{B}_n, \mathbb{B}_{N+n+1}, \ldots, \mathbb{B}_{N+n+m}$ and $\mathbb{C}_1, \ldots, \mathbb{C}_m$ from the first term on the right–hand side of (I.7), since FLSDFE output $\hat{X}_{t-n|t}$ only depends on variables $X_t, \ldots, X_{t-N-n}$, and $\hat{X}_{t|t}, \ldots, \hat{X}_{t-n+1|t-n+1}, \hat{X}_{t-n-1|t-n-1}, \ldots, \hat{X}_{t-N-n|t-N-n}$. We thus simplify (I.7) to

$$\begin{aligned}
&\mathbb{P}(\mathbb{A}_0 \cap \cdots \cap \mathbb{A}_{N+n+m} \cap \mathbb{B}_0 \cap \cdots \cap \mathbb{B}_{N+n+m} \cap \mathbb{C}_0 \cap \cdots \cap \mathbb{C}_m) \\
&= \mathbb{P}(\mathbb{C}_0 | \mathbb{A}_0 \cap \cdots \cap \mathbb{A}_{N+n} \cap \mathbb{B}_0 \cap \cdots \cap \mathbb{B}_{n-1} \cap \mathbb{B}_{n+1} \cap \cdots \cap \mathbb{B}_{N+n}) \\
&\quad \times \mathbb{P}(\mathbb{A}_0 \cap \cdots \cap \mathbb{A}_{N+n+m} \cap \mathbb{B}_0 \cap \cdots \cap \mathbb{B}_{N+n+m} \cap \mathbb{C}_1 \cap \cdots \cap \mathbb{C}_m).
\end{aligned} \tag{I.8}$$

Isolate $\mathbb{C}_1$ from the second term on the right–hand side of (I.8):

$$\begin{aligned}
&\mathbb{P}(\mathbb{A}_0 \cap \cdots \cap \mathbb{A}_{N+n+m} \cap \mathbb{B}_0 \cap \cdots \cap \mathbb{B}_{N+n+m} \cap \mathbb{C}_0 \cap \cdots \cap \mathbb{C}_m) \\
&= \mathbb{P}(\mathbb{C}_0 | \mathbb{A}_0 \cap \cdots \cap \mathbb{A}_{N+n} \cap \mathbb{B}_0 \cap \cdots \cap \mathbb{B}_{n-1} \cap \mathbb{B}_{n+1} \cap \cdots \cap \mathbb{B}_{N+n}) \\
&\quad \times \mathbb{P}(\mathbb{C}_1 | \mathbb{A}_0 \cap \cdots \cap \mathbb{A}_{N+n+m} \cap \mathbb{B}_0 \cap \cdots \cap \mathbb{B}_{N+n+m} \cap \mathbb{C}_2 \cap \cdots \cap \mathbb{C}_m) \\
&\quad \times \mathbb{P}(\mathbb{A}_0 \cap \cdots \cap \mathbb{A}_{N+n+m} \cap \mathbb{B}_0 \cap \cdots \cap \mathbb{B}_{N+n+m} \cap \mathbb{C}_2 \cap \cdots \cap \mathbb{C}_m).
\end{aligned} \tag{I.9}$$

Remove the events $\mathbb{A}_0, \mathbb{A}_{N+n+2}, \ldots, \mathbb{A}_{N+n+m}, \mathbb{B}_0, \mathbb{B}_{n+1}, \mathbb{B}_{N+n+2}, \ldots, \mathbb{B}_{N+n+m}$ and $\mathbb{C}_2, \ldots, \mathbb{C}_m$ from the second term on the right–hand side of (I.9), since FLSDFE output $\hat{X}_{t-n-1|t-1}$ only depends on variables $X_{t-1}, \ldots, X_{t-N-n-1}$, and $\hat{X}_{t-1|t-1}, \ldots, \hat{X}_{t-n|t-n}, \hat{X}_{t-n-2|t-n-2}, \ldots, \hat{X}_{t-N-n-1|t-N-n-1}$. We thus simplify (I.9) to

$$\begin{aligned}
&\mathbb{P}(\mathbb{A}_0 \cap \cdots \cap \mathbb{A}_{N+n+m} \cap \mathbb{B}_0 \cap \cdots \cap \mathbb{B}_{N+n+m} \cap \mathbb{C}_0 \cap \cdots \cap \mathbb{C}_m) \\
&= \mathbb{P}(\mathbb{C}_0 | \mathbb{A}_0 \cap \cdots \cap \mathbb{A}_{N+n} \cap \mathbb{B}_0 \cap \cdots \cap \mathbb{B}_{n-1} \cap \mathbb{B}_{n+1} \cap \cdots \cap \mathbb{B}_{N+n}) \\
&\quad \times \mathbb{P}(\mathbb{C}_1 | \mathbb{A}_1 \cap \cdots \cap \mathbb{A}_{N+n+1} \cap \mathbb{B}_1 \cap \cdots \cap \mathbb{B}_n \cap \mathbb{B}_{n+2} \cap \cdots \cap \mathbb{B}_{N+n+1}) \\
&\quad \times \mathbb{P}(\mathbb{A}_0 \cap \cdots \cap \mathbb{A}_{N+n+m} \cap \mathbb{B}_0 \cap \cdots \cap \mathbb{B}_{N+n+m} \cap \mathbb{C}_2 \cap \cdots \cap \mathbb{C}_m).
\end{aligned} \tag{I.10}$$

Isolate $\mathbb{B}_0$ from the third term on the right–hand side of (I.10):

$$\begin{aligned}
&\mathbb{P}(\mathbb{A}_0 \cap \cdots \cap \mathbb{A}_{N+n+m} \cap \mathbb{B}_0 \cap \cdots \cap \mathbb{B}_{N+n+m} \cap \mathbb{C}_0 \cap \cdots \cap \mathbb{C}_m) \\
&= \mathbb{P}(\mathbb{C}_0 | \mathbb{A}_0 \cap \cdots \cap \mathbb{A}_{N+n} \cap \mathbb{B}_0 \cap \cdots \cap \mathbb{B}_{n-1} \cap \mathbb{B}_{n+1} \cap \cdots \cap \mathbb{B}_{N+n}) \\
&\quad \times \mathbb{P}(\mathbb{C}_1 | \mathbb{A}_1 \cap \cdots \cap \mathbb{A}_{N+n+1} \cap \mathbb{B}_1 \cap \cdots \cap \mathbb{B}_n \cap \mathbb{B}_{n+2} \cap \cdots \cap \mathbb{B}_{N+n+1}) \\
&\quad \times \mathbb{P}(\mathbb{B}_0 | \mathbb{A}_0 \cap \cdots \cap \mathbb{A}_{N+n+m} \cap \mathbb{B}_1 \cap \cdots \cap \mathbb{B}_{N+n+m} \cap \mathbb{C}_2 \cap \cdots \cap \mathbb{C}_m) \\
&\quad \times \mathbb{P}(\mathbb{A}_0 \cap \cdots \cap \mathbb{A}_{N+n+m} \cap \mathbb{B}_1 \cap \cdots \cap \mathbb{B}_{N+n+m} \cap \mathbb{C}_2 \cap \cdots \cap \mathbb{C}_m).
\end{aligned} \tag{I.11}$$

Remove the events $\mathbb{A}_{N+1}, \ldots, \mathbb{A}_{N+n+m}, \mathbb{B}_{N+1}, \ldots, \mathbb{B}_{N+n+m}$ and $\mathbb{C}_2, \ldots, \mathbb{C}_m$ from the third term on the right–hand side of (I.11), as FLSDFE output variable $\hat{X}_{t|t}$ only depends on the variables $X_t, \ldots, X_{t-N}$ and $\hat{X}_{t-1|t-1}, \ldots, \hat{X}_{t-N|t-N}$. We thus simplify

(I.11) to

$$\begin{aligned}
&\mathbb{P}(\mathbb{A}_0 \cap \cdots \cap \mathbb{A}_{N+n+m} \cap \mathbb{B}_0 \cap \cdots \cap \mathbb{B}_{N+n+m} \cap \mathbb{C}_0 \cap \cdots \cap \mathbb{C}_m) \\
&= \mathbb{P}(\mathbb{C}_0 | \mathbb{A}_0 \cap \cdots \cap \mathbb{A}_{N+n} \cap \mathbb{B}_0 \cap \cdots \cap \mathbb{B}_{n-1} \cap \mathbb{B}_{n+1} \cap \cdots \cap \mathbb{B}_{N+n}) \\
&\quad \times \mathbb{P}(\mathbb{C}_1 | \mathbb{A}_1 \cap \cdots \cap \mathbb{A}_{N+n+1} \cap \mathbb{B}_1 \cap \cdots \cap \mathbb{B}_n \cap \mathbb{B}_{n+2} \cap \cdots \cap \mathbb{B}_{N+n+1}) \\
&\quad \times \mathbb{P}(\mathbb{B}_0 | \mathbb{A}_0 \cap \cdots \cap \mathbb{A}_N \cap \mathbb{B}_1 \cap \cdots \cap \mathbb{B}_N) \\
&\quad \times \mathbb{P}(\mathbb{A}_0 \cap \cdots \cap \mathbb{A}_{N+n+m} \cap \mathbb{B}_1 \cap \cdots \cap \mathbb{B}_{N+n+m} \cap \mathbb{C}_2 \cap \cdots \cap \mathbb{C}_m).
\end{aligned} \tag{I.12}$$

Observe that $\mathbb{A}_0$ is independent of the compound event

$$\mathbb{A}_1 \cap \cdots \cap \mathbb{A}_{N+n+m} \cap \mathbb{B}_1 \cap \cdots \cap \mathbb{B}_{N+n+m} \cap \mathbb{C}_2 \cap \cdots \cap \mathbb{C}_m, \tag{I.13}$$

as the current symbol X_t is independent of the previous symbols $X_{t-1}, \ldots, X_{t-N-n-m}$ and the FLSDFE outputs $\hat{X}_{t-1|t-1}, \ldots, \hat{X}_{t-N-n-m|t-N-n-m}$ and $\hat{X}_{t-n-2|t-2}, \ldots, \hat{X}_{t-n-m|t-m}$. Thus we can simplify (I.12) further to

$$\begin{aligned}
&\mathbb{P}(\mathbb{A}_0 \cap \cdots \cap \mathbb{A}_{N+n+m} \cap \mathbb{B}_0 \cap \cdots \cap \mathbb{B}_{N+n+m} \cap \mathbb{C}_0 \cap \cdots \cap \mathbb{C}_m) \\
&= \mathbb{P}(\mathbb{C}_0 | \mathbb{A}_0 \cap \cdots \cap \mathbb{A}_{N+n} \cap \mathbb{B}_0 \cap \cdots \cap \mathbb{B}_{n-1} \cap \mathbb{B}_{n+1} \cap \cdots \cap \mathbb{B}_{N+n}) \\
&\quad \times \mathbb{P}(\mathbb{C}_1 | \mathbb{A}_1 \cap \cdots \cap \mathbb{A}_{N+n+1} \cap \mathbb{B}_1 \cap \cdots \cap \mathbb{B}_n \cap \mathbb{B}_{n+2} \cap \cdots \cap \mathbb{B}_{N+n+1}) \\
&\quad \times \mathbb{P}(\mathbb{B}_0 | \mathbb{A}_0 \cap \cdots \cap \mathbb{A}_N \cap \mathbb{B}_1 \cap \cdots \cap \mathbb{B}_N) \times \mathbb{P}(\mathbb{A}_0) \\
&\quad \times \mathbb{P}(\mathbb{A}_1 \cap \cdots \cap \mathbb{A}_{N+n+m} \cap \mathbb{B}_1 \cap \cdots \cap \mathbb{B}_{N+n+m} \cap \mathbb{C}_2 \cap \cdots \cap \mathbb{C}_m).
\end{aligned} \tag{I.14}$$

Now apply (I.5) to the denominator of (I.6), and isolate $\mathbb{C}_1$:

$$\begin{aligned}
&\mathbb{P}(\mathbb{A}_1 \cap \cdots \cap \mathbb{A}_{N+n+m} \cap \mathbb{B}_1 \cap \cdots \cap \mathbb{B}_{N+n+m} \cap \mathbb{C}_1 \cap \cdots \cap \mathbb{C}_m) \\
&= \mathbb{P}(\mathbb{C}_1 | \mathbb{A}_1 \cap \cdots \cap \mathbb{A}_{N+n+m} \cap \mathbb{B}_1 \cap \cdots \cap \mathbb{B}_{N+n+m} \cap \mathbb{C}_2 \cap \cdots \cap \mathbb{C}_m) \\
&\quad \times \mathbb{P}(\mathbb{A}_1 \cap \cdots \cap \mathbb{A}_{N+n+m} \cap \mathbb{B}_1 \cap \cdots \cap \mathbb{B}_{N+n+m} \cap \mathbb{C}_2 \cap \cdots \cap \mathbb{C}_m).
\end{aligned} \tag{I.15}$$

Remove $\mathbb{A}_{N+n+2}, \ldots, \mathbb{A}_{N+n+m}$, $\mathbb{B}_{n+1}$, $\mathbb{B}_{N+n+2}, \ldots, \mathbb{B}_{N+n+m}$ and $\mathbb{C}_2, \ldots, \mathbb{C}_m$ from the first term on the right–hand side of (I.15), since FLSDFE output $\hat{X}_{t-n-1|t-1}$ only depends on variables $X_{t-1}, \ldots, X_{t-N-n-1}$, and $\hat{X}_{t-1|t-1}, \ldots, \hat{X}_{t-n|t-n}$, $\hat{X}_{t-n-2|t-n-2}, \ldots,$ $\hat{X}_{t-N-n-1|t-N-n-1}$. We thus simplify (I.15) to

$$\begin{aligned}
&\mathbb{P}(\mathbb{A}_1 \cap \cdots \cap \mathbb{A}_{N+n+m} \cap \mathbb{B}_1 \cap \cdots \cap \mathbb{B}_{N+n+m} \cap \mathbb{C}_1 \cap \cdots \cap \mathbb{C}_m) \\
&= \mathbb{P}(\mathbb{C}_1 | \mathbb{A}_1 \cap \cdots \cap \mathbb{A}_{N+n+1} \cap \mathbb{B}_1 \cap \cdots \cap \mathbb{B}_n \cap \mathbb{B}_{n+2} \cap \cdots \cap \mathbb{B}_{N+n+1}) \\
&\quad \times \mathbb{P}(\mathbb{A}_1 \cap \cdots \cap \mathbb{A}_{N+n+m} \cap \mathbb{B}_1 \cap \cdots \cap \mathbb{B}_{N+n+m} \cap \mathbb{C}_2 \cap \cdots \cap \mathbb{C}_m).
\end{aligned} \tag{I.16}$$

Finally, observe that the right–hand side of (I.16) is a factor of the right–hand side of (I.14). From (I.6), (I.14) and (I.16), we thus obtain the result

$$\begin{aligned}
&\mathbb{P}(\mathbf{S}_{t-n|t} = \mathbf{s}_{t-n|t} | \mathbf{S}_{t-n-1|t-1} = \mathbf{s}_{t-n-1|t-1} \cap \cdots \cap \mathbf{S}_{t-n-m|t-m} = \mathbf{s}_{t-n-m|t-m}) \\
&= \mathbb{P}(\mathbb{A}_0) \times \mathbb{P}(\mathbb{B}_0 | \mathbb{A}_0 \cap \cdots \cap \mathbb{A}_N \cap \mathbb{B}_1 \cap \cdots \cap \mathbb{B}_N) \\
&\quad \times \mathbb{P}(\mathbb{C}_0 | \mathbb{A}_0 \cap \cdots \cap \mathbb{A}_{N+n} \cap \mathbb{B}_0 \cap \cdots \cap \mathbb{B}_{n-1} \cap \mathbb{B}_{n+1} \cap \cdots \cap \mathbb{B}_{N+n}) \\
&= \mathbb{P}(X_t = x_t) \mathbb{P}(\hat{X}_{t|t} = \hat{x}_{t|t} | X_t = x_t \cap \cdots \cap X_{t-N} = x_{t-N} \\
&\quad \cap \hat{X}_{t-1|t-1} = \hat{x}_{t-1|t-1} \cap \cdots \cap \hat{X}_{t-N|t-N} = \hat{x}_{t-N|t-N}) \\
&\quad \times \mathbb{P}(\hat{X}_{t-n|t} = \hat{x}_{t-n|t} | X_t = x_t \cap \cdots \cap X_{t-N-n} = x_{t-N-n} \cap \hat{X}_{t|t} = \hat{x}_{t|t} \\
&\quad \cap \cdots \cap \hat{X}_{t-n+1|t-n+1} = \hat{x}_{t-n+1|t-n+1} \cap \hat{X}_{t-n-1|t-n-1} = \hat{x}_{t-n-1|t-n-1} \\
&\quad \cap \cdots \cap \hat{X}_{t-N-n|t-N-n} = \hat{x}_{t-N-n|t-N-n}).
\end{aligned} \tag{I.17}$$

Note that the right–hand side of (I.17) is independent of $m \in \mathbb{Z}^+$, and thus (I.17) holds for $m = 1$ as well as for arbitrary $m \in \mathbb{Z}^+$. Hence (I.3) is true for all $m \in \mathbb{Z}^+$ and $\mathbf{S}_{t-n|t}$ of (I.1) is a first–order Markov process [50].

Setting $m = 1$ in (I.17) we thus have the state transition probability

$$\begin{aligned}
&\mathbb{P}(\mathbf{S}_{t-n|t} = \mathbf{s}_{t-n|t} | \mathbf{S}_{t-n-1|t-1} = \mathbf{s}_{t-n-1|t-1}) \\
&= \mathbb{P}(X_t = x_t) \mathbb{P}(\hat{X}_{t|t} = \hat{x}_{t|t} | X_t = x_t \cap \cdots \cap X_{t-N} = x_{t-N} \\
&\quad \cap \hat{X}_{t-1|t-1} = \hat{x}_{t-1|t-1} \cap \cdots \cap \hat{X}_{t-N|t-N} = \hat{x}_{t-N|t-N}) \\
&\quad \times \mathbb{P}(\hat{X}_{t-n|t} = \hat{x}_{t-n|t} | X_t = x_t \cap \cdots \cap X_{t-N-n} = x_{t-N-n} \cap \hat{X}_{t|t} = \hat{x}_{t|t} \\
&\quad \cap \cdots \cap \hat{X}_{t-n+1|t-n+1} = \hat{x}_{t-n+1|t-n+1} \cap \hat{X}_{t-n-1|t-n-1} = \hat{x}_{t-n-1|t-n-1} \\
&\quad \cap \cdots \cap \hat{X}_{t-N-n|t-N-n} = \hat{x}_{t-N-n|t-N-n}).
\end{aligned} \tag{I.18}$$

The first term in the right–hand side of (I.18) is the *a priori* probability of symbol value $x_t \in \mathcal{A}_M$. $\mathbb{P}(X_t = x_t)$ is the relative frequency at which the baseband symbol x_t is transmitted into the digital transmission or data storage channel.

The second term gives the conditional probability of the filtered estimate $\hat{x}_{t|t} \in \mathcal{A}_M$, where we have fixed values $x_t, \ldots, x_{t-N}, \hat{x}_{t-1|t-1}, \ldots, \hat{x}_{t-N|t-N}$ for the variables $X_t, \ldots, X_{t-N}, \hat{X}_{t-1|t-1}, \ldots, \hat{X}_{t-N|t-N}$ of which $\hat{X}_{t|t}$ is a function. This functional dependence is illustrated in section 2.3 for the case of MPSK and MQAM signalling, and 2.4 for BPSK signalling. Note that the second term in the right-hand side of (I.18) is a probability distribution function, rather than a deterministic expression, since $\hat{X}_{t|t}$ also depends on the additive noise variate V_t, as shown in the models of chapter 2.

The third term in the right–hand side of (I.18) gives the conditional probability of the smoothed estimate $\hat{x}_{t-n|t} \in \mathcal{A}_M$, where we have fixed values $x_t, \ldots, x_{t-N-n}$, $\hat{x}_{t|t}, \ldots, \hat{x}_{t-n+1|t-n+1}, \hat{x}_{t-n-1|t-n-1}, \ldots, \hat{x}_{t-N-n|t-N-n}$ of the variates $X_t, \ldots, X_{t-N-n}$, $\hat{X}_{t|t}, \ldots, \hat{X}_{t-n+1|t-n+1}, \hat{X}_{t-n-1|t-n-1}, \ldots, \hat{X}_{t-N-n|t-N-n}$ of which $\hat{X}_{t-n|t}$ is a function. As with the second term, the third term is a probability distribution function, rather than a deterministic expression, as $\hat{X}_{t-n|t}$ also depends on the additive noise variates $V_t, \ldots, V_{t-n}$.

Appendix J

Program HANKEL

A computer program, `HANKEL`, was constructed to compute the complex acoustic pressure field within a horizontally–stratified ocean. `HANKEL` was intended to provide results that were exact to within the limits of numerical quadrature, based upon equation (6.31) of Frisk's treatise on ocean acoustics [32]. The principal use of `HANKEL` in the present work is to provide benchmark solutions for demonstrating the effectiveness of the simpler ray tracing models used in chapter 4. It is shown that ray tracing provides an adequate physical model for use in many underwater acoustic communication channel simulations.

There is an extensive literature on computational ocean acoustics, and we direct the reader to the work of Frisk [32] and Jensen *et al.* [36] as entry points. For our purposes in the present work, we provide below only the theory relevant to the pressure field formulae that are encoded in `HANKEL`. These formulae were derived from equation (6.31) of chapter 6 of Frisk's work [32].

Suppose we have a horizontal acoustic waveguide, with a stationary, uniform body of fluid (the ocean) overlying a horizontally stratified basement (the bottom). By *horizontally stratified* we mean that the only variation in material property is with depth z. We consider the domain $\{(x, y, z) \in \mathbb{R}^3\}$, but restrict our attention to the ocean, defined by $\{(x, y, z) \in \mathbb{R}^3 : 0 < z < h\}$, where $h > 0$ is the depth of the ocean. The depth co-ordinate z is measured from the surface of the ocean, and is defined to be positive downwards. We model the air only through the surface reflection coefficient R_S. The *surface* of the ocean is the plane $\{(x, y, z) \in \mathbb{R}^3 : z = 0\}$, while the ocean *bottom* is the plane $\{(x, y, z) \in \mathbb{R}^3 : z = h\}$. We assume the air, the acoustic medium occupying $\{(x, y, z) \in \mathbb{R}^3 : z < 0\}$, to be a stationary, homogeneous fluid, serving only as a reflector of acoustic energy back into the ocean. We are interested only in the complex acoustic pressure within the ocean.

Let there be a point monopole acoustic source S within the ocean at the cylindrical (r, z) co-ordinates $(0, z_S)$. Assume that S emits a continuous complex unit-amplitude monochromatic pressure wave $e^{-i\omega t}$, of angular frequency $\omega = 2\pi f$ radians per second. We say that S is a CW (continuous wave) source, otherwise known as an NB (narrowband) source.

Set the density of water to be $\rho_1 > 0\,\mathrm{kg\,m^{-3}}$ and the speed of sound to be $c_1 > 0\,\mathrm{m\,s^{-1}}$, applying uniformly over $\{(x, y, z) \in \mathbb{R}^3 : 0 < z < h\}$. We allow the water to have a material attenuation $\alpha_1 \geq 0$ decibels per wavelength (dB/λ).

Propagation of acoustic energy in the water is described in terms of the wavenumber $k = 2\pi f / c_1$, which is composed of a horizontal component k_r and a vertical component k_z. Note that the horizontal component is strictly real, with $0 \leq k_r < \infty$. The vertical component is either purely real, given by $k_z = \sqrt{k^2 - k_r^2}$ for $k_r \leq k$; or purely imaginary, given by $k_z = i\sqrt{k_r^2 - k^2}$ for $k_r > k$.

The complex acoustic pressure $p \equiv p(r; z_R, z_S)$ at a receiver R, with cylindrical coordinates (r, z_R), where $0 < z_R < h$, is given by [32, eq. (6.31)]

$$
\begin{aligned}
p = & \sum_{n=0}^{\infty} \int_0^{\infty} \frac{i}{k_z} e^{ik_z|z_R - z_S|} \left(R_S R_B\right)^n e^{2ink_z h} J_0(k_r r) k_r dk_r \\
& + \sum_{n=0}^{\infty} \int_0^{\infty} \frac{i}{k_z} R_B e^{ik_z[2h - (z_R + z_S)]} \left(R_S R_B\right)^n e^{2ink_z h} J_0(k_r r) k_r dk_r \\
& + \sum_{n=0}^{\infty} \int_0^{\infty} \frac{i}{k_z} R_S e^{ik_z(z_R + z_S)} \left(R_S R_B\right)^n e^{2ink_z h} J_0(k_r r) k_r dk_r \\
& + \sum_{n=0}^{\infty} \int_0^{\infty} \frac{i}{k_z} R_B R_S e^{ik_z[2h - |z_R - z_S|]} \left(R_S R_B\right)^n e^{2ink_z h} J_0(k_r r) k_r dk_r,
\end{aligned}
\qquad \text{(J.1)}
$$

where $R_S \equiv R_S(k_r)$ and $R_B \equiv R_B(k_r)$ are the complex surface and bottom reflection coefficients, respectively, and $J_0(.)$ is the Bessel function of the first kind of order 0 [62]. Note that (J.1) is strictly valid only for $|R_B R_S| < 1$, which is the radius of convergence of each of the four infinite series terms.

Define a complex propagation angle $\theta = \Re(\theta) + i\Im(\theta)$, with $k_r = k\sin\theta$ and $k_z = k\cos\theta$. Transform (J.1) into

$$
\begin{aligned}
p = & \sum_{n=0}^{\infty} \int_0^{\frac{\pi}{2} - i\infty} \frac{i}{k\cos\theta} e^{i(k\cos\theta)|z_R - z_S|} \left(R_S R_B\right)^n e^{2in(k\cos\theta)h} \\
& \times J_0(kr\sin\theta)(k\sin\theta)(k\cos\theta) d\theta \\
& + \sum_{n=0}^{\infty} \int_0^{\frac{\pi}{2} - i\infty} \frac{i}{k\cos\theta} R_B e^{i(k\cos\theta)[2h - (z_R + z_S)]} \left(R_S R_B\right)^n e^{2in(k\cos\theta)h} \\
& \times J_0(kr\sin\theta)(k\sin\theta)(k\cos\theta) d\theta
\end{aligned}
$$

$$+\sum_{n=0}^{\infty}\int_0^{\frac{\pi}{2}-i\infty}\frac{i}{k\cos\theta}R_S e^{i(k\cos\theta)(z_R+z_S)}\left(R_S R_B\right)^n e^{2in(k\cos\theta)h}$$
$$\times J_0(kr\sin\theta)(k\sin\theta)(k\cos\theta)d\theta$$
$$+\sum_{n=0}^{\infty}\int_0^{\frac{\pi}{2}-i\infty}\frac{i}{k\cos\theta}R_B R_S e^{i(k\cos\theta)[2h-|z_R-z_S|]}\left(R_S R_B\right)^n e^{2in(k\cos\theta)h}$$
$$\times J_0(kr\sin\theta)(k\sin\theta)(k\cos\theta)d\theta, \qquad \text{(J.2)}$$

where now $R_S \equiv R_S(\theta)$ and $R_B \equiv R_B(\theta)$.

The contour integrals throughout (J.2) are to be interpreted as over $C \equiv C_1 \cup C_2$, where $C_1 \equiv \{\theta \in \mathbb{C} : 0 \leq \Re(\theta) \leq \frac{\pi}{2}, \Im(\theta) = 0\}$ and $C_2 \equiv \{\theta \in \mathbb{C} : \Re(\theta) = \frac{\pi}{2}, -\infty < \Im(\theta) < 0\}$. The first contour, C_1, corresponds to real propagation angles $0 \leq \Re(\theta) \leq \frac{\pi}{2}$, where $\Re(\theta) = 0$ applies to propagation in the vertical direction (up or down), while $\Re(\theta) = \frac{\pi}{2}$ applies to horizontal propagation. The second contour, C_2, describes diffraction, and involves evanescent waves that propagate along the interfaces $z = 0$ and $z = h$ (the planes of discontinuity between adjacent media). We need to include contour C_2 to provide the full solution to the Helmholtz equation.

The acoustic impedance of a medium is defined to be the product of its density and speed of sound. Seawater has an impedance that is several orders of magnitude larger than that of air. To a close approximation, then, we may use the simplification $R_S(\theta) = -1$ for all $\theta \in C$.

Write the complex bottom reflection coefficient $R_B(\theta)$ in polar form,

$$R_B(\theta) = |R_B(\theta)|\, e^{i\Phi_B(\theta)}, \qquad \text{(J.3)}$$

where $|R_B(\theta)|$ is the magnitude and $\Phi_B(\theta)$ is the phase, respectively. With these substitutions, (J.2) becomes

$$p = ik\sum_{n=0}^{\infty}(-1)^n\int_0^{\frac{\pi}{2}-i\infty}|R_B(\theta)|^n e^{i(2nh+|z_R-z_S|)k\cos\theta}e^{in\Phi_B(\theta)}J_0(kr\sin\theta)\sin\theta d\theta$$
$$+ik\sum_{n=0}^{\infty}(-1)^n\int_0^{\frac{\pi}{2}-i\infty}|R_B(\theta)|^{n+1} e^{i(2(n+1)h-(z_R+z_S))k\cos\theta}e^{i(n+1)\Phi_B(\theta)}$$
$$\times J_0(kr\sin\theta)\sin\theta d\theta$$
$$+ik\sum_{n=0}^{\infty}(-1)^{n+1}\int_0^{\frac{\pi}{2}-i\infty}|R_B(\theta)|^n e^{i(2nh+z_R+z_S)k\cos\theta}e^{in\Phi_B(\theta)}J_0(kr\sin\theta)\sin\theta d\theta$$
$$+ik\sum_{n=0}^{\infty}(-1)^{n+1}\int_0^{\frac{\pi}{2}-i\infty}|R_B(\theta)|^{n+1} e^{i(2(n+1)h-|z_R-z_S|)k\cos\theta}e^{i(n+1)\Phi_B(\theta)}$$
$$\times J_0(kr\sin\theta)\sin\theta d\theta. \qquad \text{(J.4)}$$

Note that contour C_2 can be parameterized as $C_2 = \{\pi/2 - ia \in \mathbb{C} : a > 0\}$. We may thus write integrals in θ over $(0, \pi/2 - i\infty)$ as integrals in a over $(0, \infty)$. Using this parameterization of C_2, and making use of the identities

$$e^{i\phi} = \cos\phi + i\sin\phi, \tag{J.5}$$

$$\cos(\pi/2 - ia) = i\sinh a, \tag{J.6}$$

and

$$\sin(\pi/2 - ia) = \cosh a, \tag{J.7}$$

we expand (J.4) into 24 infinite series terms as follows:

$$\begin{aligned}
p = {} & ik\sum_{n=0}^{\infty}(-1)^n\int_0^{\pi/2}|R_B(\theta)|^n\cos((2nh+|z_R-z_S|)k\cos\theta)\cos(n\Phi_B(\theta)) \\
& \times J_0(kr\sin\theta)\sin\theta d\theta \\
& + k\sum_{n=0}^{\infty}(-1)^{n+1}\int_0^{\pi/2}|R_B(\theta)|^n\cos((2nh+|z_R-z_S|)k\cos\theta)\sin(n\Phi_B(\theta)) \\
& \times J_0(kr\sin\theta)\sin\theta d\theta \\
& + k\sum_{n=0}^{\infty}(-1)^{n+1}\int_0^{\pi/2}|R_B(\theta)|^n\sin((2nh+|z_R-z_S|)k\cos\theta)\cos(n\Phi_B(\theta)) \\
& \times J_0(kr\sin\theta)\sin\theta d\theta \\
& + ik\sum_{n=0}^{\infty}(-1)^{n+1}\int_0^{\pi/2}|R_B(\theta)|^n\sin((2nh+|z_R-z_S|)k\cos\theta)\sin(n\Phi_B(\theta)) \\
& \times J_0(kr\sin\theta)\sin\theta d\theta \\
& + ik\sum_{n=0}^{\infty}(-1)^n\int_0^{\infty}|R_B(\pi/2-ia)|^n e^{-(2nh+|z_R-z_S|)k\sinh a}\cos(n\Phi_B(\pi/2-ia)) \\
& \times J_0(kr\cosh a)\cosh a da \\
& + k\sum_{n=0}^{\infty}(-1)^{n+1}\int_0^{\infty}|R_B(\pi/2-ia)|^n e^{-(2nh+|z_R-z_S|)k\sinh a}\sin(n\Phi_B(\pi/2-ia)) \\
& \times J_0(kr\cosh a)\cosh a da \\
& + ik\sum_{n=0}^{\infty}(-1)^n\int_0^{\pi/2}|R_B(\theta)|^{n+1}\cos((2(n+1)h-(z_R+z_S))k\cos\theta) \\
& \times \cos((n+1)\Phi_B(\theta))J_0(kr\sin\theta)\sin\theta d\theta \\
& + k\sum_{n=0}^{\infty}(-1)^{n+1}\int_0^{\pi/2}|R_B(\theta)|^{n+1}\cos((2(n+1)h-(z_R+z_S))k\cos\theta) \\
& \times \sin((n+1)\Phi_B(\theta))J_0(kr\sin\theta)\sin\theta d\theta
\end{aligned}$$

$$+ k \sum_{n=0}^{\infty} (-1)^{n+1} \int_0^{\pi/2} |R_B(\theta)|^{n+1} \sin((2(n+1)h - (z_R + z_S))k \cos\theta)$$
$$\times \cos((n+1)\Phi_B(\theta)) J_0(kr \sin\theta) \sin\theta d\theta$$
$$+ ik \sum_{n=0}^{\infty} (-1)^{n+1} \int_0^{\pi/2} |R_B(\theta)|^{n+1} \sin((2(n+1)h - (z_R + z_S))k \cos\theta)$$
$$\times \sin((n+1)\Phi_B(\theta)) J_0(kr \sin\theta) \sin\theta d\theta$$
$$+ ik \sum_{n=0}^{\infty} (-1)^{n} \int_0^{\infty} |R_B(\pi/2 - ia)|^{n+1} e^{-(2(n+1)h-(z_R+z_S))k \sinh a}$$
$$\times \cos((n+1)\Phi_B(\pi/2 - ia)) J_0(kr \cosh a) \cosh a da$$
$$+ k \sum_{n=0}^{\infty} (-1)^{n+1} \int_0^{\infty} |R_B(\pi/2 - ia)|^{n+1} e^{-(2(n+1)h-(z_R+z_S))k \sinh a}$$
$$\times \sin((n+1)\Phi_B(\pi/2 - ia)) J_0(kr \cosh a) \cosh a da$$
$$+ ik \sum_{n=0}^{\infty} (-1)^{n+1} \int_0^{\pi/2} |R_B(\theta)|^{n} \cos((2nh + z_R + z_S)k \cos\theta) \cos(n\Phi_B(\theta))$$
$$\times J_0(kr \sin\theta) \sin\theta d\theta$$
$$+ k \sum_{n=0}^{\infty} (-1)^{n} \int_0^{\pi/2} |R_B(\theta)|^{n} \cos((2nh + z_R + z_S)k \cos\theta) \sin(n\Phi_B(\theta))$$
$$\times J_0(kr \sin\theta) \sin\theta d\theta$$
$$+ k \sum_{n=0}^{\infty} (-1)^{n} \int_0^{\pi/2} |R_B(\theta)|^{n} \sin((2nh + z_R + z_S)k \cos\theta) \cos(n\Phi_B(\theta))$$
$$\times J_0(kr \sin\theta) \sin\theta d\theta$$
$$+ ik \sum_{n=0}^{\infty} (-1)^{n} \int_0^{\pi/2} |R_B(\theta)|^{n} \sin((2nh + z_R + z_S)k \cos\theta) \sin(n\Phi_B(\theta))$$
$$\times J_0(kr \sin\theta) \sin\theta d\theta$$
$$+ ik \sum_{n=0}^{\infty} (-1)^{n+1} \int_0^{\infty} |R_B(\pi/2 - ia)|^{n} e^{-(2nh+z_R+z_S)k \sinh a} \cos(n\Phi_B(\pi/2 - ia))$$
$$\times J_0(kr \cosh a) \cosh a da$$
$$+ k \sum_{n=0}^{\infty} (-1)^{n} \int_0^{\infty} |R_B(\pi/2 - ia)|^{n} e^{-(2nh+z_R+z_S)k \sinh a} \sin(n\Phi_B(\pi/2 - ia))$$
$$\times J_0(kr \cosh a) \cosh a da$$
$$+ ik \sum_{n=0}^{\infty} (-1)^{n+1} \int_0^{\pi/2} |R_B(\theta)|^{n+1} \cos((2(n+1)h - |z_R - z_S|)k \cos\theta)$$
$$\times \cos((n+1)\Phi_B(\theta)) J_0(kr \sin\theta) \sin\theta d\theta$$

$$
\begin{aligned}
&+ k \sum_{n=0}^{\infty} (-1)^n \int_0^{\pi/2} |R_B(\theta)|^{n+1} \cos((2(n+1)h - |z_R - z_S|)k \cos\theta) \\
&\times \sin((n+1)\Phi_B(\theta)) J_0(kr \sin\theta) \sin\theta d\theta \\
&+ k \sum_{n=0}^{\infty} (-1)^n \int_0^{\pi/2} |R_B(\theta)|^{n+1} \sin((2(n+1)h - |z_R - z_S|)k \cos\theta) \\
&\times \cos((n+1)\Phi_B(\theta)) J_0(kr \sin\theta) \sin\theta d\theta \\
&+ ik \sum_{n=0}^{\infty} (-1)^n \int_0^{\pi/2} |R_B(\theta)|^{n+1} \sin((2(n+1)h - |z_R - z_S|)k \cos\theta) \\
&\times \sin((n+1)\Phi_B(\theta)) J_0(kr \sin\theta) \sin\theta d\theta \\
&+ ik \sum_{n=0}^{\infty} (-1)^{n+1} \int_0^{\infty} |R_B(\pi/2 - ia)|^{n+1} e^{-(2(n+1)h - |z_R - z_S|)k \sinh a} \\
&\times \cos((n+1)\Phi_B(\pi/2 - ia)) J_0(kr \cosh a) \cosh a da \\
&+ k \sum_{n=0}^{\infty} (-1)^n \int_0^{\infty} |R_B(\pi/2 - ia)|^{n+1} e^{-(2(n+1)h - |z_R - z_S|)k \sinh a} \\
&\times \sin((n+1)\Phi_B(\pi/2 - ia)) J_0(kr \cosh a) \cosh a da. \qquad \text{(J.8)}
\end{aligned}
$$

Each of the 24 integrals appearing throughout (J.8) is now a real integral, and may be evaluated by standard quadrature routines. For the integrals over the finite domain $\theta \in (0, \pi/2)$, we isolate all zeros of the integrand and subsequently apply Gauss–Kronrod–Patterson quadrature between each pair of zeros. Rules of order 10, 21, 43 and 87 are used in succession until a convergence criterion is met or the maximum order (87) is reached. For the integrals over the infinite domain $a \in (0, \infty)$, we also use successive zeros, but terminate the quadrature after a convergence criterion has been reached. The infinite–domain integrals decay rapidly due to the negative exponential terms.

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