

# **Aspects of stochastic control and switching: from Parrondo's games to electrical circuits**

by

**Andrew Gordon Allison**

B.Sc.(Mathematical Sciences), The University of Adelaide, 1978.

B.E. (Computer Systems Engineering, Honours), The University of Adelaide, 1995.

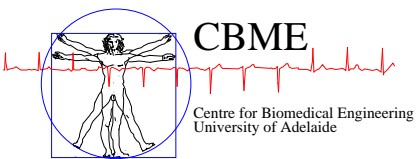
Thesis submitted for the degree of

**Doctor of Philosophy**

in

The School of Electrical and Electronic Engineering, Faculty of  
Engineering, Computer and Mathematical Sciences,  
The University of Adelaide

2009



© 2009  
Andrew Gordon Allison  
All Rights Reserved



# Contents

<b>Contents</b>	<b>iii</b>
<b>Abstract</b>	<b>ix</b>
<b>Statement of Originality</b>	<b>xi</b>
<b>Acknowledgements</b>	<b>xiii</b>
<b>Conventions</b>	<b>xv</b>
<b>Publications</b>	<b>xvii</b>
<b>List of Figures</b>	<b>xxi</b>
<b>List of Tables</b>	<b>xxv</b>
<b>Chapter 1. Introduction and motivation</b>	<b>1</b>
1.1 Introduction . . . . .	2
1.1.1 Brownian motors . . . . .	2
1.2 Motivation . . . . .	3
1.3 Thesis overview . . . . .	3
1.4 Original contributions . . . . .	7
1.5 Chapter summary . . . . .	9
<b>Chapter 2. Background for Parrondo's games</b>	<b>11</b>
2.1 Brownian motion . . . . .	12
2.2 Maxwell's demon . . . . .	14
2.3 The ratchet and pawl machine . . . . .	16
2.4 Flashing ratchets . . . . .	18
2.5 Constructed Brownian ratchets . . . . .	22
2.6 A brief history of finite discrete games of chance . . . . .	23
2.7 Chapter summary . . . . .	26

<b>Chapter 3. The physical basis of Parrondo's games</b>	<b>27</b>
3.1 Discrete transformations of continuous functions . . . . .	29
3.2 Finite difference equations and Parrondo's games . . . . .	38
3.3 Finite partial difference equations . . . . .	40
3.4 Sampling the Fokker-Planck Equation . . . . .	44
3.5 Parrondo's games, as a set of PDEs . . . . .	47
3.5.1 Game A, as a partial difference equation . . . . .	48
3.5.2 Game B, as a partial difference equation . . . . .	50
3.5.3 Conditions for convergence of the solution . . . . .	50
3.5.4 An appropriate choice of scale . . . . .	51
3.5.5 Mean position and mean velocity of drift . . . . .	53
3.5.6 An example of a simulation, including null-transitions . . . . .	54
3.5.7 A more realistic simulation . . . . .	55
3.6 Summary of results, regarding the sampling process . . . . .	58
3.7 Estimating the moments of Parrondo's games . . . . .	58
3.7.1 Evaluation of the discrete transforms of the solution function . . . . .	59
3.7.2 Evaluation of first and second moments, of the solution . . . . .	62
3.7.3 The Bernoulli process, a simple worked example . . . . .	64
3.7.4 Stochastic processes with stationary probabilities of transition . . . . .	66
3.7.5 The w-transforms of some well known distributions . . . . .	68
3.7.6 Parrondo's Game A . . . . .	69
3.7.7 Taleb's game, a game with highly asymmetrical rewards . . . . .	70
3.7.8 Difference equations with periodic coefficients . . . . .	72
3.7.9 Parrondo's game B . . . . .	74
3.7.10 The small-matrix representation of Parrondo's games . . . . .	77
3.8 Chapter summary . . . . .	80
<b>Chapter 4. Rates of return from discrete games of chance</b>	<b>81</b>
4.1 Some definitions of terms . . . . .	82
4.1.1 Phase space . . . . .	84
4.1.2 Limiting fixed-points in phase-space . . . . .	85

---

4.1.3	Parrondo's games . . . . .	86
4.1.4	A definition for Parrondo's games . . . . .	87
4.2	The unconstrained or <i>large-matrix</i> formulation . . . . .	88
4.3	The spatially-periodic case, reduced modulo $L$ . . . . .	92
4.4	Asymptotic value of the first moment of the games . . . . .	94
4.4.1	Markov Chains with Rewards . . . . .	94
4.4.2	A matrix notation for the first moment . . . . .	95
4.5	The matrix technique for the first moment . . . . .	96
4.5.1	Parrondo's original games . . . . .	96
4.5.2	The apparent paradox of Parrondo's games . . . . .	100
4.5.3	Parrondo's games, with natural diffusion . . . . .	115
4.5.4	A pair of discrete games with only two states . . . . .	118
4.5.5	Astumian's games . . . . .	121
4.5.6	Astumian's games, with absorbing boundary conditions . . . . .	125
4.5.7	Summary of common features of the discrete games . . . . .	130
4.6	Visualisation of the process . . . . .	132
4.6.1	Time-homogeneous Markov chains and notation . . . . .	132
4.6.2	Time-inhomogeneous Markov chains . . . . .	134
4.6.3	Reduction of the periodic case . . . . .	136
4.7	Long sequences of operators . . . . .	136
4.8	Phase-space visualisation and fractal properties . . . . .	138
4.8.1	Two Markov games that generate simple fractals . . . . .	138
4.8.2	The Cantor middle-third fractal . . . . .	139
4.8.3	Iterated Function Systems (IFS) . . . . .	148
4.8.4	Parrondo's fractal . . . . .	148
4.9	Equivalent representation . . . . .	149
4.9.1	The average probability vector, over time . . . . .	151
4.9.2	The average probability vector, over phase-space . . . . .	153
4.9.3	Consistency of the two averages . . . . .	155
4.10	An optimised form of Parrondo's games . . . . .	156
4.10.1	An interesting fractal object . . . . .	157
4.11	Chapter summary . . . . .	158

<b>Chapter 5. switched-mode circuits and switched Markov systems</b>	<b>161</b>
5.1 Switched-mode circuits and switched Markov systems . . . . .	162
5.1.1 Switched-mode circuits . . . . .	162
5.1.2 Switched state-space and switched Markov systems . . . . .	172
5.1.3 Fractals in the phase-space of switched-mode circuits . . . . .	175
5.1.4 The limiting case of fast switching as $\tau \rightarrow 0$ . . . . .	177
5.2 A Parrondo effect for a switched-mode circuit . . . . .	184
5.2.1 Construction of a simple switched-mode system . . . . .	186
5.2.2 A switched state-space formulation . . . . .	190
5.2.3 Internal stored energy . . . . .	192
5.2.4 Proof of instability of plants $A_1$ and $A_2$ . . . . .	197
5.2.5 Proof of stability of the stochastically mixed processes . . . . .	200
5.3 Sources of noise . . . . .	202
5.4 Chapter summary . . . . .	203
<b>Chapter 6. Langevin equations as models for noise in circuits</b>	<b>205</b>
6.1 Introduction, to noise techniques in electronics . . . . .	206
6.2 Stochastic analysis of circuits . . . . .	209
6.2.1 Outline of stochastic calculus of Itô . . . . .	209
6.2.2 The Fokker Planck equation and the Langevin SDE . . . . .	216
6.3 Modelling of electronic circuits, using SDEs . . . . .	217
6.3.1 Infinitesimal forms of Kirchhoff's laws . . . . .	217
6.3.2 Kirchhoff's current law . . . . .	217
6.3.3 Kirchhoff's voltage law . . . . .	220
6.3.4 Models for resistors . . . . .	223
6.3.5 Modelling of capacitors . . . . .	227
6.3.6 Modelling of inductors . . . . .	227
6.4 A one-dimensional Langevin equation (SDE) . . . . .	228
6.4.1 An approach, based on power spectral density . . . . .	228
6.4.2 The Langevin SDE . . . . .	234
6.5 A two-dimensional Langevin equation (SDE) . . . . .	236

6.5.1	Nyquist’s approach, based on power spectral density . . . . .	236
6.5.2	An approach, using the the Langevin stochastic differential equation	243
6.6	Noise models for the JFET . . . . .	244
6.7	A simple JFET circuit . . . . .	245
6.8	Analysis of the JFET circuit . . . . .	246
6.9	Summary and open questions . . . . .	247
 <b>Chapter 7. Conclusions and future challenges</b>		 <b>251</b>
7.1	Original contribution . . . . .	252
7.2	Future prospects . . . . .	253
7.2.1	The physical basis of Parrondo’s games . . . . .	253
7.2.2	Rates of return from discrete games of chance . . . . .	260
7.2.3	Switched-mode circuits and switched Markov systems . . . . .	261
7.2.4	Langevin equations to model noise in electronic circuits . . . . .	263
 <b>Résumé</b>		 <b>267</b>
 <b>Methods of work</b>		 <b>269</b>
 <b>Epilogue</b>		 <b>273</b>
 <b>Bibliography</b>		 <b>275</b>
 <b>Glossary</b>		 <b>289</b>
 <b>Index</b>		 <b>291</b>





# Abstract

The first half of this thesis deals with the line of thought that leads to the development of discrete games of chance as models in statistical physics, with an emphasis on analysis of Parrondo's games.

The second half of the thesis is concerned with applying discrete games of chance to the modelling of other phenomena in the discipline of electrical engineering. The important features being the element of *switching* that is implicit in discrete games of chance and the element of *uncertainty*, introduced by the random aspect of discrete games of chance.



# Statement of Originality

This work contains no material that has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published written by another person, except where due reference has been made in the text.

I give consent to this copy of the thesis, when deposited in the University Library, being available for loan, photocopying and dissemination through the library digital thesis collection, subject to the provisions of the Copyright act of 1968. Copying or publication or use of this thesis or part thereof for financial gain is not allowed without the author's written permission. Due recognition shall be given to the author, and the University of Adelaide, in any scholarly use that may be made of any material in this thesis.

---

*10th Sep. 2009*



# Acknowledgements

Whenever one writes about a scientific subject, one stands on the shoulders of giants. Some of these people are famous, and appear in the literature and are cited. Some are less famous, for their contributions to the work, and deserve special recognition here.

The period of time in which this thesis was completed was very stressful for me because both of my parents died. They deserve personal as well as professional acknowledgement. My mother, Gwen Anderson, encouraged my interest in chemistry and the natural world. My father, Robert Allison, encouraged my interest in electronics and statistics. He taught me to play cards and to take only reasonable risks, in cards or in life. If in doubt, he always encouraged me to “play it straight.”

I thank my supervisors, Professors Derek Abbott and Charles E. M. Pearce. I owe an enormous debt to Professor Derek Abbott. He has always encouraged me to think outside the square. He made every effort to bring me into contact with experts in the many fields in which he works. He has encouraged me to persevere in the face of difficulties. He had faith in my ideas when I had doubts. He brought me back to reality when my feet left the ground. He helped me to improve the precision of my prose.

I owe an equal debt to Professor Charles Pearce, who also had had faith in me and encouraged me to follow up my intuitive ideas until they were properly formulated. Charles also prevented me from committing a number of grievous mathematical errors. If any errors remain, they are certainly mine.

Thanks to Derek’s efforts, I have been able to have informal discussions with many scientists, whom I admire greatly. These include, Prof. Juan Parrondo (Universidad Computense), Prof. Dean Astumian (University of Maine), Prof. Eugene Stanley (Director, Center for Polymer Studies), Prof. Peter Hänggi (Universität Augsburg), Prof. Raúl Toral (Instituto de Física Interdisciplinaria Sistemas Complejos), Assoc. Prof. Martin Bier (East Carolina University), Prof. Charles Doering (University of Michigan), Prof. Erhard Behrends (Freie Universität Berlin) and Prof. Michael Barnsley (Australian

## Acknowledgements

---

National University). I am honoured to have met these people and I feel very humble when I claim to have extended their ideas by even a small amount.

I particularly appreciate the efforts of the long suffering editors of my open publications, who also helped me to make my ideas more logical and coherent, and fit within the page limits. I particularly appreciate the efforts of Prof. Laszlo Kish (Texas A&M University) Dr. Sergey Bezrukov (National Institutes of Health, NIH) and Prof. Lutz Schimansky-Geier (Institut für Physik Humboldt, Universität zu Berlin).

I also appreciate the contributions of many of the staff and students at the School of Electrical and Electronic Engineering at the University of Adelaide, especially my very patient Heads of School, Dr Tony Parker, Dr Ken Sarkies and Assoc. Prof. Mike Liebelt. I thank Greg Harmer for the use of his fabulous Smoluchowski-ratchet figure, Matthew Berryman, Frederic Bonnet, Tze Wei Tang, Chris Illert and Pau Amengual for being very astute and reliable co-authors. I particularly thank Matthew for the use of his PERL code. I thank Mark McDonnell for the use of his laptop, for sharing a hotel room (at a conference) and for many interesting and discursive conversations about noise, correlation, and information. I thank Withawat Withayachumanankul for his tree diagram of our scientific genealogy.

I am especially grateful to Associate Professor John van der Hoek (University of South Australia) for his advice regarding stochastic differential equations. I am grateful to Chris Illert for many fruitful discussions regarding the calculus of variations and the application of variational techniques to problems in electronics. Finally, I thank my partner Mei Sheong Wong, and family, for putting up with my absence (at a computer) for very long periods of time and for help with proofs of some of the less technical material.

*“Writing a book is a horrible, exhausting struggle, like a long bout of some painful illness. One would never undertake such a thing if one were not driven on by some demon whom one can neither resist nor understand.”* George Orwell

# Conventions

This thesis is typeset using L<sup>A</sup>T<sub>E</sub>X2e software, including the core packages tetex-base, tetex-bin and tetex-extra. All L<sup>A</sup>T<sub>E</sub>X software was obtained from the Debian Archive at: <http://www.debian.org>.

Numerical calculations were carried out in Matlab, and in the equivalent open-source packages GNU Octave and Gnuplot, which were also obtained from the Debian archive.

Many of the Figures were generated in Matlab and in GNU Octave. The other figures were drawn, or post-processed, in a number of other drawing packages, including Corel-Draw 9 under Windows 2000, Adobe Creative Suite 3 under Mac OS-X, or in xfig and Inkscape, under LINUX. The last two packages were downloaded from the Debian archive. All drawings have been converted or exported to encapsulated post-script (eps) format.

The complete editing environment, Emacs21 (Editing with MACroS, version 21.4.1) was used as an effective interface to L<sup>A</sup>T<sub>E</sub>X. The idiomatic conventions, for L<sup>A</sup>T<sub>E</sub>X, conform to standard described in (Lamport 1994).

Harvard style is used for referencing and citation in this thesis. British spelling is adopted, consistent with the Ispell package, using the British dictionary, in Emacs21. Additional words have been traced back to their original sources. Where we have needed to quote works in other languages, including works in US English, we have used the original spelling.





# Publications

## Book Chapters

ALLISON-A., ABBOTT-D. & AND PEARCE-C. E. M. (2005) State-space visualisation and fractal properties of Parrondo's games, in A. S. Nowak., and K. Szajowski. (eds.), *Proceedings of the Ninth International Symposium on Dynamic Games and Applications 2000, Advances in Dynamic Games: Applications to Economics, Finance, Optimization, and Stochastic Control.*, Vol. 7, The International Society of Dynamic Games (ISDG), Birkhauser, pp. 613–633.

## Journal Articles

BERRYMAN-M. J., ALLISON-A., WILKINSON-C. R. & ABBOTT-D. (2005) Review of signal processing in genetics, *Fluctuation and Noise Letters*, **5**(4), pp. R13–R35.

AMENGUAL-P., ALLISON-A., TORAL-R. & ABBOTT-D. (2004) Discrete-time ratchets, the Fokker-Planck equation and Parrondo's paradox, *Proceedings of the Royal Society of London*, **460**(2048), pp. 2269–2284.

BERRYMAN-M. J., ALLISON-A., & ABBOTT-D. (2004) Mutual information for examining correlations in DNA, *Fluctuation and Noise Letters*, **4**, pp. L237–L246.

ILLERT-C. & ALLISON-A. (2004) Phono-genesis and the origin of accusative syntax in proto-Australian language, *Journal of Applied Statistics*, **31**(1), pp. 73–104.

TANG-T. W., ALLISON-A. & AND ABBOTT-D. (2004) Investigation of chaotic switching strategies in Parrondo's games, *Fluctuation and Noise Letters*, **4**, pp. L585–L596.

BERRYMAN-M. J., ALLISON-A. & ABBOTT-D. (2003) Statistical techniques for text classification based on word recurrence intervals, *Fluctuation and Noise Letters*, **3**(1), pp. L1–L10.

LEE-Y., ALLISON-A. & ABBOTT-D. (2003) Minimal Brownian ratchet: An exactly solvable model, *Phys. Rev. Lett.*, **91**(22), Art. No. 220601.

ALLISON-A. & ABBOTT-D. (2002) A MEMS Brownian ratchet, *Microelectronics Journal*, **33**(3), pp. 235–243.

## Publications

---

ALLISON-A. & ABBOTT-D. (2002) The physical basis for Parrondo's games, *Fluctuation and Noise Letters*, **2**(4), pp. L327–L341.

ALLISON-A. & ABBOTT-D. (2001) Control systems with stochastic feedback, *Chaos Journal*, **11**(3), pp. 715–724.

ALLISON-A. & ABBOTT-D. (2001) Stochastic resonance in a Brownian ratchet, *Fluctuation and Noise Letters*, **1**(4), pp. L239–L244.

ALLISON-A. & ABBOTT-D. (2000) Some benefits of random variables in switched control systems, *Microelectronics Journal*, **31**, pp. 515–522.

## Conference Articles

PEARCE-C. E. M., ALLISON-A. & ABBOTT-D. (2007) Perturbing singular systems and the correlating of uncorrelated random sequences, in T. E. Simos, G. Psihoyios and C. Tsitouras(eds.), *Proc. AIP, International Conference of Numerical Analysis and Applied Mathematics*, **936** (1), pp. 699–699.

ALLISON-A., ABBOTT-D. & PEARCE-C. E. M. (2007) , Finding keywords amongst noise: Automatic text classification without parsing, in J. Kertész., S. Bornholdt., and R. Mantegna. (eds.), *Proc. SPIE, Noise and Stochastics in Complex Systems and Finance*, Vol. 6601, Art. No. 660113.

BONNET-F. D. R., ALLISON-A. & ABBOTT-D. (2006) Bubbles in a minority game setting with real financial data, in A. Bender. (ed.), *Proc. SPIE, Complex Systems*, Vol. 6039, pp. 99–103.

ALLISON-A. & ABBOTT-D. (2005) Applications of Stochastic Differential Equations in electronics, in L. Reggiani and C. Penneta and V. Akimov and E. Alfinito and M. Rosini. (eds.), *Proc. AIP, Unsolved Problems of Noise and Fluctuations in Physics, Biology and High Technology*, Vol. 800, pp. 15–23.

BERRYMAN-M. J., ALLISON-A. & ABBOTT-D. (2005) Gene network analysis and design, in D. V. Nicolau. (ed.), *Proc. SPIE, Biomedical Applications of Micro- and Nanoengineering II*, Vol. 5651, pp. 126–133.

BERRYMAN-M. J., COUSSENS-S. W., PAMULA-Y., KENNEDY-D., LUSHINGTON-K., SHALIZI-C., ALLISON-A., MARTIN-J., SAINT-D. & ABBOTT-D. (2005) Nonlinear aspects of the EEG during sleep in children, *Proc. SPIE, Fluctuations and Noise in Biological, Biophysical, and Biomedical Systems*, Vol. 5841, Austin, Texas, pp. 40–48.

- 
- BERRYMAN-M. J., ALLISON-A. & ABBOTT-D. (2004) Optimizing genetic algorithm strategies for evolving networks, *Proc. SPIE, Fluctuations and Noise in Biological, Biophysical and Biomedical Systems*, Vol. 5473, pp. 122–130.
- BERRYMAN-M. J., ALLISON-A. & ABBOTT-D. (2004) Stochastic evolution and multifractal classification of prokaryotes, in S. M. Bezrukov., H. Frauenfelder., and F. Moss. (eds.), *Proc. SPIE, Fluctuations and Noise in Biological, Biophysical and Biomedical Systems*, Vol. 5110, pp. 192–200.
- BERRYMAN-M. J., SPENCER-S. L., ALLISON-A. & ABBOTT-D. (2004) Fluctuations and noise in cancer development, in Z. Gingl. (ed.), *Proc. SPIE, Noise in Interdisciplinary Applications II, Fluctuations and Noise 2004*, Vol. 5471, pp. 322–332.
- BERRYMAN-M. J., ALLISON-A. & ABBOTT-D. (2004) Cellular automata for exploring gene regulation in drosophila segmentation, *Proc. SPIE, BioMEMS and Nanotechnology 2003*, pp. 266–277.
- BERRYMAN-M. J., KHOO-W. L., NGUYEN-H., O'NEILL-E., ALLISON-A. & ABBOTT-D. (2004) Exploring tradeoffs in pleiotrophy and redundancy using evolutionary computing, in D. V. Nicolau. (ed.), *Proc. SPIE, BioMEMS and Nanotechnology 2003*, Vol. 5275, pp. 49–58.
- BONNET-F. D. R., ALLISON-A. & ABBOTT-D., (2004) Path integrals in fluctuating markets, in Z. Gingl. (ed.), *Proc. SPIE, Noise in Complex Systems and Stochastic Dynamics II*, Vol. 5471, pp. 595–611.
- BONNET-F. D. R., ALLISON-A. & ABBOTT-D (2004) Review of quantum path integrals in fluctuating markets, in D. Abbott., and K. Eshraghian. (eds.), *Proc. SPIE, Microelectronics: Design, Technology, and Packaging*, Vol. 5274, pp. 569–580.
- TANG-T. W., ALLISON-A. & ABBOTT-D. (2004) Parrondo's games with chaotic switching, *Proc. SPIE, Fluctuations and Noise in Biological, Biophysical and Biomedical Systems*, Vol. 5471, pp. 520–530.
- ALLISON-A. & ABBOTT-D. (2003) Brownian ratchets with distributed charge, in S. M. Bezrukov., H. Frauenfelder., and F. Moss. (eds.), *Proc. SPIE, Fluctuations and Noise in Biological, Biophysical, and Biomedical Systems*, Vol. 5110, pp. 302–311.
- ALLISON-A. & ABBOTT-D. (2003) Discrete games of chance as models for continuous stochastic transport processes, in L. Schimansky-Geier., D. Abbott., A. Neiman., and C. V. den Broeck. (eds.), *Proc. SPIE, Noise in Complex Systems and Stochastic Dynamics*, Vol. 5114, pp. 363–371.
- BERRYMAN-M.J, ALLISON-A., CARPEÑA-P & ABBOTT-D. (2002) Signal processing and statistical methods in analysis of text and DNA, D. V. Nicolau (ed.), *Proc. SPIE, Biomedical Applications of Micro-*
-

and *Nanoengineering*, Vol. 4937, pp. 231–240.

**ALLISON-A.** & **ABBOTT-D.** (2002) Stochastic resonance, Brownian ratchets and the Fokker-Planck equation, in S. M. Bezrukov. (ed.), *Proc. SPIE, Unsolved Problems of Noise and Fluctuations: UPoN 2002*, Vol. 665, pp. 74–83.

**ALLISON-A.** & **ABBOTT-D.** (2001) MEMS implementation of a Brownian ratchet, in D. Abbott., V. K. Varadan., and K. F. Boehringer. (eds.), *Proc. SPIE, Smart Electronics and MEMS II*, Vol. 4236, pp. 319–329.

**HOO-T. L.**, **TING-A. S. C.**, **O'NEILL-E.**, **ALLISON-A.** & **ABBOTT-D.** (2001) Real life: A cellular automation for investigating competition between pleiotrophy and redundancy, *Proc. SPIE, Electronics and Structures for MEMS II*, Vol. 4591, pp. 380–390.

**ALLISON-A.** & **ABBOTT-D.** (2000) Stable processes in econometric time series: Are prices made out of noise?, in D. Abbott., and L. B. Kish. (eds.), *Proc. AIP, Second. Int. Conf. Unsolved Problems of Noise and Fluctuations (UPoN 99)*, Vol. 511, pp. 221–232.

**ALLISON-A.** & **ABBOTT-D.** (2000) Stochastically switched control systems, in D. Abbott. (ed.), *Proc. SPIE, Second. Int. Conf. Unsolved Problems of Noise and Fluctuations (UPoN 99)*, Vol. 511, pp. 249–254.

**ALLISON-A.** & **ABBOTT-D.** (1999) Simulation and properties of randomly switched control systems, in B. Courtois., and S. N. Demidenko. (eds.), *Proc. SPIE, Design, Characterization, and Packaging for MEMS and Microelectronics*, Vol. 3893, pp. 204–213.

# List of Figures

1.1	Probability density in a flashing ratchet . . . . .	4
<hr/>		
2.1	Maxwell's demon . . . . .	15
2.2	The ratchet and pawl machine . . . . .	18
2.3	The flashing ratchet . . . . .	20
2.4	Top view of an inter-digital flashing ratchet . . . . .	21
2.5	Side view of an inter-digital flashing ratchet . . . . .	22
2.6	Charge separation in a Brownian ratchet . . . . .	23
<hr/>		
3.1	A single sample path of Parrondo's original process . . . . .	49
3.2	Time-evolution of the mean of the distribution $p(t, x)$ . . . . .	55
3.3	The time-evolution of $p(t, x)$ . . . . .	56
3.4	The mean position of a Brownian particle in a ratchet . . . . .	57
3.5	Point probabilities and probabilities of transition for the Bernoulli process	60
<hr/>		
4.1	The decision tree for Game A . . . . .	87
4.2	The decision tree for Game B . . . . .	88
4.3	State transitions of Parrondo's games, with no limits on position . . . . .	90
4.4	Results from a simulation based on the large matrix approach . . . . .	91
4.5	State transitions of Parrondo's games, (reduced modulo $L$ ) . . . . .	93
4.6	Expected rates of return $\varrho$ for various choices of the mixing fraction $\gamma$ . . . . .	98
4.7	The zero-gain surface for Parrondo's games . . . . .	99
4.8	The quasi-stable forms in Onsager's model . . . . .	100
4.9	The winning and losing regions in the 2D version of Parrondo's games . . . . .	105
4.10	An example of a linear reward function, $\varrho_1$ . . . . .	110
4.11	An example of a non-linear reward function, $\varrho_2$ . . . . .	111

## List of Figures

---

4.12	A stereo-pair plot of the winning and losing regions of Parrondo's games	116
4.13	The state transitions of Parrondo's games, with natural diffusion . . . . .	117
4.14	The expected rates of return $\varrho$ for various choices of the mixing fraction $\gamma$	118
4.15	Definitions for a simple two-state game . . . . .	119
4.16	Simulation of a two-state version of Parrondo's games . . . . .	120
4.17	The expected rates of return $\varrho$ for various choices of the mixing fraction $\gamma$	121
4.18	State transitions for Astumian's rule-set number 1 . . . . .	124
4.19	State transitions for Astumian's rule-set number 2 . . . . .	125
4.20	The expected rates of return $\varrho$ for various choices of the mixing fraction $\gamma$	126
4.21	A rule-set for Astumian's games . . . . .	127
4.22	The phase-space of a simple Markov chain . . . . .	134
4.23	A fractal attractor generated by games S and T . . . . .	139
4.24	A histogram of a distribution in phase-space . . . . .	142
4.25	The fractal object generated by Parrondo's original games . . . . .	150
4.26	The scaling properties of the attractor generated by Parrondo's games .	151
4.27	Fractal attractor generated by a limiting case of Parrondo's games . . . . .	158
<hr/>		
5.1	A simple switched capacitor energy converting circuit . . . . .	163
5.2	Switched-capacitor, equivalent circuit, during the <i>ON</i> mode . . . . .	165
5.3	Switched-capacitor, equivalent circuit, during the <i>OFF</i> mode . . . . .	165
5.4	A sketch of $V_c$ as a function of time . . . . .	167
5.5	A detailed numerical simulation of the switched-capacitor circuit . . . . .	170
5.6	A histogram of the scaled voltage, $x = V_c/V_s - 1/2$ . . . . .	176
5.7	The output from a switched-capacitor circuit . . . . .	178
5.8	The result from an SDE model for a switched capacitor circuit . . . . .	180
5.9	The scaling of variance with switching frequency . . . . .	182
5.10	General plan of a second-order system with one feedback loop . . . . .	187
5.11	Root locus plot for a second order system . . . . .	189
5.12	A model for the open-loop transfer function, $G(s)$ . . . . .	193
5.13	A model for the feedback transfer function, $G(s)$ . . . . .	194

---

5.14	Discrete state-space simulation of the neutral system . . . . .	197
5.15	Discrete state-space simulation of system $A_1$ . . . . .	198
5.16	Discrete state-space simulation of system $A_2$ . . . . .	199
5.17	Discrete state-space simulation of the randomly switched system . . . . .	200

---

6.1	Some sample paths from Geometric Brownian Motion (GBM) . . . . .	215
6.2	The infinitesimal form of Kirchhoff's Current Law (KCL) . . . . .	218
6.3	The infinitesimal form of Kirchhoff's Voltage Law (KVL) . . . . .	221
6.4	Linear noise models for a resistor . . . . .	224
6.5	Modelling of capacitors . . . . .	227
6.6	Modelling of inductors . . . . .	228
6.7	Parallel, or Norton, representation of an RC circuit . . . . .	229
6.8	A Thévenin Equivalent Circuit for the RC circuit . . . . .	230
6.9	The Poles of the PSD function for an RC circuit . . . . .	233
6.10	SDE models for an RC parallel circuit . . . . .	235
6.11	A parallel RCL circuit . . . . .	237
6.12	Positions of poles of a second-order under-damped circuit . . . . .	238
6.13	The normalised poles of a second-order under-damped circuit . . . . .	238
6.14	Poles of the power spectral density function . . . . .	240
6.15	An RCL parallel circuit . . . . .	243
6.16	Noise model for a JFET . . . . .	245
6.17	Large-signal, schematic circuit diagram for a Colpitts oscillator . . . . .	245
6.18	Small signal equivalent circuit of a Colpitts oscillator . . . . .	246

---

7.1	The Gaussian function, as a <i>basis</i> function . . . . .	256
7.2	Fundamental limitations of computation . . . . .	273

---





# List of Tables

3.1	Notation for discrete transformation and associated calculations . . . . .	31
3.2	Notation for transformed version of $p_{m,n}$ . . . . .	37
3.3	Semantic interpretations of variously transformed versions of $p_{m,n}$ . . .	37
3.4	Transforms of solutions to the Bernoulli process . . . . .	62
3.5	Transforms of some one-dimensional probability mass functions . . . . .	69
3.6	The generators for some stationary stochastic processes . . . . .	71
4.1	Rule-set number one, for Astumian's games . . . . .	122
4.2	Rule-set number two, for Astumian's games . . . . .	122
4.3	Parameters for the rule-sets, for Astumian's games . . . . .	128
4.4	Steady state probabilities, for Astumian's games . . . . .	130
5.1	Some values for the poles, $s_1$ and $s_2$ , as functions of the loop gain $K$ . . .	188

