Expansion of Lévy Process Functionals and Its Application in Econometric Estimation

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Abstract

This research focuses on the estimation of a class of econometric models for involved unknown nonlinear functionals of nonstationary processes. The proxy of nonstationary processes studied here is Lévy processes including Brownian motion as a particular one. A Lévy process is a càdlàg[1] stochastic process which starts at zero almost surely, which has independent increments over disjoint intervals, which has stationary increment distribution meaning that under shift the distributions of increments are identical, which has stochastic continuous trajectory. Obviously, Brownian motion, Poisson process, Gamma process and Pascal process are fundamental examples of Lévy processes. Lévy processes $(Z(t), t \geq 0)$ studied in this thesis possess density or probability distribution functions which verify some properties stated in the text.

Why do we care about the functionals of Lévy processes?

Starting with Brownian motion

In the galaxy of stochastic processes used to model random phenomena in disciplines such as economics, finance and engineering, Brownian motion is undoubtedly the brightest star. Brownian motion is the most widely studied stochastic process and the mother of the modern stochastic analysis. Brownian motion, for example, and financial modelling have been tied together from the very beginning when Bachelier (1900) proposed to model the price $S(t)$ of an asset at the Paris Bourse as $S(t) = S(0) + \sigma B(t)$ where $B(t)$ is a standard Brownian motion. The multiplicative version of Bachelier's model led to the celebrated Black-Scholes option pricing model[2] where log-price $\ln S(t)$ follows a Brownian

[1]right continuous with left limits.

[2]The Black-Scholes model is one of the most important concepts in modern financial theory. It was developed in 1973 by Fisher Black, Robert Merton and Myron Scholes and is still widely used today, and

motion $S(t) = S(0)\exp(\mu t + \sigma B(t))$ (see Black and Scholes, 1973).

Of course, the Black-Scholes model is not the only continuous time model built on Brownian motion. Nonlinear Markov diffusion where instantaneous volatility can depend on the price and time via a local volatility function have been proposed by Derman and Kani (1994) and Dupire (1994): $\frac{1}{S(t)}dS(t) = \mu dt + \sigma(t, S(t))dB(t)$. Another possibility is given by stochastic volatility models (see Hull and White, 1987; Heston, 1993) where the price $S(t)$ is the component of a bivariate diffusion $(S(t), \sigma(t))$ driven by a two-dimensional Brownian motion $(B^{(1)}(t), B^{(2)}(t))$: $\frac{1}{S(t)}dS(t) = \mu dt + \sigma(t)dB^{(1)}(t)$, $\sigma(t) = f(Y(t))$, $dY(t) = \alpha(t)dt + \gamma(t)dB^{(2)}(t)$. While these models have more flexible statistical properties, they share with Brownian motion the property of continuity, which does not seem to be shared by the real price over time scales of interest. Assuming that prices move in a continuous manner amounts to neglecting the abrupt movements in which most of the risk is concentrated.

Let us take an example from economics. Let Q denote the customer's total wealth and K the value of their house. The price of housing is constant, and the service flow from a house is equal to its value. For now there is no adjustment cost, so the customer can adjust K continuously and costlessly.

There are two assets, one safe and one risky. Assume that short sales of risky asset are not allowed, and let $A > 0$ be the customer's holding of the risky asset. Then $Q - A$ is the wealth of the safe asset. The mortgage interest rate is the same as the return of the bond, so holdings of the safe asset are the sum of equity in the house and bond holdings.

Let $r > 0$ be the riskless rate of return, let $\mu > r$ and $\sigma^2 > 0$ be the mean return and variance of risky asset, and let $\delta \geq 0$ be the maintenance cost per unit of housing. Then given K and A, the law of motion for total wealth is

$$\begin{aligned} dQ &= [rQ + (\mu - r)A - (r + \delta)K]dt + \sigma A dB \\ &= a(Q, \Theta)dt + b(Q, \Theta)dB \end{aligned}$$

where $\Theta = (\mu, \sigma, r, \delta)$ and B stands for Brownian motion. In the equation, function a is the total return constituting safe assets, risky assets, mortgage payments and maintenance cost, which are considered as a function of the time in question; while function b measures

regarded as one of the best ways of determining fair prices of options. The seminal work brought a Nobel prize in economics for Robert Merton and Myron Scholes in 1997.

the risky return from risky assets due to fluctuation of the stock market. More examples can be found in Stoke (2009).

One thing of note is that, more often than not, the processes depicted by stochastic differential equations involving Brownian motion take the form of the functional of the underlying process $B(t)$ as the solutions of the equations (Mikosch, 1998).

From Brownian motion to the Lévy process

> In the end, a theory is accepted not because it is confirmed by conventional empirical tests, but because researchers persuade one another that the theory is correct and relevant.
>
> Fischer Black (1986)

The Black-Scholes model stipulates that the log returns of an asset in question follow normal distribution. However, as suggested by empirical researches, e.g. Cont (2001) and Schoutens (2003), this assumption is not supported by real-world data. The following table tells that the daily log returns have significant (negative) skewness; the daily log returns have kurtosis bigger than 3; the P-values of the $\hat{\chi}^2$ statistics in the table show that the normal distribution is always rejected. The first dataset (S& P 500 (1970-2001)) contains all daily log returns of the S& P 500 Index over the period 1970-2001. The second dataset (*S&P 500(1970-2001)) contains the same data except for the exceptional log return (-0.2280) of the crash of 19 October 1987. All other datasets are over the period 1997-1999.

Table 1 Skewness, kurtosis and P_{Normal}-value of major indices

Index	Skewness	Kurtosis	P_{Normal}-value
S&P 500(1970-2001)	-1.6663	43.36	0.0000
*S&P 500(1970-2001)	-0.1099	7.17	-
S&P 500(1997-1999)	-0.4409	6.94	0.0421
Nasdaq-Composite	-0.5439	5.78	0.0049
DAX	-0.4314	4.65	0.0366
SMI	-0.3584	5.35	0.0479
CAC-40	-0.2116	4.63	0.0285

Moreover, another failure of the Black-Scholes model is that it does not capture the feature of heavy tail for the distribution of the real-world data sets. Figure 1 compares the five-minute returns on the Yen/Deutschemark (DM) exchange rate to increments of a Brownian motion with the same average volatility. While both return series have the same variance, the Brownian model achieves it by generating returns which always have roughly the same amplitude whereas the Yen/DM returns are widely dispersed in their amplitude and manifest frequent large peaks corresponding to 'jumps' in the price. This

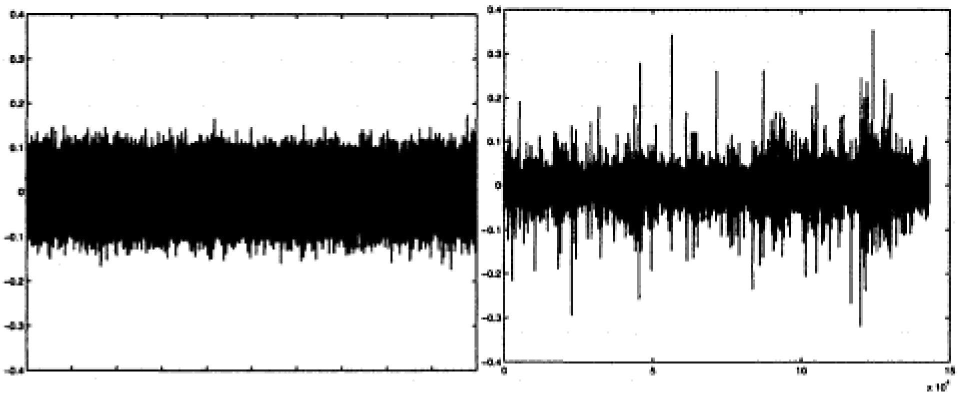

Figure 1: Five-minute log-returns for Yen/DM exchange rate, 1992-1995, compared with log-returns of a Black-Scholes model with the same annualised mean and variance

high variability is a constantly observed feature of financial asset returns. In statistical terms this results in heavy tails in the empirical distribution returns: the tails of the distribution decay slowly at infinity and very large moves have a significant probability of occurring. This well-known fact leads to a poor representation of the distribution of returns by a normal distribution. No book on financial risk is nowadays complete without a reference to the traditional six standard deviation market moves which are commonly observed on all markets, even the largest and the most liquid ones. Since for a normal random variable the probability of occurrence of a value six times the standard deviation is less than 10^{-8}, in a Gaussian model a daily return of such magnitude occurs less than once in a million years! Saying that such a model underestimates risk is a polite understatement. For detailed discussion, see Schoutens (2003, Chapter 4) and Cont and Tankov (2004).

Another observation is that many empirical datasets show non-linearity and non-stationarity. For example, in Gao (2007), there is strong evidence that the short rate is not stationary and normally distributed. The graph in Figure 2 shows the data of three

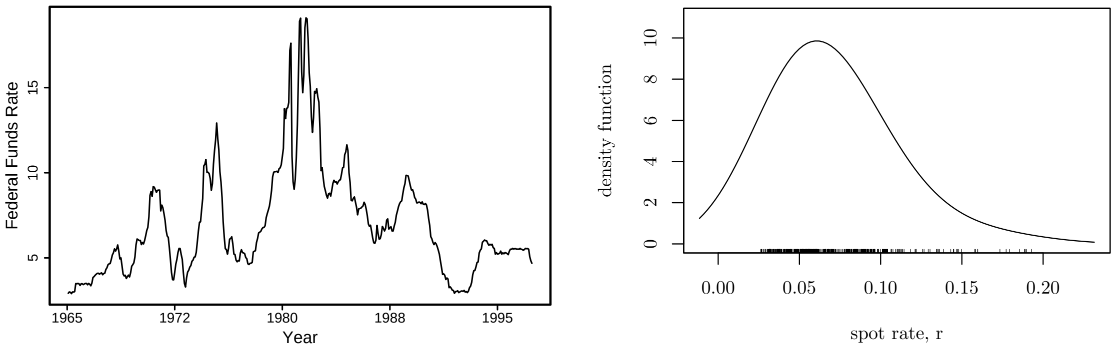

Figure 2: Left: three month Treasury bill rates 1963,1-1998,12; right: the estimated density

month Treasury bill rates between January 1963 and December 1998 (432 observations) and the estimated density function. It is clear that the density function is not normal distributed, and at 1% significance level it is acceptable that the set of data is non-stationary (see Gao et al., 2009).

Thanks to the aforementioned reasons, for a number of years, researchers have focused on developing a richer class of asset price models that include jumps as well as stochastic parameters; see Erakar et al. (2003) and Kou (2002). Meanwhile, several works realise that more sophisticated processes, Lévy processes, are able to represent skewness and excess kurtosis. See, for example, Schoutens (2003, Chapter 5) and Leblane and Yor (1998). In addition, several particular choices for non-Brownian Lévy processes have been proposed in the last few decades. Madan and Seneta (1990) have proposed a Lévy process with variance gamma distributed increments. We mention also the hyperbolic model proposed by Eberlein and Keller (1995), and in the same year the normal inverse Gaussian Lévy process proposed by Barndorff-Nielson (1995). Carr et al. (2000) introduced the CMGY model. Finally, we mention the Meixner model (see Grigelionis 1999 and Schoutens 2001).

Obviously, by Theorem 7 on Protter (2004, p.253), under some conditions, a stochastic differential equation driven by a Lévy process $(Z(t), t \geq 0)$ has a solution $f(Z(t))$. See, for example, Lim (2005) and Brockwell et al. (2007, 2011).

Both time-homogeneous and time-inhomogeneous functionals matter

It then makes sense to consider Lévy process functionals for modelling stochastic phenomena. Note that it is quite reasonable to consider time-inhomogeneous functionals of Lévy processes like $f(t, Z(t))$, instead of only dealing with the homogeneous functionals $f(Z(t))$. Since Hamilton and Susmel (1994) and Mikosch and Starica (2004) pointed out that invariant parametric specifications are often inconvenient to model long return series, in recent years the literature has naturally evolved towards the inclusion of multiple variables in continuous-time models. One example is that in Mercurio and Spokoiny (2004) the returns R_t of the asset process are stipulated as a heteroscedastic model $R_t = \sigma_t \xi_t$ where ξ_t are standard Gaussian independent innovations and σ_t is a time-varying volatility coefficient. The relevant works include Fan et al. (2003), Ait-Sahalia (2002), Hardle et al. (2003) and so forth.

About orthogonal expansions

Due to its extensive use in science, economics, finance and engineering and its central position within stochastic processes, the starting point of this research is to expand Brownian motion functionals including $f(B(t))$ and $f(t, B(t))$ where $B(t)$ is a standard Brownian motion into orthogonal series.

Notice that in the literature, albeit there exist some expansions of Brownian motion in terms of i.i.d. N(0,1) sequence, (see, for example, Yeh 1973 and Mikosch 1998), few researchers are working in the area of general form of Brownian motion functionals.

There are two papers which are close to our topic in some sense in the literature about orthogonal expansion of nonlinear functionals of some processes. To understand the relevant results, let us introduce some notations in the corresponding papers. Denote by C the space of real functions $x(t)$ which are continuous on the interval $0 \le t \le 1$ and which vanish at $t = 0$. Let $\{\alpha_p(t)\}$ be any orthonormal set of real functions in $L^2(0,1)$, and define

$$\Phi_{m,p}(x) = H_m\left(\int_0^1 \alpha_p(t)dx(t)\right); \quad m = 0, 1, 2, \cdots, \ p = 1, 2, \cdots,$$

where $H_m(\cdot)$ is the sequence of Hermite orthogonal polynomials and

$$\Psi_{m_1,\cdots,m_p}(x) \equiv \Psi_{m_1,\cdots,m_p,0,\cdots,0}(x)$$

$$=\Phi_{m_1,1}(x)\cdots\Phi_{m_p,p}(x),$$

in which the index p may be any positive number; the subscripts $m_1,\cdots,m_p$ may be any nonnegative numbers.

Using the Wiener measure on C and completeness properties of Hermite polynomials over $(-\infty,\infty)$, Cameron and Martin (1947) introduced a complete orthonormal set of functionals on C so that every real or complex valued functional $F[x(\cdot)]$ which belongs to $L^2(C)$,

$$\int_c^w |F[x]|^2 d_w x < \infty,$$

has a Fourier development in terms of this set which converges in the $L^2(C)$ sense to functional $F[x]$:

$$\int_c^w \left| F[x] - \sum_{m_1,\cdots,m_N=0}^{N} A_{m_1,\cdots,m_N}\Psi_{m_1,\cdots,m_N}(x) \right|^2 d_w x \to 0,$$

as $N\to\infty$, where $A_{m_1,\cdots,m_N}$ is the Fourier-Hermite coefficient

$$A_{m_1,\cdots,m_N} = \int_c^w F[x]\Psi_{m_1,\cdots,m_N}(x) d_w x.$$

Ogura (1972) did an analogous job as Cameron and Martin (1947) but expanded functionals of the Poisson process $F[D(\cdot)]$ in a series of multiple Poisson-Wiener integrals:

$$F[D(\cdot)] = \sum_{n=0}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f_n(t_1,\cdots,t_n)c^{(n)}[dD(t_1),\cdots,dD(t_n)],$$

where $D(\cdot)$ stands for a Poisson process.

Clearly, the bases in both papers for expansion of functionals are highly complicated since, as discussed in Ogura (1972), they are all multiple Hermite polynomials having the number of arguments increasing to infinity. By contrast, the expansions proposed in Chapter 2 and 4 in this study are quite simple thanks to the simplicity of the bases. This difference gives convenience in calculation of the coefficients and application in practice. Notice that the expansions in the literature have coefficients which are actually functions in the time variable, which would hamstring the applicability of the expansion in econometrics. Nonetheless, from the econometrical applicability perspective, we tackle this issue by expanding time-inhomogeneous functionals, so that coefficients in our expansion are all pure constants which can be estimated by econometric methods. Furthermore, another huge difference between the proposed method in this research and the literature is that we are going to expand functionals of a general class of Lévy processes, not just for Brownian motion or the Poisson process. Additionally, due to the reasons mentioned above our expansion method may be used to estimate unknown functional forms in a general class of econometric models.

The methodology undertaken here, for both Brownian motion functionals and general Lévy process functionals is about to expand the functional in some Hilbert space into Fourier series in terms of a particular orthonormal polynomial basis in the aforementioned space. The basis is actually a sequence of polynomial solutions of hypergeometric differential equations. It is noteworthy that the correspondence between the Lévy process and the orthonormal polynomial system is one–one. The key link between them is the density or probability function of the process. From the Hilbert space theory standpoint, the Fourier series expansion gives the coordinates of a functional in infinite dimensional space, and thereby characterises the functional in nature.

Econometric applications of Fourier expansion

Nevertheless, the Fourier series expansion of Lévy process functionals is by no means our destination. We are interested in estimating an unknown functional form in a general model

$$Y(t) = m(t, Z(t)) + \varepsilon(t),$$

where $Z(t)$ is a Lévy process, and $\varepsilon(t)$ is an error process with zero mean and finite variance, given that we have discrete observations of $Y(t)$.

It is known that existing literature already discusses how to estimate unknown functions of nonlinear time series using nonparametric and semiparametric methods. For the stationary case, recent studies include Fan and Yao (2003), Gao (2007) and Li and Racine (2007). It should also be pointed out that the literature shows that many economic and financial data exhibit both nonlinearity and nonstationarity. Consequently, some nonparametric and semiparametric models and kernel–based methods have been proposed to deal with both nonlinearity and nonstationarity simultaneously. Existing studies mainly discuss the employment of nonparametric kernel estimation methods. Such studies include Phillips and Park (1998), Park and Phillips (1999, 2001), Karlsen and Tjøstheim (2001), Karlsen et al. (2007), Cai et al. (2009), Phillips (2009), Wang and Phillips (2009a,b), Xiao

(2009), and Gao and Phillips (2010). Observe that such kernel–based estimation methods are not applicable to establish closed–form expansions of Brownian motion/Lévy process functionals. In the stationary case, the literature already discusses how series approximations may be used in dealing with stationary time series models, such as Ai and Chen (2003), Chapter 2 of Gao (2007) and Li and Racine (2007). Therefore, it is reasonable to seek its counterpart in the nonstationary scenario to tackle the nonstationary problems.

An inevitable question of doing so is on what time horizon we shall estimate the functional $m(\cdot, \cdot)$. The intuitive choices of time horizon are no more than two cases, viz., a compact interval $[0, T]$ and an infinite interval $(0, \infty)$. However, apart from these two options, we consider the third case, that is, on $[0, T_n]$ with T_n approaching to infinity as sample size goes to infinity. In technical terms, allowing $T = T_n \to \infty$ and $\frac{T_n}{n} \to 0$ amounts to both infill and long span asymptotics. Meanwhile, the two-fold limit theory keeps one away from the so-called aliasing problem (i.e. different continuous-time processes may be indistinguishable when sampled at discrete time). Phillips (1973) and Hansen and Sargent (1983) are early references on the aliasing phenomenon in econometric literature.

A pivotal asymptotic theory

Of the most importance is an asymptotic theory as it is a tool, also a bottleneck, for obtaining the limit distribution of estimators. Without a more general asymptotic theory, our method would be extremely restricted. In order to obtain the asymptotic distribution of the estimators of $m(\cdot, \cdot)$ estimated from the model mentioned before, we have to study an asymptotic theory for different classes of functionals $f(\cdot, \cdot)$ for their sample mean and sample covariance.

Note that in last decade or so, several studies have been devoted to developing an asymptotic theory of a general class of functionals of integrated time series. The relevant researchers have noticed that the absence of such a limit distribution theory has hamstrung time series application. See Park and Phillips (1999, 2001) and Wang and Phillips (2009a,b). However, the existing theory in the literature cannot furnish an answer for the limit problems arising from the scenarios in this study since $f(\cdot, \cdot)$ includes not only a random walk with a unit root but also the time variable, while in literature only a single random walk is involved. Whence, a new asymptotic theory needs to be established. The asymptotic theory developed in this research depends heavily on the local–time process of a Brownian motion defined as a limit by the underlying process and shows that the limit distribution of the estimators on infinity horizon $(0, \infty)$ and on compact interval $[0, T_n]$ with T_n approaching infinity are a mixed normal,

$$\left(\int_0^1 \int_{\mathbb{R}} f^2(t,x)dxdL_W(t,0)\right)^{\frac{1}{2}} N$$

where $L_W(t,0)$ is the local–time process of the limiting Brownian motion $W(r)$ on $[0,1]$ standing for the sojourn time at origin over $[0,t]$ by $W(r)$ and N is a standard normal random variable independent of W, f is some suitable function defined on $[0,1] \times \mathbb{R}$.

By contrast, in the situation where the time variable lies in $[0,T]$ with T fixed, the asymptotic distribution of the estimator is a stochastic integral,

$$\int_0^1 f(Tr, T\mu r + \sqrt{T}\sigma_z W(r))dU(r)$$

where $(W(r), U(r))$ is a vector of Brownian motion which is a limit of some process vector $(W_n(r), U_n(r))$ constructed from Lévy processes $Z(t)$ and error process $\varepsilon(t)$, $\mu = E[Z(1)]$ and $\sigma_z^2 = Var[Z(1)]$, f is some suitable function defined on $[0,T] \times \mathbb{R}$. It is noteworthy to point out that W and U may not be independent which gives more flexibility for the models used in practice.

Outline

The thesis is not presented according to the chronology of the research. We display the asymptotic theory in Chapter 1, which provides an essential tool for the following development. At the same time, as can be seen from the text, since the framework is quite general the results in asymptotic theory of Chapter 1 are applicable even beyond the ambit of this research.

Chapter 2 is devoted to a special case for expansions where Lévy process $Z(t)$ reduces to Brownian motion $B(t)$. Restricted within Brownian motion, the setup in Chapter 2 is concrete. For example, the polynomial system in terms of which we expand functionals is the Hermite polynomial system. In addition, many ideas and methods which are used in the general situation are fostered in this period.

Chapter 3 studies the estimation of an unknown functional form in a general econometric model which involves Brownian motion. The estimators are obtained according to different time horizons and sampling styles. Meanwhile, their asymptotic distributions are obtained and from the results we can see that the rates of convergence are affected by not only sample size but also many other factors.

Chapter 4 dwells on the general situation where the underlying process is a Lévy process $Z(t)$ whose density or probability function $\rho(t,x)$ satisfies the so-called boundary condition. Every such process admits a so-called classical orthonormal polynomial system with weight $\rho(t,x)$, with which the functional of $Z(t)$ can be expanded in the corresponding Hilbert space into Fourier series.

As an application of the orthogonal expansion and asymptotic theory in the previous chapters, Chapter 5 estimates the unknown functional $m(\tau,z)$ by $\widehat{m}(\tau,z)$ in the model aforementioned with the help of OLS (ordinary least squares). After obtaining the estimators in three types of time horizon, their asymptotic distributions are investigated.

The last chapter concludes what we did and discusses potential applications of the proposed expansion method for Lévy process functionals.

Appendix A, entitled Miscellaneous, states an alternative expansion method for the quadratic Brownian motion form using stochastic integral method. Without doubt, it has a kind of quaint charm although comparing with the text it is difficult to be extended to general situations.

Declaration

This work contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and beliefs, contains no material previously published, or written by another person except where due reference has been made in the text.

I give consent to the copy of my thesis, when deposited in the university library, being available for loan and photocopying.

Chaohua DONG

12 December, 2011

Acknowledgements

I had wasted a lot of time before starting my PhD journey in 2008. I did not realise that I had a strong desire to pursue knowledge until I visited Professor Jiti Gao in 2004 at the University of Western Australia, and discovered that studying for a PhD is the best way for me to do this. Given that I was not so young, this was a 'now or never' opportunity for me. I therefore sat the IELTS qualification three times in order to acquire a sufficient score in English to be an eligible applicant. This was the prelude for the journey of my PhD, which has been full of difficulties and trials.

With the eligible IELTS result and excellent academic performance, I was awarded EIPRS (Endeavor International Postgraduate Research Scholarship) from the Australian government and the University of Adelaide in 2008, triggering my PhD journey off.

I have very much enjoyed the last three years and four months of hectic study under the supervision of Professor Jiti Gao. I strived for the answer of every single question; my intelligence was tortured by the research questions again and again; my endeavours to pursue the results of the research, to achieve the degree of PhD took all of my effort and energy, exhausting both my physical and spiritual self. Now I see the lighthouse which will guide me into the harbour of destination.

At such a crucial and critical moment, it is high time to deliver my acknowledgements, both professional and personal, like a champion after winning an exciting and fierce game, for the supervisors, the peers, the staff in the school and my family members.

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