Analysis of variance of s(s-1)/2 frequencies a_{ij} ...

The analysis of variance of $\frac{1}{2}s(s-1)$ frequencies a_{ij}, where i and j are unequal and unordered, and take integral values from 1 to s.

In addition to the one degree of freedom for the total, which contributes

$$\frac{2}{s(s-1)} S^2_{ij}(a_{ij})$$

to the sum of squares for $\frac{1}{2}s(s-1)$ degrees of freedom

$$S_{ij}(a^2_{ij})$$

we need only consider linear functions with coefficients adding to zero; s of these are

$$A_i = (s-2)S_j(a_{ij}) - 2S_{jk}(a_{jk}),$$

satisfying $S(A_i) = 0$,

where i, j and k are all unequal. These must jointly specify the distribution of frequency among the s suffixes, having (s-1) degrees of freedom.

The sum of the squares of the coefficients in A is

$$(s-1)(s-2)^2 + 4.\frac{(s-1)(s-2)}{2} = s(s-1)(s-2);$$

so in the sum of the squares of the s quantities, A, the sum of the coefficients of terms such as a^2_{ij} must be

$$s^2(s-1)(s-2)$$

Hence in the expression

$$\frac{1}{s^2(s-2)} S_i(A^2_i)$$

the sum is s-1, equal to the number of degrees of freedom.

There will remain

$$\tfrac{1}{2}s(s-1) - 1 - (s-1) \text{ , } = \tfrac{1}{2}s(s-3) \text{ ,}$$

degrees of freedom for deviations among the frequencies compatible with fixed A_i.

For these we may define

$$B_{ij} = (s-2)(s-3)a_{ij} - (s-3)S_k(a_{ik}) - (s-3)S_k(a_{jk}) + 2S_{k\ell}(a_{k\ell})$$

in which i, j, k, ℓ are all unequal.

The sum of the coefficients in B_{ij} is zero.

The sum of the products of corresponding coefficients in B_{ij} and A_i is

for a_{ij}	$(s-2)^2(s-3)$	
for a_{ik}	$-(s-2)(s-3)$	$(s-2)$ terms
for a_{jk}	$2(s-3)$	$(s-2)$ terms
for $a_{k\ell}$	-4	$\tfrac{1}{2}(s-2)(s-3)$ terms

coming in all to zero; similarly the sum of the products for B_{ij} and A_k is

for a_{ij}	$-2(s-2)(s-3)$	
for a_{ik}	$-(s-2)(s-3)$	2 terms
for $a_{i\ell}$	$2(s-3)$	$2(s-3)$ terms
for $a_{k\ell}$	$2(s-2)$	$(s-3)$ terms
for $a_{\ell m}$	-4	$\tfrac{1}{2}(s-3)(s-4)$ terms

again giving a zero total, and showing that each of the components B is orthogonal to each of the components A.

But the sum of the squares of the coefficients of B is

$$(s-2)^2(s-3)^2 + 2(s-2)(s-3)^2 + 2(s-2)(s-3)$$

$$= (s-1)(s-2)^2(s-3)$$

The sum for all the $\frac{1}{2}s(s-1)$ expressions B_{ij} is

$$\tfrac{1}{2}s(s-1)^2(s-2)^2(s-3),$$

so that in

$$\frac{1}{(s-1)^2(s-2)^2}S(B^2)$$

the sum is $\frac{1}{2}s(s-3)$, the same as the number of degrees of freedom.

The orthogonal analysis so arrived at may be written

d.f.	S.S
1	$\frac{2}{s(s-1)}S^2(a_{ij})$
$s-1$	$\frac{1}{s^2(s-2)}S(A^2)$
$\frac{1}{2}s(s-3)$	$\frac{1}{(s-1)^2(s-2)^2}S(B^2)$
$\frac{1}{2}s(s-1)$	$S(a_{ij}{}^2)$

2. Effect of breeding for a single generation on the self-sterility alleles of a large population.

If p_{ij} stand for the relative frequency of the genotype (i, j), and

$$p_i = \tfrac{1}{2}\Sigma_j(p_{ij})$$

for the relative gene frequency, we may define the components

$$A_i = (s-2)\Sigma_j(p_{ij}) - 2\Sigma_{j,k}(p_{jk})$$

and

$$B_{ij} = (s-2)(s-3)p_{ij} - (s-3)\Sigma_k(p_{ik} + p_{jk}) + 2\Sigma_{k,\ell}(p_{k\ell})$$

and express the gene and genotype frequencies in terms of these components, as

$$p_i = \frac{1}{s} + \frac{1}{2s}A_i$$

$$\frac{2}{s-2}(p_i + p_j) - \frac{2}{(s-1)(s-2)}$$

$$p_{ij} = \frac{2}{s(s-1)} + \frac{1}{s(s-2)}(A_i + A_j) + \frac{1}{(s-1)(s-2)}B_{ij}$$

The genetic recursion formula for one generation is

$$p'_{ij} = \tfrac{1}{2}p_j\sum_k \frac{\overset{p_{ik}}{\cancel{p_k}}}{\underset{1-p_i-p_k}{\cancel{1-p_i-p_k}}} + \tfrac{1}{2}p_i\sum_k \frac{\overset{p_{jk}}{\cancel{p_{jk}}}}{1-p_j-p_k} \;;$$

on substitution in terms of the components A and B, the first term of these gives

$$\frac{1}{4s}(2 + A_j)\sum_k \frac{\dfrac{2}{s(s-1)} + \dfrac{1}{s(s-2)}(A_i + A_k) + \dfrac{1}{(s-1)(s-2)}B_{ik}}{\dfrac{s-2}{s} - \dfrac{1}{2s}(A_i + A_k)}$$

or, keeping only terms of the first degree in A, B

$$\frac{1}{4s}(2 + A_j)\sum_k\left\{\frac{2}{(s-1)(s-2)} + \frac{s}{(s-1)(s-2)^2}(A_i + A_k) + \frac{s}{(s-1)(s-2)^2}B_{ik}\right\}$$

$$= \frac{1}{4s}\left\{\frac{4}{s-1} + \frac{2A_j}{s-1} + \frac{2s}{(s-1)(s-2)^2}\Big\{(s-3)A_i - A_j\Big\} - \frac{2s}{(s-1)(s-2)^2}B_{ij}\right\}$$

~~and~~ whence adding the same ~~expansion~~ expression with i and j interchanged we have finally

$$p'_{ij} = \frac{2}{s(s-1)} + \frac{s^2-4s+2}{s(s-1)(s-2)^2}(A_i + A_j) - \frac{1}{(s-1)(s-2)^2}B_{ij}$$

so that all A_i are decreased in the ratio

$$\lambda = \frac{s^2-4s+2}{(s-1)(s-2)} = 1 - \frac{s}{(s-1)(s-2)}$$

and all B_{ij} in the ratio

$$\lambda = -\frac{1}{s-2}$$