PUBLISHED VERSION

http://hdl.handle.net/2440/60163

Numerical Methods for Systems Excited by White Noise

Bruce R. Davis

Department of Electrical & Electronic Engineering,
University of Adelaide, South Australia 5005 [1]

Abstract.

   This paper considers the problem of the numerical solution of continuous system equations when the excitation is white noise. Many signal generation models consist of a linear system excited by white noise, and in simulating the performance of these systems it is usually necessary to solve the system differential equations numerically.

   The simplest representation of white noise is to represent it by independent random samples at the discrete time instants used in the numerical process. However it is shown in this paper that this is not appropriate when the time step is further subdivided in the numerical integration process, and doing so can lead to significant errors in the solution. Some simple examples are included to illustrate the difficulties.

INTRODUCTION

   A general system excited by white noise can be expressed in the state space form:

$$\frac{dx(t)}{dt} \;=\; f[x(t), t] \;+\; g[x(t), t]\, w(t) \tag{1}$$

where $x(t)$ is a vector of length n with components $x_1(t), x_2(t), \ldots, x_n(t)$, $w(t)$ is in general a vector white noise process with m components of correlation function $R_{ww}(t_1, t_2) = R(t_1)\delta(t_1 - t_2)$, and $f[x(t), t]$ and $g[x(t), t]$ are functions of $x(t)$ and the time variable t.

   In general, the solution of (1) may require the use of either the Ito or Stratonovich [1] forms of integration, and the results obtained for each may be different. However, in many applications the representation may be of a simplified form with $g[x(t), t]$ not a function of $x(t)$. In this case, consistent results are obtained using either integration method, and (1) becomes:

---

[1] This work was supported by the University of Adelaide. The author would also acknowledge useful discussions with R. Mannella (Universitá di Pisa / INFM UdR Pisa).

$$\frac{dx(t)}{dt} = f[x(t), t] + g(t) w(t) \tag{2}$$

This paper will consider the numerical solution of systems of the form of (2) using a standard non-stochastic Runge-Kutta algorithm [2] in which the solution is obtained by at discrete time steps h by calculating the derivative for various values of t, $x(t)$ and some suitable approximation $\hat{w}(t)$ to $w(t)$. The accuracy compared with stochastic versions of the Runge-Kutta algorithm [3,4] will be discussed.

The step size h may be variable in order to obtain the required accuracy and stability. However in most applications to stochastic differential equations, a fixed step size will suffice, and this also avoids complications with the approximation to $w(t)$.

The simplest integration method is Euler integration which corresponds to:

$$x(t + h) = x(t) + h f[x(t), t] + h g(t) \hat{w}(t)$$

However Euler integration is usually inefficient in that a very small step size h must be used to achieve accuracy and stability, particularly with lightly damped resonant systems.

REPRESENTATIONS OF WHITE NOISE

In a numerical computation the continuous time variable t must be replaced by discrete instants of time $t = kh$, where h is the sampling interval. From the sampling theorem, a sampled version of a signal $x(t)$ represents a unique bandwidth limited signal $\hat{x}(t)$ of bandwidth $W = 1/(2h)$ Hz.

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(kh) \operatorname{sinc}\left(\frac{t - kh}{h}\right) \tag{3}$$

If the original signal has no components outside this bandwidth, then $\hat{x}(t) = x(t)$, but otherwise the sampled representation is in error due to aliasing. We must therefore choose the sampling interval h so that the important frequency components of all signals are included in the bandwidth W.

Clearly, this requirement cannot be satisfied for white noise, which by definition has infinite bandwidth. However for a system in which the response has a bandwidth which is much less than W, the response to white noise of systems of the type in (2) will be essentially the same as the response to white noise bandlimited to W. Hence for the purposes of numerical computation, we can replace $w(t)$ by bandlimited white noise $\hat{w}(t)$ of bandwidth W.

At this stage we will restrict consideration to a stationary scalar noise process $w(t)$, but the analysis can be extended to vector processes in a fairly straightforward manner.

For a scalar white noise process $\hat{w}(t)$ bandlimited to $W = 1/(2h)$ we have:

534

$$\hat{w}(t) \; = \; \sum_{k=-\infty}^{\infty} \hat{w}(kh) \, \mathrm{sinc}\left(\frac{t-kh}{h}\right) \tag{4}$$

so $\hat{w}(t)$ is completely described by samples $\hat{w}_k = \hat{w}(kh)$.

If the original noise $w(t)$ has a power spectral density $S_{ww}(f) = \alpha$ and autocorrelation function $R_{ww}(\tau) = \alpha \, \delta(\tau)$, then the bandlimited version has $S_{\hat{w}\hat{w}}(f) = \alpha \, \mathrm{rect}(fh)$ and $R_{\hat{w}\hat{w}}(\tau) = (\alpha/h) \, \mathrm{sinc}(\tau/h)$, and we see that the process $\hat{w}(t)$ can be represented by samples $\hat{w}_k = \hat{w}(kh)$ which are uncorrelated and of variance $\sigma^2 = \alpha/h$.

Alternatively, we can consider $\hat{w}_k$ to be given by:

$$\hat{w}_k \; = \; \int_{kh}^{(k+1)h} w(t) \, dt \tag{5}$$

The values of $\hat{w}(t)$ within the interval $kh \leq t \leq (k+1)h$ could be computed using (4). However this computation is impractical because it is an infinite series which converges slowly.

To simplify the interpolation procedure, it is possible to consider filtering $w(t)$ by a filter which is more easily implemented than that represented by (4). We need to consider a filter which generates a noise process $\hat{w}(t)$ which is essentially bandlimited and has a constant power spectral density at frequencies $f << W$.

Suitable filters from a computational point of view are those which have $H(f) = 1$ at low frequencies and a finite impulse response. Some suitable interpolation filters are listed below.

Rectangular - This produces a stepped noise waveform which consists of constant sections of length h equal to the noise samples $\hat{w}(kh)$. This has a spectrum $S_{\hat{w}\hat{w}}(f) = \alpha \, \mathrm{sinc}^2(fh)$ which is within 0.8% of the ideal value for $f < 0.1W$.

Triangular - This produces a noise waveform which is a linear interpolation between adjacent samples of $\hat{w}(t)$. This has a spectrum $S_{\hat{w}\hat{w}}(f) = \alpha \, \mathrm{sinc}^4(fh)$ which is within 1.6% of the ideal value for $f < 0.1W$.

The time increment h must be chosen to achieve the required accuracy and stability in the numerical integration as well as the sampling theorem requirement that the noise process $\hat{w}(t)$ has a constant power spectral density well beyond the frequencies to which the system responds. The noise samples $\hat{w}_k = \hat{w}(kh)$ can be generated by a gaussian random number generator.

Any noise values required at points in between the noise samples $\hat{w}(kh)$ must be computed using an interpolation technique. With rectangular interpolation and a fourth order Runge-Kutta method (designated RK4-Rect), the noise value is held constant at all times within the numerical integration step. When the integration step finishes on a time $t = kh$, the noise value for the next numerical integration step will be different. This leads to the noise value having two different values at $t = kh$, depending on whether the numerical integration step is just before or just

after $t = kh$. This can be somewhat difficult to program in a standard numerical integration package, so triangular interpolation (designated RK4-Tri), which avoids this complication, may be preferred although it is less accurate. The consequences of using the same noise value in rectangular interpolation is considered later (as method RK4-Var).

However what must not be done is to generate the intermediate noise values as random numbers (designated RK4-Mid), or even worse to use a random number generator each time a value is required, since in a repeated calculation this will give different values for the noise sample at the same time instant. To illustrate the errors that can occur with RK4-Mid, we will consider a simple example.

Example: Consider a system excited by white gaussian noise $w(t)$ of power spectral density $S_{ww}(f) = \alpha$.

$$\frac{dx(t)}{dt} = -\lambda x(t) + w(t) \quad ; \; x(0) = 0 \tag{6}$$

For the various integration methods, the variance of σ_x^2 of $x_k = x(kh)$ was calculated.

For $\lambda = 1$, the error in σ_x^2 plotted against the step size h for each of the integration methods is shown in Figure 1, where $\alpha = 2\lambda$ was chosen to give an exact value of unity.

It can be seen that the RK4-Rect method performs best, with an error in the variance $O(h^2)$, the Euler, RK4-Tri and RK4-Var methods have an error $O(h)$, whereas the RK4-Mid method is always 11% in error. It might seem that RK4-Tri offers no advantage over Euler integration, but usually Euler integration requires a much smaller step size than Runge-Kutta to preserve numerical stability. This is not apparent from this example.

EXPERIMENTAL RESULTS

Simulations were performed on the following system:

$$\frac{dx(t)}{dt} = Ax(t) + b\,w(t) \tag{7}$$

where

$$x(t) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ -1 & -1/Q \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1/Q \end{bmatrix} \tag{8}$$

This corresponds to white noise applied to a resonant circuit of resonant frequency 1 rad/sec and quality factor Q with a transfer function:

$$H(s) = \frac{X_2(s)}{W(s)} = \frac{s/Q}{s^2 + s/Q + 1} \tag{9}$$

536

The example was chosen because the Euler integration method performs very poorly when Q is high (ie. the required step size becomes very small).

The noise power spectral density of $w(t)$ is $S_{ww}(f) = \alpha = 1$, and direct calculation gives the steady state variance of $x_2(t)$ to be $\sigma_2^2 = \alpha/(\pi Q)$.

The equations were integrated using a fourth order Runge-Kutta method with fixed step sizes of $h = 0.05$ and $h = 0.1$ for the case $Q = 50$. Euler integration would be unstable or grossly in error with these step sizes and would require a step smaller by at least two orders of magnitude. To produce directly comparable results, the noise samples at the larger step size were derived from the noise samples at the smaller step size.

Figure 2 shows a comparison of the rectangular and triangular interpolation methods for a step size of $h = 0.1$. A smaller step size would normally be used in practice, but the larger step size enables the differences between the various methods to be seen more easily. The large plot is the output for the triangular interpolation and the smaller one is the difference between the outputs using the rectangular and triangular interpolations.
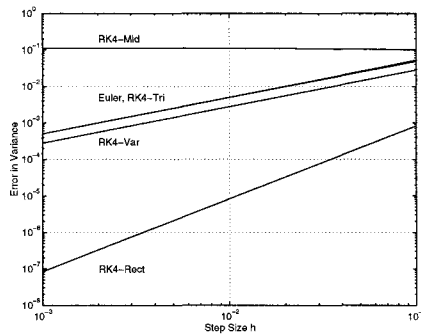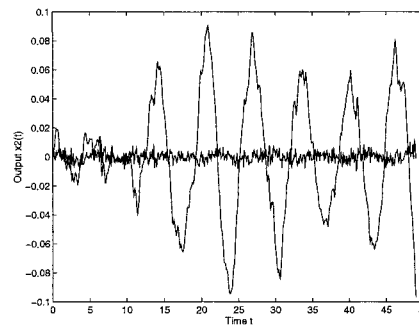

Fig 1 : Error in Steady State Variance.

Fig 2 : Response with $h = 0.1$.

Figure 3 is similar to Figure 2 and shows the comparison between the rectangular and triangular interpolation methods for a step size of $h = 0.05$.
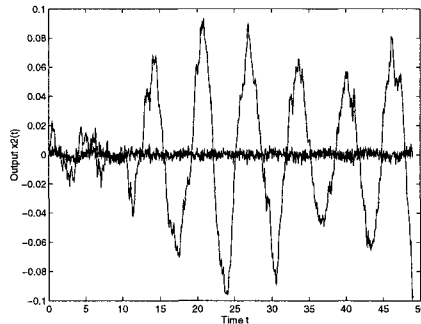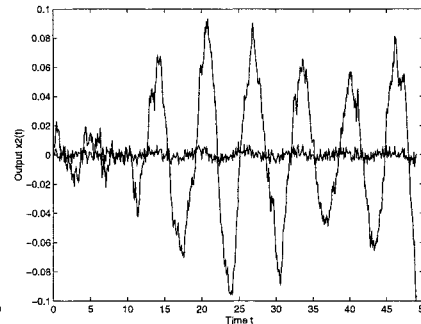

Fig 3 : Response with $h = 0.05$.

Fig 4 : Triangular Interpolation.

537

It can be seen that there is very little difference between the outputs for the rectangular and triangular interpolation functions, so either could be used in practice. Although the rectangular interpolation is more accurate in general, the triangular interpolation avoids the difficulty of having two different noise values at $t = kh$. There does not seem to be any compelling reason to use more complicated interpolation functions than these.

Figure 4 shows a comparison between $h = 0.1$ and $h = 0.05$ for the triangular interpolation method. The large plot is the output for $h == 0.05$ and the smaller plot is the difference of the outputs.

This shows that for this problem, the fourth order Runge-Kutta procedure produces reasonably accurate results for a step size of $h = 0.05$. To achieve similar accuracy with Euler integration would require a step size at least two orders of magnitude smaller.

CONCLUSIONS AND OPEN QUESTIONS

The numerical solution of system equations for systems excited by white noise can be found by replacing the white noise by a nearly bandlimited version with sample values specified at time intervals h. In order to get correct results, interpolation must be used to generate any noise samples at intermediate points used in the numerical integration process.

While the accuracy is not as high as that achievable with stochastic versions of the Runge-Kutta method, for the example in Section 2 the variance σ_x^2 is accurate to $O(h^2)$ for the recangular interpolation and $O(h)$ for the triangular interpolation. It is not known what the order of accuracy for the exact interpolation is.

Also, the accuracy of the variance is less than than the global accuracy of the integration method (which is $O(h^4)$). For linear stochastic differential equations, more accurate Runge-Kutta methods are discussed in [4], but for non-linear or multiplicative noise situations as in (1), a step accuracy of better than $O(h^2)$ [3] is difficult to achieve. While it may therefore seem that use of a Runge-Kutta method of order greater than two does not seem justified, with lightly damped resonant systems the use of a higher order method will ensure accuracy and stability of the non-stochastic part of the integration step. It can also be shown that the fourth order Runge-Kutta method has a better accuracy with multiplicative noise.

REFERENCES

1. H. Risken, *The Fokker-Planck Equation*, Springer 1996, p50.
2. R. L. Burden & J. D. Faires, *Numerical Analysis*, PWS-Kent 1989, p240.
3. W. Rümelin, *SIAM J. Numer. Anal.*, **19**, 3, (1982), pp. 604-613.
4. R. L. Honeycutt, *Phys. Rev. A*, **45**, 2, (1992), pp. 600-610.
5. A. Papoulis, *Probability, Random Variables and Stochastic Processes*, McGraw-Hill 1984, p213.