Expansion of Lévy Process Functionals and Its Application in Econometric Estimation

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Contents

List of Figures

Abstract

This research focuses on the estimation of a class of econometric models for involved unknown nonlinear functionals of nonstationary processes. The proxy of nonstationary processes studied here is Lévy processes including Brownian motion as a particular one. A Lévy process is a càdlàg[1] stochastic process which starts at zero almost surely, which has independent increments over disjoint intervals, which has stationary increment distribution meaning that under shift the distributions of increments are identical, which has stochastic continuous trajectory. Obviously, Brownian motion, Poisson process, Gamma process and Pascal process are fundamental examples of Lévy processes. Lévy processes $(Z(t), t \geq 0)$ studied in this thesis possess density or probability distribution functions which verify some properties stated in the text.

Why do we care about the functionals of Lévy processes?

Starting with Brownian motion

In the galaxy of stochastic processes used to model random phenomena in disciplines such as economics, finance and engineering, Brownian motion is undoubtedly the brightest star. Brownian motion is the most widely studied stochastic process and the mother of the modern stochastic analysis. Brownian motion, for example, and financial modelling have been tied together from the very beginning when Bachelier (1900) proposed to model the price $S(t)$ of an asset at the Paris Bourse as $S(t) = S(0) + \sigma B(t)$ where $B(t)$ is a standard Brownian motion. The multiplicative version of Bachelier's model led to the celebrated Black-Scholes option pricing model[2] where log-price $\ln S(t)$ follows a Brownian

[1] right continuous with left limits.

[2] The Black-Scholes model is one of the most important concepts in modern financial theory. It was developed in 1973 by Fisher Black, Robert Merton and Myron Scholes and is still widely used today, and

motion $S(t) = S(0)\exp(\mu t + \sigma B(t))$ (see Black and Scholes, 1973).

Of course, the Black-Scholes model is not the only continuous time model built on Brownian motion. Nonlinear Markov diffusion where instantaneous volatility can depend on the price and time via a local volatility function have been proposed by Derman and Kani (1994) and Dupire (1994): $\frac{1}{S(t)}dS(t) = \mu dt + \sigma(t, S(t))dB(t)$. Another possibility is given by stochastic volatility models (see Hull and White, 1987; Heston, 1993) where the price $S(t)$ is the component of a bivariate diffusion $(S(t), \sigma(t))$ driven by a two-dimensional Brownian motion $(B^{(1)}(t), B^{(2)}(t))$: $\frac{1}{S(t)}dS(t) = \mu dt + \sigma(t)dB^{(1)}(t)$, $\sigma(t) = f(Y(t))$, $dY(t) = \alpha(t)dt + \gamma(t)dB^{(2)}(t)$. While these models have more flexible statistical properties, they share with Brownian motion the property of continuity, which does not seem to be shared by the real price over time scales of interest. Assuming that prices move in a continuous manner amounts to neglecting the abrupt movements in which most of the risk is concentrated.

Let us take an example from economics. Let Q denote the customer's total wealth and K the value of their house. The price of housing is constant, and the service flow from a house is equal to its value. For now there is no adjustment cost, so the customer can adjust K continuously and costlessly.

There are two assets, one safe and one risky. Assume that short sales of risky asset are not allowed, and let $A > 0$ be the customer's holding of the risky asset. Then $Q - A$ is the wealth of the safe asset. The mortgage interest rate is the same as the return of the bond, so holdings of the safe asset are the sum of equity in the house and bond holdings.

Let $r > 0$ be the riskless rate of return, let $\mu > r$ and $\sigma^2 > 0$ be the mean return and variance of risky asset, and let $\delta \geq 0$ be the maintenance cost per unit of housing. Then given K and A, the law of motion for total wealth is

$$\begin{aligned} dQ &= [rQ + (\mu - r)A - (r + \delta)K]dt + \sigma A dB \\ &= a(Q, \Theta)dt + b(Q, \Theta)dB \end{aligned}$$

where $\Theta = (\mu, \sigma, r, \delta)$ and B stands for Brownian motion. In the equation, function a is the total return constituting safe assets, risky assets, mortgage payments and maintenance cost, which are considered as a function of the time in question; while function b measures

regarded as one of the best ways of determining fair prices of options. The seminal work brought a Nobel prize in economics for Robert Merton and Myron Scholes in 1997.

the risky return from risky assets due to fluctuation of the stock market. More examples can be found in Stoke (2009).

One thing of note is that, more often than not, the processes depicted by stochastic differential equations involving Brownian motion take the form of the functional of the underlying process $B(t)$ as the solutions of the equations (Mikosch, 1998).

From Brownian motion to the Lévy process

> In the end, a theory is accepted not because it is confirmed by conventional empirical tests, but because researchers persuade one another that the theory is correct and relevant.
>
> Fischer Black (1986)

The Black-Scholes model stipulates that the log returns of an asset in question follow normal distribution. However, as suggested by empirical researches, e.g. Cont (2001) and Schoutens (2003), this assumption is not supported by real-world data. The following table tells that the daily log returns have significant (negative) skewness; the daily log returns have kurtosis bigger than 3; the P-values of the $\hat{\chi}^2$ statistics in the table show that the normal distribution is always rejected. The first dataset (S& P 500 (1970-2001)) contains all daily log returns of the S& P 500 Index over the period 1970-2001. The second dataset (*S&P 500(1970-2001)) contains the same data except for the exceptional log return (-0.2280) of the crash of 19 October 1987. All other datasets are over the period 1997-1999.

Table 1 Skewness, kurtosis and P_{Normal}-value of major indices

Index	Skewness	Kurtosis	P_{Normal}-value
S&P 500(1970-2001)	-1.6663	43.36	0.0000
*S&P 500(1970-2001)	-0.1099	7.17	-
S&P 500(1997-1999)	-0.4409	6.94	0.0421
Nasdaq-Composite	-0.5439	5.78	0.0049
DAX	-0.4314	4.65	0.0366
SMI	-0.3584	5.35	0.0479
CAC-40	-0.2116	4.63	0.0285

Moreover, another failure of the Black-Scholes model is that it does not capture the feature of heavy tail for the distribution of the real-world data sets. Figure 1 compares the five-minute returns on the Yen/Deutschemark (DM) exchange rate to increments of a Brownian motion with the same average volatility. While both return series have the same variance, the Brownian model achieves it by generating returns which always have roughly the same amplitude whereas the Yen/DM returns are widely dispersed in their amplitude and manifest frequent large peaks corresponding to 'jumps' in the price. This

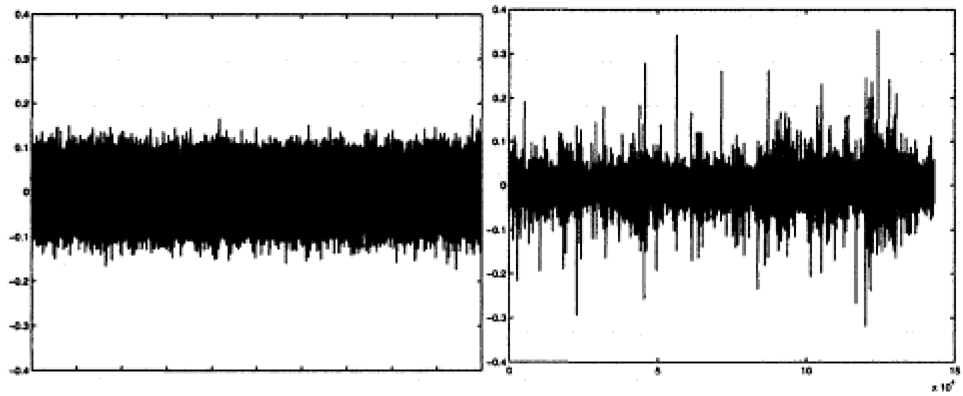

Figure 1: Five-minute log-returns for Yen/DM exchange rate, 1992-1995, compared with log-returns of a Black-Scholes model with the same annualised mean and variance

high variability is a constantly observed feature of financial asset returns. In statistical terms this results in heavy tails in the empirical distribution returns: the tails of the distribution decay slowly at infinity and very large moves have a significant probability of occurring. This well-known fact leads to a poor representation of the distribution of returns by a normal distribution. No book on financial risk is nowadays complete without a reference to the traditional six standard deviation market moves which are commonly observed on all markets, even the largest and the most liquid ones. Since for a normal random variable the probability of occurrence of a value six times the standard deviation is less than 10^{-8}, in a Gaussian model a daily return of such magnitude occurs less than once in a million years! Saying that such a model underestimates risk is a polite understatement. For detailed discussion, see Schoutens (2003, Chapter 4) and Cont and Tankov (2004).

Another observation is that many empirical datasets show non-linearity and non-stationarity. For example, in Gao (2007), there is strong evidence that the short rate is not stationary and normally distributed. The graph in Figure 2 shows the data of three

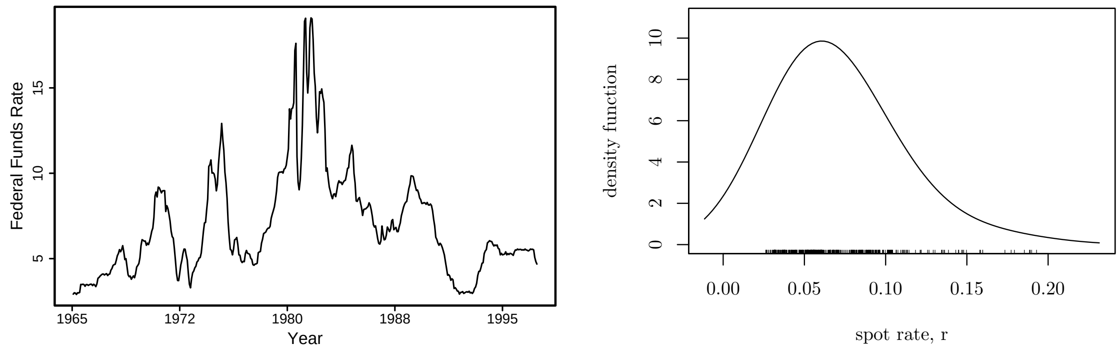

Figure 2: Left: three month Treasury bill rates 1963,1-1998,12; right: the estimated density

month Treasury bill rates between January 1963 and December 1998 (432 observations) and the estimated density function. It is clear that the density function is not normal distributed, and at 1% significance level it is acceptable that the set of data is non-stationary (see Gao et al., 2009).

Thanks to the aforementioned reasons, for a number of years, researchers have focused on developing a richer class of asset price models that include jumps as well as stochastic parameters; see Erakar et al. (2003) and Kou (2002). Meanwhile, several works realise that more sophisticated processes, Lévy processes, are able to represent skewness and excess kurtosis. See, for example, Schoutens (2003, Chapter 5) and Leblane and Yor (1998). In addition, several particular choices for non-Brownian Lévy processes have been proposed in the last few decades. Madan and Seneta (1990) have proposed a Lévy process with variance gamma distributed increments. We mention also the hyperbolic model proposed by Eberlein and Keller (1995), and in the same year the normal inverse Gaussian Lévy process proposed by Barndorff-Nielson (1995). Carr et al. (2000) introduced the CMGY model. Finally, we mention the Meixner model (see Grigelionis 1999 and Schoutens 2001).

Obviously, by Theorem 7 on Protter (2004, p.253), under some conditions, a stochastic differential equation driven by a Lévy process $(Z(t), t \geq 0)$ has a solution $f(Z(t))$. See, for example, Lim (2005) and Brockwell et al. (2007, 2011).

Both time-homogeneous and time-inhomogeneous functionals matter

It then makes sense to consider Lévy process functionals for modelling stochastic phenomena. Note that it is quite reasonable to consider time-inhomogeneous functionals of Lévy processes like $f(t, Z(t))$, instead of only dealing with the homogeneous functionals $f(Z(t))$. Since Hamilton and Susmel (1994) and Mikosch and Starica (2004) pointed out that invariant parametric specifications are often inconvenient to model long return series, in recent years the literature has naturally evolved towards the inclusion of multiple variables in continuous-time models. One example is that in Mercurio and Spokoiny (2004) the returns R_t of the asset process are stipulated as a heteroscedastic model $R_t = \sigma_t \xi_t$ where ξ_t are standard Gaussian independent innovations and σ_t is a time-varying volatility coefficient. The relevant works include Fan et al. (2003), Ait-Sahalia (2002), Hardle et al. (2003) and so forth.

About orthogonal expansions

Due to its extensive use in science, economics, finance and engineering and its central position within stochastic processes, the starting point of this research is to expand Brownian motion functionals including $f(B(t))$ and $f(t, B(t))$ where $B(t)$ is a standard Brownian motion into orthogonal series.

Notice that in the literature, albeit there exist some expansions of Brownian motion in terms of i.i.d. N(0,1) sequence, (see, for example, Yeh 1973 and Mikosch 1998), few researchers are working in the area of general form of Brownian motion functionals.

There are two papers which are close to our topic in some sense in the literature about orthogonal expansion of nonlinear functionals of some processes. To understand the relevant results, let us introduce some notations in the corresponding papers. Denote by C the space of real functions $x(t)$ which are continuous on the interval $0 \leq t \leq 1$ and which vanish at $t = 0$. Let $\{\alpha_p(t)\}$ be any orthonormal set of real functions in $L^2(0,1)$, and define

$$\Phi_{m,p}(x) = H_m \left(\int_0^1 \alpha_p(t) dx(t) \right); \quad m = 0, 1, 2, \cdots, \ p = 1, 2, \cdots,$$

where $H_m(\cdot)$ is the sequence of Hermite orthogonal polynomials and

$$\Psi_{m_1, \cdots, m_p}(x) \equiv \Psi_{m_1, \cdots, m_p, 0, \cdots, 0}(x)$$

$$=\Phi_{m_1,1}(x)\cdots\Phi_{m_p,p}(x),$$

in which the index p may be any positive number; the subscripts $m_1,\cdots,m_p$ may be any nonnegative numbers.

Using the Wiener measure on C and completeness properties of Hermite polynomials over $(-\infty,\infty)$, Cameron and Martin (1947) introduced a complete orthonormal set of functionals on C so that every real or complex valued functional $F[x(\cdot)]$ which belongs to $L^2(C)$,

$$\int_c^w |F[x]|^2 d_w x < \infty,$$

has a Fourier development in terms of this set which converges in the $L^2(C)$ sense to functional $F[x]$:

$$\int_c^w \left| F[x] - \sum_{m_1,\cdots,m_N=0}^{N} A_{m_1,\cdots,m_N}\Psi_{m_1,\cdots,m_N}(x)\right|^2 d_w x \to 0,$$

as $N\to\infty$, where $A_{m_1,\cdots,m_N}$ is the Fourier-Hermite coefficient

$$A_{m_1,\cdots,m_N} = \int_c^w F[x]\Psi_{m_1,\cdots,m_N}(x) d_w x.$$

Ogura (1972) did an analogous job as Cameron and Martin (1947) but expanded functionals of the Poisson process $F[D(\cdot)]$ in a series of multiple Poisson-Wiener integrals:

$$F[D(\cdot)] = \sum_{n=0}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f_n(t_1,\cdots,t_n)c^{(n)}[dD(t_1),\cdots,dD(t_n)],$$

where $D(\cdot)$ stands for a Poisson process.

Clearly, the bases in both papers for expansion of functionals are highly complicated since, as discussed in Ogura (1972), they are all multiple Hermite polynomials having the number of arguments increasing to infinity. By contrast, the expansions proposed in Chapter 2 and 4 in this study are quite simple thanks to the simplicity of the bases. This difference gives convenience in calculation of the coefficients and application in practice. Notice that the expansions in the literature have coefficients which are actually functions in the time variable, which would hamstring the applicability of the expansion in econometrics. Nonetheless, from the econometrical applicability perspective, we tackle this issue by expanding time-inhomogeneous functionals, so that coefficients in our expansion are all pure constants which can be estimated by econometric methods. Furthermore, another huge difference between the proposed method in this research and the literature is that we are going to expand functionals of a general class of Lévy processes, not just for Brownian motion or the Poisson process. Additionally, due to the reasons mentioned above our expansion method may be used to estimate unknown functional forms in a general class of econometric models.

The methodology undertaken here, for both Brownian motion functionals and general Lévy process functionals is about to expand the functional in some Hilbert space into Fourier series in terms of a particular orthonormal polynomial basis in the aforementioned space. The basis is actually a sequence of polynomial solutions of hypergeometric differential equations. It is noteworthy that the correspondence between the Lévy process and the orthonormal polynomial system is one–one. The key link between them is the density or probability function of the process. From the Hilbert space theory standpoint, the Fourier series expansion gives the coordinates of a functional in infinite dimensional space, and thereby characterises the functional in nature.

Econometric applications of Fourier expansion

Nevertheless, the Fourier series expansion of Lévy process functionals is by no means our destination. We are interested in estimating an unknown functional form in a general model

$$Y(t) = m(t, Z(t)) + \varepsilon(t),$$

where $Z(t)$ is a Lévy process, and $\varepsilon(t)$ is an error process with zero mean and finite variance, given that we have discrete observations of $Y(t)$.

It is known that existing literature already discusses how to estimate unknown functions of nonlinear time series using nonparametric and semiparametric methods. For the stationary case, recent studies include Fan and Yao (2003), Gao (2007) and Li and Racine (2007). It should also be pointed out that the literature shows that many economic and financial data exhibit both nonlinearity and nonstationarity. Consequently, some nonparametric and semiparametric models and kernel–based methods have been proposed to deal with both nonlinearity and nonstationarity simultaneously. Existing studies mainly discuss the employment of nonparametric kernel estimation methods. Such studies include Phillips and Park (1998), Park and Phillips (1999, 2001), Karlsen and Tjøstheim (2001), Karlsen et al. (2007), Cai et al. (2009), Phillips (2009), Wang and Phillips (2009a,b), Xiao

(2009), and Gao and Phillips (2010). Observe that such kernel–based estimation methods are not applicable to establish closed–form expansions of Brownian motion/Lévy process functionals. In the stationary case, the literature already discusses how series approximations may be used in dealing with stationary time series models, such as Ai and Chen (2003), Chapter 2 of Gao (2007) and Li and Racine (2007). Therefore, it is reasonable to seek its counterpart in the nonstationary scenario to tackle the nonstationary problems.

An inevitable question of doing so is on what time horizon we shall estimate the functional $m(\cdot, \cdot)$. The intuitive choices of time horizon are no more than two cases, viz., a compact interval $[0, T]$ and an infinite interval $(0, \infty)$. However, apart from these two options, we consider the third case, that is, on $[0, T_n]$ with T_n approaching to infinity as sample size goes to infinity. In technical terms, allowing $T = T_n \to \infty$ and $\frac{T_n}{n} \to 0$ amounts to both infill and long span asymptotics. Meanwhile, the two-fold limit theory keeps one away from the so-called aliasing problem (i.e. different continuous-time processes may be indistinguishable when sampled at discrete time). Phillips (1973) and Hansen and Sargent (1983) are early references on the aliasing phenomenon in econometric literature.

A pivotal asymptotic theory

Of the most importance is an asymptotic theory as it is a tool, also a bottleneck, for obtaining the limit distribution of estimators. Without a more general asymptotic theory, our method would be extremely restricted. In order to obtain the asymptotic distribution of the estimators of $m(\cdot, \cdot)$ estimated from the model mentioned before, we have to study an asymptotic theory for different classes of functionals $f(\cdot, \cdot)$ for their sample mean and sample covariance.

Note that in last decade or so, several studies have been devoted to developing an asymptotic theory of a general class of functionals of integrated time series. The relevant researchers have noticed that the absence of such a limit distribution theory has hamstrung time series application. See Park and Phillips (1999, 2001) and Wang and Phillips (2009a,b). However, the existing theory in the literature cannot furnish an answer for the limit problems arising from the scenarios in this study since $f(\cdot, \cdot)$ includes not only a random walk with a unit root but also the time variable, while in literature only a single random walk is involved. Whence, a new asymptotic theory needs to be established. The asymptotic theory developed in this research depends heavily on the local–time process of a Brownian motion defined as a limit by the underlying process and shows that the limit distribution of the estimators on infinity horizon $(0, \infty)$ and on compact interval $[0, T_n]$ with T_n approaching infinity are a mixed normal,

$$\left(\int_0^1 \int_{\mathbb{R}} f^2(t,x) dx dL_W(t,0) \right)^{\frac{1}{2}} N$$

where $L_W(t, 0)$ is the local–time process of the limiting Brownian motion $W(r)$ on $[0, 1]$ standing for the sojourn time at origin over $[0, t]$ by $W(r)$ and N is a standard normal random variable independent of W, f is some suitable function defined on $[0, 1] \times \mathbb{R}$.

By contrast, in the situation where the time variable lies in $[0, T]$ with T fixed, the asymptotic distribution of the estimator is a stochastic integral,

$$\int_0^1 f(Tr, T\mu r + \sqrt{T}\sigma_z W(r)) dU(r)$$

where $(W(r), U(r))$ is a vector of Brownian motion which is a limit of some process vector $(W_n(r), U_n(r))$ constructed from Lévy processes $Z(t)$ and error process $\varepsilon(t)$, $\mu = E[Z(1)]$ and $\sigma_z^2 = Var[Z(1)]$, f is some suitable function defined on $[0, T] \times \mathbb{R}$. It is noteworthy to point out that W and U may not be independent which gives more flexibility for the models used in practice.

Outline

The thesis is not presented according to the chronology of the research. We display the asymptotic theory in Chapter 1, which provides an essential tool for the following development. At the same time, as can be seen from the text, since the framework is quite general the results in asymptotic theory of Chapter 1 are applicable even beyond the ambit of this research.

Chapter 2 is devoted to a special case for expansions where Lévy process $Z(t)$ reduces to Brownian motion $B(t)$. Restricted within Brownian motion, the setup in Chapter 2 is concrete. For example, the polynomial system in terms of which we expand functionals is the Hermite polynomial system. In addition, many ideas and methods which are used in the general situation are fostered in this period.

Chapter 3 studies the estimation of an unknown functional form in a general econometric model which involves Brownian motion. The estimators are obtained according to different time horizons and sampling styles. Meanwhile, their asymptotic distributions are obtained and from the results we can see that the rates of convergence are affected by not only sample size but also many other factors.

Chapter 4 dwells on the general situation where the underlying process is a Lévy process $Z(t)$ whose density or probability function $\rho(t, x)$ satisfies the so-called boundary condition. Every such process admits a so-called classical orthonormal polynomial system with weight $\rho(t, x)$, with which the functional of $Z(t)$ can be expanded in the corresponding Hilbert space into Fourier series.

As an application of the orthogonal expansion and asymptotic theory in the previous chapters, Chapter 5 estimates the unknown functional $m(\tau, z)$ by $\widehat{m}(\tau, z)$ in the model aforementioned with the help of OLS (ordinary least squares). After obtaining the estimators in three types of time horizon, their asymptotic distributions are investigated.

The last chapter concludes what we did and discusses potential applications of the proposed expansion method for Lévy process functionals.

Appendix A, entitled Miscellaneous, states an alternative expansion method for the quadratic Brownian motion form using stochastic integral method. Without doubt, it has a kind of quaint charm although comparing with the text it is difficult to be extended to general situations.

Declaration

This work contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and beliefs, contains no material previously published, or written by another person except where due reference has been made in the text.

I give consent to the copy of my thesis, when deposited in the university library, being available for loan and photocopying.

Chaohua DONG

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12 December, 2011

Acknowledgements

I had wasted a lot of time before starting my PhD journey in 2008. I did not realise that I had a strong desire to pursue knowledge until I visited Professor Jiti Gao in 2004 at the University of Western Australia, and discovered that studying for a PhD is the best way for me to do this. Given that I was not so young, this was a 'now or never' opportunity for me. I therefore sat the IELTS qualification three times in order to acquire a sufficient score in English to be an eligible applicant. This was the prelude for the journey of my PhD, which has been full of difficulties and trials.

With the eligible IELTS result and excellent academic performance, I was awarded EIPRS (Endeavor International Postgraduate Research Scholarship) from the Australian government and the University of Adelaide in 2008, triggering my PhD journey off.

I have very much enjoyed the last three years and four months of hectic study under the supervision of Professor Jiti Gao. I strived for the answer of every single question; my intelligence was tortured by the research questions again and again; my endeavours to pursue the results of the research, to achieve the degree of PhD took all of my effort and energy, exhausting both my physical and spiritual self. Now I see the lighthouse which will guide me into the harbour of destination.

At such a crucial and critical moment, it is high time to deliver my acknowledgements, both professional and personal, like a champion after winning an exciting and fierce game, for the supervisors, the peers, the staff in the school and my family members.

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Chapter 1

Asymptotic theory

In the last few decades, nonstationary time series arising from autoregressive models with roots on the unit circle has been an intensive research interest. As a result, the asymptotic behaviour of regression statistics including integrated time series has received the most attention. Although a fairly complete theory is now available for linear time series regressions, asymptotic theory for nonlinear regressions with integrated time series is in the process of development and in a great deal of situations the demand for the theory becomes a bottleneck for both econometric theory and application. Recent studies include Park and Phillips (1999, 2001), Karlsen et al. (2007), and Wang and Phillips (2009a,b) among others.

This chapter dwells on a more general setting, that is, the asymptotic theory of statistics involving $f(s, x_{s,n})$ where $x_{s,n}$ is a triangular array constructed from some underlying time series. Clearly, this theory includes the existing literature as a special case. The results show the limit distributions are mixed normal distribution in one case, relying on local time of a limiting Brownian motion, while in another case stochastic integrals involving a correlated vector of Brownian motion.

1.1 Local time and assumptions

In what follows our asymptotic theory depends heavily on a local–time process of Brownian motion. The following three lemmas are basic definition and properties for the local–time process which can be found in a standard reference book Revuz and Yor (1999).

Lemma 1.1.1 (The Tanaka Formula)**.** *For any real number a, there exists a non-decreasing continuous process $L(\cdot, a)$ called the local time of a continuous semimartigale (SMG) M_t at a such that*

$$\begin{aligned} |M_t - a| &= |M_0 - a| + \int_0^t sgn(M_s - a) dM_s + L_M(t, a) \\ (M_t - a)^+ &= (M_0 - a)^+ + \int_0^t \mathbf{1}_{(M_s > a)} dM_s + \frac{1}{2} L_M(a, t) \\ (M_t - a)^- &= (M_0 - a)^- - \int_0^t \mathbf{1}_{(M_s \leq a)} dM_s + \frac{1}{2} L_M(a, t) \end{aligned}$$

In particular, $|M_t - a|$, $(M_t - a)^+$ and $(M_t - a)^-$ are SMGs.

Lemma 1.1.2 (Continuity of the Local Time of SMG)**.** *For any continuous SMG M_t, there exists a version of the local time such that the map $(t, a) \mapsto L_M(t, a)$ is almost surely continuous in t and càdlàg in a.*

Lemma 1.1.3 (The Occupation Time Formula)**.** *Let M_t be a continuous SMG with quadratic variation process $[M]_t$. Then,*

$$\int_0^t f(s, M_s) d[M]_s = \int_{-\infty}^{\infty} da \int_0^t f(s, a) dL_M(s, a) \tag{1.1.1}$$

for every positive Borel measurable function $f(t, x)$.

Given a triangular array $x_{s,n}$ ($x_{0,n} = 0$ by definition), $1 \leq s \leq n$, constructed from some underlying time series, we assume that $x_{[nr],n}$ $(0 \leq r \leq 1)$ converges in distribution to a stochastic process $W(r)$ on $D[0, 1]$ with respect to the Skorohod topology which admits a continuous local time process, where $D[0, 1]$ stands for the space of real-valued functions that are right continuous with left limits. It is known that there are many cases in which $\{x_{s,n}\}$ satisfies this condition, and in some suitable probability space it can be improved as $\sup_{0 \leq r \leq 1} |x_{[nr],n} - W(r)| = o_P(1)$. Readers are referred to Phillips (1987), Park and Phillips (1999, 2001) and Wang and Phillips (2009a) for detailed discussions.

We now impose the following assumption on $x_{s,n}$.

Assumption A

(a) Suppose that $x_{[nr],n}$ $(0 \leq r \leq 1)$ converges in distribution to a stochastic process $W(r)$ on $D[0, 1]$ with respect to the Skorohod topology. Let $W(r)$ admit a continuous local–time $L_W(r, s)$.

(b) In some suitable probability space there exists a stochastic process $W(r)$ that admits a continuous local–time $L_W(r,s)$ such that $\sup_{0\le r\le 1}|x_{[nr],n} - W(r)| = o_P(1)$.

(c) Denote for ϵ $(0<\epsilon<1)$ that $\Omega_n(\epsilon) = \{(l,k) : \epsilon n \le k \le (1-\epsilon)n, k+\epsilon n \le l \le n\}$. For all $0 \le k < l \le n$, there exist a sequence of constants $d_{l,k,n}$ and a sequence of σ-fields $\mathcal{F}_{n,k}$ where $\mathcal{F}_{n,0} = \{\emptyset, \Omega\}$, such that

(i) for some $m_0 > 0$ and $C > 0$, $\inf_{(l,k)\in\Omega_n(\epsilon)} d_{l,k,n} \ge \epsilon^{m_0}/C$ as $n \to \infty$,

$$\lim_{\epsilon\to 0}\lim_{n\to\infty}\frac{1}{n}\sum_{l=(1-\epsilon)n}^{n}\frac{1}{d_{l,0,n}} = 0, \tag{1.1.2}$$

$$\lim_{\epsilon\to 0}\lim_{n\to\infty}\frac{1}{n}\max_{0\le k\le(1-\epsilon)n}\sum_{l=k+1}^{k+\epsilon n}\frac{1}{d_{l,k,n}} = 0, \tag{1.1.3}$$

$$\limsup_{n\to\infty}\frac{1}{n}\max_{0\le k\le n-1}\sum_{l=k+1}^{n}\frac{1}{d_{l,k,n}} < \infty. \tag{1.1.4}$$

(ii) Suppose that $x_{k,n}$ are adapted to $\mathcal{F}_{n,k}$. Moreover, if $x_{k,n}$ are continuous variables, conditional on $\mathcal{F}_{n,k}$, $(x_{l,n} - x_{k,n})/d_{l,k,n}$ has a density $h_{l,k,n}$ which is uniformly bounded by a constant K and

$$\lim_{\delta\to 0}\lim_{n\to\infty}\sup_{(l,k)\in\Omega_n(\delta^{1/(2m_0)})}\sup_{|u|<\delta}|h_{l,k,n}(u) - h_{l,k,n}(0)| = 0. \tag{1.1.5}$$

If $x_{k,n}$ are discrete variables, conditional on $\mathcal{F}_{n,k}$, $(x_{l,n} - x_{k,n})/d_{l,k,n}$ has a probability distribution $P_{l,k,n}(x)$ and its distribution function $F_{l,k,n}(x)$ satisfies

$$\lim_{\delta\to 0}\lim_{n\to\infty}\sup_{(l,k)\in\Omega_n(\delta^{1/(2m_0)})}\sup_{|u|<\delta}|F_{l,k,n}(u) - F_{l,k,n}(0)| = 0. \tag{1.1.6}$$

Remark 1.1.1. Assumption A is almost the same as the conditions in the univariate function case in Wang and Phillips (2009a) except that we concern both continuous and discrete variables in A (c). We shall discuss the condition (1.1.6) later. Note that Assumption A (except the discrete case in A (c)) is quite weak which is discussed in the literature. As a consequence, the following theorems are generally applicable.

Also, we remark that this situation particularly accommodates any Lévy process. According to infinite divisibility, a Lévy process $Z(t)$ at point positive integer s can be rephrased as $Z(s) = \mu s + v_1 + \cdots + v_s$ in distribution where $v_i = Z(i) - Z(i-1) - \mu$ $(i = 1, \cdots, s)$ form an i.i.d. sequence, and $\mu = E(Z(1))$. Whence, define $x_{s,n} = \frac{1}{\sqrt{n}\sigma_z}Z(s)$

for $s = 1, \cdots, n$ and $n \geq 1$ where $\sigma_z^2 = Var(Z(1))$, then by virtue of functional central limit theorem $x_{s,n}$ converges in distribution to a Brownian motion $W(r)$ on $[0,1]$ as $n \to \infty$. In addition, with $d_{l,k,n} = \sqrt{(l-k)/n}$, $x_{s,n}$ and $d_{l,k,n}$ satisfy Assumption A (a) and (c), and also A (b) can be achieved by the Skorohod representation theorem.

Take an example to verify the condition (1.1.6). Suppose now that $Z(t)$ is a Poisson process, viz., $Z(t) \sim Poi(\mu t)$. Because $\frac{1}{d_{l,k,n}}(x_{l,n} - x_{k,n}) =_D \frac{1}{\sqrt{l-k}\sigma_z}(Z(l-k) - (l-k)\mu)$,

$$F_{l,k,n}(0) = \sum_{i \leq (l-k)\mu} \frac{[(l-k)\mu]^i}{i!} e^{-(l-k)\mu}$$

$$F_{l,k,n}(u) = \sum_{i \leq (l-k)\mu + u\sqrt{l-k}\sigma_z} \frac{[(l-k)\mu]^i}{i!} e^{-(l-k)\mu}.$$

Thus, if $u > 0$

$$F_{l,k,n}(u) - F_{l,k,n}(0) = e^{-(l-k)\mu} \sum_{(l-k)\mu < i \leq (l-k)\mu + u\sqrt{l-k}\sigma_z} \frac{[(l-k)\mu]^i}{i!},$$

if $u < 0$,

$$|F_{l,k,n}(u) - F_{l,k,n}(0)| = e^{-(l-k)\mu} \sum_{(l-k)\mu + u\sqrt{l-k}\sigma_z < i \leq (l-k)\mu} \frac{[(l-k)\mu]^i}{i!}.$$

Because $e^{-(l-k)\mu} \to 0$ as $(l,k) \in \Omega_n(\epsilon)$, $n \to \infty$ and the sums are less than the tail of a convergent series, the condition (1.1.6) is fulfilled.

Notice also that in some situation, for continuous process the condition (1.1.6) implies the requirement (1.1.5), so that they merge as (1.1.6) which harbours both continuous and discrete cases.

Since we study the asymptotic theory of not only the sample mean but also the sample covariance, the following assumption stipulates some necessary conditions for $x_{s,n}$ and e_s.

Assumption B

(a) There is a martingale difference sequence $(e_s, \mathcal{F}_{n,s})$ with $E(e_s^2|\mathcal{F}_{n,s-1}) = \sigma_e^2$ a.s. for all $s = 1, 2, \ldots, n$ and $\sup_{1 \leq s \leq n} E(|e_s|^p|\mathcal{F}_{n,s-1}) < \infty$ a.s. for some $p > 2$.

(b) $\{x_{s+1,n}\}$ is adapted to $\mathcal{F}_{n,s}$, $s \geq 0$.

(c) Let, for $r \in [0,1]$,

$$U_n(r) = \frac{1}{\sqrt{n}} \sum_{s=1}^{[nr]} e_s \quad \text{and} \quad W_n(r) = x_{[nr],n}.$$

Suppose that (U_n, W_n) converges in distribution to (U, W) on $D[0,1]^2$ as $n \to \infty$, where (U, W) is a correlated Brownian motion vector.

Remark 1.1.2. As mentioned for Assumption A, Assumption B is also quite general and applicable in many situations. For example, condition (c) holds when $\{e_s\}$ is a sequence of independent errors and $\mathcal{F}_{n,s} = \sigma(e_1, \cdots, e_s, x_{s+1,n})$.

The trajectories of the stochastic process (U_n, W_n) for each $\omega \in \Omega$ are in $D[0,1]^2$. The space $D[0,1]$[1] is usually equipped with the Skorohod topology. It then follows from the so-called Skorohod representation theorem that there exists a common probability space $(\Omega, \mathcal{F}, P)$ supporting (U_n^0, W_n^0) and (U, W) such that

$$(U_n, W_n) \stackrel{D}{=} (U_n^0, W_n^0), \quad \text{and} \quad (U_n^0, W_n^0) \to_{a.s.} (U, W) \tag{1.1.7}$$

in $D[0,1]^2$ with the uniform topology in a suitable space.

1.2 Time-normalised and integrable functionals

This section establishes an asymptotic theory whose results extend existing literature, such as Park and Phillips (1999, 2001) and Wang and Phillips (2009a), from the univariate case to the bivariate case.

Let us now define the class of functionals for which we will establish an important theorem. Such a theorem is of general interest.

Assumption C

(a) Suppose that $f(t,x)$ is defined on $[0,1] \times (-\infty, \infty)$. Suppose further that both $|f(t,x)|$ and $f^2(t,x)$ are Lebesgue integrable with respect to x on $(-\infty, \infty)$.

(b) There exists a function $c_f(x) : \mathbb{R} \to \mathbb{R}^+$ such that $|f(t,x)| \le c_f(x)$ uniformly in $t \in [0,1]$ and $c_f(x)$ is integrable on $\mathbb{R}$.

[1] $D[0,1]$ designates the space of càdlàg functions (French, means the function at every point who is right continuous and possesses finite left limit) on unit interval $[0,1]$.

(c) For each $x \in \mathbb{R}$, $f(t,x)$ is continuous in t and there are at most a finite number of points for t at which $\int f(t,x)dx = 0$.

Remark 1.2.1. We shall denote $G_1(t) = \int_{-\infty}^{\infty} f(t,x)dx$, $G_2(t) = \int_{-\infty}^{\infty} |f(t,x)|dx$ and $G_3(t) = \int_{-\infty}^{\infty} f^2(t,x)dx$ for universal convenience. Notice that they are all continuous functions by the dominated convergence theorem.

Condition (a) is an extension of Assumption 2.1 in Wang and Phillips (2009a). Requirement on integrability of functions is a basic need to deal with this kind of problems. Note that if $f(t,x) = f(x)$ becomes time-homogeneous, condition (a) reduces to Assumption 2.1 in Wang and Phillips (2009a).

Condition (b) requires that the function $f(t,x)$ be dominated uniformly in t over compact interval $[0,1]$ by an integrable function $c_f(x)$. In the situations where $f(t,x)$ is the product of a continuous function of t and an integrable function of x or the superposition of such products, the condition is automatically fulfilled.

Condition (c) also excludes the situation where there are infinite many points $t_j \in [0,1]$ such that $G_1(t_j) = 0$.

The functionals of interest L_n and M_n are defined as follows:

$$\begin{aligned} L_n &= \frac{c_n}{n} \sum_{s=1}^{n} f\left(\frac{s}{n}, c_n x_{s,n}\right), \\ M_n &= \sqrt{\frac{c_n}{n}} \sum_{s=1}^{n} f\left(\frac{s}{n}, c_n x_{s,n}\right) e_s, \end{aligned}$$

where c_n is a sequence of positive constants, and f satisfies Assumption C. When the underlying time series is a random walk, c_n may take an explicit form of $\sqrt{n}$. We are interested in the general situation in this section that $n \to \infty$, $c_n \to \infty$ and $n/c_n \to \infty$. Note that if $f(t,x) = f(x)$, L_n and M_n reduce to the forms of the functionals discussed in Wang and Phillips (2009a) and Jeganathan (2004) respectively.

Before stating the main result of the section, there are three crucial lemmas. One of them is the existing one in the literature, while the other two are new and rigorously proved. we introduce the following notations for any $\epsilon > 0$ and $0 \le r \le 1$,

$$L_n^{(r)} = \frac{c_n}{n} \sum_{k=1}^{[nr]} f\left(\frac{k}{n}, c_n x_{k,n}\right)$$

$$L_{n,\epsilon}^{(r)} = \frac{c_n}{n} \sum_{k=1}^{[nr]} \int_{-\infty}^{\infty} f\left(\frac{k}{n}, c_n(x_{k,n} + z\epsilon)\right) \phi(z) dz,$$

where $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$.

For later use in this section we also define $\phi_\epsilon(z) = \frac{1}{\sqrt{2\pi\epsilon}} \exp\left(-\frac{z^2}{2\epsilon^2}\right)$ for some $\epsilon > 0$.

Lemma 1.2.1. *Let Assumptions B (a) and (c) hold. We may represent* U_n^0 *introduced in* (1.1.7) *as*

$$U_n^0\left(\frac{t}{n}\right) = U\left(\frac{\tau_{nt}}{n}\right),$$

with an increasing sequence of stopping times τ_{nt} *in* $(\Omega, \mathcal{F}, P)$ *with* $\tau_{n0} = 0$ *such that as* $n \to \infty$

$$\sup_{1 \le t \le n} \left| \frac{\tau_{nt} - t}{n^\delta} \right| \to_{a.s.} 0, \tag{1.2.1}$$

for any $\delta > \max\{\frac{1}{2}, \frac{2}{p}\}$*, where* p *is the moment exponent in Assumption B for* $\{e_t\}$*.*

This lemma is exactly Lemma 2.1 in Park and Phillips (2001). Readers can find the proof there.

Lemma 1.2.2. *Suppose that Assumptions A (c) and C hold. Then*

$$\lim_{\epsilon \to 0} \lim_{n \to \infty} \sup_{0 \le r \le 1} E|L_n^{(r)} - L_{n,\epsilon}^{(r)}| = 0. \tag{1.2.2}$$

Proof. The proof consists of two parts according to $x_{k,n}$ being continuous and discrete respectively in A (c).

The following arguments about the continuous case naturally treat those used for the univariate case in Wang and Phillips (2009a) as a special case.

Denote $Y_{k,n}(z) = f\left(\frac{k}{n}, c_n x_{k,n}\right) - f\left(\frac{k}{n}, c_n(x_{k,n} + z\epsilon)\right)$. We have

$$\begin{aligned} \sup_{0 \le r \le 1} E|L_n^{(r)} - L_{n,\epsilon}^{(r)}| &= \sup_{0 \le r \le 1} E \left| \frac{c_n}{n} \int_{-\infty}^{\infty} \sum_{k=1}^{[nr]} Y_{k,n}(z) \phi(z) dz \right| \\ &\le \frac{c_n}{n} \int_{-\infty}^{\infty} \sup_{0 \le r \le 1} E \left| \sum_{k=1}^{[nr]} Y_{k,n}(z) \right| \phi(z) dz, \end{aligned}$$

by the fact that $\int \phi(z) dz = 1$. Notice that, by Assumption A (c),

$$E|Y_{k,n}(z)| = \int_{-\infty}^{\infty} \left| f\left(\frac{k}{n}, c_n d_{k,0,n} x\right) - f\left(\frac{k}{n}, c_n d_{k,0,n} x + c_n z\epsilon\right) \right| h_{k,0,n}(x) dx$$

$$\leq \frac{K}{c_n d_{k,0,n}} \left[\int_{-\infty}^{\infty} \left| f\left(\frac{k}{n}, x\right) \right| dx + \int_{-\infty}^{\infty} \left| f\left(\frac{k}{n}, x + c_n z\epsilon\right) \right| dx \right]$$
$$= \frac{2K}{c_n d_{k,0,n}} G_2\left(\frac{k}{n}\right), \tag{1.2.3}$$

where $G_2(\cdot) = \int_{-\infty}^{\infty} |f(\cdot, x)| dx$ and K is the uniform upper bound of the density $h_{l,k,n}$. Accordingly, for each $z \in \mathbb{R}$,

$$\frac{c_n}{n} \sup_{0 \leq r \leq 1} E \left| \sum_{k=1}^{[nr]} Y_{k,n}(z) \right| \leq \frac{c_n}{n} \sum_{k=1}^{n} \frac{2K}{c_n d_{k,0,n}} G_2\left(\frac{k}{n}\right) = 2KK_2 \frac{1}{n} \sum_{t=1}^{n} \frac{1}{d_{k,0,n}} < \infty$$

by virtue of (1.1.4), where $K_2 = \sup_{t \in [0,1]} G_2(t) < \infty$ due to the continuity of $G_2(t)$. It therefore follows from the dominated convergence theorem that, to prove the lemma, it suffices to show that for any fixed z,

$$\Lambda_n(\epsilon) = \frac{c_n^2}{n^2} \sup_{0 \leq r \leq 1} E \left[\sum_{k=1}^{[nr]} Y_{k,n}(z) \right]^2 \to 0,$$

as $n \to \infty$ first and then $\epsilon \to 0$. Meanwhile, we have

$$\Lambda_n(\epsilon) \leq \frac{c_n^2}{n^2} \sum_{k=1}^{n} EY_{k,n}^2(z) + \frac{2c_n^2}{n^2} \sum_{k=1}^{n-1} \sum_{l=k+1}^{n} |E[Y_{k,n}(z) Y_{l,n}(z)]|$$
$$:= \Lambda_{1n}(\epsilon) + \Lambda_{2n}(\epsilon).$$

We next investigate $\Lambda_{1n}(\epsilon)$ and $\Lambda_{2n}(\epsilon)$ separately.

In view of Assumption A (c), we have as $n \to \infty$

$$\Lambda_{1n}(\epsilon) = \frac{c_n^2}{n^2} \sum_{k=1}^{n} EY_{k,n}^2(z) = \frac{c_n^2}{n^2} \sum_{k=1}^{n} E\left[f\left(\frac{k}{n}, c_n x_{k,n}\right) - f\left(\frac{k}{n}, c_n(x_{k,n} + z\epsilon)\right) \right]^2$$
$$= \frac{c_n^2}{n^2} \sum_{k=1}^{n} \int_{-\infty}^{\infty} \left[f\left(\frac{k}{n}, c_n d_{k,0,n} x\right) - f\left(\frac{k}{n}, c_n d_{k,0,n} x + c_n z\epsilon\right) \right]^2 h_{k,0,n}(x) dx$$
$$\leq \frac{c_n^2}{n^2} \sum_{k=1}^{n} \frac{K}{c_n d_{k,0,n}} \int_{-\infty}^{\infty} \left[f\left(\frac{k}{n}, x\right) - f\left(\frac{k}{n}, x + c_n z\epsilon\right) \right]^2 dx$$
$$\leq \frac{4Kc_n}{n^2} \sum_{k=1}^{n} \frac{1}{d_{k,0,n}} \int_{-\infty}^{\infty} \left| f\left(\frac{k}{n}, x\right) \right|^2 dx = \frac{4Kc_n}{n^2} \sum_{k=1}^{n} \frac{1}{d_{k,0,n}} G_3\left(\frac{k}{n}\right)$$
$$\leq 4KK_3 \frac{c_n}{n} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{d_{k,0,n}} \to 0,$$

where $K_3 = \sup_{t\in[0,1]} G_3(t)$ and $G_3(\cdot)$ is continuous on the interval in question.

We then prove that $\Lambda_{2n}(\epsilon) \to 0$ as $n \to \infty$. Because

$$\Lambda_{2n}(\epsilon) = \frac{2c_n^2}{n^2}\sum_{k=1}^{n-1}\sum_{l=k+1}^{n} |E[Y_{k,n}(z)Y_{l,n}(z)]|$$
$$= \frac{2c_n^2}{n^2}\sum_{k=1}^{n-1}\sum_{l=k+1}^{n} |E[Y_{k,n}(z)E(Y_{l,n}(z)|\mathcal{F}_{k,n})]|,$$

For $k < l$, we begin with the following calculation of the conditional expectation:

$$|E(Y_{l,n}(z)|\mathcal{F}_{k,n})| = \left|E\left[f\left(\frac{l}{n}, c_n x_{l,n}\right) - f\left(\frac{l}{n}, c_n(x_{l,n}+z\epsilon)\right)\Big|\mathcal{F}_{k,n}\right]\right|$$
$$=\left|E\left[f\left(\frac{l}{n}, c_n x_{k,n} + c_n(x_{l,n}-x_{k,n})\right) - f\left(\frac{l}{n}, c_n x_{k,n} + c_n(x_{l,n}-x_{k,n}) + c_n z\epsilon\right)\Big|\mathcal{F}_{k,n}\right]\right|$$
$$=\left|\int_{-\infty}^{\infty}\left[f\left(\frac{l}{n}, c_n x_{k,n} + c_n d_{l,k,n} y\right) - f\left(\frac{l}{n}, c_n x_{k,n} + c_n d_{l,k,n} y + c_n z\epsilon\right)\right] h_{l,k,n}(y)dy\right|$$
$$=\frac{1}{c_n d_{l,k,n}}\left|\int_{-\infty}^{\infty}\left[f\left(\frac{l}{n}, y\right) h_{l,k,n}\left(\frac{y - c_n x_{k,n}}{c_n d_{l,k,n}}\right) - f\left(\frac{l}{n}, y\right) h_{l,k,n}\left(\frac{y - c_n x_{k,n} - c_n\epsilon z}{c_n d_{l,k,n}}\right)\right] dy\right|$$
$$=\frac{1}{c_n d_{l,k,n}}\left|\int_{-\infty}^{\infty} f\left(\frac{l}{n}, y\right)\left[h_{l,k,n}\left(\frac{y - c_n x_{k,n}}{c_n d_{l,k,n}}\right) - h_{l,k,n}\left(\frac{y - c_n x_{k,n} - c_n\epsilon z}{c_n d_{l,k,n}}\right)\right] dy\right|$$
$$\leq\frac{1}{c_n d_{l,k,n}}\int_{-\infty}^{\infty}\left|f\left(\frac{l}{n}, y\right)\right| |V(y, c_n x_{k,n})|\, dy,$$

where $V(y, c_n x_{k,n}) = h_{l,k,n}\left(\frac{y-c_n x_{k,n}}{c_n d_{l,k,n}}\right) - h_{l,k,n}\left(\frac{y-c_n x_{k,n}-c_n\epsilon z}{c_n d_{l,k,n}}\right)$.

Recall the definition of $\Omega_n(\epsilon)$ in Assumption A(c) and note that a pair (l,k) $(l > k)$ belongs to either $\Omega_n(\epsilon^{1/2m_0})$ or its complement. It follows that

$$|E(Y_{l,n}(z)|\mathcal{F}_{k,n})|$$
$$\leq \begin{cases} \frac{2K}{c_n d_{l,k,n}}\int_{-\infty}^{\infty}\left|f\left(\frac{l}{n}, y\right)\right| dy = \frac{2K}{c_n d_{l,k,n}} G_2\left(\frac{l}{n}\right), & \text{if } (l,k) \notin \Omega_n, \\ \frac{2K}{c_n d_{l,k,n}}\int_{|y|>\sqrt{c_n}}\left|f\left(\frac{l}{n}, y\right)\right| dy + \frac{1}{c_n d_{l,k,n}}\int_{|y|\leq\sqrt{c_n}}\left|f\left(\frac{l}{n}, y\right)\right| |V(y, c_n x_{k,n})| dy, & \text{otherwise.}\end{cases}$$

According to Assumption A (c), $\inf_{(l,k)\in\Omega_n(\epsilon^{1/2m_0})} d_{l,k,n} \geq \frac{\sqrt{\epsilon}}{C}$, and at the same time we can choose n large enough such that $\sqrt{c_n}\epsilon > 1$. For $|y| \leq \sqrt{c_n}$ and $|x| \leq \sqrt{c_n} + c_n|z|\epsilon$, when $(l,k) \in \Omega_n(\epsilon^{1/2m_0})$, we have

$$|V(y,x)| = \left|h_{l,k,n}\left(\frac{y-x}{c_n d_{l,k,n}}\right) - h_{l,k,n}\left(\frac{y-x-c_n\epsilon z}{c_n d_{l,k,n}}\right)\right|$$
$$\leq \left|h_{l,k,n}\left(\frac{y-x}{c_n d_{l,k,n}}\right) - h_{l,k,n}(0)\right| + \left|h_{l,k,n}\left(\frac{y-x-c_n\epsilon z}{c_n d_{l,k,n}}\right) - h_{l,k,n}(0)\right|$$

$$\leq 2 \sup_{|u|<2C(1+|z|)\sqrt{\epsilon}} |h_{l,k,n}(u) - h_{l,k,n}(0)| . \tag{1.2.4}$$

Therefore, when $|y| \leq \sqrt{c_n}$, n is large enough and $(l,k) \in \Omega_n(\epsilon^{1/2m_0})$, we have

$$\begin{aligned}
&E|Y_{k,n}(z)||V(y, c_n x_{k,n})| \\
=&\int_{-\infty}^{\infty} \left| f\left(\frac{k}{n}, c_n d_{k,0,n} x\right) - f\left(\frac{k}{n}, c_n d_{k,0,n} x + c_n z\epsilon\right)\right| |V(y, c_n d_{k,0,n} x)| h_{k,0,n}(x) dx \\
\leq&\frac{K}{c_n d_{k,0,n}} \int_{-\infty}^{\infty} \left| f\left(\frac{k}{n}, x\right) - f\left(\frac{k}{n}, x + c_n z\epsilon\right)\right| |V(y,x)| dx \\
\leq&\frac{K}{c_n d_{k,0,n}} \left[\int_{-\infty}^{\infty} \left| f\left(\frac{k}{n}, x\right)\right| |V(y,x)| dx + \int_{-\infty}^{\infty} \left| f\left(\frac{k}{n}, x + c_n z\epsilon\right)\right| |V(y,x)| dx\right] \\
=&\frac{K}{c_n d_{k,0,n}} \int_{-\infty}^{\infty} \left| f\left(\frac{k}{n}, x\right)\right| [|V(y,x)| + |V(y, x - c_n z\epsilon)|] dx \\
=&\frac{K}{c_n d_{k,0,n}} \left[\int_{|x|>\sqrt{c_n}} + \int_{|x|\leq\sqrt{c_n}} \left| f\left(\frac{k}{n}, x\right)\right| [|V(y,x)| + |V(y, x - c_n z\epsilon)|] dx\right] \\
\leq&\frac{2K^2}{c_n d_{k,0,n}} \int_{|x|>\sqrt{c_n}} \left| f\left(\frac{k}{n}, x\right)\right| dx \\
&+ \frac{K}{c_n d_{k,0,n}} \int_{|x|\leq\sqrt{c_n}} \left| f\left(\frac{k}{n}, x\right)\right| [|V(y,x)| + |V(y, x - c_n z\epsilon)|] dx \\
\leq&\frac{2K^2}{c_n d_{k,0,n}} \int_{|x|>\sqrt{c_n}} \left| f\left(\frac{k}{n}, x\right)\right| dx \\
&+ \frac{4K}{c_n d_{k,0,n}} \sup_{|u|<2C(1+|z|)\sqrt{\epsilon}} |h_{l,k,n}(u) - h_{l,k,n}(0)| \int_{|x|\leq\sqrt{c_n}} \left| f\left(\frac{k}{n}, x\right)\right| dx.
\end{aligned}$$

We summarise that if $(l,k) \notin \Omega_n$, equation (1.2.3) yields

$$\begin{aligned}
|E(Y_{k,n}(z) Y_{l,n}(z))| &= |E[Y_{k,n}(z) E(Y_{l,n}(z)|\mathcal{F}_{k,n})]| \\
&\leq \frac{2K}{c_n d_{l,k,n}} G_2\left(\frac{l}{n}\right) |EY_{k,n}(z)| \\
&\leq \frac{4K^2}{c_n^2 d_{l,k,n} d_{k,0,n}} G_2\left(\frac{l}{n}\right) G_2\left(\frac{k}{n}\right),
\end{aligned}$$

while if $(l,k) \in \Omega_n$,

$$\begin{aligned}
&|E(Y_{k,n}(z) Y_{l,n}(z))| = |E[Y_{k,n}(z) E(Y_{l,n}(z)|\mathcal{F}_{k,n})]| \\
&\leq E[|Y_{k,n}(z)||E(Y_{l,n}(z)|\mathcal{F}_{k,n})|] \\
\leq&\frac{2K}{c_n d_{l,k,n}} \int_{|y|>\sqrt{c_n}} \left| f\left(\frac{l}{n}, y\right)\right| dy E|Y_{k,n}(z)|
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{c_n d_{l,k,n}}\int_{|y|\le\sqrt{c_n}}\left|f\left(\frac{l}{n},y\right)\right|E[|Y_{k,n}(z)||V(y,c_nx_{k,n})|]dy\\
\le&\frac{4K^2}{c_n^2 d_{l,k,n}d_{k,0,n}}G_2\left(\frac{k}{n}\right)\int_{|y|>\sqrt{c_n}}\left|f\left(\frac{l}{n},y\right)\right|dy\\
&+\frac{2K^2}{c_n^2 d_{l,k,n}d_{k,0,n}}\int_{|y|\le\sqrt{c_n}}\left|f\left(\frac{l}{n},y\right)\right|dy\int_{|x|>\sqrt{c_n}}\left|f\left(\frac{k}{n},x\right)\right|dx\\
&+\frac{4K}{c_n^2 d_{l,k,n}d_{k,0,n}}\int_{|y|\le\sqrt{c_n}}\left|f\left(\frac{l}{n},y\right)\right|dy\int_{|x|\le\sqrt{c_n}}\left|f\left(\frac{k}{n},x\right)\right|dx\\
&\times\sup_{|u|<2C(1+|z|)\sqrt{\epsilon}}\left|h_{l,k,n}(u)-h_{l,k,n}(0)\right|.
\end{aligned}
$$

Finally, we have

$$
\begin{aligned}
|\Lambda_{2n}(\epsilon)|\le&\frac{2c_n^2}{n^2}\left(\sum_{l>k,\ (l,k)\notin\Omega_n}+\sum_{(l,k)\in\Omega_n}\right)E|Y_{s,n}(z)Y_{t,n}(z)|\\
\le&\frac{2c_n^2}{n^2}\sum_{k=(1-\epsilon)n}^{n}\sum_{l=k+1}^{n}E|Y_{k,n}(z)Y_{l,n}(z)|+\frac{2c_n^2}{n^2}\sum_{k=1}^{n-1}\sum_{l=k+1}^{k+\epsilon n}E|Y_{k,n}(z)Y_{l,n}(z)|\\
&+\frac{2c_n^2}{n^2}\sum_{k=1}^{\epsilon n}\sum_{l=k+1}^{n}E|Y_{k,n}(z)Y_{l,n}(z)|+\frac{2c_n^2}{n^2}\sum_{k=1}^{n-1}\sum_{l=k+\epsilon n}^{n}E|Y_{k,n}(z)Y_{l,n}(z)|\\
\le&8K^2K_2^2\frac{1}{n}\sum_{k=(1-\epsilon)n}^{n}\frac{1}{d_{k,0,n}}\max_{1\le k\le n-1}\frac{1}{n}\sum_{l=k+1}^{n}\frac{1}{d_{l,k,n}}\\
&+8K^2K_2^2\frac{1}{n}\sum_{k=1}^{n-1}\frac{1}{d_{k,0,n}}\max_{1\le k\le n-1}\frac{1}{n}\sum_{l=k+1}^{k+\epsilon n}\frac{1}{d_{l,k,n}}\\
&+8K^2K_2^2\frac{1}{n}\sum_{k=1}^{\epsilon n}\frac{1}{d_{k,0,n}}\max_{1\le k\le n-1}\frac{1}{n}\sum_{l=k+1}^{n}\frac{1}{d_{l,k,n}}\\
&+8K^2K_2\int_{|y|>\sqrt{c_n}}c_f(y)dy\frac{1}{n^2}\sum_{k=1}^{n-1}\frac{1}{d_{k,0,n}}\max_{1\le k\le n-1}\sum_{l=k+1}^{n}\frac{1}{d_{l,k,n}}\\
&+4K^2K_2\int_{|x|>\sqrt{c_n}}c_f(x)dx\frac{1}{n^2}\sum_{k=1}^{n-1}\frac{1}{d_{k,0,n}}\max_{1\le k\le n-1}\sum_{l=k+1}^{n}\frac{1}{d_{l,k,n}}\\
&+8KK_2^2\frac{1}{n^2}\sum_{k=1}^{n-1}\frac{1}{d_{k,0,n}}\max_{1\le k\le n-1}\sum_{l=k+1}^{n}\frac{1}{d_{l,k,n}}\sup_{|u|<2C(1+|z|)\sqrt{\epsilon}}\left|h_{l,k,n}(u)-h_{l,k,n}(0)\right|,
\end{aligned}
$$

in which we have used Assumption C (c) that $\left|f\left(\frac{l}{n},y\right)\right|\le c_f(y)$ and the fact that

$$
\int_{|y|\le\sqrt{c_n}}\left|f\left(\frac{l}{n},y\right)\right|dy\le G_2\left(\frac{l}{n}\right)\le K_2.
$$

In view of Assumptions A (c) and C, by virtue of the dominated convergence theorem, $\Lambda_{2n}(\epsilon) \to 0$ as $n \to \infty$ and then $\epsilon \to 0$. This finishes the proof of the continuous case.

The proof of the discrete case is quite similar to that of the continuous case. Some critical steps are shown as follows.

Let $\mathcal{A}_{k,n}$ be the set of points that $x_{k,n}$ assumes. Suppose the points are equally distributed on $\mathbb{R}$ with distance $\triangle$. In what follows, define $\mathcal{B}_{k,n} := c_n d_{k,0,n}\mathcal{A}_{k,n} := \{c_n d_{k,0,n} a : a \in \mathcal{A}_{k,n}\}$. Then,

$$\begin{aligned}
E|Y_{k,n}(z)| =& E\left|f\left(\frac{k}{n}, c_n x_{k,n}\right) - f\left(\frac{k}{n}, c_n(x_{k,n} + z\epsilon)\right)\right| \\
=& \sum_{x \in \mathcal{A}_{k,n}} \left|f\left(\frac{k}{n}, c_n d_{k,0,n} x\right) - f\left(\frac{k}{n}, c_n(d_{k,0,n} x + z\epsilon)\right)\right| P_{k,0,n}(x) \\
=& \sum_{x \in \mathcal{B}_{k,n}} \left|f\left(\frac{k}{n}, x\right) - f\left(\frac{k}{n}, x + c_n z\epsilon\right)\right| P_{k,0,n}\left(\frac{x}{c_n d_{k,0,n}}\right) \\
\leq& \sum_{x \in \mathcal{B}_{k,n}} \left|f\left(\frac{k}{n}, x\right)\right| + \sum_{x \in \mathcal{B}_{k,n}} \left|f\left(\frac{k}{n}, x + c_n z\epsilon\right)\right| \\
=& \frac{1}{c_n d_{k,0,n}\triangle} \sum_{x \in \mathcal{B}_{k,n}} \left|f\left(\frac{k}{n}, x\right)\right| c_n d_{k,0,n}\triangle \\
&+ \frac{1}{c_n d_{k,0,n}\triangle} \sum_{x \in \mathcal{B}_{k,n}} \left|f\left(\frac{k}{n}, x + c_n z\epsilon\right)\right| c_n d_{k,0,n}\triangle \\
\leq& \frac{1}{c_n d_{k,0,n}\triangle} \left(\int \left|f\left(\frac{k}{n}, x\right)\right| dx + \int \left|f\left(\frac{k}{n}, x + c_n z\epsilon\right)\right| dx\right) \\
=& \frac{2}{c_n d_{k,0,n}\triangle} \int \left|f\left(\frac{k}{n}, x\right)\right| dx = \frac{2}{c_n d_{k,0,n}\triangle} G_2\left(\frac{k}{n}\right) \\
\leq& \frac{2}{c_n d_{k,0,n}\triangle} K_2,
\end{aligned}$$

where we may modify the function f, e.g. $f^o(\cdot, x) = \max_{y \geq x} |f(\cdot, y)|$ for $x > 0$ to get the inequality in the derivation and note that the result above is similar to (1.2.3). Following the same arguments as before, to complete the proof, it suffices to show both $\Lambda_{1n}(\epsilon)$ and $\Lambda_{2n}(\epsilon)$ converge to zero. Nevertheless, $\Lambda_{1n}(\epsilon) \to 0$ is easy to obtain, while the key step in the proof of $\Lambda_{2n}(\epsilon) \to 0$ is the evaluation of the following conditional expectation.

$$\begin{aligned}
|E(Y_{l,n}(z)|\mathcal{F}_{k,n})| &= \left|E\left[f\left(\frac{l}{n}, c_n x_{l,n}\right) - f\left(\frac{l}{n}, c_n(x_{l,n} + z\epsilon)\right) \Big| \mathcal{F}_{k,n}\right]\right| \\
= &\left|E\left[f\left(\frac{l}{n}, c_n x_{k,n} + c_n(x_{l,n} - x_{k,n})\right) - f\left(\frac{l}{n}, c_n x_{k,n} + c_n(x_{l,n} - x_{k,n}) + c_n z\epsilon\right) \Big| \mathcal{F}_{k,n}\right]\right|
\end{aligned}$$

$$
\begin{aligned}
&= \left| \int \left[f\left(\frac{l}{n}, c_n x_{k,n} + c_n d_{l,k,n} y\right) - f\left(\frac{l}{n}, c_n x_{k,n} + c_n d_{l,k,n} y + c_n z\epsilon\right)\right] dF_{l,k,n}(y) \right| \\
&= \left| \int f\left(\frac{l}{n}, y\right) dF_{l,k,n}\left(\frac{y - c_n x_{k,n}}{c_n d_{l,k,n}}\right) - \int f\left(\frac{l}{n}, y\right) dF_{l,k,n}\left(\frac{y - c_n x_{k,n} - c_n \epsilon z}{c_n d_{l,k,n}}\right) \right| \\
&= \left| \int f\left(\frac{l}{n}, y\right) d\left[F_{l,k,n}\left(\frac{y - c_n x_{k,n}}{c_n d_{l,k,n}}\right) - F_{l,k,n}\left(\frac{y - c_n x_{k,n} - c_n \epsilon z}{c_n d_{l,k,n}}\right)\right] \right| \\
&= \left| \int f\left(\frac{l}{n}, y\right) dQ(y, c_n x_{k,n}) \right|,
\end{aligned}
$$

where $Q(y, c_n x_{k,n}) = F_{l,k,n}\left(\frac{y - c_n x_{k,n}}{c_n d_{l,k,n}}\right) - F_{l,k,n}\left(\frac{y - c_n x_{k,n} - c_n \epsilon z}{c_n d_{l,k,n}}\right)$.

Thus,

$$
\begin{aligned}
&|E(Y_{l,n}(z)|\mathcal{F}_{k,n})| \\
&\leq \begin{cases} \frac{2}{c_n d_{l,k,n} \triangle} \int \left| f\left(\frac{l}{n}, y\right)\right| dy = \frac{2}{c_n d_{l,k,n}} G_2\left(\frac{l}{n}\right), & \text{if } (l,k) \notin \Omega_n, \\ \frac{2}{c_n d_{l,k,n} \triangle} \int_{|y| > \sqrt{c_n}} \left| f\left(\frac{l}{n}, y\right)\right| dy + \left| \int_{|y| \leq \sqrt{c_n}} f\left(\frac{l}{n}, y\right) dQ(y, c_n x_{k,n})\right|, & \text{if } (l,k) \in \Omega_n. \end{cases}
\end{aligned}
$$

Then the important ingredient is to deal with the following expectation.

$$
\begin{aligned}
& E|Y_{k,n}| \left| \int_{|y| \leq \sqrt{c_n}} f\left(\frac{l}{n}, y\right) dQ(y, c_n x_{k,n}) \right| \\
=& E \left| f\left(\frac{k}{n}, c_n x_{k,n}\right) - f\left(\frac{k}{n}, c_n (x_{k,n} + z\epsilon)\right) \right| \left| \int_{|y| \leq \sqrt{c_n}} f\left(\frac{l}{n}, y\right) dQ(y, c_n x_{k,n}) \right| \\
=& \int \left| f\left(\frac{k}{n}, c_n d_{k,0,n} x\right) - f\left(\frac{k}{n}, c_n (d_{k,0,n} x + z\epsilon)\right) \right| \\
& \times \left| \int_{|y| \leq \sqrt{c_n}} f\left(\frac{l}{n}, y\right) dQ(y, c_n d_{k,0,n} x) \right| dF_{k,0,n}(x) \\
=& \int \left| f\left(\frac{k}{n}, x\right) - f\left(\frac{k}{n}, x + c_n z\epsilon\right) \right| \left| \int_{|y| \leq \sqrt{c_n}} f\left(\frac{l}{n}, y\right) dQ(y, x) \right| dF_{k,0,n}\left(\frac{x}{c_n d_{k,0,n}}\right) \\
\leq& \int \left| f\left(\frac{k}{n}, x\right) \right| \left| \int_{|y| \leq \sqrt{c_n}} f\left(\frac{l}{n}, y\right) dQ(y, x) \right| dF_{k,0,n}\left(\frac{x}{c_n d_{k,0,n}}\right) \\
& + \int \left| f\left(\frac{k}{n}, x + c_n z\epsilon\right) \right| \left| \int_{|y| \leq \sqrt{c_n}} f\left(\frac{l}{n}, y\right) dQ(y, x) \right| dF_{k,0,n}\left(\frac{x}{c_n d_{k,0,n}}\right) \\
=& \int \left| f\left(\frac{k}{n}, x\right) \right| \left| \int_{|y| \leq \sqrt{c_n}} f\left(\frac{l}{n}, y\right) dQ(y, x) \right| dF_{k,0,n}\left(\frac{x}{c_n d_{k,0,n}}\right) \\
& + \int \left| f\left(\frac{k}{n}, x\right) \right| \left| \int_{|y| \leq \sqrt{c_n}} f\left(\frac{l}{n}, y\right) dQ(y, x - c_n z\epsilon) \right| dF_{k,0,n}\left(\frac{x - c_n z\epsilon}{c_n d_{k,0,n}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{|x|\le\sqrt{c_n}} \left| f\left(\frac{k}{n}, x\right)\right| \left| \int_{|y|\le\sqrt{c_n}} f\left(\frac{l}{n}, y\right) dQ(y,x) \right| dF_{k,0,n}\left(\frac{x}{c_n d_{k,0,n}}\right) \\
&\quad + \int_{|x|>\sqrt{c_n}} \left| f\left(\frac{k}{n}, x\right)\right| \left| \int_{|y|\le\sqrt{c_n}} f\left(\frac{l}{n}, y\right) dQ(y,x) \right| dF_{k,0,n}\left(\frac{x}{c_n d_{k,0,n}}\right) \\
&\quad + \int_{|x|\le\sqrt{c_n}} \left| f\left(\frac{k}{n}, x\right)\right| \left| \int_{|y|\le\sqrt{c_n}} f\left(\frac{l}{n}, y\right) dQ(y,x-c_n z\epsilon) \right| dF_{k,0,n}\left(\frac{x-c_n z\epsilon}{c_n d_{k,0,n}}\right) \\
&\quad + \int_{|x|>\sqrt{c_n}} \left| f\left(\frac{k}{n}, x\right)\right| \left| \int_{|y|\le\sqrt{c_n}} f\left(\frac{l}{n}, y\right) dQ(y,x-c_n z\epsilon) \right| dF_{k,0,n}\left(\frac{x-c_n z\epsilon}{c_n d_{k,0,n}}\right) \\
&:= \sum_1^4 T_i(l,k;n).
\end{aligned}
$$

For $T_1(l,k;n)$ is similar to $T_3(l,k;n)$, and $T_2(l,k;n)$ is similar to $T_4(l,k;n)$, we only explain $T_1(l,k;n)$ and $T_2(l,k;n)$.

$$
\begin{aligned}
T_1(l,k;n) &= \int_{|x|\le\sqrt{c_n}} \left| f\left(\frac{k}{n}, x\right)\right| \left| \int_{|y|\le\sqrt{c_n}} f\left(\frac{l}{n}, y\right) dQ(y,x) \right| dF_{k,0,n}\left(\frac{x}{c_n d_{k,0,n}}\right) \\
&= \sum_{\substack{|x|\le\sqrt{c_n}\\ x\in\mathcal{B}_{k,n}}} \left| f\left(\frac{k}{n}, x\right)\right| \left| \sum_{\substack{|y|\le\sqrt{c_n}\\ y\in x+c_n z\epsilon\mathcal{B}_{l,k,n}}} f\left(\frac{l}{n}, y\right) P(y,x) \right| P_{k,0,n}\left(\frac{x}{c_n d_{k,0,n}}\right),
\end{aligned}
$$

where $\mathcal{B}_{l,k,n} = \{c_n d_{l,k,n} a : a \in \mathcal{A}_{l,k,n}\}$, in which $\mathcal{A}_{l,k,n}$ is the set of points that $(x_{l,n} - x_{k,n})/d_{l,k,n}$ assumes; meanwhile, $P(y,x) = P_{l,k,n}\left(\frac{y-c_n x_{k,n}}{c_n d_{l,k,n}}\right) - P_{l,k,n}\left(\frac{y-c_n x_{k,n}-c_n\epsilon z}{c_n d_{l,k,n}}\right)$.

Notice that when $|x| \le \sqrt{c_n}$, $|y| \le \sqrt{c_n}$ and $(l,k) \in \Omega_n(\epsilon)$, we have

$$
\begin{aligned}
|P(y,x)| &= \left| P_{l,k,n}\left(\frac{y-c_n x_{k,n}}{c_n d_{l,k,n}}\right) - P_{l,k,n}\left(\frac{y-c_n x_{k,n}-c_n\epsilon z}{c_n d_{l,k,n}}\right)\right| \\
&= \left| F_{l,k,n}\left(\frac{y-c_n x_{k,n}}{c_n d_{l,k,n}}\right) - F^-_{l,k,n}\left(\frac{y-c_n x_{k,n}}{c_n d_{l,k,n}}\right)\right. \\
&\qquad \left. - F_{l,k,n}\left(\frac{y-c_n x_{k,n}-c_n\epsilon z}{c_n d_{l,k,n}}\right) + F^-_{l,k,n}\left(\frac{y-c_n x_{k,n}-c_n\epsilon z}{c_n d_{l,k,n}}\right)\right| \\
&\le 4 \sup_{|u|<2C(1+|z|)\sqrt{\epsilon}} |F_{l,k,n}(u) - F_{l,k,n}(0)|.
\end{aligned}
$$

Here $F^-_{l,k,n}(\cdot)$ denotes the left limit of the function at the point.

Therefore, we have

$$
T_1(l,k;n) \le \frac{4}{c_n d_{k,0,n}\triangle} \frac{1}{c_n d_{l,k,n}\triangle} G_2\left(\frac{k}{n}\right) G_2\left(\frac{l}{n}\right) \sup_{|u|<2C(1+|z|)\sqrt{\epsilon}} |F_{l,k,n}(u) - F_{l,k,n}(0)|.
$$

Regarding $T_2(l,k;n)$, we directly have

$$\begin{aligned}T_2(l,k;n) \leq & \frac{2}{c_n d_{k,0,n}\triangle}\frac{1}{c_n d_{l,k,n}\triangle}\int_{|x|\geq\sqrt{c_n}}\left|f\left(\frac{k}{n},x\right)\right|dx\int_{|y|\leq\sqrt{c_n}}\left|f\left(\frac{k}{n},y\right)\right|dy\\ \leq & \frac{2}{c_n d_{k,0,n}\triangle}\frac{1}{c_n d_{l,k,n}\triangle}G_2\left(\frac{l}{n}\right)\int_{|x|\geq\sqrt{c_n}}\left|f\left(\frac{k}{n},x\right)\right|dx.\end{aligned}$$

As can be seen, every term in $\Lambda_{2n}(\epsilon)$ has the similar evaluation, so that we obtain the vanish of $\Lambda_{2n}(\epsilon)$. As yet, the whole proof is finished. □

Lemma 1.2.3. *Let Assumption C hold. Then we have for any fixed $\epsilon > 0$,*

$$L_{n,\epsilon}^{(r)} - \int_{-\infty}^{\infty}\frac{1}{n}\sum_{k=1}^{[nr]} f\left(\frac{k}{n},y\right)\phi_\epsilon(x_{k,n})dy \to_{a.s.} 0$$

uniformly in $r\in[0,1]$ as $n\to\infty$.

Proof. Observe that

$$\begin{aligned}L_{n,\epsilon}^{(r)} &= \frac{c_n}{n}\sum_{k=1}^{[nr]}\int_{-\infty}^{\infty} f\left(\frac{k}{n}, c_n(x_{t,n}+z\epsilon)\right)\phi(z)dz\\ &= \frac{c_n}{n}\sum_{k=1}^{[nr]}\int_{-\infty}^{\infty} f\left(\frac{k}{n},y\right)\phi\left(\frac{y-c_n x_{t,n}}{c_n\epsilon}\right)\frac{1}{c_n\epsilon}dy\\ &= \int_{-\infty}^{\infty}\frac{1}{n}\sum_{k=1}^{[nr]} f\left(\frac{k}{n},y\right)\phi_\epsilon\left(\frac{y}{c_n}-x_{t,n}\right)dy.\end{aligned}$$

It follows that for any $M>0$,

$$\begin{aligned}&\left|L_{n,\epsilon}^{(r)} - \int_{-\infty}^{\infty}\frac{1}{n}\sum_{k=1}^{[nr]} f\left(\frac{k}{n},y\right)\phi_\epsilon(x_{k,n})dy\right|\\ \leq & \int_{-\infty}^{\infty}\frac{1}{n}\sum_{k=1}^{n}\left|f\left(\frac{k}{n},y\right)\right|\left|\phi_\epsilon\left(\frac{y}{c_n}-x_{t,n}\right)-\phi_\epsilon(x_{k,n})\right|dy\\ = & \int_{|y|>M}+\int_{|y|\leq M}\frac{1}{n}\sum_{k=1}^{n}\left|f\left(\frac{k}{n},y\right)\right|\left|\phi_\epsilon\left(\frac{y}{c_n}-x_{t,n}\right)-\phi_\epsilon(x_{k,n})\right|dy\\ := & \Gamma_{1n}+\Gamma_{2n}.\end{aligned}$$

Notice that,

$$\Gamma_{1n} \leq \frac{2}{\sqrt{2\pi}\epsilon}\int_{|y|>M}\frac{1}{n}\sum_{k=1}^{n}\left|f\left(\frac{k}{n},y\right)\right|dy \leq \frac{2}{\sqrt{2\pi}\epsilon}\int_{|y|>M}c_f(y)dy,$$

using Assumption C (c). Due to the integrability of $c_f(y)$ on $\mathbb{R}$, one can choose large enough M such that $\Gamma_{1n} < \varepsilon$ for any given $\varepsilon > 0$.

Moreover, since $\phi'_\epsilon(x) = -\frac{x}{\sqrt{2\pi\epsilon^3}} e^{-x^2/2\epsilon^2}$ and $|\phi'_\epsilon(x)|$ is bounded by $\frac{1}{\sqrt{2\pi e\epsilon^2}}$ on $\mathbb{R}$, we have

$$\left|\phi_\epsilon\left(\frac{y}{c_n} - x_{t,n}\right) - \phi_\epsilon(x_{t,n})\right| = \left|\phi'_\epsilon(\xi)\left(-\frac{y}{c_n}\right)\right| \le \frac{|y|}{\sqrt{2\pi e\epsilon^2 c_n}},$$

where ξ is in between $x_{t,n} - \frac{y}{c_n}$ and $x_{t.n}$. Therefore,

$$\begin{aligned}\Gamma_{2n} &\le \int_{|y|\le M} \frac{1}{n}\sum_{k=1}^{n}\left|f\left(\frac{k}{n}, y\right)\right| \frac{|y|}{\sqrt{2\pi e\epsilon^2 c_n}} dy \\ &\le \frac{M}{\sqrt{2\pi e\epsilon^2 c_n}} \frac{1}{n}\sum_{k=1}^{n}\int_{|y|\le M}\left|f\left(\frac{k}{n}, y\right)\right| dy \le \frac{M}{\sqrt{2\pi e\epsilon^2 c_n}} \frac{1}{n}\sum_{k=1}^{n} G_2\left(\frac{k}{n}\right).\end{aligned}$$

As $\frac{1}{n}\sum_{k=1}^{n} G_2\left(\frac{k}{n}\right) \le K_2$ and $c_n \to \infty$ as $n \to \infty$, $\Gamma_{2n} \to 0$. The assertion follows. □

Theorem 1.2.1. *If Assumptions C and A (a) and (c) hold, we have for any $c_n \to \infty$, $n/c_n \to \infty$ and $r \in [0,1]$,*

$$\frac{c_n}{n}\sum_{s=1}^{[nr]} f\left(\frac{s}{n}, c_n x_{s,n}\right) \to_D \int_0^r G_1(t) dL_W(t,0), \tag{1.2.5}$$

where $G_1(\cdot) = \int f(\cdot, x)dx$ and $L_W(t,0)$ is the local-time process of W at origin over time interval $[0,t]$.

If, in addition, Assumption A (a) is replaced by Assumption A(b), then for any $c_n \to \infty$, $n/c_n \to \infty$, and $r \in [0,1]$,

$$\sup_{0\le r\le 1}\left|\frac{c_n}{n}\sum_{s=1}^{[nr]} f\left(\frac{s}{n}, c_n x_{s,n}\right) - \int_0^r G_1(t) dL_W(t,0)\right| = o_P(1), \tag{1.2.6}$$

under the same probability space as defined in Assumption A(b).

Moreover, suppose that $f^2(t,x)$ satisfies Assumption C, and that $\{e_s\}$ and $\{x_{s,n}\}$ satisfy Assumption B. We have for $n \to \infty$, $c_n \to \infty$ and $c_n/n \to 0$, and $r \in [0,1]$,

$$\sqrt{\frac{c_n}{n}}\sum_{s=1}^{[nr]} f\left(\frac{s}{n}, c_n x_{s,n}\right) e_s \to_D \left(\int_0^r G_3(t) dL_W(t,0)\right)^{\frac{1}{2}} N, \tag{1.2.7}$$

where $G_3(\cdot) = \int f^2(\cdot, x)dx$ and N is a standard normal random variable independent of W.

Remark 1.2.2. Note that if the function $f(t,x)$ reduces to $f(x)$, (1.2.5) and (1.2.6) reduce to Theorem 2.1 of Wang and Phillips (2009a) and with $c_n = \sqrt{n}$ to Theorem 5.1 of Park and Phillips (1999), since $G_1(t) = \int f(x)dx$ becomes a constant and $\int_0^1 dL_W(r,0) = L_W(1,0)$. Also, these reduced cases of (1.2.6) and (1.2.7) can be viewed as a special case of Theorem 3.2 in Park and Phillips (2001) by taking parameter set Π as singleton since in the situation $G_3 = \int f^2(x)dx$ is a constant.

Proof. In view of Lemmas 1.2.2 and 1.2.3, we start to investigate the convergence of

$$\int_{-\infty}^{\infty} \frac{1}{n} \sum_{k=1}^{[nr]} f\left(\frac{k}{n}, y\right) \phi_\epsilon(x_{k,n}) dy.$$

It follows from Assumptions A (a) and C, the continuous mapping theorem and the occupation time formula in Lemma 1.1.3 that

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{1}{n} \sum_{k=1}^{[nr]} f\left(\frac{k}{n}, y\right) \phi_\epsilon(x_{k,n}) dy \\
=&\frac{1}{n} \sum_{k=1}^{[nr]} \phi_\epsilon(x_{k,n}) \int_{-\infty}^{\infty} f\left(\frac{k}{n}, y\right) dy = \frac{1}{n} \sum_{k=1}^{[nr]} \phi_\epsilon(x_{k,n}) G_1\left(\frac{k}{n}\right) \\
=&\int_0^r G_1\left(\frac{[nt]}{n}\right) \phi_\epsilon(x_{[nt],n}) dt - \frac{1}{n} G_1(0)\phi_\epsilon(0) + \frac{1}{n} G_1\left(\frac{[nr]}{n}\right) \phi_\epsilon(x_{[nr],n}) \\
\to_D &\int_0^r G_1(t)\, \phi_\epsilon(W(t)) dt \quad \text{as } n \to \infty \\
=&\int_{-\infty}^{\infty} dy \int_0^r G_1(t) \phi_\epsilon(y) dL_W(t,y) \\
=&\int_{-\infty}^{\infty} dy \int_0^r G_1(t) \phi(y) dL_W(t,\epsilon y) \qquad\qquad \text{and then as } \epsilon \to 0 \\
\to_{a.s.} &\int_{-\infty}^{\infty} dy \int_0^r G_1(t) \phi(y) dL_W(t,0) \\
=&\int_0^r G_1(t) dL_W(t,0).
\end{aligned}$$

This finishes the proof of (1.2.5). To prove (1.2.6), we need only to show the following:

$$\begin{aligned}
&\sup_{0 \le r \le 1} \left| \int_{-\infty}^{\infty} \frac{1}{n} \sum_{k=1}^{[nr]} f\left(\frac{k}{n}, y\right) \phi_\epsilon(x_{k,n}) dy - \int_0^r G_1(t)\, \phi_\epsilon(W(t)) dt \right| \qquad (1.2.8) \\
=&\sup_{0 \le r \le 1} \left| \frac{1}{n} \sum_{k=1}^{[nr]} \phi_\epsilon(x_{k,n}) G_1\left(\frac{k}{n}\right) - \int_0^r G_1(t)\, \phi_\epsilon(W(t)) dt \right|
\end{aligned}$$

$$
\begin{aligned}
&= \sup_{0\le r\le 1}\left|\int_0^r G_1\left(\frac{[nt]}{n}\right)\phi_\epsilon(x_{[nt],n})dt - \frac{1}{n}G_1(0)\phi_\epsilon(0) + \frac{1}{n}G_1\left(\frac{[nr]}{n}\right)\phi_\epsilon(x_{[nr],n})\right.\\
&\qquad\left. - \int_0^r G_1(t)\,\phi_\epsilon(W(t))dt\right|\\
&\le \int_0^1 \left|G_1\left(\frac{[nt]}{n}\right)\phi_\epsilon(x_{[nt],n}) - G_1(t)\phi_\epsilon(W(t))\right| dt + \frac{A}{n},
\end{aligned}
$$

where A comes from the bounds of G_1 on $[0,1]$ and ϕ_ϵ on $\mathbb{R}$. It follows that

$$
\begin{aligned}
&\int_0^1 \left|G_1\left(\frac{[nt]}{n}\right)\phi_\epsilon(x_{[nt],n}) - G_1(t)\phi_\epsilon(W(t))\right| dt\\
&\le \int_0^1 \left|G_1\left(\frac{[nt]}{n}\right) - G_1(t)\right| \left|\phi_\epsilon(x_{[nt],n})\right| dt + \int_0^1 |G_1(t)| \left|\phi_\epsilon(x_{[nt],n}) - \phi_\epsilon(W(t))\right| dt\\
&\le \frac{1}{\sqrt{2\pi\epsilon}}\int_0^1 \left|G_1\left(\frac{[nt]}{n}\right) - G_1(t)\right| dt + \max_{0\le t\le 1}|G_1(t)| \int_0^1 \left|\phi_\epsilon(x_{[nt],n}) - \phi_\epsilon(W(t))\right| dt\\
&\le \frac{1}{\sqrt{2\pi\epsilon}}\int_0^1 \left|G_1\left(\frac{[nt]}{n}\right) - G_1(t)\right| dt + \frac{1}{\sqrt{2\pi e}\epsilon^2}\max_{0\le t\le 1}|G_1(t)| \sup_{0\le t\le 1}\left|x_{[nt],n} - W(t)\right|.
\end{aligned}
$$

Hence, using the dominated convergence theorem and Assumption A(b), as $n \to \infty$, equation (1.2.8) converges in probability to zero. Then the assertion follows as $\epsilon \to 0$.

Now we turn to prove (1.2.7). Define, for $\frac{\tau_{n,i-1}}{n} < t \le \frac{\tau_{n,i}}{n}$,

$$
\begin{aligned}
M_n(t) =& \sqrt{c_n}\sum_{k=1}^{i-1} f\left(\frac{k}{n}, c_n x_{k,n}\right)\left(U\left(\frac{\tau_{nk}}{n}\right) - U\left(\frac{\tau_{n,k-1}}{n}\right)\right)\\
&+ \sqrt{c_n} f\left(\frac{i}{n}, c_n x_{i,n}\right)\left(U(t) - U\left(\frac{\tau_{n,i-1}}{n}\right)\right),
\end{aligned} \tag{1.2.9}
$$

where τ_{nk} $(k = 1, \dots, n)$ are the stopping times in Lemma 1.2.1. It follows that, for any $n \ge 1$, $M_n(t)$ is a continuous martingale with respect to the filtration $\mathcal{F}_n(t) := \sigma(x_{1,n}, \dots, x_{i,n}, U(s)\ |s \le t, \frac{\tau_{n,i-1}}{n} < t \le \frac{\tau_{n,i}}{n})$. We can then derive that

$$
\sqrt{\frac{c_n}{n}}\sum_{k=1}^{[nr]} f\left(\frac{k}{n}, c_n x_{k,n}\right) e_k \overset{D}{=} M_n\left(\frac{\tau_{n,i}}{n}\right), \quad \text{if } \frac{\tau_{n,i-1}}{n} < r \le \frac{\tau_{n,i}}{n}, \tag{1.2.10}
$$

and deduct from (1.2.1) that

$$
\sup_{1\le k\le n}\left|\left(\frac{\tau_{nk}}{n} - \frac{\tau_{n,k-1}}{n}\right) - \frac{1}{n}\right| = o(1), \qquad a.s.. \tag{1.2.11}
$$

The quadratic variation process $[M_n]$ of $M_n(t)$ is that

$$
[M_n]_t = c_n\sum_{k=1}^{i-1} f^2\left(\frac{k}{n}, c_n x_{k,n}\right)\left(\frac{\tau_{nk}}{n} - \frac{\tau_{n,k-1}}{n}\right) + c_n f^2\left(\frac{i}{n}, c_n x_{i,n}\right)\left(t - \frac{\tau_{n,i-1}}{n}\right)
$$

$$=\frac{c_n}{n}\sum_{k=1}^{i-1} f^2\left(\frac{k}{n}, c_n x_{k,n}\right)(1+o_{a.s.}(1)) + c_n f^2\left(\frac{i}{n}, c_n x_{i,n}\right)\left(t-\frac{\tau_{n,i-1}}{n}\right).$$

Because

$$\begin{aligned}
&E\left|c_n f^2\left(\frac{i}{n}, c_n x_{i,n}\right)\left(t-\frac{\tau_{n,i-1}}{n}\right)\right| \leq \frac{c_n}{n} E f^2\left(\frac{i}{n}, c_n x_{i,n}\right)\\
=&\frac{c_n}{n}\int_{-\infty}^{\infty} f^2\left(\frac{i}{n}, c_n d_n x\right) h_{i,0,n}(x)dx \leq \frac{K}{nd_n}\int_{-\infty}^{\infty} f^2\left(\frac{i}{n}, x\right)dx\\
=&\frac{K}{nd_n} G_3\left(\frac{i}{n}\right) \leq \frac{KK_3}{nd_n} \to 0,
\end{aligned}$$

where $K_3 = \max_{0\leq t\leq 1} G_3(t)$ and by Assumption 4.3, $d_n = o(n) \to \infty$, we have that $c_n f^2\left(\frac{i}{n}, c_n x_{i,n}\right)\left(r-\frac{\tau_{n,i-1}}{n}\right) \to_P 0$.

It therefore follows from (1.2.5) that

$$[M_n]_t \to_D \int_0^t G_3(a)dL_W(a,0), \tag{1.2.12}$$

as $n\to\infty$.

Moreover, the covariance process $[M_n, W]$ of (M_n, W) is

$$\begin{aligned}
[M_n, W]_t =&\sqrt{c_n}\sum_{k=1}^{i-1} f\left(\frac{k}{n}, c_n x_{k,n}\right)\left(\frac{\tau_{nk}}{n}-\frac{\tau_{n,k-1}}{n}\right)\sigma_{uw}\\
&+\sqrt{c_n} f\left(\frac{i}{n}, c_n x_{i,n}\right)\left(t-\frac{\tau_{n,i-1}}{n}\right)\sigma_{uw}\\
=&\sigma_{uw}(1+o(1))\frac{\sqrt{c_n}}{n}\sum_{k=1}^{i-1} f\left(\frac{k}{n}, c_n x_{k,n}\right)\\
&+\sigma_{uw}\sqrt{c_n} f\left(\frac{i}{n}, c_n x_{i,n}\right)\left(t-\frac{\tau_{n,i-1}}{n}\right),
\end{aligned}$$

for any $t\in[0,1]$, where $\sigma_{uw} = \mathrm{Cov}(U,W)$. Meanwhile, using argument in Example 25.7 on Billingsley (1995, p.332),

$$\left|\frac{\sqrt{c_n}}{n}\sum_{k=1}^{i-1} f\left(\frac{k}{n}, c_n x_{k,n}\right)\right| \leq \frac{1}{\sqrt{c_n}}\frac{c_n}{n}\sum_{k=1}^{n}\left|f\left(\frac{k}{n}, c_n x_{k,n}\right)\right| \to_P 0,$$

because $c_n\to\infty$ and using (1.2.6), we have

$$\frac{c_n}{n}\sum_{k=1}^{n}\left|f\left(\frac{k}{n}, c_n x_{k,n}\right)\right| \to_D \int_0^1 G_2(t)dL_W(t,0).$$

Additionally, $\left|\sqrt{c_n} f\left(\frac{i}{n}, c_n x_{i,n}\right)\left(t - \frac{\tau_{n,i-1}}{n}\right)\right| \leq \frac{\sqrt{c_n}}{n} c_f(c_n x_{i,n}) \to 0$ a.s. by the integrability of function $c_f(\cdot)$ on $\mathbb{R}$ and $\frac{\sqrt{c_n}}{n} \to 0$ as $n \to \infty$. Thus,

$$[M_n, W]_{T_n(t)} \to_P 0, \tag{1.2.13}$$

where $T_n(t) = \inf\{s \in [0,1], [M_n]_s > t\}$ be the sequence of time changes. Then, in virtue of DDS (Dambis, Dubins-Schwarz) theorem (see, for example, Revuz and Yor, 1999, p.181), it follows that the process defined by

$$B^n(t) = M_n(T_n(t))$$

becomes a so-called DDS Brownian motion. Also, $M_n(t) = B^n([M_n]_t)$, and it follows from Theorem 2.3 of Revuz and Yor (1999, p.524) that (W, B^n) converges in distribution jointly to two independent Brownian motions (W, B). Therefore, we have as $n \to \infty$

$$\begin{aligned}
&\sqrt{\frac{c_n}{n}} \sum_{k=1}^{[nr]} f\left(\frac{k}{n}, c_n x_{k,n}\right) e_k \stackrel{D}{=} M_n\left(\frac{\tau_{n,i}}{n}\right) \\
=&M_n(r) + o_{a.s.}(1) = B^n([M_n]_r) + o_{a.s.}(1) \\
\to_D &B\left(\int_0^r G_3(a) dL_W(a,0)\right) = \left(\int_0^r G_3(a) dL_W(a,0)\right)^{\frac{1}{2}} B(1).
\end{aligned}$$

This finishes the whole proof. □

1.3 Time-homogeneous and integrable functionals

Since in most cases we encounter in reality the interested statistic quantities are $L_n = \sum_{s=1}^n F(s, c_n x_{s,n})$ and $M_n = \sum_{s=1}^n F(s, c_n x_{s,n}) e_s$, the results in the last section could not be used directly. To tackle this issue, the key point is how can we normalise the time variable in the functionals. Noting that if s in function F is in the form of some polynomial, we would be able to deal with the normalisation issue of time variable given that the F has some convenient form. Motivated by this idea, we propose the following definition of asymptotical homogeneity with respect to t.

Definition 1.3.1. Let $F(t,x)$ be defined on $t \geq 0$ and $x \in \mathbb{R}$. Suppose for every $x \in \mathbb{R}$, $\forall \eta > 0$, and $t \in [0,1]$,

$$F(\eta t, x) = \upsilon(\eta) f(t,x) + R_\eta(t,x),$$

where

(a) $f(t,x)$ satisfies Assumption C.

(b) $R_\eta(t,x)$ is such that one of the following holds:

(i) $|R_\eta(t,x)| \le q_\eta(t)P(x)$ where both $P(x)$ and $P^2(x)$ are Lebesgue integrable, and $q_\eta(t)/\upsilon(\eta) \to 0$ uniformly in $t \in [0,1]$ as $\eta \to \infty$.

(ii) $|R_\eta(t,x)| \le q_\eta(t)Q(\eta t)P(x)$ where both $P(x)$ and $P^2(x)$ are Lebesgue integrable, $\lim_{\eta\to\infty} \frac{q_\eta(t)}{\upsilon(\eta)} = l(t)$ which is bounded on [0,1] and $Q(y)$ that is bounded on any compact interval and $\lim_{y\to+\infty} Q(y) = 0$.

Such functions $F(t,x)$ are asymptotic homogeneous with respect to t and integrable with respect to x, thus called homogeneous-integrable functions, said to be in Class (HI), denoted by $\mathcal{T}(HI)$. Functions υ and f are called homogeneity power and normal function respectively. Function $F(t,x)$ with $R(t,x)$ satisfying (i) and (ii) is said to be in $\mathcal{T}(HI_1)$ and $\mathcal{T}(HI_2)$ respectively.

Theorem 1.3.1. *Suppose that $F(t,x)$ is in Class $\mathcal{T}(HI)$ with homogeneity power υ and normal function f. Then, when Assumption A(a) and (c) hold, for any $c_n \to \infty$, $n/c_n \to \infty$, and $r \in [0,1]$,*

$$\frac{c_n}{n\upsilon(n)} \sum_{s=1}^{[nr]} F(s, c_n x_{s,n}) \to_D \int_0^r G_1(t) dL_W(t,0), \tag{1.3.1}$$

where $G_1(\cdot) = \int f(\cdot, x)dx$ and L_W is the local–time process of W.

If A (a) is replaced by A (b), then for any $c_n \to \infty$, $n/c_n \to \infty$, and $r \in [0,1]$,

$$\frac{c_n}{n\upsilon(n)} \sum_{s=1}^{[nr]} F(s, c_n x_{s,n}) \to_P \int_0^r G_1(t) dL_W(t,0), \tag{1.3.2}$$

uniformly in $r \in [0,1]$ as $n \to \infty$ under the same probability space defined in Assumption A (b).

Moreover, if $\{e_s\}$ and $\{x_{s,n}\}$ satisfy Assumption B, and $f^2(t,x)$ satisfies Assumption C. We have for $n \to \infty$, $c_n \to \infty$, $c_n/n \to 0$, and $r \in [0,1]$,

$$\sqrt{\frac{c_n}{n}} \frac{1}{\upsilon(n)} \sum_{s=1}^{[nr]} F(s, c_n x_{s,n}) e_s \to_D \left(\int_0^r G_3(t) dL_W(t,0) \right)^{\frac{1}{2}} N, \tag{1.3.3}$$

where $G_3(\cdot) = \int f^2(\cdot, x)dx$ and N is a standard normal random variable independent of W.

Proof. It follows from the definition of Class $\mathcal{T}(HI)$ that

$$\begin{aligned}\frac{c_n}{n\upsilon(n)}\sum_{s=1}^{[nr]}F(s,c_nx_{s,n}) =&\frac{c_n}{n}\sum_{s=1}^{[nr]}f\left(\frac{s}{n},c_nx_{s,n}\right)+\frac{c_n}{n\upsilon(n)}\sum_{s=1}^{[nr]}R_n\left(\frac{s}{n},c_nx_{s,n}\right)\\ :=&\Pi_1+\Pi_2.\end{aligned}$$

As suggested by Theorem 1.2.1, if A (a) and (c) are fulfilled, $\Pi_1 \to_D \int_0^r G_1(t)dL_W(t,0)$; while if A (b) and (c) are fulfilled, $\Pi_1 \to_P \int_0^r G_1(t)dL_W(t,0)$ uniformly in r. It thus suffices to prove that $\Pi_2 \to_P 0$ uniformly in r under the condition A (c) in order to complete (1.3.1) and (1.3.2).

If $F(t,x)$ is in the class $\mathcal{T}(HI_1)$, $q_n(t)/\upsilon(n) \to 0$ uniformly in $t \in [0,1]$ as $n \to \infty$, then for a given $\epsilon > 0$, when n is large, $0 < q_n(t)/\upsilon(n) < \epsilon$ for all t. Thus, we have from Assumption A(c) that

$$\begin{aligned}\sup_{0\le r\le 1} E|\Pi_2| \le&\frac{c_n}{n\upsilon(n)}\sup_{0\le r\le 1}\sum_{s=1}^{[nr]}E\left|R_n\left(\frac{s}{n},c_nx_{s,n}\right)\right|\\ \le&\frac{c_n}{n\upsilon(n)}\sup_{0\le r\le 1}\sum_{s=1}^{[nr]}q_n\left(\frac{s}{n}\right)E[P(c_nx_{s,n})]\\ \le&\frac{c_n}{n\upsilon(n)}\sum_{s=1}^{n}q_n\left(\frac{s}{n}\right)E[P(c_nx_{s,n})]\\ \le&\epsilon\frac{c_n}{n}\sum_{s=1}^{n}\int_{-\infty}^{\infty}P(c_nd_{s,0,n}x)h_{s,0,n}(x)dx\\ =&\epsilon\frac{1}{n}\sum_{s=1}^{n}\frac{1}{d_{s,0,n}}\int_{-\infty}^{\infty}P(x)h_{s,0,n}\left(\frac{1}{c_nd_{s,0,n}}x\right)dx\\ \le&\epsilon K\int_{-\infty}^{\infty}P(x)dx\frac{1}{n}\sum_{s=1}^{n}\frac{1}{d_{s,0,n}},\end{aligned}$$

where K is the uniform upper bound of the densities $h_{l,k,n}(x)$. Thus, the desired result of $\Pi_2 \to_P 0$ uniformly in r follows from (1.1.4) and $\epsilon \to 0$.

If $F(t,x)$ is in the class $\mathcal{T}(HI_2)$, $|R_n\left(\frac{s}{n},c_nx_{s,n}\right)| \le q_n(t)Q(nt)P(c_nx_{s,n})$ with $P(x)$ and $P(x)^2$ integrable, $\lim_{n\to\infty} q_n(t)/\upsilon(n) = l(t)$ which is bounded on [0,1] and $Q(y)$ that is bounded on any compact interval and $\lim_{y\to+\infty} Q(y) = 0$. We have when n is large, $q_n(t)/\upsilon(n) = l(t)(1+o(1))$ and for a given $\epsilon > 0$, there exists $s_0 > 0$ such that $0 < Q(s) < \epsilon$

whenever $s > s_0$. Whence,

$$\begin{aligned}\sup_{0\le r\le 1} E|\Pi_2| \le & \frac{c_n}{n\upsilon(n)} \sup_{0\le r\le 1} \sum_{s=1}^{[nr]} E\left|R_n\left(\frac{s}{n}, c_n x_{s,n}\right)\right| \\ \le & \frac{c_n}{n\upsilon(n)} \sup_{0\le r\le 1} \sum_{s=1}^{[nr]} q_n\left(\frac{s}{n}\right) Q(s) E[P(c_n x_{s,n})] \\ \le & \frac{c_n}{n\upsilon(n)} \sum_{s=1}^{n} q_n\left(\frac{s}{n}\right) Q(s) E[P(c_n x_{s,n})] \\ \le & \frac{c_n}{n} \max_{0\le t\le 1} l(t) \sum_{s=1}^{n} Q(s) \int_{-\infty}^{\infty} P(c_n d_{s,0,n} x) h_{s,0,n}(x) dx \\ \le & K \max_{0\le t\le 1} l(t) \int_{-\infty}^{\infty} P(x) dx \frac{1}{n} \sum_{s=1}^{n} \frac{1}{d_{s,0,n}} Q(s) \\ = & K \max_{0\le t\le 1} l(t) \int_{-\infty}^{\infty} P(x) dx \left[\frac{1}{n}\sum_{s=1}^{s_0} \frac{1}{d_{s,0,n}} Q(s) + \frac{1}{n}\sum_{s=s_0}^{n} \frac{1}{d_{s,0,n}} Q(s)\right] \\ \le & K \max_{0\le t\le 1} l(t) \int_{-\infty}^{\infty} P(x) dx \left[K_Q(s_0)\frac{1}{n}\sum_{s=1}^{s_0} \frac{1}{d_{s,0,n}} + \epsilon\frac{1}{n}\sum_{s=1}^{n} \frac{1}{d_{s,0,n}}\right] \\ \to & 0,\end{aligned}$$

as $n \to \infty$ and then $\epsilon \to 0$ due to (1.1.3) and (1.1.4) where $K_Q(s_0) = \max(Q(1), \cdots, Q(s_0))$. This finishes the proof of (1.3.1) and (1.3.2). Now we turn to prove (1.3.3).

By virtue of the definition of the class $\mathcal{T}(HI)$,

$$\begin{aligned}& \sqrt{\frac{c_n}{n}} \frac{1}{\upsilon(n)} \sum_{s=1}^{[nr]} F(s, c_n x_{s,n}) e_s \\ = & \sqrt{\frac{c_n}{n}} \sum_{s=1}^{[nr]} f\left(\frac{s}{n}, c_n x_{s,n}\right) e_s + \sqrt{\frac{c_n}{n}} \frac{1}{\upsilon(n)} \sum_{s=1}^{[nr]} R_n\left(\frac{s}{n}, c_n x_{s,n}\right) e_s \\ := & \Pi_3 + \Pi_4.\end{aligned}$$

It follows from Theorem 1.2.1 that

$$\Pi_3 \to_D \left(\int_0^r G_3(t) dL_W(t,0)\right)^{\frac{1}{2}} N,$$

as $n \to \infty$ where N is a standard normal distributed variable independent of W. Hence, it is sufficient to show $\Pi_4 \to_P 0$ in order to complete the proof.

The structure of martingale difference of $(e_s, \mathcal{F}_{n.s})$ and the adaptivity between e_s and $x_{s,n}$ give

$$E[\Pi_4]^2 = \sigma_e^2 \frac{c_n}{n} \frac{1}{\upsilon(n)^2} \sum_{s=1}^{[nr]} ER_n^2 \left(\frac{s}{n}, c_n x_{s,n}\right).$$

If $F(t,x)$ is in the class $\mathcal{T}(HI_1)$ and $\frac{q_n(t)}{\upsilon(n)} \to 0$ uniformly in $t \in [0,1]$ as $n \to \infty$, for a given $\epsilon > 0$, when n is large, $0 < q_n(t)/\upsilon(n) < \epsilon$ for all t. Therefore,

$$\begin{aligned}
E[\Pi_4]^2 \leq & \sigma_e^2 \frac{c_n}{n} \frac{1}{\upsilon(n)^2} \sum_{s=1}^{n} q_n^2 \left(\frac{s}{n}\right) EP^2(c_n x_{s,n}) \\
\leq & \epsilon^2 \sigma_e^2 \frac{c_n}{n} \sum_{s=1}^{n} \int_{-\infty}^{\infty} P^2(c_n d_{s,0,n} x) h_{s,0,n}(x) dx \\
= & \epsilon^2 \sigma_e^2 \frac{1}{n} \sum_{s=1}^{n} \frac{1}{d_{s,0,n}} \int_{-\infty}^{\infty} P^2(x) h_{s,0,n} \left(\frac{1}{c_n d_{s,0,n}} x\right) dx \\
\leq & \epsilon^2 \sigma_e^2 K \int_{-\infty}^{\infty} P^2(x) dx \frac{1}{n} \sum_{s=1}^{n} \frac{1}{d_{s,0,n}} \\
\to & 0,
\end{aligned}$$

as $n \to \infty$ and $\epsilon \to 0$ on account of (1.1.4).

If $F(t,x)$ is in the class $\mathcal{T}(HI_2)$, then $|R_n\left(\frac{s}{n}, c_n x_{s,n}\right)| \leq q_n\left(\frac{s}{n}\right) Q(s) P(c_n x_{s,n})$ with $P(x)$ square integrable, $\lim_{n\to\infty} q_n(t)/\upsilon(n) = l(t)$ which is bounded on [0,1] and $Q(y)$ that is bounded on any compact interval and $\lim_{y\to+\infty} Q(y) = 0$. We have when n is large, $q_n(t)/\upsilon(n) = l(t)(1+o(1))$ and for a given $\epsilon > 0$, there exists $s_0 > 0$ such that $0 < Q(s) < \epsilon$ whenever $s > s_0$. Whence,

$$\begin{aligned}
E[\Pi_4]^2 \leq & \sigma_e^2 \frac{c_n}{n} \frac{1}{\upsilon(n)^2} \sum_{s=1}^{n} q_n^2 \left(\frac{s}{n}\right) Q^2(s) EP^2(c_n x_{s,n}) \\
\leq & \sigma_e^2 \max_{0\leq t\leq 1} l^2(t) \frac{c_n}{n} \sum_{s=1}^{n} Q^2(s) \int_{-\infty}^{\infty} P^2(c_n d_{s,0,n} x) h_{s,0,n}(x) dx \\
= & \sigma_e^2 \max_{0\leq t\leq 1} l^2(t) \frac{1}{n} \sum_{s=1}^{n} \frac{1}{d_{s,0,n}} Q^2(s) \int_{-\infty}^{\infty} P^2(x) h_{s,0,n} \left(\frac{1}{c_n d_{s,0,n}} x\right) dx \\
\leq & \sigma_e^2 K \max_{0\leq t\leq 1} l^2(t) \int_{-\infty}^{\infty} P^2(x) dx \frac{1}{n} \sum_{s=1}^{n} \frac{1}{d_{s,0,n}} Q^2(s) \\
= & \sigma_e^2 K \max_{0\leq t\leq 1} l^2(t) \int_{-\infty}^{\infty} P^2(x) dx \left[\frac{1}{n} \sum_{s=1}^{s_0} \frac{1}{d_{s,0,n}} Q^2(s) + \frac{1}{n} \sum_{s=s_0}^{n} \frac{1}{d_{s,0,n}} Q^2(s)\right]
\end{aligned}$$

$$\leq \sigma_e^2 K \max_{0\leq t\leq 1} l^2(t) \int_{-\infty}^{\infty} P^2(x)dx \left[K_Q(s_0)\frac{1}{n}\sum_{s=1}^{s_0}\frac{1}{d_{s,0,n}} + \epsilon^2 \frac{1}{n}\sum_{s=1}^{n}\frac{1}{d_{s,0,n}} \right]$$
$$\to 0,$$

when $n \to \infty$ and $\epsilon \to 0$. This completes the proof. □

1.4 Regular functionals

In this section, we discuss an asymptotic theory about sample mean and sample covariance for regular functionals $f(t,x)$ to be defined below which treats the corresponding definition and results in the literature as a special case.

The following definition of regularity of functional $f(t,x)$ extends that of $T(x)$ in Park and Phillips (1999, 2001) for univariate functions. However, it also depends on the definition of regularity for univariate functions in the literature. Therefore, when we say a univariate function is regular, we mean that in the sense of the definition in Park and Phillips (2001) with Π singleton.

Definition 1.4.1. Let $f(t,x)$ be defined on $[0,1]\times\mathbb{R}$. We say $f(t,x)$ is regular, if

(a) for each $x\in\mathbb{R}$, $f(t,x)$ is Lipschitz with respect to t, that is, there exists a constant $L(x)$ relative to x such that for any $t_1, t_2 \in [0,1]$,

$$|f(t_1,x) - f(t_2,x)| \leq L(x)|t_1 - t_2|, \tag{1.4.1}$$

where when x varies $L(x)$ is regular;

(b) for each $t\in[0,1]$, $f(t,x)$ is continuous in x in a neighbourhood of infinity;

(c) on any compact interval K of $\mathbb{R}$, for any given $\epsilon > 0$ there exist functions $\underline{f}_\epsilon(t,x)$, $\overline{f}_\epsilon(t,x)$, which are continuous in x, and δ_ϵ such that whenever $|y-x| < \delta_\epsilon$ on K, for each $t\in[0,1]$,

$$\underline{f}_\epsilon(t,x) \leq f(t,y) \leq \overline{f}_\epsilon(t,x), \tag{1.4.2}$$

and

$$\int_K \sup_{t\in[0,1]} (\overline{f}_\epsilon(t,x) - \underline{f}_\epsilon(t,x))dx \to 0, \tag{1.4.3}$$

as $\epsilon \to 0$.

Remark 1.4.1. Notice that if $f(t,x)$ reduces to $f(x)$, the regularity of $f(t,x)$ would reduce to that of $f(x)$ in the sense of Definition 3.1 in Park and Phillips (1999, 2001). In addition, since t is in [0,1], any type of functions $f(t,x) = q(t)P(x)$ with $q(t) \in C^1[0,1]$ and $P(x)$ is regular in the sense of reference in the literature is regular in this paper.

Notice also that the main difference between this definition for $f(t,x)$ and Definition 3.2 in Park and Phillips (2001) for function $F(x,\pi)$, $\pi \in \Pi$, is that π is a parameter in a compact set Π, while $t \in [0,1]$ is not a parameter, which is involved in the following asymptotic theory as a variable.

Theorem 1.4.1. *Let $f(t,x)$ be regular. For the triangular array $x_{s,n}$, $1 \le s \le n$, $n = 1, 2, \cdots$, and martingale difference $(e_s, \mathcal{F}_{n,s})$ satisfying Assumption B,*

$$\frac{1}{n}\sum_{s=1}^{n} f\left(\frac{s}{n}, x_{s,n}\right) \to_D \int_0^1 f(r, W(r))dr, \tag{1.4.4}$$

$$\frac{1}{\sqrt{n}}\sum_{s=1}^{n} f\left(\frac{s}{n}, x_{s,n}\right) e_s \to_D \int_0^1 f(r, W(r))dU(r), \tag{1.4.5}$$

as $n \to \infty$.

Proof. Observe that with the condition B (d) in Assumption B that $(U_n, W_n) \to_D (U, W)$ on $D[0,1]^2$, it follows from the so-called Skorohod-Dudley-Wichura representation theorem that there is a common probability space $(\Omega, \mathcal{F}, \mathcal{P})$ supporting (U_n^0, W_n^0) and (U_n, W_n) such that

$$(U_n^0, W_n^0) =_D (U_n, W_n) \quad \text{and} \quad (U_n^0, W_n^0) \to_{a.s.} (U, W), \tag{1.4.6}$$

in $D[0,1]^2$ with uniform topology.

Moreover, under conditions B(a), (c) in Assumption B, as indicated by Lemma 1.2.1, there exists an increasing sequence of stopping times τ_{nk} $(k = 1, \cdots, n)$ with $\tau_{n0} = 0$ on the space $(\Omega, \mathcal{F}, \mathcal{P})$ such that

$$U^0\left(\frac{k}{n}\right) =_D U\left(\frac{\tau_{nk}}{n}\right) \quad \text{and} \quad W^0\left(\frac{k}{n}\right) =_D W\left(\frac{\tau_{nk}}{n}\right), \tag{1.4.7}$$

for $k = 1, \cdots, n$, and

$$\sup_{1 \le k \le n}\left|\frac{\tau_{nk} - k}{n^\delta}\right| \to_{a.s.} 0 \tag{1.4.8}$$

as $n \to \infty$ for any $\delta > \max(\frac{1}{2}, \frac{2}{p})$ where p is the moment exponent given in condition B(a).

Such a schedule of consideration, referred to as the *embedding schedule*[2] in the sequel, allows us to rewrite any statistic about U_n and W_n equivalently in distribution into an

[2]We emphasise that the embedding schedule applies in the subsequent proofs. We shall mention it without showing the details whenever it is used.

expression of U_n^0 and W_n^0, so that we can obtain the weak convergence of the statistic by studying the latter with almost sure convergence of $(U_n^0, W_n^0) \to_{a.s.} (U, W)$. It therefore is reasonable in the sequel to assume without loss of generality that $(U_n, W_n) \to_{a.s.} (U, W)$ in order to avoid notational complication. To prove the result in (1.4.4) we first write that

$$\frac{1}{n}\sum_{s=1}^{n} f\left(\frac{s}{n}, x_{s,n}\right) = \frac{1}{n}\sum_{s=1}^{n} f\left(\frac{s-1}{n} + \frac{1}{n}, W_n\left(\frac{s-1}{n} + \frac{1}{n}\right)\right)$$
$$= \sum_{s=1}^{n} \int_{\frac{s-1}{n}}^{\frac{s}{n}} f(r + o(1), W_n(r + o(1)))dr = \int_0^1 f(r + o(1), W_n(r + o(1)))dr.$$

Thus, to complete the result in (1.4.4), it therefore suffices to show

$$\int_0^1 f(r + o(1), W_n(r + o(1)))dr \to_{a.s.} \int_0^1 f(r, W(r))dr.$$

Because of the condition (b) in regularity definition, there exists a constant $c > 0$ such that $f(t, x)$ is continuous in x whenever $|x| > c$. Let $J = [-c - 2, c + 2]$. For any given $\epsilon > 0$, it follows from the regularity of f that there exist continuous functions $\underline{f}_\epsilon(r, x)$, $\overline{f}_\epsilon(r, x)$ in x and $\delta > 0$ such that whenever $|x - y| < \delta$ on J, for each $r \in [0, 1]$,

$$\underline{f}_\epsilon(r, x) \le f(r, y) \le \overline{f}_\epsilon(r, x).$$

Note that when $x = y \in J$, we always have $\underline{f}_\epsilon(r, x) \le f(r, x) \le \overline{f}_\epsilon(r, x)$.

Since $\sup_{0\le r\le 1} |W_n(r) - W(r)| = o_{a.s.}(1)$, let n large enough such that $\sup_{0\le r\le 1} |W_n(r) - W(r)| < \frac{1}{2}\delta$ almost surely. Without loss of generality, assume that $\delta < 1$.

Observe that for large n, $|W_n(r + o(1)) - W(r)| \le |W_n(r + o(1)) - W(r + o(1))| + |W(r + o(1)) - W(r)| < \delta$ almost surely uniformly in r where we exploit the fact that the Brownian motion sample path is almost surely continuous, hence almost surely uniformly continuous on $[0, 1]$.

Denote $A(r) = \{|W(r)| < c+1\}$. It follows that on $A(r)$, when n is large, $W_n(r+o(1)) \in J$, $W(r) \in J$; while on $\bar{A}(r)$, $|W_n(r + o(1))| > c$, $|W(r)| > c$.

Notice that from Condition (a) of regularity,

$$\left|\int_0^1 f(r + o(1), W_n(r + o(1)))dr - \int_0^1 f(r, W(r))dr\right|$$
$$\le \int_0^1 |f(r + o(1), W_n(r + o(1))) - f(r, W_n(r + o(1)))|\, dr$$
$$+ \left|\int_0^1 [f(r, W_n(r + o(1))) - f(r, W(r))]dr\right|$$

$$\leq o(1)\int_0^1 L(W_n(r+o(1)))dr + \left|\int_0^1 [f(r, W_n(r+o(1))) - f(r, W(r))]dr\right|.$$

However,

$$\begin{aligned}
&\left|\int_0^1 [f(r, W_n(r+o(1))) - f(r, W(r))]dr\right| \\
\leq &\int_0^1 |f(r, W_n(r+o(1)))dr - f(r, W(r))| I(A(r))dr \\
&+ \left|\int_0^1 [f(r, W_n(r+o(1))) - f(r, W(r))] I(\bar{A}(r))dr\right| \\
\leq &\int_0^1 |\overline{f}_\epsilon(r, W(r)) - \underline{f}_\epsilon(r, W(r))| I(A(r))dr \\
&+ \left|\int_0^1 [f(r, W_n(r+o(1)))dr - f(r, W(r))] I(\bar{A}(r))dr\right| \\
:= &\Delta_1 + \Delta_2.
\end{aligned}$$

where $I(\cdot)$ is the indicator function.

Moreover, it follows from the occupation time formula for the bivariate Brownian functional that

$$\begin{aligned}
\Delta_1 = &\int_0^1 [\overline{f}_\epsilon(r, W(r)) - \underline{f}_\epsilon(r, W(r))] I(|W(r)| < c+1)dr \\
= &\int_{-c-1}^{c+1} da \int_0^1 \left[\overline{f}_\epsilon(r, a) - \underline{f}_\epsilon(r, a)\right] dL_W(r, a) \\
\leq &\int_J \sup_{0\leq r\leq 1} \left[\overline{f}_\epsilon(r, a) - \underline{f}_\epsilon(r, a)\right] da \int_0^1 dL_W(r, a) \\
= &\int_J L_W(1, a) \sup_{0\leq r\leq 1} \left[\overline{f}_\epsilon(r, a) - \underline{f}_\epsilon(r, a)\right] da \\
\leq &\sup_{a\in J} L_W(1, a) \int_J \sup_{0\leq r\leq 1} \left[\overline{f}_\epsilon(r, a) - \underline{f}_\epsilon(r, a)\right] da \to_{a.s.} 0,
\end{aligned}$$

as $\epsilon \to 0$, due to regularity of f and $\sup_{a\in J} L_W(1, a) \leq 1$ almost surely.

Furthermore, because $f(r, \cdot)$ is continuous on $|x| > c$, the continuous mapping theorem implies that $\Delta_2 \to 0$ a.s.

Regarding $\int_0^1 L(W_n(r+o(1)))dr$, since $L(\cdot)$ satisfies Condition (b) and (c) in regularity, similar derivation as above yields the result that it approaches $\int_0^1 L(W(r))dr$ almost surely. Hence, the proof of (1.4.4) is completed.

We are ready to prove (1.4.5). Once again the embedding schedule described in the first part permits us to derive it under a stronger condition that $(U_n, W_n) \to_{a.s.} (W, U)$.

Let us write

$$
\begin{aligned}
&\frac{1}{\sqrt{n}}\sum_{s=1}^{n} f\left(\frac{s}{n}, x_{s,n}\right) e_s = \sum_{s=1}^{n} f\left(\frac{s}{n}, x_{s,n}\right) \frac{1}{\sqrt{n}} e_s \\
=&\sum_{s=1}^{n} f\left(\frac{s-1}{n} + o(1), W_n\left(\frac{s-1}{n} + o(1)\right)\right)\left(U_n\left(\frac{s}{n}\right) - U_n\left(\frac{s-1}{n}\right)\right) \\
=&\sum_{s=1}^{n}\int_{\frac{s-1}{n}}^{\frac{s}{n}} f(r+o(1), W_n(r+o(1)))dU_n(r) \\
=&\int_0^1 f(r+o(1), W_n(r+o(1)))dU_n(r) := \sum_{k=1}^{4}\Pi_k,
\end{aligned}
$$

where

$$
\begin{aligned}
\Pi_1 =& \int_0^1 [f(r+o(1), W_n(r+o(1))) - f(r, W_n(r+o(1)))]dU_n(r) \\
\Pi_2 =& \int_0^1 [f(r, W_n(r+o(1))) - f_\epsilon(r, W_n(r+o(1)))]dU_n(r) \\
\Pi_3 =& \int_0^1 f_\epsilon(r, W_n(r+o(1)))dU_n(r) - \int_0^1 f_\epsilon(r, W(r))dU(r) \\
\Pi_4 =& \int_0^1 f_\epsilon(r, W(r))dU(r),
\end{aligned}
$$

in which denote $f_\epsilon(r,x) = \overline{f}_\epsilon(r,x)$ or $\underline{f}_\epsilon(r,x)$ for notational convenience. Observe that $(f_\epsilon(r, W_n(r+o(1))), U_n(r)) \to (f_\epsilon(r, W(r)), U(r))$ almost surely due to continuity in x of f_ϵ. It follows from Theorem 2.2 in Kurtz and Protter (1991) that $\Pi_3 \to_P 0$ as $n\to\infty$.

Therefore, in order to finish the proof, we need to show (1) $\Pi_1 \to_P 0$ when $n\to\infty$; (2) for all large n, $\Pi_2 \to_P 0$ and $\Pi_4 \to_P \int_0^1 f(r, W(r))dU(r)$ when $\epsilon\to 0$. Let us investigate them term by term.

It follows from Assumption B (a), (c) and regularity that

$$
\begin{aligned}
E[\Pi_1]^2 =& E\left\{\int_0^1 [f(r+o(1), W_n(r+o(1))) - f(r, W_n(r+o(1)))]dU_n(r)\right\}^2 \\
=&\sigma_e^2 E\int_0^1 [f(r+o(1), W_n(r+o(1))) - f(r, W_n(r+o(1)))]^2 dr \\
\le& o(1)\sigma_e^2 E\int_0^1 L^2(W_n(r+o(1)))dr \to 0,
\end{aligned}
$$

as $n\to\infty$ because we have $\int_0^1 L^2(W_n(r+o(1)))dr \to_{a.s.} \int_0^1 L^2(W(r))dr$ similar to the counterpart in first part, and by virtue of the regularity, $L^2(W_n(r))$ can be dominated by

$L_\epsilon^2(W(r))$ when n is large for some $\epsilon > 0$ and $L_\epsilon(\cdot)$ is continuous, $E\int_0^1 L^2(W_n(r))dr \to E\int_0^1 L^2(W(r))dr < \infty$. This finishes the proof of (1).

The convergence of Π_2 and Π_4 can be proven at the same time if we show

$$\int_0^1 [f(r, W_n(r+o(1))) - f_\epsilon(r, W_n(r+o(1)))]dU_n(r) \to_P 0,$$

as $\epsilon \to 0$ for all large n including $n = \infty$ that means conventionally $(U_\infty(r), V_\infty(r)) = (U(r), V(r))$.

Let real c be defined as before. All notations ϵ, δ, J, $A(r)$, $\overline{f}_\epsilon(t,x)$ and $\underline{f}_\epsilon(t,x)$ keep the same meanings as in the first part. In view of regularity condition (b), we may find $\overline{f}_\epsilon(r,x)$ and $\underline{f}_\epsilon(r,x)$ such that they are continuous in x on $\mathbb{R}$ for each $r \in [0,1]$, since beyond $[-c,c]$, we can take $\overline{f}_\epsilon(r,x) = \underline{f}_\epsilon(r,x) = f(t,x)$ and due to this reason, $\overline{f}_\epsilon(r,x) - \underline{f}_\epsilon(r,x)$ is bounded on $\mathbb{R}$. Consequently, $\sup_{r\in[0,1]}(\overline{f}_\epsilon(r,x) - \underline{f}_\epsilon(r,x))$ is bounded on $\mathbb{R}$ because it is continuous and beyond $[-c,c]$ it is zero. Let C be the upper bound of $\sup_{r\in[0,1]}[\overline{f}_\epsilon(r,x) - \underline{f}_\epsilon(r,x)]$.

By the adaptivity of $(U_n(r), W_n(r+o(1)))$, for large n,

$$\begin{aligned}
&E\left\{\int_0^1 [f(r, W_n(r+o(1))) - f_\epsilon(r, W_n(r+o(1)))]dU_n(r)\right\}^2 \\
=&\sigma_e^2 E\int_0^1 [f(r, W_n(r+o(1))) - f_\epsilon(r, W_n(r+o(1)))]^2 dr \\
=&\sigma_e^2 E\int_0^1 [f(r, W_n(r+o(1))) - f_\epsilon(r, W_n(r+o(1)))]^2 I(A(r))dr \\
\le&\sigma_e^2 E\int_0^1 [\overline{f}_\epsilon(r, W_n(r+o(1))) - \underline{f}_\epsilon(r, W_n(r+o(1)))]^2 I(A(r))dr \\
\to_{a.s.}&\sigma_e^2 E\int_0^1 [\overline{f}_\epsilon(r, W(r)) - \underline{f}_\epsilon(r, W(r))]^2 I(A(r))dr,
\end{aligned}$$

by virtue of continuity and boundedness of $\overline{f}_\epsilon(t,x) - \underline{f}_\epsilon(t,x)$ in x and the fact that indicator function is bounded as $n \to \infty$. Observe that by the occupation formula

$$\begin{aligned}
&\int_0^1 [\overline{f}_\epsilon(r, W(r)) - \underline{f}_\epsilon(r, W(r))]^2 I(|W(r)| \le c+1)dr \\
=&\int_{-\infty}^{\infty} da \int_0^1 [\overline{f}_\epsilon(r,a) - \underline{f}_\epsilon(r,a)]^2 I(|a| \le c+1)dL_W(r,a) \\
=&\int_{-c-1}^{c+1} da \int_0^1 [\overline{f}_\epsilon(r,a) - \underline{f}_\epsilon(r,a)]^2 dL_W(r,a)
\end{aligned}$$

$$\leq C \int_J \sup_{0\leq r\leq 1} [\overline{f}_\epsilon(r,a) - \underline{f}_\epsilon(r,a)]da \int_0^1 dL_W(r,a)$$
$$\leq C \sup_a L_W(1,a) \int_J \sup_{0\leq r\leq 1} [\overline{f}_\epsilon(r,a) - \underline{f}_\epsilon(r,a)]da \to_{a.s.} 0,$$

as $\epsilon \to 0$.

It follows from the dominated convergence theorem that $\Pi_2 \to_P 0$ and Π_4 converges to the desired variable in probability as $\epsilon \to 0$. This finishes the proof. □

The following lemma gives the closure of the usual operation: addition, multiply by a scalar and product for regular functionals.

Lemma 1.4.1. *Suppose that both $f(t,x)$ and $g(t,x)$ are regular, then $f(t,x)+g(t,x)$, $cf(t,x)$ for any $c\in\mathbb{R}$ and $f(t,x)g(t,x)$ are regular.*

Proof. For the sake of convenience, we firstly denote the components in the definition of regularity for $f(t,x)$ and $g(t,x)$. There exist function $L_f(x)$ and $L_g(x)$ which are regular such that for any $t_1,t_2\in[0,1]$ we have

$$|f(t_1,x) - f(t_2,x)| \leq L_f(x)|t_1-t_2|, \quad |g(t_1,x) - g(t_2,x)| \leq L_g(x)|t_1-t_2|.$$

On any compact interval K of $\mathbb{R}$, for any given $\epsilon>0$ there exist functions $\underline{f}_\epsilon(t,x)$, $\overline{f}_\epsilon(t,x)$, $\underline{g}_\epsilon(t,x)$, $\overline{g}_\epsilon(t,x)$ which are continuous in x, and δ_ϵ such that whenever $|y-x|<\delta_\epsilon$ on K, for each $t\in[0,1]$,

$$\underline{f}_\epsilon(t,x) \leq f(t,y) \leq \overline{f}_\epsilon(t,x), \quad \underline{g}_\epsilon(t,x) \leq g(t,y) \leq \overline{g}_\epsilon(t,x)$$

and

$$\int_K \sup_{t\in[0,1]} (\overline{f}_\epsilon(t,x) - \underline{f}_\epsilon(t,x))dx \to 0, \quad \int_K \sup_{t\in[0,1]} (\overline{g}_\epsilon(t,x) - \underline{g}_\epsilon(t,x))dx \to 0$$

as $\epsilon\to 0$.

We shall prove the statements one by one.

(1) Evidently $f(t,x)+g(t,x)$ is Lipschitz with regular function $L_f(x)+L_g(x)$ because of Lemma A1 of Park and Phillips (2001). Whenever $|y-x|<\delta_\epsilon$, we have

$$\underline{f}_\epsilon(t,x) + \underline{g}_\epsilon(t,x) \leq f(t,y)+g(t,y) \leq \overline{f}_\epsilon(t,x) + \overline{g}_\epsilon(t,x),$$

and

$$\int_K \sup_{t\in[0,1]} [\overline{f}_\epsilon(t,x) + \overline{g}_\epsilon(t,x) - \underline{f}_\epsilon(t,x) - \underline{g}_\epsilon(t,x)]dx$$

$$\leq \int_K \sup_{t\in[0,1]} (\overline{f}_\epsilon(t,x) - \underline{f}_\epsilon(t,x))dx + \int_K \sup_{t\in[0,1]} (\overline{g}_\epsilon(t,x) - \underline{g}_\epsilon(t,x))dx \to 0,$$

as $\epsilon \to 0$. Thus, $f(t,x) + g(t,x)$ is regular.

(2) It is obviously valid.

(3) For any $t_1, t_2 \in [0,1]$, we have

$$\begin{aligned}
&|f(t_1,x)g(t_1,x) - f(t_2,x)g(t_2,x)| \\
\leq &|g(t_1,x)||f(t_1,x) - f(t_2,x)| + |f(t_2,x)||g(t_1,x) - g(t_2,x)| \\
\leq &|g(t_1,x)|L_f(x)|t_1 - t_2| + |f(t_2,x)|L_g(x)|t_1 - t_2| \\
\leq &(t_1 L_g(x) + g(0,x))L_f(x)|t_1 - t_2| + (t_2 L_f(x) + f(0,x))L_g(x)|t_1 - t_2| \\
\leq &[(t_1 L_g(x) + g(0,x))L_f(x) + (t_2 L_f(x) + f(0,x))L_g(x)]|t_1 - t_2| \\
\leq &[(L_g(x) + \overline{g}_\epsilon(0,x))L_f(x) + (L_f(x) + \overline{f}_\epsilon(0,x))L_g(x)]|t_1 - t_2|,
\end{aligned}$$

and since both $\overline{g}_\epsilon(0,x)$ and $\overline{f}_\epsilon(0,x)$ are continuous function, by Lemma A1 of Park and Phillips (2001) the term in the square brackets is regular.

Meanwhile, $M_1(t,x) \leq f(t,y)g(t,y) \leq M_2(t,x)$ where

$$\begin{aligned}
M_1(t,x) =&\min(\underline{f}_\epsilon(t,x)\underline{g}_\epsilon(t,x), \underline{f}_\epsilon(t,x)\overline{g}_\epsilon(t,x), \overline{f}_\epsilon(t,x)\underline{g}_\epsilon(t,x), \overline{f}_\epsilon(t,x)\overline{g}_\epsilon(t,x)), \\
M_2(t,x) =&\max(\underline{f}_\epsilon(t,x)\underline{g}_\epsilon(t,x), \underline{f}_\epsilon(t,x)\overline{g}_\epsilon(t,x), \overline{f}_\epsilon(t,x)\underline{g}_\epsilon(t,x), \overline{f}_\epsilon(t,x)\overline{g}_\epsilon(t,x)).
\end{aligned}$$

Because all components in the min and max are continuous in x, on compact set K they are bounded in absolute value by $\Gamma(t)$, say. Meanwhile, $M_1(t,x)$ and $M_2(t,x)$ are continuous in x as well due to the same reason. Thus, $0 \leq M_2(t,x) - M_1(t,x) \leq \Gamma(t)[|\overline{g}_\epsilon(t,x) - \underline{g}_\epsilon(t,x)| + |\overline{f}_\epsilon(t,x) - \underline{f}_\epsilon(t,x)|]$. We then have

$$\begin{aligned}
0 &\leq \int_K \sup_{0\leq t\leq 1} [M_2(t,x) - M_1(t,x)]dx \\
&\leq \sup_{0\leq t\leq 1} \Gamma(t) \left[\int_K \sup_{0\leq t\leq 1} (\overline{g}_\epsilon(t,x) - \underline{g}_\epsilon(t,x))dx + \int_K \sup_{0\leq t\leq 1} (\overline{f}_\epsilon(t,x) - \underline{f}_\epsilon(t,x))dx\right] \to 0
\end{aligned}$$

as $\epsilon \to 0$. The proof is completed. □

1.5 Homogeneous regular functionals

We borrow some notations from Park and Phillips (2001) for convenience. Let $\mathscr{T}_{LB}$ denote the class of locally bounded transformations on $\mathbb{R}$; let $\mathscr{T}_{LB}^0$ be a subclass of $\mathscr{T}_{LB}$ consisting only of locally bounded transformations which are exponential bounded, i.e. transformations P such that $P(x) = O(e^{c|x|})$ for some $c > 0$; the class of bounded transformations on $\mathbb{R}$ is denoted by $\mathscr{T}_B$, and a subclass $\mathscr{T}_B^0$ of $\mathscr{T}_B$ is the collection of transformations that are bounded and vanish at infinity, i.e. transformations P such that $P(x) \to 0$ as $|x| \to \infty$. Clearly, $\mathscr{T}_B^0 \subset \mathscr{T}_B \subset \mathscr{T}_{LB}^0 \subset \mathscr{T}_{LB}$.

Definition 1.5.1. We say function $F(t, x)$ is asymptotic homogeneous regular with respect to both t and x, if for all $\xi, \eta > 0$ and $t \in [0, 1]$,

$$F(\xi t, \eta x) = \upsilon_1(\xi)\upsilon_2(\eta)f(t, x) + R(\xi, \eta; t, x), \tag{1.5.1}$$

where $f(t, x)$ is regular on $[0, 1] \times \mathbb{R}$, and $|R(\xi, \eta; t, x)| \leq A_\xi(t)a(\eta)P(x) + q(t)b(\xi)B_\eta(x)$ with positive functions A, a, P, B, q, b such that

a) $P(x) \in \mathscr{T}_{LB}^0$, $\limsup_{\eta\to\infty} \frac{a(\eta)}{\upsilon_2(\eta)} < \infty$ and either $\limsup_{\xi\to\infty} \frac{A_\xi(t)}{\upsilon_1(\xi)} = 0$ uniformly in $t \in [0, 1]$; or $\upsilon_1(\xi) \to \infty$ as $\xi \to \infty$ and $A_\xi(t) = A(t)$ which is Riemann integrable on $[0, 1]$; or $A_\xi(t) = \bar{A}_\xi(t)Q(\xi t)$ with $\limsup_{\xi\to\infty} \frac{\bar{A}_\xi(t)}{\upsilon_1(\xi)} = l(t)$ which is bounded on $[0, 1]$ and $Q(\cdot) \in \mathscr{T}_B^0$. And,

b) $q(t)$ is bounded on $[0, 1]$, $\limsup_{\xi\to\infty} \frac{b(\xi)}{\upsilon_1(\xi)} < \infty$ and either $B_\eta(x) = \bar{B}(\eta)V(x)$ with $\limsup_{\eta\to\infty} \frac{\bar{B}(\eta)}{\upsilon_2(\eta)} = 0$ and $V(x) \in \mathscr{T}_{LB}^0$, or $B_\eta(x) = \bar{B}(\eta)V(\eta x)$ where $V(\cdot) \in \mathscr{T}_B^0$ and $\limsup_{\eta\to\infty} \frac{\bar{B}(\eta)}{\upsilon_2(\eta)} < \infty$.

In the definition of asymptotic homogeneity, we denote $F(t, x) \in \mathcal{T}(HH)$ and call $f(t, x)$ the normal function of $F(t, x)$, and $\upsilon_1(\cdot)$ and $\upsilon_2(\cdot)$ the homogeneity powers with respect to t and x respectively.

Remark 1.5.1. (a) If the functions involved in the definition reduce to univariate functions without time variable, i.e., $F(t, x) \equiv F(x)$, $\upsilon_1(\xi) = 1$, $f(t, x) \equiv f(x)$ and $R(\xi, \eta; t, x) \equiv R(\eta; x)$ with $q(t) \equiv 1$, $b(\xi) = 1$, it becomes the Class (H) in Park and Phillips (1999, 2001).

(b) In practice, often one of the two dominate terms of R appears. The only appearance of the first term implies that $q(t) = 0$, while that of the second term indicates that $P(x) = 0$.

(c) There are many functions which have asymptotic homogeneity. For example,

(1). $F(t, x) = a_1 t^{m_1} x^{l_1} + \cdots + a_k t^{m_k} x^{l_k}$ with $m_1 \geq \cdots \geq m_k \geq 0$, $m_1 \geq 1$ and $l_1 \geq \cdots \geq l_m \geq 0$, is homogeneous where $f(t, x) = a_1 t^{m_1} x^{l_1}$, $\upsilon_1(\xi) = \xi^{m_1}$, $\upsilon_2(\eta) = \eta^{l_1}$, and if $m_1 > m_2$, $|R(\xi, \eta; t, x)| \leq A_\xi(t)a(\eta)P(x)$ where $A_\xi(t) = a_2\xi^{m_2}t^{m_2} + \cdots + a_k\xi^{m_k}t^{m_2}$,

$a(\eta) = \eta^{l_2}$ and $P(x) = 1 + |x|^{l_2}$. Clearly, $\lim_{\xi\to\infty} \frac{A_\xi(t)}{\upsilon_1(\xi)} = 0$ uniformly in t. If $l_2 < l_1$, $|R(\xi, \eta; t, x)| \le q(t)b(\xi)B_\eta(x)$ where $q(t) = 1 + t^{m_2}$, $b(\xi) = \xi^{m_2}$, $B_\eta(x) = \eta^{l_2}(1 + |x|^{l_2})$. Palpably, $\lim_{\eta\to\infty} \frac{\eta^{l_2}}{\upsilon_2(\eta)} = 0$ and $1 + |x|^{l_2} \in \mathscr{T}^0_{LB}$.

(2). $F(t, x) = t^\alpha \log(1 + |x|)$ with $\alpha \ge 1$. The normal function $f(t, x) = t^\alpha$ and $\upsilon_1(\xi) = \xi^\alpha$, $\upsilon_2(\eta) = \log(\eta)$, while $R(\xi, \eta; t, x) \le \xi^\alpha t^\alpha \log(1 + |x|)$. Notice that $b(\xi) = \xi^\alpha$, $q(t) = t^\alpha$, $B_\eta(x) = \log(1 + |x|)$ with $\bar{B}(\eta) = 1$ and $\log(1 + |x|) \in \mathscr{T}^0_{LB}$.

(3). $F(t, x) = t^2 x + \sqrt{1 + t^4}\frac{1}{1+|\ln t|}x$. The normal function is $f(t, x) = t^2 x$, the homogeneous powers are $\upsilon_1(\xi) = \xi^2$ and $\upsilon_2(\eta) = \eta$; while $R(\xi, \eta; t, x) \le A_\xi(t)a(\eta)P(x)$, where $a(\eta) = \eta$, $P(x) = |x|$, $A_\xi(t) = \bar{A}_\xi(t)Q(\xi t)$ with $\lim_{\xi\to\infty} \frac{\bar{A}_\xi(t)}{\upsilon_1(\xi)} = \lim_{\xi\to\infty} \frac{\sqrt{1+\xi^4 t^4}}{\xi^2} = t^2$ and $Q(y) = \frac{1}{1+|\ln y|} \to 0$ when $y \to +\infty$.

(4). $F(t, x) = t^\alpha D(x)$ where $\alpha \ge 1$ and $D(x)$ is a distribution function for any random variable. Then $f(t, x) = t^\alpha I(x \ge 0)$, $\upsilon_1(\xi) = \xi^\alpha$, $\upsilon_2(\eta) = 1$, $R(\xi, \eta; t, x) < b(\xi)q(t)Q(\eta x)$ where $b(\xi) = \xi^\alpha$, $q(t) = t^\alpha$ and $Q(y) = D(y)I(y < 0) + (1 - D(y))I(y \ge 0)$ which approaches to zero when $y \to +\infty$. □

Theorem 1.5.1. *Let $F(t, x)$ be in Class $\mathcal{T}(HH)$ with homogeneity powers $\upsilon_1(\cdot)$ and $\upsilon_2(\cdot)$ and normal function $f(t, x)$. Let martingale difference $(e_s, \mathcal{F}_{n,s})$ and $x_{s,n}$ satisfy Assumption B. We then have*

$$\frac{1}{n\upsilon_1(n)\upsilon_2(c_n)} \sum_{s=1}^{n} F(s, c_n x_{s,n}) \to_D \int_0^1 f(r, W(r))dr, \tag{1.5.2}$$

$$\frac{1}{\sqrt{n}\upsilon_1(n)\upsilon_2(c_n)} \sum_{s=1}^{n} F(s, c_n x_{s,n})e_s \to_D \int_0^1 f(r, W(r))dU(r), \tag{1.5.3}$$

where $(U(r), W(r))$ is the limit of $(U_n(r), W_n(r))$ for $r \in [0, 1]$ stipulated in Assumption B.

Remark 1.5.2. Notice that if $F(t, x)$ reduces to an univariate function $F(x)$, with $c_n = \sqrt{n}$, (1.5.2) becomes Theorem 5.3 of Park and Phillips (1999) and the first part of Theorem 3.3 with singleton Π of Park and Phillips (2001); (1.5.3) becomes the second part of Theorem 3.3 with singleton Π in Park and Phillips (2001).

Proof. Observe that, like preceding proofs, the embedding schedule allow us to work under a stronger condition $(W_n, U_n) \to (W, U)$ almost surely but still achieve the weak convergence for the assertion.

It follows from the asymptotic homogeneity of F function that

$$\begin{aligned}&\frac{1}{n\upsilon_1(n)\upsilon_2(c_n)}\sum_{s=1}^{n}F(s,c_nx_{s,n})\\=&\frac{1}{n}\sum_{s=1}^{n}f\left(\frac{s}{n},x_{s,n}\right)+\frac{1}{n\upsilon_1(n)\upsilon_2(c_n)}\sum_{s=1}^{n}R(n,c_n;s,c_nx_{s,n}).\end{aligned}$$

Note that $f(t,x)$ is regular and thus by the proof (not the result) of Theorem 1.4.1,

$$\frac{1}{n}\sum_{s=1}^{n}f\left(\frac{s}{n},x_{s,n}\right)\to_{a.s.}\int_0^1 f(r,W(r))dr,$$

as $n\to\infty$.

In order to complete the proof of (1.5.2), it thus suffices to show

$$\frac{1}{n\upsilon_1(n)\upsilon_2(c_n)}\sum_{s=1}^{n}R(n,c_n;s,c_nx_{s,n})\to_{a.s.}0.$$

Let $\lim_{n\to\infty}\frac{a(c_n)}{\upsilon_2(c_n)}=a$ and $\lim_{n\to\infty}\frac{b(n)}{\upsilon_1(n)}=b$. Let $K=[s_{\min}-1,s_{\max}+1]$ with $s_{\min}=\inf_{r\in[0,1]}W(r)$ and $s_{\max}=\sup_{r\in[0,1]}W(r)$. Note that almost surely K is a finite compact interval.

It follows from the definition that as n is large,

$$\begin{aligned}&\frac{1}{n\upsilon_1(n)\upsilon_2(c_n)}\sum_{s=1}^{n}|R(n,c_n;s,c_nx_{s,n})|\\\le&\frac{a(c_n)}{n\upsilon_1(n)\upsilon_2(c_n)}\sum_{s=1}^{n}A_n\left(\frac{s}{n}\right)P(x_{s,n})+\frac{b(n)}{n\upsilon_1(n)\upsilon_2(c_n)}\sum_{s=1}^{n}q\left(\frac{s}{n}\right)B_{c_n}(x_{s,n})\\=&\frac{a(1+o(1))}{n\upsilon_1(n)}\sum_{s=1}^{n}A_n\left(\frac{s}{n}\right)P(x_{s,n})+\frac{b(1+o(1))}{n\upsilon_2(c_n)}\sum_{s=1}^{n}q\left(\frac{s}{n}\right)B_{c_n}(x_{s,n})\\:=&\Pi_1+\Pi_2.\end{aligned}$$

If $\limsup_{n\to\infty}\frac{A_n\left(\frac{s}{n}\right)}{\upsilon_1(n)}=0$ uniformly in s, then for any given $\epsilon>0$, when n is large enough, $0<\frac{A_n\left(\frac{s}{n}\right)}{\upsilon_1(n)}<\epsilon$. Thus,

$$0\le\Pi_1<\epsilon a(1+o(1))\frac{1}{n}\sum_{s=1}^{n}P(x_{s,n})\le\epsilon a(1+o(1))\|P\|_K\to_{a.s.}0,$$

as $n\to\infty$ and $\epsilon\to0$ since $x_{s,n}=W_n(r)\in K$ due to convergence of $W_n(r)$ to $W(r)$ almost surely and $\|P\|_K$, the bound of $P(x)$ on K (in the sequel similar notations have the similar meaning), is almost surely finite. Thus, $\Pi_1\to0$, a.s.

If $\upsilon_1(n) \to \infty$ as $n \to \infty$ and $A_n(t) = A(t)$ which is Riemann integrable on $[0, 1]$, then

$$\begin{aligned} 0 < \Pi_1 =& \frac{a(1 + o(1))}{n\upsilon_1(n)} \sum_{s=1}^{n} A\left(\frac{s}{n}\right) P(x_{s,n}) \\ \leq& \frac{a(1 + o(1))}{\upsilon_1(n)} \|P\|_K \frac{1}{n} \sum_{s=1}^{n} A\left(\frac{s}{n}\right) \to_{a.s.} 0, \end{aligned}$$

since as $n \to \infty$, $\frac{1}{\upsilon_1(n)} \to 0$, $\frac{1}{n}\sum_{s=1}^{n} A\left(\frac{s}{n}\right) \to \int_0^1 A(t)dt < \infty$ and $\|P\|_K < \infty$ a.s.. We have $\Pi_1 \to_{a.s.} 0$ as well.

If $A_n(t) = \bar{A}_n(t)Q(nt)$ with $\limsup_{n\to\infty} \frac{\bar{A}_n(t)}{\upsilon_1(n)} = l(t)$ bounded on [0,1] and $Q(y)$ is bounded on $\mathbb{R}$ as well as $\lim_{y\to+\infty} Q(y) = 0$, then for any given $\epsilon > 0$, there exists a positive integer s_0 such that when $y > s_0$, $0 < Q(y) < \epsilon$. Therefore,

$$\begin{aligned} 0 < \Pi_1 =& \frac{a(1 + o(1))}{n\upsilon_1(n)} \sum_{s=1}^{n} \bar{A}_n\left(\frac{s}{n}\right) Q(s)P(x_{s,n}) \\ \leq& \frac{a(1 + o(1))}{n} \sum_{s=1}^{n} l\left(\frac{s}{n}\right) Q(s)P(x_{s,n}) \\ \leq& a(1 + o(1)) \max_{0\leq t\leq 1} l(t)\|P\|_K \frac{1}{n} \sum_{s=1}^{n} Q(s) \\ \leq& a(1 + o(1)) \max_{0\leq t\leq 1} l(t)\|P\|_K \left[\frac{1}{n}\sum_{s=1}^{s_0} Q(s) + \frac{1}{n}\sum_{s=s_0}^{n} \epsilon\right] \to_{a.s.} 0, \end{aligned}$$

as $n \to \infty$ and $\epsilon \to 0$. Thus, $\Pi_1 \to_{a.s.} 0$ too.

We are now in a position to show $\Pi_2 \to_{a.s.} 0$.

If $B_{c_n}(x_{s,n}) = \bar{B}(c_n)V(x_{s,n})$ with $\limsup_{n\to\infty} \frac{\bar{B}(c_n)}{\upsilon_2(c_n)} = 0$, $0 \leq q(t) \leq M_q < \infty$ on [0,1] and $V(x)$ is locally bounded, then for any given $\epsilon > 0$, when n is large, $0 < \frac{\bar{B}(c_n)}{\upsilon_2(c_n)} < \epsilon$. Thus,

$$\begin{aligned} 0 < \Pi_2 =& \frac{b(1 + o(1))}{n\upsilon_2(c_n)} \sum_{s=1}^{n} q\left(\frac{s}{n}\right) \bar{B}(c_n)V(x_{s,n}) \\ \leq& \epsilon b(1 + o(1))\|V\|_K M_q \to_{a.s.} 0, \end{aligned}$$

as $n \to \infty$ and $\epsilon \to 0$. Thus, $\Pi_2 \to_{a.s.} 0$.

If $B_{c_n}(x_{s,n}) = \bar{B}(c_n)V(c_n x_{s,n})$ where $\limsup_{n\to\infty} \frac{\bar{B}(c_n)}{\upsilon_2(c_n)} = l < \infty$ and $V(y)$ is bounded and vanishes at infinity, viz., $\lim_{y\to\infty} V(y) = 0$, then when n is large, $\frac{\bar{B}(c_n)}{\upsilon_2(c_n)} = l(1 + o(1))$ and when $|y| > y_0$ for some positive y_0 and a given $\epsilon > 0$, $|V(y)| < \epsilon$. Therefore,

$$0 < \Pi_2 = \frac{b(1 + o(1))\bar{B}(c_n)}{n\upsilon_2(c_n)} \sum_{s=1}^{n} q\left(\frac{s}{n}\right) V(c_n x_{s,n})$$

$$
\begin{aligned}
&= bl(1+o(1))\frac{1}{n}\sum_{s=1}^{n} q\left(\frac{s}{n}\right) V(c_n x_{s,n}) \\
&= blM_q(1+o(1))\frac{1}{n}\sum_{s=1}^{n} V(c_n x_{s,n}) \\
&\quad \times [I(|c_n x_{s,n}| \le y_0) + I(|c_n x_{s,n}| > y_0)] \\
&\le blM_q(1+o(1))\left(\|V\|\frac{1}{n}\sum_{s=1}^{n} I(c_n|x_{s,n}| \le y_0) + \epsilon\right) \\
&= blM_q(1+o(1))\left(\|V\|\int_0^1 I(c_n|W_n(r+o(1))| \le y_0)dr + \epsilon\right).
\end{aligned}
$$

Observe that for $\epsilon > 0$,

$$
\begin{aligned}
&\{c_n|W_n(r+o(1))| \le y_0\} \\
=&\left\{|W_n(r+o(1))| \le \frac{y_0}{c_n}, |W(r)| \le \frac{y_0}{c_n} + \epsilon\right\} \\
&\cup \left\{|W_n(r+o(1))| \le \frac{y_0}{c_n}, |W(r)| > \frac{y_0}{c_n} + \epsilon\right\} \\
\subset &\left\{|W(r)| \le \frac{y_0}{c_n} + \epsilon\right\} \cup \{|W_n(r+o(1)) - W(r)| > \epsilon\}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
I\{c_n|W_n(r+o(1))| \le y_0\} \le & I\left\{|W(r)| \le \frac{1}{c_n}y_0 + \epsilon\right\} \\
& + I\{|W_n(r+o(1)) - W(r)| > \epsilon\}.
\end{aligned}
$$

However, as $n \to \infty$, for every $r \in [0,1]$,

$$
\begin{aligned}
&\left\{|W(r)| \le \frac{1}{c_n}y_0 + \epsilon\right\} \downarrow \{|W(r)| \le \epsilon\}, \quad \text{and} \\
&\{|W_n(r+o(1)) - W(r)| > \epsilon\} \downarrow \emptyset,
\end{aligned}
$$

which imply that

$$
\begin{aligned}
&I\left\{|W(r)| \le \frac{1}{c_n}y_0 + \epsilon\right\} \to_{a.s.} I\{|W(r)| \le \epsilon\} \\
&I\left\{\sup_{0\le r\le 1} |W_n(r+o(1)) - W(r)| > \epsilon\right\} \to_{a.s.} 0.
\end{aligned}
$$

It follows from the dominated convergence theorem that

$$
0 \le \int_0^1 I(c_n|W_n(r+o(1))| \le y_0)dr
$$

$$\leq \int_0^1 I\left\{|W(r)| \leq \frac{1}{c_n} y_0 + \epsilon\right\} dr + \int_0^1 I\left\{|W_n(r + o(1)) - W(r)| > \epsilon\right\} dr$$
$$\to_{a.s.} \int_0^1 I\left\{|W(r)| \leq \epsilon\right\} dr.$$

Then, as $\epsilon \to 0$, $I\left\{|W(r)| \leq \epsilon\right\} \to_{a.s.} I\{|W(r)| = 0\} = 0$ almost surely except $r = 0$. Once again, the dominated convergence theorem implies that

$$\int_0^1 I\left\{|W(r)| \leq \epsilon\right\} dr \to_{a.s.} \int_0^1 I\{|W(r)| = 0\} dr = 0, \quad a.s.$$

Hence, $\Pi_2 \to_{a.s.} 0$ as $n \to \infty$ first and then $\epsilon \to 0$. This finishes the proof of (1.5.2). We are now ready to prove (1.5.3).

It follows from the asymptotic homogeneity of $F(\cdot, \cdot)$ that

$$\begin{aligned}
&\frac{1}{\sqrt{n}\upsilon_1(n)\upsilon_2(c_n)} \sum_{s=1}^n F(s, c_n x_{s,n}) e_s \\
=&\frac{1}{\sqrt{n}} \sum_{s=1}^n f\left(\frac{s}{n}, x_{s,n}\right) e_s + \frac{1}{\sqrt{n}\upsilon_1(n)\upsilon_2(c_n)} \sum_{s=1}^n R(n, c_n; s, c_n x_{s,n}) e_s \\
:=&\Pi_3 + \Pi_4.
\end{aligned}$$

According to Theorem 1.4.1,

$$\Pi_3 = \frac{1}{\sqrt{n}} \sum_{s=1}^n f\left(\frac{s}{n}, x_{s,n}\right) e_s \to_D \int_0^1 f(r, W(r)) dU(r).$$

It thus suffices to show that with the help of the embedding schedule, $\Pi_4 \to_P 0$ as $n \to \infty$ in order to finish the proof. Using martingale structure of $(e_s, \mathcal{F}_{n,s})$ we have

$$\begin{aligned}
E[\Pi_4]^2 =&\frac{1}{n\upsilon_1(n)^2\upsilon_2(c_n)^2} E\left[\sum_{s=1}^n R(n, c_n; s, c_n x_{s,n}) e_s\right]^2 \\
=&\frac{\sigma_e^2}{n\upsilon_1(n)^2\upsilon_2(c_n)^2} \sum_{s=1}^n E R^2(n, c_n; s, c_n x_{s,n}) \\
\leq&\frac{2\sigma_e^2 a^2(c_n)}{n\upsilon_1(n)^2\upsilon_2(c_n)^2} \sum_{s=1}^n A_n^2\left(\frac{s}{n}\right) E[P(x_{s,n})]^2 \\
&+ \frac{2\sigma_e^2 b^2(n)}{n\upsilon_1(n)^2\upsilon_2(c_n)^2} \sum_{s=1}^n q^2\left(\frac{s}{n}\right) E[B_{c_n}(x_{s,n})]^2 \\
:=&\Pi_{41} + \Pi_{42}.
\end{aligned}$$

Observe that if $\limsup_{n\to\infty} \frac{A_n\left(\frac{s}{n}\right)}{\upsilon_1(n)} = 0$ uniformly in s, then for any given $\epsilon > 0$, when n is large enough, $0 < \frac{A_n\left(\frac{s}{n}\right)}{\upsilon_1(n)} < \epsilon$. Thus

$$0 \le \Pi_{41} \le 2a^2\sigma_e^2(1+o(1))\epsilon^2 \frac{1}{n}\sum_{s=1}^{n} E[P(x_{s,n})]^2.$$

Since $P^2(\cdot) \in \mathscr{T}_{LB}^0$, $P^2(x_{s,n}) \le \|P^2\|_K < \infty$ almost surely and then $E[\|P^2\|_K] < \infty$, hence $\frac{1}{n}\sum_{s=1}^{n} E[P(x_{s,n})]^2 \le E[\|P^2\|_K]$, which implies that $\Pi_{41} \to 0$.

If $A_n\left(\frac{s}{n}\right) = A\left(\frac{s}{n}\right)$ and $\upsilon_1(n) \to \infty$ when $n \to \infty$,

$$\begin{aligned} 0 \le \Pi_{41} \le & 2a^2\sigma_e^2(1+o(1))\frac{1}{n\upsilon_1(n)^2}\sum_{s=1}^{n} A^2\left(\frac{s}{n}\right) E[P(x_{s,n})]^2 \\ \le & \sigma_e^2(1+o(1))E[\|P^2\|_K]\frac{1}{\upsilon_1(n)^2}\frac{1}{n}\sum_{s=1}^{n} A^2\left(\frac{s}{n}\right) \to 0, \end{aligned}$$

on account of integrability of $A^2(t)$, which implies that $\Pi_{41} \to 0$ as well.

If $A_n(t) = \bar{A}_n(t)Q(nt)$ with $\limsup_{n\to\infty} \frac{\bar{A}_n(t)}{\upsilon_1(n)} = l(t)$ which is bounded on $[0,1]$ and $Q(y)$ is bounded as well as $\lim_{y\to+\infty} Q(y) = 0$, then for any given $\epsilon > 0$, there exists a positive integer s_0 such that when $y > s_0$, $0 < Q(y) < \epsilon$.

$$\begin{aligned} 0 \le \Pi_{41} \le & 2a^2\sigma_e^2 \max_{t\in[0,1]} l^2(t)(1+o(1))\frac{1}{n}\sum_{s=1}^{n} Q^2(s)E[P(x_{s,n})]^2 \\ \le & 2a^2\sigma_e^2 E[\|P^2\|_K] \max_{t\in[0,1]} l^2(t)(1+o(1))\frac{1}{n}\sum_{s=1}^{n} Q^2(s) \\ \le & 2a^2\sigma_e^2 E[\|P^2\|_K] \max_{t\in[0,1]} l^2(t)(1+o(1))\left[\frac{1}{n}\sum_{s=1}^{s_0} Q^2(s) + \frac{1}{n}\sum_{s=s_0}^{n} \epsilon^2\right] \\ \to & 0, \end{aligned}$$

as $n \to \infty$ and $\epsilon \to 0$.

As for Π_{42}, if $B_{c_n}(x_{s,n}) = \bar{B}(c_n)V(x_{s,n})$, $V(x) \in \mathscr{T}_{LB}^0$, and $\limsup_{n\to\infty} \frac{\bar{B}(c_n)}{\upsilon_2(c_n)} = 0$, then for any given $\epsilon > 0$, when n is large,

$$\begin{aligned} 0 \le \Pi_{42} \le & 2\sigma_e^2 b^2(1+o(1))\epsilon^2\frac{1}{n}\sum_{s=1}^{n} q^2\left(\frac{s}{n}\right) E[V(x_{s,n})]^2 \\ \le & 2\sigma_e^2 b^2(1+o(1))\epsilon^2 E[\|V^2\|_K]M_q \to 0, \end{aligned}$$

as $n \to \infty$ and $\epsilon \to 0$, hence, $\Pi_{42} \to 0$.

If $B_{c_n}(x_{s,n}) = \bar{B}(c_n)V(c_n x_{s,n})$ where $\limsup_{n\to\infty} \frac{\bar{B}(c_n)}{\upsilon_2(c_n)} = l < \infty$ and $V(y)$ is bounded and vanishes at infinity, viz., $\lim_{y\to\infty} V(y) = 0$, then when n is large, $\frac{\bar{B}(c_n)}{\upsilon_2(c_n)} = l(1 + o(1))$ and for a given $\epsilon > 0$, let $y_0 > 0$ such that whenever $|y| > y_0$, $|V(y)| < \epsilon$. Consequently,

$$\begin{aligned}
0 \le \Pi_{42} =& \frac{2\sigma_e^2 b^2(n)\bar{B}^2(c_n)}{n\upsilon_1(n)^2\upsilon_2(c_n)^2} \sum_{s=1}^{n} q^2\left(\frac{s}{n}\right) E[V(c_n x_{s,n})]^2 \\
\le& 2\sigma_e^2 b^2 l^2(1 + o(1))\frac{1}{n}\sum_{s=1}^{n} q^2\left(\frac{s}{n}\right) E[V(c_n x_{s,n})]^2 \\
=& 2\sigma_e^2 b^2 l^2(1 + o(1))\frac{1}{n}\sum_{s=1}^{n} q^2\left(\frac{s}{n}\right) E\{[V(c_n x_{s,n})]^2 \\
&\times [I(|c_n x_{s,n}| \le y_0) + I(|c_n x_{s,n}| > y_0)]\} \\
\le& 2\sigma_e^2 b^2 l^2(1 + o(1))\|V^2\|M_q^2 E\frac{1}{n}\sum_{s=1}^{n} I(c_n|x_{s,n}| \le y_0) + 2\sigma_e^2 b^2 l^2(1 + o(1))\epsilon^2 M_q^2 \\
=& 2\sigma_e^2 b^2 l^2(1 + o(1))\|V^2\|M_q^2 E\int_0^1 I(c_n|W_n(r)| < y_0)dr + 2\sigma_e^2 b^2 l^2(1 + o(1))\epsilon^2 M_q^2 \\
&\to 0
\end{aligned}$$

by the result in the first part that $\int_0^1 I(c_n|W_n(r)| < y_0)dr \to_{a.s.} 0$ and the dominated convergence theorem as $n \to \infty$ and $\epsilon \to 0$.

This finishes the whole proof. □

Chapter 2

Orthogonal expansion of Brownian motion functionals

2.1 Introduction

Researchers have long employed stochastic processes to depict random phenomena in many disciplines such as economics, finance and engineering. In finance, for example, equity's price in the Black-Scholes model is formulated as a stochastic process. Meanwhile, the price of a derivative on the equity is described in terms of the so-called Black-Scholes PDE (Partial Differential Equation) in Black and Scholes (1973). As a matter of fact, many popular models in economics and finance, like those for pricing derivative securities, involve diffusion processes formulated in continuous time as solutions to stochastic differential equations. These processes have been employed to model options prices, the term structure of interest rates, and exchange rates, for example. Stochastic differential equations also have been used to model macroeconomic aggregates like consumption and investment, and systems of such equations have been studied extensively to delineate economic activities at a national level.

A diffusion process can be thought of as a strong Markov process with continuous paths. Inspired by Lévy's investigation of sample paths, Itô studied diffusions that could be represented as solutions of stochastic differential equations of the form

$$dX(t) = a(t, X(t))dt + b(t, X(t))dB(t), \quad X(0, \omega) = Y(\omega), \tag{2.1.1}$$

where $B = (B(t), t \geq 0)$ denotes a standard Brownian motion, and $a(t, x)$ and $b(t, x)$ are deterministic functions. Under sufficient conditions, such as the Lipschitz condition for a and b functions and finite second moment for initial value $X(0)$, differential equation (2.1.1) has a so-called strong solution, which can be expressed as a functional of Brownian motion, i.e. $X(t) = f(B(t))$. In the last few decades, researchers devote remarkable effort to studying solutions of such diffusion equations based on some assumptions on the forms of the drift function $a(t, X(t))$ and diffusion function $b(t, X(t))$.

When both $a(t, x)$ and $b(t, x)$ are unknown parametrically, the parameters involved in $X(t)$ can be estimated using existing estimation methods, such as Fan et al. (2003), Ait-Sahalia (2002) and Hardle et al. (2003). When both $a(t, x)$ and $b(t, x)$ are unknown nonparametrically or semiparametrically, statistical estimation of the drift and diffusion functions have been extensively discussed in the literature, such as Chapter 5 of Gao (2007). Meanwhile, stochastic diffusion models with time–inhomogeneity are also useful in modelling economic and financial data. See, for example, Hamilton and Susmel (1994), **?**, Fan et al. (2003), Ait-Sahalia (2002) and Hardle et al. (2003).

In the meantime, existing literature discusses how to estimate unknown functions of nonlinear time series using nonparametric and semiparametric methods. For the stationary case, recent studies include Fan and Yao (2003), Gao (2007) and Li and Racine (2007). It should be pointed out that the literature also shows that many economic and financial data exhibit both nonlinearity and nonstationarity. Consequently, some nonparametric and semiparametric models and kernel-based methods have been proposed to deal with both nonlinearity and nonstationarity simultaneously. Existing studies mainly discuss the employment of nonparametric kernel estimation methods. Such studies include Phillips and Park (1998), Park and Phillips (1999, 2001), Karlsen and Tjøstheim (2001), Karlsen et al. (2007), Cai et al. (2009), Phillips (2009), Wang and Phillips (2009a,b), Xiao (2009), and Gao and Phillips (2010).

However, such kernel-based estimation methods are not applicable to establish closed-form expansions of Brownian motion functionals. In the stationary case, the literature discusses how series approximations may be used in dealing with stationary time series models, such as Ai and Chen (2003), Chapter 2 of Gao (2007) and Li and Racine (2007). As discussed above, there is need to study Brownian motion functionals of both time-homogeneity and time-inhomogeneity. Note that one powerful way of dealing with such problems is to decompose the process, say $f(B(t))$ or $f(t, B(t))$, where the functional form is unknown, into an orthogonal series in some Hilbert space, such that once one has obtained observed values of the process, the coefficients involved in the series can be estimated using an econometric method. Actually, there is long history that there is a close connection between stochastic processes and orthogonal polynomials. For example, the so-called Karlin-McGregor representation expresses the transition probability of the birth and death process by means of a spectral representation in terms of orthogonal polynomials. Some people clearly feel the potential importance of orthogonal polynomials in probability theory. Schoutens (2000), for instance, gives an extensive discussion about relations between stochastic processes and orthogonal polynomials.

It is therefore clear that our first aim in this paper is to expand an unknown stochastic process into an orthogonal series. To this purpose, a suitable Hilbert space which contains objective processes should be constructed, and we need to find out the orthonormal basis for the space. It then follows from the Hilbert space theory that we can expand any element in the space into series by means of existing bases. Our second aim is to employ the expansions developed in this paper to estimate an unknown function of the form $m(t, x)$ involved in an econometric time series model. This question is dealt with by considering three types of sampling, that are, on infinite interval $(0, \infty)$, on finite interval $[0, T]$ with fixed T and on compact interval $[0, T_n]$ with T_n increasing to infinity as sample size goes to infinity. All sampling points are equally spaced. Essentially they are quite different. The first case requires more restrictions on $m(t, x)$, while the last two cases, due to the compactness of the time horizon, involve relatively less restrictions on $m(t, x)$. Our results, on the other hand, show that the limiting theory in the case of $t \in [0, T]$ with T being fixed is very unique. By contrast, the asymptotic theory for the other two cases is quite similar.

2.2 Orthogonal expansion of homogeneous functionals of Brownian motion

The aim of this section is to expand a functional of Brownian motion $f(B(t))$ into an orthogonal series. The Hermite polynomial system $\{H_i(x)\}_0^\infty$ is known orthogonal on $(-\infty, \infty)$ with respect to the density function $\phi(x)$ of standard normal distribution. Let $D = d/dx$ be the differential operator. Then Hermite polynomials are defined as

$$H_i(x) := (-1)^i \frac{1}{\phi(x)} D^i \phi(x), \quad i \geq 0.$$

In addition, Hermite polynomials $H_i(x)$ can be generated by the exponential generating function

$$\exp(xw - w^2/2) = \sum_{i=0}^{\infty} H_i(x)\frac{w^i}{i!}, \tag{2.2.1}$$

which, in some cases, is more convenient to be used than the definition.

The first 11 Hermite polynomials are

$$\begin{aligned}
H_0(x) &= 1 \\
H_1(x) &= x \\
H_2(x) &= x^2 - 1 \\
H_3(x) &= x^3 - 3x \\
H_4(x) &= x^4 - 6x^2 + 3 \\
H_5(x) &= x^5 - 10x^3 + 15x \\
H_6(x) &= x^6 - 15x^4 + 45x^2 - 15 \\
H_7(x) &= x^7 - 21x^5 + 105x^3 - 105x \\
H_8(x) &= x^8 - 28x^6 + 210x^4 - 420x^2 + 105 \\
H_9(x) &= x^9 - 36x^7 + 378x^5 - 1260x^3 + 945x \\
H_{10}(x) &= x^{10} - 45x^8 + 630x^6 - 3150x^4 + 4725x^2 - 945
\end{aligned}$$

The relationship of orthogonality of Hermite system is

$$\int_{-\infty}^{\infty} H_i(x)H_j(x)\phi(x)dx = i!\delta_{ij}, \tag{2.2.2}$$

where δ_{ij} is the Kronecker delta.

It is known, (for example, Example 1 of Shiryaev, 1996, p.268) that Hermite polynomial system $\{H_i(x)\}_{i=0}^{\infty}$ is a complete orthogonal basis in the Hilbert space $L^2(\mathbb{R}, \phi(x))$ defined as

$$L^2(\mathbb{R}, \phi(x)) = \left\{ f(x) : \int_{-\infty}^{\infty} f^2(x)\phi(x)dx < \infty \right\} \tag{2.2.3}$$

and in which the inner product is defined as

$$(f, g) = \int_{-\infty}^{\infty} f(x)g(x)\phi(x)dx, \quad f, g \in L^2(\mathbb{R}, \phi(x)). \tag{2.2.4}$$

Here the completeness means that every function in the space can be represented either as $\sum_{i=0}^{k} c_i H_i(x)$, or as a limit of these form in the sense of mean square. A necessary and sufficient condition of an orthogonal system to be complete in a complete space is that if a function in the space is orthogonal with every element in the sequence, then this function must be a zero function. See Theorem 3.17 of Kufner and Kadlec (1971, p.90).

Since we shall work with Brownian motion whose density function is $\phi_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}$, it is necessary to construct a system which is orthogonal with respect to $\phi_t(x)$. To this end, for any $t > 0$, define

$$h_i(t,x) = \frac{1}{\sqrt{i!}} H_i(x/\sqrt{t}), \quad i = 0, 1, 2, \dots. \tag{2.2.5}$$

Such defined system $h_i(t,x)$ belongs to the space

$$L^2(\mathbb{R}, \phi_t(x)) = \left\{ f(x) : \int_{-\infty}^{\infty} f^2(x)\phi_t(x)dx < \infty \right\}, \tag{2.2.6}$$

which is a Hilbert space (see, for example, p.162 of Dudley 2003) with the conventional inner-product defined by $(f,g) = \int fg\phi_t dx$ and induced norm $\|f\| = (f,f)^{1/2}$.

Lemma 2.2.1. *In the space $L^2(\mathbb{R}, \phi_t(x))$, $\{h_i(t,x)\}$ is a complete orthonormal polynomial system.*

Proof. Firstly, note that

$$(h_i(t,x), h_j(t,x))_{L^2(\mathbb{R},\phi_t(x))} = \frac{1}{\sqrt{i!j!}} (H_i(x), H_j(x))_{L^2(\mathbb{R},\phi(x))} = \delta_{ij},$$

implying the orthogonality of the system.

In addition, since $L^2(\mathbb{R}, \phi_t(x))$ is complete, the completeness of $\{h_i(t,x)\}$ is tantamount to showing that suppose $f(x) \in L^2(\mathbb{R}, \phi_t(x))$ is orthogonal with every $h_i(t,x)$, then $f(x)$ must be a zero function, viz., $f(x) = 0$ in the space. In fact,

$$\begin{aligned} 0 =& (f(x), h_i(t,x))_{L^2(\mathbb{R},\phi_t(x))} = \int f(x) h_i(t,x)\phi_t(x)dx \\ =& \frac{1}{\sqrt{i!}} \int f(x) H_i(x/\sqrt{t}) \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} dx \\ =& \frac{1}{\sqrt{i!}} \int f(\sqrt{t}x) H_i(x)\phi(x)dx = \frac{1}{\sqrt{i!}} (f(\sqrt{t}x), H_i(x))_{L^2(\mathbb{R},\phi(x))}, \end{aligned}$$

which implies that $(f(\sqrt{t}x), H_i(x))_{L^2(\mathbb{R},\phi(x))} = 0$ for every i. Whence, by the completeness of $H_i(x)$ in the space $L^2(\mathbb{R}, \phi(x))$, $f(\sqrt{t}x) = 0$, hence $f(x) = 0$. ☐

Therefore, we assert that $L^2(\mathbb{R}, \phi_t(x))$ is a Hilbert space equipping with this inner-product and corresponding induced norm as well as possessing the complete orthonormal basis $\{h_i(t,x)\}_{i=0}^{\infty}$.

Let $L^2(\Omega)$ be a Hilbert space of random variables on the probability space $(\Omega, \mathcal{F}, P)$ with finite second order moments, where the inner product is defined as $\langle X, Y\rangle = E(XY)$. Now, for every $f \in L^2(\mathbb{R}, \phi_t(x))$, we can constitute a mapping between $L^2(\mathbb{R}, \phi_t(x))$ and $L^2(\Omega)$:

$$\mathscr{T}: \quad f \to f(B(t)). \tag{2.2.7}$$

Denote by Θ the image of $L^2(\mathbb{R}, \phi_t(x))$ under mapping $\mathscr{T}$. Since $E[f(B(t))]^2 < \infty$ for all $f(x) \in L^2(\mathbb{R}, \phi_t(x))$, Θ is a subset of $L^2(\Omega)$, hence, there exists inner product operation on Θ. The following lemmas show the properties of $\mathscr{T}$ and Θ.

Lemma 2.2.2. *(1) $\mathscr{T}$ is linear;*

(2) $\mathscr{T}$ is an one-one mapping from $L^2(\mathbb{R}, \phi_t(x))$ to Θ;

(3) $\mathscr{T}$ is an isomorphism.

Proof. (1). Straightforward verification. (2). For any functions $f, g \in L^2(\mathbb{R}, \phi_t(x))$, we have,

$$\begin{aligned}
\langle \mathscr{T}(f), \mathscr{T}(g)\rangle_{L^2(\Omega)} &= \langle f(B(t)), g(B(t))\rangle \\
&= E[f(B(t))g(B(t))] = \int_{-\infty}^{\infty} f(x)g(x)\phi_t(x)dx \\
&= (f, g)_{L^2(\mathbb{R}, \phi_t(x))}.
\end{aligned}$$

Thus, the transformation $\mathscr{T}$ is inner product preserving. If $f \neq g$, which amounts that they are not in the same equivalent class, then

$$\|\mathscr{T}(f) - \mathscr{T}(g)\|_{L^2(\Omega)} = \|\mathscr{T}(f-g)\| = \|f-g\|_{L^2(\mathbb{R}, \phi_t(x))} \neq 0.$$

On the other hand, if $\mathscr{T}(f) = \mathscr{T}(g)$, then $\|f-g\|_{L^2(\mathbb{R}, \phi_t(x))} = \|\mathscr{T}(f-g)\|_{L^2(\Omega)} = \|\mathscr{T}(f) - \mathscr{T}(g)\| = 0$, therefore $f = g$. Thus, $\mathscr{T}$ is one-one.

(3). Since $\mathscr{T}$ is linear and $\|\mathscr{T}(f)\| = \|f\|$ for $f \in L^2(\mathbb{R}, \phi_t(x))$, $\mathscr{T}$ is isomorphism. $\square$

Lemma 2.2.3. *Θ is a closed subspace of $L^2(\Omega)$, hence it is a Hilbert space.*

Proof. It is easy to see that Θ is a linear space due to linearity of $\mathscr{T}$. Next, suppose in Θ_t there is a Cauchy sequence $\{\xi_n\}$. Because the mapping $\mathscr{T}$ is one-one, there is a unique sequence $\{f_n\}$ in $L^2(\mathbb{R}, \phi_t(x))$ such that $\mathscr{T}(f_n) = \xi_n$, $n = 0, 1, 2, \ldots$. Due to $\|\mathscr{T}(\cdot)\| = \|\cdot\|$ and linearity of $\mathscr{T}$, $\{f_n(x)\}$ is a Cauchy sequence in $L^2(\mathbb{R}, \phi_t(x))$ as well. Therefore there exists a function $f \in L^2(\mathbb{R}, \phi_t(x))$ such that $\{f_n(x)\}$ converges to $f(x)$ in the sense of mean square since $L^2(\mathbb{R}, \phi_t(x))$ is a Hilbert space. Thus,

$$\|\xi_n - f(B(t))\|_{L^2(\Omega)} = \|\mathscr{T}(f_n) - \mathscr{T}(f)\| = \|f_n - f\|_{L^2(\mathbb{R},\phi_t(x))} \to 0$$

as $n \to \infty$, which implies that Θ is a closed subspace of $L^2(\Omega)$. Hence it is a Hilbert space. □

Lemma 2.2.4. *If $\{p_i(x)\}_{i=0}^{\infty}$ is any orthonormal basis in $L^2(\mathbb{R}, \phi_t(x))$, then $\{\mathscr{T}(p_i)\}_{n=0}^{\infty}$ is an orthonormal basis in Θ. Particularly, $\{\mathscr{T}(h_i(t, x))\}_{i=0}^{\infty} = \{h_i(t, B(t))\}_{i=0}^{\infty}$, $t > 0$, is an orthonormal basis in Θ.*

Proof. By virtue of the properties of $\mathscr{T}$ that $\mathscr{T}$ is inner product preserving, it is valid. □

Now that Θ is a Hilbert space and $\{h_i(t, B(t))\}_{i=0}^{\infty}$ is an orthonormal basis in it, we naturally have the following theorem.

Theorem 2.2.1. *For any random variable $f(B(t)) \in \Theta$, it admits a Fourier series expansion,*

$$f(B(t)) = \sum_{i=0}^{\infty} c_i(t) h_i(t, B(t)), \tag{2.2.8}$$

where $c_i(t) = c_i(t, f) = \langle f(B(t)), h_i(t, B(t))\rangle_{\Theta}$.

Example 2.1
Here is to show expansions of some Brownian motion functionals into orthogonal series using the method of Theorem 2.2.1. The first one is from straightforward calculation; the last three are all derived from exponential generating function (2.2.1).

$$B^5(t) = 15t^{5/2}h_1(t, B(t)) + 10\sqrt{6}t^{5/2}h_3(t, B(t)) + 2\sqrt{30}t^{5/2}h_5(t, B(t)).$$

$$\exp(B(t)) = \sum_{i=0}^{\infty} h_i(t, B(t)) \frac{\sqrt{t}^i}{\sqrt{i!}} e^{t/2}.$$

$$\sin B(t) = \sum_{i=0}^{\infty} c_i h_i(t, B(t)),$$

$$\cos B(t) = \sum_{i=0}^{\infty} \widetilde{c}_i h_i(t, B(t)),$$

where

$$c_i(t) = \begin{cases} 0, & \text{if } i = 2k,\ k = 0, 1, \dots, \\ (-1)^k \dfrac{\sqrt{t}^i}{\sqrt{i!}} e^{-t/2} & \text{if } i = 2k+1,\ k = 0, 1, \dots \end{cases}$$

$$\widetilde{c}_i(t) = \begin{cases} 0, & \text{if } i = 2k+1,\ k = 0, 1, \dots, \\ (-1)^k \dfrac{\sqrt{t}^i}{\sqrt{i!}} e^{-t/2} & \text{if } i = 2k,\ k = 0, 1, \dots. \end{cases}$$

Given truncation parameter k, the truncation series for $f(B(t))$ which admits an expansion of orthogonal series $\sum_{i=0}^{\infty} c_i(t)h_i(t, B(t))$ is defined as

$$f_k(B(t)) := \sum_{i=0}^{k} c_i(t)h_i(t, B(t)). \tag{2.2.9}$$

The question that naturally arises is that what is the degree of approximation of $f_k(B(t))$ to $f(B(t))$. The following theorem answers this question.

Theorem 2.2.2. *If f and its derivatives $f^{(v)}$, $v = 1, \dots, r$, are all in the Hilbert space $L^2(\mathbb{R}, \phi_t(x))$, $(t > 0)$. Denote by $c_i(t, f^{(v)})$ the coefficients in the expansion of $f^{(v)}(B(t))$, $v = 0, 1, \dots, r$, in terms of $\{h_i(t, B(t))\}_{i=0}^{\infty}$, then for sufficient large $k \in \mathbb{N}$,*

$$\|f(B(t)) - f_k(B(t))\|_{L^2(\Omega)}^2 \le \frac{t^r}{k^r} R(k, f^{(r)}). \tag{2.2.10}$$

where $R(k, f^{(r)}) = (1 + o(1)) \sum_{i=k+1}^{\infty} \left[c_{i-r}(t, f^{(r)})\right]^2 \to 0$ as $k \to \infty$ with fixed t.

Proof. Hermite polynomials can be expressed as

$$H_i(x) = (-1)^i \exp\left(\frac{x^2}{2}\right) D^i \exp\left(-\frac{x^2}{2}\right), \quad i = 0, 1, 2, \dots. \tag{2.2.11}$$

It therefore follows from the chain rule of derivative that for all $t > 0$,

$$h_i(t, x) = (-1)^i \frac{\sqrt{t}^i}{\sqrt{i!}} \exp\left(\frac{x^2}{2t}\right) D^i \exp\left(-\frac{x^2}{2t}\right), \quad i = 0, 1, 2, \dots. \tag{2.2.12}$$

Now, under the assumption that f and its derivatives $f^{(v)}$, $v = 1, \dots, r$, are all in the Hilbert space $L^2(\mathbb{R}, \phi_t(x))$, integration by parts gives,

$$\begin{aligned} c_i(t,f) &= \langle f(B(t)), h_i(t, B(t)) \rangle_\Theta = \int_{-\infty}^{\infty} f(x) h_i(t,x) \phi_t(x) dx \\ &= \int_{-\infty}^{\infty} f(x)(-1)^i \frac{\sqrt{t}^i}{\sqrt{i!}} \exp\left(\frac{x^2}{2t}\right) \left[D^i \exp\left(-\frac{x^2}{2t}\right)\right] \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right) dx \\ &= (-1)^i \frac{\sqrt{t}^i}{\sqrt{i!}} \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} f(x) D^i \exp\left(-\frac{x^2}{2t}\right) dx \\ &= (-1)^i \frac{\sqrt{t}^i}{\sqrt{i!}} \frac{1}{\sqrt{2\pi t}} \left(f(x) D^{i-1} \exp\left(-\frac{x^2}{2t}\right) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x) D^{i-1} \exp\left(-\frac{x^2}{2t}\right) dx \right) \\ &= (-1)^{i-1} \frac{\sqrt{t}^i}{\sqrt{i!}} \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} f'(x) D^{i-1} \exp\left(-\frac{x^2}{2t}\right) dx \\ &= \frac{\sqrt{t}}{\sqrt{i}} \int_{-\infty}^{\infty} f'(x) h_{i-1}(t,x) \phi_t(x) dx \\ &= \frac{\sqrt{t}}{\sqrt{i}} c_{i-1}(t, f'). \end{aligned}$$

We have to explain two limits in the above derivation,

$$(1): \lim_{x \to \infty} f(x) D^{i-1} \exp(-\frac{x^2}{2t}) = 0,$$

and

$$(2): \lim_{x \to -\infty} f(x) D^{i-1} \exp(-\frac{x^2}{2t}) = 0.$$

To prove (1), recall that $\int_{-\infty}^{\infty} f^2(x) \phi_t(x) dx < \infty$ and rewrite

$$f(x) D^{i-1} \exp(-\frac{x^2}{2t}) = A f(x) h_{i-1}(t,x) \phi_t(x),$$

where A is a constant depending on i and t, independent of x. Since both $f(x)$ and $h_{i-1}(t,x)$ are in $L^2(\mathbb{R}, \phi_t(x))$, their inner product $\int_{\mathbb{R}} f(x) h_{i-1}(t,x) \phi_t(x) dx$ exists. Suppose $F(x)$ is the primary function of $f(x) h_{i-1}(t,x) \phi_t(x)$, hence, $\lim_{x \to \infty} F(x) = L < \infty$. Therefore, by the definition,

$$A f(x) h_{i-1}(t,x) \phi_t(x) = F'(x) = \lim_{\Delta x \to 0} \frac{F(x + \Delta x) - F(x)}{\Delta x}.$$

For any $\epsilon > 0$, fixed Δx, we can choose sufficient large x, such that $|F(x)-L| < \epsilon|\Delta x|/2$ and $|F(x+\Delta x) - L| < \epsilon|\Delta x|/2$. Thus, when x is sufficient large,

$$\frac{|F(x+\Delta x) - F(x)|}{|\Delta x|} \leq \frac{|F(x+\Delta x) - L| + |F(x) - L|}{|\Delta x|} < \epsilon,$$

which implies that $\lim_{x\to\infty} Af(x)h_{i-1}(t,x)\phi_t(x) = 0$. For the same reason (2) is valid.[1]

Using the relation $c_i(t,f) = \frac{\sqrt{t}}{\sqrt{i}} c_{i-1}(t,f')$ repeatedly, we iterate that

$$c_i(t,f) = \frac{\sqrt{t}^r}{\sqrt{i(i-1)\cdots(i-r+1)}} c_{i-r}(t,f^{(r)}). \tag{2.2.13}$$

Finally, it follows from the orthogonality of $h_i(t,B(t))$ and the relation (2.2.13) that

$$\begin{aligned}
\|f(B(t)) - f_k(B(t))\|_\Theta^2 &= \left\| \sum_{i=k+1}^{\infty} c_i(t,f) h_i(t,B(t)) \right\|_\Theta^2 = \sum_{i=k+1}^{\infty} [c_i(t,f)]^2 \\
&= \sum_{i=k+1}^{\infty} \frac{t^r}{i(i-1)\cdots(i-r+1)} \left[c_{i-r}(t,f^{(r)})\right]^2 \\
&\leq \frac{t^r}{(k+1)k\cdots(k-r+2)} \sum_{i=k+1}^{\infty} \left[c_{i-r}(t,f^{(r)})\right]^2 \\
&= \frac{t^r}{k^r} R(k,f^{(r)}).
\end{aligned}$$

□

Theorem 2.8 of Hall and Heyde (1980) points out that a sufficient condition such that orthogonal series $\sum_{n=1}^{\infty} c_n X_n$ converges almost surely is $\sum_{n=1}^{\infty} c_n^2 (\log n)^2 E X_n^2 < \infty$. One therefore has the following corollary.

Corollary 2.2.1. *If the conditions in Theorem 2.2.2 hold for $r \geq 1$, $f_k(B(t))$ converges to $f(B(t))$ almost surely in $\omega \in \Omega$ for each t as $k \to \infty$.*

---

[1] There is an alternative way to demonstrate the relations (1) and (2): Suppose that they are not valid, $\lim_{x\to\infty} f(x) D^{i-1} \exp(-\frac{x^2}{2t}) = \lim_{x\to\infty} Af(x)h_{i-1}(t,x)\phi_t(x) = b_i \neq 0$, so to speak, then when x is large

$$f(x) \approx \frac{b_i}{Ah_{i-1}(t,x)\phi_t(x)}, \Rightarrow \ f^2(x)\phi_t(x) \approx \frac{b_i^2}{A^2 h_{i-1}^2(t,x)\phi_t(x)},$$

which would lead $\int f^2(x)\phi_t(x)dx$ diverges.

Proof. Note that $Eh_i^2(t, B(t)) = 1$ for $\forall i$. We only need to check the sufficient condition for $r = 1$. In fact, by (2.2.13), we have

$$\sum_{i=2}^{\infty} c_i^2(t,f)(\log i)^2 = t \sum_{i=2}^{\infty} \frac{\log^2 i}{i} c_{i-1}^2(t,f')$$
$$\le t \sum_{i=2}^{\infty} c_{i-2}^2(t,f'') \le t\|f''\|_{L^2(\mathbb{R},\phi_t(x))} < \infty,$$

for any $t > 0$ since $(\log i)/i$ converges to zero. In view of Theorem 2.8 in Hall and Heyde (1980), the assertion holds. □

2.3 Expansion of coefficient functions

As suggested in Example 2.1, the coefficients in the expansion of $f(B(t))$ are actually functions of t. This would hamstring the application of this method in time series econometrics. One way to tackle this issue is to expand the coefficient functions again into orthogonal series given that the coefficients satisfy a certain condition, so that at the range of econometrical applications we can estimate the constant coefficients.

We would focus our attention in two scenarios where time horizons are finite and infinite respectively, since in practice time zone is always limited, and theoretically, however, it can reach infinity.

2.3.1 Infinite time horizon

The Laguerre system is engaged to expand the coefficient functions when the time zone is infinity, that is, we consider $t \in (0, \infty)$. The reason for adopting Laguerre system is that this system is orthogonal on $(0, \infty)$ which coincides with the domain of the time horizon and Brownian motion. The generalised Laguerre polynomial system $\{L_j^{(\alpha)}(t)\}_0^{\infty}$ $(\alpha > -1)$ is a complete orthogonal sequence with respect to the density $t^\alpha e^{-t}$ in the function space $L^2(\mathbb{R}^+, t^\alpha e^{-t}) := \{\varphi(t) : \int_0^\infty \varphi^2(t) t^\alpha e^{-t} dt < \infty\}$. See, for example, Szego (1975). It is defined by the Rodrigues equation that

$$L_j^{(\alpha)}(t) = \frac{t^{-\alpha} e^t}{j!} D^j (t^{j+\alpha} e^{-t}).$$

The orthogonality for generalised Laguerre polynomials is expressed as

$$\int_0^\infty t^\alpha e^{-t} L_j^{(\alpha)}(t) L_m^{(\alpha)}(t) dt = \Gamma(\alpha+1)\binom{j+\alpha}{j}\delta_{mj},$$

where $\Gamma(\cdot)$ is the Gamma function.

The simplest Laguerre polynomials are recovered from the generalised polynomials by setting $\alpha = 0$: $L_j(t) := L_j^{(0)}(t)$.

As the system of $\{L_j^{(\alpha)}(t)\}$ consists of polynomials which are unbounded on $\mathbb{R}^+$, it is difficult to obtain uniform convergence to a function in the space. Nevertheless, the uniform approximation of a function on $\mathbb{R}^+$ is crucial to the development of this study. To obtain the uniform convergence, we introduce a modified orthonormal system on $\mathbb{R}^+$.

Let for $\alpha \geq 0$

$$
\begin{aligned}
\mathscr{L}_j(t) &:= e^{-t/2}L_j(t) = \frac{1}{j!}e^{t/2}D^j(t^je^{-t}) \\
\mathscr{L}_j^{(\alpha)}(t) &:= \frac{1}{\sqrt{\Gamma(\alpha+1)C_{j+\alpha}^j}}t^{\alpha/2}e^{-t/2}L_j^{(\alpha)}(t) \\
&= \frac{1}{\sqrt{\Gamma(\alpha+1)C_{j+\alpha}^j j!}}t^{-\alpha/2}e^{t/2}D^j(t^{j+\alpha}e^{-t})
\end{aligned}
$$

Now $\mathscr{L}_j^{(\alpha)}(t), j = 0, 1, \ldots$, form an orthonormal basis in $L^2(\mathbb{R}^+) = \{\varphi(t) : \int_0^\infty \varphi(t)^2 dt < \infty\}$ (see Sansone, 1959, p.351). The space $L^2(\mathbb{R}^+)$ is apparently a Hilbert space after equipped with an inner product $(g, h) = \int_0^\infty g(t)h(t)dt$ and an induced norm $\|f\| = (f, f)^{1/2}$, by the conventional L^2 space theory, since the Lebesgue measure is σ-finite on $\mathbb{R}^+$. By virtue of Hilbert space, for any $\varphi \in L^2(\mathbb{R}^+)$, we have an unique expansion:

$$
\varphi(t) = \sum_{j=0}^{\infty} a_j^{(\alpha)}\mathscr{L}_j^{(\alpha)}(t), \tag{2.3.1}
$$

where $a_j^{(\alpha)} = \int_0^\infty \varphi(t)\mathscr{L}_j^{(\alpha)}(t)dt$. Designate $a_j = a_j^{(0)}$ for convenience. We mainly in the sequel discuss in the situation that $\alpha = 0$.

Given truncation parameter p, correspondingly we have a truncation series

$$
\varphi_p(t) = \sum_{j=0}^{p} a_j\mathscr{L}_j(t). \tag{2.3.2}
$$

It is known that $\varphi_p(t)$ converges to $\varphi(t)$ in the space $L^2(\mathbb{R}^+)$ as p approaches to infinity. The following theorem gives the convergence rates both in the sense of norm and pointwise.

Theorem 2.3.1. *Suppose $\varphi(t) \in L^2(\mathbb{R}^+)$ is differentiable until r-th ($r \geq 1$) order such that $t^{r/2}\varphi^{(v)}(t), v = 0, 1, \ldots, r$ are in the space $L^2(\mathbb{R}^+)$ as well. Then we have*

$$
\|\varphi(t) - \varphi_p(t)\|^2 \leq \frac{1}{p^r}R^2(p), \tag{2.3.3}
$$

$$|\varphi(t) - \varphi_p(t)|^2 \leq \frac{1}{r-1}\frac{1}{p^{r-1}}\left(\sup_{j\geq p+1} |\mathscr{L}_j(t)|\right)^2 R^2(p), \quad (r>1), \tag{2.3.4}$$

for sufficient large p, where $R^2(p) = (1+o(1))\sum_{j=p+1}^{\infty}[a_{j-r}^{(r)}(\tilde{\phi})]^2$ is an infinitesimal with $p \to \infty$ in which $\tilde{\varphi}(t) = t^{r/2}e^{-t/2}[\varphi(t)e^{t/2}]^{(r)}$.

Proof. Straightforward calculation yields that

$$\begin{aligned}
a_j(\varphi) &= \int_0^{\infty} \varphi(t)\mathscr{L}_j(t)dt = \int_0^{\infty} \varphi(t)\frac{1}{j!}e^{t/2}D^j(t^je^{-t})dt \\
&= \frac{1}{j!}\varphi(t)e^{t/2}D^{j-1}(t^je^{-t})\Big|_0^{\infty} - \frac{1}{j!}\int_0^{\infty}[\varphi(t)e^{t/2}]'D^{j-1}(t^je^{-t})dt \\
&= -\frac{1}{j!}\int_0^{\infty}[\varphi(t)e^{t/2}]'D^{j-1}(t^je^{-t})dt \\
&= \cdots \\
&= \frac{(-1)^r}{j!}\int_0^{\infty}[\varphi(t)e^{t/2}]^{(r)}D^{j-r}(t^je^{-t})dt \\
&= \frac{(-1)^r}{j!}\frac{\sqrt{\Gamma(r+1)C_j^{j-r}}(j-r)!}{\sqrt{\Gamma(r+1)C_j^{j-r}}(j-r)!}\int_0^{\infty}\tilde{\varphi}(t)t^{-r/2}e^{t/2}D^{j-r}(t^{j-r+r}e^{-t})dt \\
&= \frac{(-1)^r}{j!}\sqrt{\Gamma(r+1)C_j^{j-r}}(j-r)!a_{j-r}^{(r)}(\tilde{\varphi}) \\
&= (-1)^r\sqrt{\frac{(j-r)!}{j!}}a_{j-r}^{(r)}(\tilde{\varphi}),
\end{aligned}$$

where $\tilde{\phi}(t) = t^{r/2}e^{-t/2}[\phi(t)e^{t/2}]^{(r)}$.

It follows that

$$\begin{aligned}
\|\varphi(t) - \varphi_p(t)\|^2_{L^2(\mathbb{R}^+)} &= \left\|\sum_{j=p+1}^{\infty} a_j\mathscr{L}_j(t)\right\|^2 = \sum_{j=p+1}^{\infty} a_j^2 \\
&= \sum_{j=p+1}^{\infty}\frac{(j-r)!}{j!}[a_{j-r}^{(r)}(\tilde{\varphi})]^2 \leq \frac{(p+1-r)!}{(p+1)!}\sum_{j=p+1}^{\infty}[a_{j-r}^{(r)}(\tilde{\varphi})]^2 \\
&= \frac{1}{p^r}R^2(p).
\end{aligned}$$

On the other hand, using Cauchy-Schwarz inequality,

$$|\varphi(t) - \varphi_p(t)|^2 = \left|\sum_{j=p+1}^{\infty} a_j\mathscr{L}_j(t)\right|^2$$

$$\leq \left(\sum_{j=p+1}^{\infty} |a_j||\mathscr{L}_j(t)| \right)^2 \leq \left(\sup_{j\geq p+1} |\mathscr{L}_j(t)| \right)^2 \left(\sum_{j=p+1}^{\infty} |a_j| \right)^2$$

$$= \left(\sup_{j\geq p+1} |\mathscr{L}_j(t)| \right)^2 \left(\sum_{j=p+1}^{\infty} \sqrt{\frac{(j-r)!}{j!}} |a_{j-r}^{(r)}(\tilde{\varphi})| \right)^2$$

$$\leq \left(\sup_{j\geq p+1} |\mathscr{L}_j(t)| \right)^2 \sum_{j=p+1}^{\infty} \frac{(j-r)!}{j!} \sum_{j=p+1}^{\infty} |a_{j-r}^{(r)}(\tilde{\varphi})|^2$$

$$= \left(\sup_{j\geq p+1} |\mathscr{L}_j(t)| \right)^2 \sum_{j=p+1}^{\infty} |a_{j-r}^{(r)}(\tilde{\varphi})|^2 \sum_{j=p+1}^{\infty} \frac{1}{j(j-1)\cdots(j-r+1)}$$

$$\leq \left(\sup_{j\geq p+1} |\mathscr{L}_j(t)| \right)^2 \sum_{j=p+1}^{\infty} |a_{j-r}^{(r)}(\tilde{\varphi})|^2 \int_p^{\infty} \frac{1}{u(u-1)\cdots(u-r+1)} du$$

$$\leq \left(\sup_{j\geq p+1} |\mathscr{L}_j(t)| \right)^2 \sum_{j=p+1}^{\infty} |a_{j-r}^{(r)}(\tilde{\varphi})|^2 \left(\int_p^{\infty} \frac{1}{(u-r+1)^r} du \right)$$

$$= \frac{1}{r-1} \frac{1}{(p-r+1)^{r-1}} \left(\sup_{j\geq p+1} |\mathscr{L}_j(t)| \right)^2 \sum_{j=p+1}^{\infty} |a_{j-r}^{(r)}(\tilde{\varphi})|^2$$

$$= \frac{1}{r-1} \frac{1}{p^{r-1}} R^2(p),$$

where $r > 1$. □

Actually approximation of $\varphi_p(t)$ to $\varphi(t)$ in the above theorem is uniform. The table on Askey and Wainger (1965, p.699) shows that, given any $\alpha \geq 0$, there are positive constants C and γ, independent of j and t, such that for all integers $j \geq 0$,

$$|\mathscr{L}_j^{(\alpha)}(t)| \leq \begin{cases} Ct^{\alpha/2} m^{\alpha/2}, & \text{if } 0 < t \leq \frac{1}{m} \\ Ct^{-1/4} m^{-1/4}, & \text{if } \frac{1}{m} < t \leq \frac{m}{2} \\ Cm^{-3/4}(m^{1/3} + |t-m|)^{1/4}, & \text{if } \frac{m}{2} < t \leq \frac{3m}{2} \\ Ce^{-\gamma t}, & \text{if } t > \frac{3m}{2} \end{cases} \tag{2.3.5}$$

where $m = 4j + 2\alpha + 2$.

It therefore follows that $|\mathscr{L}_j(t)|$ is uniformly bounded by C and turns out that $\varphi_p(t)$ approaches $\varphi(t)$ uniformly.

2.3.2 Finite time horizon

We now concentrate on finite horizon, namely, let $t \in [0, T]$ for some fixed $T > 0$. To expand a function $g(t)$ on $[0, T]$ into an orthogonal series, a basic requirement is that $g(t) \in L^2[0, T]$ since it is a Hilbert space. Observe that there are many complete orthonormal basis in $L^[0, T]$, for example,

(I) $\varphi_0(t) = \sqrt{\frac{1}{T}}$, $\varphi_j(t) = \sqrt{\frac{2}{T}} \cos \frac{j\pi t}{T}$, $\quad j = 1, 2, \cdots;$

(II) $\psi_j(t) = \sqrt{\frac{2}{T}} \sin \frac{j\pi t}{T}$, $\quad j = 1, 2, \cdots.$

See Davis (1963). Not only trigonometrical system, there are also orthogonal polynomials system in the space and one even can form an orthogonal series based on some density and the interval (see Nikiforov and Uvarov, 1988). In what follows, we shall employ the system $\varphi_j(t)$ to our purpose. In order to emphasise this system is orthogonal on $[0, T]$, write it as $\varphi_{jT}(t)$.

It is clear that if $g(t) \in L^2[0, T]$, we have an expansion of $g(t)$ in terms of $\varphi_{jT}(t)$, namely, $g(t) = \sum_{j=0}^{\infty} b_j \varphi_{jT}(t)$, where $b_j = b_j(g) = (g(t), \varphi_{jT}(t))$, the conventional inner product on $L^2[0, T]$.

Given a truncation parameter N, let $g_N(t) = \sum_{j=0}^{N} b_j \varphi_{jT}(t)$ be the truncation series.

Theorem 2.3.2. *(1) If $g(t)$ is differentiable on $[0, T]$ and $g'(t) \in L^2[0, T]$. Then the expansion of $g(t)$ converges to $g(t)$ pointwise.*

(2) If $g(t), g'(t), g''(t)$ are all in $L^2[0, T]$, then we have

$$\|g(t) - g_N(t)\|^2 \leq C_1 \frac{a_T^2 T^3}{N^3} + C_2 \frac{a_T T^{3.5}}{N^{3.5}} R_{1N} + C_3 \frac{T^4}{N^4} R_{1N}^2, \tag{2.3.6}$$

$$|g(t) - g_N(t)| \leq C_4 \frac{a_T T}{N} + C_5 \frac{T^{1.5}}{N^{1.5}} R_{1N}, \tag{2.3.7}$$

where $C_j (j = 1, 2, \cdots 5)$ are absolutely constants; $a_T := |g'(T)| + |g'(0)|$, $R_{1N}^2 = \sum_{j=N+1}^{\infty} |b_j(g'')|^2$ which converge to zero when $N \to \infty$.

(3) If $g(t)$ and its derivatives until third order are all in $L^2[0, T]$, then we have

$$|g(t) - g_N(t)| \leq C_4 \frac{a_T T}{N} + C_5 \frac{T^{1.5}}{N^2} R_N, \tag{2.3.8}$$

where $R_{2N} = \sum_{j=N+1}^{\infty} |b_j(g'')|$ that converges to zero as $N \to \infty$.

Proof. Let us start with the calculation of the coefficients. Integration by parts gives

$$\begin{aligned} b_j(g) &= \int_0^T g(t)\varphi_{jT}(t)dt = \sqrt{\frac{2}{T}}\int_0^T g(t)\cos\frac{j\pi t}{T}dt \\ &= -\frac{T}{j\pi}\sqrt{\frac{2}{T}}\int_0^T g'(t)\sin\frac{j\pi t}{T}dt \\ &= -\frac{T}{j\pi}\beta_j(g'), \end{aligned}$$

where β_j is the j-th coefficient of the expansion of $g'(t)$ in terms of the orthonormal system $\psi_j(t)$. Therefore,

$$\sum_{j=1}^{\infty}|b_j| \leq \frac{T}{\pi}\left(\sum_{j=1}^{\infty}|\beta_j|^2\right)^{1/2}\left(\sum_{j=1}^{\infty}\frac{1}{j^2}\right)^{1/2} = \frac{T}{\pi}\|g'(t)\|\left(\sum_{j=1}^{\infty}\frac{1}{j^2}\right)^{1/2}.$$

Absolutely convergence of $\sum_{j=1}^{\infty} b_j$ indicates that the series $\sum_{j=0}^{\infty} b_j\varphi_{jT}(t)$ converges uniformly, and due to the continuity of the basis the sum function is continuous on $[0,T]$. This sum function must be $g(t)$. In fact, if we signify $\tilde{g}(t) = \sum_{j=0}^{\infty} b_j\varphi_{jT}(t)$, then $b_j = (\tilde{g}(t), \varphi_{jT}(t))$, but from the expansion, we know $b_j = (g(t), \varphi_{jT}(t))$. Thus $(\tilde{g}(t) - g(t), \varphi_{jT}(t)) = 0$ for $\forall j$. In view of completeness of the basis, $\tilde{g}(t) = g(t)$.

Suppose now that $g(t), g'(t), g''(t)$ are all in $L^2[0,T]$. We can calculate b_j further using integration by parts

$$\begin{aligned} b_j(g) &= -\frac{T}{j\pi}\sqrt{\frac{2}{T}}\int_0^T g'(t)\sin\frac{j\pi t}{T}dt \\ &= \left(\frac{T}{j\pi}\right)^2\sqrt{\frac{2}{T}}\int_0^T g'(t)d\cos\frac{j\pi t}{T} \\ &= \left(\frac{T}{j\pi}\right)^2\sqrt{\frac{2}{T}}[(-1)^j g'(T) - g'(0)] - \left(\frac{T}{j\pi}\right)^2\sqrt{\frac{2}{T}}\int_0^T g''(t)\cos\frac{j\pi t}{T}dt \\ &= \left(\frac{T}{j\pi}\right)^2\sqrt{\frac{2}{T}}[(-1)^j g'(T) - g'(0)] - \left(\frac{T}{j\pi}\right)^2 b_j(g''). \end{aligned}$$

Accordingly, denoting $a_T = |g'(T)| + |g'(0)|$,

$$\|g(t) - g_N(t)\|^2 = \left\|\sum_{j=N+1}^{\infty} b_j\varphi_{jT}(t)\right\|^2 = \sum_{j=N+1}^{\infty} b_j^2$$

$$= \sum_{j=N+1}^{\infty}\left[\left(\frac{T}{j\pi}\right)^2\sqrt{\frac{2}{T}}[(-1)^j g'(T) - g'(0)] - \left(\frac{T}{j\pi}\right)^2 b_j(g'')\right]^2$$

$$
\begin{aligned}
&= \sum_{j=N+1}^{\infty} \left(\frac{T}{j\pi}\right)^4 \left[\sqrt{\frac{2}{T}}[(-1)^j g'(T) - g'(0)] - b_j(g'')\right]^2 \\
&\leq \sum_{j=N+1}^{\infty} \left(\frac{T}{j\pi}\right)^4 \left(\frac{2}{T}a_T^2 + 2\sqrt{\frac{2}{T}}a_T|b_j(g'')| + |b_j(g'')|^2\right) \\
&\leq \frac{2T^3 a_T^2}{\pi^4} \sum_{j=N+1}^{\infty} \frac{1}{j^4} + \frac{2\sqrt{2}T^{3.5}a_T}{\pi^4} \sum_{j=N+1}^{\infty} \frac{1}{j^4}|b_j(g'')| + \frac{T^4}{\pi^4} \sum_{j=N+1}^{\infty} \frac{1}{j^4}|b_j(g'')|^2 \\
&\leq C_1 \frac{a_T^2 T^3}{N^3} + C_2 \frac{a_T T^{3.5}}{N^{3.5}} R_{1N} + C_3 \frac{T^4}{N^4} R_{1N}^2,
\end{aligned}
$$

where C_1, C_2, C_3 are $2/\pi^4$, $2\sqrt{2}/\pi^4$ and $1/\pi^4$ respectively; $R_{1N}^2 = \sum_{j=N+1}^{\infty} |b_j(g'')|^2$ which converge to zero when $N \to \infty$.

Meanwhile,

$$
\begin{aligned}
|g(t) - g_N(t)| &= \left|\sum_{j=N+1}^{\infty} b_j \varphi_{jT}(t)\right| \leq \sqrt{\frac{2}{T}} \sum_{j=N+1}^{\infty} |b_j| \\
&= \sqrt{\frac{2}{T}} \sum_{j=N+1}^{\infty} \left| \left(\frac{T}{j\pi}\right)^2 \sqrt{\frac{2}{T}}[(-1)^j g'(T) - g'(0)] - \left(\frac{T}{j\pi}\right)^2 b_j(g'') \right| \\
&\leq a_T \frac{2}{T} \left(\frac{T}{\pi}\right)^2 \sum_{j=N+1}^{\infty} \frac{1}{j^2} + \sqrt{\frac{2}{T}} \left(\frac{T}{\pi}\right)^2 \sum_{j=N+1}^{\infty} \frac{1}{j^2}|b_j(g'')| \\
&\leq C_4 \frac{a_T T}{N} + C_5 \frac{T^{1.5}}{N^{1.5}} R_{1N},
\end{aligned}
$$

where C_4 and C_5 are $2/\pi^2$ and $\sqrt{2}/\pi^2$ respectively; R_{1N} retains the same as before.

However, if $g'''(t) \in L^2[0, T]$, since $\sum_{j=0}^{\infty} |b_j(g'')|$ is convergent, we have

$$
|g(t) - g_N(t)| \leq C_4 \frac{a_T T}{N} + C_5 \frac{T^{1.5}}{N^2} R_{2N},
$$

in which all notations are the same as before except for $R_{2N} = \sum_{j=N+1}^{\infty} |b_j(g'')|$ that converges to zero as $N \to \infty$. □

2.3.3 Converse questions of expansion

Two interesting and useful converse questions arising are that (1) Given constant sequences $\{a_j\}$ and $\{b_j\}$, when the series $\sum_0^{\infty} a_j \mathscr{L}_j(t)$ and $\sum_{j=0}^{\infty} b_j \varphi_{jT}(t)$ converge to functions in the spaces respectively? (2) If any, when is it/ are they differentiable? The reason for studying these questions is to be specified later.

Fortunately question (1) has been answered by some existing theorems. See, Riesz-Fischer theorem in Dudley (2003, p.167). Roughly speaking, if the sequence $\{a_j\}$ is square summable the series $\sum_0^\infty a_j P_j(t)$ converges to a function in the corresponding space given that $P_j(t)$ is a complete orthogonal system in the space. On top of that, if the basis functions are continuous and the series converges uniformly, the sum function is continuous as well.

We are now in a position to answer the second question.

Theorem 2.3.3. *Suppose that there is a positive integer r for given sequence $\{a_j\}$ such that*

$$\sum_{j=r}^{\infty} \omega(j,r)|a_{j-r}| < \infty, \tag{2.3.9}$$

where $\omega(j,r) := \sqrt{j(j-1)\cdots(j-r+1)}$, then the function $\varphi(t) = \sum_{j=r}^\infty a_j \mathscr{L}_j(t)$, generated by $\{a_j\}$, exists and is differentiable until r-th derivative.

Remark 2.3.1. As condition (2.3.9) is much stronger than its counterpart in the Riesz-Fischer theorem, the function $\varphi(t)$ exists and, as can be seen in its proof, the convergence of the series is uniformly.

Proof. It is evident that $\varphi(t)$ exists.

Let us now prove the case of $r = 1$ for differentiability.

Let $\varphi(t) = \sum_{j=0}^\infty a_j \mathscr{L}_j(t)$ and $g_1(t) = \sum_{j=1}^\infty \sqrt{j} a_{j-1} \mathscr{L}_{j-1}^{(1)}(t)$. They are elements in $L^2(\mathbb{R})$ since both $\sum_{j=0}^\infty a_j^2$ and $\sum_{j=1}^\infty j a_{j-1}^2$ converge.

Rewrite $g_1(t) = \sum_{j=1}^\infty \sqrt{j} a_{j-1} \mathscr{L}_{j-1}^{(1)}(t) = \sum_{j=0}^\infty \sqrt{j+1} a_j \mathscr{L}_j^{(1)}(t)$. Note that

$$\begin{aligned}
\frac{\sqrt{j+1}}{\sqrt{t}} \mathscr{L}_j^{(1)}(t) &= \frac{\sqrt{j+1}}{\sqrt{t}} \frac{1}{\sqrt{\Gamma(2) C_{j+1}^j j!}} t^{-1/2} e^{t/2} D^j (t^{j+1} e^{-t}) \\
&= \frac{1}{j! t} e^{t/2} D^j (t^{j+1} e^{-t}) = \frac{1}{j! t} e^{t/2} \sum_{k=0}^{j} [t^{j+1}]^{(k)} [e^{-t}]^{(j-k)} C_j^k \\
&= \frac{1}{j! t} e^{-t/2} \sum_{k=0}^{j} (-1)^{j-k} (j+1) j \cdots (j-k+2) t^{j+1-k} C_j^k \\
&= \frac{1}{j!} e^{-t/2} \sum_{k=0}^{j} (-1)^{j-k} \frac{(j+1)!}{(j-k+1)!} \frac{j!}{k!(j-k)!} t^{j-k}.
\end{aligned} \tag{2.3.10}$$

On the other hand, since $\mathscr{L}_j(t) = \frac{1}{j!} e^{t/2} D^j(t^j e^{-t})$,

$$\begin{aligned}
\frac{d}{dt}\mathscr{L}_j(t) &= \frac{1}{j!}\frac{1}{2} e^{t/2} D^j(t^j e^{-t}) + \frac{1}{j!} e^{t/2} D^{j+1}(t^j e^{-t}) \\
&= \frac{1}{2}\mathscr{L}_j(t) + \frac{1}{j!} e^{t/2} \sum_{k=0}^{j+1} [t^j]^{(k)} [e^{-t}]^{(j+1-k)} C_{j+1}^k \\
&= \frac{1}{2}\mathscr{L}_j(t) + \frac{1}{j!} e^{-t/2} \sum_{k=0}^{j} j(j-1)\cdots(j-k+1) t^{j-k} (-1)^{j+1-k} C_{j+1}^k \\
&= \frac{1}{2}\mathscr{L}_j(t) - \frac{1}{j!} e^{-t/2} \sum_{k=0}^{j} (-1)^{j-k} \frac{j!}{(j-k)!} \frac{(j+1)!}{(j+1-k)!k!} t^{j-k} \\
&= \frac{1}{2}\mathscr{L}_j(t) - \frac{\sqrt{j+1}}{\sqrt{t}} \mathscr{L}_j^{(1)}(t),
\end{aligned}$$

in virtue of (2.3.10). It follows from the above that

$$\begin{aligned}
\sum_{j=0}^{\infty} a_j \frac{d}{dt}\mathscr{L}_j(t) &= \frac{1}{2}\sum_{j=0}^{\infty} a_j \mathscr{L}_j(t) - \frac{1}{\sqrt{t}} \sum_{j=0}^{\infty} \sqrt{j+1}\, a_j \mathscr{L}_j^{(1)}(t) \\
&= \frac{1}{2}\varphi(t) - \frac{1}{\sqrt{t}} g_1(t),
\end{aligned}$$

which is exactly $\varphi'(t)$. This is because firstly the continuity and boundedness of $\mathscr{L}_j^{(\alpha)}(t)$ and secondly $\sum_{j=0}^{\infty} |a_j| < \sum_{j=0}^{\infty} \sqrt{j+1}|a_j| < \infty$, hence both of two series involved are uniformly convergent.

Now induction is used to obtain the further result. Suppose that the derivatives $\varphi'(t), \cdots, \varphi^{(v)}(t), (v < r)$ exist. we construct $g_1(t), \cdots, g_{v+1}(t)$ as

$$g_k(t) = \sum_{j=k}^{\infty} \omega(j,k) a_{j-k} \mathscr{L}_{j-k}^{(k)}(t) = \sum_{j=0}^{\infty} \omega(j+k,k) a_j \mathscr{L}_j^{(k)}(t),$$

for $k = 1, \cdots, v+1$. Similarly, we have that

$$\begin{aligned}
&\frac{\omega(j+v+1, v+1)}{t^{(v+1)/2}} \mathscr{L}_j^{(v+1)}(t) \\
&= \frac{\omega(j+v+1, v+1)}{t^{(v+1)/2}} \frac{t^{-(v+1)/2} e^{t/2}}{\sqrt{\Gamma(v+2) C_{j+v+1}^j} j!} D^j(t^{j+v+1} e^{-t}) \\
&= \frac{e^{t/2}}{j! t^{v+1}} D^j(t^{j+v+1} e^{-t}) = \frac{e^{t/2}}{j! t^{v+1}} \sum_{u=0}^{j} (t^{j+v+1})^{(u)} (e^{-t})^{(j-u)} C_j^u
\end{aligned}$$

$$= \frac{e^{-t/2}}{j!t^{v+1}} \sum_{u=0}^{j} (-1)^{j-u}(j+v+1)(j+v)\cdots(j+v-u+2)t^{j+v-u+1}C_j^u$$

$$= \frac{e^{-t/2}}{j!} \sum_{u=0}^{j} (-1)^{j-u} \frac{(j+v+1)!}{(j+v-u+1)!} \frac{j!}{u!(j-u)!} t^{j-u}. \tag{2.3.11}$$

Meanwhile, we also have

$$\begin{aligned}
D^{v+1}\mathscr{L}_j(t) &= \frac{1}{j!} \sum_{u=0}^{v+1} C_{v+1}^u [e^{t/2}]^{(u)} D^{j+v+1-u}(t^j e^{-t}) \\
&= \frac{1}{j!} \sum_{u=0}^{v+1} C_{v+1}^u 2^{-u} e^{t/2} \sum_{l=0}^{j+v+1-u} C_{j+v+1-u}^l (t^j)^{(l)} (e^{-t})^{(j+v+1-u-l)} \\
&= \frac{1}{j!} \sum_{u=0}^{v+1} C_{v+1}^u 2^{-u} e^{-t/2} \sum_{l=0}^{j} C_{j+v+1-u}^l j(j-1)\cdots(j-l+1)t^{j-l}(-1)^{j+v+1-u-l} \\
&= \sum_{u=0}^{v+1} C_{v+1}^u 2^{-u}(-1)^{v+1-u} \frac{e^{-t/2}}{j!} \sum_{l=0}^{j} \frac{(j+v+1-u)!}{l!(j+v+1-u-l)!} \frac{j!}{(j-l)!} t^{j-l}(-1)^{j-l} \\
&:= \sum_{u=0}^{v+1} C_{v+1}^u 2^{-u}(-1)^{v+1-u} A(u).
\end{aligned}$$

Notice that

$$\begin{aligned}
A(u) &:= \frac{e^{-t/2}}{j!} \sum_{l=0}^{j} \frac{(j+v+1-u)!}{l!(j+v+1-u-l)!} \frac{j!}{(j-l)!} t^{j-l}(-1)^{j-l} \\
&= \frac{\omega(j+v+1-u, v-u+1)}{t^{(v+1-u)/2}} \mathscr{L}_j^{(v+1-u)}(t),
\end{aligned}$$

using (2.3.11), where $u = 0, 1, \cdots, v+1$. It follows from the same argument as in the case of $r = 1$ that

$$\sum_{j=0}^{\infty} a_j D^{v+1}\mathscr{L}_j(t) = \sum_{u=0}^{v+1} C_{v+1}^u 2^{-u}(-1)^{v+1-u} \frac{1}{t^{(v+1-u)/2}} g_{v+1-u}(t)$$

in which $g_0(t) = \varphi(t)$. Thus $\varphi^{(v+1)}(t)$ exists by virtue of the uniform convergence of the series involved. The proof is completed. □

A similar result for the expansion on $[0, T]$ is as follows.

Theorem 2.3.4. *For a given real sequence $\{b_j\}$ if there is a positive integer r such that $\sum_{j=0}^{\infty} j^r|b_j| < \infty$, then the function $g(t) = \sum_{j=0}^{\infty} b_j \varphi_{jT}(t)$ exists on $[0, T]$ and is differentiable until r-th order.*

Proof. We only need to prove the case $r = 1$, as the following proof indicated.

First, $g(t) = \sum_{j=0}^{\infty} b_j \varphi_{jT}(t)$ exists due to $\sum_{j=0}^{\infty} |b_j| < \infty$. Now consider series of

$$\frac{\pi}{T} \sum_{j=1}^{\infty} j b_j \psi_{jT}(t),$$

where $\psi_{jT}(t) = \sqrt{\frac{2}{T}} \sin \frac{j\pi t}{T}$. The sum makes sense because $\sum_{j=1}^{\infty} j^2 b_j^2 < \infty$. Let us designate it as $p(t)$. Moreover, the sum converges to $p(t)$ uniformly due to the boundedness of $\psi_{jT}(t)$ and absolutely convergence of $\sum_{j=1}^{\infty} j|b_j| < \infty$. In view of the relation between $\psi_{jT}(t)$ and $\varphi_{jT}(t)$, one concludes that $p(t) = g'(t)$. $\square$

2.4 Orthogonal expansion of time-inhomogeneous functionals of Brownian motion

There is a strong motivation that makes us extend the expansion technique from univariate to multivariate. Time-homogeneous models in finance and economics have undoubtedly dominated the literature on the modelling of time series datasets. For example, both Black and Scholes (1973) and Cox et al. (1985) models assume that the underlying process is a diffusion process. The reason is because economic theories do not suggest an explicit functional form for continuous-time models and researchers therefore take the simple specification of the functions involved. However, Hamilton and Susmel (1994) and Mikosch and Starica (2004) point out that invariant parametric specifications are often inconvenient to model long return series. In recent years, therefore, the literature has naturally evolved towards the inclusion of multiple variables in continuous time models. One example is that in Mercurio and Spokoiny (2004) the returns R_t of the asset process are stipulated as a heteroscedastic model $R_t = \sigma_t \xi_t$ where ξ_t are standard Gaussian independent innovations and σ_t is a time-varying volatility coefficient. The relevant works include Fan et al. (2003), Ait-Sahalia (2002), Hardle et al. (2003) and so forth.

We consider the expansion of $f(t, B(t))$ in this section for some bivariate functional f where $B(t)$ is a standard Brownian motion. The expansion eventually facilitates the application in econometric estimation.

2.4.1 Infinite time horizon

Let $\mathbb{R}$ and $\mathbb{R}^+$ be the sets of real numbers and positive real numbers respectively, $(\mathbb{R}, \mathfrak{B})$ and $(\mathbb{R}^+, \mathfrak{B}^+)$ be correspondingly the measurable spaces in which $\mathfrak{B}$ and $\mathfrak{B}^+$ are Borel σ-algebra on $\mathbb{R}$ and $\mathbb{R}^+$. Now introduce measure spaces $(\mathbb{R}, \mathfrak{B}, \mu_t)$ and $(\mathbb{R}^+, \mathfrak{B}^+, \lambda)$ where μ_t is defined for any $t > 0$

$$\mu_t((-\infty, a]) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right) dx := \int_{-\infty}^{a} \phi_t(x)dx, \tag{2.4.1}$$

a measure defined by the density of $N(0,t)$ (actually the distribution function of $N(0,t)$) and λ is the Lebesgue measure.

Consider the product space of these two measure spaces. For $\forall B \in \mathfrak{B}$ and $\forall B^+ \in \mathfrak{B}^+$ define

$$\nu(B^+ \times B) = \lambda(B^+)\mu_t(B). \tag{2.4.2}$$

Since μ_t is a finite measure and λ is σ-finite, ν can be uniquely extended to a measure on $\mathfrak{B}^+ \otimes \mathfrak{B}$, which is the σ-algebra generated by the collection of all sets $B^+ \times B$, $\forall B \in \mathfrak{B}$ and $B^+ \in \mathfrak{B}^+$. See, for example, Theorem 4.4.4 in Dudley (2003). The measure ν on $\mathfrak{B}^+ \otimes \mathfrak{B}$ is so-called the product measure $\lambda \times \mu_t$. Thus, $(\mathbb{R}^+ \times \mathbb{R}, \mathfrak{B}^+ \otimes \mathfrak{B}, \nu)$ becomes a measure space.

For measure space $(\mathbb{R}^+ \times \mathbb{R}, \mathfrak{B}^+ \otimes \mathfrak{B}, \nu)$, define function space $\mathfrak{L}^2(\mathbb{R}^+ \times \mathbb{R}, \mathfrak{B}^+ \otimes \mathfrak{B}, \nu)$ as the set of all real measurable functions such that $\int f^2 d\nu = \int_{\mathbb{R}^+} \int_{\mathbb{R}} f^2(t,x)\mu_t(dx)\lambda(dt) < \infty$. Let $\|f\|_2 := (\int f^2 d\nu)^{1/2}$ be the L^2-norm.

In order to obtain a Hilbert space, one identifies functions differing on a measure-zero set and define

$$L^2(\mathbb{R}^+ \times \mathbb{R}, \mathfrak{B}^+ \otimes \mathfrak{B}, \nu) = \left\{ f^\sim : f \in \mathfrak{L}^2(\mathbb{R}^+ \times \mathbb{R}, \mathfrak{B}^+ \otimes \mathfrak{B}, \nu) \right\} \tag{2.4.3}$$

where $f^\sim$ stands for the equivalent class of f in which all functions are equal a.e. to f. Therefore, $(L^2(\mathbb{R}^+ \times \mathbb{R}, \mathfrak{B}^+ \otimes \mathfrak{B}, \nu), \|\cdot\|_2)$ is a Banach space (see, e.g., Theorem 5.2.1 of Dudley (2003)).

Introduce

$$(f_1, f_2) = \int f_1 f_2 d\nu, \tag{2.4.4}$$

which is a semi-inner product on $(\mathfrak{L}^2(\mathbb{R}^+ \times \mathbb{R}, \mathfrak{B}^+ \otimes \mathfrak{B}, \nu), \|\cdot\|_2)$. Let $\|f\| := (f,f)^{1/2}$, which hence is a semi-norm induced by (2.4.4) on the space. However, the mapping

$(f_1^{\sim}, f_2^{\sim}) := (f_1, f_2) = \int f_1 f_2 d\nu$ defines an inner product on $L^2(\mathbb{R}^+ \times \mathbb{R}, \mathfrak{B}^+ \otimes \mathfrak{B}, \nu)$. Therefore, it becomes a Hilbert space equipped with this inner product. For the sake of convenience, we simplify the notation of the space as $L^2(\mathbb{R}^+ \times \mathbb{R}, \nu)$ and specify the norm as $\|\cdot\|_{L^2(\mathbb{R}^+\times\mathbb{R},\nu)}$ if necessary.

Since for any Hilbert space there exists a basis, we aim on finding a basis for $L^2(\mathbb{R}^+ \times \mathbb{R}, \nu)$. After obtaining the basis of a Hilbert space, any element in the space can be formulated by means of the basis.

In view of the structure of the Hilbert space, it is possible to find the bases of spaces $L^2(\mathbb{R}, \mathfrak{B}, \mu_t)$ and $L^2(\mathbb{R}^+, \mathfrak{B}^+, \lambda)$ separately, then their product will be the basis of $L^2(\mathbb{R}^+ \times \mathbb{R}, \mathfrak{B}^+ \otimes \mathfrak{B}, \nu)$, as the following lemma shown.

Lemma 2.4.1. *$\{\mathscr{L}_j(t)h_i(t,x)\}_{i,j=0}^{\infty}$ is a basis in the Hilbert space $L^2(\mathbb{R}^+ \times \mathbb{R}, \mathfrak{B}^+ \otimes \mathfrak{B}, \nu)$.*

Remark 2.4.1. A more general assertion can be found in p.173, Problem 12 of Dudley (2003). But there is no proof there.

Proof. First of all, let us check the orthogonality of $\mathscr{L}_j(t)h_i(t,x)$. For any $i, j, m, l \geq 0$,

$$(\mathscr{L}_j(t)h_i(t,x), \mathscr{L}_m(t)h_l(t,x)) = \int_{\mathbb{R}^+\times\mathbb{R}} \mathscr{L}_j(t)h_i(t,x)\mathscr{L}_m(t)h_l(t,x)d\nu$$

$$= \int_0^\infty \int_{-\infty}^\infty \mathscr{L}_j(t)h_i(t,x)\mathscr{L}_m(t)h_l(t,x)\phi_t(x)dxdt = \begin{cases} \int_0^\infty \mathscr{L}_j(t)\mathscr{L}_m(t)dt, & \text{if } i = l; \\ 0, & \text{if } i \neq l. \end{cases}$$

$$= \begin{cases} 1, & \text{if } i = l \text{ and } j = m; \\ 0, & \text{if } i \neq l, \text{or } j \neq m. \end{cases}$$

Secondly, for any $f(t,x) \in L^2(\mathbb{R}^+ \times \mathbb{R})$ since $\|f(t,x)\|_{L^2(\mathbb{R},\phi_t(x))} < \infty$ a.e.$[\lambda]$ for $t > 0$, $f(t,x) \in L^2(\mathbb{R}, \phi_t(x))$. Because $h_i(t,x)$ is a basis in $L^2(\mathbb{R}, \phi_t(x))$, there exists a sequence $c_i(t)$ for almost every t such that $f(t,x) = \lim_{k\to\infty} \sum_{i=0}^k c_i(t)h_i(t,x)$ in the sense of norm. Therefore $c_i(t) = (f(t,x), h_i(t,x))_{L^2(\mathbb{R},\phi_t(x))}$ and Bessel's inequality shows $\sum_{i=0}^k c_i^2(t) \leq \|f(t,x)\|^2_{L^2(\mathbb{R},\phi_t(x))}$ for almost every t. However, since $f(t,x) \in L^2(\mathbb{R}^+ \times \mathbb{R})$, $\sum_{i=0}^k \int_{\mathbb{R}^+} c_i^2(t)dt \leq \|f(t,x)\|^2_{L^2(\mathbb{R}^+\times\mathbb{R})} < \infty$. It follows that for every $i \leq k$, $c_i(t) \in L^2(\mathbb{R}^+)$.

Meanwhile, for almost every $t > 0$, $0 < \|f(t,x) - \sum_{i=0}^k c_i(t)h_i(t,x)\|^2 \leq \|f(t,x)\|^2 - \sum_{i=0}^k c_i^2(t) \leq \|f(t,x)\|^2_{L^2(\mathbb{R},\phi_t(x))}$, which is integrable on $\mathbb{R}^+$.

On the other hand, since for every $i \leq k$, $c_i(t) \in L^2(\mathbb{R}^+)$ and $\mathscr{L}_j(t)$ is a basis in the space, $c_i(t) = \lim_{p_i\to\infty} \sum_{j=0}^{p_i} c_{ij}\mathscr{L}_j(t)$ in the sense of norm. Hence, for any given

$\epsilon > 0$, there exists an integer N_i such that when $p_i > N_i$, $\|c_i(t) - \sum_{j=0}^{p_i} c_{ij}\mathscr{L}_j(t)\|_{L^2(\mathbb{R}^+)} < \epsilon/2^{(i+1)/2}$, $0 \leq i \leq k$.

Finally, for any $f(t,x) \in L^2(\mathbb{R}^+ \times \mathbb{R})$, there exist a sequence c_{ij} such that

$$\begin{aligned}
&\left\| f(t,x) - \sum_{i=0}^{k}\sum_{j=0}^{p_i} c_{ij}\mathscr{L}_j(t)h_i(t,x) \right\|_{L^2(\mathbb{R}^+\times\mathbb{R})} \\
\leq &\left\| f(t,x) - \sum_{i=0}^{k} c_i(t)h_i(t,x) \right\|_{L^2(\mathbb{R}^+\times\mathbb{R})} + \left\| \sum_{i=0}^{k}\left(c_i(t) - \sum_{j=0}^{p_i} c_{ij}\mathscr{L}_j(t) \right) h_i(t,x) \right\|_{L^2(\mathbb{R}^+\times\mathbb{R})} \\
= &\left(\int_0^\infty \left\| f(t,x) - \sum_{i=0}^{k} c_i(t)h_i(t,x) \right\|_{L^2(\mathbb{R})}^2 dt \right)^{1/2} \\
&+ \left(\sum_{i=0}^{k} \left\| c_i(t) - \sum_{j=0}^{p_i} c_{ij}\mathscr{L}_j(t) \right\|_{L^2(\mathbb{R}^+)}^2 \right)^{1/2} \\
\leq &\left(\int_0^\infty \left\| f(t,x) - \sum_{i=0}^{k} c_i(t)h_i(t,x) \right\|_{L^2(\mathbb{R})}^2 dt \right)^{1/2} + \epsilon \to 0
\end{aligned}$$

when $p_{\min} = \min\{p_0, \cdots, p_k\} > \max_{0\leq i\leq k}\{N_i\}$, $\epsilon \to 0$ as well as the dominated convergence theorem when $k \to \infty$. □

We now dwell on the expansion of $f(t, B(t))$. Let

$$\begin{aligned}
&\mathscr{T} : f(t,x) \mapsto f(t,B(t)), \\
&\text{for } f(t,x) \in L^2(\mathbb{R}^+ \times \mathbb{R}, \mathfrak{B}^+ \otimes \mathfrak{B}, \nu) \text{ and for any } t > 0,\ E[f(t,B(t))]^2 < \infty.
\end{aligned} \tag{2.4.5}$$

Notice that the redundant condition $E[f(t,B(t))]^2 < \infty$ rules out the possibility that on a measure-zero set of t, $E[f(t,B(t))]^2 = \infty$. Actually, even if this unfortunate thing happens, such a function is in the same equivalence class as the function in (2.4.5). Anyway, the redundant condition makes stating convenient.

Denote by Θ the image of $\mathscr{T}$. Define an operation on $\Theta \times \Theta$:

$$\langle f(t,B(t)), g(t,B(t))\rangle_\Theta = \int_0^\infty E[f(t,B_t)g(t,B_t)]dt.$$

First, this operation makes sense. In fact,

$$\left| \int_0^\infty E[f(t,B(t))g(t,B(t))]dt \right| \leq \int_0^\infty |E[f(t,B(t))g(t,B(t))]|dt$$

$$
\begin{aligned}
&\le \int_0^\infty \sqrt{Ef^2(t,B(t))}\sqrt{Eg^2(t,B(t))}dt \\
&\le \left(\int_0^\infty Ef^2(t,B(t))dt\right)^{1/2}\left(\int_0^\infty Eg^2(t,B(t))dt\right)^{1/2} \\
&= \|f(t,x)\|_{L^2(\mathbb{R}^+\times\mathbb{R},\nu)}\|g(t,x)\|_{L^2(\mathbb{R}^+\times\mathbb{R},\nu)}.
\end{aligned}
$$

Second, $\langle f(t,B(t)), g(t,B(t))\rangle_\Theta = (f(t,x), g(t,x))_{L^2(\mathbb{R}^+\times\mathbb{R},\nu)}$, which implies that the operation is an inner product in Θ and the transformation $\mathscr{T}$ preserves inner product. It is not difficult to show that $\mathscr{T}$ and Θ enjoy the properties in Lemmas 2.2.2–2.2.4. Particularly, Θ is a Hilbert space with this inner product $\langle\cdot,\cdot\rangle_\Theta$ and $\{\mathscr{L}_j(t)h_i(t,B(t))\}_{i,j=0}^\infty$ is its orthonormal basis. Accordingly, we have the following Theorem.

Theorem 2.4.1. *Any stochastic process in the form of $f(t,B(t))$ in the space Θ admits a Fourier series expression*

$$
f(t,B(t)) = \sum_{i=0}^\infty\sum_{j=0}^\infty c_{ij}\mathscr{L}_j(t)h_i(t,B(t)), \tag{2.4.6}
$$

where $c_{ij} = \langle f(t,B(t)), \mathscr{L}_j(t)h_i(t,B(t))\rangle_\Theta$.

Proof. It follows immediately from the Hilbert space theory. □

Notice that since

$$
\begin{aligned}
c_{ij} =& \langle f(t,B(t)), \mathscr{L}_j(t)h_i(t,B(t))\rangle_\Theta = \int_0^\infty E[f(t,B(t))\mathscr{L}_j(t)h_i(t,B(t))]dt \\
=& \int_0^\infty \mathscr{L}_j(t)E[f(t,B(t))h_i(t,B(t))]dt := \int_0^\infty c_i(t,f)\mathscr{L}_j(t)dt,
\end{aligned}
$$

where $c_i(t,f) := E[f(t,B(t))h_i(t,B(t))]$ and Cauchy-Schwarz inequality shows the square integrability of $c_i(t,f)$ on $\mathbb{R}^+$:

$$
\begin{aligned}
\int_0^\infty c_i^2(t,f)dt &= \int_0^\infty \left[\int_{-\infty}^\infty f(t,x)h_i(t,x)\phi_t(x)dx\right]^2 dt \\
&\le \int_0^\infty\int_{-\infty}^\infty f^2(t,x)\phi_t(x)dxdt < \infty,
\end{aligned}
$$

the expansion (2.4.6) may be viewed as a two-step expansion, viz., expand first $f(t,B(t))$ in the space $L^2(\mathbb{R},\phi_t(x))$ in terms of $h_i(t,B(t))$ obtaining coefficient function $c_i(t,f)$, then expand $c_i(t,f)$ by means of $\mathscr{L}_j(t)$ in the space $L^2(\mathbb{R}^+)$. In addition, from Parseval equality it follows that

$$\|f(t,B(t))\|_\Theta^2 = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_{ij}^2 = \sum_{i=0}^{\infty} \|c_i(t,f)\|_{L^2(\mathbb{R}^+)}^2.$$

As usual, given truncation parameters k and $p=(p_0,\ldots,p_k)$, $f_{k,p}(t,B_t)$ signifies the truncation series,

$$f_{k,p}(t,B(t)) = \sum_{i=0}^{k}\sum_{j=0}^{p_i} c_{ij}\mathscr{L}_j(t)h_i(t,B(t)).$$

Let us then establish an approximation rate for the truncation series $f_{k,p}(t,B(t))$ to $f(t,B(t))$.

Theorem 2.4.2. *Suppose that $f(t,x) \in L^2(\mathbb{R}^+\times\mathbb{R})$ and for every $t>0$, $f(t,x)$ and its partial derivatives $f_x^{(v)}(t,x)$, $v=1,\ldots,r_1$ ($r_1\geq 1$), are all in the space $L^2(\mathbb{R},\phi_t(x))$. Moreover, suppose that $t^{\frac{r_1}{2}}f_x^{(r_1)}(t,x)\in L^2(\mathbb{R}^+\times\mathbb{R})$. In addition, there exists a positive integer $r_2\geq 1$ such that $c_i(t,f)$ and $t^{\frac{r_2}{2}}\frac{d^v}{dt^v}c_i(t,f)$ ($v=1,\ldots,r_2$) are all in $L^2(\mathbb{R}^+)$ for all i. Then we have*

$$\|f(t,B(t)) - f_{k,p}(t,B(t))\|_\Theta^2 \leq \frac{1}{k^{r_1}}R_1^2(k) + \frac{k}{p_{\min}^{r_2}}R_2^2(p_{\min}),$$

where $R_1^2(k) = (1+o(1))\sum_{i=k+1}^{\infty}\|c_{i-r_1}(t,\sqrt{t}^{r_1}f_x^{(r_1)}(t,x))\|_{L^2(\mathbb{R}^+)}^2 \to 0$ as $k\to\infty$, $R_2^2(p_{\min}) = (1+o(1))\sum_{j=p_{\min}+1}^{\infty}(c_{j-r_2}^{(r_2)}(\tilde{c}_i))^2 \to 0$ as $p_{\min}=\min\{p_0,\ldots,p_k\}\to\infty$, in which $\tilde{c}_i = t^{\frac{r_2}{2}}e^{-t/2}[c_i(t,f)e^{t/2}]^{(r_2)}$ and provided that the truncation parameters satisfy that $\frac{k}{p_{\min}^{r_2}}\to 0$.

Remark 2.4.2. Firstly, note that the notation $c_{j-r_2}^{(r_2)}(\tilde{c}_i)$ signifies the coefficients of the expansion of the function involved expanded by $\{\mathscr{L}^{(r_2)}(t)\}$. Secondly, the conditions can be satisfied by many functions such as $f_1(t,x)=\frac{t}{1+t^2}\sin x$, $f_2(t,x)=t^\alpha\sin x$ ($\alpha\geq 0$), by virtue of Example 2.1. When $\sin x$ is substituted by $\cos x$ it is clear that the conditions are also satisfied. In addition, $f_3(t,x) = t^\alpha e^{-\gamma t}p_n(x)$ also satisfy the condition with $\forall\gamma>0,\alpha\geq 0$ and polynomial $p_n(x)$ of fixed degree $n\in\mathbb{N}$ on account of Example 2.1.

Proof. It follows from the orthogonality of the basis that

$$\begin{aligned}&\|f(t,B(t)) - f_{k,p}(t,B(t))\|_\Theta^2\\ =&\left\|\sum_{i=0}^{k}\sum_{j=p_i+1}^{\infty} c_{ij}\mathscr{L}_j(t)h_i(t,B(t)) + \sum_{i=k+1}^{\infty}\sum_{j=0}^{\infty} c_{ij}\mathscr{L}_j(t)h_i(t,B(t))\right\|^2\end{aligned}$$

$$=\sum_{i=0}^{k}\sum_{j=p_i+1}^{\infty} c_{ij}^2 + \sum_{i=k+1}^{\infty}\sum_{j=0}^{\infty} c_{ij}^2.$$

Notice that $\sum_{j=0}^{\infty} c_{ij}^2 = \|c_i(t,f)\|_{L^2(\mathbb{R}^+)}^2$ and $f(t,x)$, as a function of x, satisfies all the conditions in Theorem 2.2.2. Using equality (2.2.13) yields

$$\begin{aligned}
&\sum_{i=k+1}^{\infty}\sum_{j=0}^{\infty} c_{ij}^2 = \sum_{i=k+1}^{\infty} \|c_i(t,f)\|_{L^2(\mathbb{R}^+)}^2 \\
=&\sum_{i=k+1}^{\infty} \frac{1}{i(i-1)\cdots(i-r_1+1)} \|\sqrt{t}^{r_1} c_{i-r_1}(t, f_x^{(r_1)}(t,x))\|_{L^2(\mathbb{R}^+)}^2 \\
=&\sum_{i=k+1}^{\infty} \frac{1}{i(i-1)\cdots(i-r_1+1)} \|c_{i-r_1}(t, \sqrt{t}^{r_1} f_x^{(r_1)}(t,x))\|_{L^2(\mathbb{R}^+)}^2 \\
\leq&\frac{1}{(k+1)k\cdots(k-r_1+2)} \sum_{i=k+1}^{\infty} \|c_{i-r_1}(t, \sqrt{t}^{r_1} f_x^{(r_1)}(t,x))\|_{L^2(\mathbb{R}^+)}^2 \\
=&\frac{1}{k^{r_1}} R_1^2(k),
\end{aligned}$$

where $R_1^2(k) = (1+o(1))\sum_{i=k+1}^{\infty} \|c_{i-r_1}(t, \sqrt{t}^{r_1} f_x^{(r_1)}(t,x))\|_{L^2(\mathbb{R}^+)}^2$, which converges to zero as $k \to \infty$, because of $\sqrt{t}^{r_1} f_x^{(r_1)}(t,x) \in L^2(\mathbb{R}^+ \times \mathbb{R})$.

Meanwhile, since each $c_i(t,f)$ satisfies all the conditions in Theorem 2.3.1, by (2.3.3) we have

$$\begin{aligned}
&\sum_{i=0}^{k}\sum_{j=p_i+1}^{\infty} c_{ij}^2 = \sum_{i=0}^{k} \left\| c_i(t,f) - \sum_{j=0}^{p_i} c_{ij}\mathscr{L}_j(t) \right\|_{L^2(\mathbb{R}^+)}^2 \\
\leq&\sum_{i=0}^{k} \frac{1}{p_i^{r_2}} R_2^2(p_i) \leq \frac{k}{p_{\min}^{r_2}}(1+o(1))R_2^2(p_{\min}),
\end{aligned}$$

where $R_2^2(p_i) = (1+o(1))\sum_{j=p_i+1}^{\infty}(c_{j-r_2}^{(r_2)}(\tilde{c}_i))^2$ and $\tilde{c}_i = t^{r_2/2}e^{-r_2/2}[c_i(t)e^{r_2/2}]^{(r_2)}$. Observe that $R_2^2(p)$ converges to 0 as $p_{\min} \to \infty$. This finishes the proof. □

2.4.2 Finite time horizon

It is known that $([0,T]\times\mathbb{R}, \mathfrak{B}_T \otimes \mathfrak{B}, \nu)$ is a measure space, where $\mathfrak{B}_T$ and $\mathfrak{B}$ are Borel σ–algebras on $[0,T]$ and on $\mathbb{R}$ respectively, ν is the product measure of Lebesgue measure λ on $[0,T]$ and μ_t, as defined in last subsection, a measure on $\mathbb{R}$ for $t \in [0,T]$. Since both μ_t and λ are finite, ν is finite. Therefore, $L^2([0,T]\times\mathbb{R}, \mathfrak{B}_T\otimes\mathfrak{B}, \nu) = \{f(t,x) : \int_{[0,T]\times\mathbb{R}} f^2(t,x)d\nu < \infty\}$ is a Hilbert space with an inner product defined by

$$(f_1, f_2) = \int_0^T \int_{-\infty}^{\infty} f_1(t,x) f_2(t,x) \phi_t(x) dx dt, \tag{2.4.7}$$

and the induced norm. We also simplify the notation $L^2([0,T]\times\mathbb{R}, \mathfrak{B}_T\otimes\mathfrak{B}, \nu)$ as $L^2([0,T]\times\mathbb{R})$. Similar to Lemma 2.4.1, system $\{\varphi_{jT}(t)h_i(t,x)\}$ is a complete orthonormal basis in $L^2([0,T]\times\mathbb{R})$.

We can conduct the same mapping as in the last subsection to establish a space of stochastic processes that is omitted for brevity. We give the result directly. Let Ξ be the space of

$$\Xi = \{f(t,B(t)) : f(t,x)\in L^2([0,T]\times\mathbb{R}),\ \text{and } E[f(t,B(t))]^2 < \infty \text{ for } \forall\ t\in[0,T]\},$$

which is a Hilbert space with an inner product of the form $\langle f(t,B(t)), g(t,B(t))\rangle_\Xi = \int_0^T E[f(t,B(t))g(t,B(t))]dt$ and the induced norm. Note that $\{\varphi_{jT}(t)h_i(t,B(t))\}$ is an orthonormal basis in Ξ.

Theorem 2.4.3. *Any stochastic process in the form of $f(t,B(t))$ in the space Ξ admits a Fourier series expression*

$$f(t,B(t)) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} b_{ij}\varphi_{jT}(t)h_i(t,B(t)), \tag{2.4.8}$$

where $b_{ij} = \langle f(t,B(t)), \varphi_{jT}(t)h_i(t,B(t))\rangle_\Xi$.

As $b_{ij} = \int_0^T \varphi_{jT}(t)E[f(t,B(t))h_i(t,B(t))]dt$, let $b_i(t,f) := E[f(t,B(t))h_i(t,B(t))]$, which is square integrable on $[0,T]$ implied by Cauchy-Schwarz inequality, the relation $b_{ij} = \int_0^T \varphi_{jT}(t)b_i(t,f)dt$ and the same arguments as in the last subsection imply that expansion (2.4.8) can be regarded as a two-step expansion: $b_i(t,f)$ is the i-th coefficient of the expansion of $f(t,B(t))$ in terms of $\{h_i(t,B(t))\}$ and then b_{ij} is the j-th coefficient of the expansion of $b_i(t,f)$ in terms of $\{\varphi_{jT}(t)\}$. It follows from the Parseval-Bessel equality that

$$\|f(t,B(t))\|_\Xi^2 = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} b_{ij}^2 = \sum_{i=0}^{\infty} \|b_i(t,f)\|^2_{L^2([0,T])}.$$

Given a bundle of truncation parameters k for i and $(p_0, p_1, \cdots, p_k)$ for j's, we define the truncation series for (2.4.8):

$$f_{k,p}(t,B(t)) = \sum_{i=0}^{k}\sum_{j=0}^{p_i} b_{ij}\varphi_{jT}(t)h_i(t,B(t)). \tag{2.4.9}$$

The following theorem gives an approximation rate for the truncation series $f_{k,p}(t, B(t))$ to $f(t, B(t))$.

Theorem 2.4.4. *Suppose $f(t,x) \in L^2([0,T]\times\mathbb{R})$ and for every $t > 0$, $f(t,x)$ and its partial derivatives $f_x^{(v)}(t,x)$, $v = 1, \ldots, r$ $(r \geq 1)$, are all in the space $L^2(\mathbb{R}, \phi_t(x))$. Moreover, suppose $\sqrt{t}^r f_x^{(r)}(t,x)) \in L^2([0,T] \times \mathbb{R})$. In addition, $b_i(t,f) = E[f(t,B(t))h_i(t,B(t))]$ are twice differentiable in $t \in [0,T]$ and $b_i''(t,f) \in L^2[0,T]$. Furthermore, suppose $D_T = \sup_i\{|b_i'(0,f)| + |b_i'(T,f)|\} < \infty$. Then*

$$\|f(t,B(t)) - f_{k,p}(t,B(t))\|_{\Xi}^2 \leq C\frac{k}{p_{\min}^3} + \frac{1}{k^r}R_k^2, \tag{2.4.10}$$

where $C = C_T$ is a constant depending on T and D_T, $p_{\min} = \min\{p_0, p_1, \cdots, p_k\}$, and $R_k^2 = (1 + o(1))\sum_{i=k+1}^{\infty} \|b_{i-r}(t, \sqrt{t}^r f_x^{(r)}(t,x))\|_{L^2[0,T]}^2 \to 0$ as $k \to \infty$. Also, suppose that the truncation parameters satisfy $\frac{k}{p_{\min}^3} \to 0$.

Remark 2.4.3. The conditions of this theorem are quite weak since time zone is finite. Let $\alpha \geq 1$ and $n \geq 1$. All functions $f(t,x) = t^\alpha p_n(x)$ where $p_n(x)$ is a polynomial of n-th order satisfy such conditions because $b_i(t,f)$ is a power function of t with power greater than or equal to one when $i \leq n$, and 0 when $i > n$. Thus, functions of the type $t^\alpha e^{\gamma t} p_n(x)$ $(\gamma > 0)$ and their superpositions satisfy the conditions, in addition to functions like $t^\alpha e^{-\gamma t} p_n(x)$. Meanwhile, $t^\alpha \sin x$ and $t^\alpha \cos x$ are in the ambit of the conditions as well.

Proof. It follows from the orthogonality that

$$\begin{aligned}\|f(t,B(t)) - f_{k,p}(t,B(t))\|_{\Xi}^2 &= \sum_{i=0}^{k}\sum_{j=p_i+1}^{\infty} b_{ij}^2 + \sum_{i=k+1}^{\infty}\sum_{j=0}^{\infty} b_{ij}^2 \\ &= \sum_{i=0}^{k}\left\|b_i(t,f) - \sum_{j=0}^{p_i} b_{ij}\varphi_{jT}(t)\right\|_{L^2[0,T]}^2 + \sum_{i=k+1}^{\infty}\|b_i(t,f)\|_{L^2[0,T]}^2.\end{aligned}$$

It follows from (2.3.6) that

$$\left\|b_i(t,f) - \sum_{j=0}^{p_i} b_{ij}\varphi_{jT}(t)\right\|^2 \leq C_1\frac{a_{iT}^2 T^3}{(p_i+1)^3} + C_2\frac{a_{iT}T^{3.5}}{(p_i+1)^3}R_{p_i} + C_3\frac{T^4}{(p_i+1)^4}R_{p_i}^2$$

where C_j $(j = 1,2,3)$ are absolutely constants, $a_{iT} = |b_i'(0)| + |b_i'(T)|$, and $R_{p_i}^2 = \sum_{j=p_i+1}^{\infty} |b_{ij}(b_i''(t))|^2$. Therefore, denoting $D_T = \sup_i\{a_{iT}\}$,

$$\sum_{i=0}^{k}\left\|b_i(t,f) - \sum_{j=0}^{p_i} b_{ij}\varphi_{jT}(t)\right\|^2$$

$$
\begin{aligned}
&\le C_1 D_T^2 T^3 \sum_{i=0}^{k} \frac{1}{(p_i+1)^3} + C_2 D_T T^{3.5} \sum_{i=0}^{k} \frac{1}{(p_i+1)^3} R_{p_i} + C_3 T^4 \sum_{i=0}^{k} \frac{1}{(p_i+1)^4} R_{p_i}^2 \\
&\le C_1 D_T^2 T^3 \frac{k}{p_{\min}^3} + C_2 D_T T^{3.5} \frac{k}{p_{\min}^3} R_{p_{\min}} + C_3 T^4 \frac{k}{p_{\min}^4} R_{p_{\min}}^2 \\
&= (C_1 D_T^2 + C_2 D_T \sqrt{T} R_{1p_{\min}} + C_3 T p_{\min}^{-1} R_{p_{\min}}^2) \frac{kT^3}{p_{\min}^3} = C \frac{k}{p_{\min}^3},
\end{aligned}
$$

where $p_{\min} = \min\{p_0, p_1, \cdots, p_k\}$ and $C = (C_1 D_T^2 + C_2 D_T \sqrt{T} R_{p_{\min}} + C_3 T p_{\min}^{-1} R_{p_{\min}}^2) T^3 = O(1)$.

In the meantime, using (2.2.13) yields

$$
\begin{aligned}
\sum_{i=k+1}^{\infty} \|b_i(t,f)\|^2 &= \sum_{i=k+1}^{\infty} \frac{1}{i(i-1)\cdots(i-r+1)} \|\sqrt{t}^r b_{i-r}(t, f_x^{(r)}(t,x))\|_{L^2[0,T]}^2 \\
&\le \frac{1}{(k+1)k\cdots(k-r+2)} \sum_{i=k+1}^{\infty} \|b_{i-r}(t, \sqrt{t}^r f_x^{(r)}(t,x))\|_{L^2[0,T]}^2 \\
&= \frac{1}{k^r} R_k^2,
\end{aligned}
$$

where $R_k^2 = (1+o(1)) \sum_{i=k+1}^{\infty} \|b_{i-r}(t, \sqrt{t}^r f_x^{(r)}(t,x))\|_{L^2[0,T]}^2$. Because of $\sqrt{t}^r f_x^{(r)}(t,x) \in L^2([0,T]\times\mathbb{R})$, R_{2k}^2 converges to zero as $k \to \infty$. The assertion eventually follows. □

Chapter 3

Estimation of Brownian motion functionals in econometric models

Consider a continuous time model of the form

$$Y(t) = m(t, B(t)) + \varepsilon(t), \tag{3.0.1}$$

where $m(\cdot, \cdot)$ is an unknown functional defined on $[0, \infty) \times (-\infty, \infty)$ or $[0, T] \times (-\infty, \infty)$ and $\varepsilon(t)$ is an error process with mean zero and finite variance.

Since there are three types of sampling to be discussed in what follows, we divide the chapter into three sections.

3.1 Infinite time horizon

This section is devoted to the estimation where $m(\cdot, \cdot)$ is defined on $[0, \infty) \times (-\infty, \infty)$ and our sampling points are $t_s = s$, $s = 1, 2, \cdots, n$. We firstly need to impose some conditions on $m(t, x)$.

Assumption B.1[1]

(a) For every $t > 0$, $m(t, x)$ and its partial derivatives with respect to x of up to third order are all in $L^2(\mathbb{R}, \phi_t(x))$.

[1] B is initialled from Brownian motion.

(b) For each i, $c_i(t, m(t,x)) = E[m(t,B(t))h_i(t,B(t))]$, the coefficient of the expansion of m in terms of the system $\{h_i(t,B(t))\}$, and its derivatives of up to third order belong to $L^2(\mathbb{R}^+)$.

(c) For i large enough, the coefficient functions $c_i(t, m_x^{(3)}(t,x))$ of $m_x^{(3)}(t,B(t))$ expanded by the system $\{h_i(t,B(t))\}$ are chosen such that $t^3c_i^2(t, m_x^{(3)}(t,x))$ are bounded on $(0,\infty)$ uniformly in i.

Remark 3.1.1. Condition (a) and the first part of (b) are to ensure that the m function can be expanded by the method in the last chapter. The second part of condition (b) and (c) are further requirements for the expansion converging with a certain speed. Since quite weak, there are variety of functions satisfying these conditions. For example, $m_1(t,x) = \frac{t^\alpha}{1+t^\beta}\sin(x)$ and $m_2(t,x) = \frac{t^\alpha}{1+t^\beta}\cos(x)$ where $\alpha \geq 1$ and $\beta \geq \alpha + 1.25$. It follows from Example 2.1 that $c_i(t,m_1) = (-1)^k \frac{1}{\sqrt{i!}}\frac{t^\alpha \sqrt{t}^i}{1+t^\beta}e^{-t/2}$, for $i = 2k+1$; 0, for $i = 2k$, where $k = 0, 1, \ldots$ and $c_i(t,m_2) = (-1)^k \frac{1}{\sqrt{i!}}\frac{t^\alpha \sqrt{t}^i}{1+t^\beta}e^{-t/2}$ for $i = 2k$; 0, for $n = 2k+1$, where $k = 0, 1, \ldots$. It is evident that $c_i(t,m_1)$ and $c_i(t,m_2)$ as well as their derivatives of up to third order belong to $L^2(0,\infty)$.

On the other hand, because $\frac{\partial^3}{\partial x^3}m_1(t,x) = -m_2(t,x)$ and $\frac{\partial^3}{\partial x^3}m_2(t,x) = m_1(t,x)$, the condition (c) is fulfilled by these two functions. In effect,

$$t^3c_i^2\left(t, \frac{\partial^3}{\partial x^3}m_2(t,x)\right) = t^3c_i^2(t, m_1(t,x)) = \begin{cases} 0, & \text{if } i = 2k; \\ \frac{1}{i!}\frac{t^{3+2\alpha}t^i}{(1+t^\beta)^2}e^{-t} & \text{if } i = 2k+1. \end{cases}$$

In the second case that $i = 2k+1$, if $0 < t \leq 1$, it is less than 1; otherwise, making use of the fact that $i! > \sqrt{2\pi i}\left(\frac{i}{e}\right)^i$ yields

$$\begin{aligned} \frac{1}{i!}\frac{t^{3+2\alpha}t^i}{(1+t^\beta)^2}e^{-t} &\leq \frac{1}{i!}t^{3+2\alpha-2\beta}t^ie^{-t} \leq \frac{1}{i!}t^{i+\frac{1}{2}}e^{-t} \leq \frac{1}{i!}\left(i+\frac{1}{2}\right)^{i+\frac{1}{2}}e^{-i-\frac{1}{2}} \\ &\leq \frac{e^i}{\sqrt{2\pi i}i^i}\left(i+\frac{1}{2}\right)^{i+\frac{1}{2}}e^{-i-\frac{1}{2}} \leq \frac{\sqrt{i+\frac{1}{2}}}{\sqrt{2\pi i}}\left(1+\frac{1}{2i}\right)^i e^{-\frac{1}{2}} \leq \frac{1}{\sqrt{\pi}} \end{aligned}$$

Also, we can verify that $t^3c_i^2\left(t, \frac{\partial^3}{\partial x^3}m_1(t,x)\right)$ satisfies the condition in the same way.

Another interesting example is $m_3(t,x) = t^\alpha e^{-rt}P_n(x)$ where $\alpha \geq 1$, $r > 0$ and $P(x)$ stands for a polynomial of fixed degree. This kind of functions is frequently encountered in finance context which may represent the present value of an asset (see Yor, 2001, p.6). Because $P(x)$ is a polynomial, it is evident that m_3 satisfies the conditions.

Suppose that we have n observations (s, Y_s) at the discrete times $s = 1, 2, \cdots, n$, where $Y_s = Y(s)$. The resulting models now become

$$Y_s = m(s, X_s) + e_s, \quad s = 1, \ldots, n, \tag{3.1.1}$$

where $X_s = B(s)$ denotes the Brownian motion at point s, $e_s = \varepsilon(s)$ $(s = 1, \ldots, n)$ form an error sequence with mean zero and finite variance.

Note that $X_s = \sum_{i=1}^{s}(X_i - X_{i-1})$, a sum of i.i.d.N(0, 1) sequence. Let $x_{s,n} = \frac{1}{\sqrt{n}}X_s$. It follows from the functional central limit theorem, $x_{s,n}$ converges in distribution to a Brownian motion on $[0, 1]$. In addition, the triangular array $x_{s,n}$, along with $d_{l,k,n} = \sqrt{(l-k)/n}$, satisfies Assumption A in Chapter 1.

Having expanded function m at sampling points, given truncation parameters k and p_i, the model in (3.1.1) can be written as

$$\begin{aligned} Y_s = & \sum_{i=0}^{k}\sum_{j=0}^{p_i} c_{ij}\mathscr{L}_j(s)h_i(s, X_s) + \sum_{i=0}^{k}\sum_{j=p_i+1}^{\infty} c_{ij}\mathscr{L}_j(s)h_i(s, X_s) \\ & + \sum_{i=k+1}^{\infty}\sum_{j=0}^{\infty} c_{ij}\mathscr{L}_j(s)h_i(s, X_s) + e_s, \quad s = 1, 2, \ldots, n. \end{aligned} \tag{3.1.2}$$

As known from the last chapter, $\sum_{j=0}^{\infty} c_{ij}\mathscr{L}_j(s) = c_i(s, m)$. Therefore, in most cases we shall supersede this relationship into the model expression. We now rewrite equations in (3.1.2) in the following matrix form:

$$Y = X\theta + \delta + \gamma + \varepsilon, \tag{3.1.3}$$

where

$$\begin{aligned} Y' =& (Y_1, Y_2, \ldots, Y_n); \quad \theta' = (c_{00}, c_{01} \ldots, c_{0p_0}, c_{10}, \ldots, c_{1p_1}, \ldots, c_{k0}, \ldots, c_{kp_k}); \\ x_1 =& (\mathscr{L}_0(1)h_0(1, X_1), \mathscr{L}_1(1)h_0(1, X_1), \ldots, \mathscr{L}_{p_0}(1)h_0(1, X_1), \\ & \mathscr{L}_0(1)h_1(1, X_1), \mathscr{L}_1(1)h_1(1, X_1), \ldots, \mathscr{L}_{p_1}(1)h_1(1, X_1), \\ & \ldots, \mathscr{L}_0(1)h_k(1, X_1), \mathscr{L}_1(1)h_k(1, X_1), \ldots, \mathscr{L}_{p_k}(1)h_k(1, X_1)), \\ & \vdots \\ x_n =& (\mathscr{L}_0(n)h_0(n, X_n), \mathscr{L}_1(n)h_0(n, X_n), \ldots, \mathscr{L}_{p_0}(n)h_0(n, X_n), \\ & \mathscr{L}_0(n)h_1(n, X_n), \mathscr{L}_1(n)h_1(n, X_n), \ldots, \mathscr{L}_{p_1}(n)h_1(n, X_n), \\ & \ldots, \mathscr{L}_0(n)h_k(n, X_n), \mathscr{L}_1(n)h_k(n, X_n), \ldots, \mathscr{L}_{p_k}(n)h_k(n, X_n)), \end{aligned}$$

and $X = (x_1', x_2', \ldots, x_n')'$; $\varepsilon' = (e_1, e_2, \ldots, e_n)$;

$$\delta' = (\delta_1, \cdots, \delta_n), \qquad \text{with} \quad \delta_s = \sum_{i=1}^{k} \sum_{j=p_i+1}^{\infty} c_{ij} \mathscr{L}_j(s) h_i(s, X_s);$$

$$\gamma' = (\gamma_1, \gamma_2, \cdots, \gamma_n) \qquad \text{with} \quad \gamma_s = \sum_{i=k+1}^{\infty} c_i(s) h_i(s, X_s), s = 1, 2, \ldots, n.$$

The OLS (ordinary least squares) estimator of θ is given by

$$\widehat{\theta} = (X'X)^{-1}X'Y. \tag{3.1.4}$$

3.1.1 Asymptotics of the estimated coefficients

In the sequel we shall explore the asymptotics of $\widehat{\theta}$. However, the dimension of $\widehat{\theta}$ will increase to infinity with n approaching infinity. To tackle the dimension curse in asymptotic theory, a transformation of $\widehat{\theta}$ has to be introduced. The transformation introduced maps $\widehat{\theta}$ into a scalar, so that this image becomes the target of the research. This is also the reason why we consider the converse questions in the previous section. Let us first introduce the following assumptions.

Assumption Bc.1[2]

(a) Let $\mathcal{S} = \{a_0, a_1, a_2, \ldots\}$, where $a_i = \{a_{ij}\}_{j=0}^{\infty}$ is such that $\sum_{j=2}^{\infty} \sqrt{j(j-1)}|a_{i,j-2}| < \infty$ for $i = 0, 1, 2, \cdots$.

(b) Suppose further that $\sum_{i=0}^{\infty} i \sum_{j=0}^{\infty} |a_{ij}| < \infty$.

Remark 3.1.2. The condition (a) on the set of sequences ensures not only the convergence of combinations of each $\{a_i\}$ and the basis $\{\mathscr{L}_j(t)\}$, but also the differentiability of the corresponding functions. The condition (2.3.9) in last chapter guarantees this requirement can be fulfilled. Furthermore, condition (b) is a sufficient condition that secures the convergence of the combination of $\{a_{ij}\}$ and the basis $\{\mathscr{L}_j(t)h_i(t, B(t))\}$ in the product space.

If $a_{ij} = O((i+1)^{-3}(j+1)^{-3})$, all the conditions can be satisfied.

Lemma 3.1.1. *Let Assumption Bc.1 holds. Then there exists a function* $\bar{F}(t, x)$ *such that*

$$\bar{F}(t, B(t)) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij} \mathscr{L}_j(t) h_i(t, B(t)), \tag{3.1.5}$$

[2]Bc is initialled from Brownian motion and coefficients respectively.

for all $t > 0$, where the convergence is in the sense of norm.

Proof. It follows immediately from Riesz-Fischer theorem. □

Assumption Bc.1 makes the following transformation of $\widehat{\theta}$ effective. Let $k = k(n); p_i = p_i(n), i = 0, 1, \ldots k$, be the truncation parameters in the expansion which are increasing with sample size n; for $\mathcal{S}$ in Assumption Bc.1 let

$$a = (a_{00}, \cdots, a_{0p_0}, \cdots, a_{k0}, \cdots, a_{kp_k}) \tag{3.1.6}$$

be the truncated sequences corresponding to the truncation parameters. We now have the following transformation for $\widehat{\theta}$:

$$aX'X[\widehat{\theta} - \theta] = aX'(\delta + \gamma + \varepsilon). \tag{3.1.7}$$

Another crucial issue is the determination of the truncation parameters. This question has considerable impact on the convergence of the estimate.

Assumption Bc.2

(a) $k = [n^{\kappa_1}]$ where $0 < \kappa_1 < 1$.

(b) For all i, $p_i = o(n)$ and $p_{\min} = \min(p_0, \cdots, p_k) = [n^{\kappa_2}]$ where $0 < \kappa_2 < 1$.

(c) Moreover, κ_1 and κ_2 satisfy that

   (i) $\kappa_1 > \frac{1}{2}$,

   (ii) $\kappa_1 + \frac{1}{2} \leq \frac{3}{2}\kappa_2$.

Remark 3.1.3. Condition (a) and (b) are requirements for all truncation parameters which are increasing to infinity with sample size. Note that they are all stipulated to be of $o(n)$. This guarantees that we have sufficient information to estimate the coefficients in the expansion. In addition, condition (c) is the relationship between the orders that ensures the convergence in the subsequential study. The feasible selection of them is clearly considerable.

The last assumption is about the function generated from the vector a satisfying the Assumption Bc.1.[3] By virtue of (3.1.5), at each observation point we can decompose

[3] We say a, a vector in (3.1.6) truncated from the sequences, satisfies the Assumption Bc.1 rather than the sequence for simplicity.

$\bar{F}(s, X_s)$ based on the given truncation parameters as

$$\bar{F}(s, X_s) = (aX')_s + \bar{\delta}_s + \bar{\gamma}_s, \qquad s = 1, \ldots, n, \tag{3.1.8}$$

where $(aX')_s$ is the s-th entry of the vector aX', $\bar{\delta}_s$ and $\bar{\gamma}_s$ are defined in the same way as δ_s and γ_s with the coefficients being substituted by a_{ij}.

Designate $\bar{\mathbf{F}}' = (\bar{F}(1, X_1), \cdots, \bar{F}(n, X_n))$, $\bar{\delta}' = (\bar{\delta}_1, \cdots, \bar{\delta}_n)$, and $\bar{\gamma}' = (\bar{\gamma}_1, \cdots, \bar{\gamma}_n)$ for later use.

Assumption Bc.3

(a) $\bar{F}(t, x)$ is in Class $\mathcal{T}(HI)$ with homogeneity power $v(\cdot)$ and normal function $F(t, x)$.

(b) $\bar{F}^2(t, x)$ is in Class $\mathcal{T}(HI)$ with homogeneity power $v^2(\cdot)$ and normal function $F^2(t, x)$.

Theorem 3.1.1. *Suppose that $\{x_{s,n}\}_1^n$ and $\{e_s\}_1^n$ satisfy Assumptions B and A (c) in Chapter 1. Moreover, let Assumptions B.1 and Bc.1 – Bc.3 hold. Then we have*

$$\frac{1}{\sqrt[4]{n}v(n)} aX'X[\widehat{\theta} - \theta] \to_D \left(\int_0^1 G_3(u) dL_W(u, 0) \right)^{\frac{1}{2}} N \tag{3.1.9}$$

where $G_3(\cdot) = \int F^2(\cdot, x)dx$ as specified in Assumption C in Chapter one, W is the Brownian motion on $[0, 1]$ and N is a standard normal variable which is independent of W, L_W is the local time of W.

Remark 3.1.4. Due to (3.1.8), we have

$$\begin{aligned}\frac{1}{\sqrt{n}v(n)^2} aX'Xa' =& \frac{1}{\sqrt{n}v(n)^2} (\bar{\mathbf{F}}' - \bar{\delta}' - \bar{\gamma}')(\bar{\mathbf{F}} - \bar{\delta} - \bar{\gamma}) \\ =& \frac{1}{\sqrt{n}v(n)^2} (\bar{\mathbf{F}}'\bar{\mathbf{F}} + \bar{\delta}'\bar{\delta} + \bar{\gamma}'\bar{\gamma} - 2\bar{\mathbf{F}}'\bar{\delta} - 2\bar{\mathbf{F}}'\bar{\gamma} + 2\bar{\delta}'\bar{\gamma}).\end{aligned}$$

However, in view of the proof of Theorem 3.1.1, we have

$$\begin{aligned}\frac{1}{\sqrt{n}} \bar{\delta}'\bar{\delta} &= \frac{1}{\sqrt{n}} \sum_{s=1}^n \bar{\delta}_s^2 \to_P 0, \\ \frac{1}{\sqrt{n}} \bar{\gamma}'\bar{\gamma} &= \frac{1}{\sqrt{n}} \sum_{s=1}^n \bar{\gamma}_s^2 \to_P 0,\end{aligned}$$

and hence it follows from Cauchy-Schwarz inequality that $\frac{1}{\sqrt{n}} \bar{\delta}'\bar{\gamma} \to_P 0$ as well. Now, using Assumption Bc.3 and Theorem 1.3.1 we have

$$\frac{1}{\sqrt{n}v(n)^2} \bar{\mathbf{F}}'\bar{\mathbf{F}} = \frac{1}{\sqrt{n}v(n)^2} \sum_{s=1}^n \bar{F}^2(s, X_s) = \frac{1}{\sqrt{n}v(n)^2} \sum_{s=1}^n \bar{F}^2(s, \sqrt{n}x_{s,n})$$

$$\to_D \int_0^1 G_3(t) dL_W(t,0).$$

We eventually obtain that $\frac{1}{\sqrt{n}v(n)^2} aX'Xa'$ converges to a random variable in distribution with $n \to \infty$, implying that $aX'Xa' = O(\sqrt{n}v(n)^2)$.

On the other hand, by the result of the theorem we have $aX'X(\widehat{\theta} - \theta) = O(\sqrt[4]{n}v(n))$. Comparison of these two magnitudes gives us an ambiguous idea of the decay rate of $\widehat{\theta} - \theta$ since a is a constant sequence. The effect of substitution of a by $\widehat{\theta} - \theta$ plays a role of $(\sqrt[4]{n}v(n))^{-1}$, which can reach the conventional rate of $n^{-1/2}$ for the regression of stationary sequence when $v(n) = n^{1/4}$.

Proof. Denote $aX'X(\widehat{\theta} - \theta) = aX'\gamma + aX'\delta + aX'\varepsilon := Q_n + M_n + L_n$.

We firstly shall show that $\frac{1}{\sqrt[4]{n}v(n)} Q_n$ converges to 0 in probability. Notice that it can be written as

$$\begin{aligned} \frac{1}{\sqrt[4]{n}v(n)} Q_n =& \frac{1}{\sqrt[4]{n}v(n)} aX'\gamma = \frac{1}{\sqrt[4]{n}v(n)} (\bar{\mathbf{F}} - \bar{\delta} - \bar{\gamma})'\gamma \\ =& \frac{1}{\sqrt[4]{n}v(n)} (\bar{\mathbf{F}}'\gamma - \bar{\delta}'\gamma - \bar{\gamma}'\gamma). \end{aligned}$$

Cauchy-Schwarz inequality suggests that we would study the convergence of $\|\bar{\mathbf{F}}\|^2$, $\|\bar{\delta}\|^2$, $\|\bar{\gamma}\|^2$ and $\|\gamma\|^2$ in order to obtain that of Q_n. To this end, we may invoke the embedding schedule delineated in Chapter 1, so we can work under a strong condition $(W_n, U_n) \to (W, U)$ almost surely but still achieve a weak convergence.

It follows from Theorem 1.3.1 and Bc.3 that

$$\frac{1}{\sqrt{n}v^2(n)} \|\bar{\mathbf{F}}\|^2 = \frac{1}{\sqrt{n}v^2(n)} \sum_{s=1}^{n} \bar{F}^2\left(s, \sqrt{n}x_{s,n}\right) \to_P \int_0^1 G_3(u) dL_W(u,0), \tag{3.1.10}$$

as $n \to \infty$, where $G_3(\cdot) = \int F^2(\cdot, x)dx$.

Therefore, it suffices to show that

$$\|\gamma\|^2 \to_P 0, \tag{3.1.11a}$$

$$\frac{1}{\sqrt{n}v(n)^2} \|\bar{\delta}\|^2 \to_P 0, \qquad \frac{1}{\sqrt{n}v(n)^2} \|\bar{\gamma}\|^2 \to_P 0. \tag{3.1.11b}$$

In effect,

$$E\|\gamma\|^2 = E\sum_{s=1}^{n} \gamma_s^2 = \sum_{s=1}^{n} E\left(\sum_{i=k+1}^{\infty} \sum_{j=0}^{\infty} c_{ij} \mathscr{L}_j(s) h_i(s, X_s)\right)^2$$

$$=\sum_{s=1}^{n} E\left(\sum_{i=k+1}^{\infty} c_i(s,m)h_i(s,X_s)\right)^2 = \sum_{s=1}^{n}\sum_{i=k+1}^{\infty} c_i^2(s,m),$$

where $c_i(s,m) := c_i(s,m(s,x))$ for brevity. Since m function satisfies conditions of Theorem 2.2.1, by (2.2.13), $c_i(s,m) = \frac{\sqrt{s}^3}{\sqrt{i(i-1)(i-2)}} c_{i-3}(s, m_x^{(3)})$, then using Assumption B.1 (c), we have

$$E\|\gamma\|^2 = \sum_{s=1}^{n}\sum_{i=k+1}^{\infty} \frac{s^3}{i(i-1)(i-2)} c_{i-3}^2(s, m_x^{(3)})$$

$$= \sum_{i=k+1}^{\infty} \frac{1}{i(i-1)(i-2)} \sum_{s=1}^{n} s^3 c_{i-3}^2(s, m_x^{(3)}) \le An \sum_{i=k+1}^{\infty} \frac{1}{i(i-1)(i-2)}$$

$$\le A\frac{n}{k^2} = A\frac{1}{n^{2\kappa_1-1}} \to 0,$$

as $n\to\infty$, where A is the uniform bound for $s^3 c_{i-3}^2(s, m_x^{(3)})$ stipulated in Assumption B.1 (c). Hence, $\|\gamma\|^2$ converges to 0 in probability.

As for (3.1.11b), it follows that

$$\frac{1}{\sqrt{n}v(n)^2} E\|\bar{\delta}\|^2 = \frac{1}{\sqrt{n}v(n)^2}\sum_{s=1}^{n} E\bar{\delta}_s^2$$

$$=\frac{1}{\sqrt{n}v(n)^2}\sum_{s=1}^{n} E\left(\sum_{i=0}^{k}\sum_{j=p_i+1}^{\infty} a_{ij}\mathscr{L}_j(s)h_i(s,X_s)\right)^2$$

$$=\frac{1}{\sqrt{n}v(n)^2}\sum_{s=1}^{n}\sum_{i=0}^{k}\left(\sum_{j=p_i+1}^{\infty} a_{ij}\mathscr{L}_j(s)\right)^2$$

$$\le\frac{1}{\sqrt{n}v(n)^2}\sum_{s=1}^{n}\sum_{i=0}^{k}\left(\sup_{j\ge p_i+1}|\mathscr{L}_j(s)| \sum_{j=p_i+1}^{\infty}|a_{ij}|\right)^2$$

$$=\frac{1}{\sqrt{n}v(n)^2}\sum_{s=1}^{n}\sum_{i=0}^{k}\left(\sup_{j\ge p_i+1}|\mathscr{L}_j(s)|\right)^2\left(\sum_{j=p_i+1}^{\infty}\frac{\sqrt{(j+2)(j+1)}}{\sqrt{(j+2)(j+1)}}|a_{ij}|\right)^2$$

$$\le\frac{1}{\sqrt{n}v(n)^2}\sum_{s=1}^{n}\sum_{i=0}^{k}\frac{1}{\sqrt{sp_i}}\sum_{j=p_i+1}^{\infty}\frac{1}{(j+2)(j+1)}\sum_{j=p_i+1}^{\infty}(j+2)(j+1)|a_{ij}|^2$$

$$\le\frac{1}{\sqrt{n}v(n)^2}\sum_{s=1}^{n}\sum_{i=0}^{k}\frac{1}{\sqrt{s}}\frac{1}{\sqrt{p_i}}\frac{o(1)}{p_i} \le \frac{o(1)}{v(n)^2}\frac{k}{p_{\min}^{3/2}}\frac{1}{\sqrt{n}}\sum_{i=0}^{n}\frac{1}{\sqrt{s}}$$

$$=\frac{o(1)}{v(n)^2}\frac{n^{\kappa_1}}{n^{3\kappa_2/2}} \to 0,$$

as $n \to \infty$, where we have used Assumption Bc.1, Bc.2 and the bound of $|\mathscr{L}_j(s)|$ in (2.3.5). In addition,

$$\begin{aligned}
\frac{1}{\sqrt{n}v(n)^2} E\|\bar{\gamma}\|^2 &= \frac{1}{\sqrt{n}v(n)^2} \sum_{s=1}^{n} E\bar{\gamma}_s^2 \\
=&\frac{1}{\sqrt{n}v(n)^2} \sum_{s=1}^{n} E \left(\sum_{i=k+1}^{\infty} \sum_{j=0}^{\infty} a_{ij} \mathscr{L}_j(s) h_i(s, X_s) \right)^2 \\
=&\frac{1}{\sqrt{n}v(n)^2} \sum_{s=1}^{n} \sum_{i=k+1}^{\infty} \left(\sum_{j=0}^{\infty} a_{ij} \mathscr{L}_j(s) \right)^2 \\
\leq& C\frac{\sqrt{n}}{v(n)^2} \sum_{i=k+1}^{\infty} \left(\sum_{j=0}^{\infty} |a_{ij}| \right)^2 \\
\leq& C\frac{\sqrt{n}}{v(n)^2 k} \sum_{i=k+1}^{\infty} i \left(\sum_{j=0}^{\infty} |a_{ij}| \right)^2 = o(1)\frac{\sqrt{n}}{v(n)^2 k} \\
=&o(1) n^{\frac{1}{2}-\kappa_1} \to 0,
\end{aligned}$$

where we have used Assumption Bc.1, Bc.2 and $|\mathscr{L}_j(s)| \leq C$ in (2.3.5) for any j.

Thus, the results in (3.1.10) and (3.1.11) imply all ingredients in Q_n converge to zero in probability, so does Q_n in the original probability space due to the equivalence of $Q_n \to_D 0$ and $Q_n \to_P 0$.

Now we are in a position to prove that $\frac{1}{\sqrt[4]{n}v(n)} M_n$ converges to 0 in probability. Once again, we can write it as

$$\begin{aligned}
\frac{1}{\sqrt[4]{n}v(n)} M_n =& \frac{1}{\sqrt[4]{n}v(n)} aX'\delta = \frac{1}{\sqrt[4]{n}v(n)} (\bar{\mathbf{F}} - \bar{\delta} - \bar{\gamma})'\delta \\
=& \frac{1}{\sqrt[4]{n}v(n)} (\bar{\mathbf{F}}'\delta - \bar{\delta}'\delta - \bar{\gamma}'\delta).
\end{aligned}$$

By virtue of Cauchy-Schwarz inequality, in view of (3.1.11) and (3.1.10), what we need to do is only to prove that $\|\delta\|^2$ converges to 0 in probability. In fact,

$$\begin{aligned}
E\sum_{s=1}^{n} \delta_s^2 =& \sum_{s=1}^{n} E \left(\sum_{i=0}^{k} \sum_{j=p_i+1}^{\infty} c_{ij} \mathscr{L}_j(s) h_i(s, X_s) \right)^2 \\
=& \sum_{s=1}^{n} \sum_{i=0}^{k} \left(\sum_{j=p_i+1}^{\infty} c_{ij} \mathscr{L}_j(s) \right)^2 \leq \sum_{s=1}^{n} \sum_{i=0}^{k} \frac{1}{\sqrt{s}} \frac{1}{\sqrt{p_i}} \frac{o(1)}{p_i}
\end{aligned}$$

$$=o(1)\frac{k}{\sqrt{p_{\min}}^3}\sum_{s=1}^{n}\frac{1}{\sqrt{s}}=o(1)\frac{n^{\kappa_1+1/2}}{n^{3\kappa_2/2}}\to 0,$$

where we again have used the result of Theorem 2.2.2 with $r=2$, the bound of $|\mathscr{L}_j(s)|$ in (2.3.5) and Assumption Bc.2.

The last step is to demonstrate that $\frac{1}{\sqrt[4]{nv(n)}}L_n$ converges to the desired random variable in distribution. We write

$$\begin{aligned}\frac{1}{\sqrt[4]{nv(n)}}L_n=&\frac{1}{\sqrt[4]{nv(n)}}aX'\varepsilon=\frac{1}{\sqrt[4]{nv(n)}}(\bar{\mathbf{F}}-\bar{\delta}-\bar{\gamma})'\varepsilon\\ =&\frac{1}{\sqrt[4]{nv(n)}}(\bar{\mathbf{F}}'\varepsilon-\bar{\delta}'\varepsilon-\bar{\gamma}'\varepsilon).\end{aligned}$$

In view of Theorem 1.3.1, we have

$$\frac{1}{\sqrt[4]{nv(n)}}\bar{\mathbf{F}}'\varepsilon=\frac{1}{\sqrt[4]{nv(n)}}\sum_{s=1}^{n}\bar{F}(s,X_s)e_s\to_D\left(\int_0^1 G_3(u)dL_W(u,0)\right)^{\frac{1}{2}}N, \tag{3.1.12}$$

where $G_3(\cdot)=\int F^2(\cdot,x)dx$.

In addition, invoking the fact that $(e_s,\mathcal{F}_{n,s})$ is a martingale difference and $x_{s+1,n}$ is adapted to $\mathcal{F}_{n,s}$, we have

$$\begin{aligned}&\frac{1}{\sqrt{nv^2(n)}}E|\bar{\delta}'\varepsilon|^2=\frac{1}{\sqrt{nv^2(n)}}E\left(\sum_{s=1}^{n}\bar{\delta}_se_s\right)^2=\frac{1}{\sqrt{nv^2(n)}}\sum_{s_1=1}^{n}\sum_{s_2=1}^{n}E[\bar{\delta}_{s_1}e_{s_1}\bar{\delta}_{s_2}e_{s_2}]\\ =&\frac{1}{\sqrt{nv^2(n)}}\sum_{s=1}^{n}E[\bar{\delta}_s^2e_s^2]+\frac{2}{\sqrt{nv^2(n)}}\sum_{s_1=1}^{n-1}\sum_{s_2=s_1+1}^{n}E[\bar{\delta}_{s_1}e_{s_1}\bar{\delta}_{s_2}e_{s_2}]\\ =&\frac{1}{\sqrt{nv^2(n)}}\sum_{s=1}^{n}E[\bar{\delta}_s^2E(e_s^2|\mathcal{F}_{n,s})]+\frac{2}{\sqrt{nv^2(n)}}\sum_{s_1=1}^{n-1}\sum_{s_2=s_1+1}^{n}E[\bar{\delta}_{s_1}e_{s_1}\bar{\delta}_{s_2}E(e_{s_2}|\mathcal{F}_{n,s_2})]\\ =&\frac{\sigma_e^2}{\sqrt{nv(n)^2}}\sum_{s=1}^{n}E\bar{\delta}_s^2,\end{aligned}$$

and similarly

$$\frac{1}{\sqrt{nv(n)^2}}E|\bar{\gamma}'\varepsilon|^2=\frac{1}{\sqrt{nv(n)^2}}E\left(\sum_{s=1}^{n}\bar{\gamma}_se_s\right)^2=\frac{\sigma_e^2}{\sqrt{nv(n)^2}}\sum_{s=1}^{n}E\bar{\gamma}_s^2,$$

and thus (3.1.11b) gives us what we want. This finishes the proof. □

3.1.2 Asymptotics of the estimated unknown functional

After obtaining the estimation of coefficients in the expansion of functional $m(t, B(t))$, we would be able to estimate the function $m(\tau, x)$ at point (τ, x) where $\forall \tau > 0$ and $x \in \mathbb{R}$ is any point in the trajectory of $B(\tau)$, namely, we can have $\widehat{m}(\tau, x)$ by superseding $\widehat{\theta}$ in lieu of θ and getting rid of residuals in the expansion of $m(\tau, x)$.

More precisely, given that $m(\cdot, \cdot)$ satisfies Assumption B.1, $m(\tau, x)$ is decomposed in terms of $\{\mathscr{L}_j(\tau)h_i(\tau, x)\}$ as

$$\begin{aligned} m(\tau, x) =& \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_{ij}\mathscr{L}_j(\tau)h_i(\tau, x) \\ =& \sum_{i=0}^{k}\sum_{j=0}^{p_i} c_{ij}\mathscr{L}_j(\tau)h_i(\tau, x) + \sum_{i=0}^{k}\sum_{j=p_i+1}^{\infty} c_{ij}\mathscr{L}_j(\tau)h_i(\tau, x) \\ &+ \sum_{i=k+1}^{\infty}\sum_{j=0}^{\infty} c_{ij}\mathscr{L}_j(\tau)h_i(\tau, x) \\ :=& A'(\tau, x)\theta + \delta(\tau, x) + \gamma(\tau, x), \end{aligned} \tag{3.1.13}$$

where

$$\begin{aligned} A'(\tau, x) &= (\mathscr{L}_0(\tau)h_0(\tau, x), \cdots, \mathscr{L}_{p_0}(\tau)h_0(\tau, x), \cdots, \\ &\qquad \mathscr{L}_0(\tau)h_k(\tau, x), \cdots, \mathscr{L}_{p_k}(\tau)h_k(\tau, x)), \\ \delta(\tau, x) &= \sum_{i=0}^{k}\sum_{j=p_i+1}^{\infty} c_{ij}\mathscr{L}_j(\tau)h_i(\tau, x), \\ \gamma(\tau, x) &= \sum_{i=k+1}^{\infty}\sum_{j=0}^{\infty} c_{ij}\mathscr{L}_j(\tau)h_i(\tau, x). \end{aligned}$$

Thus,

$$\widehat{m}(\tau, x) = A'(\tau, x)\widehat{\theta}. \tag{3.1.14}$$

We shall investigate the limit of

$$\begin{aligned} \widehat{m}(\tau, x) - m(\tau, x) &= A'(\tau, x)(\widehat{\theta} - \theta) - \delta(\tau, x) - \gamma(\tau, x) \\ &= A'(\tau, x)(X'X)^{-1}X'(\delta + \gamma + \varepsilon) - \delta(\tau, x) - \gamma(\tau, x). \end{aligned} \tag{3.1.15}$$

The following lemma is very useful for the consequential development.

Lemma 3.1.2. *Let v be an $1 \times p$ unit column vector. Define $p \times p$ matrix $A = vv'$. Then A has eigenvalues $\lambda_1 = 1$, $\lambda_i = 0$, $i = 2, \ldots, p$.*

Proof. It is evident that A is both symmetric and nonnegative definite, so that for $i = 1, \cdots, p$, λ_i are all real and $\lambda_i \geq 0$. Moreover,

$$\sum_{i=1}^{p} \lambda_i = \text{tr}(A) = \text{tr}(vv') = \text{tr}(v'v) = \|v\|^2 = 1.$$

Nonetheless, $\lambda_1 = 1$ since $Av = vv'v = v$. Whence, the assertion follows. □

Define matrices $A_{p\times p}$ and $B_{p\times p}$ by

$$A = \frac{A(\tau, x)A'(\tau, x)}{\|A(\tau, x)\|^2} \quad \text{and} \quad B = (X'X)A(X'X)^{-1} \tag{3.1.16}$$

where $\|\cdot\|$ signifies Euclidean norm and dimension $p = p_0 + \cdots + p_k + k + 1$ indicated by the expression of $A'(\tau, x)$.

Because of similarity, A and B share the same eigenvalues. In view of Lemma 3.1.2, A has $\lambda_1 = 1$ as its eigenvalue, so does B. Let normalised vector α be the left eigenvector of B pertaining to eigenvalue $\lambda_1 = 1$, viz. $\alpha' B = \alpha'$. As α is p-dimensional vector, in accordance with $A(\tau, x)$, represent α in double-index subscript, $\alpha' = (\alpha_{00}, \cdots, \alpha_{0p_0}, \cdots, \alpha_{k0}, \cdots, \alpha_{kp_k})$. The following assumption proposes a two dimensional sequence we are working with.

Assumption Bm.1[4]

(a) Let $\mathcal{S} = \{a_0, a_1, a_2, \ldots\}$, where $a_i = \{a_{ij}\}_{j=0}^{\infty}$ is a sequence such that $\sum_{j=1}^{\infty} j|a_{i,j}| < \infty$ for $i = 0, 1, 2, \cdots$.

(b) Suppose further that $\sum_{i=1}^{\infty} i \left(\sum_{j=0}^{\infty} |a_{ij}| \right)^2 < \infty$.

Remark 3.1.5. Two conditions are independent, meaning that they do not have an inclusive relationship. This is because the first condition is the requirement of decay speed of $|a_{ij}|$ in terms of j, while the second one postulates that for each $i > 0$, $\varsigma_i = \sum_{j=0}^{\infty} |a_{i,j}|$ is approximately of $O\left(\frac{1}{i^{1+\eta}}\right)$ for some $\eta > 0$. Obviously, if there are some $\epsilon > 0$ and $\eta > 0$ such that $a_{ij} = O\left(\frac{1}{(1+j)^{2+\epsilon}(1+i)^{1+\eta}}\right)$ for $i, j \geq 0$, both conditions are fulfilled.

In the sequel, denote $p_{\min} = \min(p_0, \cdots, p_k)$ and $p_{\max} = \max(p_0, \cdots, p_k)$. Let us reshuffle the set $\mathcal{S}$ as $\widetilde{\mathcal{S}}$ by defining

[4]Bm stands for Brownian motion and m function respectively.

1) $\widetilde{\mathcal{S}} = \{\widetilde{a}_0, \cdots, \widetilde{a}_i, \cdots\}$

2) $\widetilde{a}_i = \{\widetilde{a}_{ij}\}$ where $\widetilde{a}_{ij} = \frac{1}{\sqrt{p_{\max}}}\alpha_{ij}$ for $0 \le i \le k$ and $0 \le j \le p_i$ with $p_{\max} = \max\{p_0, \cdots, p_k\}$; otherwise, $\widetilde{a}_{ij} = a_{ij}$.

Obviously, since $\widetilde{\mathcal{S}}$ satisfies the Riesz-Fischer theorem, namely $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\widetilde{a}_{ij}^2 < \infty$, there exists a function, denoted by $\widetilde{F}(t,x)$, such that

$$\widetilde{F}(t, B(t)) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\widetilde{a}_{ij}\mathscr{L}_j(t)h_i(t, B(t)), \tag{3.1.17}$$

for any $t > 0$.

Therefore, in view of (3.1.17),

$$\frac{1}{\sqrt{p_{\max}}}\alpha' X' = \widetilde{\mathbf{F}}' - \widetilde{\delta}' - \widetilde{\gamma}' \tag{3.1.18}$$

where $\widetilde{\mathbf{F}}' = (\widetilde{F}(1, X_1), \cdots, \widetilde{F}(n, X_n))$, $\widetilde{\delta}' = (\widetilde{\delta}_1, \cdots, \widetilde{\delta}_n)$ with $\widetilde{\delta}_s = \sum_{i=0}^{k}\sum_{j=p_i+1}^{\infty} a_{ij}\mathscr{L}_j(s)$ $h_i(s, X_s)$ and $\widetilde{\gamma}' = (\widetilde{\gamma}_1, \cdots, \widetilde{\gamma}_n)$ with $\widetilde{\gamma}_s = \sum_{i=k+1}^{\infty}\sum_{j=0}^{\infty} a_{ij}\mathscr{L}_j(s)h_i(s, X_s)$.

Also the above reshuffle procedure can be applied with $\frac{1}{\|A(\tau,x)\|}A(\tau, x)$ as follows. Let us denote the resulting set by $\bar{\mathcal{S}}$. Accordingly, $\bar{\mathcal{S}}$ amounts to a set of sequences $\{\bar{a}_0, \bar{a}_1, \bar{a}_2, \cdots\}$ where $\bar{a}_i = \{\bar{a}_{ij}\}$ and $\bar{a}_{ij} = \frac{1}{\sqrt{p_{\max}}\|A(\tau,x)\|}\mathscr{L}_j(\tau)h_i(\tau, x)$ if $i = 0, \cdots, k$ and $j = 0, \cdots, p_i$; otherwise, $\bar{a}_{ij} = a_{ij}$.

For the same reason, there exists a function, denoted by $\widetilde{G}(t, x)$, such that

$$\widetilde{G}(t, B(t)) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\bar{a}_{ij}\mathscr{L}_j(t)h_i(t, B(t)), \tag{3.1.19}$$

for any $t > 0$. Similarly,

$$\frac{1}{\sqrt{p_{\max}}\|A(\tau, x)\|}XA(\tau, x) = \widetilde{\mathbf{G}} - \bar{\delta} - \bar{\gamma} \tag{3.1.20}$$

where $\widetilde{\mathbf{G}}' = (\widetilde{G}(1, X_1), \cdots, \widetilde{G}(n, X_n))$, $\bar{\delta}' = (\bar{\delta}_1, \cdots, \bar{\delta}_n)$ with $\bar{\delta}_s = \sum_{i=0}^{k}\sum_{j=p_i+1}^{\infty} a_{ij}\mathscr{L}_j(s)$ $h_i(s, X_s)$ and $\bar{\gamma}' = (\bar{\gamma}_1, \cdots, \bar{\gamma}_n)$ with $\bar{\gamma}_s = \sum_{i=k+1}^{\infty}\sum_{j=0}^{\infty} a_{ij}\mathscr{L}_j(t)h_i(s, X_s)$.

Note that $\widetilde{\delta} = \bar{\delta}$ and $\widetilde{\gamma} = \bar{\gamma}$ since $\widetilde{\mathcal{S}}$ and $\bar{\mathcal{S}}$ have the same tails. We have the following lemma for the generated functions $\widetilde{F}(t, x)$ and $\widetilde{G}(t, x)$.

Lemma 3.1.3. *For any $t > 0$, (a) $E[\widetilde{G}(t, B(t))]^2 < \infty$, and (b) $E[\widetilde{F}(t, B(t))]^2 < \infty$.*

Proof. (a) It follows from the orthogonality that

$$
\begin{aligned}
E[\widetilde{G}(t, B(t))]^2 =& E\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\bar{a}_{ij}\mathscr{L}_j(t)h_i(t, B(t))\right)^2 = \sum_{i=0}^{\infty}\left(\sum_{j=0}^{\infty}\bar{a}_{ij}\mathscr{L}_j(t)\right)^2 \\
\leq& 2\sum_{i=0}^{k}\left(\sum_{j=0}^{p_i}\bar{a}_{ij}\mathscr{L}_j(t)\right)^2 + 2\sum_{i=0}^{k}\left(\sum_{j=p_i+1}^{\infty}a_{ij}\mathscr{L}_j(t)\right)^2 \\
&+ \sum_{i=k+1}^{\infty}\left(\sum_{j=0}^{\infty}a_{ij}\mathscr{L}_j(t)\right)^2 \\
:=& 2\Gamma_1 + 2\Gamma_2 + \Gamma_3.
\end{aligned}
$$

Using the boundedness of $\mathscr{L}_j(t)$ and the conditions for a_{ij} in Assumption Bm.1, we have

$$
\begin{aligned}
\Gamma_2 =& \sum_{i=0}^{k}\left(\sum_{j=p_i+1}^{\infty}a_{ij}\mathscr{L}_j(t)\right)^2 \leq c\sum_{i=0}^{k}\left(\sum_{j=p_i+1}^{\infty}|a_{ij}|\right)^2 \\
\leq& c\sum_{i=0}^{k}\frac{1}{p_i^2}\left(\sum_{j=p_i+1}^{\infty}j|a_{ij}|\right)^2 \leq o(1)\frac{k}{p_{\min}^2} = o(1)n^{\kappa_1-2\kappa_2} \to 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\Gamma_3 =& \sum_{i=k+1}^{\infty}\left(\sum_{j=0}^{\infty}a_{ij}\mathscr{L}_j(t)\right)^2 \leq c\sum_{i=k+1}^{\infty}\left(\sum_{j=0}^{\infty}|a_{ij}|\right)^2 \\
\leq& \frac{c}{k}\sum_{i=k+1}^{\infty}i\left(\sum_{j=0}^{\infty}|a_{ij}|\right)^2 = o(1)n^{-\kappa_1} \to 0.
\end{aligned}
$$

Therefore, what we need to do is to show Γ_1 is bounded. By definition of $\bar{a}_{ij}$,

$$
\begin{aligned}
\Gamma_1 =& \sum_{i=0}^{k}\left(\sum_{j=0}^{p_i}\bar{a}_{ij}\mathscr{L}_j(t)\right)^2 = \sum_{i=0}^{k}\left(\sum_{j=0}^{p_i}\frac{1}{\sqrt{p_{\max}}\|A(\tau, x)\|}\mathscr{L}_j(\tau)h_i(\tau, x)\mathscr{L}_j(t)\right)^2 \\
=& \frac{1}{p_{\max}\|A(\tau, x)\|^2}\sum_{i=0}^{k}h_i^2(\tau, x)\left(\sum_{j=0}^{p_i}\mathscr{L}_j(\tau)\mathscr{L}_j(t)\right)^2 \\
=& \begin{cases} \frac{1}{p_{\max}\|A(\tau,x)\|^2}\sum_{i=0}^{k}h_i^2(\tau, x)\left(\sum_{j=0}^{p_i}\mathscr{L}_j^2(\tau)\right)^2 & \text{if } t = \tau \\ \frac{1}{p_{\max}\|A(\tau,x)\|^2}\sum_{i=0}^{k}h_i^2(\tau, x)\left(\sum_{j=0}^{p_i}\mathscr{L}_j(\tau)\mathscr{L}_j(t)\right)^2 & \text{if } t \neq \tau. \end{cases}
\end{aligned}
$$

According to Alexits (1961, p.295), if an orthogonal polynomial system $\{P_i(x)\}$ which is orthogonal with respect to $\rho(x)$ satisfying $0 \leq \rho(x) \leq C$ (for some $C > 0$) contains a constant, then the following assertion is uniformly fulfilled on any compact interval

$$\sum_{i=0}^{m} P_i^2(x) = O(m). \tag{3.1.21}$$

If $t = \tau$, it follows that $\sum_{j=0}^{p_i} \mathscr{L}_j^2(\tau) = e^{-\tau} \sum_{j=0}^{p_i} L_j^2(\tau) = O(1)p_i$. Thus

$$\Gamma_1 = \frac{1}{p_{\max}\|A(\tau,x)\|^2} \sum_{i=0}^{k} O(1)p_i h_i^2(\tau,x) \sum_{j=0}^{p_i} \mathscr{L}_j^2(\tau) \leq O(1).$$

If $t \neq \tau$,

$$\begin{aligned}\Gamma_1 =& \frac{1}{p_{\max}\|A(\tau,x)\|^2} \sum_{i=0}^{k} h_i^2(\tau,x) \left(\sum_{j=0}^{p_i} \mathscr{L}_j(\tau)\mathscr{L}_j(t) \right)^2 \\ \leq& \frac{1}{p_{\max}\|A(\tau,x)\|^2} \sum_{i=0}^{k} h_i^2(\tau,x) \sum_{j=0}^{p_i} \mathscr{L}_j^2(\tau) \sum_{j=0}^{p_i} \mathscr{L}_j^2(t) \\ =& \frac{1}{\|A(\tau,x)\|^2} \sum_{i=0}^{k} O(1) h_i^2(\tau,x) \sum_{j=0}^{p_i} \mathscr{L}_j^2(\tau) = O(1).\end{aligned}$$

To conclude, for any $t > 0$, $E[\widetilde{G}(t,B(t))]^2 < \infty$.

(b) Similar to the part (a),

$$\begin{aligned}E[\widetilde{F}(t,B(t))]^2 =& E\left(\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \widetilde{a}_{ij} \mathscr{L}_j(t) h_i(t,B(t)) \right)^2 = \sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} \widetilde{a}_{ij} \mathscr{L}_j(t) \right)^2 \\ \leq& 2 \sum_{i=0}^{k} \left(\sum_{j=0}^{p_i} \widetilde{a}_{ij} \mathscr{L}_j(t) \right)^2 + 2 \sum_{i=0}^{k} \left(\sum_{j=p_i+1}^{\infty} a_{ij} \mathscr{L}_j(t) \right)^2 \\ &+ \sum_{i=k+1}^{\infty} \left(\sum_{j=0}^{\infty} a_{ij} \mathscr{L}_j(t) \right)^2 .\end{aligned}$$

In view of the proof in the part (a), we only need to show the boundedness of the first term. By Cauchy-Schwarz inequality,

$$\sum_{i=0}^{k} \left(\sum_{j=0}^{p_i} \widetilde{a}_{ij} \mathscr{L}_j(t) \right)^2 \leq \sum_{i=0}^{k} \sum_{j=0}^{p_i} \mathscr{L}_j^2(t) \sum_{j=0}^{p_i} |\widetilde{a}_{ij}|^2 = O(1) \sum_{i=0}^{k} \frac{p_i}{p_{\max}} \sum_{j=0}^{p_i} \alpha_{i,j}^2 \leq O(1) < \infty,$$

since α is an unit vector. The proof is finished. □

Assumption Bm.2

(a) Suppose that $\widetilde{F}(t,x)$ and $\widetilde{G}(t,x)$ are in Class $\mathcal{T}(HI)$ with homogeneity powers $v(\cdot)$ and $g(\cdot)$ and normal functions F and G respectively.

(b) Suppose further that $\widetilde{F}^2(t,x)$, $\widetilde{G}^2(t,x)$ and $\widetilde{F}(t,x)\widetilde{G}(t,x)$ are all in Class $\mathcal{T}(HI)$ with homogeneity powers $v^2(\cdot)$, $g^2(\cdot)$ and $v(\cdot)g(\cdot)$ and normal functions $F(\cdot,\cdot)$, $G(\cdot,\cdot)$ and $F(\cdot,\cdot)G(\cdot,\cdot)$ respectively.

Another crucial issue is the orders of the truncation parameters and homogeneity powers involved. This question has considerable impact on the convergence of the estimator.

Assumption Bm.3

(a) $k = [n^{\kappa_1}]$ where $0 < \kappa_1 < 1$.

(b) Suppose that $p_{\min} = [n^{\kappa_2}]$ and $p_{\max} = [n^{\bar{\kappa}_2}]$ where $0 < \kappa_2 \le \bar{\kappa}_2 < 1$.

(c) Moreover, κ_1 and κ_2 satisfy that $\kappa_1 > \frac{1}{2}$, and $\frac{2}{3}\kappa_1 + \frac{1}{3} \le \kappa_2 \le \frac{7}{5}\kappa_1$.

(d) Let $g(n) = n^\rho$. Suppose that $\frac{1}{2}(\kappa_1 - \frac{1}{2}) < \rho < \frac{1}{2}\kappa_2$.

Remark 3.1.6. Conditions (a) and (b) are requirements for all truncation parameters. Note that they are all required to be of $o(n)$. This guarantees that we have sufficient information to estimate the coefficients in the expansion. In addition, condition (c) imposes a kind of relationship between the orders to ensure the convergence in the subsequential study. If necessary, we can control the difference $\bar{\kappa}_2 - \kappa_2$ as small as possible. This is the reason that we ignore the difference in the following proof. Approximately, ρ is greater than some positive number but less than $\frac{1}{2}$. Feasible selections of them are clearly considerable, for example, $\kappa_1 = 0.6$ and $\kappa_2 = 0.8$.

Theorem 3.1.2. *Suppose that $\{x_{s,n}\}_1^n$ and $\{e_s\}_1^n$ satisfy Assumptions B and A(c). Under Assumptions B.1 and Bm.1–Bm.3 we have*

$$\begin{aligned}&\frac{1}{\sqrt[4]{n}v(n)}\frac{\alpha' X' X A(\tau,x)}{\sqrt{p_{\max}}\|A(\tau,x)\|^2}(\widehat{m}(\tau,x) - m(\tau,x))\\ &\to_D \left(\int_0^1 G_3(t)dL_W(t,0)\right)^{\frac{1}{2}} N,\end{aligned} \tag{3.1.22}$$

where $G_3(t) = \int F(t,x)^2 dx$, W is a standard Brownian motion on $[0,1]$, N is a standard normal random variable independent of W, and L_W is the local–time process of W.

Remark 3.1.7. As can be seen from the proof of the theorem, since the quantity

$$\frac{1}{\sqrt[4]{n}v(n)}\frac{\alpha' X' X A(\tau,x)}{\sqrt{p_{\max}}\|A(\tau,x)\|^2} = \Delta\frac{1}{\|A(\tau,x)\|}\sqrt[4]{n}\sqrt{p_{\max}}g(n)$$

and Δ is convergent in distribution to a random variable, the quantity is about $\frac{\sqrt[4]{n}\sqrt{p_{\max}}g(n)}{\|A(\tau,x)\|}$. To estimate the order, observe from the proof that $O(1)\sqrt{kp_{\min}} \le \|A(\tau,x)\| \le O(1)\sqrt{kp_{\max}}$. Accordingly,

$$\frac{1}{\|A(\tau,x)\|}\sqrt[4]{n}\sqrt{p_{\max}}g(n) \le O(1)\frac{1}{\sqrt{kp_{\min}}}\sqrt[4]{n}\sqrt{p_{\max}}g(n) = n^{\frac{1}{4}+\rho+\frac{1}{2}(\bar{\kappa}_2-\kappa_2)-\frac{1}{2}\kappa_1}$$

and $\frac{1}{4}+\rho+\frac{1}{2}(\bar{\kappa}_2-\kappa_2)-\frac{1}{2}\kappa_1 < \frac{1}{4}+\frac{1}{2}(\kappa_2-\kappa_1) < \frac{1}{2}$. Meanwhile, $\frac{1}{\|A(\tau,x)\|}\sqrt[4]{n}\sqrt{p_{\max}}g(n) \ge \sqrt[4]{n}g(n)k^{-\frac{1}{2}} = n^{\frac{1}{4}+\rho-\frac{1}{2}\kappa_1}$ and $\frac{1}{4}+\rho-\frac{1}{2}\kappa_1 > 0$. Thus, the convergence could be significant slow if the parameters are not appropriately selected.

Proof. It follows from the relation (3.1.15) that

$$\begin{aligned}
&\frac{1}{\sqrt[4]{n}v(n)}\frac{\alpha' X' X A(\tau,x)}{\sqrt{p_{\max}}\|A(\tau,x)\|^2}(\widehat{m}(\tau,x)-m(\tau,x))\\
=&\frac{1}{\sqrt[4]{n}v(n)}\frac{\alpha' X' X A(\tau,x)}{\sqrt{p_{\max}}\|A(\tau,x)\|^2}[A'(\tau,x)(X'X)^{-1}X'(\delta+\gamma+\varepsilon)-\delta(\tau,x)-\gamma(\tau,x)]\\
=&\frac{1}{\sqrt[4]{n}\sqrt{p_{\max}}v(n)}\alpha' X' X\frac{A(\tau,x)A'(\tau,x)}{\|A(\tau,x)\|^2}(X'X)^{-1}X'(\delta+\gamma+\varepsilon)\\
&-\frac{1}{\sqrt[4]{n}v(n)}\frac{\alpha' X' X A(\tau,x)}{\sqrt{p_{\max}}\|A(\tau,x)\|^2}[\delta(\tau,x)+\gamma(\tau,x)]\\
=&\frac{1}{\sqrt[4]{n}\sqrt{p_{\max}}v(n)}\alpha' B X'(\delta+\gamma+\varepsilon)-\frac{1}{\sqrt[4]{n}v(n)}\frac{\alpha' X' X A(\tau,x)}{\sqrt{p_{\max}}\|A(\tau,x)\|^2}[\delta(\tau,x)+\gamma(\tau,x)]\\
=&\frac{1}{\sqrt[4]{n}\sqrt{p_{\max}}v(n)}\alpha' X'(\delta+\gamma+\varepsilon)-\frac{1}{\sqrt[4]{n}v(n)}\frac{\alpha' X' X A(\tau,x)}{\sqrt{p_{\max}}\|A(\tau,x)\|^2}[\delta(\tau,x)+\gamma(\tau,x)]\\
:=&\Pi_1-\Pi_2.
\end{aligned}$$

Firstly, we shall prove that Π_1 converges to the desired random variable in distribution. Using (3.1.18), write

$$\begin{aligned}
\Pi_1 =&\frac{1}{\sqrt[4]{n}\sqrt{p_{\max}}v(n)}\alpha' X'(\delta+\gamma+\varepsilon)\\
=&\frac{1}{\sqrt[4]{n}v(n)}(\widetilde{\mathbf{F}}'-\widetilde{\delta}'-\widetilde{\gamma}')(\delta+\gamma+\varepsilon).
\end{aligned}$$

Observe that it follows from Assumption Bm.2 and Theorem 1.3.1 that

$$\begin{aligned}\frac{1}{\sqrt[4]{n}v(n)}\widetilde{\mathbf{F}}'\varepsilon =&\frac{1}{\sqrt[4]{n}v(n)}\sum_{s=1}^{n}\widetilde{F}(s,X_s)e_s\\ =&\frac{1}{\sqrt[4]{n}v(n)}\sum_{s=1}^{n}\widetilde{F}(s,\sqrt{n}x_{s,n})e_s \to_D \left(\int_0^1 G_3(u)dL_W(u,0)\right)^{\frac{1}{2}} N,\end{aligned} \tag{3.1.23}$$

where $G_3(\cdot)=\int F^2(\cdot,x)dx$.

Accordingly, our aim now is to show all the other terms in Π_1 converge to zero in probability in which we may invoke the embedding schedule that allows us to work under a stronger condition: almost surely convergence of (W_n,U_n) in Assumption B but in an expanded probability space. The reason is that convergence in probability is preserved under operations like addition and product. To this purpose, Cauchy-Schwarz inequality suggests it suffices to show that

$$\|\delta\|^2 \to_P 0, \qquad \|\gamma\|^2 \to_P 0, \tag{3.1.24}$$

$$\frac{1}{\sqrt{n}v(n)^2}\|\bar{\delta}\|^2 \to_P 0, \qquad \frac{1}{\sqrt{n}v(n)^2}\|\bar{\gamma}\|^2 \to_P 0. \tag{3.1.25}$$

$$\frac{1}{\sqrt[4]{n}v(n)}\widetilde{\delta}'\varepsilon \to_P 0, \qquad \frac{1}{\sqrt[4]{n}v(n)}\widetilde{\gamma}'\varepsilon \to_P 0, \tag{3.1.26}$$

because it follows from Assumption Bm.2 and Theorem 1.3.1 with $c_n=\sqrt{n}$ that

$$\frac{1}{\sqrt{n}v(n)^2}\|\widetilde{\mathbf{F}}\|^2 = \frac{1}{\sqrt{n}v(n)^2}\sum_{s=1}^{n}\widetilde{F}^2(s,\sqrt{n}x_{s,n}) \to_P \int_0^1 G_3(u)dL_W(u,0), \tag{3.1.27}$$

as $n\to\infty$, where $G_3(\cdot)=\int F^2(\cdot,x)dx$.

We begin with (3.1.24). Straightforward calculation implies

$$\begin{aligned}E\|\gamma\|^2 =&E\left[\sum_{s=1}^{n}\gamma_s^2\right] = \sum_{s=1}^{n}E\left(\sum_{i=k+1}^{\infty}\sum_{j=0}^{\infty}c_{ij}\mathscr{L}_j(s)h_i(s,X_s)\right)^2\\ =&\sum_{s=1}^{n}E\left(\sum_{i=k+1}^{\infty}c_i(s,m)h_i(s,X_s)\right)^2 = \sum_{s=1}^{n}\sum_{i=k+1}^{\infty}c_i^2(s),\end{aligned}$$

where $c_i(s):=c_i(s,m)$ for brevity. Since m function satisfies conditions of Theorem 2.2.1, by (2.2.13) with $r=3$, $c_i(s)=\frac{\sqrt{s}^3}{\sqrt{i(i-1)(i-2)}}c_{i-3}(s,m_x^{(3)})$, then using Assumption B.1, we have

$$E\|\gamma\|^2 = \sum_{s=1}^{n}\sum_{i=k+1}^{\infty}\frac{s^3}{i(i-1)(i-2)}c_{i-3}^2(s,m_x^{(3)})$$

$$
\begin{aligned}
&= \sum_{i=k+1}^{\infty} \frac{1}{i(i-1)(i-2)} \sum_{s=1}^{n} s^3 c_{i-3}^2(s, m_x^{(3)}) \le An \sum_{i=k+1}^{\infty} \frac{1}{i(i-1)(i-2)} \\
&\le A\frac{n}{k^2} = A\frac{1}{n^{2\kappa_1 - 1}} \to 0,
\end{aligned}
$$

as $n \to \infty$, where A is the uniform bound for $s^3 c_{i-3}^2(s, m_x^{(3)})$ imposed in Assumption B.1. Hence, $\|\gamma\|^2$ converges to 0 in probability. Similarly, we have

$$
\begin{aligned}
E\|\delta\|^2 =& E\left[\sum_{s=1}^{n} \delta_s^2\right] = \sum_{s=1}^{n} E\left(\sum_{i=0}^{k}\sum_{j=p_i+1}^{\infty} c_{ij}\mathscr{L}_j(s)h_i(s, X_s)\right)^2 \\
=& \sum_{s=1}^{n}\sum_{i=0}^{k}\left(\sum_{j=p_i+1}^{\infty} c_{ij}\mathscr{L}_j(s)\right)^2 \le \sum_{s=1}^{n}\sum_{i=0}^{k} \frac{1}{\sqrt{s}}\frac{1}{\sqrt{p_i}}\frac{o(1)}{p_i} \\
=& o(1)\frac{k}{\sqrt{p_{\min}}^3}\sum_{s=1}^{n}\frac{1}{\sqrt{s}} = o(1)\frac{n^{\kappa_1+1/2}}{n^{3\kappa_2/2}} \to 0,
\end{aligned}
$$

where we again have used the result of Theorem 2.3.1 with $r = 2$, the bound of $|\mathscr{L}_j(s)| \le C(sj)^{-\frac{1}{4}}$ in (2.3.5) and Assumption Bm.3.

Regarding (3.1.25), it follows that

$$
\begin{aligned}
&\frac{1}{\sqrt{n}v(n)^2}\sum_{s=1}^{n} E\bar{\delta}_s^2 = \frac{1}{\sqrt{n}v(n)^2}\sum_{s=1}^{n} E\left(\sum_{i=0}^{k}\sum_{j=p_i+1}^{\infty} a_{ij}\mathscr{L}_j(s)h_i(s, X_s)\right)^2 \\
=& \frac{1}{\sqrt{n}v(n)^2}\sum_{s=1}^{n}\sum_{i=0}^{k}\left(\sum_{j=p_i+1}^{\infty} a_{ij}\mathscr{L}_j(s)\right)^2 \le \frac{C^2}{\sqrt{n}v(n)^2}\sum_{s=1}^{n}\sum_{i=0}^{k}\left(\sum_{j=p_i+1}^{\infty}|a_{ij}|\right)^2 \\
\le& \frac{C^2\sqrt{n}}{v(n)^2}\sum_{i=0}^{k}\frac{1}{p_i^2}\left(\sum_{j=p_i+1}^{\infty} j|a_{ij}|\right)^2 \\
\le& \frac{C^2\sqrt{n}}{v(n)^2}\sum_{i=0}^{k}\frac{o(1)}{p_i^2} \le \frac{o(1)}{v(n)^2}\frac{k\sqrt{n}}{p_{\min}^2} = \frac{o(1)}{v(n)^2} n^{\kappa_1+\frac{1}{2}-2\kappa_2} \to 0,
\end{aligned}
$$

as $n \to \infty$, where we have used Assumptions Bm.1 and Bm.3, and the implication of (2.3.5) that $|\mathscr{L}_j(s)| < C$ for some $C > 0$.

Additionally, we have as $n \to \infty$

$$
\frac{1}{\sqrt{n}v(n)^2}\sum_{s=1}^{n} E\left[\bar{\gamma}_s^2\right] = \frac{1}{\sqrt{n}v(n)^2}\sum_{s=1}^{n} E\left(\sum_{i=k+1}^{\infty}\sum_{j=0}^{\infty} a_{ij}\mathscr{L}_j(s)h_i(s, X_s)\right)^2
$$

$$
\begin{aligned}
=&\frac{1}{\sqrt{n}v(n)^2}\sum_{s=1}^{n}\sum_{i=k+1}^{\infty}\left(\sum_{j=0}^{\infty}a_{ij}\mathscr{L}_j(s)\right)^2\\
\le&\frac{C^2}{\sqrt{n}v(n)^2}\sum_{s=1}^{n}\sum_{i=k+1}^{\infty}\left(\sum_{j=0}^{\infty}|a_{ij}|\right)^2 \le \frac{C^2\sqrt{n}}{kv(n)^2}\sum_{i=k+1}^{\infty} i\left(\sum_{j=0}^{\infty}|a_{ij}|\right)^2\\
=&\frac{o(1)}{v(n)^2}\frac{\sqrt{n}}{k}=\frac{o(1)}{v(n)^2}n^{1/2-\kappa_1}\to 0,
\end{aligned}
$$

where again the fact that $|\mathscr{L}_j(s)|<C$ is exploited; in addition, we have used Assumptions Bm.1 and Bm.3, as Assumption Bm.1 implies that $\sum_{i=k+1}^{\infty} i\left(\sum_{j=0}^{\infty}|a_{ij}|\right)^2=o(1)$.

Furthermore, invoking the fact that $(e_s,\mathcal{F}_{n,s})$ is a martingale difference and $x_{s+1,n}$ is adapted to $\mathcal{F}_{n,s}$ as well as $E(e_s^2|\mathcal{F}_{n,s-1})=\sigma_e^2$ a.s., we have

$$
\begin{aligned}
\frac{1}{\sqrt{n}v(n)^2}E|\bar{\delta}'\varepsilon|^2=&\frac{1}{\sqrt{n}v(n)^2}E\left(\sum_{s=1}^{n}\bar{\delta}_s e_s\right)^2=\frac{\sigma_e^2}{\sqrt{n}v(n)^2}\sum_{s=1}^{n}E\left[\bar{\delta}_s^2\right],\\
\frac{1}{\sqrt{n}v(n)^2}E|\bar{\gamma}'\varepsilon|^2=&\frac{1}{\sqrt{n}v(n)^2}E\left(\sum_{s=1}^{n}\bar{\gamma}_s e_s\right)^2=\frac{\sigma_e^2}{\sqrt{n}v(n)^2}\sum_{s=1}^{n}E\left[\bar{\gamma}_s^2\right],
\end{aligned}
$$

and using (3.1.25), the assertions in (3.1.26) are obtained. Therefore, we conclude that Π_1 converges to the limit of (3.1.23) in distribution.

Now we are in a position to prove that Π_2 is convergent to zero in probability.

To begin with, let us find out the limit of

$$
\Delta:=\frac{1}{\sqrt{n}v(n)g(n)}\frac{1}{p_{\max}\|A(\tau,x)\|}\alpha'X'XA(\tau,x).
$$

It follows from (3.1.18) and (3.1.20) that

$$
\begin{aligned}
\Delta=&\frac{1}{\sqrt{n}v(n)g(n)}\frac{\alpha'X'XA(\tau,x)}{p_{\max}\|A(\tau,x)\|}=\frac{1}{\sqrt{n}v(n)g(n)}(\widetilde{\mathbf{F}}'-\widetilde{\delta}'-\widetilde{\gamma}')(\widetilde{\mathbf{G}}-\bar{\delta}-\bar{\gamma})\\
=&\frac{1}{\sqrt{n}v(n)g(n)}(\widetilde{\mathbf{F}}'-\bar{\delta}'-\bar{\gamma}')(\widetilde{\mathbf{G}}-\bar{\delta}-\bar{\gamma})\\
=&\frac{1}{\sqrt{n}v(n)g(n)}(\widetilde{\mathbf{F}}'\widetilde{\mathbf{G}}-\bar{\delta}'\widetilde{\mathbf{G}}-\bar{\gamma}'\widetilde{\mathbf{G}}-\widetilde{\mathbf{F}}'\bar{\delta}-\widetilde{\mathbf{F}}'\bar{\gamma}+2\bar{\gamma}'\bar{\delta}+\bar{\delta}'\bar{\delta}+\bar{\gamma}'\bar{\gamma}). \qquad (3.1.28)
\end{aligned}
$$

We now investigate term by term. Firstly, using Assumption Bm.2 it follows from Theorem 1.3.1 with $c_n=\sqrt{n}$ and Assumption Bm.2 that, as $n\to\infty$

$$
\frac{1}{\sqrt{n}v(n)g(n)}\widetilde{\mathbf{F}}'\widetilde{\mathbf{G}}\to_P\int_0^1 J_1(u)dL_W(u,0), \qquad (3.1.29)
$$

$$\frac{1}{\sqrt{n}g(n)^2}\|\widetilde{\mathbf{F}}\|^2 \to_P \int_0^1 J_2(u)dL_W(u,0), \tag{3.1.30}$$

$$\frac{1}{\sqrt{n}g(n)^2}\|\widetilde{\mathbf{G}}\|^2 \to_P \int_0^1 J_3(u)dL_W(u,0), \tag{3.1.31}$$

where $J_1(u) = \int F(u,x)G(u,x)dx$,$J_2(u) = \int F^2(u,x)dx$, $J_3(u) = \int G^2(u,x)dx$, W is a standard Brownian motion on $[0,1]$ and $L_W(u,0)$ is the local time of W.

Secondly, apropos of the terms $\bar{\delta}'\widetilde{\mathbf{G}}$, $\bar{\gamma}'\widetilde{\mathbf{G}}$, $\widetilde{\mathbf{F}}'\bar{\delta}$ and $\widetilde{\mathbf{F}}'\bar{\gamma}$, we use Cauchy-Schwarz inequality

$$\begin{aligned}
\frac{1}{nv(n)^2g(n)^2}|\bar{\delta}'\widetilde{\mathbf{G}}|^2 &\le \frac{1}{\sqrt{n}v(n)^2}\|\bar{\delta}'\|^2\frac{1}{\sqrt{n}g(n)^2}\|\widetilde{\mathbf{G}}\|^2,\\
\frac{1}{nv(n)^2g(n)^2}|\bar{\gamma}'\widetilde{\mathbf{G}}|^2 &\le \frac{1}{\sqrt{n}v(n)^2}\|\bar{\gamma}'\|^2\frac{1}{\sqrt{n}g(n)^2}\|\widetilde{\mathbf{G}}\|^2,\\
\frac{1}{nv(n)^2g(n)^2}|\bar{\delta}'\widetilde{\mathbf{F}}|^2 &\le \frac{1}{\sqrt{n}g(n)^2}\|\bar{\delta}'\|^2\frac{1}{\sqrt{n}v(n)^2}\|\widetilde{\mathbf{F}}\|^2,\\
\frac{1}{nv(n)^2g(n)^2}|\bar{\delta}'\widetilde{\mathbf{F}}|^2 &\le \frac{1}{\sqrt{n}g(n)^2}\|\bar{\delta}'\|^2\frac{1}{\sqrt{n}v(n)^2}\|\widetilde{\mathbf{F}}\|^2.
\end{aligned}$$

Notice that in (3.1.25), we can remove $v(n)^2$ since it does not play any role for the convergence. Therefore, the limits (3.1.25), (3.1.30) and (3.1.31) indicate that all the remaining terms in (3.1.28) converge in probability to zero. Thus, Δ converges to $\int_0^1 J_1(u)dL_W(u,0)$.

Because

$$\frac{1}{\sqrt[4]{n}v(n)}\frac{\alpha'X'XA(\tau,x)}{\sqrt{p_{\max}}\|A(\tau,x)\|^2}[\delta(\tau,x)+\gamma(\tau,x)] = \Delta\frac{\sqrt[4]{n}\sqrt{p_{\max}}g(n)}{\|A(\tau,x)\|}[\delta(\tau,x)+\gamma(\tau,x)],$$

in order to prove that $\Pi_2 \to_P 0$, it suffices to show

$$\frac{\sqrt[4]{n}\sqrt{p_{\max}}g(n)}{\|A(\tau,x)\|}\delta(\tau,x) \to 0 \quad \text{and} \quad \frac{\sqrt[4]{n}\sqrt{p_{\max}}g(n)}{\|A(\tau,x)\|}\gamma(\tau,x) \to 0$$

as $n \to \infty$.

Let us first estimate $\|A(\tau,x)\|$. Recall that by definition $\mathscr{L}_j(\tau) = e^{-\tau/2}L_j(\tau)$ where $\{L_j(\cdot)\}$ is the Laguerre orthogonal polynomial system. According to Alexits (1961, p.295), if an orthogonal polynomial system $\{P_i(x)\}$ which is orthogonal with respect to $\rho(x)$ satisfying $0 \le \rho(x) \le C$ (for some $C > 0$) contains a constant, then the following assertion is uniformly fulfilled on any compact interval

$$\sum_{i=0}^{m} P_i^2(x) = O(m). \tag{3.1.32}$$

It follows that

$$\begin{aligned}\|A(\tau,x)\|^2 =&\sum_{i=0}^{k}\sum_{j=0}^{p_i}\mathscr{L}_j^2(\tau)h_i^2(\tau,x)=e^{-\tau}\sum_{i=0}^{k}\sum_{j=0}^{p_i}L_j^2(\tau)h_i^2(\tau,x)\\ =&O(1)\sum_{i=0}^{k}p_ih_i^2(\tau,x)\geq O(1)p_{\min}\sum_{i=0}^{k}h_i^2(\tau,x)=O(1)kp_{\min}.\end{aligned}$$

We then assert that $\|A(\tau,x)\|\geq O(1)\sqrt{kp_{\min}}$. Palpably, $\|A(\tau,x)\|\leq O(1)\sqrt{kp_{\max}}$.

Accordingly, due to Assumption B.1 (b), using the result in Theorem 2.3.1 with $r=3$ and the relation of (2.3.5) gives

$$\begin{aligned}&\frac{\sqrt[4]{n}\sqrt{p_{\max}}g(n)}{\|A(\tau,x)\|}|\delta(\tau,x)|=\frac{\sqrt[4]{n}\sqrt{p_{\max}}g(n)}{\|A(\tau,x)\|}\left|\sum_{i=0}^{k}\sum_{j=p_i+1}^{\infty}c_{ij}\mathscr{L}_j(\tau)h_i(\tau,x)\right|\\ \leq&\frac{\sqrt[4]{n}\sqrt{p_{\max}}g(n)}{\|A(\tau,x)\|}\sum_{i=0}^{k}|h_i(\tau,x)|\left|\sum_{j=p_i+1}^{\infty}c_{ij}\mathscr{L}_j(\tau)\right|\\ \leq&\frac{\sqrt[4]{n}\sqrt{p_{\max}}g(n)}{\|A(\tau,x)\|}\left(\sum_{i=0}^{k}h_i^2(\tau,x)\right)^{\frac{1}{2}}\left[\sum_{i=0}^{k}\left(\sum_{j=p_i+1}^{\infty}c_{ij}\mathscr{L}_j(\tau)\right)^2\right]^{\frac{1}{2}}\\ \leq&O(1)\frac{\sqrt[4]{n}\sqrt{p_{\max}}n^{\rho}}{\sqrt{kp_{\min}}}\sqrt{k}\left[\sum_{i=0}^{k}\frac{1}{\sqrt{\tau p_i}}\frac{o(1)}{p_i^2}\right]^{\frac{1}{2}}\leq o(1)\frac{n^{\frac{1}{4}+\rho}\sqrt{p_{\max}}k^{\frac{1}{2}}}{\sqrt{p_{\min}}p_{\min}^{\frac{5}{4}}}\\ =&o(1)n^{\frac{1}{4}+\rho+\frac{1}{2}\kappa_1+\frac{1}{2}(\bar{\kappa}_2-\kappa_2)-\frac{5}{4}\kappa_2}\to 0,\end{aligned}$$

where we have used Assumption Bm.3 (c) and (d).

Meanwhile, exploiting an asymptotic property of Hermite polynomials that, for large i, $|h_i(\tau,x)\|\leq Ci^{-\frac{1}{4}}$ where C is independent of i (see Nikiforov and Uvarov, 1988, p.54), and on account of Assumption B.1 using (2.2.13) with $r=3$, we have

$$\begin{aligned}&\frac{\sqrt[4]{n}\sqrt{p_{\max}}g(n)}{\|A(\tau,x)\|}|\gamma(\tau,x)|=\frac{\sqrt[4]{n}\sqrt{p_{\max}}g(n)}{\|A(\tau,x)\|}\left|\sum_{i=k+1}^{\infty}\sum_{j=0}^{\infty}c_{ij}\mathscr{L}_j(\tau)h_i(\tau,x)\right|\\ =&\frac{\sqrt[4]{n}\sqrt{p_{\max}}g(n)}{\|A(\tau,x)\|}\left|\sum_{i=k+1}^{\infty}c_i(\tau,m)h_i(\tau,x)\right|\\ =&\frac{\sqrt[4]{n}\sqrt{p_{\max}}g(n)}{\|A(\tau,x)\|}\left|\sum_{i=k+1}^{\infty}\frac{\sqrt{\tau}^3}{\sqrt{i(i-1)(i-2)}}c_{i-3}(\tau,m_x^{(3)})h_i(\tau,x)\right|\end{aligned}$$

$$
\begin{aligned}
&\le O(1)\frac{\sqrt[4]{n}\sqrt{p_{\max}}g(n)}{\|A(\tau,x)\|}\left|\sum_{i=k+1}^{\infty}c_{i-3}^2(\tau,m_x^{(3)})\right|^{\frac{1}{2}}\left|\sum_{i=k+1}^{\infty}\frac{1}{i(i-1)(i-2)\sqrt{i}}\right|^{\frac{1}{2}}\\
&\le o(1)\frac{\sqrt[4]{n}n^{\rho}\sqrt{p_{\max}}}{\sqrt{kp_{\min}}}\frac{1}{k^{5/4}}\\
&=o(1)n^{\frac{1}{4}+\rho+\frac{1}{2}(\bar{\kappa}_2-\kappa_2)-\frac{7}{4}\kappa_1}\to 0,
\end{aligned}
$$

where we have utilised Assumption Bm.3 (c) and (d). Therefore $\Pi_2 \to_P 0$, which finishes the proof. □

3.2 Finite time horizon

Assume time variable t lies in $[0,T]$ with T fixed. In this section function m is defined on $[0,T]\times\mathbb{R}$. Therefore, conditions on m would be weakened since square integrability on $[0,T]$ is much weaker than that on the half line. We make the following assumptions about $m(t,x)$ in model (3.0.1).

Assumption B.2

(a) Let $m(t,x)\in L^2([0,T]\times\mathbb{R},\nu)$. Moreover, $m(t,x)$ has partial derivatives with respect to x of up to second order such that $m_x', m_x''\in L^2([0,T]\times\mathbb{R},\nu)$ and $m_x''(t,x)$ is continuous in both t and x.

(b) For each i, $b_i(t,m)=E[m(t,B_t)h_i(t,B_t)]$, the coefficient of the expansion of m in terms of the system $\{h_i(t,B(t))\}$, and its derivatives of up to second order belong to $L^2[0,T]$.

(c) Furthermore, $b_i(t,m)$ satisfies that $|b_i'(0,m)|+|b_i'(T,m)|$ is bounded by $M(T)$ uniformly in i.

Remark 3.2.1. Apart from condition (c), Assumption B.2 is quite weak and can cover a variate of functions as discussed in the remark for Theorem 2.4.4. In addition, all functions satisfying Assumption B.1 is in the ambit of Assumption B.2(a) and (b). Moreover, condition (c) is fulfilled when $m(t,x)=g(t)tp_n(x)$ with $p_n(x)$ being a polynomial of degree n and $g(t)$ being arbitrary but continuously differentiable on $[0,T]$, since $b_i=0$ if $i>n$. One more example is $m(t,x)=g(t)\cos(x)$, where $g(t)\in C^1[0,T]$. In this case, it follows from Example 2.1 that $b_i(t,m) = (-1)^k g(t) t^k \cdot \frac{1}{\sqrt{(2k)!}} e^{-\frac{t}{2}}$ when $i = 2k$, and 0 when $i = 2k+1$ for $k = 0, 1, \ldots$. Therefore,

$$\begin{aligned}|b_i'(0)| + |b_i'(T)| \leq & |g(0)| \\ & + \frac{1}{\sqrt{(2k)!}} \left[|g'(T)| + (k - 1/2)|g(T)| \right] T^k e^{-T/2} \quad \text{for} \quad i = 2k, \ k \geq 1,\end{aligned}$$

which is bounded uniformly in $i = 2k$ because $\frac{kT^k}{\sqrt{(2k)!}}$ converges to zero as $k \to \infty$.

Suppose that we have n observations for the process $Y(t)$ on $[0, T]$ and the observations are $Y_{s,n} = Y(t_{s,n})$ at $t_{s,n} = T\frac{s}{n}$ for $s = 1, 2, \cdots, n$. At the sampling points, we have the following models

$$Y_{s,n} = m(t_{s,n}, X_{s,n}) + e_s, \quad s = 1, \ldots, n, \tag{3.2.1}$$

where $X_{s,n} = B(T\frac{s}{n})$ denotes the Brownian motion at point $t_{s,n}$, $e_s = \varepsilon(T\frac{s}{n})$ $(s = 1, \ldots, n)$ form an error sequence with mean zero and finite variance.

Note that $X_{s,n} = \sum_{i=1}^{s} (X_{i,n} - X_{i-1,n}) = \sqrt{T}\frac{1}{\sqrt{n}} \sum_{i=1}^{s} w_i$, where $w_i = \frac{1}{\sqrt{T}}\sqrt{n}(X_{i,n} - X_{i-1,n})$ forms an i.i.d. N(0,1) sequence. Let $x_{s,n} = \frac{1}{\sqrt{n}} \sum_{i=1}^{s} w_i$. It follows from the functional central limit theorem that $x_{s,n}$ converges to a standard Brownian motion in distribution as $n \to \infty$. It also is clear that $x_{s,n}$, along with $d_{l,k,n} = \sqrt{(l-k)/n}$, satisfies Assumption A.

Due to the expansion of functional $m(t, B(t))$, given truncation parameters k and p_i $(0 \leq i \leq k)$, equation (3.2.1) can be rephrased as

$$\begin{aligned}Y_{s,n} = & \sum_{i=0}^{k} \sum_{j=0}^{p_i} b_{ij} \varphi_{jT}(t_{s,n}) h_i(t_{s,n}, X_{s,n}) + \sum_{i=0}^{k} \sum_{j=p_i+1}^{\infty} b_{ij} \varphi_{jT}(t_{s,n}) h_i(t_{s,n}, X_{s,n}) \\ & + \sum_{i=k+1}^{\infty} \sum_{j=0}^{\infty} b_{ij} \varphi_{jT}(t_{s,n}) h_i(t_{s,n}, X_{s,n}) + e_s, \quad s = 1, 2, \ldots, n.\end{aligned} \tag{3.2.2}$$

Equivalently, (3.2.2) in matrix form is

$$Y = X\beta + \delta + \gamma + \varepsilon, \tag{3.2.3}$$

where all notations remain the same as in the last section so that we omit reciting them. The OLS estimator of β is given by

$$\widehat{\beta} = (X'X)^{-1} X'Y. \tag{3.2.4}$$

3.2.1 Asymptotics of the estimated coefficients

As in last section, to tackle the dimension curse we introduce the following assumption.

Assumption Bc.4

(a) Let $\mathcal{S} = \{a_0, a_1, a_2, \ldots\}$, where $a_i = \{a_{ij}\}_{j=0}^{\infty}$ is a sequence such that $\sum_{j=1}^{\infty} j|a_{i,j}| < \infty$ for $i = 0, 1, 2, \cdots$.

(b) Suppose further that $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} |a_{ij}| < \infty$.

Remark 3.2.2. Condition (a) on the set of sequences ensures not only the convergence of combinations of each $\{a_i\}$ and the basis $\{\varphi_{jT}(t)\}$, but also differentiability of the corresponding function that is guaranteed by Theorem 2.3.4. Furthermore, condition (b) is a sufficient condition that secures the convergence of the combination of $\{a_{ij}\}$ and the basis $\{\varphi_{jT}(t)h_i(t, B(t))\}$ in the product space, because under which $a_{ij}^2 < |a_{ij}|$.

The Assumption Bc.4 makes the following transformation of $\widehat{\beta}$ effective. Let $k = k(n); p_i = p_i(n), i = 0, 1, \ldots k$, be the truncation parameters in the expansion which are increasing with sample size n; for $\mathcal{S}$ in Assumption Bc.4 let

$$a = (a_{00}, \cdots, a_{0p_0}, \cdots, a_{k0}, \cdots, a_{kp_k})$$

be the truncated series corresponding to the truncation parameters. We now have the following transformation for $\widehat{\beta}$:

$$aX'X[\widehat{\beta} - \beta] = aX'(\delta + \gamma + \varepsilon). \tag{3.2.5}$$

In order to obtain asymptotic behavior of $\widehat{\beta}$, we make the following assumptions for the truncation parameters.

Assumption Bc.5

(a) $k = n^{\kappa_1}$ and $1/2 \leq \kappa_1 < 1$

(b) For any i, $p_i = o(n)$ and $p_{\min} = n^{\kappa_2}$ with $0 < \kappa_2 < 1$ and $2\kappa_2 > 1 + \kappa_1$.

Observe that given a satisfying the Assumption Bc.4,[5] as $n \to \infty$, there is a function $F(t, x)$ such that

$$F(t, B(t)) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij}\varphi_{jT}(t)h_i(t, B(t)), \tag{3.2.6}$$

[5]Actually, it is a set of sequences $\mathcal{S}$ which satisfies Assumption Bc.4. We simplify the stating by neglecting the difference between a and $\mathcal{S}$.

in the sense of the norm in the space. Also at each observation point we can decompose $F(t_{s,n}, X_{s,n})$ based on the given truncation parameters as

$$F(t_{s,n}, X_{s,n}) = (aX')_s + \bar{\delta}_s + \bar{\gamma}_s, \qquad s = 1, \dots, n, \tag{3.2.7}$$

where $(aX')_s$ is the s-th entry of the vector aX', $\bar{\delta}_s$ and $\bar{\gamma}_s$ are defined in the same way as δ_s and γ_s.

Denote $\mathbf{F}' = (F(t_{1,n}, X_{1,n}), \cdots, F(t_{n,n}, X_{n,n}))$, $\bar{\delta}' = (\bar{\delta}_1, \cdots, \bar{\delta}_n)$ and $\bar{\gamma}' = (\bar{\gamma}_1, \cdots, \bar{\gamma}_n)$. Thus, $aX' = \mathbf{F}' - \bar{\delta}' - \bar{\gamma}'$. The last assumption is about function $F(t, x)$.

Assumption Bc.6

(a) Function $F(t, x)$ is continuous in both t and x. In addition, both $F(t, x)$ and $F^2(t, x)$ are in $L^2(\mathbb{R}, \phi_t(x))$ for any $t > 0$.

Theorem 3.2.1. *Suppose that $\{x_{s,n}\}_1^n$ and $\{e_s\}_1^n$ satisfy Assumptions B and A (c) in Chapter 1. Let Assumption B.2 and Bc.4–Bc.6 hold. Then we have*

$$\frac{1}{\sqrt{n}} aX'X[\widehat{\beta} - \beta] \to_D \int_0^1 F(rT, \sqrt{T}W(r))dU(r) \tag{3.2.8}$$

where (W, U) is a vector of Brownian motions on $[0, 1]$ specified in Assumption B.

Remark 3.2.3. Because $aX' = \mathbf{F}' - \bar{\delta}' - \bar{\gamma}'$, it follows that

$$\begin{aligned} \frac{1}{n} aX'Xa' =& \frac{1}{n}(\mathbf{F}' - \bar{\delta}' - \bar{\gamma}')(\mathbf{F} - \bar{\delta} - \bar{\gamma}) \\ =& \frac{1}{n}(\mathbf{F}'\mathbf{F} + \bar{\delta}'\bar{\delta} + \bar{\gamma}'\bar{\gamma} - 2\mathbf{F}'\bar{\delta} - 2\mathbf{F}'\bar{\gamma} + 2\bar{\delta}'\bar{\gamma}). \end{aligned}$$

However, in view of the result and proof of Theorem 3.2.1, we have

$$\begin{aligned} &\frac{1}{n}\mathbf{F}'\mathbf{F} = \frac{1}{n}\sum_{s=1}^{n} F^2\left(\frac{s}{n}T, \sqrt{T}x_{s,n}\right) \to_D \int_0^1 F^2(rT, \sqrt{T}W_r)dr, \\ &\frac{1}{n}\bar{\delta}'\bar{\delta} = \frac{1}{n}\sum_{s=1}^{n}\bar{\delta}_s^2 \to_P 0, \quad \text{and} \quad \frac{1}{n}\bar{\gamma}'\bar{\gamma} = \frac{1}{n}\sum_{s=1}^{n}\bar{\gamma}_s^2 \to_P 0, \end{aligned}$$

and consequently, by Cauchy-Schwarz inequality, $\frac{1}{n}\mathbf{F}'\bar{\delta}$, $\frac{1}{n}\mathbf{F}'\bar{\gamma}$ and $\frac{1}{n}\bar{\delta}'\bar{\gamma}$ are all convergent in probability to zero as well. Therefore, $\frac{1}{n}aX'Xa'$ converges with $n \to \infty$ in distribution to a random variable, implying that $aX'Xa' = O(n)$. The result of Theorem 3.2.1 indicates that $aX'X(\widehat{\beta} - \beta) = O(\sqrt{n})$. Comparison of these two magnitudes shows the effect of supersedence of a by $\widehat{\beta} - \beta$ plays a role of about $n^{-1/2}$.

Proof. In view of (3.2.7), we have

$$\begin{aligned}\frac{1}{\sqrt{n}}aX'X[\widehat{\beta}-\beta]=&\frac{1}{\sqrt{n}}aX'(\delta+\gamma+\varepsilon)\\=&\frac{1}{\sqrt{n}}(\mathbf{F}-\bar{\delta}-\bar{\gamma})'(\delta+\gamma+\varepsilon).\end{aligned}\tag{3.2.9}$$

We can write $X_{k,n} = \sqrt{T}x_{k,n}$ where $x_{k,n} = \frac{1}{\sqrt{n}}\sum_{j=1}^{k} w_j$ and w_j is an i.i.d.N(0,1) sequence. Because $\{x_{k,n}\}$ and $\{e_k\}$ satisfy Assumption B, the embedding schedule permits us to assume $(U_n(r), W_n(r)) \to_{a.s.} (U(r), W(r))$, in a suitable probability space but we still achieve the weak convergence for the theorem.

Notice that, because of predictability of $x_{k,n}$, we have

$$\begin{aligned}\frac{1}{\sqrt{n}}\mathbf{F}'\varepsilon =&\frac{1}{\sqrt{n}}\sum_{k=1}^{n} F(t_{k,n},\sqrt{T}x_{k,n})e_k\\=&\sum_{k=1}^{n} F\left(\frac{k}{n}T,\sqrt{T}x_{k,n}\right)\left(\frac{1}{\sqrt{n}}e_k\right)\\=&\sum_{k=1}^{n} F\left(\frac{k-1}{n}T+\frac{1}{n}T,\sqrt{T}W_n\left(\frac{k-1}{n}+\frac{1}{n}\right)\right)(U_n(k/n)-U_n((k-1)/n))\\=&\int_0^1 F\left(rT+o(1),\sqrt{T}W_n(r+o(1))\right)dU_n(r).\end{aligned}$$

Observe that since $(W_n(r+o(1)), U_n(r)) \to_{a.s.} (W, U)$ as shown in Chapter 1, it follows from the continuity of $F(\cdot,\cdot)$ and continuous mapping theorem that

$$(F(rT+o(1),\sqrt{T}W_n(r+o(1))), U_n(r)) \to_{a.s.} (F(rT,\sqrt{T}W(r)), U(r)).$$

Using Theorem 2.2 in Kurtz and Protter (1991) yields

$$\frac{1}{\sqrt{n}}\mathbf{F}'\varepsilon = \frac{1}{\sqrt{n}}\sum_{k=1}^{n} F(t_{k,n},\sqrt{T}x_{k,n})e_k \to_P \int_0^1 F(rT,\sqrt{T}W(r))dU(r).$$

Next, we shall prove all the rest terms in (3.2.19) converge in probability to zero. In effect,

$$\begin{aligned}\frac{1}{\sqrt{n}}|\mathbf{F}'\delta| =&\frac{1}{\sqrt{n}}\left|\sum_{s=1}^{n} F\left(T\frac{s}{n},\sqrt{T}x_{k,n}\right)\delta_s\right|\\\le&\left(\frac{1}{n}\sum_{s=1}^{n} F^2\left(T\frac{s}{n},\sqrt{T}x_{k,n}\right)\right)^{1/2}\left(\sum_{s=1}^{n}\delta_s^2\right)^{1/2},\end{aligned}$$

$$\frac{1}{\sqrt{n}}|\mathbf{F}'\gamma| = \frac{1}{\sqrt{n}}\left|\sum_{s=1}^{n} F\left(T\frac{s}{n}, \sqrt{T}x_{k,n}\right)\gamma_s\right|$$
$$\leq \left(\frac{1}{n}\sum_{s=1}^{n} F^2\left(T\frac{s}{n}, \sqrt{T}x_{k,n}\right)\right)^{1/2}\left(\sum_{s=1}^{n}\gamma_s^2\right)^{1/2},$$

and

$$\frac{1}{\sqrt{n}}|\bar{\delta}'\delta| \leq \left(\frac{1}{n}\sum_{s=1}^{n}\bar{\delta}_s^2\right)^{1/2}\left(\sum_{s=1}^{n}\delta_s^2\right)^{1/2}, \qquad \frac{1}{\sqrt{n}}|\bar{\delta}'\gamma| \leq \left(\frac{1}{n}\sum_{s=1}^{n}\bar{\delta}_s^2\right)^{1/2}\left(\sum_{s=1}^{n}\gamma_s^2\right)^{1/2},$$
$$\frac{1}{\sqrt{n}}|\bar{\gamma}'\delta| \leq \left(\frac{1}{n}\sum_{s=1}^{n}\bar{\gamma}_s^2\right)^{1/2}\left(\sum_{s=1}^{n}\delta_s^2\right)^{1/2}, \qquad \frac{1}{\sqrt{n}}|\bar{\gamma}'\gamma| \leq \left(\frac{1}{n}\sum_{s=1}^{n}\bar{\gamma}_s^2\right)^{1/2}\left(\sum_{s=1}^{n}\gamma_s^2\right)^{1/2}.$$

In addition,

$$\frac{1}{n}\sum_{k=1}^{n} F^2\left(T\frac{k}{n}, \sqrt{T}x_{k,n}\right)$$
$$=\sum_{k=0}^{n-1}\int_{\frac{k}{n}}^{\frac{k+1}{n}} F^2\left(T\frac{[nr]}{n}, \sqrt{T}x_{[nr],n}\right)dr - \frac{1}{n}F(0,0) + \frac{1}{n}F(T, \sqrt{T}x_{n,n})$$
$$=\int_0^1 F^2\left(T\frac{[nr]}{n}, \sqrt{T}x_{[nr],n}\right)dr - \frac{1}{n}F(0,0) + \frac{1}{n}F(T, \sqrt{T}x_{n,n})$$
$$\to_P \int_0^1 F^2(rT, \sqrt{T}W(r))dr.$$

using continuity of F. Therefore, to complete the proof it suffices to show that

$$\sum_{s=1}^{n}\delta_s^2 \to_P 0, \qquad \sum_{s=1}^{n}\gamma_s^2 \to_P 0, \qquad \frac{1}{n}\sum_{s=1}^{n}\bar{\delta}_s^2 \to_P 0,$$
$$\frac{1}{n}\sum_{s=1}^{n}\bar{\gamma}_s^2 \to_P 0, \qquad \frac{1}{\sqrt{n}}\bar{\gamma}'\varepsilon \to_P 0, \qquad \frac{1}{\sqrt{n}}\bar{\delta}'\varepsilon \to_P 0.$$

In fact, using (2.3.7), Assumption B.2, Bc.5 yields

$$E\sum_{s=1}^{n}\delta_s^2 = \sum_{s=1}^{n} E\left[\sum_{i=1}^{k}\sum_{j=p_i+1}^{\infty} b_{ij}\varphi_{jT}(sT/n)h_i(sT/n, X_{sT/N})\right]^2$$
$$=\sum_{s=1}^{n}\sum_{i=1}^{k}\left(\sum_{j=p_i+1}^{\infty} b_{ij}\varphi_{jT}(sT/n)\right)^2$$
$$\leq CT^2\sum_{s=1}^{n}\sum_{i=1}^{k}\frac{(|b_i'(0)| + |b_i'(T)|)^2}{p_i^2}$$

$$\leq CT^2 M(T)^2 \frac{nk}{p_{\min}^2} = CT^2 M(T)^2 n^{1+\kappa_1-2\kappa_2} \to 0$$

as $n \to \infty$, which implies that $\sum_{s=1}^n \delta_s^2 \to_P 0$. Meanwhile,

$$\begin{aligned} E\sum_{s=1}^n \gamma_s^2 =& \sum_{s=1}^n E\left[\sum_{i=k+1}^\infty \sum_{j=0}^\infty b_{ij}\varphi_{jT}(sT/n)h_i(sT/n, X_{sT/N})\right]^2 \\ =& \sum_{s=1}^n \sum_{i=k+1}^\infty b_i^2(sT/n). \end{aligned}$$

However, by virtue of (2.2.13) with $r = 2$, $b_i(sT/n) = \dfrac{\sqrt{sT/n}^2}{\sqrt{i(i-1)}} b_{i-2}(m_x^{(2)})$. Thus,

$$\begin{aligned} &\sum_{s=1}^n \sum_{i=k+1}^\infty b_i^2(sT/n) \leq \frac{T^2}{n^2} \sum_{s=1}^n s^2 \sum_{i=k+1}^\infty \frac{1}{i(i-1)} b_{i-2}^2(m_x^{(2)}(sT/n, x)) \\ &\leq \frac{T^2}{n^2k^2} \sum_{s=1}^n s^2 \sum_{i=k+1}^\infty b_{i-2}^2(m_x^{(2)}(sT/n, x)) \leq T^2 \frac{n}{3k^2} \max_{0\leq t\leq T} E[m''(t, B_t)]^2 \\ &= T^2 \max_{0\leq t\leq T} E[m''(t, B_t)]^2 n^{1-2\kappa_1} \to 0, \end{aligned}$$

in view of Assumption Bc.5, and we have invoked the fact that $\sum_{i=2}^\infty b_{i-2}^2(m_x^{(2)}(t, x)) = E[m''(t, B_t)]^2$ as well as the continuity of $E[m''(t, B_t)]^2$ in $[0, T]$. One hence obtains $\sum_{s=1}^n \gamma_s^2 \to_P 0$. Moreover, since

$$\begin{aligned} \frac{1}{n} E\sum_{s=1}^n \bar{\delta}_s^2 =& \frac{1}{n}\sum_{s=1}^n E\left[\sum_{i=1}^k \sum_{j=p_i+1}^\infty a_{ij}\varphi_{jT}(sT/n)h_i(sT/n, X_{sT/N})\right]^2 \\ =& \frac{1}{n}\sum_{s=1}^n \sum_{i=1}^k \left(\sum_{j=p_i+1}^\infty a_{ij}\varphi_{jT}(sT/n)\right)^2 \leq \frac{1}{n}\sum_{s=1}^n \sum_{i=1}^k \left(\frac{\sqrt{2}}{\sqrt{T}} \sum_{j=p_i+1}^\infty |a_{ij}|\right)^2 \\ =& \frac{1}{n}\sum_{s=1}^n \sum_{i=1}^k \left(\frac{\sqrt{2}}{\sqrt{T}} \sum_{j=p_i+1}^\infty \frac{1}{j}|a_{ij}|j\right)^2 \leq \frac{1}{n}\sum_{s=1}^n \sum_{i=1}^k \frac{2}{Tp_i^2}\left(\sum_{j=p_i+1}^\infty |a_{ij}|j\right)^2 \\ \leq& \frac{2k}{Tp_{\min}^2}\left(\sum_{j=p_{\min}+1}^\infty |a_{ij}|j\right)^2 = \frac{o(1)}{T} n^{\kappa_1-2\kappa_2} \to 0 \end{aligned}$$

as $n \to \infty$, where we use Assumption Bc.5 and the implication of Assumption Bc.4 that

$\sum_{j=p_i+1}^{\infty}|a_{ij}|j = o(1)$. Thereby, $\sum_{s=1}^{n}\bar{\delta}_s^2 \to_P 0$. At the meantime,

$$
\begin{aligned}
\frac{1}{n}E\sum_{s=1}^{n}\bar{\gamma}_s^2 =& \frac{1}{n}\sum_{s=1}^{n}E\left[\sum_{i=k+1}^{\infty}\sum_{j=0}^{\infty}a_{ij}\varphi_{jT}(sT/n)h_i(sT/n, X_{sT/N})\right]^2 \\
=& \frac{1}{n}\sum_{s=1}^{n}\sum_{i=k+1}^{\infty}\left[\sum_{i=k+1}^{\infty}\sum_{j=0}^{\infty}a_{ij}\varphi_{jT}(sT/n)\right]^2 \\
\le& \frac{2}{T}\sum_{i=k+1}^{\infty}\left[\sum_{j=0}^{\infty}|a_{ij}|\right]^2 \le \frac{2}{T}\sum_{i=k+1}^{\infty}\sum_{j=0}^{\infty}|a_{ij}| \to 0,
\end{aligned}
$$

by Assumption Bc.4, which leads that $\frac{1}{n}\sum_{s=1}^{n}\bar{\gamma}_s^2 \to_P 0$.

Furthermore, invoking that $x_{k,n}$ is adapted to $\mathcal{F}_{n,k-1}$ and $(e_k, \mathcal{F}_{n,k})$ is a martingale difference sequence satisfying Assumption B,

$$
\begin{aligned}
&E\left(\frac{1}{\sqrt{n}}\bar{\gamma}'\varepsilon\right)^2 = \frac{1}{n}E\left(\sum_{s=1}^{n}\bar{\gamma}_s e_s\right)^2 \\
&= \frac{1}{n}\sum_{s=1}^{n}E[\bar{\gamma}_s^2 e_s^2] + \frac{2}{n}\sum_{s_1=1}^{n-1}\sum_{s_2=s_1+1}^{n}E[\bar{\gamma}_{s_1}e_{s_1}\bar{\gamma}_{s_2}e_{s_2}] \\
&= \frac{1}{n}\sum_{s=1}^{n}E[\bar{\gamma}_s^2 E(e_s^2|\mathcal{F}_{n,s-1})] + \frac{2}{n}\sum_{s_1=1}^{n-1}\sum_{s_2=s_1+1}^{n}E[\bar{\gamma}_{s_1}e_{s_1}\bar{\gamma}_{s_2}E(e_{s_2}|\mathcal{F}_{n,s_2-1})] \\
&= \frac{1}{n}\sum_{s=1}^{n}E[\bar{\gamma}_s^2] \to 0
\end{aligned}
$$

and similar derivation gives

$$
E\left(\frac{1}{\sqrt{n}}\bar{\delta}'\varepsilon\right)^2 = \frac{1}{n}\sum_{s=1}^{n}E[\bar{\delta}_s^2] \to 0
$$

which imply that $\bar{\gamma}'\varepsilon$ and $\bar{\delta}'\varepsilon$ converge in probability to zero as well. This completes the proof. □

3.2.2 Asymptotics of the estimated unknown functional

Having obtained the estimation of coefficients, the most desirable result is to find the estimator of function $m(\tau, x)$ where $\tau \in (0, T]$ and $x \in \mathbb{R}$ is any point on the path of B_τ, and its asymptotic distribution. Given that $m(\cdot,\cdot)$ satisfies Assumption B.2, $m(\tau, x)$ is decomposed in terms of $\{\varphi_{jT}(\tau)h_i(\tau,x)\}$ as

$$m(\tau,x)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}b_{ij}\varphi_{jT}(\tau)h_i(\tau,x):=A'(\tau,x)\beta+\delta(\tau,x)+\gamma(\tau,x), \tag{3.2.10}$$

where

$$\begin{aligned}A'(\tau,x) &= (\varphi_{0T}(\tau)h_0(\tau,x),\cdots,\varphi_{p_0T}(\tau)h_0(\tau,x),\cdots,\\ &\qquad \cdots,\varphi_{0T}(\tau)h_k(\tau,x),\cdots,\varphi_{p_kT}(\tau)h_k(\tau,x)),\\ \delta(\tau,x) &= \sum_{i=0}^{k}\sum_{j=p_i+1}^{\infty}b_{ij}\varphi_{jT}(\tau)h_i(\tau,x),\\ \gamma(\tau,x) &= \sum_{i=k+1}^{\infty}\sum_{j=0}^{\infty}b_{ij}\varphi_{jT}(\tau)h_i(\tau,x).\end{aligned}$$

Thus, superseding β with its estimation $\widehat{\beta}$ and abandoning all residues, we have

$$\widehat{m}(\tau,x)=A'(\tau,x)\widehat{\beta}. \tag{3.2.11}$$

We shall investigate the limit of

$$\begin{aligned}\widehat{m}(\tau,x)-m(\tau,x) &= A'(\tau,x)(\widehat{\beta}-\beta)-\delta(\tau,x)-\gamma(\tau,x)\\ &= A'(\tau,x)(X'X)^{-1}X'(\delta+\gamma+\varepsilon)-\delta(\tau,x)-\gamma(\tau,x).\end{aligned} \tag{3.2.12}$$

Similarly, set up matrices $A_{p\times p}$ and $B_{p\times p}$ defined by

$$A=\frac{A(\tau,x)A'(\tau,x)}{\|A(\tau,x)\|^2}\quad\text{and}\quad B=(X'X)A(X'X)^{-1}. \tag{3.2.13}$$

As shown in Lemma 3.1.2, B has eigenvalues $\lambda_1=1$, $\lambda_2=\cdots=\lambda_p=0$. Let normalised α be the left eigenvector of B pertaining to λ_1. Hence, we have $\alpha'B=\alpha'$ and $\|\alpha\|=1$. In accordance with the notation of $A(\tau,x)$, the subscript of α is specified of double-index, viz., $\alpha'=(\alpha_{00},\cdots,\alpha_{0p_0},\cdots,\alpha_{k0}\cdots,\alpha_{kp_k})$.

Let us apply the reshuffle procedure for the set $\mathcal{S}$ from Assumption Bm.1 by α. Denote by $\widetilde{\mathcal{S}}$ the resulting set:

1) $\widetilde{\mathcal{S}}=\{\widetilde{a}_0,\cdots,\widetilde{a}_i,\cdots\}$.

2) $\widetilde{a}_i=\{\widetilde{a}_{ij}\}$ where $\widetilde{a}_{ij}=\frac{1}{\sqrt{p_{\max}}}\alpha_{ij}$ for $0\le i\le k$ and $0\le j\le p_i$; otherwise, $\widetilde{a}_{ij}=a_{ij}$.

Since the Riesz-Fischer theorem is satisfied by $\widetilde{\mathcal{S}}$, there exists a function, denoted by $F(t,x)$, such that

$$F(t, B(t)) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \widetilde{a}_{ij}\varphi_{jT}(t)h_i(t, B(t)), \tag{3.2.14}$$

for any $t \in [0, T]$.

Therefore, by virtue of (3.2.14),

$$\frac{1}{\sqrt{p_{\max}}}\alpha' X' = \mathbf{F}' - \widetilde{\delta}' - \widetilde{\gamma}' \tag{3.2.15}$$

where

$$\begin{aligned}
\mathbf{F}' =& (F(t_{1,n}, X_{1,n}), \cdots, F(t_{n,n}, X_{n,n}));\\
\widetilde{\delta}' =& (\widetilde{\delta}_1, \cdots, \widetilde{\delta}_n), \quad \text{with} \quad \widetilde{\delta}_s = \sum_{i=0}^{k}\sum_{j=p_i+1}^{\infty} a_{ij}\varphi_{jT}(t_{s,n})h_i(t_{s,n}, X_{s,n});\\
\widetilde{\gamma}' =& (\widetilde{\gamma}_1, \cdots, \widetilde{\gamma}_n), \quad \text{with} \quad \widetilde{\gamma}_s = \sum_{i=k+1}^{\infty}\sum_{j=0}^{\infty} a_{ij}\varphi_{jT}(t_{s,n})h_i(t_{s,n}, X_{s,n}).
\end{aligned}$$

Also the above reshuffle procedure can be applied with $\frac{1}{\|A(\tau,x)\|}A(\tau, x)$ as follows. Let us denote the resulting set by $\bar{\mathcal{S}}$. Accordingly, $\bar{\mathcal{S}}$ amounts to a set of sequence $\{\bar{a}_0, \bar{a}_1, \bar{a}_2, \cdots\}$ where $\bar{a}_i = \{\bar{a}_{ij}\}$ and $\bar{a}_{ij} = \frac{1}{\sqrt{p_{\max}}\|A(\tau,x)\|}\varphi_{jT}(\tau)h_i(\tau, x)$ if $i = 0, \cdots, k$ and $j = 0, \cdots, p_i$; otherwise, $\bar{a}_{ij} = a_{ij}$.

Similarly, by the Riesz-Fischer theorem there exists a function, denoted by $G(t,x)$, such that

$$G(t, B(t)) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \bar{a}_{ij}\varphi_{jT}(t)h_i(t, B(t)) \tag{3.2.16}$$

for any $t \in [0, T]$. Consequently, it follows from (3.2.16) that

$$\frac{1}{\|A(\tau, x)\|\sqrt{p_{\max}}} X A(\tau, x) = \mathbf{G} - \bar{\delta} - \bar{\gamma} \tag{3.2.17}$$

where

$$\begin{aligned}
\mathbf{G}' =& (G(t_{1,n}, X_{1,n}), \cdots, G(t_{n,n}, X_{n,n}));\\
\bar{\delta}' =& (\bar{\delta}_1, \cdots, \bar{\delta}_n), \quad \text{with} \quad \bar{\delta}_s = \sum_{i=0}^{k}\sum_{j=p_i+1}^{\infty} a_{ij}\varphi_{jT}(t_{s,n})h_i(t_{s,n}, X_{s,n});\\
\bar{\gamma}' =& (\bar{\gamma}_1, \cdots, \bar{\gamma}_n), \quad \text{with} \quad \bar{\gamma}_s = \sum_{i=k+1}^{\infty}\sum_{j=0}^{\infty} a_{ij}\varphi_{jT}(t_{s,n})h_i(t_{s,n}, X_{s,n}).
\end{aligned}$$

Notice that $\widetilde{\delta} = \bar{\delta}$ and $\widetilde{\gamma} = \bar{\gamma}$ since $\widetilde{\mathcal{S}}$ and $\bar{\mathcal{S}}$ have the same tails. The following lemma demonstrates the finiteness of second moment of $F(t, B_t)$ and $G(t, B_t)$.

Lemma 3.2.1. *For any $t \in [0, T]$, (a) $E[G^2(t, B_t)] < \infty$, and (b) $E[F^2(t, B_t)] < \infty$.*

Proof. (a) From the orthogonality of $h_i(t, B(t))$, it follows that

$$E[G^2(t, B(t))] = E\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \bar{a}_{ij}\varphi_{Tj}(t)h_i(t, B(t))\right)^2 = \sum_{i=0}^{\infty}\left(\sum_{j=0}^{\infty} \bar{a}_{ij}\varphi_{Tj}(t)\right)^2.$$

In view of the proof of Lemma 3.1.3 and the structure of $\bar{a}_{ij}$, we only need consider the main part of the series. It follows from the definition of $\bar{a}_{ij}$ that if $t = \tau$,

$$\begin{aligned}
\sum_{i=0}^{k}\left(\sum_{j=0}^{p_i} \bar{a}_{ij}\varphi_{Tj}(t)\right)^2 &= \frac{1}{\|A(\tau, x)\|^2 p_{\max}} \sum_{i=0}^{k} h_i^2(\tau, x)\left(\sum_{j=0}^{p_i} \varphi_{Tj}^2(\tau)\right)^2 \\
=& O(1)\frac{1}{\|A(\tau, x)\|^2}\sum_{i=0}^{k}\frac{p_i}{p_{\max}} h_i^2(\tau, x)\sum_{j=0}^{p_i}\varphi_{Tj}^2(\tau) \le O(1),
\end{aligned}$$

and if $t \neq \tau$,

$$\begin{aligned}
\sum_{i=0}^{k}\left(\sum_{j=0}^{p_i} \bar{a}_{ij}\varphi_{Tj}(t)\right)^2 &\le \sum_{i=0}^{k}\sum_{j=0}^{p_i}\varphi_{Tj}^2(t)\sum_{j=0}^{p_i}|\bar{a}_{ij}|^2 \\
\le& O(1)\sum_{i=0}^{k} p_i \sum_{j=0}^{p_i}|\bar{a}_{ij}|^2 = O(1)\sum_{i=0}^{k}\frac{p_i}{p_{\max}\|A(\tau, x)\|^2}\sum_{j=0}^{p_i}\varphi_{Tj}^2(\tau)h_i^2(\tau, x) \\
\le& O(1)\frac{1}{\|A(\tau, x)\|^2}\sum_{i=0}^{k}\sum_{j=0}^{p_i}\varphi_{Tj}^2(\tau)h_i^2(\tau, x) = O(1) < \infty.
\end{aligned}$$

(b) It follows similarly as the part (b) of Lemma 3.1.3. □

In order to obtain asymptotic behavior of $\widehat{m}$, we make the following assumptions for the truncation parameters.

Assumption Bm.4

(a) $k = [n^{\kappa_1}]$ and $\frac{1}{2} < \kappa_1 < 1$

(b) Let $p_{\min} = [n^{\kappa_2}]$ and $p_{\max} = [n^{\bar{\kappa}_2}]$ with $0 < \kappa_2 \le \bar{\kappa}_2 < 1$ and $0 \le \bar{\kappa}_2 - \kappa_2 < 2\kappa_2 - \kappa_1 - 1$.

Clearly, feasible solutions of truncation parameters do exist. Additionally, Condition (b) implies that $\kappa_2 > \kappa_1$. The last assumption is about the function $F(t,x)$, $G(t,x)$.

Assumption Bm.5

(a) Both $F(t,x)$ and $G(t,x)$ are continuous in t and x.

Theorem 3.2.2. *Suppose that $\{x_{s,n}\}_1^n$ and $\{e_s\}_1^n$ satisfy Assumption B in Chapter 1. Under Assumption B.2, Bm.4 and Bm.5 we have*

$$\frac{1}{\sqrt{n}}\frac{\alpha' X' X A(\tau,x)}{\sqrt{p_{\max}}\|A(\tau,x)\|^2}(\widehat{m}(\tau,x)-m(\tau,x)) \to_D \int_0^1 F(Tr,\sqrt{T}W(r))dU(r) \tag{3.2.18}$$

where $(U(r),W(r))$ is the vector of Brownian motion in Assumption B.

Remark 3.2.4. As can be seen from the proof of the theorem, since Δ is convergent to a random variable in distribution, the quantity $\frac{1}{\sqrt{n}}\frac{\alpha' X' X A(\tau,x)}{\sqrt{p_{\max}}\|A(\tau,x)\|^2} = \Delta\frac{\sqrt{np_{\max}}}{\|A(\tau,x)\|}$ is about $\frac{\sqrt{np_{\max}}}{\|A(\tau,x)\|}$. Notice also from the proof that, $O(1)\sqrt{kp_{\min}} \le \|A(\tau,x)\| \le O(1)\sqrt{kp_{\max}}$. Therefore, $O(1)\sqrt{\frac{n}{k}} \le \frac{\sqrt{np_{\max}}}{\|A(\tau,x)\|} \le \sqrt{\frac{n}{k}}\sqrt{\frac{p_{\max}}{p_{\min}}}$ in which the order of the left hand side is $0 < \frac{1}{2}(1-\kappa_1) < \frac{1}{4}$, while that of the right hand side is $\frac{1}{2}(1-\kappa_1)+\frac{1}{2}(\bar{\kappa}_2-\kappa_2)$, a slight bigger than the former. Roughly speaking, the order of the convergence is about $\frac{1}{2}(1-\kappa_1)$ which is less than a quarter.

Proof. We shall exploit the embedding schedule to achieve our aim since, as mentioned before, it gives us much convenient framework to work with.

It follows from the relation (3.2.12) that

$$\begin{aligned}
&\frac{1}{\sqrt{n}}\frac{\alpha' X' X A(\tau,x)}{\sqrt{p_{\max}}\|A(\tau,x)\|^2}(\widehat{m}(\tau,x)-m(\tau,x))\\
=&\frac{1}{\sqrt{n}}\frac{\alpha' X' X A(\tau,x)}{\sqrt{p_{\max}}\|A(\tau,x)\|^2}[A'(\tau,x)(X'X)^{-1}X'(\delta+\gamma+\varepsilon)-\delta(\tau,x)-\gamma(\tau,x)]\\
=&\frac{1}{\sqrt{n}\sqrt{p_{\max}}}\alpha' B X'(\delta+\gamma+\varepsilon)-\frac{1}{\sqrt{n}}\frac{\alpha' X' X A(\tau,x)}{\sqrt{p_{\max}}\|A(\tau,x)\|^2}[\delta(\tau,x)+\gamma(\tau,x)]\\
=&\frac{1}{\sqrt{n}\sqrt{p_{\max}}}\alpha' X'(\delta+\gamma+\varepsilon)-\frac{1}{\sqrt{n}}\frac{\alpha' X' X A(\tau,x)}{\sqrt{p_{\max}}\|A(\tau,x)\|^2}[\delta(\tau,x)+\gamma(\tau,x)]\\
:=&\Pi_1-\Pi_2.
\end{aligned}$$

First and foremost, let us find out the limit of Π_1. In view of (3.2.15) and noting that $\widetilde{\delta}=\bar{\delta}$, $\widetilde{\gamma}=\bar{\gamma}$, we have

$$\Pi_1 = \frac{1}{\sqrt{np_{\max}}}\alpha' X'(\delta+\gamma+\varepsilon) = \frac{1}{\sqrt{n}}(\mathbf{F}-\bar{\delta}-\bar{\gamma})'(\delta+\gamma+\varepsilon). \tag{3.2.19}$$

Notice that

$$
\begin{aligned}
\frac{1}{\sqrt{n}}\mathbf{F}'\varepsilon =& \frac{1}{\sqrt{n}}\sum_{s=1}^{n} F(t_{s,n}, \sqrt{T}x_{s,n})e_s = \sum_{s=1}^{n} F\left(\frac{s}{n}T, \sqrt{T}x_{s,n}\right)\left(\frac{1}{\sqrt{n}}e_s\right) \\
=& \sum_{s=1}^{n} F\left(\frac{s-1}{n}T + o(1), \sqrt{T}W_n\left(\frac{s-1}{n} + o(1)\right)\right)\left(U_n\left(\frac{s}{n}\right) - U_n\left(\frac{s-1}{n}\right)\right) \\
=& \sum_{s=1}^{n} \int_{\frac{s-1}{n}}^{\frac{s}{n}} F\left(rT + o(1), \sqrt{T}W_n(r + o(1))\right) dU_n(r) \\
=& \int_0^1 F\left(rT + o(1), \sqrt{T}W_n(r + o(1))\right) dU_n(r).
\end{aligned}
$$

Since $(W_n(r), U_n(r)) \to_{a.s.} (W, U)$, it follows from the continuity of $F(\cdot,\cdot)$ in Assumption Bm.5 that $(F(rT+o(1), \sqrt{T}W_n(r+o(1))), U_n(r)) \to_{a.s.} (F(rT, \sqrt{T}W(r)), U(r))$. Using Theorem 2.2 in Kurtz and Protter (1991) yields

$$
\frac{1}{\sqrt{n}}\mathbf{F}'\varepsilon = \frac{1}{\sqrt{n}}\sum_{s=1}^{n} F(t_{s,n}, \sqrt{T}x_{s,n})e_s \to_P \int_0^1 F(rT, \sqrt{T}W(r))dU(r).
$$

Next, we shall prove that each of the rest terms in (3.2.19) converges in probability to zero. Note that

$$
\begin{aligned}
\frac{1}{\sqrt{n}}|\mathbf{F}'\delta| \le \left(\frac{1}{n}\sum_{s=1}^{n} F^2\left(T\frac{s}{n}, \sqrt{T}x_{k,n}\right)\right)^{1/2}\left(\sum_{s=1}^{n}\delta_s^2\right)^{1/2}, \\
\frac{1}{\sqrt{n}}|\mathbf{F}'\gamma| \le \left(\frac{1}{n}\sum_{s=1}^{n} F^2\left(T\frac{s}{n}, \sqrt{T}x_{k,n}\right)\right)^{1/2}\left(\sum_{s=1}^{n}\gamma_s^2\right)^{1/2},
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{\sqrt{n}}|\bar{\delta}'\delta| \le \left(\frac{1}{n}\sum_{s=1}^{n}\bar{\delta}_s^2\right)^{1/2}\left(\sum_{s=1}^{n}\delta_s^2\right)^{1/2}, \quad \frac{1}{\sqrt{n}}|\bar{\delta}'\gamma| \le \left(\frac{1}{n}\sum_{s=1}^{n}\bar{\delta}_s^2\right)^{1/2}\left(\sum_{s=1}^{n}\gamma_s^2\right)^{1/2}, \\
&\frac{1}{\sqrt{n}}|\bar{\gamma}'\delta| \le \left(\frac{1}{n}\sum_{s=1}^{n}\bar{\gamma}_s^2\right)^{1/2}\left(\sum_{s=1}^{n}\delta_s^2\right)^{1/2}, \quad \frac{1}{\sqrt{n}}|\bar{\gamma}'\gamma| \le \left(\frac{1}{n}\sum_{s=1}^{n}\bar{\gamma}_s^2\right)^{1/2}\left(\sum_{s=1}^{n}\gamma_s^2\right)^{1/2}.
\end{aligned}
$$

In addition, we have as $n \to \infty$

$$
\begin{aligned}
&\frac{1}{n}\sum_{s=1}^{n} F^2\left(T\frac{s}{n}, \sqrt{T}x_{s,n}\right) \\
=& \int_0^1 F^2\left(T\frac{[nr]}{n}, \sqrt{T}x_{[nr],n}\right) dr - \frac{1}{n}F^2(0,0) + \frac{1}{n}F^2(T, \sqrt{T}x_{n,n})
\end{aligned}
$$

$$\to_P \int_0^1 F^2(rT, \sqrt{T}W(r))dr \tag{3.2.20}$$

using the continuity of F in Assumption Bm.5 and $E[F^2(T, \sqrt{T}x_{n,n})] = E[F^2(T, X_T)] < \infty$ by Lemma 3.2.1.

Therefore, in view of the above equations, in order to complete the convergence of Π_1 it suffices to show that as $n \to \infty$

$$\sum_{s=1}^n \delta_s^2 \to_P 0, \qquad \sum_{s=1}^n \gamma_s^2 \to_P 0, \qquad \frac{1}{n}\sum_{s=1}^n \bar{\delta}_s^2 \to_P 0,$$

$$\frac{1}{n}\sum_{s=1}^n \bar{\gamma}_s^2 \to_P 0, \qquad \frac{1}{\sqrt{n}}\bar{\gamma}'\varepsilon \to_P 0, \qquad \frac{1}{\sqrt{n}}\bar{\delta}'\varepsilon \to_P 0.$$

Firstly, using the result in Theorem 2.3.2 as well as Assumptions B.2 and Bm.4, we have

$$\begin{aligned}
E\left[\sum_{s=1}^n \delta_s^2\right] =& \sum_{s=1}^n E\left[\sum_{i=1}^k \sum_{j=p_i+1}^\infty b_{ij}\varphi_{jT}\left(\frac{sT}{n}\right) h_i\left(\frac{sT}{n}, X_{sT/n}\right)\right]^2 \\
=& \sum_{s=1}^n \sum_{i=1}^k \left(\sum_{j=p_i+1}^\infty b_{ij}\varphi_{jT}\left(\frac{sT}{n}\right)\right)^2 \\
\le& CT^2 \sum_{s=1}^n \sum_{i=1}^k \frac{(|b_i'(0)| + |b_i'(T)|)^2}{p_i^2} \\
\le& CT^2 M(T)^2 \frac{nk}{p_{\min}^2} = CT^2 M(T)^2 n^{1+\kappa_1 - 2\kappa_2} \to 0
\end{aligned}$$

as $n \to \infty$, which implies $\sum_{s=1}^n \delta_s^2 \to_P 0$.

Secondly, straightforward calculation gives

$$\begin{aligned}
E\left[\sum_{s=1}^n \gamma_s^2\right] =& \sum_{s=1}^n E\left[\sum_{i=k+1}^\infty \sum_{j=0}^\infty b_{ij}\varphi_{jT}\left(\frac{sT}{n}\right) h_i\left(\frac{sT}{n}, X_{sT/n}\right)\right]^2 \\
=& \sum_{s=1}^n \sum_{i=k+1}^\infty b_i^2\left(\frac{sT}{n}\right).
\end{aligned}$$

Meanwhile, by virtue of (2.2.13) with $r = 2$, we have $b_i\left(\frac{sT}{n}\right) = \frac{\sqrt{sT/n}^2}{\sqrt{i(i-1)}} b_{i-2}\left(\frac{sT}{n}, m_x''\right)$, which implies

$$\sum_{s=1}^n \sum_{i=k+1}^\infty b_i^2\left(\frac{sT}{n}\right) \le \frac{T^2}{n^2}\sum_{s=1}^n s^2 \sum_{i=k+1}^\infty \frac{1}{i(i-1)} b_{i-2}^2\left(\frac{sT}{n}, m_x''\right)$$

$$\leq \frac{T^2}{n^2k^2}\sum_{s=1}^{n} s^2 \sum_{i=k+1}^{\infty} b_{i-2}^2\left(\frac{sT}{n}, m_x''\right) \leq T^2\frac{n}{3k^2}\max_{0\leq t\leq T} E[m''(t,B_t)]^2$$
$$= T^2 \max_{0\leq t\leq T} E[m''(t,B_t)]^2 n^{1-2\kappa_1} \to 0,$$

in view of Assumption Bm.4, and we have invoked the fact that $\sum_{i=2}^{\infty} b_{i-2}^2(t, m_x'') = E[m''(t,B_t)]^2$ as well as the continuity of $E[m''(t,B_t)]^2$ in $[0,T]$, since $E[m''(t,B_t)]^2 = \int [m_x''(t,x)]^2\phi_t(x)dx = \int [m_x''(t,\sqrt{t}x)]^2\phi(x)dx$, then by Assumption B.2(a) it is continuous. This finally proves $\sum_{s=1}^{n}\gamma_s^2 \to_P 0$.

Thirdly, we have

$$\begin{aligned}
\frac{1}{n}E\left[\sum_{s=1}^{n}\bar{\delta}_s^2\right] =& \frac{1}{n}\sum_{s=1}^{n} E\left[\sum_{i=0}^{k}\sum_{j=p_i+1}^{\infty} a_{ij}\varphi_{jT}\left(\frac{sT}{n}\right) h_i\left(\frac{sT}{n}, X_{sT/n}\right)\right]^2 \\
=& \frac{1}{n}\sum_{s=1}^{n}\sum_{i=0}^{k}\left(\sum_{j=p_i+1}^{\infty} a_{ij}\varphi_{jT}\left(\frac{sT}{n}\right)\right)^2 \leq \frac{1}{n}\sum_{s=1}^{n}\sum_{i=0}^{k}\left(\frac{\sqrt{2}}{\sqrt{T}}\sum_{j=p_i+1}^{\infty}|a_{ij}|\right)^2 \\
\leq& \sum_{i=0}^{k}\frac{2}{Tp_i^2}\left(\sum_{j=p_i+1}^{\infty} j|a_{ij}|\right)^2 \leq \frac{2k}{Tp_{\min}^2}\left(\sum_{j=p_{\min}+1}^{\infty} j|a_{ij}|\right)^2 \\
=& \frac{o(1)}{T}n^{\kappa_1-2\kappa_2} \to 0
\end{aligned}$$

as $n \to \infty$, where we have used Assumption Bm.4 and the implication of Assumption Bm.1 that $\sum_{j=p_i+1}^{\infty}|a_{ij}|j = o(1)$. Hence, $\sum_{s=1}^{n}\bar{\delta}_s^2 \to_P 0$.

Fourthly, we may also have

$$\begin{aligned}
\frac{1}{n}E\left[\sum_{s=1}^{n}\bar{\gamma}_s^2\right] =& \frac{1}{n}\sum_{s=1}^{n} E\left[\sum_{i=k+1}^{\infty}\sum_{j=0}^{\infty} a_{ij}\varphi_{jT}\left(\frac{sT}{n}\right) h_i\left(\frac{sT}{n}, X_{sT/n}\right)\right]^2 \\
=& \frac{1}{n}\sum_{s=1}^{n}\sum_{i=k+1}^{\infty}\left(\sum_{j=0}^{\infty} a_{ij}\varphi_{jT}\left(\frac{sT}{n}\right)\right)^2 \leq \frac{2}{nT}\sum_{s=1}^{n}\sum_{i=k+1}^{\infty}\left(\sum_{j=0}^{\infty}|a_{ij}|\right)^2 \\
\leq& \frac{2}{kT}\sum_{i=k+1}^{\infty} i\left(\sum_{j=0}^{\infty}|a_{ij}|\right)^2 \to 0
\end{aligned}$$

due to Assumption Bm.1. We thus obtain that $\frac{1}{n}\sum_{s=1}^{n}\bar{\gamma}_s^2 \to_P 0$.

Finally, invoking that $x_{k,n}$ is adapted to $\mathcal{F}_{n,k-1}$ and $(e_k, \mathcal{F}_{n,k})$ is a martingale difference sequence satisfying Assumption B, as well as $E(e_s^2|\mathcal{F}_{n,s-1}) = \sigma^2$ a.s., we can deduce that

$$
\begin{aligned}
E\left(\frac{1}{\sqrt{n}}\bar{\gamma}'\varepsilon\right)^2 &= \frac{1}{n}E\left(\sum_{s=1}^{n}\bar{\gamma}_s e_s\right)^2 \\
&= \frac{1}{n}\sum_{s=1}^{n}E[\bar{\gamma}_s^2 E(e_s^2|\mathcal{F}_{n,s-1})] + \frac{2}{n}\sum_{s_1=1}^{n-1}\sum_{s_2=s_1+1}^{n}E[\bar{\gamma}_{s_1}e_{s_1}\bar{\gamma}_{s_2}E(e_{s_2}|\mathcal{F}_{n,s_2-1})] \\
&= \frac{\sigma^2}{n}\sum_{s=1}^{n}E[\bar{\gamma}_s^2] \to 0,
\end{aligned}
$$

and a similar derivation gives

$$
E\left(\frac{1}{\sqrt{n}}\bar{\delta}'\varepsilon\right)^2 = \frac{\sigma^2}{n}\sum_{s=1}^{n}E[\bar{\delta}_s^2] \to 0,
$$

and thus both of which imply that $\frac{1}{\sqrt{n}}\bar{\gamma}'\varepsilon$ and $\frac{1}{\sqrt{n}}\bar{\delta}'\varepsilon$ converge in probability to zero as well.

We are now in a position to prove that $\Pi_2 \to_P 0$.

To begin, let us find out the limit of $\Delta := \frac{1}{np_{\max}\|A(\tau,x)\|}\alpha'X'XA(\tau,x)$. It follows from (3.2.15) and (3.2.17) that

$$
\begin{aligned}
\Delta =& \frac{1}{n}\frac{\alpha'X'XA(\tau,x)}{p_{\max}\|A(\tau,x)\|} = \frac{1}{n}(\mathbf{F}' - \bar{\delta}' - \bar{\gamma}')(\mathbf{G} - \bar{\delta} - \bar{\gamma}) \\
=& \frac{1}{n}(\mathbf{F}'\mathbf{G} - \mathbf{F}'\bar{\delta} - \mathbf{F}'\bar{\gamma} - \bar{\delta}'\mathbf{G} - \bar{\gamma}'\mathbf{G} + 2\bar{\delta}'\bar{\gamma} + \bar{\delta}'\bar{\delta} + \bar{\gamma}'\bar{\gamma}).
\end{aligned}
$$

We investigate term by term. Using continuous mapping theorem gives

$$
\begin{aligned}
\frac{1}{n}\mathbf{F}'\mathbf{G} =& \frac{1}{n}\sum_{s=1}^{n}F(t_{s,n},X_{s,n})G(t_{s,n},X_{s,n}) \\
=& \frac{1}{n}\sum_{s=1}^{n}F\left(T\frac{s}{n},\sqrt{T}x_{s,n}\right)G\left(T\frac{s}{n},\sqrt{T}x_{s,n}\right) \\
=& \sum_{s=1}^{n}\int_{\frac{s-1}{n}}^{\frac{s}{n}}F\left(T\frac{[nr]}{n},\sqrt{T}x_{[nr],n}\right)G\left(T\frac{[nr]}{n},\sqrt{T}x_{[nr],n}\right)dr \\
& - \frac{1}{n}F(0,0)G(0,0) + \frac{1}{n}F(T,\sqrt{T}x_{n,n})G(T,\sqrt{T}x_{n,n}) \\
\to_P & \int_0^1 F(Tr,\sqrt{T}W(r))G(Tr,\sqrt{T}W(r))dr
\end{aligned}
$$

since $\frac{1}{n}F(T,\sqrt{T}x_{n,n})G(T,\sqrt{T}x_{n,n}) \to_P 0$ as $n \to \infty$ by Lemma 3.2.1.

Meanwhile,

$$
\begin{aligned}
\frac{1}{n^2}|\mathbf{F}'\bar{\delta}|^2 \leq & \frac{1}{n^2}\|\mathbf{F}\|^2\|\bar{\delta}\|^2 = \frac{1}{n}\sum_{s=1}^{n} F^2\left(T\frac{s}{n}, \sqrt{T}x_{s,n}\right)\frac{1}{n}\sum_{s=1}^{n}\bar{\delta}_s^2, \\
\frac{1}{n^2}|\mathbf{F}'\bar{\gamma}|^2 \leq & \frac{1}{n^2}\|\mathbf{F}\|^2\|\bar{\gamma}\|^2 = \frac{1}{n}\sum_{s=1}^{n} F^2\left(T\frac{s}{n}, \sqrt{T}x_{s,n}\right)\frac{1}{n}\sum_{s=1}^{n}\bar{\gamma}_s^2, \\
\frac{1}{n^2}|\bar{\delta}'\mathbf{G}|^2 \leq & \frac{1}{n^2}\|\mathbf{G}\|^2\|\bar{\delta}\|^2 = \frac{1}{n}\sum_{s=1}^{n} G^2\left(T\frac{s}{n}, \sqrt{T}x_{s,n}\right)\frac{1}{n}\sum_{s=1}^{n}\bar{\delta}_s^2, \\
\frac{1}{n^2}|\bar{\gamma}'\mathbf{G}|^2 \leq & \frac{1}{n^2}\|\mathbf{G}\|^2\|\bar{\gamma}\|^2 = \frac{1}{n}\sum_{s=1}^{n} G^2\left(T\frac{s}{n}, \sqrt{T}x_{s,n}\right)\frac{1}{n}\sum_{s=1}^{n}\bar{\gamma}_s^2, \\
\frac{1}{n^2}|\bar{\delta}'\bar{\gamma}|^2 \leq & \frac{1}{n}\sum_{s=1}^{n}\bar{\delta}_s^2\frac{1}{n}\sum_{s=1}^{n}\bar{\gamma}_s^2.
\end{aligned}
$$

Notice that we have shown that $\frac{1}{n}\sum_{s=1}^{n}\bar{\delta}_s^2 \to_P 0$ and $\frac{1}{n}\sum_{s=1}^{n}\bar{\gamma}_s^2 \to_P 0$ and we can have a similar result for G as that of (3.2.20) for F.

Whence all the above arguments indicate that $\Delta \to_P \int_0^1 F(Tr, \sqrt{T}W_r)G(Tr, \sqrt{T}W_r)dr$. Next, because

$$
\frac{1}{\sqrt{n}}\frac{\alpha' X' X A(\tau, x)}{\sqrt{p_{\max}}\|A(\tau, x)\|^2}[\delta(\tau, x) + \gamma(\tau, x)] = \Delta\frac{\sqrt{np_{\max}}}{\|A(\tau, x)\|}[\delta(\tau, x) + \gamma(\tau, x)],
$$

up to now what we need to show is $\frac{\sqrt{np_{\max}}}{\|A(\tau,x)\|}\delta(\tau, x) \to 0$ and $\frac{\sqrt{np_{\max}}}{\|A(\tau,x)\|}\gamma(\tau, x) \to 0$.

By aforementioned reason, it is easy to obtain that

$$
O(1)kp_{\min} \leq \|A(\tau, x)\|^2 = \sum_{i=0}^{k} h_i^2(\tau, x)\sum_{j=0}^{p_i}\varphi_{jT}^2(\tau) \leq O(1)kp_{\max}.
$$

It follows from Assumption B.2 and Theorem 2.3.2 that

$$
\begin{aligned}
\frac{\sqrt{np_{\max}}}{\|A(\tau, x)\|}|\delta(\tau, x)| = & \frac{\sqrt{np_{\max}}}{\|A(\tau, x)\|}\left|\sum_{i=0}^{k}\sum_{j=p_i+1}^{\infty} b_{ij}\varphi_{jT}(\tau)h_i(\tau, x)\right| \\
\leq & \frac{\sqrt{np_{\max}}}{\|A(\tau, x)\|}\sum_{i=0}^{k}|h_i(\tau, x)|\left|\sum_{j=p_i+1}^{\infty} b_{ij}\varphi_{jT}(\tau)\right| \\
\leq & O(1)\frac{\sqrt{np_{\max}}}{\sqrt{kp_{\min}}}\left(\sum_{i=0}^{k}|h_i(\tau, x)|^2\right)^{\frac{1}{2}}\left(\sum_{i=0}^{k}\left|\sum_{j=p_i+1}^{\infty} b_{ij}\varphi_{jT}(\tau)\right|^2\right)^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&\le O(1)\frac{\sqrt{np_{\max}}}{\sqrt{kp_{\min}}}\sqrt{k}\left(\sum_{i=0}^{k}\frac{a_T^2T^2}{p_i^2}\right)^{\frac{1}{2}} \le O(1)TM(T)\frac{\sqrt{np_{\max}}\sqrt{k}}{p_{\min}\sqrt{p_{\min}}}\\
&=O(1)n^{\frac{1}{2}+\frac{1}{2}\kappa_1+\frac{1}{2}(\bar{\kappa}_2-\kappa_2)-\kappa_2} \to 0
\end{aligned}
$$

as $n \to \infty$ using Assumption Bm.4.

Additionally, using (2.2.13) with $r = 2$ and $|h_i(\tau, x)| \le i^{-\frac{1}{4}}$ for large i (see Nikiforov and Uvarov, 1988, p.54) gives

$$
\begin{aligned}
\frac{\sqrt{np_{\max}}}{\|A(\tau,x)\|}|\gamma(\tau,x)| &= \frac{\sqrt{np_{\max}}}{\|A(\tau,x)\|}\left|\sum_{i=k+1}^{\infty}\sum_{j=0}^{\infty} b_{ij}\varphi_{jT}(\tau)h_i(\tau,x)\right|\\
&= \frac{\sqrt{np_{\max}}}{\|A(\tau,x)\|}\left|\sum_{i=k+1}^{\infty} b_i(\tau)h_i(\tau,x)\right|\\
&= \frac{\sqrt{np_{\max}}}{\|A(\tau,x)\|}\left|\sum_{i=k+1}^{\infty} \frac{\tau}{\sqrt{i(i-1)}} b_{i-2}(\tau,m_x'')h_i(\tau,x)\right|\\
&\le O(1)\frac{\sqrt{np_{\max}}}{\sqrt{kp_{\min}}}\left(\sum_{i=k+1}^{\infty}|b_{i-2}(\tau,m_x'')|^2\right)^{\frac{1}{2}}\left(\sum_{i=k+1}^{\infty}\frac{1}{i(i-1)\sqrt{i}}\right)^{\frac{1}{2}}\\
&\le o(1)\frac{\sqrt{np_{\max}}}{\sqrt{kp_{\min}}}\frac{1}{k^{3/4}}\\
&= o(1)n^{\frac{1}{2}+\frac{1}{2}(\bar{\kappa}_2-\kappa_2)-\frac{5}{4}\kappa_1} \to 0
\end{aligned}
$$

as $n \to \infty$ by virtue of Assumption Bm.4. The proof is finished. □

3.3 Compact time horizon approaching infinity

The interesting situation is no more than that the interval we work with is $[0, T_n]$ with $T_n \to \infty$. Because of this, we need to constrain the divergence of T_n. Our strategy is to require $\frac{T_n}{n} \to 0$ as $n \to \infty$ so that comparing with the sample size, the increase of time span of observation is negligible meaning that we get sufficient information from the sample path. This will help us avoid two drawbacks in the previous situations, that is, on $(0, \infty)$ we could not shrink the time span of observation lengths, whereas on $[0, T]$ with fixed T we ignore considerable information beyond the time zone that may be helpful for our estimation. In technical terms, allowing $T = T_n \to \infty$ and $\frac{T_n}{n} \to 0$ amounts to both infill and long span asymptotics. Meanwhile, the two-fold limit theory keeps one away from the so-called aliasing problem (i.e. different continuous-time processes may be indistinguishable when sampled at discrete times). Phillips (1973) and Hansen and Sargent (1983) were among the first discussing the aliasing phenomenon in the econometric literature. Recent studies include Bandi and Phillips (2003) and Bandi and Phillips (2007).

Let $m(t,x)$ be the function in model (3.0.1) defined on $[0,\infty)\times\mathbb{R}$. Intuitively, we need only to require that $m(t,x)$ satisfies Assumption B.2. Nonetheless, for $T = T_n \to \infty$, $m(t,x)$ has to satisfy more stringent restrictions than that in Assumption B.2.

Assumption B.3

(a) For every $t > 0$, $m(t,x)$ and its partial derivatives with respect to x of up to third order are all in $L^2(\mathbb{R},\phi_t(x))$.

(b) For each i, $b_i(t,m) = E[m(t,B(t))h_i(t,B(t))]$, the coefficient of the expansion of m in terms of the system $\{h_i(t,B(t))\}$, and its derivatives of up to second order belong to $L^2[0,T]$ for any $T > 0$.

(c) For i large enough, the coefficient functions $b_i(t, m_x^{(3)}(t,x))$ of $m_x^{(3)}(t,B(t))$ expanded by the system $\{h_i(t,B(t))\}$ are such that $t^3 b_i^2(t, m_x^{(3)}(t,x))$ are bounded on $(0,\infty)$ uniformly in i.

(d) $b_i(t,m(t,x))$ is such that its derivative $b_i'(t,m)$ is bounded in absolute value on $[0,\infty)$ by $M > 0$ uniformly in i.

Remark 3.3.1. Conditions (a), (b) and (c) are almost the same as those in Assumption B.1. This is because we now, on the one hand, confine the time variable on a compact interval, and on the other we let the time span go to infinity. Condition (d) is similar to its counterpart in Assumption B.2. There are many functions that satisfy these four conditions at the same time. For instance, $m(t,x) = t^\eta e^{-rt} P(x)$ with $\eta \geq 1$, $r > 0$ and $P(x)$ being any polynomial of fixed degree; $m(t,x) = \frac{t}{1+t^\eta}\cos x$ with $\eta \geq 3$, and so on.

For the truncation parameters and time span T_n, we make the following assumption.

Assumption B.4

(a) Let $k = [n^{\kappa_1}]$ and $p_i = o(n)$ for $0 \leq i \leq k$, $p_{\min} = [n^{\kappa_2}]$ and $T_n = [n^{\kappa_3}]$, where $0 < \kappa_i < 1$ $(i = 1,2,3)$.

(b) Let $2\kappa_3 + \kappa_1 + 1 < 2\kappa_2$ and $2\kappa_1 > 1$.

Remark 3.3.2. Feasible solutions for κ_i $(i = 1, 2, 3)$ do exist. For instance, $\kappa_1 = 0.55$, $\kappa_2 = 0.95$ and $\kappa_3 = 0.15$. Meanwhile, condition (b) implies that $\kappa_1 < \kappa_2$ and $2\kappa_3 < \kappa_1$.

Given the observation number n, one can choose $T = T_n$ according to Assumption B.4. Let us sample on $[0, T_n]$ at equally spaced points: $t_{s,n} = T_n \frac{s}{n}$ $(s = 1, \cdots, n)$ for model (3.0.1). Denote by $Y_{s,n}$ for the process $Y(t)$ at $t_{s,n}$, $X_{s,n} = B(t_{s,n})$ for the Brownian motion at the discrete points and $e_s = \varepsilon(t_{s,n})$. Note that $X_{s,n} = \sum_{i=1}^{s}(X_{i,n} - X_{i-1,n}) = \sqrt{T_n} \cdot \frac{1}{\sqrt{n}} \sum_{i=1}^{s} w_i$, where $w_i = \sqrt{\frac{n}{T_n}}(X_{i,n} - X_{i-1,n})$ forms an i.i.d N(0,1) sequence.

Let $x_{s,n} = \frac{1}{\sqrt{n}} \sum_{i=1}^{s} w_i$. It therefore follows from the functional central limit theorem that $x_{s,n}$ converges in distribution to a Brownian motion on $[0, 1]$ as $n \to \infty$. In addition, it is clear that $x_{s,n}$, along with $d_{l,k,n} = \sqrt{(l-k)/n}$, satisfies Assumption A in Chapter 1.

We expand $m(t, B(t))$ using an orthonormal basis of the form $\{\varphi_{jT_n}(t)h_i(t, B(t))\}$ at each sampling point, and then obtain n equations. The n equations can be written in the following matrix form with the similar notations as before

$$Y = X\beta + \delta + \gamma + \varepsilon. \tag{3.3.1}$$

The OLS estimator of β is given by

$$\widehat{\beta} = (X'X)^{-1}X'Y. \tag{3.3.2}$$

3.3.1 Asymptotics of the estimated coefficients

Let $\mathcal{S}$ be the set of sequences in Assumption Bc.4. Then we have proposition below.

Lemma 3.3.1. *Let Assumption Bc.4 hold. There exists a function $\bar{F}(t, x)$ such that*

$$\bar{F}(t, B(t)) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij}\varphi_{jT_n}(t)h_i(t, B(t)), \tag{3.3.3}$$

for all $t \in [0, T_n]$. Meanwhile, functions $a_i(t) = \sum_{j=0}^{\infty} a_{ij}\varphi_{jT_n}(t)$ for every i exist and are differentiable.

Proof. The existence of $\bar{F}(t, x)$ is due to the Riesz-Fischer theorem, while the existence and differentiability of $a_i(t)$ are attributed to Theorem 2.3.4. □

Let a be the row vector truncated from sequences satisfying Assumption Bc.4, that is, $a = (a_{00}, \cdots, a_{0p_0}, \cdots, a_{k0}, \cdots, a_{kp_k})$. We can apply a transformation to $\widehat{\beta}$:

$$aX'X(\widehat{\beta} - \beta) = aX'(\delta + \gamma + \varepsilon),$$

while vector aX' can be expressed by virtue of (3.3.3) as follows

$$(aX')_s = \bar{F}(t_{s,n}, X_{s,n}) - \bar{\delta}_s - \bar{\gamma}_s, \qquad s = 1, \dots, n, \tag{3.3.4}$$

where $(aX')_s$ is the s-th entry of the vector aX', $\bar{\delta}_s$ and $\bar{\gamma}_s$ are defined in the same way as δ_s and γ_s. In the vector version, $aX' = \bar{\mathbf{F}}' - \bar{\delta}' - \bar{\gamma}'$ where $\bar{\mathbf{F}} = (\bar{F}(t_{1,n}, X_{1,n}), \cdots, \bar{F}(t_{n,n}, X_{n,n}))'$.

Assumption Bc.7

(a) $\bar{F}(t,x)$ is in Class $\mathcal{T}(HI)$ with homogeneity power $v(\cdot)$ and normal function $F(t,x)$.

(b) $\bar{F}^2(t,x)$ is in Class $\mathcal{T}(HI)$ as well with homogeneity power $v^2(\cdot)$ and normal function $F^2(t,x)$.

Theorem 3.3.1. *Suppose that $\{x_{s,n}\}_{s=1}^n$ and $\{e_s\}_{s=1}^n$ satisfy Assumption B. Under Assumptions B.3, B.4 and Bc.7, we have*

$$\frac{\sqrt[4]{T_n}}{\sqrt{n}v(T_n)} aX'X[\widehat{\beta} - \beta] \to_D \left(\int_0^1 G_3(u) dL_W(u,0) \right)^{1/2} N \tag{3.3.5}$$

where $G_3(\cdot) = \int F^2(\cdot, x)dx$ as specified in Assumption 4.1, W is the Brownian motion on $[0,1]$ and N is a standard normal random variable which is independent of W, L_W is the local time of W.

Remark 3.3.3. It follows from (3.3.4) that

$$\begin{aligned}\frac{\sqrt{T_n}}{nv(T_n)^2} aX'Xa' =& \frac{\sqrt{T_n}}{nv(T_n)^2}(\bar{\mathbf{F}}' - \bar{\delta}' - \bar{\gamma}')(\bar{\mathbf{F}} - \bar{\delta} - \bar{\gamma}) \\ =& \frac{\sqrt{T_n}}{nv(T_n)^2}(\bar{\mathbf{F}}'\bar{\mathbf{F}} + \bar{\delta}'\bar{\delta} + \bar{\gamma}'\bar{\gamma} - 2\bar{\mathbf{F}}'\bar{\delta} - 2\bar{\mathbf{F}}'\bar{\gamma} + 2\bar{\delta}'\bar{\gamma}).\end{aligned}$$

However, in view of (3.3.8b) in the proof of Theorem 3.3.1, we have

$$\frac{\sqrt{T_n}}{nv(T_n)^2}\bar{\delta}'\bar{\delta} = \frac{\sqrt{T_n}}{nv(T_n)^2}\sum_{s=1}^n \bar{\delta}_s^2 \to_P 0, \quad \text{and} \quad \frac{\sqrt{T_n}}{nv(T_n)^2}\bar{\gamma}'\bar{\gamma} = \frac{\sqrt{T_n}}{nv(T_n)^2}\sum_{s=1}^n \bar{\gamma}_s^2 \to_P 0,$$

and consequently, Cauchy-Schwarz inequality gives that $\frac{\sqrt{T_n}}{nv(T_n)^2}\bar{\delta}'\bar{\gamma} \to_P 0$. Now, using Assumption Bc.7 and Theorem 1.3.1 we have

$$\frac{\sqrt{T_n}}{nv(T_n)^2}\bar{\mathbf{F}}'\bar{\mathbf{F}} = \frac{\sqrt{T_n}}{nv(T_n)^2}\sum_{s=1}^n \bar{F}^2\left(\frac{s}{n}T_n, \sqrt{T_n}x_{s,n}\right) \to_D \int_0^1 G_3(t)dL_W(t,0).$$

We therefore obtain that $\frac{\sqrt{T_n}}{nv(T_n)^2} aX'Xa'$ converges with $n \to \infty$ in distribution to a random variable, implying that $aX'Xa' = O\left(\frac{nv(T_n)^2}{\sqrt{T_n}}\right)$. But according to the result of Theorem 3.3.1, $aX'X(\widehat{\beta} - \beta) = O\left(\frac{\sqrt{n}v(T_n)}{\sqrt[4]{T_n}}\right)$. These two magnitudes imply the effect of supersedence of a by $\widehat{\beta} - \beta$ is about $\frac{\sqrt[4]{T_n}}{\sqrt{n}v(T_n)}$. The effect can be $n^{-1/2}$ as that of stationary process when $v(T_n) = \sqrt[4]{T_n}$.

Proof. Notice that

$$\begin{aligned}\frac{\sqrt[4]{T_n}}{\sqrt{n}v(T_n)} aX'X[\widehat{\beta} - \beta] =& \frac{\sqrt[4]{T_n}}{\sqrt{n}v(T_n)} aX'(\delta + \gamma + \varepsilon) \\ =& \frac{\sqrt[4]{T_n}}{\sqrt{n}v(T_n)} (\mathbf{F} - \bar{\delta} - \bar{\gamma})'(\delta + \gamma + \varepsilon),\end{aligned} \tag{3.3.6}$$

where vector $\mathbf{F} = (\bar{F}(t_{1,n}, X_{1,n}), \cdots, \bar{F}(t_{n,n}, X_{n,n}))'$.

Moreover, in view of Theorem 1.3.1 with $c_n = \sqrt{T_n}$, we have

$$\begin{aligned}\frac{\sqrt[4]{T_n}}{\sqrt{n}v(T_n)} \mathbf{F}'\varepsilon =& \frac{\sqrt[4]{T_n}}{\sqrt{n}v(T_n)} \sum_{s=1}^{n} \bar{F}(t_{s,n}, X_{s,n}) e_s \\ =& \frac{\sqrt[4]{T_n}}{\sqrt{n}v(T_n)} \sum_{s=1}^{n} \bar{F}(T_n \frac{s}{n}, \sqrt{T_n} x_{s,n}) e_s \\ \to_D & \left(\int_0^1 G_3(u) dL_W(u, 0) \right)^{1/2} N,\end{aligned} \tag{3.3.7}$$

where $G_3(\cdot) = \int F^2(\cdot, x)dx$ as specified in the Assumption C, W, N and L_W remain the previous meanings.

To facilitate the following proof, we invoke the embedding schedule again. Remember we now can use the almost surely convergence of (W_n, U_n) but to get a weak convergence for the theorem.

Next, we are about to prove that all the rest terms are convergent in probability to zero. Nonetheless, by Cauchy-Schwarz inequality, to this purpose, it is sufficient to show that

$$\|\delta\|^2 \to_P 0, \qquad \|\gamma\|^2 \to_P 0, \tag{3.3.8a}$$

$$\frac{\sqrt{T_n}}{nv(T_n)^2} \|\bar{\delta}\|^2 \to_P 0, \qquad \frac{\sqrt{T_n}}{nv(T_n)^2} \|\bar{\gamma}\|^2 \to_P 0, \tag{3.3.8b}$$

$$\frac{\sqrt[4]{T_n}}{\sqrt{n}v(T_n)} |\bar{\delta}'\varepsilon| \to_P 0, \qquad \frac{\sqrt[4]{T_n}}{\sqrt{n}v(T_n)} |\bar{\gamma}'\varepsilon| \to_P 0, \tag{3.3.8c}$$

because once again Theorem 1.3.1 with $c_n = \sqrt{T_n}$ and Assumption Bc.7 imply that

$$\frac{\sqrt{T_n}}{nv(T_n)^2}\|\bar{\mathbf{F}}\|^2 \to_P \int_0^1 G_3(r)dL_W(r,0).$$

In fact, using the result in Theorem 2.3.2 yields

$$\begin{aligned}
E\sum_{s=1}^n \delta_s^2 =& \sum_{s=1}^n E\left[\sum_{i=0}^k \sum_{j=p_i+1}^\infty b_{ij}\varphi_{jT_n}(sT_n/n)h_i(sT_n/n, X_{sT_n/n})\right]^2 \\
=& \sum_{s=1}^n \sum_{i=0}^k \left(\sum_{j=p_i+1}^\infty b_{ij}\varphi_{jT_n}(sT_n/n)\right)^2 \\
\leq& CT_n^2 \sum_{s=1}^n \sum_{i=0}^k \frac{(|b_i'(0)| + |b_i'(T_n)|)^2}{p_i^2}(1+o(1)) \\
\leq& CT_n^2 M^2 \frac{nk}{p_{\min}^2}(1+o(1)) = CM^2 n^{1+2\kappa_3+\kappa_1-2\kappa_2}(1+o(1)) \to 0
\end{aligned}$$

as $n \to \infty$ using Assumption B.4, which in turn implies $\sum_{s=1}^n \delta_s^2 \to_P 0$. Meanwhile, invoking (2.2.13) with $r=3$, $b_i(sT_n/n) = \frac{\sqrt{sT_n/n}^3}{\sqrt{i(i-1)(i-2)}} b_{i-3}(m_x^{(3)})$ and therefore

$$\begin{aligned}
E\sum_{s=1}^n \gamma_s^2 =& \sum_{s=1}^n E\left[\sum_{i=k+1}^\infty \sum_{j=0}^\infty b_{ij}\varphi_{jT_n}(sT_n/n)h_i(sT_n/n, X_{sT_n/n})\right]^2 \\
=& \sum_{s=1}^n \sum_{i=k+1}^\infty b_i(sT_n/n)^2 = \sum_{s=1}^n \sum_{i=k+1}^\infty \frac{(sT_n/n)^3}{i(i-1)(i-2)} b_{i-3}^2(m_x^{(3)}) \\
=& \sum_{i=k+1}^\infty \frac{1}{i(i-1)(i-2)} \sum_{s=1}^n (sT_n/n)^3 b_{i-3}^2(m_x^{(3)}) \\
\leq& An \sum_{i=k+1}^\infty \frac{1}{i(i-1)(i-2)} \leq A(1+o(1))\frac{n}{k^2} \\
=& A(1+o(1))n^{1-2\kappa_1} \to 0,
\end{aligned}$$

by Assumption B.3 and B.4, where A is the uniform bound of $t^3 b_{i-3}^2(t, m_x^{(3)}(t,x))$.

Regarding (3.3.8b),

$$\frac{\sqrt{T_n}}{nv(T_n)^2}\sum_{s=1}^n E\bar{\delta}_s^2 = \frac{\sqrt{T_n}}{nv(T_n)^2}\sum_{s=1}^n E\left[\sum_{i=0}^k \sum_{j=p_i+1}^\infty a_{ij}\varphi_{jT_n}(sT_n/n)h_i(sT_n/n, X_{sT_n/n})\right]^2$$

$$
\begin{aligned}
&=\frac{\sqrt{T_n}}{nv(T_n)^2}\sum_{s=1}^{n}\sum_{i=0}^{k}\left(\sum_{j=p_i+1}^{\infty}a_{ij}\varphi_{jT_n}(sT_n/n)\right)^2\\
&\le\frac{2}{n\sqrt{T_n}v(T_n)^2}\sum_{s=1}^{n}\sum_{i=0}^{k}\left(\sum_{j=p_i+1}^{\infty}|a_{ij}|\right)^2\\
&\le\frac{2}{n\sqrt{T_n}v(T_n)^2}\sum_{s=1}^{n}\sum_{i=0}^{k}\frac{1}{p_i^2}\left(\sum_{j=p_i+1}^{\infty}j|a_{ij}|\right)^2\\
&\le\frac{2ko(1)}{\sqrt{T_n}p_{\min}^2v(T_n)^2}=\frac{o(1)}{v(T_n)^2}n^{\kappa_1-2\kappa_2-\kappa_3/2}\to 0
\end{aligned}
$$

as $n\to\infty$ by Assumption B.4 and Bc.4. Moreover,

$$
\begin{aligned}
\frac{\sqrt{T_n}}{nv(T_n)^2}\sum_{s=1}^{n}E\bar{\gamma}_s^2 &=\frac{\sqrt{T_n}}{nv(T_n)^2}\sum_{s=1}^{n}E\left[\sum_{i=k+1}^{\infty}\sum_{j=0}^{\infty}a_{ij}\varphi_{jT_n}(sT_n/n)h_i(sT_n/n,X_{sT_n/n})\right]^2\\
&=\frac{\sqrt{T_n}}{nv(T_n)^2}\sum_{s=1}^{n}\sum_{i=k+1}^{\infty}\left[\sum_{j=0}^{\infty}a_{ij}\varphi_{jT_n}(sT_n/n)\right]^2\\
&\le\frac{1}{nv(T_n)^2\sqrt{T_n}}\sum_{s=1}^{n}\sum_{i=k+1}^{\infty}\left[\sum_{j=0}^{\infty}|a_{ij}|\right]^2\\
&=\frac{1}{v(T_n)^2\sqrt{T_n}}\sum_{i=k+1}^{\infty}\left[\sum_{j=0}^{\infty}|a_{ij}|\right]^2\to 0,
\end{aligned}
$$

because $\frac{1}{\sqrt{T_n}}\to 0$ and $\sum_{i=k+1}^{\infty}\left[\sum_{j=0}^{\infty}|a_{ij}|\right]^2<\sum_{i=k+1}^{\infty}\sum_{j=0}^{\infty}|a_{ij}|\to 0$ by Assumption Bc.4.

The last step is to show the convergence of (3.3.8c). These hold because using the property of martingale difference of e_s and adaptivity of $x_{s,n}$ we can deduce

$$
\begin{aligned}
E\left(\frac{\sqrt[4]{T_n}}{\sqrt{n}v(T_n)}\bar{\delta}'\varepsilon\right)^2 &=E\left(\frac{\sqrt[4]{T_n}}{\sqrt{n}v(T_n)}\sum_{s=1}^{n}\bar{\delta}_se_s\right)^2=\frac{\sqrt{T_n}\sigma_e^2}{nv(T_n)^2}\sum_{s=1}^{n}E\bar{\delta}_s^2\to 0,\\
E\left(\frac{\sqrt[4]{T_n}}{\sqrt{n}v(T_n)}\bar{\gamma}'\varepsilon\right)^2 &=E\left(\frac{\sqrt[4]{T_n}}{\sqrt{n}v(T_n)}\sum_{s=1}^{n}\bar{\gamma}_se_s\right)^2=\frac{\sqrt{T_n}\sigma_e^2}{nv(T_n)^2}\sum_{s=1}^{n}E\bar{\gamma}_s^2\to 0,
\end{aligned}
$$

which finishes the proof. □

3.3.2 Asymptotics of the estimated unknown functional

Having obtained $\widehat{\beta}$, we may be able to have $\widehat{m}(\tau,x)$, estimation of $m(\tau,x)$ for fixed $\tau>0$ and fixed $x\in\mathbb{R}$ on the path of $B(\tau)$. Thus, one desired result is the asymptotic distribution of $\widehat{m}(\tau,x)-m(\tau,x)$.

Given that $m(\cdot,\cdot)$ satisfies Assumption B.3, using orthogonal system $\{\varphi_{jT_n}(\tau)h_i(\tau,x)\}$, $m(\tau,x)$ is expanded as

$$m(\tau,x)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}b_{ij}\varphi_{jT_n}(\tau)h_i(\tau,x):=A'(\tau,x)\beta+\delta(\tau,x)+\gamma(\tau,x), \tag{3.3.9}$$

here

$$\begin{aligned}A'(\tau,x) &= (\varphi_{0T_n}(\tau)h_0(\tau,x),\cdots\varphi_{p_0T_n}(\tau)h_0(\tau,x),\cdots\varphi_{0T_n}(\tau)h_k(\tau,x),\cdots\\ &\qquad \varphi_{p_kT_n}(\tau)h_k(\tau,x)),\\ \delta(\tau,x) &= \sum_{i=0}^{k}\sum_{j=p_i+1}^{\infty}b_{ij}\varphi_{jT_n}(\tau)h_i(\tau,x),\\ \gamma(\tau,x) &= \sum_{i=k+1}^{\infty}\sum_{j=0}^{\infty}b_{ij}\varphi_{jT_n}(\tau)h_i(\tau,x).\end{aligned}$$

Whence, after substituting $\widehat{\beta}$ in lieu of β and getting rid off all residues, we have

$$\widehat{m}(\tau,x)=A'(\tau,x)\widehat{\beta}. \tag{3.3.10}$$

We shall investigate the limit of

$$\begin{aligned}\widehat{m}(\tau,x)-m(\tau,x)&=A'(\tau,x)(\widehat{\beta}-\beta)-\delta(\tau,x)-\gamma(\tau,x)\\ &=A'(\tau,x)(X'X)^{-1}X'(\delta+\gamma+\varepsilon)-\delta(\tau,x)-\gamma(\tau,x).\end{aligned} \tag{3.3.11}$$

Similarly, put

$$A_{p\times p}=\frac{A(\tau,x)A'(\tau,x)}{\|A(\tau,x)\|^2}\quad\text{and}\quad B_{p\times p}=(X'X)A(X'X)^{-1}. \tag{3.3.12}$$

According to Lemma 3.1.2, B has eigenvalues $\lambda_1=1$, $\lambda_2=\cdots=\lambda_p=0$. Let normalised α be the unit left eigenvector of B pertaining to λ_1. Hence, we have $\alpha'B=\alpha'$ and $\|\alpha\|=1$. Denote $\alpha'=(\alpha_{00},\cdots,\alpha_{0p_0},\cdots,\alpha_{k0}\cdots,\alpha_{kp_k})$ in concert with $A(\tau,x)$.

Let us apply the reshuffle procedure for the set $\mathcal{S}$ from Assumption Bm.1 by α. Denote by $\widetilde{\mathcal{S}}$ the resulting set:

1) $\widetilde{\mathcal{S}} = \{\widetilde{a}_0, \cdots, \widetilde{a}_i, \cdots\}$

2) $\widetilde{a}_i = \{\widetilde{a}_{ij}\}$ where $\widetilde{a}_{ij} = \sqrt{\frac{T_n}{p_{\max}}}\alpha_{ij}$ for $0 \le i \le k$ and $0 \le j \le p_i$ where $p_{\max} = \max\{p_0, \cdots, p_k\}$; otherwise, $\widetilde{a}_{ij} = a_{ij}$.

Obviously, there exists a function, denoted by $\widetilde{F}(t, x)$, such that

$$\widetilde{F}(t, B(t)) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \widetilde{a}_{ij}\varphi_{jT_n}(t)h_i(t, B(t)), \tag{3.3.13}$$

for any $t \in [0, T_n]$. Therefore, by virtue of (3.3.13), we have

$$\sqrt{\frac{T_n}{p_{\max}}}\alpha' X' = \widetilde{\mathbf{F}}' - \widetilde{\delta}' - \widetilde{\gamma}', \tag{3.3.14}$$

where $\widetilde{\mathbf{F}}' = (\widetilde{F}(t_{1,n}, X_{1,n}), \cdots, \widetilde{F}(t_{n,n}, X_{n,n}))$; $\widetilde{\delta}' = (\widetilde{\delta}_1, \cdots, \widetilde{\delta}_n)$, $\widetilde{\delta}_s = \sum_{i=0}^{k}\sum_{j=p_i+1}^{\infty} a_{ij}$ $\varphi_{jT_n}(t_{s,n})h_i(t_{s,n}, X_{s,n})$; $\widetilde{\gamma}' = (\widetilde{\gamma}_1, \cdots, \widetilde{\gamma}_n)$, $\widetilde{\gamma}_s = \sum_{i=k+1}^{\infty}\sum_{j=0}^{\infty} a_{ij}\varphi_{jT_n}(t_{s,n})h_i(t_{s,n}, X_{s,n})$.

Also the above reshuffle procedure can be applied with $\frac{1}{\|A(\tau,x)\|}A(\tau, x)$. Let us denote the resulting set by $\bar{\mathcal{S}}$. Accordingly, $\bar{\mathcal{S}}$ amounts to a set of sequences $\{\bar{a}_0, \bar{a}_1, \bar{a}_2, \cdots\}$ where $\bar{a}_i = \{\bar{a}_{ij}\}$ and $\bar{a}_{ij} = \frac{1}{\|A(\tau,x)\|}\sqrt{\frac{T_n}{p_{\max}}}\varphi_{jT_n}(\tau)h_i(\tau, x)$ if $i = 0, \cdots, k$ and $j = 0, \cdots, p_i$; otherwise, $\bar{a}_{ij} = a_{ij}$.

Similarly, there exists a function, denoted by $\widetilde{G}(t, x)$, such that

$$\widetilde{G}(t, B(t)) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \bar{a}_{ij}\varphi_{jT_n}(t)h_i(t, B(t)), \tag{3.3.15}$$

for any $t \in [0, T_n]$. Consequently, by (3.3.15),

$$\frac{1}{\|A(\tau, x)\|}\sqrt{\frac{T_n}{p_{\max}}}XA(\tau, x) = \widetilde{\mathbf{G}} - \bar{\delta} - \bar{\gamma} \tag{3.3.16}$$

where $\widetilde{\mathbf{G}}' = (\widetilde{G}(t_{1,n}, X_{1,n}), \cdots, \widetilde{G}(t_{n,n}, X_{n,n}))$; $\bar{\delta}' = (\bar{\delta}_1, \cdots, \bar{\delta}_n)$, $\bar{\delta}_s = \sum_{i=0}^{k}\sum_{j=p_i+1}^{\infty} a_{ij}$ $\varphi_{jT_n}(t_{s,n})h_i(t_{s,n}, X_{s,n})$; $\bar{\gamma}' = (\bar{\gamma}_1, \cdots, \bar{\gamma}_n)$, $\bar{\gamma}_s = \sum_{i=k+1}^{\infty}\sum_{j=0}^{\infty} a_{ij}\varphi_{jT_n}(t_{s,n})h_i(t_{s,n}, X_{s,n})$.

Notice that $\widetilde{\delta} = \bar{\delta}$ and $\widetilde{\gamma} = \bar{\gamma}$ since $\widetilde{\mathcal{S}}$ and $\bar{\mathcal{S}}$ have the same tails. The following proposition shows the finiteness of the second moment of $\widetilde{G}(t, B(t))$ and $\widetilde{F}(t, B(t))$.

Lemma 3.3.2. *For $t \in [0, T_n]$, (a) $E[\widetilde{G}(t, B_t)]^2 < \infty$, and (b) $E[\widetilde{F}(t, B_t)]^2 < \infty$.*

Proof. (a) From the orthogonality of $h_i(t, B(t))$,

$$E[\widetilde{G}(t, B(t))]^2 = E\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\bar{a}_{ij}\varphi_{jT_n}(t)h_i(t, B(t))\right)^2 = \sum_{i=0}^{\infty}\left(\sum_{j=0}^{\infty}\bar{a}_{ij}\varphi_{jT_n}(t)\right)^2.$$

We mainly focus on the partial sum $\sum_{i=0}^{k}\left(\sum_{j=0}^{p_i}\bar{a}_{ij}\varphi_{jT_n}(t)\right)^2$ in view of the proof of Lemma 3.1.3. Notice that if $t = \tau$,

$$\sum_{i=0}^{k}\left(\sum_{j=0}^{p_i}\bar{a}_{ij}\varphi_{jT_n}(\tau)\right)^2 = \frac{T_n}{\|A(\tau, x)\|^2 p_{\max}}\sum_{i=0}^{k}h_i^2(\tau, x)\left(\sum_{j=0}^{p_i}\varphi_{jT_n}^2(\tau)\right)^2.$$

Observe from the definition of $\varphi_{jT_n}(\cdot)$ that

$$\begin{aligned}\sum_{j=0}^{p_i}\varphi_{jT_n}^2(\tau) =& \frac{1}{T_n} + \frac{2}{T_n}\sum_{j=1}^{p_i}\cos^2\frac{j\pi\tau}{T_n} = \frac{1}{T_n}\left(1 + 2\sum_{j=0}^{p_i}\cos^2\frac{j\pi\tau}{T_n}\right)\\ =& \frac{1}{T_n}\left[1 + \sum_{j=1}^{p_i}\left(1 + \cos\frac{2j\pi\tau}{T_n}\right)\right] = \frac{1}{T_n}\left(1 + p_i + \sum_{j=1}^{p_i}\cos\frac{2j\pi\tau}{T_n}\right)\\ =& \frac{1}{T_n}\left(\frac{1}{2} + p_i\right) + \frac{\sin(p_i + \frac{1}{2})\frac{2\pi\tau}{T_n}}{2T_n\sin\frac{\pi\tau}{T_n}},\end{aligned}$$

where we have employed a trigonometric formula $\frac{1}{2} + \cos u + \cdots + \cos mu = \frac{\sin(m+\frac{1}{2})u}{2\sin\frac{u}{2}}$. Notice that by Assumption B.4 $p_{\min}/T_n \to \infty$, hence $\sin(p_i + \frac{1}{2})\frac{2\pi\tau}{T_n}$ fluctuates between -1 and 1, while the denominator $2T_n\sin\frac{\pi\tau}{T_n} \to 2\pi\tau$ as sample size increases. Thus, we can assert that $\sum_{j=0}^{p_i}\varphi_{jT_n}^2(\tau) = \frac{1}{T_n}p_i(1 + o(1))$. Consequently, $\sum_{i=0}^{k}\left(\sum_{j=0}^{p_i}\bar{a}_{ij}\varphi_{jT_n}(\tau)\right)^2 \leq \frac{1+o(1)}{\|A(\tau,x)\|^2}\sum_{i=0}^{k}h_i^2(\tau, x)\sum_{j=0}^{p_i}\varphi_{jT_n}^2(\tau) = 1 + o(1)$.

When $t \neq \tau$, from Cauchy-Schwarz inequality and the above derivation it follows that

$$\begin{aligned}\sum_{i=0}^{k}\left(\sum_{j=0}^{p_i}\bar{a}_{ij}\varphi_{jT_n}(t)\right)^2 =& \frac{1}{\|A(\tau, x)\|^2}\sum_{i=0}^{k}\frac{T_n}{p_{\max}}h_i^2(\tau, x)\left(\sum_{j=0}^{p_i}\varphi_{jT_n}(\tau)\varphi_{jT_n}(t)\right)^2\\ \leq& \frac{1}{\|A(\tau, x)\|^2}\sum_{i=0}^{k}\frac{T_n}{p_{\max}}h_i^2(\tau, x)\sum_{j=0}^{p_i}\varphi_{jT_n}^2(\tau)\sum_{j=0}^{p_i}\varphi_{jT_n}^2(t)\\ =& \frac{1}{\|A(\tau, x)\|^2}\sum_{i=0}^{k}\frac{T_n}{p_{\max}}\frac{p_i}{T_n}(1 + o(1))h_i^2(\tau, x)\sum_{j=0}^{p_i}\varphi_{jT_n}^2(\tau)\end{aligned}$$

$$\leq \frac{1+o(1)}{\|A(\tau,x)\|^2} \sum_{i=0}^{k} h_i^2(\tau,x) \sum_{j=0}^{p_i} \varphi_{jT_n}^2(\tau) = 1 + o(1).$$

In conclusion, for $t > 0$, $E[\widetilde{G}(t, B_t)]^2 < \infty$.

(b) Since $E[\widetilde{F}(t, B(t))]^2$ has the similar expression as $E[\widetilde{G}(t, B(t))]^2$, we focus on the estimation of the partial sum. In effect,

$$\sum_{i=0}^{k} \left(\sum_{j=0}^{p_i} \widetilde{a}_{ij} \varphi_{jT_n}(t) \right)^2 = \sum_{i=0}^{k} \frac{T_n}{p_{\max}} \left(\sum_{j=0}^{p_i} \alpha_{ij} \varphi_{jT_n}(t) \right)^2$$
$$\leq \sum_{i=0}^{k} \frac{T_n}{p_{\max}} \sum_{j=0}^{p_i} \alpha_{ij}^2 \sum_{j=0}^{p_i} \varphi_{jT_n}^2(t) = \sum_{i=0}^{k} \frac{T_n}{p_{\max}} \frac{p_i}{T_n} (1 + o(1)) \sum_{j=0}^{p_i} \alpha_{ij}^2 \leq 1 + o(1),$$

because α is an unit vector. The proof is finished. □

We propose the following assumptions in order to obtain the asymptotic behaviour of $\widehat{m}(\tau, x)$.

Assumption Bm.6

(a) Both $\widetilde{F}(t, x)$ and $\widetilde{G}(t, x)$ are in Class $\mathcal{T}(HI)$ with normal functions $F(t, x)$, $G(t, x)$, and homogeneity powers $v(\cdot)$ and $g(\cdot)$ respectively.

(b) Let $p_{\max} = [n^{\bar{\kappa}_2}]$ and $g(n) = n^{\rho}$. Suppose that $\frac{1}{2} + \frac{1}{2}\kappa_1 + (\rho + \frac{3}{4})\kappa_3 < \kappa_2$.

(c) Suppose further that $\widetilde{F}^2(t, x)$ and $\widetilde{G}^2(t, x)$ and $\widetilde{F}(t, x)\widetilde{G}(t, x)$ are all in Class $\mathcal{T}(HI)$ with normal functions $F^2(t, x)$, $G^2(t, x)$ and $F(t, x)G(t, x)$ and homogeneity powers $v^2(\cdot)$ and $g^2(\cdot)$, $v(\cdot)g(\cdot)$ respectively.

Remark 3.3.4. Since by Assumption B.4(b) $\frac{1}{2} + \frac{1}{2}\kappa_1 + \kappa_3 < \kappa_2$, it is easy to see that if $0 < \rho \leq \frac{1}{4}$ then Condition (b) of Assumption Bm.6 is guaranteed by B.4 (b); if however, $\rho > \frac{1}{4}$, Assumption Bm.6 implies that B.4 (b) holds automatically. Meanwhile, as we always can control the difference $\bar{\kappa}_2 - \kappa_2$ as small as we wish, the requirement for $\bar{\kappa}_2 - \kappa_2$ is ignored.

Note also that the ambit of ρ is able to be extended if we impose a more rigorous condition on Assumption B.3(d), viz., assuming that the derivative of $b_i(t, m)$ is dominated by $t^{-\eta}$ with some $\eta > 0$. Thus, in the proof of the following theorem $a(T_n) < T_n^{-\eta} = n^{-\eta\kappa_3}$, hence $\rho - \eta$ would be in lieu of ρ, namely $0 < \rho < \eta + \frac{1}{4}$.

Theorem 3.3.2. *Suppose that $\{x_{s,n}\}_{s=1}^n$ and $\{e_s\}_{s=1}^n$ satisfy Assumptions B and A (c) in Chapter 1. Under Assumptions B.3, B.4 and Bm.6 we have*

$$\begin{aligned}&\frac{\sqrt[4]{T_n}^3}{\sqrt{n}v(T_n)}\frac{\alpha' X' X A(\tau,x)}{\sqrt{p_{\max}}\|A(\tau,x)\|^2}(\widehat{m}(\tau,x)-m(\tau,x))\\&\to_D \left(\int_0^1 G_3(u)dL_W(u,0)\right)^{\frac{1}{2}} N,\end{aligned} \tag{3.3.17}$$

where $G_3(\cdot)=\int F^2(\cdot,x)dx$, W is the standard Brownian motion on $[0,1]$ and N is a standard normal random variable independent of W, and L_W is the local–time process of W.

Remark 3.3.5. As can be seen from the proof of the theorem, since the Δ converges to a random variable in distribution, $\frac{\sqrt[4]{T_n}^3}{\sqrt{n}v(T_n)}\frac{\alpha' X' X A(\tau,x)}{\sqrt{p_{\max}}\|A(\tau,x)\|^2}$ is equivalent to $\frac{\sqrt{np_{\max}}g(T_n)}{\sqrt[4]{T_n}^3\|A(\tau,x)\|}$. In view of the fact that $O(1)\frac{1}{T_n}kp_{\min}\le\|A(\tau,x)\|^2\le O(1)\frac{1}{T_n}kp_{\max}$, we have $O(1)\sqrt{\frac{n}{k}}T_n^{\rho-\frac{1}{4}}\le\frac{\sqrt{np_{\max}}g(T_n)}{\sqrt[4]{T_n}^3\|A(\tau,x)\|}\le O(1)\sqrt{\frac{n}{k}}\sqrt{\frac{p_{\max}}{p_{\min}}}T_n^{\rho-\frac{1}{4}}$. Moreover, taking account of Assumption Bm.6 (b), the left hand side is $n^{\frac{1}{2}(1-\kappa_1)+(\rho-\frac{1}{4})\kappa_3}$ of which the exponent is greater than zero but lower than $\frac{1}{4}$, while the right hand side is $n^{\frac{1}{2}(1-\kappa_1)+\frac{1}{2}(\bar{\kappa}_2-\kappa_2)+(\rho-\frac{1}{4})\kappa_3}$ of which the order is less than $\frac{1}{2}$.

Proof. It follows from (3.3.11) that

$$\begin{aligned}&\frac{\sqrt[4]{T_n}^3}{\sqrt{n}v(T_n)}\frac{\alpha' X' X A(\tau,x)}{\sqrt{p_{\max}}\|A(\tau,x)\|^2}(\widehat{m}(\tau,x)-m(\tau,x))\\
=&\frac{\sqrt[4]{T_n}^3}{\sqrt{n}v(T_n)}\frac{\alpha' X' X A(\tau,x)}{\sqrt{p_{\max}}\|A(\tau,x)\|^2}[A'(\tau,x)(X'X)^{-1}X'(\delta+\gamma+\varepsilon)-\delta(\tau,x)-\gamma(\tau,x)]\\
=&\frac{\sqrt[4]{T_n}^3}{\sqrt{n}v(T_n)\sqrt{p_{\max}}}\alpha' B X'(\delta+\gamma+\varepsilon)-\frac{\sqrt[4]{T_n}^3}{\sqrt{n}v(T_n)}\frac{\alpha' X' X A(\tau,x)}{\sqrt{p_{\max}}\|A(\tau,x)\|^2}[\delta(\tau,x)+\gamma(\tau,x)]\\
=&\frac{\sqrt[4]{T_n}}{\sqrt{n}v(T_n)}\sqrt{\frac{T_n}{p_{\max}}}\alpha' X'(\delta+\gamma+\varepsilon)-\frac{\sqrt[4]{T_n}^3}{\sqrt{n}v(T_n)}\frac{\alpha' X' X A(\tau,x)}{\sqrt{p_{\max}}\|A(\tau,x)\|^2}[\delta(\tau,x)+\gamma(\tau,x)]\\
:=&\Pi_1-\Pi_2.\end{aligned} \tag{3.3.18}$$

We are about to show that Π_1 converges to the desired variable in distribution and Π_2 converges to zero in probability. To make life easier, we shall invoke the embedding schedule. Bearing in mind, albeit using almost surely convergence of (W_n, U_n), the assertion is still a weak convergence.

First and foremost, let us prove the convergence of Π_1. It follows from (3.3.14) that

$$\Pi_1 = \frac{\sqrt[4]{T_n}}{\sqrt{n}v(T_n)}(\widetilde{\mathbf{F}} - \bar{\delta} - \bar{\gamma})'(\delta + \gamma + \varepsilon), \tag{3.3.19}$$

where vector $\widetilde{\mathbf{F}}' = (\widetilde{F}(t_{1,n}, X_{1,n}), \cdots, \widetilde{F}(t_{n,n}, X_{n,n}))$.

From Assumption Bm.6 (a) and (c) and Theorem 1.3.1 it follows that

$$\begin{aligned}
\frac{\sqrt[4]{T_n}}{\sqrt{n}v(T_n)}\widetilde{\mathbf{F}}'\varepsilon =& \frac{\sqrt[4]{T_n}}{\sqrt{n}v(T_n)}\sum_{s=1}^{n}\widetilde{F}(t_{s,n}, X_{s,n})e_s \\
=& \frac{\sqrt[4]{T_n}}{\sqrt{n}v(T_n)}\sum_{s=1}^{n}\widetilde{F}\left(T_n\frac{s}{n}, \sqrt{T_n}x_{s,n}\right)\ e_s \\
\to_D& \left(\int_0^1 G_3(u)dL_W(u,0)\right)^{\frac{1}{2}}\ N, && (3.3.20a) \\
\frac{\sqrt{T_n}}{nv(T_n)^2}\|\widetilde{\mathbf{F}}\|^2 =& \frac{\sqrt{T_n}}{nv(T_n)^2}\sum_{s=1}^{n}\widetilde{F}^2(t_{s,n}, X_{s,n}) \\
=& \frac{\sqrt{T_n}}{nv(T_n)^2}\sum_{s=1}^{n}\widetilde{F}^2\left(T_n\frac{s}{n}, \sqrt{T_n}x_{s,n}\right) \\
\to_P& \int_0^1 G_3(u)dL_W(u,0), && (3.3.20b)
\end{aligned}$$

due to asymptotic homogeneity, where $G_3(\cdot) = \int F^2(\cdot, x)dx$ is as specified in Assumption C, W, N and L_W are the same as defined before. At the meantime, Cauchy-Schwarz inequality gives

$$\begin{aligned}
\frac{\sqrt{T_n}}{nv(T_n)^2}|\widetilde{\mathbf{F}}'\delta|^2 \le& \frac{\sqrt{T_n}}{nv(T_n)^2}\|\widetilde{\mathbf{F}}\|^2\|\delta\|^2, & \frac{\sqrt{T_n}}{nv(T_n)^2}|\widetilde{\mathbf{F}}'\gamma|^2 \le& \frac{\sqrt{T_n}}{nv(T_n)^2}\|\widetilde{\mathbf{F}}\|^2\|\gamma\|^2, \\
\frac{\sqrt{T_n}}{nv(T_n)^2}|\bar{\delta}'\delta|^2 \le& \frac{\sqrt{T_n}}{nv(T_n)^2}\|\bar{\delta}\|^2\|\delta\|^2, & \frac{\sqrt{T_n}}{nv(T_n)^2}|\bar{\delta}'\gamma|^2 \le& \frac{\sqrt{T_n}}{nv(T_n)^2}\|\bar{\delta}\|^2\|\gamma\|^2, \\
\frac{\sqrt{T_n}}{nv(T_n)^2}|\bar{\gamma}'\delta|^2 \le& \frac{\sqrt{T_n}}{nv(T_n)^2}\|\bar{\gamma}\|^2\|\delta\|^2, & \frac{\sqrt{T_n}}{nv(T_n)^2}|\bar{\gamma}'\gamma|^2 \le& \frac{\sqrt{T_n}}{nv(T_n)^2}\|\bar{\gamma}\|^2\|\gamma\|^2.
\end{aligned}$$

Moreover, invoking Assumption B that $x_{s,n}$ is adapted to $\mathcal{F}_{n,s-1}$ and $(e_s, \mathcal{F}_{n,s})$ is a martingale difference sequence, as well as $E(e_s^2|\mathcal{F}_{n,s-1}) = \sigma^2$ a.s., noting the expressions of $\bar{\delta}_s$, we have

$$E\left(\frac{\sqrt[4]{T_n}}{\sqrt{n}v(T_n)}\bar{\delta}\varepsilon\right)^2 = \frac{\sqrt{T_n}}{nv(T_n)^2}E\left(\sum_{s=1}^{n}\bar{\delta}_s e_s\right)^2 = \frac{\sqrt{T_n}}{nv(T_n)^2}\sum_{s_1=1}^{n}\sum_{s_2=1}^{n}E[\bar{\delta}_{s_1}e_{s_1}\bar{\delta}_{s_2}e_{s_2}]$$

$$
\begin{aligned}
=&\frac{\sqrt{T_n}}{nv(T_n)^2}\sum_{s=1}^{n}E[\bar{\delta}_s^2E(e_s^2|\mathcal{F}_{n,s-1})]\\
&+2\frac{\sqrt{T_n}}{nv(T_n)^2}\sum_{s_1=1}^{n-1}\sum_{s_2=s_1+1}^{n}E[\bar{\delta}_{s_1}e_{s_1}\bar{\delta}_{s_2}E(e_{s_2}|\mathcal{F}_{n,s_2-1})]\\
=&\frac{\sqrt{T_n}\sigma^2}{nv(T_n)^2}\sum_{s=1}^{n}E[\bar{\delta}_s^2]=\frac{\sqrt{T_n}\sigma^2}{nv(T_n)^2}E\|\bar{\delta}\|^2,
\end{aligned}
$$

and similarly

$$
E\left(\frac{\sqrt[4]{T_n}}{\sqrt{n}v(T_n)}\bar{\gamma}\varepsilon\right)^2=\frac{\sqrt{T_n}\sigma^2}{nv(T_n)^2}E\|\bar{\gamma}\|^2.
$$

Therefore, in order to prove that all the rest terms in Π_1 converge in probability to zero, it suffices to show

$$
\|\delta\|^2\to_P 0,\qquad\qquad \|\gamma\|^2\to_P 0, \tag{3.3.21a}
$$

$$
\frac{\sqrt{T_n}}{nv(T_n)^2}E\|\bar{\delta}\|^2\to 0,\qquad\qquad \frac{\sqrt{T_n}}{nv(T_n)^2}E\|\bar{\gamma}\|^2\to 0. \tag{3.3.21b}
$$

In fact, using the result in Theorem 2.3.2 yields that as $n\to\infty$

$$
\begin{aligned}
E\left[\sum_{s=1}^{n}\delta_s^2\right]=&\sum_{s=1}^{n}E\left[\sum_{i=0}^{k}\sum_{j=p_i+1}^{\infty}b_{ij}\varphi_{jT_n}\left(\frac{sT_n}{n}\right)h_i\left(\frac{sT_n}{n},X_{sT_n/n}\right)\right]^2\\
=&\sum_{s=1}^{n}\sum_{i=0}^{k}\left(\sum_{j=p_i+1}^{\infty}b_{ij}\varphi_{jT_n}\left(\frac{sT_n}{n}\right)\right)^2\\
\le&CT_n^2\sum_{s=1}^{n}\sum_{i=0}^{k}\frac{(|b_i'(0)|+|b_i'(T_n)|)^2}{p_i^2}(1+o(1))\\
\le&CT_n^2M^2\frac{nk}{p_{\min}^2}(1+o(1))=CM^2n^{1+2\kappa_3+\kappa_1-2\kappa_2}(1+o(1))\to 0
\end{aligned}
$$

by virtue of Assumption B.4, which in turn implies $\|\delta\|^2\to_P 0$.

Similarly, invoking (2.2.13) with $r=3$ and $b_i\left(\frac{sT_n}{n}\right)=\frac{\sqrt{sT_n/n}^3}{\sqrt{i(i-1)(i-2)}}b_{i-3}\left(\frac{sT_n}{n},m_x^{(3)}\right)$, we have as $n\to\infty$

$$
E\left[\sum_{s=1}^{n}\gamma_s^2\right]=\sum_{s=1}^{n}E\left[\sum_{i=k+1}^{\infty}\sum_{j=0}^{\infty}b_{ij}\varphi_{jT_n}\left(\frac{sT_n}{n}\right)h_i\left(\frac{sT_n}{n},X_{sT_n/n}\right)\right]^2
$$

$$
\begin{aligned}
=&\sum_{s=1}^{n}\sum_{i=k+1}^{\infty} b_i\left(\frac{sT_n}{n}\right)^2 = \sum_{s=1}^{n}\sum_{i=k+1}^{\infty}\frac{(sT_n/n)^3}{i(i-1)(i-2)}b_{i-3}^2\left(\frac{sT_n}{n},m_x^{(3)}\right)\\
=&\sum_{i=k+1}^{\infty}\frac{1}{i(i-1)(i-2)}\sum_{s=1}^{n}\left(\frac{sT_n}{n}\right)^3 b_{i-3}^2\left(\frac{sT_n}{n},m_x^{(3)}\right)\\
\leq& An\sum_{i=k+1}^{\infty}\frac{1}{i(i-1)(i-2)}\leq A(1+o(1))\frac{n}{k^2}=A(1+o(1))n^{1-2\kappa_1}\to 0
\end{aligned}
$$

by Assumption B.3, B.4, where A is the uniform bound of $t^3 b_{i-3}^2(t, m_x^{(3)}(t,x))$.

In a very similar fashion, we have

$$
\begin{aligned}
\frac{\sqrt{T_n}}{nv(T_n)^2}\sum_{s=1}^{n}E\left[\bar{\delta}_s^2\right]=&\frac{\sqrt{T_n}}{nv(T_n)^2}\sum_{s=1}^{n}E\left[\sum_{i=0}^{k}\sum_{j=p_i+1}^{\infty}a_{ij}\varphi_{jT_n}\left(\frac{sT_n}{n}\right)h_i\left(\frac{sT_n}{n},X_{sT_n/n}\right)\right]^2\\
=&\frac{\sqrt{T_n}}{nv(T_n)^2}\sum_{s=1}^{n}\sum_{i=0}^{k}\left(\sum_{j=p_i+1}^{\infty}a_{ij}\varphi_{jT_n}\left(\frac{sT_n}{n}\right)\right)^2\\
\leq&\frac{2}{n\sqrt{T_n}v(T_n)^2}\sum_{s=1}^{n}\sum_{i=0}^{k}\left(\sum_{j=p_i+1}^{\infty}|a_{ij}|\right)^2\\
\leq&\frac{2}{\sqrt{T_n}v(T_n)^2}\sum_{i=0}^{k}\frac{1}{p_i^2}\left(\sum_{j=p_i+1}^{\infty}j|a_{ij}|\right)^2\\
\leq&\frac{o(1)k}{\sqrt{T_n}p_{\min}^2 v(T_n)^2}=\frac{o(1)}{v(T_n)^2}n^{\kappa_1-2\kappa_2-\kappa_3/2}\to 0
\end{aligned}
$$

as $n\to\infty$ by Assumptions B.4 and Bm.1.

Analogously, we have as $n\to\infty$

$$
\begin{aligned}
\frac{\sqrt{T_n}}{nv(T_n)^2}\sum_{s=1}^{n}E\left[\bar{\gamma}_s^2\right]=&\frac{\sqrt{T_n}}{nv(T_n)^2}\sum_{s=1}^{n}E\left[\sum_{i=k+1}^{\infty}\sum_{j=0}^{\infty}a_{ij}\varphi_{jT_n}\left(\frac{sT_n}{n}\right)h_i\left(\frac{sT_n}{n},X_{sT_n/n}\right)\right]^2\\
=&\frac{\sqrt{T_n}}{nv(T_n)^2}\sum_{s=1}^{n}\sum_{i=k+1}^{\infty}\left[\sum_{j=0}^{\infty}a_{ij}\varphi_{jT_n}\left(\frac{sT_n}{n}\right)\right]^2\\
\leq&\frac{2\sqrt{T_n}}{nv(T_n)^2T_n}\sum_{s=1}^{n}\sum_{i=k+1}^{\infty}\left(\sum_{j=0}^{\infty}|a_{ij}|\right)^2\\
\leq&\frac{2}{\sqrt{T_n}v(T_n)^2k}\sum_{i=k+1}^{\infty}i\left(\sum_{j=0}^{\infty}|a_{ij}|\right)^2
\end{aligned}
$$

$$=\frac{o(1)}{v(T_n)^2} n^{-\kappa_1-\kappa_3/2} \to 0$$

on account of Assumption Bm.1 (b).

Therefore, we assert that Π_1 converges in distribution to the limit of (3.3.20a).

We are now in a position to prove that $\Pi_2 \to_P 0$. To this end, we begin with finding the limit of

$$\Delta := \frac{\sqrt{T_n}^3}{nv(T_n)g(T_n)} \frac{\alpha' X' X A(\tau, x)}{p_{\max}\|A(\tau, x)\|}. \tag{3.3.22}$$

On account of (3.3.14) and (3.3.16), we have

$$\begin{aligned}\Delta =& \frac{\sqrt{T_n}^3}{nv(T_n)g(T_n)} \frac{\alpha' X' X A(\tau, x)}{p_{\max}\|A(\tau, x)\|} = \frac{\sqrt{T_n}}{nv(T_n)g(T_n)} (\widetilde{\mathbf{F}}' - \widetilde{\delta}' - \widetilde{\gamma}')(\widetilde{\mathbf{G}} - \bar{\delta} - \bar{\gamma}) \\ =& \frac{\sqrt{T_n}}{nv(T_n)g(T_n)} (\widetilde{\mathbf{F}}'\widetilde{\mathbf{G}} - \widetilde{\mathbf{F}}'\bar{\delta} - \widetilde{\mathbf{F}}'\bar{\gamma} - \widetilde{\delta}'\widetilde{\mathbf{G}} - \widetilde{\gamma}'\widetilde{\mathbf{G}} + \widetilde{\delta}'\bar{\delta} + \widetilde{\delta}'\bar{\gamma} + \widetilde{\gamma}'\bar{\delta} + \widetilde{\gamma}'\bar{\gamma}) \\ =& \frac{\sqrt{T_n}}{nv(T_n)g(T_n)} (\widetilde{\mathbf{F}}'\widetilde{\mathbf{G}} - \widetilde{\mathbf{F}}'\bar{\delta} - \widetilde{\mathbf{F}}'\bar{\gamma} - \bar{\delta}'\widetilde{\mathbf{G}} - \bar{\gamma}'\widetilde{\mathbf{G}} + \|\bar{\delta}\|^2 + 2\bar{\delta}'\bar{\gamma} + \|\bar{\gamma}\|^2).\end{aligned}$$

However, using Assumption Bm.6 for $\widetilde{F}(\cdot, \cdot)$ and $\widetilde{G}(\cdot, \cdot)$ and Theorem 1.3.1 gives

$$\begin{aligned}\frac{\sqrt{T_n}}{nv(T_n)g(T_n)} \widetilde{\mathbf{F}}'\widetilde{\mathbf{G}} =& \frac{\sqrt{T_n}}{nv(T_n)g(T_n)} \sum_{s=1}^{n} \widetilde{F}(t_{s,n}, X_{s,n}) \widetilde{G}(t_{s,n}, X_{s,n}) \\ =& \frac{\sqrt{T_n}}{nv(T_n)g(T_n)} \sum_{s=1}^{n} \widetilde{F}\left(T_n \frac{s}{n}, \sqrt{T_n} x_{s,n}\right) \widetilde{G}\left(T_n \frac{s}{n}, \sqrt{T_n} x_{s,n}\right) \\ \to_P& \int_0^1 J(u) dL_W(u, 0),\end{aligned} \tag{3.3.23}$$

where $J(u) = \int F(u, x) G(u, x) dx$ and W is a standard Brownian motion on $[0, 1]$ and $L_W(u, 0)$ is the local time of W, and

$$\frac{\sqrt{T_n}}{ng(T_n)^2} \|\widetilde{\mathbf{G}}\|^2 \to_P \int_0^1 \int G^2(u, x) dx dL_W(u, 0).$$

Hence, in view of (3.3.21), using Cauchy-Schwarz inequality, Δ of (3.3.22) converges to the same limit as (3.3.23).

We are now ready to prove that Π_2 in (3.3.18) converges to zero in probability.

Because $\Pi_2 = \Delta \frac{\sqrt{np_{\max}} g(T_n)}{\sqrt[4]{T_n}^3 \|A(\tau, x)\|} [\delta(\tau, x) + \gamma(\tau, x)]$, in order to show $\Pi_2 \to_P 0$, it suffices to prove that

$$\frac{\sqrt{np_{\max}} g(T_n)}{\sqrt[4]{T_n}^3 \|A(\tau, x)\|} \delta(\tau, x) \to 0 \quad \text{and} \quad \frac{\sqrt{np_{\max}} g(T_n)}{\sqrt[4]{T_n}^3 \|A(\tau, x)\|} \gamma(\tau, x) \to 0 \tag{3.3.24}$$

as $n \to \infty$.

Notice that by the proof of Lemma 3.3.2, we have $O(1)kp_{\max} \geq T_n\|A(\tau,x)\|^2 \geq O(1)kp_{\min}$. Using Assumption B.3, B.4, Bm.6 and the result in Theorem 2.3.2,

$$\begin{aligned}
&\frac{\sqrt{np_{\max}}g(T_n)}{\sqrt[4]{T_n}^3\|A(\tau,x)\|}|\delta(\tau,x)| \leq O(1)\frac{\sqrt{np_{\max}}g(T_n)}{\sqrt[4]{T_n}\sqrt{kp_{\min}}}\left|\sum_{i=0}^{k}\sum_{j=p_i+1}^{\infty} b_{ij}\varphi_{jT_n}(\tau)h_i(\tau,x)\right| \\
\leq &O(1)\frac{\sqrt{np_{\max}}g(T_n)}{\sqrt[4]{T_n}\sqrt{kp_{\min}}}\sum_{i=0}^{k}|h_i(\tau,x)|\left|\sum_{j=p_i+1}^{\infty} b_{ij}\varphi_{jT_n}(\tau)\right| \\
\leq &O(1)\frac{\sqrt{np_{\max}}g(T_n)}{\sqrt[4]{T_n}\sqrt{kp_{\min}}}\left(\sum_{i=0}^{k}|h_i(\tau,x)|^2\right)^{\frac{1}{2}}\left(\sum_{i=0}^{k}\left|\sum_{j=p_i+1}^{\infty} b_{ij}\varphi_{jT_n}(\tau)\right|^2\right)^{\frac{1}{2}} \\
\leq &O(1)\frac{\sqrt{np_{\max}}g(T_n)}{\sqrt[4]{T_n}\sqrt{kp_{\min}}}\sqrt{k}\left(\sum_{i=0}^{k}\frac{a^2(T_n)T_n^2}{p_i^2}\right)^{\frac{1}{2}} \leq O(1)\frac{\sqrt{np_{\max}}T_n^{\rho}}{\sqrt[4]{T_n}}\frac{\sqrt{k}T_n}{\sqrt{p_{\min}}^3} \\
= &O(1)n^{\frac{1}{2}+\frac{1}{2}\kappa_1+(\rho+\frac{3}{4})\kappa_3+\frac{1}{2}(\bar{\kappa}_2-\kappa_2)-\kappa_2} \to 0
\end{aligned}$$

as $n \to \infty$ where we use Assumption B.3(d) that $a^2(T_n) < M$ and Assumption B.4 and Bm.6 for ρ and truncation parameters.

Meanwhile, using (2.2.13) with $r=2$ and $|h_i(\tau,x)| \leq i^{-\frac{1}{4}}$ for large i (see Nikiforov and Uvarov, 1988, p.54) gives

$$\begin{aligned}
&\frac{\sqrt{np_{\max}}g(T_n)}{\sqrt[4]{T_n}^3\|A(\tau,x)\|}|\gamma(\tau,x)| \leq O(1)\frac{\sqrt{np_{\max}}g(T_n)}{\sqrt[4]{T_n}\sqrt{kp_{\min}}}\left|\sum_{i=k+1}^{\infty}\sum_{j=0}^{\infty} b_{ij}\varphi_{jT_n}(\tau)h_i(\tau,x)\right| \\
= &O(1)\frac{\sqrt{np_{\max}}g(T_n)}{\sqrt[4]{T_n}\sqrt{kp_{\min}}}\left|\sum_{i=k+1}^{\infty} b_i(\tau)h_i(\tau,x)\right| \\
= &O(1)\frac{\sqrt{np_{\max}}g(T_n)}{\sqrt[4]{T_n}\sqrt{kp_{\min}}}\left|\sum_{i=k+1}^{\infty}\frac{\tau}{\sqrt{i(i-1)}}b_{i-2}(\tau,m_x'')h_i(\tau,x)\right| \\
\leq &O(1)\frac{\sqrt{np_{\max}}T_n^{\rho}}{\sqrt[4]{T_n}k\sqrt{kp_{\min}}}\left(\sum_{i=k+1}^{\infty}|b_{i-2}(\tau,m_x'')|^2\right)^{\frac{1}{2}}\left(\sum_{i=k+1}^{\infty}\frac{1}{i(i-1)\sqrt{i}}\right)^{\frac{1}{2}} \\
\leq &o(1)\frac{\sqrt{np_{\max}}T_n^{\rho}}{\sqrt[4]{T_n}k\sqrt{kp_{\min}}}\frac{1}{k^{3/4}} \\
= &o(1)n^{\frac{1}{2}+\frac{1}{2}(\bar{\kappa}_2-\kappa_2)-(\frac{1}{4}-\rho)\kappa_3-\frac{5}{4}\kappa_1} \to 0
\end{aligned}$$

as $n \to \infty$ in view of Assumption B.4 and Bm.6. The proof is finished. □

Chapter 4

Orthogonal expansion of Lévy process functionals

4.1 Introduction

Stochastic differential equations driven by a Lévy process under some conditions have solutions in the form of functionals of the underlying process. Such equations are used extensively in economics, finance and engineering disciplines to describe random phenomena in both theory and practice. Meanwhile, some empirical studies show that many datasets admit nonlinearity and nonstationarity. Consequently, a number of nonparametric and semiparametric models and kernel-based methods have been proposed to deal with both nonlinearity and nonstationarity simultaneously. Existing studies mainly discuss the employment of nonparametric kernel estimation methods. Such studies include Phillips and Park (1998), Park and Phillips (1999), Park and Phillips (2001), Karlsen and Tjøstheim (2001), Karlsen et al. (2007), Cai et al. (2009), Phillips (2009), Wang and Phillips (2009a,b), Xiao (2009), and Gao and Phillips (2010).

However, such kernel-based estimation methods are not applicable to establish closed-form expansions of functionals of Lévy processes. In the stationary case, the literature already discusses how series approximations may be used in dealing with stationary time series models, such as Ai and Chen (2003), Chapter 2 of Gao (2007) and Li and Racine (2007). In addition, although the celebrated Black-Scholes option pricing formula described the price of the financial product as a functional of Brownian motion, literature has pointed out that there are some significant drawbacks to this formula. For example, empirical evidence suggests that log returns do not behave according to a normal distribution (see Schoutens, 2003). Hence, the researchers realise that one would need to include other stochastic processes (not just Brownian motion) when one needs to formulate a continuous-time stochastic models in order to depict some stochastic phenomena or scientific datasets.

Therefore, there is need to study functionals of the Lévy process, $Z(t)$, in the both cases of time-homogeneity and time–inhomogeneity. Note that one powerful way of dealing with such problems is to decompose the process, say $f(Z(t))$ or $f(t, Z(t))$, where the functional form is unknown, into an orthogonal series in some Hilbert space, such that once one has obtained observed values of the process, the coefficients involved in the series can be estimated using an econometric method. Actually, the literature has found for a long time that there exists a close connection between stochastic processes and orthogonal polynomials. For example, the so-called Karlin-McGregor representation expresses the transition probability of the birth and death process by means of a spectral representation in terms of orthogonal polynomials. Some people clearly feel the potential importance of orthogonal polynomials in probability theory. Schoutens (2000), for instance, gives an extensive discussion about relations between stochastic processes and orthogonal polynomials.

In what follows, we shall establish some general theory and methodology for the expansion of a class of functionals of Lévy processes. As an application, we shall estimate an unknown function of the form $m(t, z)$ involved in the following model:

$$Y(t) = m(t, Z(t)) + \varepsilon(t), \quad t \in [0, \infty), \tag{4.1.1}$$

where $Z(t)$ is a Lévy process that covers both the continuous (such as Brownian motion) and the discrete (such as the Poisson process) cases, $\varepsilon(t)$ is an error process with zero mean and finite variance, and $m(t, z)$ is an unknown function of (t, z).

As far as we are aware, there is no discussion about how to estimate $m(t, z)$ by a non- or semi-parametric method in the literature. Even in the discrete case where $t = 1, 2, \cdots$, it is not clear whether a nonparametric kernel method can provide a consistent estimator for $m(t, z)$. Part of the contribution of this paper is to establish an asymptotically consistent estimator of $m(t, z)$ and the resulting asymptotic theory in each of the three sampling situations: a) the case where $Z_t = Z(t)$ at $t = 1, 2, \cdots$, b) the case where $Z_{t,n} = Z\left(\frac{tT}{n}\right)$ at $t = 1, 2, \cdots$, and c) the case where $Z_{t,n} = Z\left(\frac{tT_n}{n}\right)$ at $t = 1, 2, \cdots$ and with $T_n \to \infty$.

The estimation methodology proposed in the sequel is summarised as follows. We shall employ an appropriate polynomial sequence that is orthogonal with respect to either the probability density or the probability distribution of $Z(t)$ depending on whether $Z(\cdot)$ is continuous or discrete. We then expand the unknown function $m(t, Z(t))$ into an orthogonal series in some Hilbert space in terms of the polynomial sequence. We then propose using a semiparametric least squares (SLS) estimation method to estimate $m(t, z)$ by $\widehat{m}(t, z)$. To establish an asymptotic theory for $\widehat{m}$, we introduce a general asymptotic theory to deal with the sample mean and sample covariance of four classes of functionals of Lévy processes. It is noteworthy to point out that the established asymptotic theory considerably extends some existing results, such as Park and Phillips (1999, 2001), and Wang and Phillips (2009a).

With the advantage of expanding an unknown functional into an orthogonal series, the proposed method can be used to deal with some estimation problems in economics, finance and engineering. For example, there are a number of studies involving models with conditional moment restriction containing an unknown functional, such as Ai and Chen (2003, 2007), and Chen and Ludvigson (2009). Since existing theory for expansions of functionals of stationary processes is not directly applicable, the proposed expansion and estimation method in this paper is useful and significant in both theory and applications.

4.2 Lévy processes and infinite divisibility

The term "Lévy process" honours the work of the French mathematician Paul Lévy who, although not alone in his contribution, played an instrumental role in bringing together an understanding and characterisation of processes with stationary independent increments. In earlier literature, Lévy processes can be found under a number of different names. In the 1940s, Lévy himself referred to them as a sub-class of *processus additif* (additive processes), that is processes with independent increments. For the most part however, research literature from the 1960s and 1970s refers to Lévy processes simply as *processes with stationary independent increments.* One sees a change in language through the 1980s and by the 1990s the use of the term "Lévy process" had become standard.

Definition 4.2.1 (Lévy process)**.** A process $Z = (Z(t), t \geq 0)$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a Lévy process if it possesses the following properties

(i) The paths of Z are $\mathbb{P}$-almost surely right continuous with left limits.

(ii) $\mathbb{P}(Z(0) = 0) = 1$.

(iii) For $0 \leq s \leq t$, $Z(t) - Z(s)$ is equal in distribution to $Z(t - s)$.

(iv) For $0 \leq s \leq t$, $Z(t) - Z(s)$ is independent of $\{Z(u), u \leq s\}$.

(v) Stochastic continuity: for $\forall\, \varepsilon > 0$, $\lim_{h\to 0} \mathbb{P}(|Z(t+h) - Z(t)| > \varepsilon) = 0$.

It is evident that both Brownian motion and the Poisson process are Lévy processes. However, from the definition alone it is difficult to see how rich a class of processes the class of Lévy processes forms.

If we sample a Lévy process $Z(t)$ at regular time intervals $0, \triangle, 2\triangle, \cdots$, we obtain a random work: defining $S_n(\triangle) \equiv Z(n\triangle)$, we can write $S_n(\triangle) = Y_1 + \cdots + Y_n$ where $Y_i = Z(i\triangle) - Z((i-1)\triangle)$ are i.i.d. random variables whose distribution is the same as that of $Z(\triangle)$. Since this can be done for any sampling interval $\triangle$, we can see by specifying a Lévy process we have specified a whole family of random walks $S_n(\triangle)$: these models simply correspond to sampling the Lévy process $Z(t)$ at different frequencies.

Choosing $n\triangle = t$, we see that for any $t > 0$ and any $n \geq 1$, $Z(t) = S_n(\triangle)$ can be represented as a sum of n i.i.d. random variables whose distribution is that of $Z(\frac{t}{n})$: $Z(t)$ can be "divided" into n i.i.d. parts. A distribution having this property is said to be infinitely divisible. Finetti (1929) introduced the notion of an infinitely divisible distribution and showed that they have an intimate relationship with Lévy processes. This relationship gives a reasonably good impression of how varied the class of Lévy processes really is.

Definition 4.2.2 (Infinitely divisible distribution)**.** We say that a real-valued random variable ξ has an infinitely divisible distribution if for each $n = 1, 2, \cdots$, there exists a sequence of i.i.d. random variables $\xi_{1,n}, \cdots, \xi_{n,n}$ such that

$$\xi \stackrel{D}{=} \xi_{1,n} + \cdots + \xi_{n,n}$$

where $\stackrel{D}{=}$ is equality in distribution.

Define the characteristic function $\Phi_t(\theta)$ of $Z(t)$

$$\Phi_t(\theta) \equiv \Phi_{Z(t)}(\theta) \equiv E[\exp(i\theta Z(t))], \quad \theta \in \mathbb{R}.$$

For $t, s > 0$, noting that $Z(t+s) = Z(t+s) - Z(s) + Z(s)$ and the fact that $Z(t+s) - Z(s)$ and $Z(s)$ are mutually independent, we obtain that $t \mapsto \Phi_t(\theta)$ is a multiplicative function

$$\begin{aligned}\Phi_{t+s}(\theta) =& \Phi_{Z(t+s)}(\theta) = \Phi_{Z(t+s)-Z(s)}(\theta)\Phi_{Z(s)}(\theta)\\ =& \Phi_{Z(t)}(\theta)\Phi_{Z(s)}(\theta) = \Phi_t(\theta)\Phi_s(\theta).\end{aligned}$$

Meanwhile, stochastic continuity implies in particular that $Z(t) \to Z(s)$ in distribution as $t \to s$. Thus $t \mapsto \Phi_t(\theta)$ is a continuous function, which, together with multiplicative property, implies that $\Phi_t(\theta)$ is an exponential function.

$$\Phi_t(\theta) = e^{-t\psi(\theta)},$$

where $\psi(\theta) := \psi_1(\theta) = -\log E\left[e^{i\theta Z(1)}\right]$ is the characteristic exponent of $Z(1)$. Hence, $\Phi_t(\theta) = [\Phi_1(\theta)]^t$. The law of $Z(t)$ is therefore determined by the knowledge of the law of $Z(1)$: the only degree of freedom when we are specifying a Lévy process is to specify the distribution of $Z(t)$ for a single time (say $t = 1$). Please consult Kyprianou (2006) and Cont and Tankov (2004).

4.3 Existence of orthogonal polynomials associated with the Lévy process

Let $(Z(t), t \geq 0)$ be any Lévy process. Let $\Phi(\theta)$ be the characteristic function of $Z(1)$. According to Schoutens (2000, p.50), it can be shown that

$$\frac{\Phi'(\theta)}{\Phi(\theta)} = i(\mu + \sigma^2\tau(i\theta)) \tag{4.3.1}$$

for some function $\tau(\cdot)$ with $\tau(0) = 0$ where $\mu = EZ(1)$ and $\sigma^2 = Var(Z(1))$.

Let $u(\cdot)$ be the inverse function of $\tau(\cdot)$. Define $\pi(z) = [\phi(-iu(z))]^{-1}$.

Definition 4.3.1 (Lévy-Meixner system)**.** A polynomial set $\{q_i(t, x), i \geq 0, t \geq 0\}$ is called a Lévy-Meixner system if it is defined by a generating function of the form

$$\sum_{i=0}^{\infty} q_i(t, x)\frac{z^i}{i!} = (\pi(z))^t \exp(xu(z)). \tag{4.3.2}$$

This definition is a general case of Lévy-Meixner system in Schoutens (2000) since time variable t is involved. Such a change makes it possible to obtain the existence of a polynomial system which is orthogonal with respect to the distribution of Lévy process $Z(t)$. That is, such defined $q_i(t,x)$ is orthogonal with respect to the distribution $\Psi_t(x)$ of $Z(t)$:

$$\int_{-\infty}^{\infty} q_i(t,x)q_j(t,x)d\Psi_t(x) = \delta_{ij}\widetilde{d_i^2}(t). \tag{4.3.3}$$

where $\widetilde{d_i^2}(t) = \int_{-\infty}^{\infty} q_i^2(t,x)d\Psi_t(x) > 0$.

Remark 4.3.1. When process $Z(t)$ is specified as Brownian motion, $q_i(t,x)$ becomes Hermite polynomial with weight the density of normal distribution $N(0,t)$; when $Z(t)$ is specified as Gamma process, $q_i(t,x)$ will be Laguerre polynomial system $L_i^{(\alpha t)}(x)$; if $Z(t) = N(t)$ a Poisson process, $q_i(t,x)$ will be Charlier polynomial system $C_i(\mu t, x)$; if $Z(t)$ is a Pascal process, $q_i(t,x)$ will be a Meixner polynomial system.

Note that the Lévy-Meixner system is a subclass of the Lévy-Sheffer system, so that we have the following martingale equality

$$E[q_i(t,Z(t))|Z(s)] = q_i(s,Z(s)), \qquad 0 \le s \le t,\ i \ge 0. \tag{4.3.4}$$

In what follows we are about to find the explicit expressions of $q_i(t,x)$. Frankly speaking, the following arguments parallel the counterpart in Nikiforov and Uvarov (1988). Since there are two scenarios to be considered, continuous and discrete, this section is split into two subsections.

4.3.1 Orthogonal polynomials of a continuous variable

Let $(Z(t), t \ge 0)$ be a Lévy process. Suppose that $Z(t)$ is a continuous random variable with distribution function $\Psi_t(x)$ for every $t > 0$, and $\frac{d}{dx}\Psi_t(x) = \rho(t,x)$.

Let us now consider differential equation of hypergeometric type with parameter $t > 0$:

$$s(t,x)y''(t,x) + v(t,x)y'(t,x) + \lambda(t)y(t,x) = 0. \tag{4.3.5}$$

where $s(t,x)$ and $v(t,x)$ are polynomials in x of degree at most 2 and 1 respectively, while $\lambda(t)$ is independent of x. We shall refer to the solutions of (4.3.5) as functions of hypergeometric type. Note that in the equation (4.3.5), and also in the sequel, all derivatives are conducted with respect to x, not to t.

Lemma 4.3.1. *All derivatives of the solutions of* (4.3.5) *are still of hypergeometric type; meanwhile, any function of hypergeometric type is a derivative of some function of hypergeometric type with* $\lambda(t) \ne 0$ *in the differential equation.*

Proof. Differentiating (4.3.5) and denoting $z_1(t,x) = y'(t,x)$ entail that

$$s(t,x)z_1''(t,x) + v_1(t,x)z_1'(t,x) + \eta_1(t)z_1(t,x) = 0 \tag{4.3.6}$$

where $v_1(t,x) = v(t,x) + s'(t,x)$ and $\eta_1(t) = \lambda(t) + v'(t,x)$.

Since $v_1(t,x)$ is a polynomial in x of degree at most 1 and η_1 is independent of x, equation (4.3.6) is of hypergeometric type.

Now let $z_1(t,x)$ be a solution of (4.3.6) with $\lambda(t) = \eta_1(t) - v_1'(t,x) + s''(t,x) \neq 0$. Construct a function $y(t,x) = -\frac{1}{\lambda(t)}(s(t,x)z_1'(t,x) + v(t,x)z_1(t,x))$. Then

$$\begin{aligned}\lambda(t)y'(t,x) =& -\left(s(t,x)z_1'(t,x) + v(t,x)z_1(t,x)\right)' \\ =& - s(t,x)z_1''(t,x) - s'(t,x)z_1'(t,x) - v(t,x)z_1'(t,x) - v'(t,x)z_1(t,x) \\ =& \lambda(t)z_1(t,x),\end{aligned}$$

by virtue of (4.3.6). Hence, $y'(t,x) = z_1(t,x)$. The construction of $y(t,x)$ therefore implies that $y(t,x)$ is a solution of equation (4.3.5).

It follows from induction that $z_i(t,x) = y^{(i)}(t,x)$ is a solution of

$$s(t,x)z_i''(t,x) + v_i(t,x)z_i'(t,x) + \eta_i(t)z_i(t,x) = 0 \tag{4.3.7}$$

where $v_i(t,x) = v(t,x) + is'(t,x)$ and $\eta_i(t) = \lambda(t) + iv'(t,x) + \frac{i(i-1)}{2}s''(t,x)$.

Moreover, every solution of (4.3.7) for $\eta_k \neq 0$ $(k = 0, 1, \cdots, i-1)$ can be represented as $z_i(t,x) = y^{(i)}(t,x)$, where $y(t,x)$ is a solution of (4.3.5). □

Observe that Lemma 4.3.1 provides with us a possibility to find out a family of particular solutions of (4.3.5) according to a given $\lambda(t)$. Indeed, if $\eta_i(t) = \lambda(t) + iv'(t,x) + \frac{i(i-1)}{2}s''(t,x) = 0$, it is evident that equation (4.3.7) has a solution $z_i(t,x) = z_i(t)$ independent of x. Since $z_i(t) = y^{(i)}(t,x)$, we assert that when $\lambda(t) \equiv \lambda_i(t) = -iv'(t,x) - \frac{i(i-1)}{2}s''(t,x)$, $y(t,x) = y_i(t,x)$, as a solution of (4.3.5), is a polynomial in x of degree exactly i.

In order to find out the explicit expression of $y_i(t,x)$, where i is any fixed positive integer, noting that (4.3.7) is valid for $k = 0, 1, \cdots, i$ (that is, i can be substituted by any k), we multiply equations (4.3.5) and (4.3.7) by appropriate functions $\rho(t,x)$ and $\rho_k(t,x)$ such that they can be written in self-adjoint form

$$(s(t,x)\rho(t,x)y'(t,x))' + \lambda(t)\rho(t,x)y(t,x) = 0 \tag{4.3.8a}$$

$$(s(t,x)\rho_k(t,x)z_k'(t,x))' + \eta_k(t)\rho_k(t,x)z_k(t,x) \quad = \quad 0 \tag{4.3.8b}$$

where $\rho(t,x)$ and $\rho_k(t,x)$ satisfy that

$$(s(t,x)\rho(t,x))' \quad = \quad v(t,x)\rho(t,x) \tag{4.3.9a}$$

$$(s(t,x)\rho_k(t,x))' \quad = \quad v_k(t,x)\rho_k(t,x) \tag{4.3.9b}$$

It follows from (4.3.9) that

$$\frac{(s(t,x)\rho_k(t,x))'}{\rho_k(t,x)} = v_k(t,x) = v(t,x) + ks'(t,x) = \frac{(s(t,x)\rho(t,x))'}{\rho(t,x)} + ks'(t,x)$$

which entails that

$$\frac{\rho_k'(t,x)}{\rho_k(t,x)} = \frac{\rho'(t,x)}{\rho(t,x)} + k\frac{s'(t,x)}{s(t,x)}.$$

Whence we have

$$\rho_k(t,x) = s^k(t,x)\rho(t,x), \quad k = 0, 1, \cdots \tag{4.3.10}$$

where $\rho_0(t,x) = \rho(t,x)$ by definition.

Observing that $s(t,x)\rho_k(t,x) = \rho_{k+1}(t,x)$ and $z_k'(t,x) = z_{k+1}(t,x)$, it follows from (4.3.8b) that

$$\rho_k(t,x)z_k(t,x) = -\frac{1}{\eta_k}(s(t,x)\rho_k(t,x)z_k'(t,x))' = -\frac{1}{\eta_k}(\rho_{k+1}(t,x)z_{k+1}(t,x))'. \tag{4.3.11}$$

Hence, when $k < i$ we obtain successively

$$\begin{aligned}\rho_k(t,x)z_k(t,x) =& -\frac{1}{\eta_k}(\rho_{k+1}(t,x)z_{k+1}(t,x))' \\ =& \frac{(-1)^2}{\eta_k\eta_{k+1}}(\rho_{k+2}(t,x)z_{k+2}(t,x))'' \\ =& \cdots = (-1)^{i-k}\frac{1}{\eta_k\cdots\eta_{i-1}}(\rho_i(t,x)z_i(t,x))^{(i-k)} \\ =& \frac{A_k}{A_i}(\rho_i(t,x)z_i(t,x))^{(i-k)}\end{aligned}$$

where we denote

$$A_i = (-1)^i\prod_{j=0}^{i-1}\eta_j, \qquad A_0 = 1. \tag{4.3.12}$$

By virtue of that $y_i(t,x)$ is a polynomial of degree i and $z_k = y^{(k)}$, $z_i = y^{(i)} = \text{const.}$, we finally have

$$y_i^{(k)}(t,x) = \frac{A_{ki}B_i}{\rho_k(t,x)}[\rho_i(t,x)]^{(i-k)} \tag{4.3.13}$$

where $A_{ki} := A_k(\lambda)|_{\lambda=\lambda_i}$ and $B_i = \frac{1}{A_{ii}} y_i^{(i)}(t,x)$.

Hence, in particular, when $k = 0$, we have an explicit representation of $y_i(t,x)$ of hypergeometric type

$$y_i(t,x) = \frac{B_i}{\rho(t,x)} \frac{d^i}{dx^i}[s^i(t,x)\rho(t,x)]. \tag{4.3.14}$$

We call (4.3.14) the Rodrigues formula, since it was established in 1814 by B. O. Rodrigues for special polynomials of hypergeometric type, namely the Legendre polynomials, for which $s(x) = 1 - x^2$, $\rho(x) = 1$.

Notice that for many Lévy processes $(Z(t), t \geq 0)$, $s''(t,x) = 0$ (Brownian motion with $s(x) = 1$, Gamma process and Poisson process with $s(x) = x$). See Appendix B of Schoutens (2000). In this case A_{ii} and B_i reduce to

$$A_{ii} = (v'(t,x))^i i!, \quad B_i = \frac{1}{(v'(t,x))^i i!} y_i^{(i)}(t,x).$$

Similar to the theorem of Nikiforov and Uvarov (1988, p.29), we have the following lemma.

Lemma 4.3.2. *Suppose that polynomials $y_i(t,x)$ in x are the solutions of equation of hypergeometric type*

$$s(t,x)y''(t,x) + v(t,x)y'(t,x) + \lambda_i(t)y(t,x) = 0 \tag{4.3.15}$$

where $\lambda_i(t) = -iv'(t,x) - \frac{i(i-1)}{2}s''(t,x)$. In addition, suppose a density function $\rho(t,x)$ satisfies that $(s(t,x)\rho(t,x))' = v(t,x)\rho(t,x)$ and the so-called boundary condition for any $t > 0$

$$s(t,x)\rho(t,x)x^k|_{x=a,b} = 0, \qquad k = 0, 1, \cdots \tag{4.3.16}$$

where a and b are the boundary points of the support of $\rho(t,x)$ relative to x. Then the $y_i(t,x)$ are orthogonal on (a,b) with respect to $\rho(t,x)$.

The orthogonal polynomial system $\{y_i(t,x)\}$ with $\rho(t,x)$ satisfying conditions in Lemma 4.3.2 is called *classic orthogonal polynomial system* of continuous variable. It is usually considered the auxiliary conditions $\rho(t,x) > 0$ and $s(t,x) > 0$ on (a,b) for any $t > 0$.

Proof. Observe that $y_m(t,x)$ and $y_i(t,x)$ satisfy the following differential equations in self-adjoint form respectively

$$(s(t,x)\rho(t,x)y_i'(t,x))' + \lambda_i(t)\rho(t,x)y_i(t,x) = 0, \tag{4.3.17a}$$

$$(s(t,x)\rho(t,x)y_m'(t,x))' + \lambda_m(t)\rho(t,x)y_m(t,x) \quad = \quad 0. \tag{4.3.17b}$$

Operation (4.3.17b)$\times y_i$–(4.3.17a)$\times y_m$ yields

$$\begin{aligned}&(\lambda_m(t) - \lambda_i(t))\rho(t,x)y_m(t,x)y_i(t,x)\\=&(s(t,x)\rho(t,x)y_i'(t,x))'y_m(t,x) - (s(t,x)\rho(t,x)y_m'(t,x))'y_i(t,x).\end{aligned}$$

Integration by parts gives

$$\begin{aligned}&(\lambda_m(t) - \lambda_i(t))\int_a^b \rho(t,x)y_m(t,x)y_i(t,x)dx\\=&s(t,x)\rho(t,x)y_i'(t,x)y_m(t,x)|_a^b - s(t,x)\rho(t,x)y_m'(t,x)y_i(t,x)|_a^b = 0\end{aligned}$$

due to the boundary condition (4.3.16). Hence, if $\lambda_m(t) \neq \lambda_i(t)$, then $y_i(t,x)$ are of orthogonality with weight $\rho(t,x)$. □

Note that (1) Interval (a,b) is infinite in this study. (2) $\lambda_i \neq \lambda_m \Leftrightarrow m \neq i$. In fact, $\lambda_i(t) - \lambda_m(t) = (m-i)[v'(t,x) + \frac{1}{2}(m+i-1)s''(t,x)]$, but $v'(t,x) + \frac{1}{2}(m+i-1)s''(t,x) = 0$ implies both $v'(t,x) = 0$ and $s''(t,x) = 0$. Nonetheless, vanishes of the two simultaneously will not result in a density $\rho(t,x)$ which satisfies boundary conditions.

Example 4.1

We are going to find out the differential equation of hypergeometric type for orthogonal polynomials with weight $\rho(t,x) = \frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}$.

Starting from $(s(t,x)\rho(t,x))' = v(t,x)\rho(t,x)$, we have $s' - s\frac{x}{t} = v$. Noting that s is a polynomial at most of second order while v is a polynomial at most of first order, in order to meet the equation, we must have $s = g(t)$, hence $v = -g(t)\frac{x}{t}$. Consequently, $\lambda_i = ig(t)\frac{1}{t}$. Therefore, the orthogonal polynomials satisfy the following differential equation of hypergeometric type

$$ty''(t,x) - xy'(t,x) + iy(t,x) = 0. \tag{4.3.18}$$

It is easy to verify that $y_0 = 1$, $y_1 = \frac{x}{\sqrt{t}}$, $y_2 = \frac{1}{\sqrt{2}}\left(\frac{x^2}{t} - 1\right)$, and $y_3 = \frac{1}{\sqrt{6}}\left(\frac{x^3}{\sqrt{t}^3} - 3\frac{x}{\sqrt{t}}\right)$ satisfy the equation (4.3.18). □

Example 4.2

In this example, we are about to find out the differential equation of hypergeometric type of orthogonal polynomials with respect to $\rho(t,x) = \frac{1}{\Gamma(1+\alpha t)}x^{\alpha t}e^{-x}$, which corresponds to a Gamma process $Z(t)$ with $\alpha > -1$.

It follows from $(s(t,x)\rho(t,x))' = \rho(t,x)v(t,x)$ that $s'(t,x)x + \alpha ts(t,x) - xs(t,x) = v(t,x)x$. Palpably, $s(t,x)$ is at most a polynomial in x of one degree to satisfy this equation since $v(t,x)$ is a polynomial of degree at most 1.

Suppose that $s(t,x) = g(t)x + h(t)$ and $v(t,x) = g_1(t)x + h_1(t)$. Substituting into the above equation yields

$$g(t)x + \alpha tg(t)x + \alpha th(t) - g(t)x^2 - h(t)x = g_1(t)x^2 + h_1(t)x.$$

Identifying the coefficients gives that $h(t) = 0$, $g_1(t) = -g(t)$ and $h_1(t) = (\alpha t + 1)g(t)$. Therefore, $s(t,x) = g(t)x$, $v(t,x) = (\alpha t+1-x)g(t)$ and $\lambda_i = ig(t)$. Whence, the differential equation is

$$xy''(t,x) + (\alpha t + 1 - x)y'(t,x) + iy(t,x) = 0. \tag{4.3.19}$$

□

Lemma 4.3.3. *For the classical orthogonal polynomial $y_i(t,x)$ of hypergeometric type with weight $\rho(t,x)$*

$$s(t,x)y''(t,x) + v(t,x)y'(t,x) + \lambda_i(t)y(t,x) = 0, \tag{4.3.20}$$

where $\lambda_i(t) = -iv'(t,x) - \frac{1}{2}i(i-1)s''(t,x)$, the derivatives $y_i^{(k)}(t,x)$ are orthogonal with respect to $\rho_k(t,x) = s^k(t,x)\rho(t,x)$ on (a,b).

Proof. According to Lemma 4.3.1, because $y_i(t,x)$ satisfies equation (4.3.20), $y_i^{(k)}(t,x)$ is a solution of the following differential equation

$$s(t,x)[y_i^{(k)}(t,x)]'' + v_k(t,x)[y_i^{(k)}(t,x)]' + \eta_{in}y_i^{(k)}(t,x) = 0, \tag{4.3.21}$$

where $v_k(t,x) = v(t,x) + ks'(t,x)$ and $\eta_{ki}(t) = \lambda_i(t) + kv'(t,x) + \frac{1}{2}k(k-1)s''(t,x)$.

It is easy to verify that given $(s'(t,x)\rho(t,x))' = v(t,x)\rho(t,x)$, $\rho_k(t,x)$ satisfies

$$(s(t,x)\rho_k(t,x))' = v_k(t,x)\rho_k(t,x). \tag{4.3.22}$$

We therefore have a self-adjoint form of equation (4.3.21)

$$\{s(t,x)\rho_k(t,x)[y_i^{(k)}(t,x)]'\}' + \eta_{ki}\rho_k(t,x)y_i^{(k)}(t,x) = 0. \tag{4.3.23}$$

Suppose, on the other hand, that $y_m^{(k)}(t,x)$ is the k-th derivative of $y_m(t,x)$ of hypergeometric type corresponding to λ_m, then we also have a similar equation to (4.3.23). Aligning them together gives

$$\{s(t,x)\rho_k(t,x)[y_i^{(k)}(t,x)]'\}' + \eta_{ki}\rho_k(t,x)y_i^{(k)}(t,x) \quad = \quad 0, \tag{4.3.24a}$$

$$\{s(t,x)\rho_k(t,x)[y_m^{(k)}(t,x)]'\}' + \eta_{km}\rho_k(t,x)y_m^{(k)}(t,x) \quad = \quad 0. \tag{4.3.24b}$$

A similar derivation as in Lemma 4.3.2 yields

$$\begin{aligned}
&(\eta_{km} - \eta_{ki}) \int_a^b \rho_k(t,x)y_m^{(k)}(t,x)y_i^{(k)}(t,x)dx \\
=&\int_a^b \{s(t,x)\rho_k(t,x)[y_i^{(k)}(t,x)]'\}'y_m^{(k)}(t,x)dx \\
&- \int_a^b \{s(t,x)\rho_k(t,x)[y_m^{(k)}(t,x)]'\}'y_i^{(k)}(t,x)dx \\
=&s(t,x)\rho_k(t,x)[y_i^{(k)}(t,x)]'y_m^{(k)}(t,x)|_a^b \\
&- s(t,x)\rho_k(t,x)[y_m^{(k)}(t,x)]'y_i^{(k)}(t,x)|_a^b \\
=&0
\end{aligned}$$

in virtue of the boundary condition of $\rho(t,x)$.

Therefore, when $\eta_{km} \neq \eta_{ki}$, viz., $m \neq i$, we have $\int_a^b \rho_k(t,x)y_m^{(k)}(t,x)y_i^{(k)}(t,x)dx = 0$. In other words,

$$\int_a^b \rho_k(t,x)y_m^{(k)}(t,x)y_i^{(k)}(t,x)dx = \delta_{mi}d_{ki}^2$$

where $d_{ki}^2(t) := \int_a^b \rho_k(t,x)[y_i^{(k)}(t,x)]^2dx$. □

Let us find out the relationship between the squared norm $d_{ki}^2(t)$ of $y_i^{(k)}(t,x)$ and the squared norm $d_i^2(t) := d_{0i}^2(t)$ of $y_i(t,x)$. Rewrite equation (4.3.24a) as

$$[\rho_{k+1}(t,x)y_i^{(k+1)}(t,x)]' + \eta_{ki}\rho_k(t,x)y_i^{(k)}(t,x) = 0. \tag{4.3.25}$$

Multiplying (4.3.25) by $y_i^{(k)}(t,x)$ and integrating by parts over (a,b) give

$$\begin{aligned}
\eta_{ki}d_{ki}^2(t) =&\eta_{ki} \int_a^b \rho_k(t,x)[y_i^{(k)}(t,x)]^2dx \qquad (4.3.26)\\
=&- \int_a^b [\rho_{k+1}(t,x)y_i^{(k+1)}(t,x)]'y_i^{(k)}(t,x)dx \\
=&- \rho_{k+1}(t,x)y_i^{(k+1)}(t,x)y_i^{(k)}(t,x)|_a^b + \int_a^b \rho_{k+1}(t,x)[y_i^{(k+1)}(t,x)]^2dx \\
=&d_{k+1,i}^2(t). \qquad (4.3.27)
\end{aligned}$$

Whence, by induction, we obtain

$$d_{mi}^2(t) = d_i^2(t) \prod_{k=0}^{m-1} \eta_{ki}(t). \tag{4.3.28}$$

where $\eta_{0i}(t) = \lambda_i(t)$.

4.3.2 Orthogonal polynomials of a discrete variable

First, we study the solutions of difference equation of hypergeometric type with parameter $t > 0$. Define difference operation $\triangle f(x) = f(x+1) - f(x)$, $\nabla f(x) = f(x) - f(x-1)$. In what follows, the following identities are frequently utilised

$$\begin{cases} \triangle f(x) =& \nabla f(x+1), \\ \triangle\nabla f(x) =& \nabla\triangle f(x) = f(x+1) - 2f(x) + f(x-1) \\ \triangle[f(x)g(x)] =& f(x)\triangle g(x) + g(x+1)\triangle f(x) \end{cases} \tag{4.3.29}$$

Note that in the sequel all difference operations are applied with respect to x only.

The difference equation of hypergeometric type takes the form

$$s(t,x)\triangle\nabla y(t,x) + v(t,x)\triangle y(t,x) + \lambda(t)y(t,x) = 0 \tag{4.3.30}$$

where $s(t,x)$ and $v(t,x)$ are polynomials in x at most second and first degree respectively, and λ is a constant relative to t.

Lemma 4.3.4. *If $y(t,x)$ is a solution to* (4.3.30)*, then $z_1(t,x) = \triangle y(t,x)$ is also a solution of some difference equation of hypergeometric type. If $\lambda(t) \neq 0$, the converse is also true.*

Proof. Applying $\triangle$ on (4.3.30) yields

$$\triangle[s(t,x)\nabla z_1(t,x)] + \triangle[v(t,x)z_1(t,x)] + \lambda(t)z_1(t,x) = 0 \tag{4.3.31}$$

Moreover, by (4.3.29),

$$\begin{aligned} \triangle[s(t,x)\nabla z_1(t,x)] =& s(t,x)\triangle\nabla z_1(t,x) + \triangle s(t,x)\nabla z_1(t,x+1) \\ =& s(t,x)\triangle\nabla z_1(t,x) + \triangle s(t,x)\triangle z_1(t,x), \\ \triangle[v(t,x)z_1(t,x)] =& z_1(t,x)\triangle v(t,x) + v(t,x+1)\triangle z_1(t,x). \end{aligned}$$

It thus follows that

$$s(t,x)\triangle\nabla z_1(t,x) + v_1(t,x)\triangle z_1(t,x) + \eta_1(t)z_1(t,x) = 0, \tag{4.3.32}$$

where $v_1(t,x) = v(t,x+1) + \triangle s(t,x)$ and $\eta_1(t) = \lambda(t) + \triangle v(t,x)$.

Palpably, $v_1(t,x)$ is a polynomial in x of degree at most 1 and $\eta_1(t)$ is independent of x. Therefore, (4.3.32) is of the same form as (4.3.30).

If $\lambda(t) \neq 0$ (which can be represented by $\eta_1(t)$, $v_1(t,x)$ and $s(t,x)$), for each solution $z_1(t,x)$ of (4.3.32), construct

$$y(t,x) = -\frac{1}{\lambda(t)}[s(t,x)\nabla z_1(t,x) + v(t,x)z_1(t,x)].$$

Then, it is easy to verify that $\triangle y(t,x) = z_1(t,x)$ and $y(t,x)$ satisfies equation (4.3.30). In fact, since (4.3.32) is equivalent to (4.3.31), $-\lambda(t)\triangle y(t,x) = \triangle[s(t,x)\nabla z_1(t,x) + v(t,x)z_1(t,x)] = -\lambda(t)z_1(t,x)$, hence, $\triangle y(t,x) = z_1(t,x)$. Moreover, $s(t,x)\triangle\nabla y(t,x) + v(t,x)\triangle y(t,x) + \lambda(t)y(t,x) = s(t,x)\nabla z_1(t,x) + v(t,x)z_1(t,x) + \lambda(t)y(t,x) = -\lambda(t)y(t,x) + \lambda(t)y(t,x) = 0$ by the definition of $y(t,x)$. The proof is finished. □

It follows from induction that $z_i(t,x) = \triangle^i y(t,x)$ satisfies a difference equation of hypergeometric type:

$$s(t,x)\triangle\nabla z_i(t,x) + v_i(t,x)\triangle z_i(t,x) + \eta_i(t)z_i(t,x) = 0, \tag{4.3.33}$$

where

$$v_i(t,x) = v_{i-1}(t,x+1) + \triangle s(t,x), \qquad v_0(t,x) = v(t,x) \tag{4.3.34a}$$

$$\eta_i(t) = \eta_{i-1}(t) + \triangle v_{i-1}(t,x), \qquad \eta_0(t) = \lambda(t). \tag{4.3.34b}$$

The converse is also correct, viz., every solution $z_i(t,x)$ of (4.3.33) with $\eta_k \neq 0$ $(k = 0, 1, \cdots, i-1)$ can be rephrased as $z_i(t,x) = \triangle^i y(t,x)$ where $y(t,x)$ is a solution of (4.3.30).

Since the first part of (4.3.34a) may be rewritten as

$$v_i(t,x) + s(t,x) = v_{i-1}(t,x+1) + s(t,x+1), \tag{4.3.35}$$

it is clear that $v_i(t,x) = v(t,x+i) + s(t,x+i) - s(t,x)$.

Observe that $\triangle v_i(t,x)$ and $\triangle^2 s(t,x)$ are independent of x. Thus, it follows from once again the first part of (4.3.34a) that $\triangle v_i(t,x) = \triangle v_{i-1}(t,x) + \triangle^2 s(t,x) = \cdots = \triangle v(t,x) + i\triangle^2 s(t,x)$. Whence, the first part of (4.3.34b) reads $\eta_i(t) = \eta_{i-1}(t) + \triangle v_{i-1}(t,x) = \eta_{i-1}(t) + \triangle v(t,x) + (i-1)\triangle^2 s(t,x)$, which gives

$$\begin{aligned}
\eta_i(t) &= \eta_0(t) + \sum_{k=1}^{i}[\eta_k(t) - \eta_{k-1}(t)] \\
&= \lambda(t) + \sum_{k=1}^{i}[\triangle v(t,x) + (k-1)\triangle^2 s(t,x)] \\
&= \lambda(t) + i\triangle v(t,x) + \frac{1}{2}i(i-1)\triangle^2 s(t,x) \\
&= \lambda(t) + iv'(t,x) + \frac{1}{2}i(i-1)s''(t,x).
\end{aligned} \tag{4.3.36}$$

Note that apparently if $\eta_i(t) = 0$ in equation (4.3.33), then $z_i(t, x)$=const. is a solution of equation (4.3.33). Note also that $z_i(t, x) = \triangle^i y(t, x)$. That means that when $\lambda(t) = \lambda_i(t) = -iv'(t, x) - \frac{1}{2}i(i-1)s''(t, x)$, equation (4.3.30) has a solution $y(t, x) = y_i(t, x)$ which is a polynomial in x of degree exactly i provided that $\eta_k \neq 0$ for $k = 0, 1, \cdots, i-1$. Indeed, the equation

$$s(t, x)\triangle\nabla z_k(t, x) + v_k(t, x)\triangle z_k(t, x) + \eta_k(t)z_k(t, x) = 0$$

can be rephrased as

$$z_k(t, x) = -\frac{1}{\eta_k}[s(t, x)\nabla z_{k+1}(t, x) + v_k(t, x)z_{k+1}(t, x)]$$

which clearly implies that if $z_{k+1}(t, x)$ is a polynomial in x then $z_k(t, x)$ is a polynomial in x as well when $\eta_k \neq 0$.

In order to obtain explicit solution $y_i(t, x)$, exploiting functions $\rho(t, x)$ and $\rho_k(t, x)$ satisfying

$$\triangle(s(t, x)\rho(t, x)) = v(t, x)\rho(t, x), \tag{4.3.37}$$

$$\triangle(s(t, x)\rho_k(t, x)) = v_k(t, x)\rho_k(t, x), \tag{4.3.38}$$

the equations (4.3.30) and (4.3.33) are written in a self-adjoint form

$$\triangle(s(t, x)\rho(t, x)\nabla y(t, x)) + \lambda(t)\rho(t, x)y(t, x) = 0, \tag{4.3.39}$$

$$\triangle(s(t, x)\rho_k(t, x)\nabla z_k(t, x)) + \eta_k(t)\rho_k(t, x)z_k(t, x) = 0. \tag{4.3.40}$$

Let us find out $\rho_k(t, x)$. It follows from (4.3.38) that

$$\frac{s(t, x+1)\rho_k(t, x+1)}{\rho_k(t, x)} = v_k(t, x) + s(t, x). \tag{4.3.41}$$

By virtue of (4.3.35), we have

$$\frac{s(t, x+1)\rho_k(t, x+1)}{\rho_k(t, x)} = \frac{s(t, x+2)\rho_{k-1}(t, x+2)}{\rho_{k-1}(t, x+1)}, \tag{4.3.42}$$

which is equivalent to

$$\frac{\rho_k(t, x+1)}{s(t, x+2)\rho_{k-1}(t, x+2)} = \frac{\rho_k(t, x)}{s(t, x+1)\rho_{k-1}(t, x+1)} := c_k(t, x)$$

where $c_k(t,x)$ is any periodic function in x with period 1. Since we only need to find out a particular $\rho_k(t,x)$ in (4.3.38), we may take $c_k(t,x) = 1$. Thus, $\rho_k(t,x) = s(t,x+1)\rho_{k-1}(t,x+1)$, which entails

$$\rho_k(t,x) = \rho(t,x+k)\prod_{j=1}^{k} s(t,x+j). \tag{4.3.43}$$

With the help of (4.3.43), equation (4.3.40) is rephrased as

$$\begin{aligned}\rho_k(t,x)z_k(t,x) =& -\frac{1}{\eta_k(t)}\triangle(s(t,x)\rho_k(t,x)\nabla z_k(t,x))\\ =& -\frac{1}{\eta_k(t)}\nabla(s(t,x+1)\rho_k(t,x+1)\triangle z_k(t,x))\\ =& -\frac{1}{\eta_k(t)}\nabla(\rho_{k+1}(t,x)z_{k+1}(t,x)).\end{aligned}$$

For $k < i$, we obtain successively

$$\begin{aligned}\rho_k(t,x)z_k(t,x) =& -\frac{1}{\eta_k(t)}\nabla(\rho_{k+1}(t,x)z_{k+1}(t,x))\\ =&\frac{(-1)^2}{\eta_k(t)\eta_{k+1}(t)}\nabla^2(\rho_{k+2}(t,x)z_{k+2}(t,x))\\ =&\cdots\\ =&\frac{A_k}{A_i}\nabla^{i-k}(\rho_i(t,x)z_i(t,x)),\end{aligned} \tag{4.3.44}$$

where

$$A_i = (-1)^i\prod_{j=0}^{i-1}\eta_j(t), \qquad A_0 = 1. \tag{4.3.45}$$

Noting that $z_i(t,x) = \triangle^i y_i(t,x)$=const. and denoting $z_{mi}(t,x) = \triangle^m y_i(t,x)$, we have

$$z_{mi}(t,x) = \triangle^m y_i(t,x) = \frac{A_{mi}B_i}{\rho_m(t,x)}\nabla^{i-m}[\rho_i(t,x)], \tag{4.3.46}$$

where

$$\begin{aligned}&A_{mi} = A_m(\lambda)|_{\lambda=\lambda_i} = \frac{i!}{(i-m)!}\prod_{k=0}^{m-1}\left(v'(t,x)+\frac{k+i-1}{2}s''(t,x)\right),\\ &A_{0i} = 1\\ &B_i = \frac{1}{A_{ii}}\triangle^i y_i(t,x) = \frac{1}{A_{ii}}y_i^{(i)}(t,x).\end{aligned} \tag{4.3.47}$$

Particularly, an explicit expression, Rodrigues formula, of $y_i(x,t)$ is

$$y_i(t,x) = \frac{B_i}{\rho(t,x)} \nabla^i [\rho_i(t,x)]. \tag{4.3.48}$$

Apparently, (4.3.48) is equivalent to

$$y_i(t,x) = \frac{B_i}{\rho(t,x)} \triangle^i [\rho_i(t,x-i)] = \frac{B_i}{\rho(t,x)} \triangle^i \left[\rho(t,x) \prod_{k=0}^{i-1} s(t,x-k) \right]. \tag{4.3.49}$$

Lemma 4.3.5. *Given that $\rho(t,x) > 0$ satisfies $\triangle(s(t,x)\rho(t,x)) = v(t,x)\rho(t,x)$ and the boundary conditions for any $t > 0$,*

$$s(t,x)\rho(t,x)x^k|_{x=a,b} = 0, \qquad k = 0, 1, \cdots, \tag{4.3.50}$$

where a and b are left and right endpoints of the support of $\rho(t,x)$ relative to x, the polynomial solutions $y_i(t,x)$ of the difference equation (4.3.30) *are orthogonal on* $[a, b-1]$ *with weight $\rho(t,x)$*

$$\sum_{x_j=a}^{b-1} y_m(t,x_j) y_i(t,x_j) \rho(t,x_j) = \delta_{mi} d_i^2(t). \tag{4.3.51}$$

Similarly, $\triangle^k y_i(t,x)$ are orthogonal with respect to $\rho_k(t,x)$:

$$\sum_{x_j=a}^{b-k} \triangle^k y_m(t,x_j) \triangle^k y_j(t,x_j) \rho_k(t,x_j) = \delta_{mi} d_{ki}^2(t). \tag{4.3.52}$$

The orthogonal polynomial system $\{y_i(t,x)\}$ with $\rho(t,x)$ satisfying conditions in Lemma 4.3.5 is called *classic orthogonal polynomial system* of discrete variable.

Proof. The equations for $y_i(t,x)$ and $y_m(t,x)$ in self-adjoint form are

$$\triangle(s(t,x)\rho(t,x)\nabla y_i(t,x)) + \lambda_i(t)\rho(t,x)y_i(t,x) = 0 \tag{4.3.53}$$

$$\triangle(s(t,x)\rho(t,x)\nabla y_m(t,x)) + \lambda_m(t)\rho(t,x)y_m(t,x) = 0. \tag{4.3.54}$$

Multiply (4.3.53) by $y_m(t,x)$ and (4.3.54) by $y_i(t,x)$, subtract the second from the first. We have

$$\begin{aligned}
&(\lambda_m(t) - \lambda_i(t))\rho(t,x)y_m(t,x)y_i(t,x) \\
=& y_m(t,x)\triangle(s(t,x)\rho(t,x)\nabla y_i(t,x)) - y_i(t,x)\triangle(s(t,x)\rho(t,x)\nabla y_m(t,x)) \\
=& \triangle(s(t,x)\rho(t,x)y_m(t,x)\nabla y_i(t,x)) - s(t,x+1)\rho(t,x+1)\nabla y_i(t,x+1)\triangle y_m(t,x)
\end{aligned}$$

$$
\begin{aligned}
&-\triangle(s(t,x)\rho(t,x)y_i(t,x)\nabla y_m(t,x)) + s(t,x+1)\rho(t,x+1)\nabla y_m(t,x+1)\triangle y_i(t,x)\\
=&\triangle[s(t,x)\rho(t,x)(y_m(t,x)\nabla y_i(t,x) - y_i(t,x)\nabla y_m(t,x))],
\end{aligned}
$$

on account of (4.3.29).

If we put $x = x_j$ and sum them over j

$$
\begin{aligned}
&(\lambda_m(t) - \lambda_i(t))\sum_j y_m(t,x_j)y_i(t,x_j)\rho(t,x_j)\\
=&s(t,x)\rho(t,x)(y_m(t,x)\nabla y_n(t,x) - y_n(t,x)\nabla y_m(t,x))|_a^b = 0
\end{aligned}
$$

by virtue of the boundary condition. Hence, (4.3.51) is valid.

We are now in a position to demonstrate (4.3.52). We use induction. Notice that $\triangle y_i(t,x)$ is a solution of difference equation which can be written in self-adjoint form with $\rho_1(t,x) = s(t,x+1)\rho(t,x+1) = (v(t,x)+s(t,x))\rho(t,x)$ by the condition that $\rho(t,x)$ satisfies.

Because $\rho(t,x)$ satisfies the boundary condition (4.3.50), $\rho_1(t,x)$ satisfies a similar condition, viz.,

$$
s(t,x)\rho_1(t,x)x^k|_{x=a,b} = 0, \qquad k = 0,1,\cdots
$$

Whence the polynomials $\triangle y_i(t,x)$ have orthogonality

$$
\sum_{x_j=a}^{b-2}\triangle y_m(t,x_j)\triangle y_i(t,x_j)\rho_1(t,x_j) = \delta_{mi}d_{1i}^2(t).
$$

Palpably, we have the relationship of orthogonality for $\triangle^k y_i(t,x)$,

$$
\sum_{x_j=a}^{b-k-1}\triangle^k y_m(t,x_j)\triangle^k y_i(t,x_j)\rho_k(t,x_j) = \delta_{mi}d_{ki}^2(t).
$$

□

Example 4.3

Let $Z(t)$ be a Poisson process with intensity μ. Then, $\rho(t,x) = e^{-\mu t}\frac{(\mu t)^x}{x!}$, $x = 0,1,2,\cdots$. In this example, we are to find out the difference equation of hypergeometric type that polynomials orthogonal with respect to $\rho(t,x)$ satisfy.

From the condition $\triangle(s(t,x)\rho(t,x)) = v(t,x)\rho(t,x)$, it follows that $s(t,x+1)\rho(t,x+1) - s(t,x)\rho(t,x) = v(t,x)\rho(t,x)$, which is

$$
s(t,x+1)\mu t = [s(t,x)+v(t,x)](x+1). \tag{4.3.55}
$$

Obviously, $s(t,x)$ could not take a quadratic form. Suppose therefore that $s(t,x) = g(t)x + h(t)$ and $v(t,x) = g_1(t)x + h_1(t)$. Plugging these representations into (4.3.55) yields

$$\mu t g(t)x + [g(t)+h(t)]\mu t = [g(t)+g_1(t)]x^2 + [h(t)+h_1(t)+g(t)+g_1(t)]x + h(t) + h_1(t).$$

Equating the coefficients of powers of x, we have

$$\begin{aligned} g(t)+g_1(t) &= 0, \\ h(t)+h_1(t)+g(t)+g_1(t) &= \mu t g(t), \\ \mu t[g(t)+h(t)] &= h(t)+h_1(t). \end{aligned}$$

Thus, $g_1(t) = -g(t)$, $h(t) = 0$ and $h_1(t) = \mu t g(t)$. Hence, $\lambda = \lambda_i(t) = -nv'(t,x) = ig(t)$. Therefore, the difference equation of hypergeometric type is

$$x\triangle\nabla y(t,x) + (\mu t - x)\triangle y(t,x) + iy(t,x) = 0.$$

□

In order to obtain the squared norm $d_i^2(t)$, we first establish the connection between $d_{ki}^2(t)$ and $d_{k+1,i}^2(t)$ where

$$d_{ki}^2(t) = \sum_{x_j=a}^{b-k-1} z_{ki}^2(t,x_j)\rho_k(t,x_j), \quad d_{0i}^2(t) = d_i^2(t), \quad z_{ki}(t,x) = \triangle^k y_i(t,x).$$

The self-adjoint equation for $z_{ki}(t,x)$ is

$$\triangle(s(t,x)\rho_k(t,x)\nabla z_{ki}(t,x)) + \eta_{ki}(t)\rho_k(t,x)z_{ki}(t,x) = 0,$$

where $\eta_{ki}(t) = \lambda_i(t) - \lambda_k(t)$.

Multiply by $z_{ki}(t,x)$, sum up over the values $x = x_j$ for which $a \le x_j \le b-k-1$:

$$\sum_j z_{ki}(t,x_j)\triangle(s(t,x_j)\rho_k(t,x_j)\nabla z_{ki}(t,x_j)) + \eta_{ki}(t)d_{ki}^2(t) = 0$$

Note that $\triangle z_{ki}(t,x) = z_{k+1,i}(t,x)$, $s(t,x+1)\rho_k(t,x+1) = \rho_{k+1}(t,x)$. Using difference identity for product in (4.3.29) gives

$$\begin{aligned} &\sum_j z_{ki}(t,x_j)\triangle(s(t,x_j)\rho_k(t,x_j)\nabla z_{ki}(t,x_j)) \\ =&\sum_j [\triangle(s(t,x_j)\rho_k(t,x_j)z_{ki}(t,x_j)\nabla z_{ki}(t,x_j)) \end{aligned}$$

$$
\begin{aligned}
&\quad - s(t, x_j+1)\rho_k(t, x_j+1)\nabla z_{ki}(t, x_j+1)\triangle z_{ki}(t, x_j)] \\
=&\sum_j \left[\triangle(s(t, x_j)\rho_k(t, x_j)z_{ki}(t, x_j)\nabla z_{ki}(t, x_j)) - \rho_{k+1}(t, x_j)z^2_{k+1,i}(t, x_j)\right] \\
=&s(t, x)\rho_k(t, x)z_{ki}(t, x)\nabla z_{ki}(t, x)|_a^{b-k} - d^2_{k+1,i}(t) \\
=&- d^2_{k+1,i}(t).
\end{aligned}
$$

We thus have

$$
d^2_{ki}(t) = \frac{1}{\eta_{ki}} d^2_{k+1,i}(t). \tag{4.3.56}
$$

And iterating the formula gives

$$
\begin{aligned}
d^2_i(t) =& d^2_{0i}(t) = \frac{1}{\eta_{0i}} d^2_{1,i}(t) = \frac{1}{\eta_{0i}\eta_{1i}} d^2_{2,i}(t) = \cdots \\
=& \frac{1}{\prod_{k=0}^{i-1}\eta_{ki}} d^2_{ii}(t) = \frac{1}{\prod_{k=0}^{i-1}\eta_{ki}} z^2_{ii}(x, t)S_i(t) \\
=& (-1)^i A_{ii} B_i^2 S_i(t)
\end{aligned} \tag{4.3.57}
$$

where $S_i(t) = \sum_{x_j=a}^{b-i-1} \rho_i(x_j, t)$.

4.3.3 Three important remarks

Remark 4.3.2. The interval and the weight of orthogonality determine the polynomial uniquely, up to a constant multiple. See Nikiforov and Uvarov (1988, p.34). We therefore have $q_i(t, x) = a_i(t)y_i(t, x)$. Palpably, $a_i(t)$ can be determined by orthogonality:

$$
\int_I q_i^2(t, x)d\Psi_t(x) = a_i^2(t)\int_I y_i^2(t, x)d\Psi_t(x) \qquad \Rightarrow \quad \widetilde{d}_i^2(t) = a_i^2(t)d_i^2(t),
$$

where $\widetilde{d}_i^2(t)$ and $d_i^2(t)$ are squared norms of $q_i(t, x)$ and $y_i(t, x)$ respectively. Hence, $a_i(t) = \widetilde{d}_i(t)/d_i(t)$ and

$$
q_i(t, x) = \frac{\widetilde{d}_i(t)}{d_i(t)} y_i(t, x). \tag{4.3.58}
$$

Remark 4.3.3. It follows from (4.3.26) and (4.3.56) that in both continuous and discrete situations, $\eta_{ki} > 0$ for all $i > 0$ and $k = 0, 1, \cdots, i-1$. Observe that $\eta_{0i} = \lambda_i = -iv'(t, x) - \frac{1}{2}i(i-1)s''(t, x) > 0$ entails that $v'(t, x) + \frac{1}{2}(i-1)s''(t, x) < 0$ for all $i > 0$.

In what follows we discuss the possible signs of $v'(t, x)$ and $s''(t, x)$. (1) $s''(t, x) > 0$. No matter what the sign of $v'(t, x)$ is, because of the arbitrariness of i, $\lambda_i < 0$ for i being large. (2) $v'(t, x) \geq 0$. It follows that $\lambda_1 = -v'(t, x) \leq 0$. Thus, the above two cases are impossible.

The possible cases are: (3) $v'(t,x) < 0$ and $s''(t,x) = 0$. Example 4.1, 4.2 and 4.3 all belong to this situation. (4) $v'(t,x) < 0$ and $s''(t,x) < 0$. This situation corresponds to Legendre polynomials, where, after variable changing, $s(t,x)$ becomes $a^2 - x^2$ implying that the support of $\rho(t,x)$ is $[-a,a]$ with fixed a. However, this scenario is beyond the scope of this paper since we are interested in the scenario that $Z(t)$ assumes values on $\mathbb{R}$, $\mathbb{R}^+$, or $\mathbb{N}$. Therefore, in the sequel, our development will rely on the case 3, viz., $v'(t,x) < 0$ and $s''(t,x) = 0$.

Remark 4.3.4. Sometimes we may need the asymptotic property of orthogonal polynomials. In the sequel, the following inequalities for Hermite polynomials and Laguerre polynomials are useful which can be found on Nikiforov and Uvarov (1988, p.54),

$$\frac{1}{d_i}|H_i(x)| \le C_1 i^{-\frac{1}{4}} \quad \text{and} \quad \frac{1}{d_i}|L_i^{(\alpha)}(x)| \le C_2 i^{-\frac{1}{4}},$$

where d_i are the norm of Hermite and Laguerre polynomials in different inequalities respectively; C_1 and C_2 only depend on fixed x.

In view of the relation $C_i(\mu,x) = C_x(\mu,i) = x!L_x^{(i-x)}(\mu)$, the above inequality is true for Charlier polynomials as well. Thus, we may assert that within the ambit of our study, all classical orthonormal polynomials $Q_i(t,x)$ satisfy that $|Q_i(t,x)| \le Ci^{-\frac{1}{4}}$ for fixed t and x where C is independent of i.

4.4 Orthogonal expansion of homogeneous functionals of the Lévy process

Let $(Z(t), t \ge 0)$ be a Lévy process with distribution function $\Psi_t(x)$, associated with a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose that the density or probability distribution $\rho(t,x)$ of $Z(t)$ satisfies the boundary condition in the preceding section, hence, there exists a polynomial system $y_i(t,x)$ orthogonal with respect to $\rho(t,x)$. Let

$$Q_i(t,x) = \frac{1}{d_i(t)} y_i(t,x) \tag{4.4.1}$$

be the orthonormal polynomial system with the measure of orthogonality $\Psi_t(x)$ where $d_i^2(t)$ is the squared norm of $y_i(t,x)$. In such a situation, we shall say that the Lévy process $Z(t)$ admits a *classical orthonormal polynomial system* $Q_i(t,x)$.

Let I be the support of the density or probability distribution $\rho(t, x)$ of $Z(t)$. Consider a function space for $t > 0$

$$L^2(I, d\Psi_t(x)) = \{f(x) : \int_I f^2(x) d\Psi_t(x) < \infty\}. \tag{4.4.2}$$

According to Billingsley (1995, p.249), $L^2(I, d\Psi_t(x))$ is a Hilbert space. Given that $\Psi_t(x)$ satisfies a sufficient condition, viz., there exists a constant $c > 0$, such that

$$\int e^{c|x|} d\Psi_t(x) < \infty, \tag{4.4.3}$$

the system $Q_i(t, x)$ is not only orthonormal but also complete in $L^2(I, d\Psi_t(x))$. See Nikiforov and Uvarov (1988, p.57). Indeed, there are many Lévy processes satisfying this sufficient condition, for example, Laguerre polynomial system associated to Gamma process satisfies (4.4.3) with $c < 1$, and both Hermite polynomial system with the density of Brownian motion as its weight and Charlier polynomial system orthogonal to the probability distribution function of the Poisson process satisfy it with any $c > 0$.

Additionally, in the Hilbert space $L^2(I, d\Psi_t(x))$, the scalar product and the induced norm are defined as follows

$$(f, g) = \int f(x) g(x) d\Psi_t(x), \qquad \|f\| = \sqrt{(f, f)}.$$

Construct a mapping for $f(x) \in L^2(I, d\Psi_t(x))$

$$\mathcal{T} : \quad f \mapsto f(Z(t)).$$

Since $E[f^2(Z(t))] = \int_I f^2(x) d\Psi_t(x) < \infty$, $f(Z(t))$ is an element of $L^2(\Omega)$, a collection of all random variables with finite second moment. Accordingly, the image of $\mathcal{T}$, denoted by Θ, is a subset of $L^2(\Omega)$. Hence, all elements in Θ admit the norms and scalar products, as the elements in $L^2(\Omega)$, namely, $\langle f(Z(t)), g(Z(t)) \rangle_\Theta = E[f(Z(t)) g(Z(t))]$ and $\|f(Z(t))\|_\Theta = \sqrt{\langle f(Z(t)), f(Z(t)) \rangle}$, the induced norm. The following lemma gives the properties of $\mathcal{T}$ and Θ.

Lemma 4.4.1. *The mapping $\mathcal{T}$ has the following properties:*

(1) $\mathcal{T}$ is linear;

(2) $\mathcal{T}$ is a one-to-one mapping from $L^2(I, d\Psi_t(x))$ to Θ;

(3) $\mathcal{T}$ is an isomorphism.

Proof. (1) Straightforward verification. (2) For any functions $f, g \in L^2(I, d\Psi_t(x))$, we have,

$$\begin{aligned}\langle \mathcal{T}(f), \mathcal{T}(g)\rangle_\Theta &= \langle f(Z(t)), g(Z(t))\rangle = E[f(Z(t))g(Z(t))] \\ &= \int_I f(x)g(x)d\Psi_t(x) = (f, g)_{L^2(I, d\Psi_t(x))}.\end{aligned}$$

That means the transformation is inner product preserving. Therefore, $f \neq g \Leftrightarrow \mathcal{T}(f) \neq \mathcal{T}(g)$. Thus $\mathcal{T}$ is one-one. (3) Since $\mathcal{T}$ is linear and $\|\mathcal{T}(f)\| = \|f\|$ for $f \in L^2(I, d\Psi_t(x))$, $\mathcal{T}$ is isomorphism. □

Lemma 4.4.2. *Θ is a closed subspace of $L^2(\Omega)$, hence it is a Hilbert space.*

Proof. Apparently, Θ is a linear space due to linearity of $\mathcal{T}$. Because $\mathcal{T}$ is one-to-one and inner product preserving, $\{\xi_n\}$ is a Cauchy sequence in Θ if and only if there is a unique sequence $\{f_n\}$ in $L^2(I, d\Psi_t(x))$ such that $\mathcal{T}(f_n) = \xi_n$, $n = 0, 1, 2, \ldots$, and $\{f_n(x)\}$ is a Cauchy sequence in $L^2(I, d\Psi_t(x))$. Therefore, due to the completeness of $L^2(I, d\Psi_t(x))$, Θ is a closed subspace of $L^2(\Omega)$. Hence it is a Hilbert space. □

Lemma 4.4.3. *If $\{p_i(x)\}_{i=0}^\infty$ is any orthonormal basis in $L^2(I, d\Psi_t(x))$, then $\{\mathcal{T}(p_i)\}_{n=0}^\infty$ is an orthonormal basis in Θ. Particularly, $\{\mathcal{T}(Q_i(t, x))\}_{i=0}^\infty = \{Q_i(t, Z(t))\}_{i=0}^\infty$, $t > 0$, is an orthonormal basis in Θ.*

Proof. By virtue of the properties of $\mathcal{T}$ that $\mathcal{T}$ is one-to-one, inner product preserving, it is valid. □

The following theorem is a consequence of the above lemmas.

Theorem 4.4.1. *Suppose that Lévy process $(Z(t), t > 0)$ admits a classical orthonormal polynomial system $Q_i(t, x)$ with weight $\rho(t, x)$. For any element $f(Z(t)) \in \Theta$, it has a Fourier series expansion*

$$f(Z(t)) = \sum_{i=0}^\infty c_i(t, f)Q_i(t, Z(t)), \tag{4.4.4}$$

where $c_i(t, f) = \langle f(Z(t)), Q_i(t, Z(t))\rangle_\Theta$.

Proof. In view of the facts that Θ is a Hilbert space and $\{Q_i(t, Z(t))\}$ is an orthonormal basis in Θ, it follows. □

Note that from Parseval equality it follows that $\|f(Z(t))\|_\Theta^2 = \sum_{i=0}^\infty c_i^2(t,f)$ for all $t > 0$.

Example 4.4

We are about to show some examples of expansion of Lévy process functionals.

1. $f(Z(t)) = a_0 + a_1 Z(t) + \cdots + a_k Z(t)^k$. Obviously, $f(Z(t)) = c_0(t)Q_0(t, Z(t)) + c_1(t)Q_1(t, Z(t)) + \cdots + c_k(t)Q_k(t, Z(t))$. The coefficients $c_i(t) = E[f(Z(t))Q_i(t, Z(t))]$, $i = 0, 1, \cdots, k$.

We have two particular examples for $Z(t) = B(t)$ and $Z(t) = N(t)$:

$$\begin{aligned} B^5(t) =& 15t^{5/2}h_1(t, B(t)) + 10\sqrt{6}t^{5/2}h_3(t, B(t)) + 2\sqrt{30}t^{5/2}h_5(t, B(t)), \\ N^2(t) =& \mu t(1+\mu t)\mathscr{C}_0(\mu t; N(t)) - \sqrt{\mu t}(2\mu t + 1)\mathscr{C}_1(\mu t; N(t)) + \sqrt{2}\mu t\mathscr{C}_2(\mu t; N(t)), \end{aligned}$$

where $h_i(t, B(t)) = \frac{1}{\sqrt{i!}}H_i(B(t)/\sqrt{t})$ with $H_i(\cdot)$ being Hermite polynomials, μ is the intensity of Poisson process $N(t)$ and $\mathscr{C}_i(t; N(t)) = \frac{\sqrt{\mu t}^i}{\sqrt{i!}}c_i(\mu t; n)$ with $c_i(\cdot;\cdot)$ being Charlier polynomials.

2. $f(Z(t)) = \exp(Z(t))$ and $g(Z(t)) = \exp(-Z(t))$. It follows from (4.3.2) that

$$\sum_{j=0}^\infty Q_j(t,x)\widetilde{d}_j(t)\frac{z^j}{j!} = (\pi(z))^t \exp(xu(z)),$$

where $\widetilde{d}_j(t)$ are norm of $q_j(t,x)$ in (4.3.2).

Note that $\pi(z) = [\phi(-iu(z))]^{-1}$ in which $\phi(\theta) = E[e^{i\theta Z(1)}]$ and $u(z)$ is the inverse function of $\tau(\cdot)$ determined by (4.3.1) and i is the imaginary unit.

Let $u(z) = 1$. Then $\pi(z) = [\phi(-i)]^{-1}$ and therefore

$$\exp(Z(t)) = [\phi(-i)]^t \sum_{j=0}^\infty Q_j(t, Z(t))\widetilde{d}_j(t)\frac{\tau(1)^j}{j!}.$$

Similarly,

$$\exp(-Z(t)) = [\phi(i)]^t \sum_{j=0}^\infty \widetilde{d}_j(t)\frac{\tau(-1)^j}{j!}Q_j(t, Z(t))$$

3. $f(Z(t)) = \cos Z(t)$ and $g(Z(t)) = \sin Z(t)$.

First, as above, we may obtain the expansion of $\exp(iZ(t))$ and $\exp(-iZ(t))$ by letting $u(z) = i$ and $u(z) = -i$ respectively, where i is the imaginary unit,

$$\exp(iZ(t)) \quad = \quad [\phi(1)]^t \sum_{j=0}^\infty \widetilde{d}_j(t)\frac{\tau(i)^j}{j!}Q_j(t, Z(t)),$$

$$\exp(-iZ(t)) = [\phi(-1)]^t \sum_{j=0}^{\infty} \widetilde{d}_j(t) \frac{\tau(-i)^j}{j!} Q_j(t, Z(t)).$$

It follows from the formulae $e^{ix} = \cos x + i \sin x$ and $e^{-ix} = \cos x - i \sin x$ that

$$\cos Z(t) = \frac{1}{2}[\exp(iZ(t)) + \exp(-iZ(t))] = \sum_{j=0}^{\infty} b_j(t) Q_j(t, Z(t)),$$
$$\sin Z(t) = \frac{1}{2i}[\exp(iZ(t)) - \exp(-iZ(t))] = \sum_{j=0}^{\infty} c_j(t) Q_j(t, Z(t)),$$

where

$$b_j(t) = \frac{1}{j!} \widetilde{d}_j(t) \frac{1}{2} \left\{ \tau(i)^j [\phi(1)]^t + \tau(-i)^j [\phi(-1)]^t \right\},$$
$$c_j(t) = \frac{1}{j!} \widetilde{d}_j(t) \frac{1}{2i} \left\{ \tau(i)^j [\phi(1)]^t - \tau(-i)^j [\phi(-1)]^t \right\}.$$

□

It is noteworthy to point out that when Lévy process $Z(t)$ is specified as Brownian motion, $\phi(\theta) = e^{-\frac{\theta^2}{2}}$ and $\tau(s) = s$. Hence $E(e^{-iZ(1)}) = E(e^{iZ(1)}) = e^{-\frac{1}{2}}$. In addition, $\widetilde{d}_j(t) = \sqrt{t}^j \sqrt{j!}$ (see Schoutens, 2000). If we supersede these relations into the expansions in last example, the basis $Q_i(t, Z(t))$ is substituted by $h_i(t, B(t))$, then the expansions of the $f(Z(t))$, $e^{Z(t)}$, $e^{-Z(t)}$, $\sin Z(t)$ and $\cos Z(t)$ would reduce to the expansions of $f(B(t))$, $e^{B(t)}$, $e^{-B(t)}$, $\sin B(t)$ and $\cos B(t)$ in the examples of Chapter 2. The coincidence in a sense implies the correctness of the method in Theorem 4.4.1. For the convenience of the following development, we coin the notations as follows.

Notations: We continuously use the notations in §2, such as, $\rho(t, x)$, $\rho_k(t, x)$, $\lambda_n(t)$, $\eta_{mn}(t)$, $d_n(t)$, $d_{mn}(t)$; Meanwhile, as Lemma 4.3.3 (Lemma 4.3.5) indicated, since for fixed integer r, $Q_{ri}(t, x) = \frac{1}{d_{ri}(t)} y_i^{(r)}(t, x)$ (or $Q_{ri}(t, x) = \frac{1}{d_{ri}(t)} \triangle^r y_i(t, x)$ in the discrete case) also form an orthonormal system, we may in what follows expand the function $f^{(r)}(Z(t))$ (or $\triangle^r f(Z(t))$) in terms of $Q_{ri}(t, Z(t))$. In this case, for the sake of convenience we use notations $c_i(t, D^r f)$ to stand for the corresponding coefficients without specifying the basis, or the notations allude the basis is $Q_{ri}(t, Z(t))$. In other cases, we would specify what basis we are using.

Moreover, because $v(t, x)$, as discussed in Remark 4.3.3, is a polynomial in x of degree exactly 1 and $v'(t, x) < 0$, it is notationally convenient to signify in the rest of the paper $v(t) := -\frac{1}{v'_x(t,x)} > 0$.

Since there are always two operations of differentiation and difference to be dealt with, we unify them as D, viz., D stands for either differentiation or difference operation, which would not be ambiguous in context. □

Let k be a truncation parameter for i. The truncation series of (4.4.4) is defined as

$$f_k(Z(t)) = \sum_{i=0}^{k} c_i(t,f)Q_i(t,Z(t)).$$

Theorem 4.4.2. *Let $(Z(t), t > 0)$ be a Lévy process satisfying conditions in Theorem 4.4.1. Suppose further that $D^h f(x) \in L^2(I, \rho_h(t,x))$ for $h = 0, 1, \cdots, r$. Then*

$$\|f(Z(t)) - f_k(Z(t))\|_\Theta^2 \leq \frac{1}{k^r} R_k^2(t, D^r f) \tag{4.4.5}$$

where $R_k^2(t, D^r f) = (1 + o(1))[v(t)]^r \sum_{i=k+1}^{\infty} c_i^2(t, D^r f) \to 0$ as $k \to \infty$ for every $t > 0$.

Proof. We begin with the calculation of the coefficients $c_i(t,f)$. If $Z(t)$ is a continuous variable with density function $\rho(t,x)$, the polynomials $y_i(t,x)$ orthogonal with respect to $\rho(t,x)$ satisfy the differential equation

$$s(t,x)y_i''(t,x) + v(t,x)y_i'(t,x) + \lambda_i(t)y_i(t,x) = 0$$

where $s(t,x) > 0$, $v(t,x)$ and $\rho(t,x)$ satisfy conditions (4.3.16) and $\lambda_i(t) = -iv'(t,x)$.

The self-adjoint form of the equation is

$$(s(t,x)\rho(t,x)y_i'(t,x))' + \lambda_i(t)\rho(t,x)y_i(t,x) = 0.$$

Multiplying by $f(x)$, integrating by part on (a,b), we have

$$\begin{aligned} &f(x)s(t,x)\rho(t,x)y_i'(t,x)|_a^b - \int_a^b s(t,x)\rho(t,x)y_i'(t,x)f'(x)dx \\ =& -\lambda_i(t)\int_a^b \rho(t,x)y_i(t,x)f(x)dx. \end{aligned} \tag{4.4.6}$$

Let us prove that $f(x)s(t,x)\rho(t,x)y_i'(t,x)|_a^b = 0$.

Suppose $\lim_{x\to b} f(x)s(t,x)\rho(t,x)y_i'(t,x) = b_i \neq 0$. Then, as $x \to b$,

$$f(x) \approx \frac{b_i}{s(t,x)\rho(t,x)y_i'(t,x)} \;\Rightarrow\; f^2(x)\rho(t,x) \sim \frac{b_i^2}{s^2(t,x)\rho(t,x)[y_i'(t,x)]^2}.$$

Because of boundary condition, $f^2(x)\rho(t,x)$ will approach to positive infinity as $x \to b$, which leads to the infiniteness of the integral $\int_a^b f^2(x)\rho(t,x)dx$.

The above discussion applies to the situation where $a = -\infty$ as well.

When a is finite, according to Nikiforov and Uvarov (1988, p.21),

$$s(t,x) \sim x - a, \quad \text{and} \quad \rho(t,x) \sim (x-a)^{\alpha}, \text{ where } \alpha > -1.$$

Hence, when $x \to +a$,

$$f(x) \sim \frac{1}{(x-a)^{1+\alpha}}, \quad \text{and} \quad f^2(x)\rho(t,x) \sim \frac{1}{(x-a)^{2+\alpha}}$$

which implies the infiniteness of $\int_a^b f^2(x)\rho(t,x)dx$.

Thus, the relation (4.4.6) reduces to

$$\int_a^b \rho_1(t,x)y_i'(t,x)f'(x)dx = \lambda_i(t)\int_a^b \rho(t,x)y_i(t,x)f(x)dx$$

which is exactly the following relationship: $d_{1i}(t)c_i(t,f') = \lambda_i(t)d_i(t)c_i(t,f)$, or equivalently

$$c_i(t,f) = \frac{d_{1i}(t)}{\lambda_i(t)d_i(t)}c_i(t,f').$$

We can iterate the relation until r-th derivative,

$$\begin{aligned} c_i(t,f) =& \frac{d_{ri}(t)}{d_i(t)\lambda_i(t)\eta_{1i}(t)\cdots\eta_{r-1,i}(t)}c_i(t,f^{(r)}) \\ =& \sqrt{v(t)}^r\sqrt{\frac{(i-r)!}{i!}}c_i(t,f^{(r)}) \end{aligned} \tag{4.4.7}$$

where we have used the relationship (4.3.28) with $\eta_{ji}(t) = \lambda_i(t) - \lambda_j(t) = -v'(t,x)(i-j) = \frac{1}{v(t)}(i-j)$.

If $Z(t)$ is a discrete variable for each $t > 0$, $\rho(t,x)$ is the probability distribution of $Z(t)$. The polynomials $y_i(t,x)$ orthogonal with respect to $\rho(t,x)$ satisfy the following difference equation

$$s(t,x)\triangle\nabla y_i(t,x) + v(t,x)\triangle y_i(t,x) + \lambda_i(t)y_i(t,x) = 0$$

where $s(t,x) > 0$, $s''(t,x) = 0$ and $v'(t,x) < 0$. The self-adjoint form of the difference equation is

$$\triangle(s(t,x)\rho(t,x)\nabla y_i(t,x)) + \lambda_i(t)\rho(t,x)y_i(t,x) = 0.$$

Multiplying by $f(x)$ and summing up over the support of $\rho(t,x)$,

$$\sum_m f(x_m)\triangle(s(t,x_m)\rho(t,x_m)\nabla y_i(t,x_m)) = -\lambda_i(t)\sum_m f(x_m)\rho(t,x_m)y_i(t,x_m).$$

Summation by parts gives

$$f(x)s(t,x)\rho(t,x)\nabla y_i(t,x)|_a^b - \sum_m s(t,x_{m+1})\rho(t,x_{m+1})\nabla y_i(t,x_{m+1})\triangle f(x_m)$$
$$= -\lambda_i(t)\sum_m f(x_m)\rho(t,x_m)y_i(t,x_m).$$

It is easy to prove that $f(x)s(t,x)\rho(t,x)\nabla y_i(t,x)|_a^b = 0$ similar to the continuous case. Note that $\nabla y_i(t,x_{m+1}) = \triangle y_i(t,x_m)$ and $s(t,x_{m+1})\rho(t,x_{m+1}) = \rho_1(t,x_m)$. Therefore,

$$\sum_m \rho_1(t,x_m)\triangle y_i(t,x_m)\triangle f(x_m) = \lambda_i(t)\sum_m f(x_m)\rho(t,x_m)y_i(t,x_m)$$

which reads $c_i(t,f) = \frac{d_{1i}(t)}{\lambda_i(t)d_i(t)}c_i(t,\triangle f)$. We iterate this relationship and obtain again (4.4.7) with derivative being substituted by difference. Adopting our operator D, we have

$$c_i(t,f) = \sqrt{v(t)}^r\sqrt{\frac{(i-r)!}{i!}}c_i(t,D^r f). \tag{4.4.8}$$

Now we are ready to obtain the result. Using (4.4.8),

$$\begin{aligned}\|f(Z(t)) - f_k(Z(t))\|_\Theta^2 = \sum_{i=k+1}^\infty c_i^2(t,f) = \sum_{i=k+1}^\infty [v(t)]^r\frac{(i-r)!}{i!}c_i^2(t,D^r f)\\ \leq [v(t)]^r\frac{(k-r)!}{k!}\sum_{i=k+1}^\infty c_i^2(t,D^r f) = \frac{1}{k^r}R_k^2(t,D^r f),\end{aligned}$$

where $R_k^2(t,D^r f) = (1+o(1))[v(t)]^r\sum_{i=k+1}^\infty c_i^2(t,D^r f) \to 0$ as $k\to\infty$ for every $t>0$. □

4.5 Orthogonal expansion of time-inhomogeneous functionals of Lévy process

In this section we discuss the expansion of $f(t,Z(t))$ for $t\in[0,T]$ and $t\in[0,\infty)$ where $(Z(t),t\geq 0)$ is a Lévy process. Let $\Psi_t(x)$ be the distribution function of $Z(t)$, and $\rho(t,x)$ be the density or probability distribution function of $Z(t)$ depending on whether $Z(t)$ is continuous or discrete variable. Let I be the support of $\rho(t,x)$. In addition, μ signifies Lebesgue measure on line.

4.5.1 Finite time horizon

Let $t \in [0,T]$. Consider function space

$$L^2(I \times [0,T], \nu) = \left\{ f(t,x) : \int_I f^2(t,x) d\Psi_t(x) < \infty, \text{ for each } t \in [0,T], \right.$$
$$\left. \text{and } \int_0^T \int_I f^2(t,x) d\nu < \infty \right\},$$

where ν is the product measure of probability measure $\Psi_t(x)$ and Lebesgue measure μ.

For the sake of convenience, we abbreviate the notation of the space defined above as $L^2(I \times [0,T])$. Space $L^2(I \times [0,T])$ is actually a conventional L^2 space. Therefore, $L^2(I \times [0,T])$ is a Hilbert space with scalar product

$$(f_1(t,x), f_2(t,x)) = \int_0^T \int_I f_1(t,x) f_2(t,x) d\Psi_t(x) dt.$$

As we know,$\{Q_i(t,x)\}$ defined in last section is an orthonormal basis for $L^2(I, d\Psi_t(x))$ and $\{\varphi_{jT}(t)\}$, with $\varphi_{0T} = \sqrt{\frac{1}{T}}$ and $\varphi_{jT} = \sqrt{\frac{2}{T}} \cos \frac{j\pi t}{T}$ for $j \geq 1$, is an orthonormal basis in $L^2([0,T], \mu)$. Whence, according to Problem 12 of Dudley (2003, p173), $\{Q_i(t,x)\varphi_{jT}(t)\}_{i,j=0}^{\infty}$ is an orthonormal basis in $L^2(I \times [0,T])$.

Construct a mapping $\mathcal{T}$ from $L^2(I \times [0,T])$ to a set of stochastic processes,

$$\mathcal{T} : \quad f(t,x) \mapsto f(t, Z(t)), \quad \text{for } f(t,x) \in L^2(I \times [0,T]).$$

Denote the image of $\mathcal{T}$ by Ξ. Evidently, $\mathcal{T}$ is a linear mapping, so that Ξ is a linear space. Define $\langle f_1(t, Z(t)), f_2(t, Z(t)) \rangle_\Xi = \int_0^T E[f_1(t, Z(t)) f_2(t, Z(t))] dt$, an operation on Ξ. Obviously, $\langle \cdot, \cdot \rangle_\Xi$ is an inner product on Ξ, which can induce a norm as well. Meanwhile, it can be shown that $\mathcal{T}$ and Ξ enjoy the properties in Lemma 4.4.1–4.4.3, hence Ξ is a Hilbert space and $\{Q_i(t, Z(t))\varphi_{jT}(t)\}_{i,j=0}^{\infty}$ is an orthonormal basis in Ξ. The following theorem is easily obtained from Hilbert space theory.

Theorem 4.5.1. *In Ξ, any element $f(t, Z(t))$ admits a Fourier series expansion*

$$f(t, Z(t)) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{ij} Q_i(t, Z(t)) \varphi_{jT}(t), \tag{4.5.1}$$

where $c_{ij} = \langle f(t, Z(t)), Q_i(t, Z(t))\varphi_{jT}(t) \rangle_\Xi$.

Because actually

$$\begin{aligned}c_{ij} =&\langle f(t,Z(t)), Q_i(t,Z(t))\varphi_{jT}(t)\rangle_\Xi = \int_0^T E[f(t,Z(t))Q_i(t,Z(t))]\varphi_{jT}(t)dt\\ :=&\int_0^T c_i(t,f)\varphi_{jT}(t)dt,\end{aligned}$$

where $c_i(t,f) := E[f(t,Z(t))Q_i(t,Z(t))]$, the expansion (4.5.1) can be regarded as two-step expansion, that is, expand $f(t,Z(t))$ first in terms of $\{Q_i(t,Z(t))\}$ obtaining coefficients $c_i(t,f) = E[f(t,Z(t))Q_i(t,Z(t))]$, then expand $c_i(t,f)$ in terms of $\{\varphi_{jT}(t)\}$ on $[0,T]$.

Notice that from Parseval equality it follows that

$$\|f(t,Z(t))\|_\Xi^2 = \sum_{i=0}^\infty\sum_{j=0}^\infty c_{ij}^2 = \sum_{i=0}^\infty \|c_i(t,f)\|_{L^2[0,T]}^2.$$

Given a bundle of truncation parameters k for i and p_i for j's, we define the truncation series of (4.5.1) as follows

$$f_{k,p}(t,Z(t)) = \sum_{i=0}^k\sum_{j=0}^{p_i} c_{ij}Q_i(t,Z(t))\varphi_{jT}(t). \tag{4.5.2}$$

Denote $p_{\min} = \min\{p_1,\cdots,p_k\}$ and $p_{\max} = \max\{p_1,\cdots,p_k\}$ throughout the paper.

Theorem 4.5.2. *Let $(Z(t), t\geq 0)$ be a Lévy process which admits a classical orthonormal polynomial system $\{Q_i(t,x)\}$ defined by (4.4.1). Suppose that functional $f(t,Z(t))\in\Xi$, that $D^h f(t,x)$, $h=1,\cdots,r$, are in the space $L^2(I,\rho_h(t,x))$ for each $t>0$. Moreover, $\sqrt{v(t)}^r D^r f(t,x)\in L^2(I\times[0,T])$. In addition, for each $i\geq 0$, $c_i(t,f)\in C^3[0,T]$ and $\|c_i''(t,f)\|_{L^2[0,T]}$ is uniformly bounded in i. Then,*

$$\|f(t,Z(t)) - f_{k,p}(t,Z(t))\|_\Xi^2 \leq \frac{1}{k^r}R_k^2 + C(k,p)\frac{k}{p_{\min}^4}, \tag{4.5.3}$$

where $R_k^2 = (1+o(1))\sum_{i=k+1}^\infty \left\|c_i\left(t,\sqrt{v(t)}^r D^r f\right)\right\|_{L^2[0,T]}^2 \to 0$ as $k\to\infty$, $C(k,p) = T^4\pi^{-4}\max_{0\leq i\leq k}\sum_{j=p_{\min}+1}^\infty b_j^2(c_i'')$ in which $b_j(c_i'')$ stands for the j-th coefficient in the expansion of $c_i''(t,f)$.

Remark 4.5.1. Basically, the error of approximation $f_{k,p}(t,Z(t))$ to $f(t,Z(t))$ consists of two types because the expansion is of two-step, that is, the first term in the right hand side of (4.5.3) is incurred since we abandon the residue in the first step expansion, while the second term is due to giving up the residues in the second step.

Because for each $i : 0 \le i \le k$, $\sum_{j=p_{\min}+1}^{\infty} b_j^2(c_i'')$ is an infinitesimal when $p_{\min}$ goes to infinity, for fixed k, $C(k,p)$ is an infinitesimal as well. However, when both k and $p_{\min}$ approach infinity, $C(k,p)$ could not be guaranteed to be infinitesimal. One sufficient condition that $C(k,p)$ is bounded is that the norm $\|c_i''(t,f)\|_{L^2[0,T]}$ is uniformly bounded in i for we always have $\sum_{j=p_{\min}+1}^{\infty} b_j^2(c_i'') \le \|c_i''(t,f)\|^2_{L^2[0,T]}$.

Proof. From orthogonality we have

$$\begin{aligned}
&\|f(t,Z(t)) - f_{k,p}(t,Z(t))\|_{\Xi}^2 \\
=&\left\| \sum_{i=0}^{k} \sum_{j=p_i+1}^{\infty} c_{ij} Q_i(t,Z(t))\varphi_{jT}(t) + \sum_{i=k+1}^{\infty} \sum_{j=0}^{\infty} c_{ij} Q_i(t,Z(t))\varphi_{jT}(t) \right\|_{\Xi}^2 \\
=&\sum_{i=0}^{k} \sum_{j=p_i+1}^{\infty} c_{ij}^2 + \sum_{i=k+1}^{\infty} \sum_{j=0}^{\infty} c_{ij}^2.
\end{aligned}$$

Since $c_i(t,f) \in C^3[0,T]$, the expansion of $c_i''(t,f)$ in terms of $\varphi_{jT}(t)$ is convergent uniformly on $[0,T]$ (Davis, 1963, p.142). We thus have $c_{ij} = \left(\frac{T}{j\pi}\right)^2 b_j(c_i'')$ where $b_j(c_i'')$ stands for the j-th coefficient in the expansion of $c''(t,f)$ in terms of $\varphi_{jT}(t)$. Then

$$\begin{aligned}
\sum_{i=0}^{k} \sum_{j=p_i+1}^{\infty} c_{ij}^2 =& \sum_{i=0}^{k} \sum_{j=p_i+1}^{\infty} \left(\frac{T}{j\pi}\right)^4 b_j^2(c_i'') \le \left(\frac{T}{\pi}\right)^4 \sum_{i=0}^{k} \frac{1}{p_i^4} \sum_{j=p_i+1}^{\infty} b_j^2(c_i'') \\
\le& \frac{T^4}{\pi^4} \frac{k}{p_{\min}^4} \sum_{j=p_{\min}+1}^{\infty} b_j^2(c_i'') \le C(k,p) T^4 \frac{k}{p_{\min}^4},
\end{aligned}$$

where $C(k,p) = T^4 \pi^{-4} \max_{0\le i\le k} \sum_{j=p_{\min}+1}^{\infty} b_j^2(c_i'')$.

On the other hand, invoking (4.4.8), We have

$$\begin{aligned}
\sum_{i=k+1}^{\infty} \sum_{j=0}^{\infty} c_{ij}^2 = \sum_{i=k+1}^{\infty} \|c_i(t,f)\|^2_{L^2[0,T]} =& \sum_{i=k+1}^{\infty} \frac{(i-r)!}{i!} \left\| c_i\left(t, \sqrt{v(t)}^r D^r f\right) \right\|^2 \\
\le& \frac{(k-r)!}{k!} \sum_{i=k+1}^{\infty} \left\| c_i\left(t, \sqrt{v(t)}^r D^r f\right) \right\|^2_{L^2[0,T]} = \frac{1}{k^r} R_k^2.
\end{aligned}$$

The proof is completed. □

4.5.2 Infinite time horizon

Let $t \in (0,\infty)$. Consider function space defined by

$$L^2(I\times\mathbb{R}^+,\nu) = \left\{ f(t,x) : \int_I f^2(t,x) d\Psi_t(x) < \infty, \text{ for each } t \in (0,\infty), \right.$$

$$\text{and} \int_0^\infty \int_I f^2(t,x)d\nu < \infty, \Big\}$$

where ν is the product of probability measure $\Psi_t(x)$ and Lebesgue measure μ.

We abbreviate the notation of the space as $L^2(I \times \mathbb{R}^+)$ for brevity. Apparently it is a L^2 space so that it is a Hilbert space. The inner product is conventional

$$(f_1(t,x), f_2(t,x)) = \int_0^\infty \int_I f_1(t, Z(t)) f_2(t, Z(t)) d\Psi_t(x) dt,$$

and $\|f(t,x)\| = \sqrt{(f(t,x), f(t,x))}$ is the induced norm.

Since $\{Q_i(t,x)\}$ and $\{\mathscr{L}_j(t)\}$ (see the definition in Chapter 2) are orthonormal bases in $L^2(I, d\Psi_t(x))$ and $L^2(\mathbb{R}^+, \mu)$ respectively, $\{Q_i(t,x)\mathscr{L}_j(t)\}$ is an orthonormal basis in $L^2(I \times \mathbb{R}^+)$ (see Dudley, 2003, Problem 12, p.173).

Similarly, construct a mapping from $L^2(I \times \mathbb{R}^+)$ to a set of stochastic process

$$\mathcal{T}: \quad f(t,x) \mapsto f(t, Z(t)).$$

It is easy to show that $\mathcal{T}$ is linear, so that the image set, denoted by Λ, is a linear real vector space. Note that after defining

$$\langle f_1(t, Z(t)), f_2(t, Z(t)) \rangle_\Lambda = \int_0^\infty E[f_1(t, Z(t)) f_2(t, Z(t))] dt,$$

Λ becomes an inner product space equipped with induced norm. Analogous to the counterpart in the preceding subsection, $\mathcal{T}$ and Λ possess the properties in Lemma 4.4.1–4.4.3. Whence, Λ is a Hilbert space and $\{Q_i(t, Z(t))\mathscr{L}_j(t)\}$ is an orthonormal basis in Λ. Consequently, we have the following theorem.

Theorem 4.5.3. *In Λ, any element $f(t, Z(t))$ admits a Fourier series expansion*

$$f(t, Z(t)) = \sum_{i=0}^\infty \sum_{j=0}^\infty b_{ij} Q_i(t, Z(t)) \mathscr{L}_j(t), \tag{4.5.4}$$

where $b_{ij} = \langle f(t, Z(t)), Q_i(t, Z(t))\mathscr{L}_j(t) \rangle_\Lambda$.

Proof. For Λ is a Hilbert space with orthonormal basis $\{Q_i(t, Z(t))\mathscr{L}_j(t)\}$, it follows immediately. □

Indeed,

$$b_{ij} = \langle f(t, Z(t)), Q_i(t, Z(t))\mathscr{L}_j(t) \rangle_\Lambda = \int_0^\infty E[f(t, Z(t)) Q_i(t, Z(t))] \mathscr{L}_j(t) dt$$

$$:= \int_0^\infty b_i(t,f)\mathscr{L}_j(t)dt,$$

where $b_i(t,f) = E[f(t,Z(t))Q_i(t,Z(t))]$, the expansion (4.5.4) can be regarded as two-step expansion, that is, expand $f(t,Z(t))$ first in terms of $\{Q_i(t,Z(t))\}$ obtaining coefficients $b_i(t,f) = E[f(t,Z(t))Q_i(t,Z(t))]$, then expand $b_i(t,f)$ in terms of $\{\mathscr{L}_j(t)\}$ on $(0,\infty)$.

Notice that from Parseval equality it follows that

$$\|f(t,Z(t))\|_\Lambda^2 = \sum_{i=0}^\infty \sum_{j=0}^\infty b_{ij}^2 = \sum_{i=0}^\infty \|b_i(t,f)\|_{L^2(\mathbb{R}^+)}^2.$$

Given a bundle of truncation parameters k for i and p_i for j's, we define the truncation series of (4.5.4) as follows

$$f_{k,p}(t,Z(t)) = \sum_{i=0}^k \sum_{j=0}^{p_i} b_{ij} Q_i(t,Z(t))\mathscr{L}_j(t). \tag{4.5.5}$$

Theorem 4.5.4. *Let $(Z(t), t \geq 0)$ be a Lévy process which admits a classical orthonormal polynomial system $\{Q_i(t,x)\}$ defined by (4.4.1). Suppose that functional $f(t,Z(t)) \in \Lambda$, that $D^h f(t,x)$, $h = 1, \cdots, r_1$, are in the space $L^2(I, \rho_h(t,x))$ for each $t > 0$. Moreover, $\sqrt{v(t)}^{r_1} D^{r_1} f(t,x) \in L^2(I \times \mathbb{R}^+)$. In addition, for each $i \geq 0$, $b_i(t,f) = E[f(t,Z_t)Q_i(t,Z(t))]$ is differentiable up to r_2-th order and $b_i(t,f)$ and $\sqrt{v(t)}^{r_2} \frac{d^h}{dt^h} b_i(t,f)$ are all in $L^2(\mathbb{R}^+)$ for $h = 1, \cdots, r_2$. Then,*

$$\|f(t,Z(t)) - f_{k,p}(t,Z(t))\|_\Lambda^2 \leq \frac{1}{k^{r_1}} R^2(k) + C(k,p)\frac{k}{p_{\min}^{r_2}}, \tag{4.5.6}$$

where $R^2(k) = (1+o(1))\sum_{i=k+1}^\infty \|b_i(t, \sqrt{v(t)}^{r_1} D^{r_1} f)\|_{L^2(\mathbb{R}^+)}^2 \to 0$, as $k \to \infty$, $C(k,p) = (1+o(1))\max_{0 \leq i \leq k} \sum_{j=p_{\min}+1}^\infty [a_{j-r_1}^{(r_1)}(\widetilde{b}_i(t))]^2$, in which $\widetilde{b}_i(t) = t^{r_2/2} e^{-t/2}[b_i(t,f)e^{t/2}]^{(r_2)}$ and $a_{j-r_2}^{(r_2)}(\widetilde{b}_i(t))$ are the coefficients of the expansion of $\widetilde{b}_i(t)$ in terms of $\mathscr{L}_j^{(r_2)}(t)$. Here we assume that $C(k,p)\frac{k}{p_{\min}^{r_2}} \to 0$.

Remark 4.5.2. Here a similar remark as that to Theorem 4.5.2 on the error of the approximation and on $C(k,p)$ can be addressed.

Proof. It follows from the orthogonality that

$$\begin{aligned}&\|f(t,Z(t)) - f_{k,p}(t,Z(t))\|_\Lambda^2 \\ =&\left\| \sum_{i=0}^k \sum_{j=p_i+1}^\infty b_{ij} Q_i(t,Z(t))\mathscr{L}_j(t) + \sum_{i=k+1}^\infty \sum_{j=0}^\infty b_{ij} Q_i(t,Z(t))\mathscr{L}_j(t) \right\|_\Lambda^2\end{aligned}$$

$$=\sum_{i=0}^{k}\sum_{j=p_i+1}^{\infty} b_{ij}^2 + \sum_{i=k+1}^{\infty}\sum_{j=0}^{\infty} b_{ij}^2.$$

Due to (4.4.8), $b_i(t,f) = \sqrt{v(t)}^{r_1}\sqrt{\frac{(i-r_1)!}{i!}}b_i(t, D^{r_1}f)$. Accordingly,

$$\begin{aligned}
\sum_{i=k+1}^{\infty}\sum_{j=0}^{\infty} b_{ij}^2 = \sum_{i=k+1}^{\infty} \|b_i(t,f)\|^2_{L^2(\mathbb{R}^+)} = \sum_{i=k+1}^{\infty} \frac{(i-r_1)!}{i!}\|b_i(t, \sqrt{v(t)}^{r_1} D^{r_1} f)\|^2 \\
\le \frac{(k+1-r_1)!}{(k+1)!} \sum_{i=k+1}^{\infty} \|b_i(t, \sqrt{v(t)}^{r_1} D^{r_1} f)\|^2 \\
=(1+o(1))\frac{1}{k^{r_1}} \sum_{i=k+1}^{\infty} \|b_i(t, \sqrt{v(t)}^{r_1} D^{r_1} f)\|^2 = \frac{1}{k^{r_1}} R_1^2(k),
\end{aligned}$$

where $R_1^2(k) = (1+o(1))\sum_{i=k+1}^{\infty} \|b_i(t, \sqrt{v(t)}^{r_1} D^{r_1} f)\|^2_{L^2(\mathbb{R}^+)}$.

On the other hand, according to Theorem 2.3.1,

$$\sum_{j=p_i+1}^{\infty} b_{ij}^2 \le \frac{(p_i+1-r_2)!}{(p_i+1)!} \sum_{j=p_i+1}^{\infty} [a_{j-r_2}^{(r_2)}(\widetilde{b}_i(t))]^2$$

where $\widetilde{b}_i(t) = t^{r_2/2}e^{-t/2}[b_i(t,f)e^{t/2}]^{(r_2)}$ and $a_{j-r_2}^{(r_2)}(\widetilde{b}_i(t))$ are the coefficients of the expansion of $\widetilde{b}_i(t)$ in terms of $\mathscr{L}_j^{(r_2)}(t)$. Thus,

$$\begin{aligned}
\sum_{i=0}^{k}\sum_{j=p_i+1}^{\infty} b_{ij}^2 \le \sum_{i=0}^{k} \frac{(p_i+1-r_2)!}{(p_i+1)!} \sum_{j=p_i+1}^{\infty} [a_{j-r_2}^{(r_2)}(\widetilde{b}_i(t))]^2 \\
\le \frac{(p_{\min}+1-r_2)!}{(p_{\min}+1)!} \sum_{i=0}^{k}\sum_{j=p_{\min}+1}^{\infty} [a_{j-r_2}^{(r_2)}(\widetilde{b}_i(t))]^2 \le C(k,p)\frac{k}{p_{\min}^{r_2}}.
\end{aligned}$$

where $C(k,p) = (1+o(1))\max_{0\le i\le k}\sum_{j=p_{\min}+1}^{\infty}[a_{j-r_2}^{(r_2)}(\widetilde{b}_i(t))]^2$. This finishes the proof. □

Chapter 5

Estimation of Lévy process functionals in econometric models

We consider a general econometric model which involves the Lévy process as follows

$$Y(t) = m(t, Z(t)) + \varepsilon(t), \tag{5.0.1}$$

where $m(\cdot, \cdot)$ is an unknown functional, and $\varepsilon(t)$ is an error process with zero mean and finite variance.

Suppose that $Z(t)$ admits a classical orthonormal polynomial system $Q_i(t, x)$ with weight $\rho(t, x)$, the density function or probability distribution function of $Z(t)$. Let the support of $\rho(t, x)$ be denoted by I, which can be $\mathbb{R}$, $\mathbb{R}^+$ or $\mathbb{N}$. Note that, as before, the operator D signifies either differentiation or difference operation and it is conducted only with respect to x.

This chapter dwells on the estimation of $m(\cdot, \cdot)$ in the model (5.0.1) given discrete observations of $Y(t)$. We divide the chapter into three sections according to the different types of time horizons, viz., on $(0, \infty)$, $[0, T]$ with fixed T and $[0, T_n]$ where T_n is increasing with sample size n.

5.1 Infinite time horizon

Suppose t is in the interval $(0, \infty)$. In this section we work with the situation where $m(\cdot, \cdot)$ is defined on $[0, \infty) \times I$ and our sampling points are $t_s = s$, $s = 1, 2, \cdots, n$. Given that we have observations (s, Y_s) where $Y_s = Y(s)$, $s = 1, 2, \cdots, n$, our aim is to estimate $m(\cdot, \cdot)$.

At each point of observations, the model (5.0.1) now becomes

$$Y_s = m(s, X_s) + e_s, \quad s = 1, \dots, n, \tag{5.1.1}$$

where $X_s = Z(s)$ denotes the Lévy process at point s, $e_s = \varepsilon(s)$ $(s = 1, \dots, n)$ form an error sequence with mean zero and finite variance.

Note that because $Z(t)$ is a Lévy process, $E[Z(t)] = t\mu$ where $\mu = E(Z(1))$ and $Var(Z(t)) = t\sigma_z^2$ where $\sigma_z^2 = Var(Z(1))$. Observe that $X_s = s\mu + X_s - s\mu = s\mu + \sum_{i=1}^s [(X_i - i\mu) - (X_{i-1} - (i-1)\mu)] = s\mu + \sqrt{n}\sigma_z x_{s,n}$ where $x_{s,n} = \frac{1}{\sqrt{n}\sigma_z} \sum_{i=1}^s [(X_i - i\mu) - (X_{i-1} - (i-1)\mu)]$. Since $(X_i - i\mu) - (X_{i-1} - (i-1)\mu)$ form an i.i.d $(0, \sigma_z^2)$ sequence, it follows from the functional central limit theorem, $x_{s,n}$ converges in distribution to a Brownian motion on $[0, 1]$. In addition, $x_{s,n}$, along with $d_{l,k,n} = \sqrt{(l-k)/n}$, satisfies Assumption A in Chapter 1.

We firstly need to impose some conditions on $m(t, x)$.

Assumption L.1

(a) For every $t > 0$, $m(t, x)$ and $D^r m(t, x)$ are in $L^2(I, \rho_r(t, x))$, $r = 1, 2, 3$.

(b) For each i, the coefficient function $c_i(t, m) = E[m(t, Z(t))Q_i(t, Z(t))]$, and its derivatives of up to third order belong to $L^2(\mathbb{R}^+)$.

(c) For i large enough, the coefficient functions $c_i(t, D^3 m)$ of $D^3 m(t, Z(t))$ expanded by the system $\{Q_{3i}(t, Z(t))\}$ verify that $v(t)^3 c_i^2(t, D^3 m)$ are bounded on $(0, \infty)$ uniformly in i.

Remark 5.1.1. Note that the notations $\rho_r(t, x)$, $v(t)$ and $\{Q_{3i}(t, Z(t))\}$ are defined in the preceding chapter.

Condition (a) is some basic requirements under which we can expand not only $m(t, Z(t))$ but also $D^r m(t, x)|_{x=Z(t)}$. Condition (b) and (c) give the necessary conditions for the coefficient functions in order to obtain some kind of convergent speed on the expansions.

There are many functionals satisfying all the conditions. (1) Let $m_1(t, x) = t^a e^{-bt} P_k(x)$ with $a \geq 3, b > 0$ and $P_k(x)$ being a polynomial of fixed degree k $(k \geq 1)$. $m_1(t, x)$ satisfies Condition (a) due to the boundary condition on $\rho(t, x)$; the reason that $m_1(t, x)$ satisfies Condition (b) is that the coefficients $c_i(t, m_1)$ are all in form of $e^{-bt} q(t)$ where $q(\cdot)$ is a polynomial in t; Condition (c) is fulfilled because when $i > k$, $c_i(t, m_1) = 0$. (2) $m_2(t, x) = \frac{t^\alpha}{1+t^\beta} \sin(x)$ and $m_3(t, x) = \frac{t^\alpha}{1+t^\beta} \cos(x)$ where $\alpha \geq 1$ and $\beta \geq \alpha + 1.25$. In the

Brownian motion case, from Example 3.1 we have explicit expression of the coefficients $c_i(t, m_2) = (-1)^k \frac{1}{\sqrt{i!}} \frac{t^\alpha \sqrt{t}^i}{1+t^\beta} e^{-t/2}$, for $i = 2k+1$; 0, for $i = 2k$, where $k = 0, 1, \dots$ and $c_i(t, m_3) = (-1)^k \frac{1}{\sqrt{i!}} \frac{t^\alpha \sqrt{t}^i}{1+t^\beta} e^{-t/2}$ for $i = 2k$; 0, for $n = 2k+1$, where $k = 0, 1, \dots$. It is not difficult to verify the conditions. (3) In the case that $Z(t) = N(t)$ a Poisson process with intensity 1, $m_4(t, x) = t^\xi 2^{-x}$ where $\xi \geq 2$, $m_5(t, x) = \frac{t^\xi}{1+t^\eta} \sin x$ and $m_6(t, x) = \frac{t^\xi}{1+t^\eta} \cos x$ with $\xi \geq 1$ and $\eta \geq \xi + 1.25$. Since $c_i(t, m_4) = t^\xi e^{-t/2} \frac{1}{2^i} \sqrt{\frac{t^i}{i!}}$, the conditions are easy to be verified for m_4. Meanwhile, from example 3.1 we can have the explicit expressions of $c_i(t, m_5)$ and $c_i(t, m_6)$

$$
\begin{aligned}
c_i(t, m_5) &= (-1)^i \frac{t^\xi}{1+t^\eta} \sqrt{\frac{t^i}{i!}} e^{-t\beta} \sqrt{2\beta}^i \sin(\alpha i + t \sin 1) \\
c_i(t, m_6) &= (-1)^i \frac{t^\xi}{1+t^\eta} \sqrt{\frac{t^i}{i!}} e^{-t\beta} \sqrt{2\beta}^i \cos(\alpha i + t \sin 1)
\end{aligned}
$$

where α is a constant and $\beta = 1 - \cos 1$. Thus, the condition can be verified.

Having expanded function m at sampling points, given truncation parameters k and p_i, model (5.1.1) can be written as

$$
\begin{aligned}
Y_s =& \sum_{i=0}^{k} \sum_{j=0}^{p_i} c_{ij} \mathscr{L}_j(s) Q_i(s, X_s) + \sum_{i=0}^{k} \sum_{j=p_i+1}^{\infty} c_{ij} \mathscr{L}_j(s) Q_i(s, X_s) \\
&+ \sum_{i=k+1}^{\infty} \sum_{j=0}^{\infty} c_{ij} \mathscr{L}_j(s) Q_i(s, X_s) + e_s, \qquad s = 1, 2, \dots, n.
\end{aligned} \tag{5.1.2}
$$

As we know from the last chapter, $\sum_{j=0}^{\infty} c_{ij} \mathscr{L}_j(s) = c_i(s, m)$ or more simply, $c_i(s)$ if there is no confusion occurred. Therefore, in most cases we shall supersede this relationship into the model expression. We now may rewrite equations (5.1.2) in the following matrix form:

$$
Y = X\theta + \delta + \gamma + \varepsilon, \tag{5.1.3}
$$

where

$$
\begin{aligned}
Y' =& (Y_1, Y_2, \dots, Y_n); \quad \theta' = (c_{00}, c_{01} \dots, c_{0p_0}, c_{10}, \dots, c_{1p_1}, \dots, c_{k0}, \dots, c_{kp_k}); \\
x_1 =& (\mathscr{L}_0(1) Q_0(1, X_1), \mathscr{L}_1(1) Q_0(1, X_1), \dots, \mathscr{L}_{p_0}(1) Q_0(1, X_1), \\
& \mathscr{L}_0(1) Q_1(1, X_1), \mathscr{L}_1(1) Q_1(1, X_1), \dots, \mathscr{L}_{p_1}(1) Q_1(1, X_1), \\
& \dots, \mathscr{L}_0(1) Q_k(1, X_1), \mathscr{L}_1(1) Q_k(1, X_1), \dots, \mathscr{L}_{p_k}(1) Q_k(1, X_1)), \\
& \vdots
\end{aligned}
$$

$$\begin{aligned}x_n =&(\mathscr{L}_0(n)Q_0(n,X_n),\mathscr{L}_1(n)Q_0(n,X_n),\ldots,\mathscr{L}_{p_0}(n)Q_0(n,X_n),\\&\mathscr{L}_0(n)Q_1(n,X_n),\mathscr{L}_1(n)Q_1(n,X_n),\ldots,\mathscr{L}_{p_1}(n)Q_1(n,X_n),\\&\ldots,\mathscr{L}_0(n)Q_k(n,X_n),\mathscr{L}_1(n)Q_k(n,X_n),\ldots,\mathscr{L}_{p_k}(n)Q_k(n,X_n)),\end{aligned}$$

and $X = (x_1', x_2', \cdots, x_n')'$; $\delta' = (\delta_1, \cdots, \delta_n)$, $\gamma' = (\gamma_1, \gamma_2, \cdots, \gamma_n)$, $\varepsilon' = (e_1, e_2, \cdots, e_n)$, with $\delta_s = \sum_{i=1}^k \sum_{j=p_i+1}^\infty c_{ij}\mathscr{L}_j(s)Q_i(s,X_s)$, $\gamma_s = \sum_{i=k+1}^\infty c_i(s)Q_i(s,X_s)$, $s = 1,2,\cdots,n$.

The OLS estimator of θ is given by

$$\widehat{\theta} = (X'X)^{-1}X'Y. \tag{5.1.4}$$

With the help of the estimation of coefficients in the expansion of functional $m(t, Z(t))$, we are able to estimate the function $m(\tau, x)$ at point (τ, x) where $\forall\tau > 0$ and $x \in \mathbb{R}$ is any point on the trajectory of $X_\tau = Z(\tau)$, namely, we can have $\widehat{m}(\tau, x)$ by superseding $\widehat{\theta}$ in lieu of θ and getting rid of residues in the expansion of $m(\tau, x)$.

More precisely, as $m(\cdot,\cdot)$ satisfies Assumption L.1, $m(\tau, x)$ is decomposed using orthonormal basis $\{\mathscr{L}_j(\tau)Q_i(\tau, x)\}$ as follows

$$\begin{aligned}m(\tau,x) =&\sum_{i=0}^\infty\sum_{j=0}^\infty c_{ij}\mathscr{L}_j(\tau)Q_i(\tau,x)\\=&\sum_{i=0}^k\sum_{j=0}^{p_i} c_{ij}\mathscr{L}_j(\tau)Q_i(\tau,x)+\sum_{i=0}^k\sum_{j=p_i+1}^\infty c_{ij}\mathscr{L}_j(\tau)Q_i(\tau,x)\\&+\sum_{i=k+1}^\infty\sum_{j=0}^\infty c_{ij}\mathscr{L}_j(\tau)Q_i(\tau,x)\\:=&A'(\tau,x)\theta+\delta(\tau,x)+\gamma(\tau,x),\end{aligned} \tag{5.1.5}$$

where θ is defined as before and

$$\begin{aligned}\delta(\tau,x) &= \sum_{i=0}^k\sum_{j=p_i+1}^\infty c_{ij}\mathscr{L}_j(\tau)Q_i(\tau,x),\\\gamma(\tau,x) &= \sum_{i=k+1}^\infty\sum_{j=0}^\infty c_{ij}\mathscr{L}_j(\tau)Q_i(\tau,x),\\A'(\tau,x) &= (\mathscr{L}_0(\tau)Q_0(\tau,x),\cdots,\mathscr{L}_{p_0}(\tau)Q_0(\tau,x),\\&\qquad\cdots,\mathscr{L}_0(\tau)Q_k(\tau,x),\cdots,\mathscr{L}_{p_k}(\tau)Q_k(\tau,x)).\end{aligned}$$

Thus,

$$\widehat{m}(\tau,x) = A'(\tau,x)\widehat{\theta}. \tag{5.1.6}$$

We shall investigate the limit of

$$\begin{aligned}\widehat{m}(\tau,x)-m(\tau,x)&=A'(\tau,x)(\widehat{\theta}-\theta)-\delta(\tau,x)-\gamma(\tau,x)\\&=A'(\tau,x)(X'X)^{-1}X'(\delta+\gamma+\varepsilon)-\delta(\tau,x)-\gamma(\tau,x).\end{aligned}\tag{5.1.7}$$

To this end, denote $A_{p\times p}$ and $B_{p\times p}$ by

$$A=\frac{A(\tau,x)A'(\tau,x)}{\|A(\tau,x)\|^2}\quad\text{and}\quad B=(X'X)A(X'X)^{-1}$$

where $\|\cdot\|$ signifies Euclidean norm and dimension $p=p_0+\cdots+p_k+k+1$.

On account of Lemma 3.1.2, we can assert that matrix B has eigenvalues $\lambda_1=1$,$\lambda_2=\cdots=\lambda_p=0$.

Let α be the unit left eigenvector of B pertaining to eigenvalue 1, viz., $\alpha'B=\alpha'$ and $\|\alpha\|=1$. As α is p-dimensional vector, in concert with $A(\tau,x)$, represent α in double-index subscript, that is, $\alpha'=(\alpha_{00},\cdots,\alpha_{0p_0},\cdots,\alpha_{k0},\cdots,\alpha_{kp_k})$. The following assumption proposes a double-index sequence we are working with.

Assumption L.2

(a) Let $\mathcal{S}=\{a_0,a_1,a_2,\ldots\}$, where $a_i=\{a_{ij}\}_{j=0}^{\infty}$ is a sequence such that $\sum_{j=1}^{\infty}j|a_{i,j}|<\infty$ for $i=0,1,2,\cdots$.

(b) Suppose further that $\sum_{i=1}^{\infty}i\left(\sum_{j=0}^{\infty}|a_{ij}|\right)^2<\infty$.

Remark 5.1.2. This assumption is actually exactly Assumption Bm.1. For completeness and independence of this chapter, we recite it here.

Using α and $\frac{1}{\|A'(\tau,x)\|}A'(\tau,x)$, let us reshuffle the set $\mathcal{S}$ as $\widetilde{\mathcal{S}}$ and $\bar{\mathcal{S}}$ by defining

1) $\widetilde{\mathcal{S}}=\{\widetilde{a}_0,\cdots,\widetilde{a}_i,\cdots\}$,

2) $\widetilde{a}_i=\{\widetilde{a}_{ij}\}$ where $\widetilde{a}_{ij}=\frac{1}{\sqrt{p_{\max}}}\alpha_{ij}$ for $0\le i\le k$ and $0\le j\le p_i$; otherwise, $\widetilde{a}_{ij}=a_{ij}$.

3) $\bar{\mathcal{S}}=\{\bar{a}_0,\cdots,\bar{a}_i,\cdots\}$,

4) $\bar{a}_i=\{\bar{a}_{ij}\}$ where $\bar{a}_{ij}=\frac{1}{\sqrt{p_{\max}}\|A(\tau,x)\|}\mathscr{L}_j(\tau)Q_i(\tau,x)$ for $0\le i\le k$ and $0\le j\le p_i$; otherwise, $\bar{a}_{ij}=a_{ij}$.

Obviously, $\widetilde{a}_{ij}=a_{ij}=\bar{a}_{ij}$ if $i>k$ or $j>p_i$. Meanwhile, since $\widetilde{\mathcal{S}}$ and $\bar{\mathcal{S}}$ satisfy Riesz-Fischer theorem, there exist functions, denoted by $\widetilde{F}(t,x)$ and $\widetilde{G}(t,x)$, such that

$$\widetilde{F}(t,Z(t))=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\widetilde{a}_{ij}\mathscr{L}_j(t)Q_i(t,Z(t)),\tag{5.1.8}$$

$$\widetilde{G}(t, Z(t)) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \bar{a}_{ij}\mathscr{L}_j(t)Q_i(t, Z(t)), \tag{5.1.9}$$

for any $t > 0$.

Therefore, in view of (5.1.8) and (5.1.9), we have

$$\frac{1}{\sqrt{p_{\max}}}\alpha' X' = \widetilde{\mathbf{F}}' - \widetilde{\delta}' - \widetilde{\gamma}', \tag{5.1.10}$$

$$\frac{1}{\sqrt{p_{\max}}\|A'(\tau, x)\|} A'(\tau, x)X' = \widetilde{\mathbf{G}}' - \widetilde{\delta}' - \widetilde{\gamma}', \tag{5.1.11}$$

where

$$\begin{aligned}
\widetilde{\mathbf{F}}' &= (\widetilde{F}(1, X_1), \cdots, \widetilde{F}(n, X_n)),\\
\widetilde{\mathbf{G}}' &= (\widetilde{G}(1, X_1), \cdots, \widetilde{G}(n, X_n)),\\
\widetilde{\delta}' &= (\widetilde{\delta}_1, \cdots, \widetilde{\delta}_n) \text{ with } \widetilde{\delta}_s = \sum_{i=0}^{k}\sum_{j=p_i+1}^{\infty} a_{ij}\mathscr{L}_j(s)Q_i(s, X_s),\\
\widetilde{\gamma}' &= (\widetilde{\gamma}_1, \cdots, \widetilde{\gamma}_n) \text{ with } \widetilde{\gamma}_s = \sum_{i=k+1}^{\infty}\sum_{j=0}^{\infty} a_{ij}\mathscr{L}_j(s)Q_i(s, X_s).
\end{aligned}$$

We have the following proposition for the generated functions $\widetilde{F}(t, x)$ and $\widetilde{G}(t, x)$.

Lemma 5.1.1. *For any $t > 0$, (a) $E[\widetilde{G}(t, Z(t))]^2 < \infty$, and (b) $E[\widetilde{F}(t, Z(t))]^2 < \infty$.*

Proof. (a) It follows from the orthogonality of $Q_i(t, Z(t))$ and the boundedness in (2.3.5) for $\mathscr{L}_j(t)$ that

$$\begin{aligned}
E[\widetilde{G}(t, Z(t))]^2 &= \sum_{i=0}^{\infty}\left(\sum_{j=0}^{\infty} \bar{a}_{ij}\mathscr{L}_j(t)\right)^2 \le C\sum_{i=0}^{\infty}\left(\sum_{j=0}^{\infty} |\bar{a}_{ij}|\right)^2\\
&= C\sum_{i=0}^{\infty}\left(\sum_{j=0}^{p_i} |\bar{a}_{ij}| + \sum_{j=p_i+1}^{\infty} |a_{ij}|\right)^2\\
&\le 2C\sum_{i=0}^{\infty}\left(\sum_{j=0}^{p_i} |\bar{a}_{ij}|\right)^2 + 2C\sum_{i=0}^{\infty}\left(\sum_{j=p_i+1}^{\infty} |a_{ij}|\right)^2\\
&= 2C\sum_{i=0}^{k}\left(\sum_{j=0}^{p_i} |\bar{a}_{ij}|\right)^2 + 2C\sum_{i=k+1}^{\infty}\left(\sum_{j=0}^{p_i} |a_{ij}|\right)^2
\end{aligned}$$

$$
\begin{aligned}
&+2C\sum_{i=0}^{\infty}\left(\sum_{j=p_i+1}^{\infty}|a_{ij}|\right)^2\\
\leq & 2C\frac{1}{p_{\max}\|A'(\tau,x)\|^2}\sum_{i=0}^{k}Q_i^2(\tau,x)\left(\sum_{j=0}^{p_i}\mathscr{L}_j(\tau)\right)^2+O(1)\\
\leq & 2C\frac{1}{p_{\max}\|A'(\tau,x)\|^2}\sum_{i=0}^{k}Q_i^2(\tau,x)p_i\sum_{j=0}^{p_i}\mathscr{L}_j^2(\tau)+O(1)=O(1).
\end{aligned}
$$

where we have used Assumption L.2 for a_{ij}, the definition of $\bar{a}_{ij}$ and $\|A'(\tau,x)\|^2 = \sum_{i=0}^{k} Q_i^2(\tau,x)\sum_{j=0}^{p_i}\mathscr{L}_j^2(\tau)$.

(b) Similar to the proof of (a). □

Notice that $E[Z(t)] = \mu t$. Denote $F(t, x-\mu t) = \widetilde{F}(t,x)$ and $G(t, x-\mu t) = \widetilde{G}(t,x)$. This is only a change in the form of functions since the process $Z(t)$ has to be centralised in order to acquire the limit distribution of $\widehat{m}$.

The following assumption is stipulated for the truncation parameters, which is crucial for obtaining the limit distribution of the estimator.

Assumption L.3

(a) $k = [n^{\kappa_1}]$ with $\frac{1}{2} < \kappa_1 < 1$;

(b) $p_{\min} = [n^{\kappa_2}]$ and $p_{\max} = [n^{\bar{\kappa}_2}]$ with $0 < \kappa_2 \leq \bar{\kappa}_2 < 1$;

(c) $2 + 2\kappa_1 < 5\kappa_2$.

Remark 5.1.3. There are obviously a great deal of feasible options for κ_1, κ_2 and $\bar{\kappa}_2$ satisfying the conditions. Note that Condition (c) is not harsh since when $\kappa_2 > 0.8$ it follows automatically.

The next assumption describes the families of functionals F and G we are studying in the asymptotic distribution of the estimator.

Assumption L.4

(a) Suppose that $F(\cdot,\cdot)$ and $G(\cdot,\cdot)$ are in class $\mathcal{T}(HI)$ with homogeneity powers $\upsilon(\cdot)$ and $\varrho(\cdot)$ and normal functions $f(\cdot,\cdot)$ and $g(\cdot,\cdot)$ respectively. Let $\upsilon(n) = n^{\varsigma}$ and $\varrho(n) = n^{\iota}$ with $\varsigma \geq 0$ and $\iota \geq 0$ satisfying $\frac{1}{2}(\kappa_1 - \frac{1}{2}) < \iota < \min\{\frac{5}{4}\kappa_2 - \frac{1}{4} - \frac{1}{2}\kappa_1, \frac{7}{4}\kappa_1 - \frac{1}{4}\}$.

(b) Suppose also that $F^2(\cdot,\cdot)$, $G^2(\cdot,\cdot)$ and $F(\cdot,\cdot)G(\cdot,\cdot)$ are all in class $\mathcal{T}(HI)$ with homogeneity powers $\upsilon^2(\cdot)$, $\varrho^2(\cdot)$ and $\upsilon(\cdot)\varrho(\cdot)$, and normal functions $f^2(\cdot,\cdot)$, $g^2(\cdot,\cdot)$ and $f(\cdot,\cdot)g(\cdot,\cdot)$ respectively.

(c) Suppose that $F(\cdot,\cdot)$ and $G(\cdot,\cdot)$ are in class $\mathcal{T}(HH)$ with homogeneity powers $\upsilon_1(\cdot)$, $\upsilon_2(\cdot)$ and $\varrho_1(\cdot)$, $\varrho_2(\cdot)$ and normal functions $f(\cdot,\cdot)$ and $g(\cdot,\cdot)$ respectively. Let $\upsilon_1(n) = n^{\varsigma_1}$, $\upsilon_2(n) = n^{\varsigma_2}$, $\varrho_1(n) = n^{\iota_1}$, and $\varrho_2(n) = n^{\iota_2}$ with $\varsigma_i \geq 0$, $\iota_i \geq 0$, $i = 1, 2$, satisfying that $\iota_1 + \frac{1}{2}\iota_2 < \min\{\frac{5}{4}\kappa_2 - \frac{1}{2}(1+\kappa_1), \frac{7}{4}\kappa_1 - \frac{1}{2}\}$.

(d) Suppose also that $F^2(\cdot,\cdot)$, $G^2(\cdot,\cdot)$ and $F(\cdot,\cdot)G(\cdot,\cdot)$ are all in class $\mathcal{T}(HH)$ with homogeneity powers $\upsilon_1^2(\cdot)$ and $\upsilon_2^2(\cdot)$; $\varrho_1^2(\cdot)$ and $\varrho_2^2(\cdot)$; $\upsilon_1(\cdot)\varrho_1(\cdot)$ and $\upsilon_2(\cdot)\varrho_2(\cdot)$ as well as normal functions $f^2(\cdot,\cdot)$, $g^2(\cdot,\cdot)$ and $f(\cdot,\cdot)g(\cdot,\cdot)$ respectively.

Remark 5.1.4. Assumption L.3 ensures that two upper bounds for ι and $\iota_1 + \frac{1}{2}\iota_2$ are positive. Of course, we can make these conditions in (a) and (c) for parameters tractable and tidy if we impose more constraints on κ_1 and κ_2. However, these raw conditions give them more options.

Notice that in the proof of the following theorem, what conditions for ι and $\iota_1 + \frac{1}{2}\iota_2$ we actually use involve $\bar{\kappa}_2 - \kappa_2$. Since we may require that $\bar{\kappa}_2$ be much closer to κ_2 such that $\bar{\kappa}_2 - \kappa_2$ is as small as we wish, conditions in (a) and (c) implicitly indicates what we require in the proof. Obviously, this does not harm any thing else and applies to the subsequential subsections.

Note also that the ambit for both ι and $\iota_1 + \frac{1}{2}\iota_2$ can be enlarged at the price of enhancing the order of differentiability for the coefficient functions in the expansion of the m function, as can be seen in the proof of the following theorem.

We are now ready to state the main result in the section.

Theorem 5.1.1. *Suppose that $\{x_{s,n}\}_1^n$ and $\{e_s\}_1^n$ satisfy Assumptions B and A (c). Let Assumptions L.1–L.3 hold.*

If Assumption L.4 (a) and (b) hold, then

$$\begin{aligned}&\frac{\sqrt{\sigma_z}}{\sqrt[4]{n}\upsilon(n)}\frac{\alpha' X' X A(\tau,x)}{\sqrt{p_{\max}}\|A(\tau,x)\|^2}(\widehat{m}(\tau,x) - m(\tau,x))\\ &\quad \to_D \left(\int_0^1 G_3(t)dL_W(t,0)\right)^{\frac{1}{2}} N,\end{aligned} \tag{5.1.12}$$

where $G_3(t) = \int f(t,x)^2 dx$, W *is a standard Brownian motion on* $[0,1]$, N *is a standard normal random variable independent of* W, *and* L_W *is the local–time process of* W.

If Assumption L.4 (c) and (d) hold, then

$$\begin{aligned}\frac{1}{\sqrt{n}\upsilon_1(n)\upsilon_2(\sqrt{n}\sigma_z)}\frac{\alpha' X' X A(\tau,x)}{\sqrt{p_{\max}}\|A(\tau,x)\|^2}(\widehat{m}(\tau,x)-m(\tau,x))\\ \to_D \int_0^1 f(r,W(r))dU(r),\end{aligned} \tag{5.1.13}$$

where $(W(r),U(r))$ *is the vector of Brownian motion in Assumption B.*

Remark 5.1.5. As can be seen from the proof, the order of the convergence of (5.1.12) is $\frac{\sqrt[4]{n}\sqrt{p_{\max}}\varrho(n)}{\|A(\tau,x)\|}$. By virtue of the calculation of $\|A(\tau,x)\|$ in the proof, we can estimate

$$n^{\frac{1}{4}+\iota-\frac{1}{2}\kappa_1} \le \frac{\sqrt[4]{n}\sqrt{p_{\max}}\varrho(n)}{\|A(\tau,x)\|} \le n^{\frac{1}{4}+\frac{1}{2}(\bar{\kappa}_2-\kappa_2)+\iota-\frac{1}{2}\kappa_1} \le n^{\frac{5}{4}\kappa_2-\kappa_1}.$$

This means when ι reaches its upper bound, the convergence rate is at maximum $n^{\frac{5}{4}\kappa_2-\kappa_1}$, while when ι is close to $\frac{1}{2}(\kappa_1-\frac{1}{2})$ the convergence is very slow.

Meanwhile, the convergence rate for (5.1.13) is $\frac{\sqrt{n}\sqrt{p_{\max}}\varrho_1(n)\varrho_2(\sqrt{n}\sigma_x)}{\|A(\tau,x)\|}$ which similarly has the following approximation

$$n^{\frac{1}{2}(1-\kappa_1)+\iota_1+\frac{1}{2}\iota_2} \le \frac{\sqrt{n}\sqrt{p_{\max}}\varrho_1(n)\varrho_2(\sqrt{n}\sigma_x)}{\|A(\tau,x)\|} \le n^{\frac{1}{2}(1-\kappa_1)+\iota_1+\frac{1}{2}\iota_2+\frac{1}{2}(\bar{\kappa}_2-\kappa_2)} \le n^{\frac{5}{4}\kappa_2-\kappa_1}.$$

As $\iota_1+\frac{1}{2}\iota_2$ reaches its upper bound, the rate is as high as $n^{\frac{5}{4}\kappa_2-\kappa_1}$, while as $\iota_1+\frac{1}{2}\iota_2$ closes to $\frac{1}{2}(\kappa_1-\frac{1}{2})$ the convergence is very slow, the same as in the first situation.

Proof. We firstly prove (5.1.12). By virtue of (5.1.7), we may write

$$\begin{aligned}&\frac{\sqrt{\sigma_z}}{\sqrt[4]{n}\upsilon(n)}\frac{\alpha' X' X A(\tau,x)}{\sqrt{p_{\max}}\|A(\tau,x)\|^2}(\widehat{m}(\tau,x)-m(\tau,x))\\ =&\frac{\sqrt{\sigma_z}}{\sqrt[4]{n}\upsilon(n)}\frac{\alpha' X' X A(\tau,x)}{\sqrt{p_{\max}}\|A(\tau,x)\|^2}[A'(\tau,x)(X'X)^{-1}X'(\delta+\gamma+\varepsilon)-\delta(\tau,x)-\gamma(\tau,x)]\\ =&\frac{\sqrt{\sigma_z}}{\sqrt[4]{n}\upsilon(n)\sqrt{p_{\max}}}\alpha' B X'(\delta+\gamma+\varepsilon)-\frac{\sqrt{\sigma_z}}{\sqrt[4]{n}\upsilon(n)}\frac{\alpha' X' X A(\tau,x)}{\sqrt{p_{\max}}\|A(\tau,x)\|^2}[\delta(\tau,x)+\gamma(\tau,x)]\\ =&\frac{\sqrt{\sigma_z}}{\sqrt[4]{n}\upsilon(n)\sqrt{p_{\max}}}\alpha' X'(\delta+\gamma+\varepsilon)-\frac{\sqrt{\sigma_z}}{\sqrt[4]{n}\upsilon(n)}\frac{\alpha' X' X A(\tau,x)}{\sqrt{p_{\max}}\|A(\tau,x)\|^2}[\delta(\tau,x)+\gamma(\tau,x)]\\ =&\sum_{i=1}^{3}\Pi_i-\Pi_4,\end{aligned}$$

where

$$\Pi_1 = \frac{\sqrt{\sigma_z}}{\sqrt[4]{n}\upsilon(n)\sqrt{p_{\max}}}\alpha' X'\delta, \qquad \Pi_2 = \frac{\sqrt{\sigma_z}}{\sqrt[4]{n}\upsilon(n)\sqrt{p_{\max}}}\alpha' X'\gamma,$$

$$\Pi_3 = \frac{\sqrt{\sigma_z}}{\sqrt[4]{n}\upsilon(n)\sqrt{p_{\max}}}\alpha' X'\varepsilon, \qquad \Pi_4 = \frac{\sqrt{\sigma_z}}{\sqrt[4]{n}\upsilon(n)}\frac{\alpha' X' X A(\tau, x)}{\sqrt{p_{\max}}\|A(\tau, x)\|^2}[\delta(\tau, x) + \gamma(\tau, x)].$$

We are about to show that $\Pi_i \to_P 0$, $i = 1, 2, 4$, and Π_3 converges to the desired variable in distribution as $n \to \infty$.

Firstly, it follows from (5.1.10) that

$$\begin{aligned}
\Pi_3 =& \frac{\sqrt{\sigma_z}}{\sqrt[4]{n}\upsilon(n)}(\widetilde{\mathbf{F}}' - \widetilde{\delta}' - \widetilde{\gamma}')\varepsilon = \frac{\sqrt{\sigma_z}}{\sqrt[4]{n}\upsilon(n)}(\widetilde{\mathbf{F}}'\varepsilon - \widetilde{\delta}'\varepsilon - \widetilde{\gamma}'\varepsilon) \\
=& \frac{\sqrt{\sigma_z}}{\sqrt[4]{n}\upsilon(n)}\sum_{s=1}^{n}\widetilde{F}(s, X_s)e_s - \frac{\sqrt{\sigma_z}}{\sqrt[4]{n}\upsilon(n)}\sum_{s=1}^{n}\widetilde{\delta}_s e_s - \frac{\sqrt{\sigma_z}}{\sqrt[4]{n}\upsilon(n)}\sum_{s=1}^{n}\widetilde{\gamma}_s e_s \\
=& \frac{\sqrt{\sigma_z}}{\sqrt[4]{n}\upsilon(n)}\sum_{s=1}^{n}\widetilde{F}(s, s\mu + \sqrt{n}\sigma_z x_{s,n})e_s - \frac{\sqrt{\sigma_z}}{\sqrt[4]{n}\upsilon(n)}\sum_{s=1}^{n}\widetilde{\delta}_s e_s - \frac{\sqrt{\sigma_z}}{\sqrt[4]{n}\upsilon(n)}\sum_{s=1}^{n}\widetilde{\gamma}_s e_s \\
=& \frac{\sqrt{\sigma_z}}{\sqrt[4]{n}\upsilon(n)}\sum_{s=1}^{n}F(s, \sqrt{n}\sigma_z x_{s,n})e_s - \frac{\sqrt{\sigma_z}}{\sqrt[4]{n}\upsilon(n)}\sum_{s=1}^{n}\widetilde{\delta}_s e_s - \frac{\sqrt{\sigma_z}}{\sqrt[4]{n}\upsilon(n)}\sum_{s=1}^{n}\widetilde{\gamma}_s e_s \\
:=& \Pi_{31} - \Pi_{32} - \Pi_{33}.
\end{aligned}$$

In view of Theorem 1.3.1 with $c_n = \sqrt{n}\sigma_z$, we have

$$\Pi_{31} \to_D \left(\int_0^1 G_3(t)dL_W(t, 0)\right)^{\frac{1}{2}} N, \tag{5.1.14}$$

where $G_3(t) = \int f(t, x)^2 dx$, W is a standard Brownian motion on $[0, 1]$, N is a standard normal random variable independent of W, and L_W is the local–time process of W.

Meanwhile, the martingale difference structure $(e_s, \mathcal{F}_{n,s})$ and the adaptivity $x_{s+1,n}$ with $\mathcal{F}_{n,s}$ yield

$$E(\Pi_{32})^2 = \frac{\sigma_z\sigma_e^2}{\sqrt{n}\upsilon(n)^2}\sum_{s=1}^{n}E\widetilde{\delta}_s^2, \tag{5.1.15a}$$

$$E(\Pi_{33})^2 = \frac{\sigma_z\sigma_e^2}{\sqrt{n}\upsilon(n)^2}\sum_{s=1}^{n}E\widetilde{\gamma}_s^2, \tag{5.1.15b}$$

where $E(e_s^2|\mathcal{F}_{n,s-1}) = \sigma_e^2$ a.s.

Regarding (5.1.15a), using the expression of $\widetilde{\delta}_s$ we have

$$E(\Pi_{32})^2 = \frac{\sigma_z\sigma_e^2}{\sqrt{n}\upsilon(n)^2}\sum_{s=1}^{n}E\widetilde{\delta}_s^2$$

$$
\begin{aligned}
&= \frac{\sigma_z \sigma_e^2}{\sqrt{n}\upsilon(n)^2} \sum_{s=1}^{n} E \left(\sum_{i=0}^{k} \sum_{j=p_i+1}^{\infty} a_{ij} \mathscr{L}_j(s) Q_i(s, X_s) \right)^2 \\
&= \frac{\sigma_z \sigma_e^2}{\sqrt{n}\upsilon(n)^2} \sum_{s=1}^{n} \sum_{i=0}^{k} \left(\sum_{j=p_i+1}^{\infty} a_{ij} \mathscr{L}_j(s) \right)^2 \\
&\leq \frac{\sigma_z \sigma_e^2}{\sqrt{n}\upsilon(n)^2} \sum_{s=1}^{n} \sum_{i=0}^{k} \left(\max_{j \geq p_i} |\mathscr{L}_j(s)| \right)^2 \left(\sum_{j=p_i+1}^{\infty} |a_{ij}| \right)^2 \\
&\leq o(1) \frac{\sigma_z \sigma_e^2}{\sqrt{n}\upsilon(n)^2} \sum_{s=1}^{n} \frac{1}{\sqrt{s}} \sum_{i=0}^{k} \frac{1}{\sqrt{p_i}} \frac{1}{p_i^2} \leq o(1) \frac{\sigma_z \sigma_e^2}{\upsilon(n)^2} \frac{k}{p_{\min}^{5/2}} \\
&= o(1) \frac{\sigma_z \sigma_e^2}{\upsilon(n)^2} n^{\kappa_1 - \frac{5}{2}\kappa_2} \to 0,
\end{aligned}
$$

as $n \to \infty$ due to Assumption L.2 and L.3, and the upper bound of $|\mathscr{L}_j(t)|$ in (2.3.5).

Similarly, using the expression of $\widetilde{\gamma}_s$ we have

$$
\begin{aligned}
E(\Pi_{33})^2 &= \frac{\sigma_z \sigma_e^2}{\sqrt{n}\upsilon(n)^2} \sum_{s=1}^{n} E \widetilde{\gamma}_s^2 \\
&= \frac{\sigma_z \sigma_e^2}{\sqrt{n}\upsilon(n)^2} \sum_{s=1}^{n} E \left(\sum_{i=k+1}^{\infty} \sum_{j=0}^{\infty} a_{ij} \mathscr{L}_j(s) Q_i(s, X_s) \right)^2 \\
&= \frac{\sigma_z \sigma_e^2}{\sqrt{n}\upsilon(n)^2} \sum_{s=1}^{n} \sum_{i=k+1}^{\infty} \left(\sum_{j=0}^{\infty} a_{ij} \mathscr{L}_j(s) \right)^2 \\
&\leq C^2 \frac{\sigma_z \sigma_e^2}{\sqrt{n}\upsilon(n)^2} \sum_{s=1}^{n} \sum_{i=k+1}^{\infty} \left(\sum_{j=0}^{\infty} |a_{ij}| \right)^2 \\
&\leq C^2 o(1) \frac{\sigma_z \sigma_e^2}{\sqrt{n}\upsilon(n)^2} \sum_{s=1}^{n} \frac{1}{k} = C^2 o(1) \frac{\sigma_z \sigma_e^2}{\upsilon(n)^2} \frac{\sqrt{n}}{k} \\
&= C^2 o(1) \frac{\sigma_z \sigma_e^2}{\upsilon(n)^2} n^{\frac{1}{2} - \kappa_1} \to 0,
\end{aligned}
$$

as $n \to \infty$ due to Assumption L.2 and L.4.

Hence, we obtain

$$
\Pi_3 \to_D \left(\int_0^1 G_3(t) dL_W(t, 0) \right)^{\frac{1}{2}} N. \tag{5.1.16}
$$

Secondly, for Π_1 and Π_2, from (5.1.10) it follows that

$$\begin{aligned}
\Pi_1 =& \frac{\sqrt{\sigma_z}}{\sqrt[4]{n}\upsilon(n)\sqrt{p_{\max}}}\alpha' X'\delta = \frac{\sqrt{\sigma_z}}{\sqrt[4]{n}\upsilon(n)}(\widetilde{\mathbf{F}}' - \widetilde{\delta}' - \widetilde{\gamma}')\delta \\
=& \frac{\sqrt{\sigma_z}}{\sqrt[4]{n}\upsilon(n)}(\widetilde{\mathbf{F}}'\delta - \widetilde{\delta}'\delta - \widetilde{\gamma}'\delta), \\
\Pi_2 =& \frac{\sqrt{\sigma_z}}{\sqrt[4]{n}\upsilon(n)\sqrt{p_{\max}}}\alpha' X'\gamma = \frac{\sqrt{\sigma_z}}{\sqrt[4]{n}\upsilon(n)}(\widetilde{\mathbf{F}}' - \widetilde{\delta}' - \widetilde{\gamma}')\gamma \\
=& \frac{\sqrt{\sigma_z}}{\sqrt[4]{n}\upsilon(n)}(\widetilde{\mathbf{F}}'\gamma - \widetilde{\delta}'\gamma - \widetilde{\gamma}'\gamma).
\end{aligned}$$

Thus, by Cauchy-Schwarz inequality, in order to obtain $\Pi_1 \to_P 0$ and $\Pi_2 \to_P 0$, we only need to show $\|\delta\| \to_P 0$ and $\|\gamma\| \to_P 0$ since the convergence of (5.1.15) indicates that

$$\frac{1}{\sqrt{n}}\|\widetilde{\delta}\|^2 \to_P 0, \quad \text{and} \quad \frac{1}{\sqrt{n}}\|\widetilde{\gamma}\|^2 \to_P 0, \tag{5.1.17}$$

and because of Theorem 1.3.1 and Assumption L.4 (b), we have

$$\begin{aligned}
\frac{\sigma_z}{\sqrt{n}\upsilon(n)^2}\|\widetilde{\mathbf{F}}'\|^2 =& \frac{\sigma_z}{\sqrt{n}\upsilon(n)^2}\sum_{s=1}^{n}\widetilde{F}^2(s, X_s) \\
=& \frac{\sigma_z}{\sqrt{n}\upsilon(n)^2}\sum_{s=1}^{n}F^2(s, \sqrt{n}\sigma_x x_{s,n}) \\
\to_P & \int_0^1\int_{-\infty}^{\infty} f^2(r, x)dxdL_W(r, 0).
\end{aligned} \tag{5.1.18}$$

In fact, by orthogonality of Q_i and Theorem 2.3.1 with $r = 3$,

$$\begin{aligned}
E\|\delta\|^2 =& E\sum_{s=1}^{n}\delta_s^2 = E\sum_{s=1}^{n}\left(\sum_{i=0}^{k}\sum_{j=p_i+1}^{\infty}c_{ij}\mathscr{L}_j(s)Q_i(s, X_s)\right)^2 \\
=& \sum_{s=1}^{n}\sum_{i=0}^{k}\left(\sum_{j=p_i+1}^{\infty}c_{ij}\mathscr{L}_j(s)\right)^2 \\
=& \sum_{s=1}^{n}\sum_{i=0}^{k}\left(c_i(t, m) - \sum_{j=0}^{p_i+1}c_{ij}\mathscr{L}_j(s)\right)^2 \\
\le& \sum_{s=1}^{n}\sum_{i=0}^{k}o(1)\frac{1}{\sqrt{sp_i}}\frac{1}{p_i^2} \le o(1)\frac{k}{p_{\min}^{\frac{5}{2}}}\sum_{s=1}^{n}\frac{1}{\sqrt{s}} \\
=& o(1)n^{\frac{1}{2}+\kappa_1-\frac{5}{2}\kappa_2} \to 0
\end{aligned}$$

as $n \to \infty$, where we have used the bound for $\mathscr{L}_j(s)$ and Assumption L.4 (a).

Similarly, in view of (4.4.8), $c_i(t,m) = \sqrt{v(t)}^r \sqrt{\frac{(i-r)!}{i!}} c_i(t, D^r m)$. It follows from Assumption L.1 (a) and (c) with $r = 3$ that

$$\begin{aligned} E\|\gamma\|^2 =& E\sum_{s=1}^n \gamma_s^2 = E\sum_{s=1}^n \left(\sum_{i=k+1}^\infty c_i(s,m) Q_i(s, X_s)\right)^2 \\ =& \sum_{s=1}^n \sum_{i=k+1}^\infty (c_i(s,m))^2 \\ =& \sum_{s=1}^n \sum_{i=k+1}^\infty v(s)^3 \frac{(i-3)!}{i!} c_i(s, D^3 m)^2 \\ =& \sum_{i=k+1}^\infty \frac{(i-3)!}{i!} \sum_{s=1}^n v(s)^3 c_i(s, D^3 m)^2 \\ \le& An\frac{1}{k^2}(1+o(1)) = An^{1-2\kappa_1} \to 0, \end{aligned}$$

where A is the uniform bound of $v(s)^3 c_i(s, D^3 m)^2$ on account of Assumption L.1 and we have used Assumption L.3.

Now we are ready to prove that $\Pi_4 \to_P 0$ as $n \to \infty$. We may rewrite

$$\begin{aligned} \Pi_4 =& \frac{\sqrt{\sigma_z}}{\sqrt[4]{n} v(n)} \frac{\alpha' X' X A(\tau, x)}{\sqrt{p_{\max}} \|A(\tau,x)\|^2} [\delta(\tau, x) + \gamma(\tau, x)] \\ =& \frac{\sqrt{\sigma_z}}{\sqrt{n} v(n) \varrho(n)} \frac{\alpha' X' X A(\tau, x)}{p_{\max}\|A(\tau,x)\|} \frac{\sqrt[4]{n}\sqrt{p_{\max}}\varrho(n)}{\|A(\tau,x)\|} [\delta(\tau, x) + \gamma(\tau, x)] \\ :=& \Pi_{41} \times (\Pi_{42} + \Pi_{43}), \end{aligned}$$

where we denote

$$\begin{aligned} \Pi_{41} =& \frac{\sqrt{\sigma_z}}{\sqrt{n} v(n) \varrho(n)} \frac{\alpha' X' X A(\tau, x)}{p_{\max}\|A(\tau,x)\|}, \\ \Pi_{42} =& \frac{\sqrt[4]{n}\sqrt{p_{\max}}\varrho(n)}{\|A(\tau,x)\|}\delta(\tau,x) \quad \text{and} \quad \Pi_{43} = \frac{\sqrt[4]{n}\sqrt{p_{\max}}\varrho(n)}{\|A(\tau,x)\|}\gamma(\tau,x). \end{aligned}$$

We shall demonstrate that Π_{41} converges to a random variable in probability and Π_{42} and Π_{43} approach to zero as $n \to \infty$. To begin with, it follows from (5.1.10) and (5.1.11) that

$$\begin{aligned} \Pi_{41} =& \frac{\sqrt{\sigma_z}}{\sqrt{n} v(n) \varrho(n)} (\widetilde{\mathbf{F}}' - \widetilde{\delta}' - \widetilde{\gamma}')(\widetilde{\mathbf{G}} - \widetilde{\delta} - \widetilde{\gamma}) \\ =& \frac{\sqrt{\sigma_z}}{\sqrt{n} v(n) \varrho(n)} (\widetilde{\mathbf{F}}'\widetilde{\mathbf{G}} - \widetilde{\delta}'\widetilde{\mathbf{G}} - \widetilde{\gamma}'\widetilde{\mathbf{G}} - \widetilde{\mathbf{F}}'\widetilde{\delta} - \widetilde{\mathbf{F}}'\widetilde{\gamma} + \|\widetilde{\delta}\|^2 + \|\widetilde{\gamma}\|^2 + 2\widetilde{\gamma}'\widetilde{\delta}). \end{aligned}$$

Therefore, by virtue of (5.1.17) and (5.1.18), to find out the limit of Π_{41}, our remaining task is to prove the convergence of $\frac{\sigma_z}{\sqrt{n}\varrho(n)^2}\|\widetilde{\mathbf{G}}\|^2$, and $\frac{\sigma_z}{\sqrt{n}\upsilon(n)\varrho(n)}\widetilde{\mathbf{F}}'\widetilde{\mathbf{G}}$ due to Cauchy-Schwarz inequality.

Similar to (5.1.18),

$$\begin{aligned}\frac{\sigma_z}{\sqrt{n}\upsilon(n)^2}\|\widetilde{\mathbf{G}}\|^2 =&\frac{\sigma_z}{\sqrt{n}\upsilon(n)^2}\sum_{s=1}^n \widetilde{\mathbf{G}}(s,X_s)^2 = \frac{\sigma_z}{\sqrt{n}\upsilon(n)^2}\sum_{s=1}^n G(s,X_s-EX_s)^2\\ =&\frac{\sigma_z}{\sqrt{n}\upsilon(n)^2}\sum_{s=1}^n G(s,\sqrt{n}\sigma_z x_{s,n})^2\\ \to_P&\int_0^1\int_{-\infty}^{\infty} g^2(t,x)dxdL_W(t,0),\end{aligned}$$

and

$$\begin{aligned}\frac{\sigma_z}{\sqrt{n}\upsilon(n)\varrho(n)}\widetilde{\mathbf{F}}'\widetilde{\mathbf{G}} =&\frac{\sigma_z}{\sqrt{n}\upsilon(n)\varrho(n)}\sum_{s=1}^n \widetilde{\mathbf{F}}(s,X_s)\widetilde{\mathbf{G}}(s,X_s)\\ =&\frac{\sigma_z}{\sqrt{n}\upsilon(n)\varrho(n)}\sum_{s=1}^n F(s,X_s-EX_s)G(s,X_s-EX_s)\\ =&\frac{\sigma_z}{\sqrt{n}\upsilon(n)\varrho(n)}\sum_{s=1}^n F(s,\sqrt{n}\sigma_z x_{s,n})G(s,\sqrt{n}\sigma_z x_{s,n})\\ \to_P&\int_0^1\int_{-\infty}^{\infty} f(t,x)g(t,x)dxdL_W(t,0),\end{aligned}$$

using Assumption L.4 (b).

Whence, $\Pi_{41}\to_P \int_0^1\int_{-\infty}^{\infty} f(t,x)g(t,x)dxdL_W(t,0)$ as $n\to\infty$.

For the proof of the vanish of Π_{42} and Π_{43}, we first estimate $\|A(\tau,x)\|$:

$$\begin{aligned}\|A(\tau,x)\|^2 =&\sum_{i=0}^k\sum_{j=0}^{p_i}\mathscr{L}_j^2(\tau)Q_i^2(\tau,x) = e^{-\tau}\sum_{i=0}^k Q_i^2(\tau,x)\sum_{j=0}^{p_i} L_j^2(\tau)\\ =&O(1)e^{-\tau}\sum_{i=0}^k p_iQ_i^2(\tau,x),\end{aligned}$$

which leads to $O(1)kp_{\min}\le \|A(\tau,x)\|^2\le O(1)kp_{\max}$ where we have used the fact that $\sum_{i=0}^k H_i^2(x)=O(1)k$ uniformly in x for any orthogonal polynomial $H_i(x)$ on any compact interval (see Alexits, 1961, p.295).

Accordingly, due to Assumption L.1(b), using the result in Theorem 2.3.1 with $r=3$

and the upper bound in (2.3.5) gives

$$
\begin{aligned}
|\Pi_{42}| =& \frac{\sqrt[4]{n}\sqrt{p_{\max}}\varrho(n)}{\|A(\tau,x)\|}|\delta(\tau,x)| = \frac{\sqrt[4]{n}\sqrt{p_{\max}}\varrho(n)}{\|A(\tau,x)\|}\left|\sum_{i=0}^{k}\sum_{j=p_i+1}^{\infty} c_{ij}\mathscr{L}_j(\tau)Q_i(\tau,x)\right| \\
\le& \frac{\sqrt[4]{n}\sqrt{p_{\max}}\varrho(n)}{\|A(\tau,x)\|}\sum_{i=0}^{k}|Q_i(\tau,x)|\left|\sum_{j=p_i+1}^{\infty} c_{ij}\mathscr{L}_j(\tau)\right| \\
\le& \frac{\sqrt[4]{n}\sqrt{p_{\max}}\varrho(n)}{\|A(\tau,x)\|}\left(\sum_{i=0}^{k}Q_i^2(\tau,x)\right)^{\frac{1}{2}}\left[\sum_{i=0}^{k}\left(\sum_{j=p_i+1}^{\infty} c_{ij}\mathscr{L}_j(\tau)\right)^2\right]^{\frac{1}{2}} \\
\le& O(1)\frac{\sqrt[4]{n}\sqrt{p_{\max}}n^{\iota}}{\sqrt{kp_{\min}}}\sqrt{k}\left[\sum_{i=0}^{k}\frac{1}{\sqrt{\tau p_i}}\frac{o(1)}{p_i^2}\right]^{\frac{1}{2}} \le o(1)\frac{n^{\frac{1}{4}+\iota}\sqrt{p_{\max}}k^{\frac{1}{2}}}{\sqrt{p_{\min}}p_{\min}^{\frac{5}{4}}} \\
=& o(1)n^{\frac{1}{4}+\iota+\frac{1}{2}\kappa_1+\frac{1}{2}(\bar{\kappa}_2-\kappa_2)-\frac{5}{4}\kappa_2} \to 0,
\end{aligned}
$$

where we have used the condition in Assumption L.4 (b) for parameters.

Meanwhile, on account of Assumption L.1 using (4.4.8) with $r = 3$ and the asymptotic inequality for Q_i in Remark 4.3.4, we have

$$
\begin{aligned}
|\Pi_{43}| =& \frac{\sqrt[4]{n}\sqrt{p_{\max}}\varrho(n)}{\|A(\tau,x)\|}|\gamma(\tau,x)| = \frac{\sqrt[4]{n}\sqrt{p_{\max}}\varrho(n)}{\|A(\tau,x)\|}\left|\sum_{i=k+1}^{\infty} c_i(\tau,m)Q_i(\tau,x)\right| \\
=& \frac{\sqrt[4]{n}\sqrt{p_{\max}}\varrho(n)}{\|A(\tau,x)\|}\left|\sum_{i=k+1}^{\infty}\frac{\sqrt{v(\tau)}^3}{\sqrt{i(i-1)(i-2)}}c_i(\tau,D^3m)Q_i(\tau,x)\right| \\
\le& o(1)\frac{\sqrt[4]{n}\sqrt{p_{\max}}\varrho(n)}{\|A(\tau,x)\|}\left(\sum_{i=k+1}^{\infty}\frac{1}{i(i-1)(i-2)}Q_i(\tau,x)^2\right)^{\frac{1}{2}} \\
\le& o(1)\frac{\sqrt[4]{n}n^{\iota}\sqrt{p_{\max}}}{\sqrt{kp_{\min}}}\left(\sum_{i=k+1}^{\infty}\frac{1}{i(i-1)(i-2)\sqrt{i}}\right)^{\frac{1}{2}} \\
\le& o(1)\frac{\sqrt[4]{n}n^{\iota}\sqrt{p_{\max}}}{\sqrt{kp_{\min}}}\frac{1}{k^{5/4}} = o(1)n^{\frac{1}{4}+\iota+\frac{1}{2}(\bar{\kappa}_2-\kappa_2)-\frac{7}{4}\kappa_1} \to 0,
\end{aligned}
$$

by the condition in Assumption L.4 (a) for parameters. The proof of (5.1.12) is completed.

We are now in a position to prove (5.1.13). In view of (5.1.7),

$$
\begin{aligned}
& \frac{1}{\sqrt{n}\upsilon_1(n)\upsilon_2(\sqrt{n}\sigma_z)}\frac{\alpha' X' X A(\tau,x)}{\sqrt{p_{\max}}\|A(\tau,x)\|^2}(\widehat{m}(\tau,x)-m(\tau,x)) \\
=& \frac{1}{\sqrt{n}\upsilon_1(n)\upsilon_2(\sqrt{n}\sigma_z)}\frac{\alpha' X' X A(\tau,x)}{\sqrt{p_{\max}}\|A(\tau,x)\|^2}
\end{aligned}
$$

$$
\begin{aligned}
&\times [A'(\tau,x)(X'X)^{-1}X'(\delta+\gamma+\varepsilon)-\delta(\tau,x)-\gamma(\tau,x)]\\
=&\frac{1}{\sqrt{n}\upsilon_1(n)\upsilon_2(\sqrt{n}\sigma_z)}\frac{\alpha'X'}{\sqrt{p_{\max}}}(\delta+\gamma+\varepsilon)\\
&-\frac{1}{\sqrt{n}\upsilon_1(n)\upsilon_2(\sqrt{n}\sigma_z)}\frac{\alpha'X'XA(\tau,x)}{\sqrt{p_{\max}}\|A(\tau,x)\|^2}[\delta(\tau,x)+\gamma(\tau,x)]\\
:=&\sum_{i=1}^{3}\Gamma_i-\Gamma_4,
\end{aligned}
$$

where we by Γ_i $(i=1,2,3,4)$ signify that

$$
\begin{aligned}
\Gamma_1=&\frac{1}{\sqrt{n}\upsilon_1(n)\upsilon_2(\sqrt{n}\sigma_z)}\frac{\alpha'X'}{\sqrt{p_{\max}}}\delta=\frac{1}{\sqrt{n}\upsilon_1(n)\upsilon_2(\sqrt{n}\sigma_z)}(\widetilde{\mathbf{F}}'-\widetilde{\delta}'-\widetilde{\gamma}')\delta,\\
\Gamma_2=&\frac{1}{\sqrt{n}\upsilon_1(n)\upsilon_2(\sqrt{n}\sigma_z)}\frac{\alpha'X'}{\sqrt{p_{\max}}}\gamma=\frac{1}{\sqrt{n}\upsilon_1(n)\upsilon_2(\sqrt{n}\sigma_z)}(\widetilde{\mathbf{F}}'-\widetilde{\delta}'-\widetilde{\gamma}')\gamma,\\
\Gamma_3=&\frac{1}{\sqrt{n}\upsilon_1(n)\upsilon_2(\sqrt{n}\sigma_z)}\frac{\alpha'X'}{\sqrt{p_{\max}}}\varepsilon=\frac{1}{\sqrt{n}\upsilon_1(n)\upsilon_2(\sqrt{n}\sigma_z)}(\widetilde{\mathbf{F}}'-\widetilde{\delta}'-\widetilde{\gamma}')\varepsilon,\\
\Gamma_4=&\frac{1}{\sqrt{n}\upsilon_1(n)\upsilon_2(\sqrt{n}\sigma_z)}\frac{\alpha'X'XA(\tau,x)}{\sqrt{p_{\max}}\|A(\tau,x)\|^2}[\delta(\tau,x)+\gamma(\tau,x)].
\end{aligned}
$$

We are going to prove that $\Gamma_i\to_P 0$, $i=1,2,4$ and $Gamma_3$ converges to the desired result in probability in the embedding framework. Observe that due to Assumption L.4 (c) and (d), we have

$$
\begin{aligned}
&\frac{1}{\sqrt{n}\upsilon_1(n)\upsilon_2(\sqrt{n}\sigma_z)}\widetilde{\mathbf{F}}'\varepsilon=\frac{1}{\sqrt{n}\upsilon_1(n)\upsilon_2(\sqrt{n}\sigma_z)}\sum_{s=1}^{n}\widetilde{F}(s,X_s)e_s\\
=&\frac{1}{\sqrt{n}\upsilon_1(n)\upsilon_2(\sqrt{n}\sigma_z)}\sum_{s=1}^{n}F(s,\sqrt{n}\sigma_z x_{s,n})e_s\to_P\int_0^1 f(r,W(r))dU(r),
\end{aligned}
\tag{5.1.19}
$$

and

$$
\begin{aligned}
&\frac{1}{n\upsilon_1(n)^2\upsilon_2(\sqrt{n}\sigma_z)^2}\|\widetilde{\mathbf{F}}\|^2=\frac{1}{n\upsilon_1(n)^2\upsilon_2(\sqrt{n}\sigma_z)^2}\sum_{s=1}^{n}\widetilde{F}(s,X_s)^2\\
=&\frac{1}{n\upsilon_1(n)^2\upsilon_2(\sqrt{n}\sigma_z)^2}\sum_{s=1}^{n}F(s,\sqrt{n}\sigma_z x_{s,n})^2\to_{a.s.}\int_0^1 f^2(r,W(r))dr,
\end{aligned}
\tag{5.1.20}
$$

by the proof (not the result) of Theorem 1.5.1.

Note that in first part we have shown that

$$
\frac{1}{\sqrt{n}}\|\widetilde{\delta}'\|^2\to_P 0,\qquad \frac{1}{\sqrt{n}}\|\widetilde{\gamma}'\|^2\to_P 0,\qquad \|\delta\|^2\to_P 0,\tag{5.1.21a}
$$

$$\|\gamma\|^2 \to_P 0, \qquad \frac{1}{\sqrt[4]{n}}\widetilde{\delta}'\varepsilon \to_P 0, \qquad \frac{1}{\sqrt[4]{n}}\widetilde{\gamma}'\varepsilon \to_P 0. \tag{5.1.21b}$$

All results in (5.1.21) remain true since all conditions for δ, γ, $\widetilde{\delta}$, $\widetilde{\gamma}$ and ε have not changed. Therefore, (5.1.19),(5.1.20) and (5.1.21) imply that $\Gamma_1 \to_P 0$ and $\Gamma_2 \to_P 0$, as well as $\Gamma_3 \to_P \int_0^1 f(r, W(r))dU(r)$ as $n \to \infty$. Thus, our remaining task is to prove $\Gamma_4 \to_P 0$ as $n \to \infty$.

To this end, let us rewrite

$$\begin{aligned}
\Gamma_4 =& \frac{1}{\sqrt{n}\upsilon_1(n)\upsilon_2(\sqrt{n}\sigma_z)} \frac{\alpha' X' X A(\tau, x)}{\sqrt{p_{\max}}\|A(\tau, x)\|^2}[\delta(\tau, x) + \gamma(\tau, x)] \\
:=& \Gamma_{41} \times (\Gamma_{42} + \Gamma_{43}),
\end{aligned}$$

where we denote

$$\begin{aligned}
\Gamma_{41} =& \frac{1}{n\upsilon_1(n)\upsilon_2(\sqrt{n}\sigma_z)\varrho_1(n)\varrho_2(\sqrt{n}\sigma_z)} \frac{\alpha' X' X A(\tau, x)}{p_{\max}\|A(\tau, x)\|}, \\
\Gamma_{42} =& \frac{\sqrt{n}\sqrt{p_{\max}}\varrho_1(n)\varrho_2(\sqrt{n}\sigma_z)}{\|A(\tau, x)\|}\delta(\tau, x), \quad \text{and} \\
\Gamma_{43} =& \frac{\sqrt{n}\sqrt{p_{\max}}\varrho_1(n)\varrho_2(\sqrt{n}\sigma_z)}{\|A(\tau, x)\|}\gamma(\tau, x).
\end{aligned}$$

It follows from (5.1.10) and (5.1.11) that

$$\begin{aligned}
\Gamma_{41} =& \frac{1}{n\upsilon_1(n)\upsilon_2(\sqrt{n}\sigma_z)\varrho_1(n)\varrho_2(\sqrt{n}\sigma_z)}(\widetilde{\mathbf{F}}' - \widetilde{\delta}' - \widetilde{\gamma}')(\widetilde{\mathbf{G}}' - \widetilde{\delta}' - \widetilde{\gamma}') \\
=& \frac{1}{n\upsilon_1(n)\upsilon_2(\sqrt{n}\sigma_z)\varrho_1(n)\varrho_2(\sqrt{n}\sigma_z)} \\
& \times (\widetilde{\mathbf{F}}'\widetilde{\mathbf{G}} - \widetilde{\delta}'\widetilde{\mathbf{G}} - \widetilde{\gamma}'\widetilde{\mathbf{G}} - \widetilde{\mathbf{F}}'\widetilde{\delta} - \widetilde{\mathbf{F}}'\widetilde{\gamma} + \|\widetilde{\delta}\|^2 + \|\widetilde{\gamma}\|^2 + 2\widetilde{\gamma}'\widetilde{\delta}).
\end{aligned}$$

Once again, due to Theorem 1.5.1 and Assumption L.4 (d), we similarly have

$$\begin{aligned}
\frac{1}{n\varrho_1(n)^2\varrho_2(\sqrt{n}\sigma_z)^2}\sum_{s=1}^n \widetilde{G}^2(s, X_s) =& \frac{1}{n\varrho_1(n)^2\varrho_2(\sqrt{n}\sigma_z)^2}\sum_{s=1}^n G^2(s, \sqrt{n}\sigma_z x_{s,n}) \\
& \to_{a.s.} \int_0^1 g^2(r, W(r))dr.
\end{aligned} \tag{5.1.22}$$

Thus, Cauchy-Schwarz inequality as well as (5.1.20), (5.1.21) and (5.1.22) suggest that to find out the limit of Γ_{41} it suffices to find that of the term in Γ_{41} involving $\widetilde{\mathbf{F}}'\widetilde{\mathbf{G}}$. In fact, by Assumption L.4 (d),

$$\frac{1}{n\upsilon_1(n)\upsilon_2(\sqrt{n}\sigma_z)\varrho_1(n)\varrho_2(\sqrt{n}\sigma_z)}\widetilde{\mathbf{F}}'\widetilde{\mathbf{G}}$$

$$
\begin{aligned}
&= \frac{1}{n\upsilon_1(n)\upsilon_2(\sqrt{n}\sigma_z)\varrho_1(n)\varrho_2(\sqrt{n}\sigma_z)} \sum_{s=1}^{n} \widetilde{F}(s, X_s)\widetilde{G}(s, X_s) \\
&= \frac{1}{n\upsilon_1(n)\upsilon_2(\sqrt{n}\sigma_z)\varrho_1(n)\varrho_2(\sqrt{n}\sigma_z)} \sum_{s=1}^{n} F(s, \sqrt{n}\sigma_z x_{s,n})G(s, \sqrt{n}\sigma_z x_{s,n}) \\
&\to_{a.s.} \int_0^1 f(r, W(r))g(r, W(r))dr,
\end{aligned}
$$

as $n \to \infty$, so that Γ_{41} converges to the same limit in probability.

We are ready to prove both $\Gamma_{42} \to 0$ and $\Gamma_{43} \to 0$, as $n \to \infty$. By virtue of the estimate of $\|A(\tau, x)\|$, due to Assumption L.1(b), using Theorem 2.3.1 with $r = 3$ gives

$$
\begin{aligned}
|\Gamma_{42}| =& \frac{\sqrt{n}\sqrt{p_{\max}}\varrho_1(n)\varrho_2(\sqrt{n}\sigma_z)}{\|A(\tau, x)\|}|\delta(\tau, x)| \\
=& \frac{\sqrt{n}\sqrt{p_{\max}}\varrho_1(n)\varrho_2(\sqrt{n}\sigma_z)}{\|A(\tau, x)\|} \left| \sum_{i=0}^{k} \sum_{j=p_i+1}^{\infty} c_{ij}\mathscr{L}_j(\tau)Q_i(\tau, x) \right| \\
=& \frac{\sqrt{n}\sqrt{p_{\max}}\varrho_1(n)\varrho_2(\sqrt{n}\sigma_z)}{\|A(\tau, x)\|} \left| \sum_{i=0}^{k} Q_i(\tau, x) \sum_{j=p_i+1}^{\infty} c_{ij}\mathscr{L}_j(\tau) \right| \\
\le& \frac{\sqrt{n}\sqrt{p_{\max}}\varrho_1(n)\varrho_2(\sqrt{n}\sigma_z)}{\|A(\tau, x)\|} \left(\sum_{i=0}^{k} Q_i^2(\tau, x) \right)^{\frac{1}{2}} \left[\sum_{i=0}^{k} \left(\sum_{j=p_i+1}^{\infty} c_{ij}\mathscr{L}_j(\tau) \right)^2 \right]^{\frac{1}{2}} \\
\le& O(1) \frac{\sqrt{n}\sqrt{p_{\max}} n^{\iota_1} n^{\frac{1}{2}\iota_2}}{\sqrt{k p_{\min}}} \sqrt{k} \left[\sum_{i=0}^{k} \frac{1}{\sqrt{\tau p_i}} \frac{o(1)}{p_i^2} \right]^{\frac{1}{2}} \\
\le& o(1)\sqrt{n} n^{\frac{1}{2}(\bar{\kappa}_2 - \kappa_2)} n^{\iota_1 + \frac{1}{2}\iota_2} \sqrt{k} p_{\min}^{-\frac{5}{4}} \\
=& o(1) n^{\frac{1}{2} + \frac{1}{2}(\bar{\kappa}_2 - \kappa_2) + \iota_1 + \frac{1}{2}\iota_2 + \frac{1}{2}\kappa_1 - \frac{5}{4}\kappa_2} \to 0,
\end{aligned}
$$

as $n \to \infty$ by Assumption L.1(b) and condition (ii) of Assumption L.4 (b).

Meanwhile, once again on account of Assumption L.1 using (4.4.8) with $r = 3$, we have

$$
\begin{aligned}
|\Gamma_{43}| =& \frac{\sqrt{n}\sqrt{p_{\max}}\varrho_1(n)\varrho_2(\sqrt{n}\sigma_z)}{\|A(\tau, x)\|}|\gamma(\tau, x)| \\
=& \frac{\sqrt{n}\sqrt{p_{\max}}\varrho_1(n)\varrho_2(\sqrt{n}\sigma_z)}{\|A(\tau, x)\|} \left| \sum_{i=k+1}^{\infty} \sum_{j=0}^{\infty} c_{ij}\mathscr{L}_j(\tau)Q_i(\tau, x) \right| \\
=& \frac{\sqrt{n}\sqrt{p_{\max}}\varrho_1(n)\varrho_2(\sqrt{n}\sigma_z)}{\|A(\tau, x)\|} \left| \sum_{i=k+1}^{\infty} c_i(\tau, m)Q_i(\tau, x) \right|
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{\sqrt{n}\sqrt{p_{\max}}\varrho_1(n)\varrho_2(\sqrt{n}\sigma_z)}{\|A(\tau,x)\|}\left|\sum_{i=k+1}^{\infty}\frac{v(\tau)^3}{\sqrt{i(i-1)(i-2)}}c_i(\tau,D^3m)Q_i(\tau,x)\right|\\
\le&O(1)\frac{\sqrt{n}\sqrt{p_{\max}}n^{\iota_1}n^{\frac{1}{2}\iota_2}}{\sqrt{kp_{\min}}}\left(\sum_{i=k+1}^{\infty}|c_i(\tau,D^3m)|^2\right)^{\frac{1}{2}}\\
&\times\left(\sum_{i=k+1}^{\infty}\frac{1}{i(i-1)(i-2)}|Q_i(\tau,x)|^2\right)^{\frac{1}{2}}\\
\le&o(1)\frac{\sqrt{n}\sqrt{p_{\max}}n^{\iota_1}n^{\frac{1}{2}\iota_2}}{\sqrt{kp_{\min}}}\left(\sum_{i=k+1}^{\infty}\frac{1}{i(i-1)(i-2)\sqrt{i}}\right)^{\frac{1}{2}}\\
\le&o(1)\frac{\sqrt{n}\sqrt{p_{\max}}n^{\iota_1}n^{\frac{1}{2}\iota_2}}{\sqrt{kp_{\min}}}\frac{1}{k^{5/4}}\\
=&o(1)n^{\frac{1}{2}+\frac{1}{2}(\bar\kappa_2-\kappa_2)+\iota_1+\frac{1}{2}\iota_2-\frac{7}{4}\kappa_1}\to 0
\end{aligned}
$$

by Assumption L.4 (c). This finishes the proof. □

5.2 Finite time horizon

Assume time variable t lies in $[0,T]$ with T fixed. In this section function m is defined on $[0,T]\times I$. Therefore, conditions on m would be weakened since square integrability on $[0,T]$ is much weaker than that on the half line. We make the following assumptions about $m(t,x)$ in the model (5.0.1).

Assumption L.5

(a) Let $D^r m(t,x)\in L^2(I,\rho_r(t,x))$ for any $t\in[0,T]$ and $r=0,1,2$. Moreover, the expansion of $D^2m(t,Z(t))$ in terms of $Q_{2i}(t,Z(t))$ is convergent in the sense of mean square uniformly on $[0,T]$.

(b) For each i, $b_i(t,m)=E[m(t,Z_t)Q_i(t,Z_t)]$ and its derivatives of up to third order belong to $C[0,T]$.

(c) Furthermore, $\|b_i''(t,m)\|_{L^2[0,T]}$ are bounded uniformly in i.

Remark 5.2.1. The conditions are quite weak. Condition (a) can be satisfied by all functions $m(t,x)$ as long as m,Dm,D^2m have finite second moment and the moment function is integrable on $[0,T]$. Condition (b) is not restrictive as well, albeit we request the derivatives are continuous. The continuities make sure their expansions are convergent uniformly on the interval $[0,T]$.

Suppose that we have n observations for the process $Y(t)$ on $[0,T]$ and the observations are $Y_{s,n} = Y(t_{s,n})$ at $t_{s,n} = T\frac{s}{n}$ for $s = 1, 2, \cdots, n$. At the sampling points, we have the following model

$$Y_{s,n} = m(t_{s,n}, X_{s,n}) + e_s, \quad s = 1, \ldots, n, \tag{5.2.1}$$

where $X_{s,n} = Z(T\frac{s}{n})$ denote the Lévy process $Z(t)$ at point $t_{s,n}$, $e_s = \varepsilon(T\frac{s}{n})$ $(s = 1, \ldots, n)$ form an error sequence with mean zero and finite variance.

Note that $X_{s,n} = \sum_{i=1}^{s}(X_{i,n} - X_{i-1,n}) = \frac{s}{n}T\mu + \sqrt{T}\sigma_z\frac{1}{\sqrt{n}}\sum_{i=1}^{s} w_i$, where $w_i = \frac{\sqrt{n}}{\sqrt{T}\sigma_z}(X_{i,n} - X_{i-1,n} - \frac{1}{n}T\mu)$ form an i.i.d.(0,1) sequence. Let $x_{s,n} = \frac{1}{\sqrt{n}}\sum_{i=1}^{s} w_i$. It follows from the functional central limit theorem that $x_{s,n}$ converges to a standard Brownian motion in distribution as $n \to \infty$. It also is clear that $x_{s,n}$, along with $d_{l,k,n} = \sqrt{(l-k)/n}$, satisfies Assumption A.

Under Assumption L.5 we can expand $m(t, Z(t))$ at every point $t \in [0,T]$ using basis $\varphi_{jT}(t)Q_i(t, Z(t))$. Let k and p_i be truncation parameters for i and j. Thus, the models (5.2.1) become

$$\begin{aligned} Y_{s,n} =& \sum_{i=0}^{k}\sum_{j=0}^{p_i} b_{ij}\varphi_{jT}(t_{s,n})Q_i(t_{s,n}, X_{s,n}) + \sum_{i=0}^{k}\sum_{j=p_i+1}^{\infty} b_{ij}\varphi_{jT}(t_{s,n})Q_i(t_{s,n}, X_{s,n}) \\ &+ \sum_{i=k+1}^{\infty}\sum_{j=0}^{\infty} b_{ij}\varphi_{jT}(t_{s,n})Q_i(t_{s,n}, X_{s,n}) + e_s, \quad s = 1, 2, \cdots, n. \end{aligned} \tag{5.2.2}$$

Equivalently, the matrix form of (5.2.2) is

$$Y = X\beta + \delta + \gamma + \varepsilon, \tag{5.2.3}$$

where all notations remain the same as in the last subsection so that we omit reciting them. The OLS estimator of β is given by

$$\widehat{\beta} = (X'X)^{-1}X'Y. \tag{5.2.4}$$

With the help of $\widehat{\beta}$ we are able to estimate $m(\cdot,\cdot)$ at (τ, x) where τ is any point in $[0,T]$ and x is any point on the path of $Z(\tau)$. On account of Assumption L.5, $m(\tau, x)$ can be expanded into an orthogonal series,

$$
\begin{aligned}
m(\tau,x) =& \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} b_{ij}\varphi_{jT}(\tau)Q_i(\tau,x) \\
:=& A'(\tau,x)\beta + \delta(\tau,x) + \gamma(\tau,x),
\end{aligned}
\tag{5.2.5}
$$

where

$$
\begin{aligned}
\delta(\tau,x) &= \sum_{i=0}^{k}\sum_{j=p_i+1}^{\infty} b_{ij}\varphi_{jT}(\tau)Q_i(\tau,x), \\
\gamma(\tau,x) &= \sum_{i=k+1}^{\infty}\sum_{j=0}^{\infty} b_{ij}\varphi_{jT}(\tau)Q_i(\tau,x), \\
A'(\tau,x) &= (\varphi_{0T}(\tau)Q_0(\tau,x),\cdots,\varphi_{p_0T}(\tau)Q_0(\tau,x), \\
&\qquad \cdots,\varphi_{0T}(\tau)Q_k(\tau,x),\cdots,\varphi_{p_kT}(\tau)Q_k(\tau,x)).
\end{aligned}
$$

Then $\widehat{m}(\tau,x)$ is obtained by substituting β with $\widehat{\beta}$ and getting rid of the residues,

$$
\widehat{m}(\tau,x) = A'(\tau,x)\widehat{\beta}. \tag{5.2.6}
$$

We shall investigate the limit of

$$
\begin{aligned}
\widehat{m}(\tau,x) - m(\tau,x) &= A'(\tau,x)(\widehat{\beta}-\beta) - \delta(\tau,x) - \gamma(\tau,x) \\
&= A'(\tau,x)(X'X)^{-1}X'(\delta+\gamma+\varepsilon) - \delta(\tau,x) - \gamma(\tau,x).
\end{aligned}
\tag{5.2.7}
$$

For this purpose, let us put

$$
A = \frac{A(\tau,x)A'(\tau,x)}{\|A(\tau,x)\|^2} \quad \text{and} \quad B = (X'X)A(X'X)^{-1}. \tag{5.2.8}
$$

Once again, by virtue of Lemma 3.1.2, B has eigenvalues $\lambda_1 = 1$, $\lambda_2 = \cdots = \lambda_p = 0$. Let unit column vector α be the left eigenvector of B pertaining to λ_1, viz., $\alpha' B = \alpha'$ and $\|\alpha\| = 1$. In accordance with the notation of $A(\tau,x)$, the subscript of α is specified in double-index, that is, $\alpha' = (\alpha_{00},\cdots,\alpha_{0p_0},\cdots,\alpha_{k0},\cdots,\alpha_{kp_k})$.

Let us apply the reshuffle procedure for the set $\mathcal{S}$ from Assumption L.2 by α and $\frac{1}{\|A(\tau,x)\|}A(\tau,x)$. Denote by $\widetilde{\mathcal{S}}$ and $\bar{\mathcal{S}}$ the resulting sets:

1) $\widetilde{\mathcal{S}} = \{\widetilde{a}_0,\cdots,\widetilde{a}_i,\cdots\}$, and $\bar{\mathcal{S}} = \{\bar{a}_0,\cdots,\bar{a}_i,\cdots\}$.

2) $\widetilde{a}_i = \{\widetilde{a}_{ij}\}$ where $\widetilde{a}_{ij} = \frac{1}{\sqrt{p_{\max}}}\alpha_{ij}$ for $0 \le i \le k$ and $0 \le j \le p_i$; otherwise, $\widetilde{a}_{ij} = a_{ij}$.

3) $\bar{a}_i = \{\bar{a}_{ij}\}$ where $\bar{a}_{ij} = \frac{1}{\sqrt{p_{\max}}\|A(\tau,x)\|}\varphi_{jT}(\tau)Q_i(\tau,x)$ for $0 \leq i \leq k$ and $0 \leq j \leq p_i$; otherwise, $\bar{a}_{ij} = a_{ij}$.

Since the Riesz-Fischer theorem is satisfied by both $\widetilde{\mathcal{S}}$ and $\bar{\mathcal{S}}$, there exist two functions, denoted by $F(t,x)$ and $G(t,x)$, such that

$$\begin{aligned} F(t,Z(t)) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\widetilde{a}_{ij}\varphi_{jT}(t)Q_i(t,Z(t)), \\ G(t,Z(t)) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\bar{a}_{ij}\varphi_{jT}(t)Q_i(t,Z(t)), \end{aligned} \tag{5.2.9}$$

for any $t \in [0,T]$.

Therefore, by virtue of equations in (5.2.9),

$$\frac{1}{\sqrt{p_{\max}}}\alpha' X' = \mathbf{F}' - \widetilde{\delta}' - \widetilde{\gamma}', \tag{5.2.10a}$$

$$\frac{1}{\sqrt{p_{\max}}\|A(\tau,x)\|}A(\tau,x)'X' = \mathbf{G}' - \widetilde{\delta}' - \widetilde{\gamma}', \tag{5.2.10b}$$

where

$$\begin{aligned} \mathbf{F}' &= (F(t_{1,n},X_{1,n}),\cdots,F(t_{n,n},X_{n,n})), \\ \mathbf{G}' &= (G(t_{1,n},X_{1,n}),\cdots,G(t_{n,n},X_{n,n})), \\ \widetilde{\delta}' &= (\widetilde{\delta}_1,\cdots,\widetilde{\delta}_n), \text{ with } \widetilde{\delta}_s = \sum_{i=0}^{k}\sum_{j=p_i+1}^{\infty} a_{ij}\varphi_{jT}(t_{s,n})Q_i(t_{s,n},X_{s,n}), \\ \widetilde{\gamma}' &= (\widetilde{\gamma}_1,\cdots,\widetilde{\gamma}_n), \text{ with } \widetilde{\gamma}_s = \sum_{i=k+1}^{\infty}\sum_{j=0}^{\infty} a_{ij}\varphi_{jT}(t_{s,n})Q_i(t_{s,n},X_{s,n}). \end{aligned}$$

The following lemma demonstrates the finiteness of second moment of $F(t,Z(t))$ and $G(t,Z(t))$.

Lemma 5.2.1. *For any $t \in [0,T]$, (a) $E[F^2(t,Z(t))] < \infty$, and (b) $E[G^2(t,Z(t))] < \infty$.*

Proof. (a) It follows from the orthogonality of $Q_i(t,Z(t))$ that

$$\begin{aligned} E[F^2(t,Z(t))] &= \sum_{i=0}^{\infty}\left(\sum_{j=0}^{\infty}\widetilde{a}_{ij}\varphi_{jT}(t)\right)^2 \\ &= \sum_{i=0}^{\infty}\left(\sum_{j=0}^{p_i}\widetilde{a}_{ij}\varphi_{jT}(t) + \sum_{j=p_i+1}^{\infty} a_{ij}\varphi_{jT}(t)\right)^2 \end{aligned}$$

$$
\begin{aligned}
\leq & 2\sum_{i=0}^{\infty}\left(\sum_{j=0}^{p_i}\widetilde{a}_{ij}\varphi_{jT}(t)\right)^2 + 2\sum_{i=0}^{\infty}\left(\sum_{j=p_i+1}^{\infty}a_{ij}\varphi_{jT}(t)\right)^2 \\
= & 2\frac{1}{p_{\max}}\sum_{i=0}^{k}\left(\sum_{j=0}^{p_i}\alpha_{ij}\varphi_{jT}(t)\right)^2 + 2\sum_{i=k+1}^{\infty}\left(\sum_{j=0}^{p_i}a_{ij}\varphi_{jT}(t)\right)^2 \\
& + 2\sum_{i=0}^{\infty}\left(\sum_{j=p_i+1}^{\infty}a_{ij}\varphi_{jT}(t)\right)^2 \\
\leq & \frac{2}{p_{\max}}\sum_{i=0}^{k}\sum_{j=0}^{p_i}\alpha_{ij}^2\sum_{j=0}^{p_i}\varphi_{jT}^2(t) + \frac{4}{T}\sum_{i=k+1}^{\infty}\left(\sum_{j=0}^{p_i}|a_{ij}|\right)^2 \\
& + \frac{4}{T}\sum_{i=0}^{\infty}\left(\sum_{j=p_i+1}^{\infty}|a_{ij}|\right)^2 \\
\leq & \frac{O(1)}{p_{\max}}\sum_{i=0}^{k}p_i\sum_{j=0}^{p_i}\alpha_{ij}^2 + \frac{4}{Tk}\sum_{i=k+1}^{\infty}i\left(\sum_{j=0}^{\infty}|a_{ij}|\right)^2 + \frac{4}{T}\sum_{i=0}^{\infty}\left[\sum_{j=0}^{\infty}|a_{ij}|\right]^2 \\
< & \infty,
\end{aligned}
$$

since $\sum_{j=0}^{p_i}\varphi_{jT}^2(t) = O(1)p_i$, α is a unite vector and $\{a_{ij}\}$ satisfies Assumption L.2.

(b) Similar to Part (a). □

In order to obtain asymptotic behaviour of $\widehat{m}$, we make the following assumptions for the truncation parameters.

Assumption L.6

(a) Let $k = [n^{\kappa_1}]$ and $\frac{1}{2} < \kappa_1 < 1$

(b) Let $p_{\min} = [n^{\kappa_2}]$, $p_{\max} = [n^{\bar{\kappa}_2}]$ with $0 < \kappa_2 \leq \bar{\kappa}_2 < 1$ and $0 \leq \bar{\kappa}_2 - \kappa_2 < 3\kappa_2 - \kappa_1 - 1$.

Clearly, feasible solutions of truncation parameters do exist. The last assumption is about the functions $F(t,x)$, $G(t,x)$.

Assumption L.7

(a) Both $F(t,x)$ and $G(t,x)$ are continuous in t and x.

Theorem 5.2.1. *Suppose that $\{x_{s,n}\}_1^n$ and $\{e_s\}_1^n$ satisfy Assumption B. Under Assumption L.5–L.7 we have*

$$\begin{aligned}&\frac{1}{\sqrt{n}}\frac{\alpha' X' X A(\tau,x)}{\sqrt{p_{\max}}\|A(\tau,x)\|^2}(\widehat{m}(\tau,x)-m(\tau,x))\\ &\quad\to_D \int_0^1 F(Tr, T\mu r+\sqrt{T}\sigma_z W(r))dU(r),\end{aligned} \tag{5.2.11}$$

as $n\to\infty$ where $(U(r),W(r))$ is the vector of Brownian motion in Assumption B.

Remark 5.2.2. As can be seen from the proof, the convergence rate of $\widehat{m}(\tau,x)-m(\tau,x)$ is about $\frac{\sqrt{np_{\max}}}{\|A(\tau,x)\|}$. In view of the estimation of $\|A(\tau,x)\|$, the rate is between $n^{\frac{1}{2}(1-\kappa_1)}$ and $n^{\frac{1}{2}(1-\kappa_1)+\frac{1}{2}(\bar{\kappa}_2-\kappa_2)}$. The minimum order is less than $\frac{1}{4}$, while the maximum order is a little bit bigger than the minimum order.

Proof. It is evident that we shall make use of the embedding schedule to facilitate the proof. Note that we still use the old notations but with strong convergence $(W_n,U_n)\to_{a.s.}(W,U)$. However, the convergence of (5.2.11) is in the weak sense remaining unchanged. As mentioned before, this only makes our proof easier.

It follows from (5.2.7) that

$$\begin{aligned}&\frac{1}{\sqrt{n}}\frac{\alpha' X' X A(\tau,x)}{\sqrt{p_{\max}}\|A(\tau,x)\|^2}(\widehat{m}(\tau,x)-m(\tau,x))\\ =&\frac{1}{\sqrt{n}\sqrt{p_{\max}}}\alpha' B X'(\delta+\gamma+\varepsilon)-\frac{1}{\sqrt{n}}\frac{\alpha' X' X A(\tau,x)}{\sqrt{p_{\max}}\|A(\tau,x)\|^2}[\delta(\tau,x)+\gamma(\tau,x)]\\ =&\frac{1}{\sqrt{n}\sqrt{p_{\max}}}\alpha' X'(\delta+\gamma+\varepsilon)-\frac{1}{\sqrt{n}}\frac{\alpha' X' X A(\tau,x)}{\sqrt{p_{\max}}\|A(\tau,x)\|^2}[\delta(\tau,x)+\gamma(\tau,x)]\\ :=&\Pi_1-\Pi_2.\end{aligned}$$

We shall show that Π_1 converges to the desired variable in probability and $\Pi_2\to_P 0$ as $n\to\infty$.

In view of (5.2.10a), Π_1 can be rephrased as

$$\begin{aligned}\Pi_1=&\frac{1}{\sqrt{n}\sqrt{p_{\max}}}\alpha' X'(\delta+\gamma+\varepsilon)=\frac{1}{\sqrt{n}}(\mathbf{F}'-\widetilde{\delta}'-\widetilde{\gamma}')(\delta+\gamma+\varepsilon)\\ =&\frac{1}{\sqrt{n}}(\mathbf{F}'\delta-\widetilde{\delta}'\delta-\widetilde{\gamma}'\delta+\mathbf{F}'\gamma-\widetilde{\delta}'\gamma-\widetilde{\gamma}'\gamma+\mathbf{F}'\varepsilon-\widetilde{\delta}'\varepsilon-\widetilde{\gamma}'\varepsilon).\end{aligned}$$

In order to complete the convergence of Π_1, we are going to demonstrate that

$$\frac{1}{\sqrt{n}}\mathbf{F}'\varepsilon\to_P\int_0^1 F(Tr,T\mu r+\sqrt{T}\sigma_z W(r))dU(r), \tag{5.2.12}$$

and

$$\frac{1}{\sqrt{n}}\mathbf{F}'\delta \to_P 0, \qquad \frac{1}{\sqrt{n}}\mathbf{F}'\gamma \to_P 0, \tag{5.2.13}$$

$$\frac{1}{\sqrt{n}}\widetilde{\delta}'\delta \to_P 0, \qquad \frac{1}{\sqrt{n}}\widetilde{\gamma}'\delta \to_P 0, \qquad \frac{1}{\sqrt{n}}\widetilde{\delta}'\gamma \to_P 0, \qquad \frac{1}{\sqrt{n}}\widetilde{\gamma}'\gamma \to_P 0, \tag{5.2.14}$$

$$\frac{1}{\sqrt{n}}\widetilde{\delta}'\varepsilon \to_P 0, \qquad \frac{1}{\sqrt{n}}\widetilde{\gamma}'\varepsilon \to_P 0. \tag{5.2.15}$$

In fact, (5.2.12) is valid because from Assumption B it follows that

$$\begin{aligned}
\frac{1}{\sqrt{n}}\mathbf{F}'\varepsilon =& \frac{1}{\sqrt{n}}\sum_{s=1}^{n} F(t_{s,n}, X_{s,n})e_s = \frac{1}{\sqrt{n}}\sum_{s=1}^{n} F\left(\frac{s}{n}T, \frac{s}{n}T\mu + \sqrt{T}\sigma_z x_{s,n}\right) e_s \\
=& \sum_{s=1}^{n} F\left(\frac{s}{n}T, \frac{s}{n}T\mu + \sqrt{T}\sigma_z x_{s,n}\right)\frac{1}{\sqrt{n}}e_s \\
=& \sum_{s=1}^{n} F\left(\frac{s-1}{n}T + \frac{1}{n}T, \frac{s-1}{n}T\mu + \frac{1}{n}T\mu + \sqrt{T}\sigma_z W_n\left(\frac{s-1}{n} + \frac{1}{n}\right)\right) \\
& \qquad \times \left(U_n\left(\frac{s}{n}\right) - U_n\left(\frac{s-1}{n}\right)\right) \\
=& \sum_{s=1}^{n} \int_{\frac{s-1}{n}}^{\frac{s}{n}} F(rT + o(1), rT\mu + o(1) + \sqrt{T}\sigma_z W_n(r + o(1)))dU_n(r) \\
=& \int_0^1 F(rT + o(1), rT\mu + o(1) + \sqrt{T}\sigma_z W_n(r) + o_P(1))dU_n(r),
\end{aligned}$$

and since $(W_n(r + o(1)), U_n(r)) \to_{a.s.} (W(r), U(r))$, as we shown in the proof of Theorem 1.4.1, it follows from the continuity of $F(\cdot, \cdot)$ that

$$\begin{aligned}
&(F(rT + o(1), rT\mu + o(1) + \sqrt{T}\sigma_x W_n(r + o(1))), U_n(r)) \\
&\to_{a.s.} (F(rT, rT\mu + \sqrt{T}\sigma_x W(r)), U(r)).
\end{aligned}$$

Using Theorem 2.2 in Kurtz and Protter (1991) yields the result.

We now turn to prove (5.2.13), (5.2.14) and (5.2.15). Due to Cauchy-Schwarz inequality we have

$$\frac{1}{n}|\mathbf{F}'\delta|^2 \le \frac{1}{n}\|\mathbf{F}'\|^2\|\delta\|^2, \qquad \frac{1}{n}|\mathbf{F}'\gamma|^2 \le \frac{1}{n}\|\mathbf{F}'\|^2\|\gamma\|^2, \qquad \frac{1}{n}|\widetilde{\delta}'\delta|^2 \le \frac{1}{n}\|\widetilde{\delta}'\|^2\|\delta\|^2,$$

$$\frac{1}{n}|\widetilde{\delta}'\gamma|^2 \le \frac{1}{n}\|\widetilde{\delta}'\|^2\|\gamma\|^2, \qquad \frac{1}{n}|\widetilde{\gamma}'\delta|^2 \le \frac{1}{n}\|\widetilde{\gamma}'\|^2\|\delta\|^2, \qquad \frac{1}{n}|\widetilde{\gamma}'\gamma|^2 \le \frac{1}{n}\|\widetilde{\gamma}'\|^2\|\gamma\|^2.$$

where $\|\cdot\|$ signifies the Euclidean norm.

In addition, using martingale difference structure of $(e_s, \mathcal{F}_{n,s})$ and adaptivity of $x_{s+1,n}$ with $\mathcal{F}_{n,s}$ yields

$$\begin{aligned}\frac{1}{n}E(\widetilde{\delta}'\varepsilon)^2 =&\frac{1}{n}E\left(\sum_{s=1}^{n}\widetilde{\delta}_s e_s\right)^2 = \frac{1}{n}\sum_{s=1}^{n}E[\widetilde{\delta}_s^2 e_s^2] + 2\frac{1}{n}\sum_{s=2}^{n}\sum_{l=1}^{s-1}E[\widetilde{\delta}_s e_s \widetilde{\delta}_l e_l]\\ =&\frac{1}{n}\sum_{s=1}^{n}E[\widetilde{\delta}_s^2 E(e_s^2|\mathcal{F}_{n,s-1})] + 2\frac{1}{n}\sum_{s=2}^{n}\sum_{l=1}^{s-1}E[\widetilde{\delta}_l e_l \widetilde{\delta}_s E(e_s|\mathcal{F}_{n,s-1})]\\ =&\sigma_e^2\frac{1}{n}\sum_{s=1}^{n}E[\widetilde{\delta}_s^2] = \sigma_e^2\frac{1}{n}E\|\widetilde{\delta}'\|^2,\end{aligned}$$

and similarly

$$\frac{1}{n}E(\widetilde{\gamma}'\varepsilon)^2 =\sigma_e^2\frac{1}{n}E\|\widetilde{\gamma}'\|^2.$$

Therefore, it is sufficient to show that as $n \to \infty$,

$$\|\delta\|^2 \to_P 0, \qquad \|\gamma\|^2 \to_P 0, \qquad \frac{1}{n}E\|\widetilde{\delta}'\|^2 \to 0, \qquad \frac{1}{n}E\|\widetilde{\gamma}'\|^2 \to 0, \tag{5.2.16}$$

since

$$\begin{aligned}\frac{1}{n}\|\mathbf{F}'\|^2 =&\frac{1}{n}\sum_{s=1}^{n}F^2(t_{s,n}, X_{s,n}) = \frac{1}{n}\sum_{s=1}^{n}F^2\left(\frac{s}{n}T, \frac{s}{n}T\mu + \sqrt{T}\sigma_z x_{s,n}\right)\\ =&\sum_{s=1}^{n}\int_{\frac{s-1}{n}}^{\frac{s}{n}}F^2(rT + o(1), rT\mu + o(1) + \sqrt{T}\sigma_z W_n(r) + o_P(1))dr\\ &-\frac{1}{n}F^2(0,0) + \frac{1}{n}F^2(T, T\mu + \sqrt{T}\sigma_z W_n(1))\\ =&\int_0^1 F^2(rT + o(1), rT\mu + o(1) + \sqrt{T}\sigma_z W_n(r) + o_P(1))dr\\ &-\frac{1}{n}F^2(0,0) + \frac{1}{n}F^2(T, X_T)\\ \to_P&\int_0^1 F^2(Tr, T\mu r + \sqrt{T}\sigma_z W(r))dr,\end{aligned} \tag{5.2.17}$$

using the continuity of $F(\cdot,\cdot)$, $W_n(r) \to_{a.s.} W(r)$ and $E[F^2(T, X_T)] < \infty$ by Lemma 5.2.1.

Let us prove the results in (5.2.16) one by one.

Firstly, because of Assumptions L.5 (b), $b_i(t) := b_i(t, m)$ is differentiable up to third order, hence all expansions of $b_i(t)$, $b_i'(t)$ and $b_i''(t)$ in terms of $\varphi_{jT}(t)$ are convergent uniformly on $[0, T]$. Whence $b_{ij} = \left(\frac{T}{j\pi}\right)^2 c_j(b_i'')$ where $c_j(b_i'')$ stands for the j-th coefficient in the expansion of $b_i''(t)$. We have

$$\begin{aligned}
E\|\delta\|^2 =& \sum_{s=1}^{n}\sum_{i=1}^{k}\left(\sum_{j=p_i+1}^{\infty} b_{ij}\varphi_{jT}(t_{s,n})\right)^2 \leq \frac{2T^3}{\pi^4}\sum_{s=1}^{n}\sum_{i=1}^{k}\left(\sum_{j=p_i+1}^{\infty}\frac{1}{j^2}|c_j(b_i'')|\right)^2 \\
\leq& \frac{2T^3}{\pi^4}\sum_{s=1}^{n}\sum_{i=1}^{k}\sum_{j=p_i+1}^{\infty}\frac{1}{j^4}\sum_{j=p_i+1}^{\infty}|c_j(b_i'')|^2 \\
\leq& o(1)\frac{nk}{p_{\min}^3} = o(1)n^{1+\kappa_1-3\kappa_2} \to 0
\end{aligned}$$

as $n \to \infty$, which implies $\|\delta\|^2 \to_P 0$.

Secondly, by virtue of (4.4.8) with $r = 2$, $b_i(t, m) = \sqrt{v(t)}^2\sqrt{\frac{(i-2)!}{i!}}b_i(t, D^2m)$. Thus,

$$\begin{aligned}
E\|\gamma\|^2 =& \sum_{s=1}^{n} E[\gamma_s^2] = \sum_{s=1}^{n} E\left(\sum_{i=k+1}^{\infty}\sum_{j=0}^{\infty} b_{ij}\varphi_{jT}\left(t_{s,n}\right)Q_i\left(t_{s,n}, X_{s,n}\right)\right)^2 \\
=& \sum_{s=1}^{n}\sum_{i=k+1}^{\infty} b_i^2\left(t_{s,n}\right) = \sum_{s=1}^{n}\sum_{i=k+1}^{\infty} v(t_{s,n})^2\frac{(i-2)!}{i!}b_i^2(t_{s,n}, D^2m) \\
\leq& \frac{1}{k^2}\sum_{s=1}^{n} v(t_{s,n})^2\sum_{i=k+1}^{\infty} b_i^2(t_{s,n}, D^2m) \leq o(1)\frac{1}{k^2}\sum_{s=1}^{n} v(t_{s,n})^2 \\
\leq& o(1)\frac{n}{k^2}\max_{0\leq t\leq T} v(t)^2 = o(1)n^{1-2\kappa_1} \to 0,
\end{aligned}$$

due to Assumption L.6, and we have invoked Assumption L.5 that the convergence of expansion of D^2m is uniformly on $[0, T]$, hence $\sum_{i=k+1}^{\infty} b_i^2(t_{s,n}, D^2m) = o(1)$ independent of s. In addition, in the scope of this study, $v(t) \in C[0, T]$ (for example, when $Z(t)$ reduces to Brownian motion, $v(t) = t$), therefore, it is bounded on the interval. Whence, $\|\gamma\|^2 \to_P 0$.

Thirdly, it follows from the expression of $\widetilde{\delta}_s$

$$\begin{aligned}
&\frac{1}{n}E\|\widetilde{\delta}'\|^2 = \frac{1}{n}\sum_{s=1}^{n} E[\widetilde{\delta}_s^2] \\
=& \frac{1}{n}\sum_{s=1}^{n} E\left(\sum_{i=0}^{k}\sum_{j=p_i+1}^{\infty} a_{ij}\varphi_{jT}(t_{s,n})Q_i(t_{s,n}, X_{s,n})\right)^2 \\
=& \frac{1}{n}\sum_{s=1}^{n}\sum_{i=0}^{k}\left(\sum_{j=p_i+1}^{\infty} a_{ij}\varphi_{jT}(t_{s,n})\right)^2 \leq \frac{1}{n}\sum_{s=1}^{n}\sum_{i=0}^{k}\left(\frac{\sqrt{2}}{\sqrt{T}}\sum_{j=p_i+1}^{\infty}|a_{ij}|\right)^2
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{i=0}^{k} \frac{2}{Tp_i^2} \left(\sum_{j=p_i+1}^{\infty} j|a_{ij}| \right)^2 \leq \frac{2k}{Tp_{\min}^2} \left(\sum_{j=p_{\min}+1}^{\infty} j|a_{ij}| \right)^2 \\
&= \frac{o(1)}{T} n^{\kappa_1 - 2\kappa_2} \to 0
\end{aligned}
$$

as $n \to \infty$, where we have used Assumption L.6 and the implication of Assumption L.2 that $\sum_{j=p_i+1}^{\infty} |a_{ij}| j = o(1)$. Hence, $\frac{1}{n} E \|\widetilde{\delta}'\|^2 \to 0$.

Lastly, we may also have

$$
\begin{aligned}
\frac{1}{n} E \|\widetilde{\gamma}\|^2 =& \frac{1}{n} \sum_{s=1}^{n} E[\widetilde{\gamma}_s^2] = \frac{1}{n} \sum_{s=1}^{n} E \left(\sum_{i=k+1}^{\infty} \sum_{j=0}^{\infty} a_{ij} \varphi_{jT}(t_{s,n}) Q_i(t_{s,n}, X_{s,n}) \right)^2 \\
=& \frac{1}{n} \sum_{s=1}^{n} \sum_{i=k+1}^{\infty} \left(\sum_{j=0}^{\infty} a_{ij} \varphi_{jT}(t_{s,n}) \right)^2 \leq \frac{2}{nT} \sum_{s=1}^{n} \sum_{i=k+1}^{\infty} \left(\sum_{j=0}^{\infty} |a_{ij}| \right)^2 \\
\leq& \frac{2}{kT} \sum_{i=k+1}^{\infty} i \left(\sum_{j=0}^{\infty} |a_{ij}| \right)^2 \to 0
\end{aligned}
$$

due to Assumption L.2. We thus obtain that $\frac{1}{n} E \|\widetilde{\gamma}\|^2 \to 0$.

Now we are in a position to prove $\Pi_2 \to_P 0$ as $n \to \infty$.

Since Π_2 can be rephrased as

$$
\begin{aligned}
\Pi_2 =& \frac{1}{\sqrt{n}} \frac{\alpha' X' X A(\tau, x)}{\sqrt{p_{\max}} \|A(\tau, x)\|^2} [\delta(\tau, x) + \gamma(\tau, x)] \\
=& \frac{1}{n} \frac{\alpha' X' X A(\tau, x)}{p_{\max} \|A(\tau, x)\|} \cdot \frac{\sqrt{n}\sqrt{p_{\max}}}{\|A(\tau, x)\|} [\delta(\tau, x) + \gamma(\tau, x)] \\
:=& \Pi_{21} \cdot \Pi_{22},
\end{aligned}
$$

we shall show that Π_{21} converges to some random variable in probability and $\Pi_{22} \to 0$.

To begin with, by virtue of (5.2.10a) and (5.2.10b) we have

$$
\begin{aligned}
\Pi_{21} =& \frac{1}{n} \frac{\alpha' X' X A(\tau, x)}{p_{\max} \|A(\tau, x)\|} = \frac{1}{n} (\mathbf{F}' - \widetilde{\delta}' - \widetilde{\gamma}')(\mathbf{G} - \widetilde{\delta} - \widetilde{\gamma}) \\
=& \frac{1}{n} (\mathbf{F}'\mathbf{G} - \mathbf{F}'\widetilde{\delta} - \mathbf{F}'\widetilde{\gamma} - \widetilde{\delta}'\mathbf{G} - \widetilde{\gamma}'\mathbf{G} + 2\widetilde{\delta}'\widetilde{\gamma} + \|\widetilde{\delta}'\|^2 + \|\widetilde{\gamma}'\|^2).
\end{aligned}
$$

Noting that (5.2.16) and (5.2.17) imply that $\frac{1}{n}\mathbf{F}'\widetilde{\delta} \to_P 0$, $\frac{1}{n}\mathbf{F}'\widetilde{\gamma} \to_P 0$, $\frac{1}{n}\widetilde{\delta}'\widetilde{\gamma} \to_P 0$, $\frac{1}{n}\|\widetilde{\delta}'\|^2 \to_P 0$ and $\frac{1}{n}\|\widetilde{\gamma}'\|^2 \to_P$, it suffices to show that the convergence of

$$
\frac{1}{n}\mathbf{F}'\mathbf{G} \quad \text{and} \quad \frac{1}{n}\|\mathbf{G}\|^2, \tag{5.2.18}
$$

as $n \to \infty$.

In effect, similar to (5.2.17),

$$\begin{aligned}
\frac{1}{n}\mathbf{F}'\mathbf{G} =& \frac{1}{n}\sum_{s=1}^{n} F(t_{s,n}, X_{s,n})G(t_{s,n}, X_{s,n}) \\
=& \int_0^1 F(Tr, T\mu r + \sqrt{T}\sigma_z W_n(r))G(Tr, T\mu r + \sqrt{T}\sigma_z W_n(r))dr \\
& - \frac{1}{n}F(0,0)G(0,0) + \frac{1}{n}F(T, X_T)G(T, X_T) \\
\to_P & \int_0^1 F(Tr, T\mu r + \sqrt{T}\sigma_z W(r))G(Tr, T\mu r + \sqrt{T}\sigma_z W(r))dr, \\
\frac{1}{n}\|\mathbf{G}\|^2 =& \frac{1}{n}\sum_{s=1}^{n} G^2(t_{s,n}, X_{s,n}) \\
=& \int_0^1 G^2(Tr, T\mu r + \sqrt{T}\sigma_z W_n(r))dr - \frac{1}{n}G^2(0,0) + \frac{1}{n}G^2(T, X_T) \\
\to_P & \int_0^1 G^2(Tr, T\mu r + \sqrt{T}\sigma_z W(r))dr
\end{aligned}$$

by continuous mapping theorem and Lemma 5.2.1 as $n \to \infty$.

Therefore, as $n \to \infty$,

$$\Pi_{21} \to_P \int_0^1 F(Tr, T\mu r + \sqrt{T}\sigma_z W(r))G(Tr, T\mu r + \sqrt{T}\sigma_z W(r))dr.$$

As for Π_{22}, since $\Pi_{22} = \Delta_1 + \Delta_2$ where

$$\Delta_1 = \frac{\sqrt{n}\sqrt{p_{\max}}}{\|A(\tau, x)\|}\delta(\tau, x) \quad \text{and} \quad \Delta_2 = \frac{\sqrt{n}\sqrt{p_{\max}}}{\|A(\tau, x)\|}\gamma(\tau, x),$$

we shall show both Δ_1 and Δ_2 are approaching to zero.

It is known that $O(1)kp_{\min} \leq \|A(\tau, x)\|^2 \leq O(1)kp_{\max}$. It follows from Assumption L.5 that

$$\begin{aligned}
|\Delta_1| =& \frac{\sqrt{np_{\max}}}{\|A(\tau, x)\|}|\delta(\tau, x)| \leq \frac{\sqrt{np_{\max}}}{\|A(\tau, x)\|}\sum_{i=0}^{k}|Q_i(\tau, x)|\left|\sum_{j=p_i+1}^{\infty} b_{ij}\varphi_{jT}(\tau)\right| \\
\leq & O(1)\frac{\sqrt{np_{\max}}}{\sqrt{kp_{\min}}}\left(\sum_{i=0}^{k}|Q_i(\tau, x)|^2\right)^{\frac{1}{2}}\left(\sum_{i=0}^{k}\frac{2T^3}{\pi^4}\left|\sum_{j=p_i+1}^{\infty}\frac{1}{j^2}|c_j(b_i'')|\right|^2\right)^{\frac{1}{2}} \\
\leq & O(1)\frac{\sqrt{np_{\max}}}{\sqrt{kp_{\min}}}\sqrt{k}\left(\sum_{i=0}^{k}\sum_{j=p_i+1}^{\infty}\frac{1}{j^4}\sum_{j=p_i+1}^{\infty}|c_j(b_i'')|^2\right)^{\frac{1}{2}} \leq o(1)\frac{\sqrt{np_{\max}}\sqrt{k}}{\sqrt{p_{\min}}p_{\min}^{3/2}}
\end{aligned}$$

$$=o(1)n^{\frac{1}{2}+\frac{1}{2}\kappa_1+\frac{1}{2}(\bar{\kappa}_2-\kappa_2)-\frac{3}{2}\kappa_2} \to 0$$

as $n \to \infty$ using Assumption L.6.

In addition, using (4.4.8) with $r = 2$ and the asymptotic property of Q_i in Remark 4.3.4 gives

$$\begin{aligned}
|\Delta_2| =& \frac{\sqrt{np_{\max}}}{\|A(\tau,x)\|}|\gamma(\tau,x)| = \frac{\sqrt{np_{\max}}}{\|A(\tau,x)\|}\left|\sum_{i=k+1}^{\infty} b_i(\tau,m)Q_i(\tau,x)\right| \\
=& \frac{\sqrt{np_{\max}}}{\|A(\tau,x)\|}\left|\sum_{i=k+1}^{\infty} \frac{v(\tau)}{\sqrt{i(i-1)}} b_i(\tau,D^2m)Q_i(\tau,x)\right| \\
\leq& O(1)\frac{\sqrt{np_{\max}}}{\sqrt{kp_{\min}}}\left(\sum_{i=k+1}^{\infty} |b_i(\tau,D^2m)|^2\right)^{\frac{1}{2}}\left(\sum_{i=k+1}^{\infty} \frac{1}{i(i-1)\sqrt{i}}\right)^{\frac{1}{2}} \\
=& o(1)\frac{\sqrt{np_{\max}}}{\sqrt{kp_{\min}}}\frac{1}{k^{3/4}} = o(1)n^{\frac{1}{2}+\frac{1}{2}(\bar{\kappa}_2-\kappa_2)-\frac{5}{4}\kappa_1} \to 0
\end{aligned}$$

as $n \to \infty$ by virtue of Assumption L.6. The proof is finished. □

5.3 Time horizon approaching infinity

We are also interested in the scenario where time variable lies in $[0, T_n]$ and $T_n \to \infty$ when sample size $n \to \infty$ for the same reason as in section 3 of Chapter 3.

We propose the following assumptions for the function $m(t,x)$ in the model (5.0.1).

Assumption L.8

(a) For every $t > 0$, $m(t,x)$ and $D^r m(t,x)$ are all in $L^2(I, \rho_r(t,x))$, $r = 1, 2, 3$.

(b) For each i, $b_i(t,m) = E[m(t,Z(t))Q_i(t,Z(t))]$, belongs to $C^3[0,T]$ for any $T > 0$.

(c) For i large enough, the coefficient functions $b_i(t,D^3m)$ of $D^3m(t,Z(t))$ expanded by the system $\{Q_{3i}(t,Z(t))\}$ are such that $v(t)^3 b_i^2(t,D^3m)$ are bounded on $(0,\infty)$ uniformly in i.

(d) $\|b_i''(t,m(t,x))\|_{L^2[0,T]}$ are bounded for any $T > 0$ uniformly in i.

Remark 5.3.1. Since the framework in this section is a combination of the first two, the requirements for $m(t,x)$ contains the basic conditions in Assumption L.1 and L.5.

There are many functions that satisfy these four conditions at the same time. For instance, $m(t,x) = t^\eta e^{-ct} P(x)$ with $\eta \geq 1$, $c > 0$ and $P(x)$ being any polynomial of fixed degree; $m(t,x) = \frac{t}{1+t^\eta} \cos x$ with $\eta \geq 3$, and so on.

For the truncation parameters and time span T_n, we make the following assumption.

Assumption L.9

(a) Let $k = [n^{\kappa_1}]$, $p_{\min} = [n^{\kappa_2}]$, $p_{\max} = [n^{\bar{\kappa}_2}]$ and $T_n = [n^{\kappa_3}]$, where $0 < \kappa_i < 1$ $(i = 1,2,3)$, $\kappa_2 \leq \bar{\kappa}_2 < 1$ and $\kappa_1 > 0.5$.

(b) Let $3\kappa_3 + \kappa_1 + 1 < 3\kappa_2$.

Remark 5.3.2. Feasible solutions for κ_i $(i = 1,2,3)$ do exist. For instance, $\kappa_1 = 0.6$, $\kappa_2 = 0.8$ and $\kappa_3 = 0.2$. Meanwhile, condition (b) implies that $\kappa_2 > \frac{1}{2} + \kappa_3$.

Given the observation number n, one can choose $T = T_n$ according to Assumption L.9. Let us sample on $[0, T_n]$ at equally spaced points: $t_{s,n} = T_n \frac{s}{n}$ $(s = 1, \cdots, n)$ for model (5.0.1). Denote by $Y_{s,n}$ the process $Y(t)$ at $t_{s,n}$, $X_{s,n} = Z(t_{s,n})$ for the Lévy process at the discrete points and $e_s = \varepsilon(t_{s,n})$. Observe that from infinite divisibility and homogeneous distribution of the Lévy process it follows that $E(Z(t)) = t\mu$ and $Var(Z(t)) = t\sigma_z^2$ for any real $t \geq 0$. Thus,

$$\begin{aligned} X_{s,n} =& \sum_{i=1}^{s} (X_{i,n} - X_{i-1,n}) \\ =& EX_{s,n} + \sum_{i=1}^{s} [X_{i,n} - X_{i-1,n} - E(X_{i,n} - X_{i-1,n})] \\ =& \frac{s}{n} T_n \mu + \sqrt{T_n} \sigma_z \frac{1}{\sqrt{n}} \sum_{i=1}^{s} w_i, \end{aligned}$$

where $w_i = \frac{\sqrt{n}}{\sqrt{T_n}\sigma_z}(X_{i,n} - X_{i-1,n} - \frac{1}{n} T_n \mu)$ form an i.i.d (0,1) sequence.

Let $x_{s,n} = \frac{1}{\sqrt{n}} \sum_{i=1}^{s} w_i$. It therefore follows from the functional central limit theorem that $x_{s,n}$ converges in distribution to a Brownian motion on $[0,1]$ as $n \to \infty$. In addition, it is clear that $x_{s,n}$, along with $d_{l,k,n} = \sqrt{(l-k)/n}$, satisfies Assumption A.

The following procedure is similar to the preceding subsections. The $m(t, Z(t))$ is expanded using an orthonormal basis $\{\varphi_{jT_n}(t) Q_i(t, Z(t))\}$ at each sampling point, and then obtain n equations. The n equations can be written in the following matrix form

$$Y = X\beta + \delta + \gamma + \varepsilon, \tag{5.3.1}$$

where all notations remain the similar meanings as before, so that we spare our effort to recite them.

The OLS estimator of β is given by

$$\widehat{\beta} = (X'X)^{-1}X'Y. \tag{5.3.2}$$

Obtaining $\widehat{\beta}$ enables us to estimate $m(\tau, x)$ for fixed $\tau > 0$ and fixed x on the path of $Z(\tau)$, viz., $\widehat{m}(\tau, x)$. $\widehat{m}(\tau, x)$ is generated from the expansion of $m(\tau, x)$ by superseding β by $\widehat{\beta}$ and removing all residues. Whence, we have

$$\widehat{m}(\tau, x) = A'(\tau, x)\widehat{\beta}, \tag{5.3.3}$$

where $A'(\tau, x) = (\varphi_{0T_n}(\tau)Q_0(\tau, x), \cdots, \varphi_{p_0T_n}(\tau)Q_0(\tau, x), \cdots, \varphi_{0T_n}(\tau)Q_k(\tau, x), \cdots,$ $\varphi_{p_kT_n}(\tau)Q_k(\tau, x))$.

The difference between $\widehat{m}(\tau, x)$ and $m(\tau, x)$ is

$$\begin{aligned} \widehat{m}(\tau, x) - m(\tau, x) &= A'(\tau, x)(\widehat{\beta} - \beta) - \delta(\tau, x) - \gamma(\tau, x) \\ &= A'(\tau, x)(X'X)^{-1}X'(\delta + \gamma + \varepsilon) - \delta(\tau, x) - \gamma(\tau, x), \end{aligned} \tag{5.3.4}$$

where

$$\begin{aligned} \delta(\tau, x) &= \sum_{i=0}^{k} \sum_{j=p_i+1}^{\infty} b_{ij}\varphi_{jT_n}(\tau)Q_i(\tau, x); \\ \gamma(\tau, x) &= \sum_{i=k+1}^{\infty} \sum_{j=0}^{\infty} b_{ij}\varphi_{jT_n}(\tau)Q_i(\tau, x). \end{aligned}$$

Thus, one desired result is the asymptotic distribution of $\widehat{m}(\tau, x) - m(\tau, x)$. To this end, put

$$A = \frac{A(\tau, x)A'(\tau, x)}{\|A(\tau, x)\|^2} \quad \text{and} \quad B = (X'X)A(X'X)^{-1}. \tag{5.3.5}$$

Once again, by Lemma 3.1.2, B has eigenvalues $\lambda_1 = 1$, $\lambda_2 = \cdots = \lambda_p = 0$. Let unit vector α be the unit left eigenvector of B pertaining to λ_1. Hence, we have $\alpha'B = \alpha'$ and $\|\alpha\| = 1$. Denote $\alpha' = (\alpha_{00}, \cdots, \alpha_{0p_0}, \cdots, \alpha_{k0} \cdots, \alpha_{kp_k})$ in concert with $A(\tau, x)$.

Let us apply the reshuffle procedure for the set $\mathcal{S}$ from Assumption L.2 by α and $\frac{1}{\|A(\tau,x)\|}A(\tau, x)$. Denote by $\widetilde{\mathcal{S}}$ and $\bar{\mathcal{S}}$ the resulting sets:

1) $\widetilde{\mathcal{S}} = \{\widetilde{a}_0, \cdots, \widetilde{a}_i, \cdots\}$, and $\bar{\mathcal{S}} = \{\bar{a}_0, \cdots, \bar{a}_i, \cdots\}$.

2) $\widetilde{a}_i = \{\widetilde{a}_{ij}\}$ where $\widetilde{a}_{ij} = \sqrt{\frac{T_n}{p_{\max}}}\alpha_{ij}$ for $0 \le i \le k$ and $0 \le j \le p_i$; otherwise, $\widetilde{a}_{ij} = a_{ij}$.

3) $\bar{a}_i = \{\bar{a}_{ij}\}$ where $\bar{a}_{ij} = \sqrt{\frac{T_n}{p_{\max}}} \frac{1}{\|A(\tau,x)\|} \varphi_{jT_n}(\tau) Q_i(\tau, x)$ for $0 \leq i \leq k$ and $0 \leq j \leq p_i$; otherwise, $\bar{a}_{ij} = a_{ij}$.

Due to the Riesz-Fischer theorem, for two sequences $\widetilde{\mathcal{S}}$ and $\bar{\mathcal{S}}$, there exist two functions, denoted by $\widetilde{F}(t, x)$ and $\widetilde{G}(t, x)$, such that

$$\begin{aligned} \widetilde{F}(t, Z(t)) &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \widetilde{a}_{ij} \varphi_{jT_n}(t) Q_i(t, Z(t)), \\ \widetilde{G}(t, Z(t)) &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \bar{a}_{ij} \varphi_{jT_n}(t) Q_i(t, Z(t)), \end{aligned} \tag{5.3.6}$$

for any $t \in [0, T_n]$.

In view of the expressions of $\widetilde{a}_{ij}$ and $\bar{a}_{ij}$, rewrite (5.3.6) as

$$\sqrt{\frac{T_n}{p_{\max}}} \alpha' X' = \widetilde{\mathbf{F}}' - \widetilde{\delta}' - \widetilde{\gamma}', \tag{5.3.7a}$$

$$\frac{1}{\|A(\tau, x)\|} \sqrt{\frac{T_n}{p_{\max}}} A(\tau, x)' X' = \widetilde{\mathbf{G}}' - \widetilde{\delta}' - \widetilde{\gamma}', \tag{5.3.7b}$$

where

$$\widetilde{\mathbf{F}}' = (\widetilde{F}(t_{1,n}, X_{1,n}), \cdots, \widetilde{F}(t_{n,n}, X_{n,n})), \quad \widetilde{\mathbf{G}}' = (\widetilde{G}(t_{1,n}, X_{1,n}), \cdots, \widetilde{G}(t_{n,n}, X_{n,n})),$$

$$\widetilde{\delta}' = (\widetilde{\delta}_1, \cdots, \widetilde{\delta}_n), \text{ with } \widetilde{\delta}_s = \sum_{i=0}^{k} \sum_{j=p_i+1}^{\infty} a_{ij} \varphi_{jT_n}(t_{s,n}) Q_i(t_{s,n}, X_{s,n}),$$

$$\widetilde{\gamma}' = (\widetilde{\gamma}_1, \cdots, \widetilde{\gamma}_n), \text{ with } \widetilde{\gamma}_s = \sum_{i=k+1}^{\infty} \sum_{j=0}^{\infty} a_{ij} \varphi_{jT_n}(t_{s,n}) Q_i(t_{s,n}, X_{s,n}).$$

Lemma 5.3.1. *For any $t \in [0, T_n]$, (a) $E[\widetilde{G}(t, Z(t))]^2 < \infty$, and (b) $E[\widetilde{F}(t, Z(t))]^2 < \infty$.*

Proof. (a) Invoking the orthogonality of $Q_i(t, Z(t))$ gives

$$\begin{aligned} E[\widetilde{G}(t, Z(t))]^2 &= E\left(\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \bar{a}_{ij} \varphi_{jT_n}(t) Q_i(t, Z(t))\right)^2 \\ &= \sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} \bar{a}_{ij} \varphi_{jT_n}(t)\right)^2 = \sum_{i=0}^{\infty} \left(\sum_{j=0}^{p_i} \bar{a}_{ij} \varphi_{jT_n}(t) + \sum_{j=p_i+1}^{\infty} a_{ij} \varphi_{jT_n}(t)\right)^2 \\ &\leq 2 \sum_{i=0}^{\infty} \left(\sum_{j=0}^{p_i} \bar{a}_{ij} \varphi_{jT_n}(t)\right)^2 + 2 \sum_{i=0}^{\infty} \left(\sum_{j=p_i+1}^{\infty} a_{ij} \varphi_{jT_n}(t)\right)^2 \end{aligned}$$

$$
\begin{aligned}
=&2\sum_{i=0}^{k}\left(\sum_{j=0}^{p_i}\bar{a}_{ij}\varphi_{jT_n}(t)\right)^2+2\sum_{i=k+1}^{\infty}\left(\sum_{j=0}^{p_i}a_{ij}\varphi_{jT_n}(t)\right)^2\\
&+2\sum_{i=0}^{\infty}\left(\sum_{j=p_i+1}^{\infty}a_{ij}\varphi_{jT_n}(t)\right)^2\\
\le&2\sum_{i=0}^{k}\left(\sum_{j=0}^{p_i}\bar{a}_{ij}\varphi_{jT_n}(t)\right)^2+\frac{4}{T_n}\sum_{i=k+1}^{\infty}\left(\sum_{j=0}^{p_i}|a_{ij}|\right)^2\\
&+\frac{4}{T_n}\sum_{i=0}^{\infty}\left(\sum_{j=p_i+1}^{\infty}|a_{ij}|\right)^2.
\end{aligned}
$$

It is easy by Assumption L.2 to see that the last two terms are finite. We thus deal with the first one in what follows. Notice that in the preceding chapter we have shown that $\sum_{j=0}^{p_i}\varphi_{jT_n}^2(t)=O(1)\frac{1}{T_n}p_i(1+o(1))$. As a result,

$$
\begin{aligned}
\sum_{i=0}^{k}\left(\sum_{j=0}^{p_i}\bar{a}_{ij}\varphi_{jT_n}(t)\right)^2=&\frac{T_n}{p_{\max}\|A(\tau,x)\|^2}\sum_{i=0}^{k}Q_i^2(\tau,x)\left(\sum_{j=0}^{p_i}\varphi_{jT_n}(\tau)\varphi_{jT_n}(t)\right)^2\\
\le&\frac{T_n}{p_{\max}\|A(\tau,x)\|^2}\sum_{i=0}^{k}Q_i^2(\tau,x)\sum_{j=0}^{p_i}\varphi_{jT_n}^2(\tau)\sum_{j=0}^{p_i}\varphi_{jT_n}^2(t)\\
=&\frac{O(1)(1+o(1))T_n}{p_{\max}\|A(\tau,x)\|^2}\sum_{i=0}^{k}\frac{p_i}{T_n}Q_i^2(\tau,x)\sum_{j=0}^{p_i}\varphi_{jT_n}^2(\tau)\\
\le&O(1)(1+o(1))\frac{1}{\|A(\tau,x)\|^2}\sum_{i=0}^{k}Q_i^2(\tau,x)\sum_{j=0}^{p_i}\varphi_{jT_n}^2(\tau)\\
=&O(1).
\end{aligned}
$$

(b) Similar to (a). □

Let $\widetilde{G}(t,x)=\widetilde{G}(t,\mu t+x-\mu t)=:G(t,x-\mu t)$ and $\widetilde{F}(t,x)=\widetilde{F}(t,\mu t+x-\mu t)=:F(t,x-\mu t)$. These reforms are because we are working on the centralised underlying process.

Assumption L.10

(a) Both $F(t,x)$ and $G(t,x)$ are in Class (HI) with normal functions $f(t,x)$, $g(t,x)$ and homogeneity powers $v(\cdot)$ and $\varrho(\cdot)$ respectively. Let $v(n)=n^{\varsigma}$ and $\varrho(n)=n^{\iota}$ satisfying (i) $1+\kappa_1+(2\iota+\frac{5}{2})\kappa_3<3\kappa_2$; (ii) $1+(2\iota-\frac{1}{2})\kappa_3<\frac{5}{2}\kappa_1$.

(b) Suppose further that $F^2(t,x)$, $G^2(t,x)$ and $F(t,x)G(t,x)$ are in Class (HI) with normal functions $f^2(t,x)$, $g^2(t,x)$ and $f(t,x)g(t,x)$ and homogeneity powers $\upsilon^2(\cdot)$, $\varrho^2(\cdot)$ and $\upsilon(\cdot)\varrho(\cdot)$ respectively.

(c) Both $F(t,x)$ and $G(t,x)$ are in Class (HH) with normal functions $f(t,x)$, $g(t,x)$ and homogeneity powers $\upsilon_1(\cdot)$, $\upsilon_2(\cdot)$ and $\varrho_1(\cdot)$, $\varrho_2(\cdot)$ respectively. Let $\upsilon_1(n) = n^{\varsigma_1}$, $\upsilon_2(n) = n^{\varsigma_2}$ and $\varrho_1(n) = n^{\iota_1}$, $\varrho_2(n) = n^{\iota_2}$ satisfying (i) $1 + \kappa_1 + (2\iota_1 + \iota_2 + 3)\kappa_3 < 3\kappa_2$; (ii) $1 + (2\iota_1 + \iota_2)\kappa_3 < \frac{5}{2}\kappa_1$.

(d) Suppose further that $F^2(t,x)$, $G^2(t,x)$ and $F(t,x)G(t,x)$ are in Class (HH) with normal functions $f^2(t,x)$, $g^2(t,x)$ and $f(t,x)g(t,x)$ and homogeneity powers $\upsilon_1^2(\cdot)$, $\upsilon_2^2(\cdot)$; $\varrho_1^2(\cdot)$, $\varrho_2^2(\cdot)$; $\upsilon_1(\cdot)\varrho_1(\cdot)$, $\upsilon_2(\cdot)\varrho_2(\cdot)$ respectively.

Remark 5.3.3. Note that the conditions in (a) and (c) are untidy and annoying since we would like to show the original requirements for the parameters.

It is clear that if $0 < \iota < \frac{1}{4}$, Assumption L.9 (b) implies the condition (i) of Assumption L.10 (a); conversely, when $\iota \geq \frac{1}{4}$ the latter always implies the former. Of course, there are feasible options for them to satisfy all the requirements. For example, if $\kappa_1 = 0.7$, $\kappa_2 = 0.9$ and $\kappa_3 = 0.1$, then ι can be chosen from $(0, 3.5)$. By the way, if we impose some relationship among κ_i $(i = 1, 2, 3)$, such as $\kappa_2 < \frac{7}{6}\kappa_1 + \kappa_3$, (i) implies (ii) in (a).

Let $\zeta = 2\iota_1 + \iota_2$ for the time being. Since $\zeta \geq 0$, the condition (i) in (c) always implies Assumption 4.9 (b). Evidently, if a relationship is imposed among κ_i $(i = 1, 2, 3)$, (i) and (ii) in (c) may substitute each other, depending what relationship is adding. Note that there are feasible choices for all parameters. For instance, $\kappa_1 = 0.6$, $\kappa_2 = 0.8$, $\kappa_3 = 0.1$, $\zeta \in (0, 2.5)$.

The following theorem is the main result for the section.

Theorem 5.3.1. *Suppose that $\{x_{s,n}\}_{s=1}^n$ and $\{e_s\}_{s=1}^n$ satisfy Assumptions B and A (c). Let Assumptions L.8 and L.9 hold.*

If Assumption L.10 (a) and (b) are true, then

$$\begin{aligned} &\frac{\sqrt[4]{T_n}^3 \sqrt{\sigma_z}}{\sqrt{n}\upsilon(T_n)} \frac{\alpha' X' X A(\tau, x)}{\sqrt{p_{\max}} \|A(\tau, x)\|^2} (\widehat{m}(\tau, x) - m(\tau, x)) \\ &\qquad \to_D \left(\int_0^1 G_3(u) dL_W(u, 0) \right)^{\frac{1}{2}} N, \end{aligned} \tag{5.3.8}$$

where $G_3(\cdot) = \int f^2(\cdot, x)dx$, *W is the standard Brownian motion on* $[0,1]$ *and N is a standard normal random variable independent of W, and* L_W *is the local–time process of W.*

If Assumption L.10 (c) and (d) are true, then

$$\begin{aligned}\frac{\sqrt{T_n}}{\sqrt{n}\upsilon_1(T_n)\upsilon_2(\sqrt{T_n}\sigma_z)}&\frac{\alpha' X' X A(\tau,x)}{\sqrt{p_{\max}}\|A(\tau,x)\|^2}(\widehat{m}(\tau,x)-m(\tau,x))\\ &\to_D \int_0^1 f(r,W(r))dU(r),\end{aligned} \tag{5.3.9}$$

where the vector $(W(r), U(r))$ *of Brownian motions is from Assumption B.*

Remark 5.3.4. The convergence rate of $\widehat{m}(\tau,x)-m(\tau,x)$ in the first case, as can be seen in its proof, is about $\frac{\sqrt{n}\sqrt{p_{\max}}\varrho(T_n)}{\sqrt[4]{T_n}^3\|A(\tau,x)\|}$. With help of the estimation of $\|A(\tau,x)\|$, the rate is between $n^{\frac{1}{2}(1-\kappa_1)+(\iota-\frac{1}{4})\kappa_3}$ and $n^{\frac{1}{2}(1-\kappa_1)+(\iota-\frac{1}{4})\kappa_3+\frac{1}{2}(\bar{\kappa}_2-\kappa_2)}$. The order of the lower bound is less than $\frac{1}{4}$, while the order of upper bound is less than $\frac{1}{2}$.

In the second case, the convergence rate is $\frac{\sqrt{n}\sqrt{p_{\max}}\varrho_1(T_n)\varrho_2(\sqrt{T_n}\sigma_z)}{\sqrt{T_n}\|A(\tau,x)\|}$ revealed by its proof. Approximately, it is between $n^{\frac{1}{2}(1-\kappa_1)+(\iota_1+\frac{1}{2}\iota_2)\kappa_3}$ and $n^{\frac{1}{2}(1-\kappa_1)+(\iota_1+\frac{1}{2}\iota_2)\kappa_3+\frac{1}{2}(\bar{\kappa}_2-\kappa_2)}$.

Comparing the upper bounds and the lower bounds in the two scenarios, roughly speaking, the second situation is faster than the first.

Proof. Let us prove (5.3.8) first. It follows from (5.3.4) that

$$\begin{aligned}&\frac{\sqrt[4]{T_n}^3\sqrt{\sigma_z}}{\sqrt{n}\upsilon(T_n)}\frac{\alpha' X' X A(\tau,x)}{\sqrt{p_{\max}}\|A(\tau,x)\|^2}(\widehat{m}(\tau,x)-m(\tau,x))\\ =&\frac{\sqrt[4]{T_n}^3\sqrt{\sigma_z}}{\sqrt{n}\upsilon(T_n)}\frac{\alpha' X' X A(\tau,x)}{\sqrt{p_{\max}}\|A(\tau,x)\|^2}[A'(\tau,x)(X'X)^{-1}X'(\delta+\gamma+\varepsilon)-\delta(\tau,x)-\gamma(\tau,x)]\\ =&\frac{\sqrt[4]{T_n}^3\sqrt{\sigma_z}}{\sqrt{n}\upsilon(T_n)\sqrt{p_{\max}}}\alpha' B X'(\delta+\gamma+\varepsilon)-\frac{\sqrt[4]{T_n}^3\sqrt{\sigma_z}}{\sqrt{n}\upsilon(T_n)}\frac{\alpha' X' X A(\tau,x)}{\sqrt{p_{\max}}\|A(\tau,x)\|^2}[\delta(\tau,x)+\gamma(\tau,x)]\\ =&\frac{\sqrt[4]{T_n}^3\sqrt{\sigma_z}}{\sqrt{n}\upsilon(T_n)\sqrt{p_{\max}}}\alpha' X'(\delta+\gamma+\varepsilon)-\frac{\sqrt[4]{T_n}^3\sqrt{\sigma_z}}{\sqrt{n}\upsilon(T_n)}\frac{\alpha' X' X A(\tau,x)}{\sqrt{p_{\max}}\|A(\tau,x)\|^2}[\delta(\tau,x)+\gamma(\tau,x)]\\ :=&\Pi_1-\Pi_2.\end{aligned}$$

We are about to show that Π_1 converges to the desired random variable in distribution, while $\Pi_2 \to_P 0$ as $n \to \infty$.

Using (5.3.7a) we can write

$$\Pi_1 = \frac{\sqrt[4]{T_n}\sqrt{\sigma_z}}{\sqrt{n}\upsilon(T_n)}(\widetilde{\mathbf{F}}' - \widetilde{\delta}' - \widetilde{\gamma}')(\delta+\gamma+\varepsilon)$$

$$=\frac{\sqrt[4]{T_n}\sqrt{\sigma_z}}{\sqrt{n}\upsilon(T_n)}(\widetilde{\mathbf{F}}'\delta-\widetilde{\delta}'\delta-\widetilde{\gamma}'\delta+\widetilde{\mathbf{F}}'\gamma-\widetilde{\delta}'\gamma-\widetilde{\gamma}'\gamma+\widetilde{\mathbf{F}}'\varepsilon-\widetilde{\delta}'\varepsilon-\widetilde{\gamma}'\varepsilon).$$

It follows from Theorem 1.3.1 and Assumption L.10 (a) that

$$\begin{aligned}\frac{\sqrt[4]{T_n}\sqrt{\sigma_z}}{\sqrt{n}\upsilon(T_n)}\widetilde{\mathbf{F}}'\varepsilon=&\frac{\sqrt[4]{T_n}\sqrt{\sigma_z}}{\sqrt{n}\upsilon(T_n)}\sum_{s=1}^{n}\widetilde{F}(t_{s,n},X_{s,n})e_s\\ =&\frac{\sqrt[4]{T_n}\sqrt{\sigma_z}}{\sqrt{n}\upsilon(T_n)}\sum_{s=1}^{n}\widetilde{F}\left(\frac{s}{n}T_n,\frac{s}{n}T_n\mu+\sqrt{T_n}\sigma_z x_{s,n}\right)e_s\\ =&\frac{\sqrt[4]{T_n}\sqrt{\sigma_z}}{\sqrt{n}\upsilon(T_n)}\sum_{s=1}^{n}F\left(\frac{s}{n}T_n,\sqrt{T_n}\sigma_z x_{s,n}\right)e_s\\ \to_D&\left(\int_0^1 G_3(u)dL_W(u,0)\right)^{\frac{1}{2}}N,\end{aligned}\tag{5.3.10}$$

where $G_3(\cdot)=\int f^2(\cdot,x)dx$, W is the standard Brownian motion on $[0,1]$ and N is a standard normal random variable independent of W, and L_W is the local–time process of W.

Meanwhile, using Cauchy-Schwarz inequality gives

$$\begin{aligned}&|\widetilde{\mathbf{F}}'\delta|^2\leq\|\widetilde{\mathbf{F}}'\|^2\|\delta\|^2, && |\widetilde{\mathbf{F}}'\gamma|^2\leq\|\widetilde{\mathbf{F}}'\|^2\|\gamma\|^2, && |\widetilde{\delta}'\delta|^2\leq\|\widetilde{\delta}'\|^2\|\delta\|^2,\\ &|\widetilde{\delta}'\gamma|^2\leq\|\widetilde{\delta}'\|^2\|\gamma\|^2, && |\widetilde{\gamma}'\delta|^2\leq\|\widetilde{\gamma}'\|^2\|\delta\|^2, && |\widetilde{\gamma}'\gamma|^2\leq\|\widetilde{\gamma}'\|^2\|\gamma\|^2.\end{aligned}$$

where $\|\cdot\|$ signifies the Euclidean norm.

In addition, it follows once again from Theorem 1.3.1 that

$$\begin{aligned}\frac{\sqrt{T_n}\sigma_z}{n\upsilon(T_n)^2}\|\widetilde{\mathbf{F}}'\|^2=&\frac{\sqrt{T_n}\sigma_z}{n\upsilon(T_n)^2}\sum_{s=1}^{n}\widetilde{F}^2(t_{s,n},X_{s,n})\\ =&\frac{\sqrt{T_n}\sigma_z}{n\upsilon(T_n)^2}\sum_{s=1}^{n}\widetilde{F}^2\left(\frac{s}{n}T_n,\frac{s}{n}T_n\mu+\sqrt{T_n}\sigma_z x_{s,n}\right)\\ =&\frac{\sqrt{T_n}\sigma_z}{n\upsilon(T_n)^2}\sum_{s=1}^{n}F^2\left(\frac{s}{n}T_n,\sqrt{T_n}\sigma_z x_{s,n}\right)\\ \to_P&\int_0^1\int_{-\infty}^{\infty}f^2(t,x)dxdL_W(t,0),\end{aligned}\tag{5.3.11}$$

since $F^2(\cdot,\cdot)$ is in Class (HI) with normal function $f^2(t,x)$ and homogeneity power $\upsilon(\cdot)^2$ due to Assumption 6.10 (b).

Also, since $x_{s,n}$ and e_s $(s = 1, \cdots, n)$ satisfy Assumption B, using martingale difference structure, similar to the proof of Theorem 5.2.1, we have

$$\frac{\sqrt{T_n}\sigma_z}{n\upsilon(T_n)^2}E(\widetilde{\delta}'\varepsilon)^2 = \frac{\sqrt{T_n}\sigma_z}{n\upsilon(T_n)^2}\sigma_e^2 E\|\widetilde{\delta}'\|^2,$$
$$\frac{\sqrt{T_n}\sigma_z}{n\upsilon(T_n)^2}E(\widetilde{\gamma}'\varepsilon)^2 = \frac{\sqrt{T_n}\sigma_z}{n\upsilon(T_n)^2}\sigma_e^2 E\|\widetilde{\gamma}'\|^2.$$

Therefore, in order to complete the convergence of Π_1 it suffices to demonstrate that as $n \to \infty$,

$$\|\delta\|^2 \to_P 0, \qquad \|\gamma\|^2 \to_P 0, \tag{5.3.12a}$$

$$\frac{\sqrt{T_n}}{n\upsilon(T_n)^2}E\|\widetilde{\delta}'\|^2 \to 0, \qquad \frac{\sqrt{T_n}}{n\upsilon(T_n)^2}E\|\widetilde{\gamma}'\|^2 \to 0. \tag{5.3.12b}$$

To begin with the proof of (5.3.12a), similar to the counterpart in the proof of Theorem 5.2.1,

$$\begin{aligned}
E\|\delta\|^2 =& \sum_{s=1}^n E\left(\sum_{i=0}^k \sum_{j=p_i+1}^\infty b_{ij}\varphi_{jT_n}(t_{s,n})Q_i(t_{s,n}, X_{s,n})\right)^2 \\
=& \sum_{s=1}^n \sum_{i=0}^k \left(\sum_{j=p_i+1}^\infty b_{ij}\varphi_{jT_n}(t_{s,n})\right)^2 \le o(1)T_n^3 \sum_{s=1}^n \sum_{i=0}^k \frac{1}{p_i^3} \\
\le& o(1)T_n^3 \frac{nk}{p_{\min}^3} = o(1)n^{1+3\kappa_3+\kappa_1-3\kappa_2} \to 0,
\end{aligned}$$

by virtue of Assumption L.9, which in turn implies $\|\delta\|^2 \to_P 0$.

In addition, it follows from (4.4.8) with $r = 3$ that as $n \to \infty$

$$\begin{aligned}
E\|\gamma\|^2 =& \sum_{s=1}^n E[\gamma_s^2] = \sum_{s=1}^n E\left(\sum_{i=k+1}^\infty \sum_{j=0}^\infty b_{ij}\varphi_{jT_n}(t_{s,n})Q_i(t_{s,n}, X_{s,n})\right)^2 \\
=& \sum_{s=1}^n \sum_{i=k+1}^\infty b_i^2(t_{s,n}) = \sum_{s=1}^n \sum_{i=k+1}^\infty \frac{v^3(t_{s,n})}{i(i-1)(i-2)} b_{i-3}^2(t_{s,n}, D^3 m) \\
=& \sum_{i=k+1}^\infty \frac{1}{i(i-1)(i-2)} \sum_{s=1}^n v^3(t_{s,n}) b_{i-3}^2(t_{s,n}, D^3 m) \\
\le& An \sum_{i=k+1}^\infty \frac{1}{i(i-1)(i-2)} \le A(1+o(1))\frac{n}{k^2} \\
=& A(1+o(1))n^{1-2\kappa_1} \to 0,
\end{aligned}$$

by Assumption L.8, L.9, where A is the uniform bound of $\upsilon(t)^3 b_{i-3}^2(t, D^3 m)$.

Now we are to prove (5.3.12b). It follows from the expressions of $\widetilde{\delta}_s$ that

$$\begin{aligned}
&\frac{\sqrt{T_n}}{n\upsilon(T_n)^2} E\|\widetilde{\delta}'\|^2 = \frac{\sqrt{T_n}}{n\upsilon(T_n)^2} \sum_{s=1}^{n} E[\widetilde{\delta}_s^2] \\
=&\frac{\sqrt{T_n}}{n\upsilon(T_n)^2} \sum_{s=1}^{n} E\left(\sum_{i=0}^{k} \sum_{j=p_i+1}^{\infty} a_{ij}\varphi_{jT_n}(t_{s,n}) Q_i(t_{s,n}, X_{s,n})\right)^2 \\
=&\frac{\sqrt{T_n}}{n\upsilon(T_n)^2} \sum_{s=1}^{n}\sum_{i=0}^{k} \left(\sum_{j=p_i+1}^{\infty} a_{ij}\varphi_{jT_n}(t_{s,n})\right)^2 \\
\leq&\frac{2}{n\sqrt{T_n}\upsilon(T_n)^2} \sum_{s=1}^{n}\sum_{i=0}^{k} \left(\sum_{j=p_i+1}^{\infty} |a_{ij}|\right)^2 \\
\leq&\frac{2}{\sqrt{T_n}\upsilon(T_n)^2} \sum_{i=0}^{k} \frac{1}{p_i^2} \left(\sum_{j=p_i+1}^{\infty} j|a_{ij}|\right)^2 \\
\leq&\frac{o(1)k}{\sqrt{T_n}p_{\min}^2\upsilon(T_n)^2} = \frac{o(1)}{\upsilon(T_n)^2} n^{\kappa_1 - 2\kappa_2 - \kappa_3/2} \to 0,
\end{aligned}$$

as $n \to \infty$ by Assumptions L.10 (a) and L.2.

Analogously, we have as $n \to \infty$

$$\begin{aligned}
&\frac{\sqrt{T_n}}{n\upsilon(T_n)^2} E\|\widetilde{\gamma}'\|^2 = \frac{\sqrt{T_n}}{n\upsilon(T_n)^2} \sum_{s=1}^{n} E[\widetilde{\gamma}_s^2] \\
=&\frac{\sqrt{T_n}}{n\upsilon(T_n)^2} \sum_{s=1}^{n} E\left(\sum_{i=k+1}^{\infty} \sum_{j=0}^{\infty} a_{ij}\varphi_{jT_n}(t_{s,n}) Q_i(t_{s,n}, X_{s,n})\right)^2 \\
=&\frac{\sqrt{T_n}}{n\upsilon(T_n)^2} \sum_{s=1}^{n}\sum_{i=k+1}^{\infty} \left(\sum_{j=0}^{\infty} a_{ij}\varphi_{jT_n}(t_{s,n})\right)^2 \\
\leq&\frac{2\sqrt{T_n}}{n\upsilon(T_n)^2 T_n} \sum_{s=1}^{n}\sum_{i=k+1}^{\infty} \left(\sum_{j=0}^{\infty} |a_{ij}|\right)^2 \\
\leq&\frac{2}{\sqrt{T_n}\upsilon(T_n)^2 k} \sum_{i=k+1}^{\infty} i \left(\sum_{j=0}^{\infty} |a_{ij}|\right)^2 \\
=&\frac{o(1)}{\upsilon(T_n)^2} n^{-\kappa_1 - \kappa_3/2} \to 0
\end{aligned}$$

on account of Assumption L.2 as $n \to \infty$.

We are now in a position to prove $\Pi_2 \to_P 0$ as $n \to \infty$.

Notice that Π_2 can be rephrased as

$$\begin{aligned}\Pi_2 =& \frac{\sqrt[4]{T_n}^3 \sqrt{\sigma_z}}{\sqrt{n}\upsilon(T_n)} \frac{\alpha' X' X A(\tau, x)}{\sqrt{p_{\max}} \|A(\tau, x)\|^2} [\delta(\tau, x) + \gamma(\tau, x)] \\ =& \frac{\sqrt{T_n}\sigma_z}{n\upsilon(T_n)\varrho(T_n)} \frac{T_n}{p_{\max}\|A(\tau, x)\|} \alpha' X' X A(\tau, x) \\ & \times \frac{\sqrt{n}\sqrt{p_{\max}}\varrho(T_n)}{\sqrt{\sigma_z}\sqrt[4]{T_n}^3 \|A(\tau, x)\|} [\delta(\tau, x) + \gamma(\tau, x)] \\ :=& \Pi_{21} \times (\Pi_{22} + \Pi_{23}),\end{aligned}$$

where

$$\begin{aligned}\Pi_{21} =& \frac{\sqrt{T_n}\sigma_z}{n\upsilon(T_n)\varrho(T_n)} \frac{T_n}{p_{\max}\|A(\tau, x)\|} \alpha' X' X A(\tau, x), \\ \Pi_{22} =& \frac{\sqrt{n}\sqrt{p_{\max}}\varrho(T_n)}{\sqrt{\sigma_z}\sqrt[4]{T_n}^3 \|A(\tau, x)\|} \delta(\tau, x), \qquad \Pi_{23} = \frac{\sqrt{n}\sqrt{p_{\max}}\varrho(T_n)}{\sqrt{\sigma_z}\sqrt[4]{T_n}^3 \|A(\tau, x)\|} \gamma(\tau, x).\end{aligned}$$

To complete the convergence of Π_2, we are going to show that Π_{21} converges to some random variable in probability, while both Π_{22} and Π_{23} are convergent to zero.

It follows from (5.3.7) that

$$\begin{aligned}\Pi_{21} =& \frac{\sqrt{T_n}\sigma_z}{n\upsilon(T_n)\varrho(T_n)} \frac{T_n}{p_{\max}\|A(\tau, x)\|} \alpha' X' X A(\tau, x) \\ =& \frac{\sqrt{T_n}\sigma_z}{n\upsilon(T_n)\varrho(T_n)} (\widetilde{\mathbf{F}}' - \widetilde{\delta}' - \widetilde{\gamma}')(\widetilde{\mathbf{G}} - \widetilde{\delta} - \widetilde{\gamma}) \\ =& \frac{\sqrt{T_n}\sigma_z}{n\upsilon(T_n)\varrho(T_n)} (\widetilde{\mathbf{F}}'\widetilde{\mathbf{G}} - \widetilde{\delta}'\widetilde{\mathbf{G}} - \widetilde{\gamma}'\widetilde{\mathbf{G}} + \widetilde{\mathbf{F}}'\widetilde{\delta} + \widetilde{\mathbf{F}}'\widetilde{\gamma} - \|\widetilde{\delta}'\|^2 - 2\widetilde{\gamma}'\widetilde{\delta} - \|\widetilde{\gamma}'\|^2).\end{aligned}$$

Nevertheless, (5.3.11) and (5.3.12b) as well as Cauchy-Schwarz inequality suggest that in order to obtain the limit of Π_{21}, one only needs to find the limits of

$$\frac{\sqrt{T_n}\sigma_z}{n\upsilon(T_n)\varrho(T_n)} \widetilde{\mathbf{F}}'\widetilde{\mathbf{G}} \quad \text{and} \quad \frac{\sqrt{T_n}\sigma_z}{n\varrho(T_n)^2} \|\widetilde{\mathbf{G}}\|^2.$$

Indeed, similar to (5.3.11) we have

$$\frac{\sqrt{T_n}\sigma_z}{n\varrho(T_n)^2} \|\widetilde{\mathbf{G}}'\|^2 = \frac{\sqrt{T_n}\sigma_z}{n\varrho(T_n)^2} \sum_{s=1}^{n} \widetilde{G}^2(t_{s,n}, X_{s,n})$$

$$
\begin{aligned}
=&\frac{\sqrt{T_n}\sigma_z}{n\varrho(T_n)^2}\sum_{s=1}^n \widetilde{G}^2\left(\frac{s}{n}T_n, \frac{s}{n}T_n\mu+\sqrt{T_n}\sigma_z x_{s,n}\right)\\
=&\frac{\sqrt{T_n}\sigma_z}{n\varrho(T_n)^2}\sum_{s=1}^n G^2\left(\frac{s}{n}T_n, \sqrt{T_n}\sigma_z x_{s,n}\right)\\
\to_P&\int_0^1\int_{-\infty}^{\infty} g^2(t,x)dxdL_W(t,0),
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{\sqrt{T_n}\sigma_z}{n\upsilon(T_n)\varrho(T_n)}\widetilde{\mathbf{F}}'\widetilde{\mathbf{G}}=\frac{\sqrt{T_n}\sigma_z}{n\upsilon(T_n)\varrho(T_n)}\sum_{s=1}^n\widetilde{F}(t_{s,n},X_{s,n})\widetilde{F}(t_{s,n},X_{s,n})\\
=&\frac{\sqrt{T_n}\sigma_z}{n\upsilon(T_n)\varrho(T_n)}\sum_{s=1}^n\widetilde{F}\left(\frac{s}{n}T_n, \frac{s}{n}T_n\mu+\sqrt{T_n}\sigma_z x_{s,n}\right)\widetilde{G}\left(\frac{s}{n}T_n, \frac{s}{n}T_n\mu+\sqrt{T_n}\sigma_z x_{s,n}\right)\\
=&\frac{\sqrt{T_n}\sigma_z}{n\upsilon(T_n)\varrho(T_n)}\sum_{s=1}^n F\left(\frac{s}{n}T_n, \sqrt{T_n}\sigma_z x_{s,n}\right)G\left(\frac{s}{n}T_n, \sqrt{T_n}\sigma_z x_{s,n}\right)\\
\to_P&\int_0^1\int_{-\infty}^{\infty} f(t,x)g(t,x)dxdL_W(t,0),
\end{aligned}
$$

by virtue of Theorem 1.3.1 and Assumption L.10 (b).

Hence, we have $\Pi_{21}\to_P\int_0^1\int_{-\infty}^{\infty}f(t,x)g(t,x)dxdL_W(t,0)$.

Regarding Π_{22}, because σ_z is a constant, and in view of the estimation $O(1)kp_{\min}\le T_n\|A(\tau,x)\|^2\le O(1)kp_{\max}$ and $b_{ij}=\left(\frac{T_n}{j\pi}\right)^2 c_j(b_i'')$ where $c_j(b_i'')$ is the j-th coefficient in the expansion of b_i'', we have,

$$
\begin{aligned}
|\Pi_{22}|=&\frac{\sqrt{n}\sqrt{p_{\max}}\varrho(T_n)}{\sqrt{\sigma_z}\sqrt[4]{T_n}^3\|A(\tau,x)\|}|\delta(\tau,x)|\\
=&\frac{\sqrt{n}\sqrt{p_{\max}}\varrho(T_n)}{\sqrt{\sigma_z}\sqrt[4]{T_n}^3\|A(\tau,x)\|}\left|\sum_{i=0}^k\sum_{j=p_i+1}^{\infty}b_{ij}\varphi_{jT_n}(\tau)Q_i(\tau,x)\right|\\
\le&\frac{\sqrt{n}\sqrt{p_{\max}}\varrho(T_n)}{\sqrt{\sigma_z}\sqrt[4]{T_n}^3\|A(\tau,x)\|}\left(\sum_{i=0}^k|Q_i(\tau,x)|^2\right)^{\frac{1}{2}}\left(\sum_{i=0}^k\left|\sum_{j=p_i+1}^{\infty}b_{ij}\varphi_{jT_n}(\tau)\right|^2\right)^{\frac{1}{2}}\\
\le&o(1)\frac{\sqrt{n}\sqrt{p_{\max}T_n}\varrho(T_n)}{\sqrt{kp_{\min}}\sqrt[4]{T_n}^3}\sqrt{k}\left(\sum_{i=0}^k\frac{T_n^3}{p_i^3}\right)^{\frac{1}{2}}\\
\le&o(1)\frac{\sqrt{np_{\max}}T_n^{\iota}}{\sqrt{p_{\min}}\sqrt[4]{T_n}}\frac{\sqrt{k}\sqrt{T_n}^3}{\sqrt{p_{\min}}^3}\\
=&O(1)n^{\frac{1}{2}+\frac{1}{2}\kappa_1+(\iota+\frac{5}{4})\kappa_3+\frac{1}{2}(\bar{\kappa}_2-\kappa_2)-\frac{3}{2}\kappa_2}\to 0,
\end{aligned}
$$

as $n \to \infty$ where we use Assumption L.8(d) , Assumption L.9 and L.10 (a) for ι and truncation parameters.

Analogously, by (4.4.8) with $r = 2$,

$$
\begin{aligned}
|\Pi_{23}| =& \frac{\sqrt{n}\sqrt{p_{\max}}\varrho(T_n)}{\sqrt{\sigma_z}\sqrt[4]{T_n}^3\|A(\tau,x)\|}|\gamma(\tau,x)| \\
=& \frac{\sqrt{n}\sqrt{p_{\max}}\varrho(T_n)}{\sqrt{\sigma_z}\sqrt[4]{T_n}^3\|A(\tau,x)\|}\left|\sum_{i=k+1}^{\infty} b_i(\tau)Q_i(\tau,x)\right| \\
=& \frac{\sqrt{n}\sqrt{p_{\max}}\varrho(T_n)}{\sqrt{\sigma_z}\sqrt[4]{T_n}^3\|A(\tau,x)\|}\left|\sum_{i=k+1}^{\infty} \frac{v(\tau)}{\sqrt{i(i-1)}}b_i(\tau,D^2m)Q_i(\tau,x)\right| \\
\le& O(1)\frac{\sqrt{np_{\max}}T_n^{\iota}}{\sqrt[4]{T_n}\sqrt{kp_{\min}}}\left(\sum_{i=k+1}^{\infty}\frac{1}{i(i-1)\sqrt{i}}\right)^{\frac{1}{2}} \le o(1)\frac{\sqrt{np_{\max}}T_n^{\iota}}{\sqrt[4]{T_n}\sqrt{kp_{\min}}}\frac{1}{k^{3/4}} \\
=& o(1)n^{\frac{1}{2}+\frac{1}{2}(\bar{\kappa}_2-\kappa_2)+(\iota-\frac{1}{4})\kappa_3-\frac{5}{4}\kappa_1} \to 0,
\end{aligned}
$$

as $n \to \infty$ in view of Assumption L.10 (a).

Up to now, the first part of the theorem is finished. In what follows we shall prove the second part.

It follows from (5.3.4) that

$$
\begin{aligned}
& \frac{\sqrt{T_n}}{\sqrt{n}\upsilon_1(T_n)\upsilon_2(\sqrt{T_n}\sigma_z)}\frac{\alpha'X'XA(\tau,x)}{\sqrt{p_{\max}}\|A(\tau,x)\|^2}(\widehat{m}(\tau,x)-m(\tau,x)) \\
=& \frac{\sqrt{T_n}}{\sqrt{n}\upsilon_1(T_n)\upsilon_2(\sqrt{T_n}\sigma_z)}\frac{\alpha'X'XA(\tau,x)}{\sqrt{p_{\max}}\|A(\tau,x)\|^2} \\
& \times [A'(\tau,x)(X'X)^{-1}X'(\delta+\gamma+\varepsilon)-\delta(\tau,x)-\gamma(\tau,x)] \\
=& \frac{1}{\sqrt{n}\upsilon_1(T_n)\upsilon_2(\sqrt{T_n}\sigma_z)}\sqrt{\frac{T_n}{p_{\max}}}\alpha'X'(\delta+\gamma+\varepsilon) \\
& - \frac{\sqrt{T_n}}{\sqrt{n}\upsilon_1(T_n)\upsilon_2(\sqrt{T_n}\sigma_z)}\frac{\alpha'X'XA(\tau,x)}{\sqrt{p_{\max}}\|A(\tau,x)\|^2}[\delta(\tau,x)+\gamma(\tau,x)] \\
:=& \Pi_3 - \Pi_4.
\end{aligned}
$$

We are about to show that Π_3 is convergent in probability to the desired stochastic integral and $\Pi_4 \to_P 0$.

Observe that in view of (5.3.7a) we have

$$
\Pi_3 = \frac{1}{\sqrt{n}\upsilon_1(T_n)\upsilon_2(\sqrt{T_n}\sigma_z)}\sqrt{\frac{T_n}{p_{\max}}}\alpha'X'(\delta+\gamma+\varepsilon)
$$

$$
\begin{aligned}
=&\frac{1}{\sqrt{n}\upsilon_1(T_n)\upsilon_2(\sqrt{T_n}\sigma_z)}(\widetilde{\mathbf{F}}' - \widetilde{\delta}' - \widetilde{\gamma}')(\delta + \gamma + \varepsilon)\\
=&\frac{1}{\sqrt{n}\upsilon_1(T_n)\upsilon_2(\sqrt{T_n}\sigma_z)}\\
&\times (\widetilde{\mathbf{F}}'\delta - \widetilde{\delta}'\delta - \widetilde{\gamma}'\delta + \widetilde{\mathbf{F}}'\gamma - \widetilde{\delta}'\gamma - \widetilde{\gamma}'\gamma + \widetilde{\mathbf{F}}'\varepsilon - \widetilde{\delta}'\varepsilon - \widetilde{\gamma}'\varepsilon).
\end{aligned}
$$

It follows from the proof (not the result) of Theorem 1.5.1 that

$$
\begin{aligned}
&\frac{1}{\sqrt{n}\upsilon_1(T_n)\upsilon_2(\sqrt{T_n}\sigma_z)}\widetilde{\mathbf{F}}'\varepsilon = \frac{1}{\sqrt{n}\upsilon_1(T_n)\upsilon_2(\sqrt{T_n}\sigma_z)}\sum_{s=1}^{n}\widetilde{F}(t_{s,n}, X_{s,n})e_s\\
=&\frac{1}{\sqrt{n}\upsilon_1(T_n)\upsilon_2(\sqrt{T_n}\sigma_z)}\sum_{s=1}^{n}\widetilde{F}(t_{s,n}, \mu t_{s,n} + \sqrt{T_n}\sigma_z x_{s,n})e_s\\
=&\frac{1}{\sqrt{n}\upsilon_1(T_n)\upsilon_2(\sqrt{T_n}\sigma_z)}\sum_{s=1}^{n}F(t_{s,n}, \sqrt{T_n}\sigma_z x_{s,n})e_s\\
=&\frac{1}{\sqrt{n}\upsilon_1(T_n)\upsilon_2(\sqrt{T_n}\sigma_z)}\sum_{s=1}^{n}F\left(\frac{s}{n}T_n, \sqrt{T_n}\sigma_z x_{s,n}\right)e_s\\
\to_P&\int_0^1 f(r, W(r))dU(r), \qquad (5.3.13)
\end{aligned}
$$

on account of Assumption L.10 (c) where $(W(r), U(r))$ is the limit of $(W_n(r), U_n(r))$ in Assumption B.

Also, for the same reason we have

$$
\begin{aligned}
&\frac{1}{n\upsilon_1^2(T_n)\upsilon_2^2(\sqrt{T_n}\sigma_z)}\|\widetilde{\mathbf{F}}'\|^2 = \frac{1}{n\upsilon_1^2(T_n)\upsilon_2^2(\sqrt{T_n}\sigma_z)}\sum_{s=1}^{n}\widetilde{F}^2(t_{s,n}, X_{s,n})\\
=&\frac{1}{n\upsilon_1^2(T_n)\upsilon_2^2(\sqrt{T_n}\sigma_z)}\sum_{s=1}^{n}\widetilde{F}^2(t_{s,n}, \mu t_{s,n} + \sqrt{T_n}\sigma_z x_{s,n})\\
=&\frac{1}{n\upsilon_1^2(T_n)\upsilon_2^2(\sqrt{T_n}\sigma_z)}\sum_{s=1}^{n}F^2(t_{s,n}, \sqrt{T_n}\sigma_z x_{s,n})\\
=&\frac{1}{n\upsilon_1^2(T_n)\upsilon_2^2(\sqrt{T_n}\sigma_z)}\sum_{s=1}^{n}F^2\left(\frac{s}{n}T_n, \sqrt{T_n}\sigma_z x_{s,n}\right)\\
\to_P&\int_0^1 f^2(r, W(r))dr, \qquad (5.3.14)
\end{aligned}
$$

on account of Assumption L.10 (d).

Notice that the parameters involved in both δ and γ are k and p_i $(i = 0, \cdots, k)$ which satisfy Assumption L.9, as the case in the first part of the theorem, and the coefficients of the expansion of $m(t,x)$ in both δ and γ remain unchanged; meanwhile, $\widetilde{\delta}'$ and $\widetilde{\gamma}'$ are the same as in the first part since $\{a_{ij}\}$ is still the sequence in Assumption L.2 and the truncation parameters are the same. Whence, (5.3.12) is still valid in this part with a modification that $\upsilon(T_n)$ is superceded by $\upsilon_1(T_n)$. Therefore, Cauchy-Schwarz inequality and (5.3.14) imply all the terms of Π_3 except for (5.3.13) are approaching to zero in probability, hence $\Pi_3 \to_P \int_0^1 f(r, W(r))dU(r)$.

Now we are ready to prove that $\Pi_4 \to_P 0$. Rewrite

$$\begin{aligned}\Pi_4 =& \frac{\sqrt{T_n}}{\sqrt{n}\upsilon_1(T_n)\upsilon_2(\sqrt{T_n}\sigma_z)} \frac{\alpha' X' X A(\tau,x)}{\sqrt{p_{\max}}\|A(\tau,x)\|^2}[\delta(\tau,x)+\gamma(\tau,x)] \\ :=& \Pi_{41} \times (\Pi_{42} + \Pi_{43})\end{aligned}$$

where

$$\begin{aligned}\Pi_{41} =& \frac{T_n}{n\upsilon_1(T_n)\upsilon_2(\sqrt{T_n}\sigma_z)\varrho_1(T_n)\varrho_2(\sqrt{T_n}\sigma_z)} \frac{\alpha' X' X A(\tau,x)}{p_{\max}\|A(\tau,x)\|}, \\ \Pi_{42} =& \frac{\sqrt{n}\sqrt{p_{\max}}\varrho_1(T_n)\varrho_2(\sqrt{T_n}\sigma_z)}{\sqrt{T_n}\|A(\tau,x)\|}\delta(\tau,x),\end{aligned}$$

and

$$\Pi_{43} = \frac{\sqrt{n}\sqrt{p_{\max}}\varrho_1(T_n)\varrho_2(\sqrt{T_n}\sigma_z)}{\sqrt{T_n}\|A(\tau,x)\|}\gamma(\tau,x).$$

It follows from (5.3.7) that

$$\begin{aligned}\Pi_{41} =& \frac{1}{n\upsilon_1(T_n)\upsilon_2(\sqrt{T_n}\sigma_z)\varrho_1(T_n)\varrho_2(\sqrt{T_n}\sigma_z)}(\widetilde{\mathbf{F}}' - \widetilde{\delta}' - \widetilde{\gamma}')(\widetilde{\mathbf{G}} - \widetilde{\delta} - \widetilde{\gamma}) \\ =& \frac{1}{n\upsilon_1(T_n)\upsilon_2(\sqrt{T_n}\sigma_z)\varrho_1(T_n)\varrho_2(\sqrt{T_n}\sigma_z)} \\ & \times (\widetilde{\mathbf{F}}'\widetilde{\mathbf{G}} - \widetilde{\delta}'\widetilde{\mathbf{G}} - \widetilde{\gamma}'\widetilde{\mathbf{G}} - \widetilde{\mathbf{F}}'\widetilde{\delta} - \widetilde{\mathbf{F}}'\widetilde{\gamma} + \|\widetilde{\delta}\|^2 + \|\widetilde{\gamma}\|^2 + 2\widetilde{\gamma}'\widetilde{\delta}).\end{aligned}$$

Once again, due to the proof of Theorem 1.5.1, Assumption L.10 (d), we similarly have

$$\begin{aligned}& \frac{1}{n\varrho_1(T_n)^2\varrho_2(\sqrt{T_n}\sigma_z)^2}\|\widetilde{\mathbf{G}}\|^2 \\ =& \frac{1}{n\varrho_1(T_n)^2\varrho_2(\sqrt{T_n}\sigma_z)^2}\sum_{s=1}^n \widetilde{G}^2(t_{s,n}, X_{s,n}) \\ =& \frac{1}{n\varrho_1(T_n)^2\varrho_2(\sqrt{T_n}\sigma_z)^2}\sum_{s=1}^n G^2\left(\frac{s}{n}T_n, \sqrt{T_n}\sigma_z x_{s,n}\right)\end{aligned}$$

$$\to_P \int_0^1 g^2(r, W(r))dr. \tag{5.3.15}$$

Thus, Cauchy-Schwarz inequality as well as (5.3.12b),(5.3.14), (5.3.15) suggests that all the terms in Π_{41} except for the one containing $\widetilde{\mathbf{F}}'\widetilde{\mathbf{G}}$ converge in probability to zero. Hence, to find out the limit of Γ_{41} it suffices to find that of that term. In fact,

$$\begin{aligned}
&\frac{1}{n\upsilon_1(T_n)\upsilon_2(\sqrt{T_n}\sigma_z)\varrho_1(T_n)\varrho_2(\sqrt{T_n}\sigma_z)}\widetilde{\mathbf{F}}'\widetilde{\mathbf{G}}\\
=&\frac{1}{n\upsilon_1(T_n)\upsilon_2(\sqrt{T_n}\sigma_z)\varrho_1(T_n)\varrho_2(\sqrt{T_n}\sigma_z)}\sum_{s=1}^{n}\widetilde{F}(t_{s,n}, X_{s,n})\widetilde{G}(t_{s,n}, X_{s,n})\\
=&\frac{1}{n\upsilon_1(T_n)\upsilon_2(\sqrt{T_n}\sigma_z)\varrho_1(T_n)\varrho_2(\sqrt{T_n}\sigma_z)}\\
&\times\sum_{s=1}^{n}\widetilde{F}\left(t_{s,n}, \mu t_{s,n}+\sqrt{T_n}\sigma_z x_{s,n}\right)\widetilde{G}\left(t_{s,n}, \mu t_{s,n}+\sqrt{T_n}\sigma_z x_{s,n}\right)\\
=&\frac{1}{n\upsilon_1(T_n)\upsilon_2(\sqrt{T_n}\sigma_z)\varrho_1(T_n)\varrho_2(\sqrt{T_n}\sigma_z)}\\
&\times\sum_{s=1}^{n}F\left(\frac{s}{n}T_n, \sqrt{T_n}\sigma_z x_{s,n}\right)G\left(\frac{s}{n}T_n, \sqrt{T_n}\sigma_z x_{s,n}\right)\\
\to_P&\int_0^1 f(r, W(r))g(r, W(r))dr,
\end{aligned}$$

as $n\to\infty$ by the proof of Theorem 1.5.1 and Assumption L.10 (d), so that Π_{41} converges to the same limit as above in probability.

Now let us turn to prove both $\Pi_{42}\to 0$ and $\Pi_{43}\to 0$, as $n\to\infty$. Recall that $O(1)kp_{\min}\le T_n\|A(\tau, x)\|^2\le O(1)kp_{\max}$. Because $\delta(\tau, x)$ and $\gamma(\tau, x)$ remain the same as in the first part, we have

$$\begin{aligned}
|\Pi_{42}| =&\frac{\sqrt{n}\sqrt{p_{\max}}\varrho_1(T_n)\varrho_2(\sqrt{T_n}\sigma_z)}{\sqrt{T_n}\|A(\tau, x)\|}|\delta(\tau, x)|\\
=&\frac{\sqrt{n}\sqrt{p_{\max}}\varrho_1(T_n)\varrho_2(\sqrt{T_n}\sigma_z)}{\sqrt{T_n}\|A(\tau, x)\|}\left|\sum_{i=0}^{k}\sum_{j=p_i+1}^{\infty}b_{ij}\varphi_{jT_n}(\tau)Q_i(\tau, x)\right|\\
=&\frac{\sqrt{n}\sqrt{p_{\max}}\varrho_1(T_n)\varrho_2(\sqrt{T_n}\sigma_z)}{\sqrt{T_n}\|A(\tau, x)\|}\left|\sum_{i=0}^{k}Q_i(\tau, x)\sum_{j=p_i+1}^{\infty}b_{ij}\varphi_{jT_n}(\tau)\right|\\
\le&\frac{\sqrt{n}\sqrt{p_{\max}}\varrho_1(T_n)\varrho_2(\sqrt{T_n}\sigma_z)}{\sqrt{T_n}\|A(\tau, x)\|}\left(\sum_{i=0}^{k}Q_i^2(\tau, x)\right)^{\frac{1}{2}}\left[\sum_{i=0}^{k}\left(\sum_{j=p_i+1}^{\infty}b_{ij}\varphi_{jT_n}(\tau)\right)^2\right]^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
\leq & O(1)\frac{\sqrt{n}\sqrt{p_{\max}}T_n^{\iota_1}T_n^{\frac{1}{2}\iota_2}}{\sqrt{kp_{\min}}}\sqrt{k}\left(\sum_{i=0}^{k}T_n^3\frac{1}{p_i^3}\right)^{\frac{1}{2}}\\
\leq & o(1)n^{\frac{1}{2}+\frac{1}{2}\kappa_1+\frac{1}{2}(\bar{\kappa}_2-\kappa_2)+(\iota_1+\frac{1}{2}\iota_2+\frac{3}{2})\kappa_3-\frac{3}{2}\kappa_2}\to 0,
\end{aligned}
$$

as $n\to\infty$ by the condition for the parameters of Assumption L.10 (c).

Meanwhile, once again on account of Assumption L.8 using (4.4.8) with $r=2$, we have

$$
\begin{aligned}
|\Pi_{43}|=&\frac{\sqrt{n}\sqrt{p_{\max}}\varrho_1(T_n)\varrho_2(\sqrt{T_n}\sigma_z)}{\sqrt{T_n}\|A(\tau,x)\|}|\gamma(\tau,x)|\\
=&\frac{\sqrt{n}\sqrt{p_{\max}}\varrho_1(T_n)\varrho_2(\sqrt{T_n}\sigma_z)}{\sqrt{T_n}\|A(\tau,x)\|}\left|\sum_{i=k+1}^{\infty}\sum_{j=0}^{\infty}b_{ij}\varphi_{jT_n}(\tau)Q_i(\tau,x)\right|\\
=&\frac{\sqrt{n}\sqrt{p_{\max}}\varrho_1(T_n)\varrho_2(\sqrt{T_n}\sigma_z)}{\sqrt{T_n}\|A(\tau,x)\|}\left|\sum_{i=k+1}^{\infty}b_i(\tau,m)Q_i(\tau,x)\right|\\
=&\frac{\sqrt{n}\sqrt{p_{\max}}\varrho_1(T_n)\varrho_2(\sqrt{T_n}\sigma_z)}{\sqrt{T_n}\|A(\tau,x)\|}\left|\sum_{i=k+1}^{\infty}\frac{v(\tau)}{\sqrt{i(i-1)}}b_{i-2}(\tau,D^2m)Q_i(\tau,x)\right|\\
\leq & o(1)\frac{\sqrt{n}\sqrt{p_{\max}}T_n^{\iota_1}T_n^{\frac{1}{2}\iota_2}}{\sqrt{kp_{\min}}}\frac{1}{k^{3/4}}\\
=&o(1)n^{\frac{1}{2}+\frac{1}{2}(\bar{\kappa}_2-\kappa_2)+(\iota_1+\frac{1}{2}\iota_2)\kappa_3-\frac{5}{4}\kappa_1}\to 0,
\end{aligned}
$$

where Assumption L.10 (c) are used.

This finishes the whole proof. □

Chapter 6

Conclusions and discussion

What has been done

Homogeneous functional $f(B(t))$ and time-inhomogeneous functional $f(t, B(t))$, given that f satisfies some not quite rigorous conditions respectively, have been expanded into orthogonal series, the so-called Fourier series in respective Hilbert space. Subsequently, everything happens automatically by virtue of Hilbert space. Different time horizons are considered from the application point of view. The key point behind doing so is to find appropriate Hilbert space which contains the functionals we are interested in and then to find some complete orthogonal polynomial sequence in the Hilbert space. What orthogonal polynomial systems we utilise in this situation are Hermite and Laguerre polynomial sequences as well as a trigonometric sequence on a fixed interval. These expansions enable us to estimate an unknown functional form in a class of general econometric models. The estimated form is explicit and the asymptotic distributions of the estimators according to different time horizons and sampling styles are shown as mixed normal for two cases where either the time zone is infinity or the time zone is compact associated with sample size and moving to infinity, and a stochastic integral in the case where the time zone is fixed.

For the same purposes, we consider a more general scenario where the underlying process is replaced by the Lévy process. Definitely, we face more challenges since we cannot rely on any particular distribution of the process. One thing is to illuminate the existence of orthogonal polynomial system which has the density function or probability distribution function of the underlying process as its weight. We thus define the classical polynomial system for some Lévy processes which encompass not only continuous processes such as Brownian motion but also discrete processes like the Poisson process. By virtue of hypergeometric differential/difference equations, we have shown the explicit expressions of the orthogonal polynomial system and their squared norms of the polynomials. These are crucial for the following developments.

Similar to the first part, homogeneous functional $f(Z(t))$ and time-inhomogeneous functional $f(t, Z(t))$ where $Z(t)$ is a Lévy process are expanded into Fourier series in respective Hilbert spaces. We also estimate unknown functionals of the Lévy processes in a general class of econometric models. Similarly, asymptotic distributions for different estimators according to the different time horizons are derived as well. The limits of the estimators on the infinite interval and compact interval which approaches infinity are different from the limit of the estimator on the fixed time zone. Moreover, the convergence rates of the estimators depend on sample size (as conventional regression estimator) and function forms produced by values of the basis at sampling points.

The existing results in the literature regarding of the asymptotic theory are not applicable for this research and it is mainly because a time variable is involved in the regressor. The exploration of asymptotic theory is exhibited in Chapter 1, which contains two large classes functionals and displays asymptotic theorems for both sample mean and sample covariance. As a result, the existing studies become a special case of the asymptotic theory developed in this paper.

As shown in Chapter 3 and 5, asymptotic theory plays a vital role for the application of the proposed expansions. More precisely, it determines for what kinds of functional forms involved in our derivation we can obtain their asymptotic distribution. Therefore, asymptotic theory amounts to a bottleneck. We establish asymptotic theory for two rudimentary classes of functionals, namely, functionals in Assumption C and regular functionals, and two extended classes of functionals, viz., Class $\mathcal{T}(HI)$ and Class $\mathcal{T}(HH)$ in Chapter 1. Our results indicate that the asymptotic theory enables us to apply the expansion in relative general functional forms.

More potential applications

Apart from the econometric estimation displayed in this thesis, the proposed method of Lévy process functional expansion may also be used in some relevant fields of economics and finance to tackle the nonlinear and nonstationary problems.

We would consider a traditional maximum utility question on agent consumption and habits. The form of utility, according to a prominent explanation of aggregate stock market behaviour, is a power function of the difference between aggregate consumption and a habit level. Thus, it is the habit function that plays a central role in such a theory. The question is addressed to maximise the utility

$$U = E \sum_{t=0}^{\infty} \delta^t \frac{(C_t - X_t)^{\gamma-1} - 1}{\gamma - 1},$$

where δ is the time preference factor, X_t is the level of habit such that $0 \leq X_t \leq C_t$.

However, the theory does not provide precise guidelines about the functional form of the habit. In the literature such as Chen and Ludvigson (2009), Campbell and Cochrane (1999, 2000) and Constantinides (1990), the habit function is formulated as a function of past and contemporaneous consumption levels, viz., $X_t = f(C_t, C_{t-1}, \cdots, C_{t-L})$. Moreover, to tackle the nonstationarity of the data, researchers presume that the habit function is homogeneous of order one which allows to rephrase the function as

$$X_t = C_t g\left(\frac{C_{t-1}}{C_t}, \cdots, \frac{C_{t-L}}{C_t}\right).$$

Definitely, such a pre-assumption restricts the applicabilities of the theory and real-world data sets. We can do better by relaxing such a particular form as a general form and rewrite $X_t = f(C_t, C_{t-1}, \cdots, C_{t-L}) = g(\triangle C_{t-1}, \cdots, \triangle C_{t-L+1}, C_{t-L})$ where $\triangle$ is the forward difference operation. Noting that the difference sequence of consumption is i.i.d., we then expand the function g and estimate the coefficients and hence obtain the habit function.

Furthermore, in economics there are a great deal of models with conditional moment restrictions containing unknown functionals in nonstationary processes. See Ai and Chen (2003, 2007). Such models take a general form as

$$E(\rho(Z, g)|W, g) = 0,$$

where (Z^T, W^T) is a vector of observable random variables, and W may or may not be included in Z. Here ρ is a one-dimensional residual function known up to g. The conditional expectation is taken with respect to conditional distribution Z given W and g, assumed unknown. The parameter of interest is g which is infinite dimensional. Interestingly, this model covers several commonly encountered nonparametric and semiparametric models:

Regular nonparametric regression

$$Y = g(W) + \varepsilon,$$

given $E(\varepsilon|W) = 0$. Let $Z = (Y, W)$. With $\rho(Z, g) = Y - g(W)$ it becomes the parent model.

Single index model

$$Y = h(W^T\theta) + \varepsilon,$$

given $E(\varepsilon|W) = 0$. The parameter of interest is (h, θ^T) where h is nonparametric. Let $Z = (Y, W)$, $g = (h, \theta^T)$ and $\rho(Z, g) = Y - h(W^T\theta)$. It returns to the parent model.

Nonparametric IV regression

$$Y = g(X) + \varepsilon,$$

where X is an endogenous regressor such that $E(\varepsilon|X) \neq 0$. Suppose there is an instrumental variable W observable for which $E(\varepsilon|W) = 0$. Define $Z = (Y, X)$ and $\rho(Z, g) = Y - g(X)$. It is rewritten as the parent model.

Nonparametric quantile IV regression

$$Y = g(X) + \varepsilon, \quad P(\varepsilon \leq 0|W) = \gamma,$$

where the unknown function g is of interest and $\gamma \in (0, 1)$ is known and fixed. With $\rho(Z, g) = I(Y \leq g(X)) - \gamma$ where $Z = (Y, X)$, we have the parent model.

However, in the literature researchers assumed that the observations of (Z, W) are identical and independent distributed data (Z_i, W_i) $(i = 1, \cdots, n)$. See, for example, Liao and Jiang (2011).

In finance, more often than not, the derivative pricing problem is associated with a functional, very popular nowadays, in a general Lévy process rather than only a Brownian motion. It can be expected that our expansion method is applicable in complete financial markets to deal with the perfect hedging problems and in incomplete financial markets to tackle the mean-variance hedging problems.

Note that two significant features of our proposed method are dealing with nonstationary data sets and unknown functional forms. The orthogonal expansion of the unknown functionals involving in the above models would be a sharper weapon to obtain the estimations of parameters and unknown function of interest.

Future work

There remain some researches about this topic to do in the future. We would study the expansion of Lévy process functionals for more general forms. In particular, the functional including time variable and two independent different Lévy processes is worthy to be studied. In addition, more classes of functionals need to be investigated in asymptotic theory that will widen our scope of applicability of proposed method and theory. Moreover, there is an urgent need to do some simulations on the convergence of the orthogonal expansions of Lévy process functionals and the estimators of unknown functionals in econometric models.

As an application of the proposed method and theory, the aforementioned problems in economics and finance would be investigated in next step, which may resolve the long-standing theoretical issues. All application studies will provide evidence of the necessity of proposing the method and theory in this thesis. Hopefully, reasonable and sound results can be achieved.

Appendix A

Miscellaneous

This chapter reports an alternative method for the expansion of the Brownian motion functional, particularly, the quadratic Brownian motion. More precisely, the proposed method adopts stochastic integral to construct a complete orthogonal basis for a complete subspace of $L^2(\Omega)$, then any element in the subspace including the quadratic Brownian motion can be expanded as a Fourier series.

A.1 Background and motivation

Recall that there are several existing expansions of Brownian motion in literature exploiting different choices of bases in L^2 space. Orthogonal decomposition of Brownian motion gives a convenient way to simulate Brownian motion, and theoretically it is helpful to understand what the Brownian motion is.

In Yeh (1973), noting that there is a complete orthonormal system $\{\phi_n(t)\}$, $n = 0, 1, \ldots$, in $L^2[0, \pi]$, where

$$\phi_0(t) = \frac{1}{\sqrt{\pi}}, \quad \text{and} \quad \phi_n(t) = \sqrt{\frac{2}{\pi}}\cos(nt), \quad n = 1, 2, \ldots, \quad t \in [0, \pi], \tag{A.1.1}$$

and using the transformation $I(f) = \int_0^\pi f(x)\mathrm{d}B(x)$, $f \in L^2[0, \pi]$, this basis $\{\phi_n(t)\}$ is mapped from $L^2[0, \pi]$ into a complete subspace of $L^2(\Omega)$ and the image forms a complete orthonormal system in the subspace. Denote by $\{Z_n, n = 0, 1, \ldots\}$ the image of $\{\phi_n\}$, $n = 0, 1, \ldots$, hence $Z_n = \int_0^\pi \phi_n(x)\mathrm{d}B(x)$. Then for every n, Z_n follows normal distribution $N(0, 1)$ and they are independent each other. Based on this orthonormal system, Brownian motion is expanded as

$$B(t) = \frac{t}{\sqrt{\pi}} Z_0 + \sum_{n=1}^{\infty} \sqrt{\frac{2}{\pi}} \frac{\sin(nt)}{n} Z_n, \qquad t \in [0, \pi]. \tag{A.1.2}$$

In Mikosch (1998), there are another two versions of Brownian motion expansions. One is the Paley-Wiener representation

$$B(t, \omega) = Z_0(\omega) \frac{t}{\sqrt{2\pi}} + \frac{2}{\sqrt{\pi}} \sum_{n=1}^{\infty} Z_n(\omega) \frac{\sin(nt/2)}{n}, \quad t \in [0, 2\pi], \tag{A.1.3}$$

where Z_n are i.i.d. $N(0, 1)$ random variables.

Another one, more generally, is the Levy-Ciesielski representation

$$B(t, \omega) = \sum_{n=1}^{\infty} Z_n(\omega) \int_0^t \phi_n(x) dx, \quad t \in [0.1], \tag{A.1.4}$$

where Z_n are i.i.d $N(0, 1)$ random variables and (ϕ_n) is a complete orthogonal system on $[0, 1]$.

However, to the best of my knowledge, there is no existing expansion for Brownian motion functionals into orthogonal series. Note that such expansions potentially have a variety application in solving stochastic differential equations and in modeling two random variables with unknown relationships. Nevertheless, the difficulty of decomposing function $f(B(t))$ into orthogonal series is overwhelming because of the arbitrariness of $f(\cdot)$. This chapter dwells on a simple function $f(x) = x^2$, and it can be seen that this method is applicable for any power function $f(x) = x^n$.

Suppose that Brownian motion is defined on probability space $(\Omega, \mathcal{F}, P)$ and interval $[0, \infty)$. To begin with, we introduce the following definition of a kind of stochastic integral.

Definition A.1.1. Suppose $B(t)$ is a standard Brownian motion on $(\Omega, \mathcal{F}, P)$. Let $F(t, \cdot)$ be a function defined on $[a, b] \times \Omega$ satisfying integrability conditions

- $F(t, B(t))$ is $\mathfrak{B} \times \mathcal{F}$-measurable, where $\mathfrak{B}$ denotes the Borel σ-field on $[a, b]$;
- $F(t, \cdot)$ is adapted with natural filtration $\mathcal{F}_t = \sigma(B(t))$ generated by Brownian motion;
- $$\int_a^b E[F^2(t, B(t))] dt = \int_a^b \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi x}} e^{-\frac{y^2}{2t}} F^2(t, y) dy dx < \infty.$$

Suppose that an arbitrary partition is described by a sequence of points on $[a, b]$, $\tau_n[a, b]$: $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$. Denote by $\Delta t_i = t_i - t_{i-1}$ and $\Delta t = \max_i\{\Delta x_i\}$. Define a stochastic integral

$$I(F) := \int_a^b F(t, B(t))dB(t), \tag{A.1.5}$$

as a mean square limit of the following summation

$$\sum_{i=1}^{n} F(t_{i-1}, B(t_{i-1}))(B(t_i) - B(x_{i-1})).$$

Remark A.1.1. The integrability conditions are normal requirements in usual text book about Itô integral. In addition, this definition is almost the same as that in general stochastic integral text book (Mikosch, 1998); the only difference is that here the integrand stochastic process is of the particular form $F(t, B(t))$. This is only for the sake of later use in this study. Furthermore, it can be seen that this integral of F with respect to Brownian motion is actually a mean square limit of stochastic integrals of a sequence of simple processes.

A.2 Expansion using stochastic integrals

In what follows, our aim is finding a transformation between $L^2[a, b]$ and $L^2(\Omega)$ and then studying expansion of $B^2(t)$, a particular functional of Brownian motion. To this end, we will focus on the case of $F(t, B(t)) = \frac{1}{\sqrt{t}} f(t)B(t)$ and introduce the subsequent definition.

Definition A.2.1. Suppose $f(t) \in L^2[a, b]$ and $a > 0$. Define a transformation $\mathscr{T}$ between $L^2[a, b]$ and $L^2(\Omega)$ as

$$\mathscr{T}(f; [a, b]) = \int_a^b \frac{f(t)}{\sqrt{t}} B(t)dB(t). \tag{A.2.1}$$

Remark A.2.1. When it is necessary to stress the interval $[a, b]$ we put it into the notation, but mostly we use the notation without interval. It is crucial to ensure that the transformation we defined exists for every $f \in L^2[a, b]$. We can verify this fact by checking whether it satisfies the three integrability conditions in Definition A.1.1. Obviously we only need to check the last one:

$$\int_a^b E\left(\frac{f(t)}{\sqrt{t}} B(t)\right)^2 dt = \int_a^b \frac{f^2(t)}{t} E[B(t)^2]dt = \int_a^b f^2(t)dt < \infty.$$

Therefore the transformation $\mathscr{T}(f;[a,b])$ is a mapping from $L^2[a,b]$ into random variables. It can be seen that $\mathscr{T}(f) \in L^2(\Omega)$ from the following theorem.

Theorem A.2.1. *Suppose $f(x), g(x) \in L^2[a,b]$ and $a > 0$. The transformation in Definition* (A.2.1) *satisfies*

(a) $E[\mathscr{T}(f)] = 0$;

(b) $\langle \mathscr{T}(f), \mathscr{T}(g) \rangle = (f,g)$;

(c) $\|\mathscr{T}(f)\|^2_{L^2(\Omega)} = \|f\|_2^2$. *hence* $\mathscr{T}(f) \in L^2(\Omega)$;

(d) $\mathscr{T}(c_1 f + c_2 g)(\omega) = c_1 \mathscr{T}(f)(\omega) + c_2 \mathscr{T}(g)(\omega)$ *for every* $\omega \in \Omega$, *where* c_1, c_2 *are real constants.*

Proof. Denote by $\tau_n[a,b]$ an arbitrary partition on interval $[a,b]$: $a = t_0 < t_1 < \cdots < t_n = b$, where n is any positive integer number. Let $\triangle t = \max_i\{t_i - t_{i-1}\}$. For $f \in L^2[a,b]$, denote by $S^{\tau_n}(f)$ the sum corresponding to the partition $\tau_n[a,b]$

$$S^{\tau_n}(f) := \sum_{i=1}^{n} \frac{f(t_{i-1})}{\sqrt{t_{i-1}}} B(t_{i-1})(B(x_i) - B(t_{i-1})).$$

By the independence of increments of Brownian motion and its distribution, it follows easily that

$$E[S^{\tau_n}(f)] = \sum_{i=1}^{n} \frac{f(t_{i-1})}{\sqrt{t_{i-1}}} E[B(t_{i-1})]E[B(t_i) - B(t_{i-1})] = 0.$$

(a). Since $E[S^{\tau_n}(f)] = 0$, by Jensen inequality we have

$$\begin{aligned}\{E[\mathscr{T}(f)]\}^2 &= \{E[\mathscr{T}(f)] - E[S^{\tau_n}(f)]\}^2 = \{E[\mathscr{T}(f) - S^{\tau_n}(f)]\}^2 \\ &\leq E\left[\mathscr{T}(f) - S^{\tau_n}(f)\right]^2 \to 0, \quad \text{as } n \to \infty,\end{aligned}$$

which implies $E[\mathscr{T}(f)] = 0$.

(b). Since $\mathscr{T}(f)$ and $\mathscr{T}(g)$ are mean square limit of sequence $S^{\tau_n}(f)$ and $S^{\tau_n}(g)$ respectively, it follows from continuity of inner product in Hilbert space $L^2(\Omega)$ that

$$\begin{aligned}&\langle \mathscr{T}(f), \mathscr{T}(g) \rangle = \left\langle \lim_{\Delta t \to 0} S^{\tau_n}(f), \lim_{\Delta t \to 0} S^{\tau_n}(g) \right\rangle \\ &= \lim_{\triangle t \to 0} \langle S^{\tau_n}(f), S^{\tau_n}(g) \rangle = \lim_{\triangle t \to 0} E[S^{\tau_n}(f) S^{\tau_n}(g)]\end{aligned}$$

$$= \lim_{\Delta t \to 0} E\left(\sum_{i=1}^{n} \frac{f(t_{i-1})}{\sqrt{t_{i-1}}} B(t_{i-1})(B(t_i) - B(t_{i-1})) \cdot \sum_{i=1}^{n} \frac{g(t_{i-1})}{\sqrt{t_{i-1}}} B(t_{i-1})(B(t_i) - B(t_{i-1}))\right)$$

$$= \lim_{\Delta t \to 0} E\left(\sum_{i=1,j=1}^{n} \frac{f(t_{i-1})}{\sqrt{t_{i-1}}} B(t_{i-1})(B(t_i) - B(t_{i-1})) \frac{g(t_{j-1})}{\sqrt{t_{j-1}}} B(t_{j-1})(B(t_j) - B(t_{j-1}))\right)$$

$$= \lim_{\Delta t \to 0} \sum_{i=1}^{n} \frac{f(t_{i-1})g(t_{i-1})}{t_{i-1}} E[B^2(t_{i-1})] E[(B(t_i) - B(t_{i-1}))^2]$$

$$= \lim_{\Delta t \to 0} \sum_{i=1}^{n} f(t_{i-1})g(t_{i-1})(t_i - t_{i-1}) = \int_a^b f(x)g(x)dx = (f, g).$$

(c). This is a particular case of property (b) when $f = g$.

(d). For arbitrary constants c_1, c_2,

$$\begin{aligned}
\mathscr{T}(c_1 f + c_2 g) &= \lim_{\triangle t \to 0} S^{\tau_n}(c_1 f + c_2 g) \\
&= \lim_{\triangle t \to 0} \sum_{i=1}^{n} \frac{c_1 f(t_{i-1}) + c_2 g(t_{i-1})}{\sqrt{t_{i-1}}} B(t_{i-1})(B(t_i) - B(t_{i-1})) \\
&= \lim_{\Delta t \to 0} \sum_{i=1}^{n} \frac{c_1 f(t_{i-1})}{\sqrt{t_{i-1}}} B(t_{i-1})(B(t_i) - B(t_{i-1})) \\
&\quad + \lim_{\Delta t \to 0} \sum_{i=1}^{n} \frac{c_2 g(t_{i-1})}{\sqrt{t_{i-1}}} B(t_{i-1})(B(t_i) - B(t_{i-1})) \\
&= c_1 \lim_{\Delta t \to 0} \sum_{i=1}^{n} \frac{f(t_{i-1})}{\sqrt{t_{i-1}}} B(t_{i-1})(B(t_i) - B(t_{i-1})) \\
&\quad + c_2 \lim_{\Delta t \to 0} \sum_{i=1}^{n} \frac{g(t_{i-1})}{\sqrt{t_{i-1}}} B(t_{i-1})(B(t_i) - B(t_{i-1})) \\
&= c_1 \int_a^b \frac{f(t)}{\sqrt{t}} B(t)dB(t) + c_2 \int_a^b \frac{g(t)}{\sqrt{t}} B(t)dB(t) \\
&= c_1 \mathscr{T}(f) + c_2 \mathscr{T}(g).
\end{aligned}$$

□

Apart from the properties listed in the previous theorem, $\mathscr{T}(f)$ possesses other characteristics of the usual integral, for example, linearity on adjacent intervals and continuity about lower limit a and upper limit b. Linearity on adjacent intervals says that if $a < c < b$, $\mathscr{T}(f; [a, b]) = \mathscr{T}(f; [a, c]) + \mathscr{T}(f; [c, b])$; while continuity about a means that if $a' \to a$ the integral $\mathscr{T}(f; [a', b])$ will converge to the integral $\mathscr{T}(f; [a, b])$ in mean square sense.

Now concentration moves to the interval $[0, T], T > 0$ and fixed. The main reason why we shift to this case is simply that customarily the standard Brownian motion starts at point zero. Moreover this relaxes the restriction that $a > 0$. However, the movement is not trivial. Since $1/\sqrt{x}$ is undefined at the point zero, the mapping $\mathscr{T}$ cannot be used on interval $[0, T]$ directly. It is therefore reasonable to apply improper integration to it. The transformation $\mathscr{T}$ from $L^2[0, T]$ to $L^2(\Omega)$ is defined as

$$\mathscr{T}(f; [0, T]) = \lim_{\varepsilon \to +0} \mathscr{T}(f; [\varepsilon, T]) = \lim_{\varepsilon \to +0} \int_{\varepsilon}^{T} \frac{f(t)}{\sqrt{t}} B(t) dB(t) \quad \text{(in norm)}. \tag{A.2.2}$$

It raises the question of whether the improper integrations exist for the functions we are studying, i.e. for $h(t) \in L^2[0, T]$, does $\mathscr{T}(h; [0, T])$ exist? To answer this question, let us define

$$\mathscr{T}_{\varepsilon}(h) = \mathscr{T}(h; [\varepsilon, T]) = \int_{\varepsilon}^{T} \frac{h(t)}{\sqrt{t}} B(t) dB(t).$$

As has been shown, $\mathscr{T}_{\varepsilon}(h)$ is always well defined for $0 < \varepsilon < T$, and $\mathscr{T}_{\varepsilon}(h) \in L^2(\Omega)$.

Lemma A.2.1. *For $\forall h(t) \in L^2[0, T]$, the improper integral $\mathscr{T}(h; [0, T])$ exists.*

Proof. To begin with, suppose $\{\delta_n\}$ is a positive sequence which converges to zero. Consequently there is a sequence $\{\mathscr{T}_{\delta_n}(h)\}$ in $L^2(\Omega)$ for $h \in L^2[0, T]$. The aim is to show that $\{\mathscr{T}_{\delta_n}(h)\}$ is a Cauchy sequence. Actually,

$$\begin{aligned}
\|\mathscr{T}_{\delta_n}(h) - \mathscr{T}_{\delta_m}(h)\|^2_{L^2(\Omega)} &= \left\| \int_{\delta_n}^{T} \frac{h(t)}{\sqrt{t}} B(t) dB(t) - \int_{\delta_m}^{T} \frac{h(t)}{\sqrt{t}} B(t) dB(t) \right\|^2 \\
&= \left\| \int_{\delta_n \wedge \delta_m}^{\delta_n \vee \delta_m} \frac{h(t)}{\sqrt{t}} B(t) dB(t) \right\|^2 \\
&= \int_{\delta_n \wedge \delta_m}^{\delta_n \vee \delta_m} h^2(t) dt \to 0, \quad (\text{as } n, m \to \infty)
\end{aligned}$$

which implies the sequence $\mathscr{T}_{\delta_n}(h)$ is a Cauchy sequence in $L^2(\Omega)$.

Since $L^2(\Omega)$ is a Hilbert space, every Cauchy sequence has a limit in the space. Suppose $\lim_{n\to\infty} \mathscr{T}_{\delta_n}(h) = \xi$ in L^2-norm for the function h and the sequence $\{\delta_n\}$, where ξ is a random variable in $L^2(\Omega)$. The problem is, for any other positive sequence, $\{\epsilon_n\}$ say, which converges to zero, whether the corresponding sequence $\mathscr{T}_{\epsilon_n}(h)$ converges in norm to another random variable η? In fact,

$$\|\xi - \eta\|_{L^2(\Omega)} \leq \|\xi - \mathscr{T}_{\delta_n}(h)\| + \|\mathscr{T}_{\delta_n}(h) - \mathscr{T}_{\epsilon_n}(h)\| + \|\mathscr{T}_{\epsilon_n}(h) - \eta\|$$

$$
\begin{aligned}
&= \|\xi - \mathscr{T}_{\delta_n}(h)\| + \int_{\delta_n \wedge \epsilon_n}^{\delta_n \vee \epsilon_n} h^2(t)dt + \|\mathscr{T}_{\epsilon_n}(h) - \eta\| \\
&\to 0, \quad (\text{as } n \to \infty)
\end{aligned}
$$

which implies $\xi = \eta$ in $L^2(\Omega)$. Now that for every positive sequence δ_n which converges to zero, the integral sequence $\mathscr{T}_{\delta_n}(h)$ converges to the same random variable, ξ say, the limit of $\mathscr{T}_\varepsilon(h)$ as $\varepsilon \to 0$ also should be ξ. Actually it follows from following inequality,

$$
\begin{aligned}
\|I_\varepsilon(h) - \xi\|_{L^2(\Omega)} \leq& \|I_\varepsilon(h) - I_{\delta_n}(h)\| + \|I_{\delta_n}(h) - \xi\| \\
=& \int_{\delta_n \wedge \varepsilon}^{\delta_n \vee \varepsilon} h^2(t)dt + \|I_{\delta_n}(h) - \xi\| \to 0,
\end{aligned}
$$

as $\varepsilon \to 0, n \to \infty$. This finishes the proof. □

Theorem A.2.2. *For any $f, g \in L^2[0,T]$ and constants c_1, c_2, the following hold:*

(a) $E[\mathscr{T}(f;[0,T])] = 0$*;*

(b) $\langle \mathscr{T}(f;[0,T]), \mathscr{T}(g;[0,T]) \rangle = (f,g)$*;*

(c) $\|\mathscr{T}(f)\|_2^2(\Omega) = \|f\|_2^2$*;*

(d) $\mathscr{T}(c_1 f + c_2 g;[0,T])(\omega) = c_1 \mathscr{T}(f;[0,T])(\omega) + c_2 \mathscr{T}(g;[0,T])(\omega)$ *for every* $\omega \in \Omega$*.*

Proof. For $\forall \varepsilon : 0 < \varepsilon < T$, $\mathscr{T}_\varepsilon(\cdot)$ is defined as before, and $\mathscr{T}(\cdot)$ is the mean square limit of $\mathscr{T}_\varepsilon(\cdot)$ as $\varepsilon \to 0$.

(a). It follows from $E[\mathscr{T}_\varepsilon(f)] = 0$ and Jensen inequality that

$$
\begin{aligned}
\{E[\mathscr{T}(f)]\}^2 &= \{E[\mathscr{T}(f)] - E[\mathscr{T}_\varepsilon(f)]\}^2 = \{E[\mathscr{T}(f) - \mathscr{T}_\varepsilon(f)]\}^2 \\
&\leq E[\mathscr{T}(f) - \mathscr{T}_\varepsilon(f)]^2 \to 0 \quad \text{as } \varepsilon \to +0,
\end{aligned}
$$

hence $E[I(f)] = 0$.

(b). Using the definition of transformation $\mathscr{T}(\cdot)$ and the continuity of inner product in Hilbert space, we have

$$
\begin{aligned}
\langle \mathscr{T}(f), \mathscr{T}(g) \rangle &= \left\langle \lim_{\varepsilon \to +0} \mathscr{T}_\varepsilon(f), \lim_{\varepsilon \to +0} \mathscr{T}_\varepsilon(g) \right\rangle = \lim_{\varepsilon \to +0} \langle \mathscr{T}_\varepsilon(f), \mathscr{T}_\varepsilon(g) \rangle \\
&= \lim_{\varepsilon \to +0} \int_\varepsilon^T f(t)g(t)dt = \int_0^T f(t)g(t)dt = (f,g),
\end{aligned}
$$

where the property (b) in theorem A.2.1 is used to derive $\langle \mathscr{T}_\varepsilon(f), \mathscr{T}_\varepsilon(g) \rangle$.

(c). This equality is the particular case of (b).

(d). This property can be proved similarly as the counterpart in theorem A.2.1 except that here it is needed to take limit for $\varepsilon \to +0$. Therefore the proof is neglected. $\square$

In order to continue the investigation on the transformation $\mathscr{T}$, denote by Θ the image of $L^2[0,T]$ under the mapping. Then the properties of $\mathscr{T}$ mapping $L^2[0,T]$ onto Θ are studied in the sequel.

Theorem A.2.3. *The stochastic integral $\mathscr{T}$ of* (A.2.2) *defines an one-to-one mapping from $L^2[0,T]$ to Θ. The image Θ of $L^2[0,T]$ under the mapping $\mathscr{T}$ is a closed linear subspace of $L^2(\Omega)$. Furthermore, the transformation $\mathscr{T}$ is an isomorphism as well between $L^2[0,T]$ and Hilbert space Θ.*

Proof. To begin with, if $f, g \in L^2[0,T]$ and $f \neq g$, then $\|f-g\|_2 \neq 0$. This implies $\mathscr{T}(f-g) = \mathscr{T}(f) - \mathscr{T}(g) \neq 0$ since $\mathscr{T}(f-g)$ is a random variable with zero mean and variance $\|f-g\|_2$, so that $\mathscr{T}(f) \neq \mathscr{T}(g)$. On the other hand, if $\xi \in \Theta$ and $\mathscr{T}(f) = \xi$ and $\mathscr{T}(g) = \xi$ for $f, g \in L^2[0,T]$. Because $\mathscr{T}(f-g) = \mathscr{T}(f) - \mathscr{T}(g) = \xi - \xi = 0$, the variance of $\mathscr{T}(f-g)$ is definitely zero, i.e. $\|f-g\|_2 = 0$ and this means $f = g$. Therefore $\mathscr{T}$ is one-to-one.

It is evident that Θ is a linear subspace of $L^2(\Omega)$ as the transformation $\mathscr{T}$ is linear. To show Θ is closed, suppose that $\{\xi_n\}$ is a sequence in Θ and ξ is an element in $L^2(\Omega)$ such that $\|\xi_n - \xi\| \to 0$. As $\mathscr{T}$ is one-to-one there is a sequence $\{f_n\} \in L^2[0,T]$ such that $\mathscr{T}(f_n) = \xi_n$. Moreover, $\mathscr{T}$ preserves metric, which indicates $\|f_n\|$ is a Cauchy sequence since $\|f_n - f_m\|_2 = \|\mathscr{T}(f_n - f_m)\|_{L^2(\Omega)} = \|\mathscr{T}(f_n) - \mathscr{T}(f_m)\| = \|\xi_n - \xi_m\|_{L^2(\Omega)}$ and $\{\xi_n\}$ is of Cauchy. Thus, there exists a function $f \in L^2[0,T]$ such that $\|f_n - f\|_2 \to 0$. Once again by the property of the mapping we have $\|\xi_n - \mathscr{T}(f)\|_2(\Omega) = \|\mathscr{T}(f_n) - \mathscr{T}(f)\|_{L^2(\Omega)} = \|f_n - f\|_2 \to 0$. Then the inequality

$$\|\xi - \mathscr{T}(f)\|_{L^2(\Omega)} \leq \|\xi_n - \xi\|_{L^2(\Omega)} + \|\xi_n - \mathscr{T}(f)\|_{L^2(\Omega)} \to 0,$$

as $n \to \infty$ implies $\xi = \mathscr{T}(f) \in \Theta$. This proves that Θ is closed.

Finally, as a closed linear subspace of Hilbert space $L^2(\Omega)$, Θ is Hilbert space as well. It follows from (b) and (d) in Theorem A.2.2 that $\mathscr{T}$ is isomorphism. $\square$

Corollary A.2.1. *If $\{f_n\}$ is a full orthonormal system in $L^2[0,T]$, then $\{\mathscr{T}(f_n)\}$ is a full orthonormal system in Θ. The inverse is also true.*

Proof. What needs to prove is merely that when $\{f_n\}$ is full in $L^2[0,T]$, $\{\mathscr{T}(f_n)\}$ is complete in $L^2(\Omega)$. Denote $\xi_n = \mathscr{T}(f_n), n = 1, \cdots, \infty$. For any $\xi \in \Theta$, there is one and only one $f \in L^2[0,T]$ such that $\xi = \mathscr{T}(f)$. This f can be uniquely represented as $f = \sum_{n=1}^{\infty} c_n f_n$, so that $\xi = \mathscr{T}(f) = \sum_{n=1}^{\infty} c_n \mathscr{T}(f_n) = \sum_{n=1}^{\infty} c_n \xi_n$ and the representation is unique. Thus, $\{\xi_n\}$ is a complete orthogonal system.

The inverse is true as $\mathscr{T}$ is an one-to-one mapping. □

Theorem A.2.4. *Let $\mathscr{T}$ be a transformation from $L^2[0,T]$ into $L^2(\Omega)$ defined by equation* (A.2.2) *and $\{f_n\}$ be an orthonormal system in $L^2[0,T]$. Let $\xi_n = \mathscr{T}(f_n), n = 1, 2, \cdots$. For any $\xi \in \Theta$ with $\xi = \mathscr{T}(f)$, we have*

$$\xi = \sum_{n=1}^{\infty} \langle \xi, \xi_n \rangle \xi_n = \sum_{n=1}^{n} (f, f_n) \xi_n,$$

where the convergence of the infinite series is in the sense of norm in $L^2(\Omega)$. Furthermore,

$$\xi(\omega) = \sum_{n=1}^{\infty} \langle \xi, \xi_n \rangle \xi_n(\omega) = \sum_{n=1}^{n} (f, f_n) \xi_n(\omega), \qquad \textit{in probability}$$

Proof. According to the theory of Fourier expansion, the representation is evident and the coefficients $\langle \xi, \xi_n \rangle$ are called Fourier coefficients. Since $\mathscr{T}$ is isomorphism, $\langle \xi, \xi_n \rangle = (f, f_n)$.

Because convergence in norm implies convergence in probability, the second expression is valid. □

Theorem A.2.5. *Suppose $B(t)$ is standard Brownian motion on $[0,\infty)$ and $\{f_n\}$ is an arbitrary full orthonormal system in $L^2[0,T]$ where $T > 0$ is a finite real number. The stochastic process $B^2(t)$ for $t \in [0,T]$ can be expanded as*

$$B^2(t) = t + 2 \sum_{n=1}^{\infty} (\sqrt{t} \mathcal{X}_{[0,t]}, f_n) \xi_n, \tag{A.2.3}$$

where $\mathcal{X}_{[0,t]}$ is the indicator function on $[0,t]$, and $\xi_n = \mathscr{T}(f_n)$.

Proof. This is a particular case of Theorem A.2.4. For any $0 < t \le T$, let $f(s) = \sqrt{s} \mathcal{X}_{[0,t]}(s)$. Then

$$\mathscr{T}(f) = \int_0^t B(s) dB(s) = \frac{1}{2}(B_t^2 - t).$$

On the other hand, $\mathscr{T}(f) = \sum_{n=1}^{\infty} (f, f_n) \xi_n = \sum_{n=1}^{\infty} (\sqrt{s} \mathcal{X}_{[0,t]}, f_n) \xi_n$. Thus the expansion follows. □

Example A.1 Let $T = \pi$. On $[0, \pi]$, a full orthonormal system is $\{f_n\}$ where

$$f_0(x) = \frac{1}{\sqrt{\pi}}, \quad \text{and} \quad f_n(x) = \sqrt{\frac{2}{\pi}} \cos(nx), \qquad n = 1, 2, \cdots .$$

To get the decomposition of $B^2(t)$, $(0 < t \leq \pi)$, let $f(s) = \sqrt{s}\mathcal{X}_{[0,t]}(s)$ for $0 < t \leq \pi$. Compute

$$c_0 = c_0(t) = (f, f_0) = \int_0^\pi f(s) f_0(s) ds = \int_0^t \sqrt{s} \frac{1}{\sqrt{\pi}} ds = \frac{2}{3\sqrt{\pi}} t^{3/2}$$
$$c_n = c_n(t) = (f, f_n) = \int_0^\pi f(s) f_n(s) dx = \sqrt{\frac{2}{\pi}} \int_0^t \sqrt{s} \cos(ns) ds.$$

The corresponding orthonormal system in $L^2(\Omega)$ is

$$\xi_0 = \mathscr{T}(f_0) = \int_0^\pi \frac{1}{\sqrt{\pi s}} B(s) dB(s),$$
$$\xi_n = \mathscr{T}(f_n) = \sqrt{\frac{2}{\pi}} \int_0^\pi \frac{\cos(ns)}{\sqrt{s}} B(s) dB(s).$$

Thus

$$B^2(t) = t + 2 \sum_{n=0}^\infty c_n(t) \xi_n.$$

Example A.2 In $L^2[0, T]$ there is an orthonormal system consisting of

$$f_n(t) = \sqrt{\frac{2}{T}} \sin\left(\left(n + \frac{1}{2}\right) \frac{\pi}{T} t\right), \quad t \in [0, T], \quad n = 0, 1, 2, \cdots$$

Again, to expand $B^2(t)$ $(0 < t \leq T)$, let $f(s) = \sqrt{s}\mathcal{X}_{[0,t]}(s)$, where $0 < t \leq T$.

$$c_0 = c_0(t) = (f, f_0) = \int_0^T f(s) f_0(s) ds = \sqrt{\frac{2}{T}} \int_0^t \sqrt{s} \sin\left(\frac{\pi}{2T} s\right) ds$$
$$c_n = c_n(t) = (f, f_n) = \int_0^T f(s) f_n(s) dx = \sqrt{\frac{2}{T}} \int_0^t \sqrt{s} \sin\left(\left(n + \frac{1}{2}\right) \frac{\pi}{T} s\right) ds.$$

The corresponding orthonormal system in $L^2(\Omega)$ is

$$\xi_0 = \mathscr{T}(f_0) = \sqrt{\frac{2}{T}} \int_0^T \frac{1}{\sqrt{s}} \sin\left(\frac{\pi}{2T} s\right) B(s) dB(s),$$
$$\xi_n = \mathscr{T}(f_n) = \sqrt{\frac{2}{T}} \int_0^T \frac{1}{\sqrt{s}} \sin\left(\left(n + \frac{1}{2}\right) \frac{\pi}{T} s\right) B(s) dB(s).$$

Thus

$$B^2(t) = t + 2 \sum_{n=0}^\infty c_n(t) \xi_n \tag{A.2.4}$$

Remark A.2.2. All studies on interval $[0, T]$ in this part can revert to $[a, b]$. This means all results about interval $[0, T]$ are valid for $[a, b]$,$a > 0$, under mapping $\mathscr{T}(f; [a, b])$ from $L^2[a, b]$ into $L^2(\Omega)$. The only change is that Brownian motion starts at a and almost surely is zero at point a.

Additionally, the drawback of the method, as can be seen in the examples, is that the coefficients in the expansions cannot be calculated precisely and the basis ξ_n is only phrased as stochastic integrals. Therefore, in practice in order to utilise such an expansion, one has to make use of a computer for its powerful computation ability.

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