Stochastically Switched Control Systems

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Abstract. Stability is a global property of a system. It is concerned with the behaviour of whole systems over indefinitely long periods of time, for all admissible inputs and uncertainties. Stability and instability are ultimately topological properties. They depend on the topology of the space defined by the equations that govern the system. It follows that that instability is not linear. It is possible to construct a linear combination of two unstable systems which will be stable. The operation of linear combination can be performed using time averaging. The switching can be periodic or stochastic. In the stochastic case, the variance plays an important role. It is possible to drive a system into instability by making the variance sufficiently large. The behaviour near the limit of stability is quite complex, even for very simple "toy" systems.

The stochastically switched system is not the same as a stationary linear filter, although we show that the power spectral densities of the two systems can appear to be very similar.

We show that variation in the strength of a feedback loop is a new mechanism introducing noise into a system.

DESIGN OF A SIMPLE SWITCHED SYSTEM WITH A NON-CONVEX UNSTABLE REGION

The central task of control theory [1,2] is to direct and regulate the behaviour of a system[1] in order to make it conform to a specified set of standards.

Our immediate aim is to design a simple "toy" system in the s domain [2] which has two unstable modes of operation that can be combined, using switching, to create a single stable mode of operation.

If a linear plant is placed inside a feedback control loop then a new system with new properties is created. A simple system topology is shown in Figure 1.

We can write the equations for this new system as:

$$F(s)^{-1} = G(s)^{-1} + K \tag{1}$$

[1] In control theory the system is often called the "plant".
[2] The variable, s, is a generalised complex frequency. It is used in the Laplace transform. Multiplication by s in this domain corresponds to differentiation in the time domain [3] as long as we have zero initial conditions.

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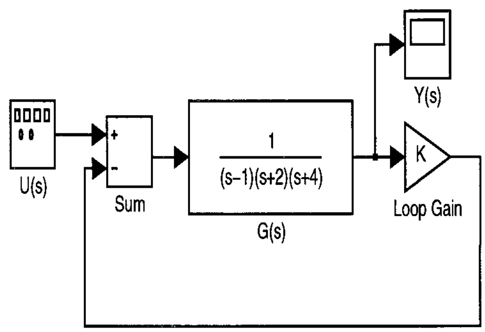

FIGURE 1. General plan of a simple system with one feedback loop

$G(s)$ is called the open loop transfer function and $F(s)$ is called the closed loop transfer function. The loop gain, K, is a free [3] parameter and is usually a positive real number.

It is customary to analyse the stability of closed loop systems in terms of the poles [4] of the closed loop transfer function, F(s). These poles will generally move about in the complex plane in response to changes in the loop gain, K. A graph of the positions of the poles, as a function of gain, is called a "root locus" plot. An example is shown in Figure 2. Some choices of gain may cause one, or more, of the poles to move into the unstable region, which would mean that the closed loop system would then be unstable. In general, there will be stable and unstable values for the gain, K.

We make a rather arbitrary choice for $G(s)$. It is an unstable system with three real poles.

$$G(s) = \frac{1}{(s-1)(s+2)(s+4)} \tag{2}$$

The open loop system is unstable because one of the poles lies in the unstable region [5]. A root locus plot for this system is shown in Figure 2.

The general behaviour of the closed loop system changes as we vary K. We can apply classical control theory and we can determine a number of critical values for, K. If we examine the locations of the poles of $F(s)$, as we vary K, then we find that the stable region for K is the union of two open intervals, $(-\infty, 8) \cup (18, \infty)$. It is not a convex set, so we could choose one value of gain from each sub-interval and form a time average, using switching, and the resulting expected value of gain, $\langle K \rangle$ would be in the *stable* region. We could switch rapidly between two *unstable* control systems and the result would be a *stable* control system!

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[3] The designer is *free* to choose a value.
[4] The poles of $F(s)$ are the zeros of $F(s)^{-1}$.
[5] The unstable region is the right is the hand side of the s plane. Any pole in this region will cause an exponentially increasing term to occur in the time domain response of the system.

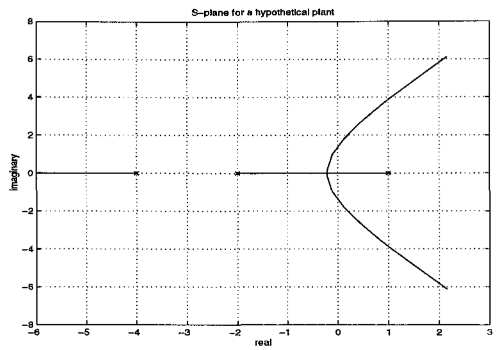

FIGURE 2. Root locus for an unstable three pole system

PERIODICALLY SWITCHED SYSTEM

The most simple switching regime is periodic. The expected value of the gain, $\langle K \rangle$, is controlled by altering the duty cycle of the switching waveform [6].

Periodic switching is frequently used in electronic systems. Switched mode power supplies use switching to control an output voltage without dissipating excessive power [2]. Some microelectronic integrated circuits use switched capacitors in order to avoid the difficult fabrication of large integrated resistors, [4] and to allow the properties of circuits to be altered through *program* changes rather than hardware changes. In this sense, switched mode control is only an incremental change to classical control.

STOCHASTICALLY SWITCHED SYSTEM WITH "SMALL" VARIANCE

One problem with traditional periodic switched mode systems is that they tend to generate uniform harmonics which are very undesirable from the point of view of electromagnetic compatibility. There has been some investigation of random pulse width modulation and random switching techniques for inverters and drive systems [5]. A natural extension of this idea is to use a random binary variable to perform the switching. We could still control the expected value of K by controlling the probability of switching "high," p. If the high value of the gain is K_h and the low value is K_l then the expected value and variance are given by:

$$\langle K \rangle = \mu = p \cdot K_h + (1 - p) \cdot K_l \tag{3}$$

$$\sigma^2 = p \cdot (1 - p) \cdot (K_h - K_l)^2 \tag{4}$$

[6] The *period* of the switching waveform must be much smaller than the smallest time constant of the system or time "averaging" will no longer occur. The system could become unstable again.

We could, for example, choose $K_l = 1$ and $K_h = 21$. The exclusive use of either of these values of gain would result in an unstable system but if we switched randomly between these possibilities then the result would be stable. This can be verified using simulation. Some results are shown in Figure 3.

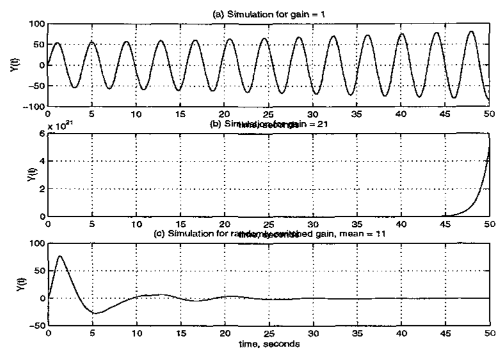

FIGURE 3. Outputs from periodically switched systems.

This result could be generalised still further by allowing K to be a continuous random variable with a specific distribution. If the variance is small then the deviation from the classical result is also small. The response is still very similar to the classical result, but it now looks as though it has some narrow band noise [7] imposed on top of it.

STOCHASTICALLY SWITCHED SYSTEM WITH "LARGE" VARIANCE

If a signal with increased variance is fed into a linear system then the result, at the output, will be a signal with larger variance. The system will not suddenly become unstable.

If the feedback variable, K, is chosen as a *random* variable and we make the variance large enough then we can drive the system into instability using only variance [8]. This can be shown by simulation. The output from the system appears to be made up of "bursts" of oscillation. The size of the "bursts" increases without limit if we allow the simulation to run for long enough,

If we choose a value of the variance which is near the limit of stability then we get an output dynamic which is very complex.

[7] Note that there is *no* noise signal being fed into the system at the input. The only "noise" in the system is due to the randomised switching policy. The actual input is a Kronecker delta function, $\delta(t)$ which is introduced in order to prevent the system from settling into the, degenerate, zero solution.

[8] We make *no* change to the mean value of K.

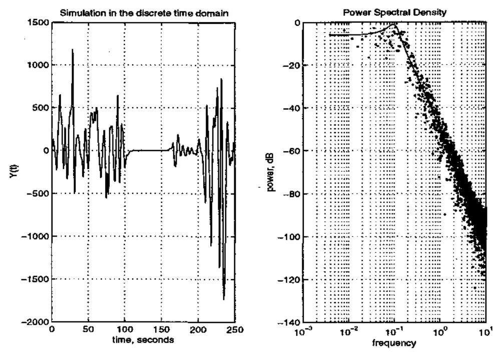

FIGURE 4. Response of a stocahstically switched system near the stability limit

It is difficult to reconcile this type of output with the narrow band noise that we would expect from a linear system with stochastic input. In particular, the oscillation seem to "die" completely, only to return again in "bursts" at later times. It would appear that the randomly variable gain is a non-linear element that fundamentally alters the behaviour of the system.

The power spectral density of the response was calculated using the Fast Fourier Transform, FFT. This was compared with the response which was calculated from the closed loop transfer function $F(s)$. The results are similar.

The stochastically switched system is *different* to the stationary filter with a white noise input, but it may be possible to use similar techniques to identify the open loop transfer function of an unknown system.

If we choose the loop gain to have the same average value as the simulation in Figure 4 and we keep this value fixed during the simulation then we have a stationary filter. We can examine the response of this stationary filter to white noise. It is shown in Figure 5.

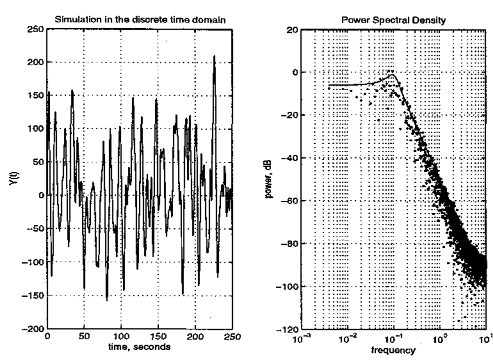

FIGURE 5. Response of a stationary filter to white noise

The power spectral density is also very similar to the theoretical transfer function of the closed loop system. The similarity of the power spectra suggests that it could be possible to use the theory of the Autoregressive Moving Average, ARMA, to identify the transfer function of the closed loop system.

The time domain responses in Figures 4 and 5 are qualitatively very different. This is further evidence that the two systems are different.

CONCLUSIONS AND OPEN QUESTIONS

The following questions require further investigation:

- Is it possible to derive exact criteria for the limits of stability as the mean and variance of the loop gain are varied? The system was simulated using a state-space formulation. Sufficient conditions for the stability of switched state-space controller systems have been stated in the literature [6].
- Can the theory of stochastic signal processing be applied to stochastically switched control? Given the similarity in the power spectral densities, it is quite possible that we use autoregressive, AR, models [9] to identify the closed loop system [2].
- Is this type of model useful for modelling systems in the real world systems, such as climate or the business cycle?
- Can we learn anything about other mathematical systems, such as Brownian ratchets through studying systems which have variable structure and a randomised policy [10]?

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[9] The Yule-Walker equations can be used to identify a system, given estimates of the autocorrelation functions.

[10] The the state-space formulation of the "toy" system is identical in form to the matrix formulation of the finite Markov chains employed in Parrondo's games.