Twisted Analytic Torsion

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Abstract

In [28], Mathai and Wu extended the notion of analytic torsion, as first conceived by Ray and Singer [34], to $\mathbb{Z}_2$-graded complexes. The main example of this is the de Rham complex with the flux-twisted differential $d_H = d + H$, where H is a closed three form, a complex that arises in geometric situations where there is twisting by a gerbe. We review the formalism required to construct this torsion, and present the key results. We generalise the analysis found in Farber [12] and Forman [14] to the $\mathbb{Z}_2$-graded situation to study the behaviour of the torsion of families of complexes near points at which the cohomology jumps. By studying analytical deformations of these complexes, we provide results showing that in some cases the torsion and some related invariants of this twisted operator are related to the untwisted torsion only through maps of a cohomological nature.

Signed Statement

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Introduction

Analytic torsion for the de Rham complex was introduced in [34], building on earlier work of [32], as a smooth analogue of the classical Reidemester Torsion. This analytic object is constructed out of certain *regularized* determinants of Laplacians of certain smooth chain complexes, so a priori it is somewhat different from its simplicial predecessor. The study of Ray and Singer's analytic torsion culminated in the magnificent monograph of Bismut-Zhang, where it was shown that it in fact agreed with the simplicial Reidemeister torsion. This lead to many interesting avenues of study relating differential and simplicial invariants of manifolds. The Ray-Singer torsion reappeared later in the seminal work of Witten [44], where it was shown that this invariant was related to certain topological field theories, an idea developed by Schwarz [40]. Recently, Mathai and Wu [28, 29] have developed an extension of analytic torsion to the case of $\mathbb{Z}_2$ graded complexes. In this case, special care needs to be taken with the analysis to show that the torsion exists, and it is here that the subtle non-commutative residue of Wodzicki-Guillemin makes an appearance.

The 'twist' of the analytic torsion that we are interested comes from a modification of the de Rham complex of forms by a total odd-degree closed form H, known as the *flux*. The de Rham differential d is modified to $d_H = d + H$, an operator that also squares to zero, and thus a flux-twisted complex of forms can be considered. This example was the original motivation for Mathai-Wu's definition of analytic torsion for $\mathbb{Z}_2$ graded complexes. The goal of this thesis is to partially answer some questions raised in [28], relating to this special case of the torsion. Specifically to compute the value of the zeta function at zero, the so called *derived Euler characteristic*, of the flux twisted de Rham complex. Mathai and Wu also show that when the flux form is of top degree, the twisted and untwisted torsions are related by a canonical cohomological isomorphism, the Knudsen-Mumford map. We give an alternate proof of this result, and extend it to the case where the twisted cohomology vanishes. Very recently, Mathai-Benameur [26] have also explored a signature-type invariant for the flux twisted de Rham complex. We leave it as a future project to extend the results obtained in this thesis to the case of the signature complex.

In Chapter 1, we review the necessary analytical tools used in the construction of regularized determinants. We show how to construct the heat kernel $k(e^{-tQ})$ for Laplacian-like operators Q, and that there exists an important asymptotic expansion near $t = 0$ (Prop 2)

$$k(e^{-tQ}) \sim \sum_{i=0}^{\infty} a_i t^{i-n/2}$$

The mildness of this singularity at the origin allows for us to define the regularized determinant of Q in terms of a trace of this heat kernel. This process involved the construction of a Zeta function, also known as a power sum, a construct that dates back to early work of Riemann in his famous Hypothesis. We also consider the case of asymptotic expansions of the operator Ae^{-tQ}, where A is a auxiliary pseudodifferential operator, and show how Grubb-Seeley were able to produce an asymptotic expansion in this case (Prop 12)

$$k(Ae^{-tQ}) \sim \sum_{i=0}^{\infty} a_i t^{i-n/2} + \sum b_i t^i \log t$$

The presence of these $b_0 \log t$ factors in the expansion prevents a straightforward definition of a determinant in this case. The obstruction to this construction is related to a quantity known as the Non-commutative residue of Wodzicki-Guillemin. We show that in the applications we are interested in, namely the construction of the Mathai-Wu torsion, the operator A is a projection and that the residue vanishes on this type of operator. This result was used in [28] to define the determinant for partial Laplacians that form the twisted analytic torsion.

In Chapter 2, we begin with a review of classical material on chain complexes and their $\mathbb{Z}_2$-graded analogues, and the language of *super*-algebra that describes them. We discuss an important map, the Knudsen-Mumford isomorphism, which is a canonical map between the determinant line of a complex and the determinant line of its cohomology. We introduce analytic torsion for these complexes, as it was defined by Mathai-Wu, and show how in the case of a folded up $\mathbb{Z}$-graded complex we recover the original definition of analytic torsion due to Ray-Singer. We discuss a somewhat more elementary object constructed out of zeta functions, which first appeared in the seminal work of Bismut-Zhang, known as the derived Euler characteristic. We demonstrate the key property of analytic torsion, namely its invariance under change of metric. In $\mathbb{Z}_2$-graded setting, we can show a further invariance under a large class of gauge transformations. We also introduce perhaps the most basic non-trivial example of a $\mathbb{Z}_2$-graded complex, that of the twisted de Rham complex. This complex is the usual complex of forms, graded by degree modulo 2, equipped with the differential $D = d + H$, where H is a closed total-odd degree form. It is this basic complex on which we will focus the majority of our analysis.

In Chapter 3, we introduce deformations of elliptic complexes, that is, a base complex $(C^\bullet, d_0)$, and a one parameter family of differentials d_t that extends d_0 such that $d_t^2 = 0$ for all t. We restrict to the class of families that satisfy certain analyticity constraints on the t-dependence. We use results of Kato to show that under certain conditions, the spectrum of such a family of operators behaves in a similarly analytic fashion. These results were first used by Farber in [12] to analyse the t dependence of the analytic torsion of the family d_t. We extend Farber's results to the $\mathbb{Z}_2$ graded situation. One of the main objectives is to compute the difference

$$\dim_{\mathbb{C}} H^{\bar{k}}(C, d_0) - \dim_{\mathbb{C}} H^{\bar{k}}(C, d_t)$$

for small t. We apply this analysis to a simple family of operators that passed through both d and $d + H$, the operators introduced in the previous chapter. We use these results to answer a question posed in Mathai-Wu concerning the derived Euler characteristic of the $d + H$ complex.

In Chapter 4, we discuss how the determinant line varies along an analytic family near the base point. We show that the de Rham cohomology of a deformation of an elliptic complex can be computed from a certain spectral sequence that is built out of taking successively better approximations to the kernel of the differential. We present a generalisation of the works of Forman [14] on this spectral sequence to the case of $\mathbb{Z}_2$-graded case, and show that similar results follow. We apply this generalisation to the case of the twisted de Rham complex with operator d_H. We apply the machinery developed in this thesis to the case of torsion of Lie groups twisted by the canonical 3-form.

Chapter 1

Heat Kernels and Regularised Determinants

The determinant of an invertible linear map $A \in GL(V)$ acting on a finite dimensional vector space V measures the scaling of the volume element induced by A. If $\{\lambda_i\}$, is a complete set of eigenvalues (with multiplicities) for A, then we have the classical formula $\det A = \prod_i \lambda_i$. In [32, 34], it was asked if this notion of determinant could be extended to certain differential operators acting on the space of sections of vector bundles over compact manifolds, in particular, the Laplace operator Δ. This generalisation was far from straightforward, since the spectrum of such operators is usually infinite and divergent. Let Q be an operator such that its eigenvalues diverge $\lambda_i \to \infty$ as $i \to \infty$, then the naive limit $\lim_{N\to\infty} \prod_{i\leq N} \lambda_i$ cannot be finite. However, it has been known since the early works of Riemann on his famous Zeta function that in certain cases there is a technique that allows us to assign finite values to certain divergent products. In particular, we can show that a generalisation of this procedure assigns a finite number for the 'determinant' of the operators we are interested in, and thus it is called a *regularised* determinant. Here we illustrate the outline of this procedure in the finite dimensional case and in this chapter we will show how this method extends, via regularization, to certain elliptic operators on compact manifolds.

- First we construct the solution to the heat equation for a matrix $A \in GL(V)$, that is, the time dependent matrix $a(t)$, called the *heat kernel*, that satisfies the system of equations

  $$(\tfrac{\partial}{\partial t} + A)a(t) = 0, \quad a(0) = \mathbf{1}$$

  The solution given by the matrix exponential, $a(t) = \exp(-tA)$. This matrix valued function exists for all time t.

- Secondly, take the matrix trace of the heat kernel, $\operatorname{tr} a(t) = \sum_i e^{-t\lambda_i}$. We see that this function only depends on the spectrum of A.

- Take the Mellin transform of this trace, known as the *zeta function* of A

$$\zeta(A,s) := \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \left[\operatorname{tr} a(t)\right] dt$$

Using the the gamma function identity

$$\int_0^\infty t^{s-1} e^{-t\lambda} dt = \Gamma(s)\lambda^{-s}$$

we have the 'power sum' realization of the zeta function

$$\zeta(A,s) := \sum_i \lambda_i^{-s} \tag{1.0.1}$$

- Determine the *small time asymptotic behavior* of the trace, i.e., its Laurent expansion near $t = 0$

$$\operatorname{tr} a(t) \sim \operatorname{rk} A - t \operatorname{tr} A + O(t^2)$$

By some general theory, this tells us that $\zeta(A,s)$ is smooth in a neighbourhood of $s = 0$, and the power sum identity yeilds

$$\left(\tfrac{\partial}{\partial s}\zeta(A,s)\right)_{s=0} = -\sum_i \log \lambda_i$$

So we can recover the determinant as

$$\det A = \exp(-\zeta'(A,0)) \tag{1.0.2}$$

This might seem like a rather complicated process to arrive at the determinant, but the key observation of [32] is that we can repeat each of the above steps for the geometric operators we are interested in. The relation (1.0.2) then becomes the definition of the *zeta-regularised* determinant for such operators. However, we need to be careful that the various quantities at each step are well defined and finite. To attempt to repeat the above process for an operator Q on a Hilbert space H, the difficulties that arise are

- The heat equation

$$\left(\tfrac{\partial}{\partial t} + Q\right)k(t) = 0$$

might not have a solution for all t. For the case we are interested in, namely some particular elliptic operators, we find that a unique solution $k(t)$ exists, and is a smoothing operator for $t > 0$, and at $t = 0$ it limits to the identity operator.

- We have to define what kind of trace we wish to take, and to see when (if at all) the heat kernel has a well defined trace. We will see that it is of trace class for $t > 0$, but this trace is singular as $t \to 0$, unlike the finite dimensional case. We will need to determine the structure of this singularity.

- Determine how these singularities in the trace contribute to the singularities of the Zeta function, and thus dictate the finiteness of the determinant.

The operators we will be most interested in are modelled on the Laplacians acting on sections of some vector bundle over a manifold. The analysis of these operators is performed by studying Clifford modules, and their associated Dirac operators.

The outline of this chapter is as follows. First we introduce Dirac operators and their generalisations, and show that these operators satisfy certain analytical properties that simplify our investigation. We then describe how to construct the heat kernel for such operators, and compute its *small time asymptotic expansion*. We then introduce zeta functions for elliptic operators, via the heat kernel, and discuss the relation between the poles of the zeta function and the expansion of the heat kernel. We then complete the analogy with the finite dimensional case and show the regularised determinant for our elliptic operator is well defined. Lastly, we present a different approach to heat kernel expansions where we use a resolvent approach due to Grubb and Seeley that allows for the incorporation of a pseudodifferential auxiliary operator. We show that when this auxiliary operator is a projection, we can also define a determinant, a result that will be crucial to our analysis of the twisted de Rham complex in the later chapters.

1.1 Clifford Modules and Dirac operators

We will be studying geometric and spectral properties of operators similar to the scalar Laplacian. This operator is of fundamental importance to the study of classical vibration and dissipation problems and harmonic analysis. A key observation is that the classical Laplacian on forms has a square root, which is the so called called *Dirac operator*. We begin with some background on Dirac operators, and the analysis of the heat equation for the squares of such operators.

1.1.1 Dirac Operators

We construct the heat kernels for the class of operators we are most interested in, the so called *Dirac operators*. The main examples we have in mind is the de Rham-Dirac operator which will be discussed later. We closely follow the material presented in [5] and [37].

The set up for our situation is a follows: (M, g) is a Riemannian manifold, and $\mathrm{Cliff}(TM_{\mathbb{C}})$ is the associated bundle of complex Clifford algebras (see [37], for a discussion of Clifford algebra bundles). Let $\mathbb{S}$ be a $\mathbb{Z}/2\mathbb{Z}$-graded bundle of complex left Clifford modules over M, i.e. the fiber $\mathbb{S}_p$ is a left $\mathrm{Cliff}(T_pM_{\mathbb{C}})$-module for each $p \in M$. Assume that $\mathbb{S}$ comes equipped with a hermitian metric h and connection $\nabla^{\mathbb{S}}$, which are compatible with the Riemannian structure on M in the following sense:

1. Clifford multiplication by $v \in T_pM$ is a skew adjoint endomorphism of $\mathbb{S}_p$ for each $p \in M$,

$$h(v.s_1, s_2) + h(s_1, v.s_2) = 0$$

2. The spinor connection $\nabla^{\mathbb{S}}$ is compatible with the Levi-Civita connection ∇ on M

$$\nabla^{\mathbb{S}}(c(X)s) = c(\nabla X)s + c(X)\nabla^{\mathbb{S}} s$$

   where c is the Clifford action of the vector field X the spinor field $s \in \Gamma(\mathbb{S})$.

With this data, we can construct the fundamental operator which is the focus of this chapter.

Definition 1. *The* Dirac operator D *for the above data is the odd, first order differential operator given by the composition*

$$\Gamma(\mathbb{S}^{\pm}) \xrightarrow{\nabla^{\mathbb{S}}} \Gamma(T^*M \otimes \mathbb{S}^{\pm}) \xrightarrow{g} \Gamma(TM \otimes \mathbb{S}^{\pm}) \xrightarrow{c} \Gamma(\mathbb{S}^{\mp})$$

Where the first map is the connection, the second map is dualization via the metric, and the last map is Clifford multiplication.

In local coordinates, the Dirac operator has the expression

$$Ds = \sum_i c(e^i)\nabla^{\mathbb{S}}_{e_i} s$$

where e^i is the basis for T_pM, and e_i is the dual basis. It is clear that the principle symbol of the Dirac operator is Clifford multiplication $\sigma_1(D)(\xi) = c(\xi)$, which is invertible for $\xi \neq 0$, i.e. $\sigma_1(D)(\xi)^2 = -|\xi|^2 \mathbf{1}_{\mathbb{S}}$. This implies that the Dirac operator is an elliptic, which enables much of the analysis that follows. It can be shown that the square of the Dirac operator D^2 has the same symbol as that of the scalar Laplacian $\sigma_2(D_E^2) = \sigma_2(\Delta_{\mathbb{S}}) = -|\xi|^2\mathbf{1}_{\mathbb{S}}$. More specifically, the Dirac operator satisfies a fundamental identity which relates its square to another geometrical operator of the same principal symbol.

Proposition 1 (Weitzenbock Identity)**.** *Let D be a Dirac operator on M, we have*

$$D^2 = \nabla^*\nabla + K$$

where $\nabla : C^\infty(M, \mathbb{S}) \to C^\infty(M, T^*M \otimes \mathbb{S})$ is the spinor connection, ∇^* is its adjoint with respect to the natural inner products on $C^\infty(M, \Lambda^k T^*M \otimes \mathbb{S})$, and $K \in \operatorname{End}\mathbb{S}$ is the endomorphism given in a local coordinate chart $\{e_i\} \in TM$ as

$$K = \sum_{i,j} c(e_i)c(e_j)R(e_i, e_j)$$

where $R \in \Omega^2(\operatorname{End}\mathbb{S})$ is the curvature of the spin connection.

Proof. This follows from the Clifford algebra relations and some simple algebraic manipulation in local coordinates, c.f. [37]. □

Using a few simple identities, we can show that this operator is self-dual in an appropriate sense.

Lemma 1. *The Dirac operator D, is formally self adjoint, i.e.*

$$\langle Ds_1, s_2 \rangle = \langle s_1, Ds_2 \rangle$$

for all $s_i \in C^\infty(\mathbb{S})$.

We can easily see that the symbol of the leading term $\sigma_2(\nabla^*\nabla)$ is scalar and proportional to the metric. We often consider a slightly more general notion of Dirac operator. we say that a first order operator T is a *generalised Dirac operator* if it satisfies a relation of the form $T^2 = \nabla^*\nabla + A$, where A is a first order operator.

We make sense of these definitions by considering the basic example: the Dirac operator for the de Rham complex. The canonical example of a Clifford module is that of the exterior bundle, $\Lambda T^*M = \oplus_{i=0}^n \Lambda^i T^*M$. This is fiberwise isomorphic to the clifford bundle $\mathrm{Cliff}(T^*M)$, with the isomorphism given by the quantization map $q : \mathrm{Cliff}(T^*M) \to \Lambda T^*M$, $q : v_1 v_2 \ldots \mapsto v_1 \wedge v_2 \wedge \ldots$. Under this map clifford multiplication by a vector v gets mapped to the operator $c(v) = e(v) + \iota(v)$, where $e(v)$ is exterior multiplication and ι is contraction with v given by the metric. This bundle has a natural flat connection, the exterior derivative d, which lifts canonically to a spin connection. The Dirac operator for this data is called the *de Rham-Dirac operator*. With some more simple algebraic manipulation, and the identities satisfied by the hodge dual, we find

Lemma 2. *The de Rham-Dirac operator D is given by the simple expression $D = d + d^*$.*

This Dirac operator is closely connected to the geometry of the underlying manifold. Its square $D^2 = d^*d + dd^* =: \Delta$ is the familiar Hodge Laplacian on forms. It is this Laplacian for which we wish to compute a determinant, and the generalisation to all Dirac operators follows without too much extra work.

1.1.2 Heat Kernels

The Heat Equation

Associated to any second order elliptic differential operator H acting on sections of a rank r vector bundle E over a compact manifold M of dimension n, there is a fundamental parabolic differential equation known as the *heat equation*, which controls heat-like dissipation over the manifold:

$$(\tfrac{\partial}{\partial t} - H)u(x,t) = 0$$

where $u \in L^2(M \times \mathbb{R}^{\geq 0}, E)$ is a time dependent distribution with compactly supported initial value

$$u(x,0) = f(x)$$

The solutions of this equation can be used to compute important geometrical and topological properties of the manifold because they encode information about the operator H. In particular, we will use the heat equation to define the regularised determinant of H. The scalar Laplacian Δ is a fundamental second order elliptic operator on any Riemannian manifold M. On $M = \mathbb{R}^n$, this equation takes the form

$$(\tfrac{\partial}{\partial t} - \Delta)u(x,t) = 0, \qquad \text{where } \Delta = \sum_i \tfrac{\partial^2}{\partial x^{i\,2}}$$

This equation is solved by introducing the *(scalar) heat kernel*

$$g_t(x,y) = (4\pi t)^{-n/2} e^{-\|x-y\|^2/4t} \tag{1.1.3}$$

The solution is then given by acting with the integral operator G_t with kernel function $g_t(x,y)$

$$u(x,t) = (G_t f)(x) := \int_{\mathbb{R}^n} g_t(x,y) f(y) dy$$

so that

$$\lim_{t\to 0} u(x,t) = f(x)$$

The fact that $u(x,t)$ as defined above solves the heat equation for Δ follows from the fundamental properties of the heat kernel

$$(\tfrac{\partial}{\partial t} - \Delta_x) g_t(x,y) = 0, \text{ for } t > 0$$

$$\lim_{t\to 0} g_t(x,y) = \delta_M(x,y)$$

where δ_M is the Dirac delta distribution on $M \times M$. Notice that the heat kernel is a smooth function on M for $t > 0$, but is a distribution with support on the diagonal at time $t = 0$. We now extend this definition to a more general situation. The operators we are interested in, namely the squares of the Dirac-type that were introduced before, have the property that their principal symbols are scalar and proportional to the Riemannian metric, e.g. $\sigma_2(D_E^2)(\xi) = -\|\xi\|^2 \mathbf{1}_E$.

Definition 2. *A* generalised Laplacian *is a second order differential operator H acting on the space of smooth sections of a Euclidean vector bundle E, such that the principal symbol of H is proportional to the Euclidean metric, i.e.*

$$\sigma_2(H)(x,\xi) = -\|\xi\|_x^2 \mathbf{1}_{E_x}$$

for $\xi \in T_x M$.

This definition implies that the heat kernels of generalised Laplacians are closely approximated by the heat kernel of the scalar Laplacian. We now formalise in what sense this truly is an approximation, and how it can be used to build the full heat kernel for generalised Laplacians. We start with the formal definition of the heat kernel.

Definition 3. *Let H be a generalised Laplacian for a vector bundle E over a manifold M. A* heat kernel *for H, is a continuous section $p_t(x,y)$ of the bundle $E \boxtimes E^*$ over $M \times M \times R^{>0}$, that is C^1 in t, and C^2 in the coordinates x^i, such that*

1) $p_t(x,y)$ satisfies the heat equation

$$(\frac{\partial}{\partial t} + H_x)p_t(x,y) = 0 \tag{1.1.4}$$

for $t > 0$.

2) for any smooth section $s \in \Gamma(E)$, the integral operator $P_t : s \mapsto \int_M p_t(x,y)s(y)dy$, satisfies

$$\lim_{t \to 0} P_t s = s$$

in the sup *norm on sections. i.e., the heat kernel satisfies the distributional equation*

$$(\frac{\partial}{\partial t} + H_x)p_t(x,y) = \delta_{\mathbb{R}}(t)\delta_M(x,y)$$

With only this definition, it is not at all obvious that such a heat kernel exists for every second order elliptic operator. We will show this existence by use of the functional calculus.

1.1.3 Construction of the Heat Kernel

The Functional Calculus

Here we summarise the fundamental facts about elliptic differential operators on compact manifolds. We closely follow the treatment presented in [37]. We begin with the classical *spectral theorem* of elliptic operator theory, many proofs of which exist in the literature.

Proposition 2. *Let D be a formally self-adjoint elliptic operator acting on sections of a vector bundle E. There is a decomposition of $H = L^2(E)$ into countably many finite-dimensional orthogonal subspaces H_λ, each of which consists of smooth eigensections with eigenvalue $\lambda \in \mathbb{R}$. Furthermore, the eigenvalues form a discrete subset of $\mathbb{R}$.*

We now discuss how this clear spectral decomposition allows for a simplified analysis of operators related to D. A square integrable section $s \in L^2(E)$ can be decomposed in the eigenfunction basis of D

$$s = \sum_{\lambda \in \sigma(H)} s_\lambda$$

where s_λ lies in the λ-eigenspace of D. Since we want to talk about smooth sections, we need to describe them in terms of this decomposition. We say that a sequence x_λ is *rapidly decreasing in* λ if $x_\lambda = O(|\lambda|^{-k})$ for every $k \geq 0$.

Proposition 3. *A section $s \in L^2(E)$ is smooth if and only if $\|s_\lambda\|_0$ is rapidly decreasing in λ.*

Proof. See [37]. □

This spectral decomposition allows us to construct a calculus of operators out of our main operator D. Given a bounded function f defined on a set containing $\sigma(D)$, we can define a operator $f(D)$ by

$$f(D)s := \sum_{\lambda \in \sigma(D)} f(\lambda)s_\lambda$$

Defining operators in this way is called the *functional calculus*. The following result is easily proven, c.f. [37]

Proposition 4. *The functional calculus defines a unital ring homomorphism from bounded functions on $\sigma(D)$ to bounded operators on $L^2(\mathbb{S})$. Furthermore, the operator norm $\|f(D)\|$ is bounded above by* $\sup f$

If the function f is also rapidly decreasing, then the previous argument shows that $f(D)s$ is smooth for any $s \in L^2(E)$. Since the spectrum of D was discrete, for each eigenvalue $\lambda \in \sigma(D)$ we can construct a smooth, compactly supported function p_λ on $\sigma(D)$ that vanishes on $\sigma(D) - \{\lambda\}$, and $p_\lambda(\lambda) = 1$. The corresponding operator $P_\lambda = p_\lambda(D)$ is the projection operator onto the λ-eigenspace of D, i.e. $P_\lambda s = s_\lambda$. We will now describe how operators defined via the functional calculus using rapidly decreasing functions can be described as integral operators with smooth kernels.

Definition 4. *A bounded operator $T : L^2(E) \to L^2(F)$ is a* smoothing operator *if there exists a smooth section k of the bundle $F \boxtimes E^*$ over $M \times M$, called the* Schwartz kernel, *such that*

$$(Ts)(x) = \int_M k(x,y)s(y)dy$$

where $s \in L^2(\mathbb{S})$. Notice that the smoothness of k ensures that the image of T consists of smooth sections of F, hence the name.

We begin by showing that the projection operators defined before are smoothing operators. Since each eigenspace is finite dimensional, we can choose an orthonormal basis of smooth sections $\varphi_i(x)$ of the λ-eigenspace. The projection operator has the kernel defined by

$$k(x,y)s(y) = \sum_i \varphi_i(x)\langle \varphi_i(y), s(y)\rangle$$

since the sections φ_i are smooth, this is a smooth kernel, and hence P_λ is a smoothing operator.

Lemma 3. *If f is a rapidly decreasing smooth function, then the operator $f(D)$ is smoothing.*

Proof. To show that the sum $f(D) = \sum_\lambda f(\lambda)P_\lambda$ converges in the space of smoothing kernels, i.e. in the Frechet topology on $Y = \Gamma(M \times M, F \boxtimes E^*)$, we need to provide estimates on the Sobolev k-norms of the smoothing kernels of P_λ for each k. This is done in [37] pp85. $\square$

We can now use the functional calculus to construct the heat kernel. The function $f(\lambda) = e^{-t\lambda^2}$ is rapidly decreasing in λ for $t > 0$ and so the corresponding operator $e^{-tD^2} := f(D)$ is smoothing for t in this range.

Proposition 5 ([37]). *The operator e^{-tD^2} solves the heat problem, that is, given initial data $s \in L^2(E)$, the time-dependent section $s_t = e^{-tD^2}s$ is a solution to the equation (1.1.4), its Schwartz kernel $k_t(x, y)$ is the aforementioned heat kernel.*

These uniqueness theorems and the functional calculus provide an obvious yet very useful semi-group property for these operators.

Corollary 1. *For all $t, t' > 0$, we have $e^{-tD^2}e^{-t'D^2} = e^{-(t+t')D^2}$.*

Proof. See [37] $\square$

This has the obvious physical interpretation, if we dissipate heat for t units of time, and then dissipate it for a further t' units of time, the resulting heat profile is the same as if we had dissipated heat for $t + t'$ units of time.

Asymptotic Expansion

Now we know that the heat kernel for a generalised Laplacian H exists, and is smoothing for all times $t > 0$, we would like to investigate its behaviour as $t \to 0$. We already know that in this limit the support (as a distribution) of the heat kernel reduces to the diagonal

$$\lim_{t \to 0} k_t(x, y) = \delta(x, y)$$

and so we look for a expansion of this kernel is a neighbourhood of the diagonal $\Delta \subset M \times M$. We do this in the following way

1. Linearizing the manifold near a point on the diagonal
2. Take successively finer approximations to the kernel starting from the Euclidean heat kernel (1.1.3) in a geodesic coordinate system.

First, choose geodesic coordinates around a point $p \in M$, i.e. an open set $V \subset T_pM$ (with coordinates $\mathbf{x}^i$) such that the exponential map $\exp_p(V) \to U$ (with local coordinates x^i) is a diffeomorphism onto its image. We write the map $\pi : V \to U$. Since H was a generalised Laplacian, we know that

$\sigma_2(H)_p(\xi) = g_p(\xi, \xi)$. If we forget about the terms of first order or lower, this operator would have heat kernel given by (1.1.3), namely for $\exp_p(x) = q \in U$, the heat kernel would be

$$h_t(p, q) = (4\pi t)^{n/2} \exp\left(-g(x, x)/4t\right) \mathbf{1}_{\mathbb{S}}$$

where $g(x, x)$ is the Riemannian metric on T_pM. Motivated by this, we take $h_t(p, q)$ as a *first approximation* to the heat kernel in a neighbourhood of the diagonal. We now perform an iterative procedure, which yields full *asymptotic expansion* of the heat kernel in this regime

Definition 5. *A formal power series $\sum_{i=-N}^{\infty} t^i w_i(x, y, t)$ with coefficients w_i that are all smooth sections of $E \boxtimes E^*$ defined in a neighbourhood of the diagonal $\Delta \subset M \times M$, is called an* asymptotic expansion *of the heat kernel $k_t(p, q)$ near $t = 0$, denoted*

$$k_t(p, q) \sim \sum_{i=-N}^{\infty} t^i w_i(x, y, t)$$

if the following estimates hold: for each $n > 0$, there exists an ℓ_n, such that for all $\ell \geq \ell_n$, and $r \geq 0$ there is a constant $c_{\ell,n,r}$ such that

$$\left\| k_t(p, q) - \sum_{i=-N}^{\ell} t^i w_i(p, q, t) \right\|_{C^r} \leq c_{\ell,n,r} |t|^n$$

for sufficiently small t, where norm is the standard one on the Banach space $C^r(E \boxtimes E^*)$.

We now show that such an expansion exists, following the approach in [37].

Proposition 6. *Let $k_t(p, q)$ be the heat kernel for D^2, there exists an asymptotic expansion of the form*

$$k_t(p, q) = h_t(p, q) \sum_{i=0}^{\infty} \Theta_i(p, q) t^i \tag{1.1.5}$$

where $h_t(p, q)$ is the first approximation introduced earlier.

Proof. We will construct a series of smooth sections Θ_j of $E \boxtimes E^*$ so that for each $m > 0$, there is a $k_m > 0$ so that the partial sum

$$h_t(p, q) \sum_{j=0}^{k_m} t^i \Theta_i(p, q)$$

approximates the heat kernel to order m in a precise way, which in turn will give us the desired estimates. Fix a geodesic coordinate system around $p \in M$. Let h_t be the function near p given by $h_t(., q)$, where $q = \exp_p(x)$. Let $r = \sqrt{g(x, x)}$ be the radial distance function. We need a few easily proved preliminary results, for f a smooth function on M and $s \in \Gamma(\mathbb{S})$, we will show

- $[D, f]s = c(\nabla f)s$
- $[D^2, f]s = (\Delta f)s - 2\nabla_{\nabla f} s$

- $\nabla h = -\frac{h}{2t} r\partial_r$

- $(\partial_t + \nabla)h = \frac{h}{4gt}(r\partial_r g)$, where $g = \det(g_{ij})$

The first two follow from simple algebraic manipulation and the rules of Clifford multiplication, c.f. [37]. These two capture the property that the symbol of the Dirac operator is given Clifford multipliation, and that the principal symbol of its square is the same as that of the scalar Lapacian. For the second two statements, notice that $dh = (-h/2t)rdr$, and then we obtain ∇h by using the metric to dualise the one-form dh. Since we are in a geodesic coordinate system, dr is dual to ∂_r. Recall that on functions $\Delta f = \nabla^* \nabla f$. So we find $\Delta h = \nabla^*(-\frac{h}{2t} r\partial_r)$. Using the identity $\nabla^*(fX) = f\nabla^* X - \langle \nabla f, X\rangle$, we obtain

$$\begin{aligned} \Delta h &= -\frac{h}{2t}\nabla^*\left(r\partial_r\right) + \frac{1}{2t}\langle \nabla h, r\partial_r\rangle && (1.1.6) \\ &= -\frac{h}{2t}\nabla^*\left(r\partial_r\right) + \frac{r}{2t}(\partial_r h) && (1.1.7) \end{aligned}$$

So we just need to compute $\nabla^*(r\partial_r)$, by using a standard formula for dual divergences, and using $r = g_{ij}x^i x^j$, we have

$$\begin{aligned} \nabla^*(r\partial_r) &= -\frac{1}{\sqrt{g}}\sum_j \partial_{x^j}(x^j\sqrt{g}) && (1.1.8) \\ &= -n - \frac{r}{2g}(\partial_r g) && (1.1.9) \end{aligned}$$

This yields a explicit formula for Δh, and we can compare these terms with and evaluation of the simple expression $\partial_t h$, and we get the result.

It now follows easily from these preliminary results that for a section $s \in \Gamma(\mathbb{S})$, we have

$$\left[\left(\partial_t + D^2\right), h\right]s = \left(\frac{h}{t}\nabla_{r\partial_r} + \frac{h}{4gt}(r\partial_r g)\right)s$$

This key formula tells us how to commute the heat operator past scalar functions, at the cost of incurring the action of a first order radial operator. Now we assume $s \equiv s(t) = \sum_{i=0}^{\infty} u_i t^i$ is a time dependent section, and we will inductively try to solve the heat equation

$$\left(\partial_t + D^2\right)(hs) = 0$$

by looking at the resulting system of equation for each coefficient in the t expansion. Using the key result above, we have

$$\left(\partial_t + D^2\right)(hs(t)) = h\left(\partial_t + D^2\right)s + \left(\frac{h}{t}\nabla_{r\partial_r} + \frac{h}{4gt}(r\partial_r g)\right)s \qquad (1.1.10)$$

if we ignore the overall factor of h, we see that the terms $\partial_t + \left(\frac{h}{t}\nabla_{r\partial_r} + \frac{h}{4gt}(r\partial_r g)\right)$ all act to decrease a power of t by one, whereas the D^2 term does not involve t. i.e.

$$\left(\partial_t + \frac{1}{t}\nabla_{r\partial_r} + \frac{1}{4gt}(r\partial_r g)\right) u_j t^j = (j + \nabla_{r\partial_r} + \frac{1}{4g}(r\partial_r g))u_j t^{j-1}$$

For the r.h.s of (1.1.10) to vanish, we must solve the system of equations

$$(j + \nabla_{r\partial_r} + \frac{1}{4g}(r\partial_r g))u_j = -D^2 u_{j-1}$$

We see that this is just a system of DE's, which we can simplify with the integrating factor $g^{1/4}$, i.e, for $j > 0$ it reduces to

$$\nabla_{r\partial_r}(r^j g^{1/4} u_j) = -r^j g^{1/4} D^2 u_{j-1}$$

however, for $j = 0$, we get the simple first order ODE

$$\nabla_{r\partial_r}(g^{1/4} u_0) = 0$$

Which by standard theory, has a unique solution determined by its initial value $u_0(0)$, which we choose to be the identity endomorphism of $\mathbb{S}_q$. The higher u_j are each determined by u_{j-1}, up to terms of the form cr^{-j}, where c is a constant, which lie in the kernel of the l.h.s. Since we require each u_j to be smooth however, this freedom is removed. Thus, each u_j is unique determined by the single initial condition $u_0(0) = \mathbf{1}_{\mathbb{S}}$. We now define our coefficients $\Theta_j(p, q)$ to be the functions taking values in $\mathbb{S} \boxtimes \mathbb{S}^*$, which are represented in the local coordinate chart near q by the functions $u_j(p)$. Thus we construct the partial sum

$$k_t^N(p, q) = h_t(p, q) \sum_{i=0}^{N} t^j \Theta_j(p, q)$$

and our goal is to show that these partial sums approximate our heat kernel in an appropriate sense. Firstly notice that by our choice of initial condition, $u_0(0) = \mathbf{1}_{\mathbb{S}}$, we have $\Theta_0(p, p) = \mathbf{1}_{\mathbb{S}_p}$. As $t \to 0$, all the other terms in the sum become negligible, and it is not hard to see that our first approximation $h_t(p, q)$ approaches a delta function $\delta(p, q)$, and thus so does $k_t^N(p, q)$. Notice now, we have

$$\left(\partial_t + D_p^2\right) k_t^N(p, q) = t^N h_t(p, q) f_N(t, p, q)$$

where $f_N(t, p, q)$ is some smooth remainder function. Now the t-exponent of the leading term $t^N h_t$ is $N - n/2$. So for $m < N - n/2$, this leading term tends to zero in the C^m topology as $t \to 0$. We take this as our definition for an approximate heat kernel or order m: a time dependent section $w_t(p, q)$, such that $\left(\partial_t + D_p^2\right) w_t(p, q) = t^m e(t, p, q)$, where $e(t, p, q)$ is a C^m section which is C^0 in t. They are approximations in the following sense, which we will prove separately: If k_t is the full heat kernel, then for every m, there exists $m' \geq m$ such that for all approximate heat kernels k'_t of order m', we have $k_t(p, q) - k'_t(p, q) = t^m e_t(p, q)$ where e_t is a C^m section and is C^0 in t. Clearly, once we have shown this, the estimates for the asymptotic expansion will follow. ◻

We have thus constructed an asymptotic expansion of our heat kernel

$$k_t(p,q) \sim h_t(p,q) \sum_{i=0}^{\infty} t^i \Theta_i(p,q) \tag{1.1.11}$$

We will show that this expansion determines the singular behaviour of the zeta function of D^2, which in turn, allows us to define a determinant. We have still to prove the final estimate in the proof of proposition (6). To do this, we need an important first result.

Lemma 4 (Duhamel's Principal)**.** *Let s_t be a C^2 section which is C^0 in t. Let r_t be the smooth section, C^1 in t with $r_0 = 0$, given by*

$$r_t = \int_0^t e^{-(t-t')D^2} s_{t'} dt'$$

Then we have

$$\left(\partial_t + D^2\right) r_t = s_t$$

and r_t is the unique solution to this equation. Furthermore, there are Sobolev estimates of the form

$$\|r_t\|_k \leq t\, C_k \sup\{\|s_{t'}\|_k : 0 \leq t' \leq t\}$$

Proof. We differentiate the formula for r_t to get $\partial_t r_t = s_t - \int_0^t D^2 e^{-(t-t')D^2} s_{t'} dt' = s_t - D^2 r_t$. Uniqueness follows from the proof given for the ordinary heat equation [37]. Its not hard to show that the operator e^{-tD^2} in uniformly bounded operator on the space L^2 sections, and it commutes with D^k, we know that it is also bounded on each Sobolev space. Thus

$$\begin{aligned}
\left\| \int_0^t e^{-(t-t')D^2} s_{t'} dt' \right\|_k &\leq t \sup\{\|e^{-(t-t')D^2} s_{t'}\|_k : 0 \leq t' \leq t\} && (1.1.12)\\
&\leq t \sup\{C_k \|s_{t'}\|_k : 0 \leq t' \leq t\} && (1.1.13)
\end{aligned}$$

where the last line follows from the uniform boundedness. ◻

We can now conclude the last section of the proof of (6).

Lemma 5. *For every m, there exists an approximate heat kernel k'_t to the full heat kernel k_t, of order $m' \geq m$, such that*

$$k_t(p,q) - k'_t(p,q) = t^m e_t(p,q)$$

where e_t is a C^m section of $S \boxtimes S^*$, C^0 in $t \geq 0$.

Proof. We follow [37]. By definition, the approximate heat kernel k'_t satisfies $(\frac{\partial}{\partial t} + D_p^2) k'_t(p,q) = t^m r_m(p,q)$, where r_m is a C^m error term, and tends to a delta function as $t \to 0$. By lemma (4), the inhomogeneous heat equation

$$\left(\tfrac{\partial}{\partial t} + D_p^2\right) s_t(p,q) = -t^m r_t(p,q)$$

has a unique solution $s_t(p,q)$, with $s_0(p,q) = 0$. The sum $k'_t(p,q) + s_t(p,q)$ then satisfies the heat equation, and thus by uniques of the heat kernel, must equal the heat kernel, i.e. $k'_t(p,q) + s_t(p,q) = k_t(p,q)$. Now using the estimate given in lemma (4), we have

$$\|s_t\|_\ell \leq tC_\ell \sup\{\|t^m r_t(p,q)\|_\ell : 0 \leq t' \leq t\}$$

for some constant C. If we take $m' = \ell > m + \frac{1}{2}\dim M$, then the Sobolev embedding theorem says that s_t/t^m is C^m, and the result follows. □

1.1.4 The Trace of the Heat Kernel

The next step in our process to define the determinant, we must consider the trace of our heat operator. Unlike the finite dimensional situation, not all bounded operators have a well defined trace, and it not even true that there is a unique notion of 'trace'. We will specify the class of operators for which we can take a trace. In finite dimensions, the trace is given as the sum of the eigenvalues. In infinite dimensions we should be looking for operators with 'summable eigenvalues', however we need to be careful with this notion because the spectral theory is quite subtle. We begin with the notion of operators which we can loosely thing of as having 'absolutely square summable eigenvalues'. We continue to follow the excellent treatment in [37].

Definition 6. *An operator $T \in B(H, H')$ between separable Hilbert spaces H, H', with orthonormal basis $\{e_i\}, \{e'_j\}$, is called Hilbert-Schmidt if it is finite in the norm*

$$\|T\|^2_{HS} := \sum_{i,j} |\langle Te_i, e'_j\rangle|^2$$

This norm is basis independent, since by Parseval's theorem, we have

$$\begin{aligned} \sum_{i,j} |\langle Te_i, e'_j\rangle|^2 &= \langle Te_i, e'_j\rangle\overline{\langle Te_i, e'_j\rangle} \\ &= \sum_{i,j}\langle Te_i, e'_j\rangle\langle e'_j, Te_i\rangle \\ &= \sum_i \|Te_i\| \end{aligned}$$

Which is clearly independent of the basis for H', and we also have $\langle Te_i, e'_j\rangle = \overline{\langle T^*e'_j, e_i\rangle}$, so $\|T\|_{HS} = \|T^*\|_{HS}$ which is independent of the basis for H. So we can think of Hilbert-Schmidt operators as those such that the eigenvalues of the symmetric operator T^*T are absolutely summable. It can easily be shown that the Hilbert-Schmidt operators form a two-sided ideal in the algebra of bounded operators, and form a Hilbert space with the inner product $\langle B, A\rangle_{HS} := \sum_i (Be_i, Ae_i)$.

Definition 7. *A bounded operator Q is of* trace class *if it can be expressed as $Q = AB$, where A, B are Hilbert-Schmidt operators. For such operators, we define the* trace *as* $\operatorname{Tr} Q := \sum_i (Qe_i, e_i) =$

$\sum_i (Be_i, A^* e_i) = \langle B, A^* \rangle_{HS}$.

This definition is clearly independent of the basis used, but with a little work it can be shown to be independent of the decomposition $Q = AB$, c.f. [37]. We also need to know how this definition of the trace compares with the definition of the trace of finite rank operators.

Proposition 7 (Lidskii's Theorem)**.** *If T is of trace class, then* $\mathrm{Tr}(T)$ *is equal to the sum of the eigenvalues of T.*

remark. This theorem is easily proved if T is compact and self adjoint, as we are then free to choose an orthonormal basis by the spectral theorem for such operators. □

The trace class operators also form an ideal of the bounded operators, and this trace enjoys the 'co-cycle' property shared by its finite dimensional version

Lemma 6. *Let A, B be two Hilbert-Schmidt operators, or A is bounded and B is of trace class, then we have*

$$\mathrm{Tr}([A, B]) = 0$$

Proof. This is again an easy application of Parseval's theorem. □

We now want to show that our integral operators on manifolds defined via smooth kernels are of trace class

For an integral operator K with square-integrable kernel $k(x, y) \in L^2(M \times M, E \boxtimes E^*)$, we see

$$\begin{aligned} \|K\|_{HS}^2 &= \sum_i \int_{M \times M} \langle k(x,y) e_i(y), k(x,y) e_i(x) \rangle dx dy \\ &= \int_{M \times M} \mathrm{tr}(k(x,y)^* k(x,y)) dx \\ &< \infty \end{aligned}$$

where tr is the point-wise trace on endomorphisms of E, so K is Hilbert-Schimdt. So for $t > 0$, the heat kernel $K_t = e^{-tD^2}$ is Hilbert-Schmidt.

Proposition 8. *For all $t > 0$, the heat kernel on a compact manifold M is trace class. The trace is given by the formula*

$$\mathrm{Tr}(k_t) = \int_{x \in M} \mathrm{tr}(k_t(x, x)) dx$$

Proof. (We follow [5] Proposition 2.32) From the semi-group property (1), we have $K_t = (K_{t/2})^2$. By the previous result, we know that each $K_{t/2}$ is Hilbert-Schmidt, thus we have

$$\begin{aligned} \mathrm{Tr}(K_t) &= \langle K_{t/2}, K_{t/2}^* \rangle_{HS} \\ &= \int_{M \times M} \mathrm{tr}(k_{t/2}(x,y) k_{t/2}(y,x)) dx dy \\ &= \int_M \mathrm{tr}(k_t(x,x)) dx \end{aligned}$$

$\square$

Now we know that the heat operator is of trace class, we use the asymptotic expansion that we calculated earlier to yield an expansion for the trace. Recall that in a geodesic coordinate system around $p \in M$, the heat kernel has an asymptotic expansion (1.1.11)

$$\begin{aligned} k_t(x,y) &\sim h_t(x,y) \sum_{i=0}^{\infty} \Theta_i(x,y) t^i \\ &= (4\pi t)^{-n/2} e^{-d(x,y)^2/4t} \sum_{i=0}^{\infty} \Theta_i(x,y) t^i \end{aligned}$$

From this, we can deduce a very important trace expansion

Corollary 2.

$$\begin{aligned} \mathrm{Tr}(e^{-tD^2}) &= \int_M \mathrm{tr}(k_t(x,x)) \\ &\sim (4\pi t)^{-n/2} \sum_{i=0}^{\infty} a_i t^i \qquad \textit{as } t \to 0 \end{aligned}$$

where $a_i = \mathrm{Tr}(\Theta_i(x,x))$.

We also would like to know large t behaviour of the heat kernel. We can see that the largest eigenvalue of e^{-tD^2} that is not equal to one, is $e^{-t\lambda_1}$, where λ_1 is the smallest positive eigenvalue of D^2. Thus we expect that this determines the large t behaviour.

Proposition 9. *Let P_0 denote the projection onto the kernel of D^2, p_0 its Schwartz kernel, and k_t be the heat kernel. The following estimates hold*

$$\|k_t(x,y) - p_0(x,y)\|_\ell \leq C_\ell e^{-t\lambda_1/2}$$

where λ_1 is the smallest positive eigenvalue of H.

Proof. See [5], Proposition 2.37 $\square$

Thus we can see that we have the following 'large time' estimate on the trace of the heat kernel.

$$\mathrm{Tr}(e^{-tD^2} - P_0) \leq C_0 \mathrm{Vol}(M) e^{-t\lambda_1/2} \tag{1.1.14}$$

We will also need to analyse the dependence of the heat kernel trace $\mathrm{Tr}(e^{-tD^2})$ as we smoothly vary the operator $H = D^2$.

Lemma 7. *Let H_s be a smooth one-parameter family of generalised Laplacians on a vector bundle $\mathscr{E}$ over a compact manifold, then*

$$\frac{\partial}{\partial s} \mathrm{Tr}\left(e^{-tH_s}\right) = -t\, \mathrm{Tr}\left(\frac{\partial H_s}{\partial s} e^{-tH_s}\right) \tag{1.1.15}$$

Proof. For a moment, assume that H_s is a matrix valued family, then it is easy to show the formula

$$\frac{\partial}{\partial s}e^{-tH_s} = -\int_0^t e^{-(t-z)H_s}\frac{\partial H_s}{\partial s}e^{-zH_s}dz$$

holds. To show that this formula extends to generalised Laplacians, some more detailed analysis of the heat kernel and its estimates is required. We refer to the proof in ([5] Theorem 2.48) where it is referenced as 'Duhamel's Formula'. Knowing that this formula holds, we take the trace of both sides

$$\begin{aligned}\frac{\partial}{\partial s}\operatorname{Tr}\left(e^{-tH_s}\right) &= -\int_0^t \operatorname{Tr}\left(e^{-(t-z)H_s}\frac{\partial H_s}{\partial s}e^{-zH_s}\right)dz \\ &= -\int_0^t \operatorname{Tr}\left(\frac{\partial H_s}{\partial s}e^{-tH_s}\right)dz \\ &= -t\operatorname{Tr}\left(\frac{\partial H_s}{\partial s}e^{-tH_s}\right)\end{aligned}$$

We have made use of the cocycle property of the trace, as well as the semi-group property of the heat-kernel. □

1.2 The Zeta Function

Our goal is to define a notion of a determinant for certain elliptic operators. We have already gone to some lengths to define the trace of the heat kernel, to which we will now feed into the Mellin transform, yielding the Zeta function. It is from this function that we will be able to define our determinant.

In the finite dimensional analogue, we constructed the power sum of the eigenvalues $a(s) = \sum_\lambda \lambda^{-s}$. The Laurent expansion of this function at $s = 0$ contained the determinant as the coefficient of the first order term. To construct an analogous function for an elliptic operator like D^2, we will use the heat kernel. The first problem we encounter is that now the spectrum is countably infinite and divergent, so the naively defined function $q(s) = \sum_{\lambda\in\mathrm{Spec}(Q)} \lambda^{-s}$ can only possibly be convergent for $\Re(s) > 0$ if at all, and may contain some singularities on the real line other than the one at $s = 0$. Using the heat kernel and the Mellin transform, we can get a global meromorphic function on the complex plane, the *zeta function*, which agrees with the function $q(s)$ for $\Re(s) >> 0$. The second problem is that this meromorphic function might have a pole at $s = 0$, which prohibits us from having a well defined derivative there. We will see that for the operators we consider, the zeta function is in fact analytic at $s = 0$, and the determinant is well defined.

1.2.1 The Mellin Transform

The tool that converts the heat kernel trace of an operator into its zeta function, is the Mellin Transform. The key observation is that the asymptotic behaviour of the trace of the heat kernel as $t \to 0$ determines the pole structure of the zeta function over the whole complex plane. This correspondence is represented concisely as

$$\text{Asymptotic expansion of } \mathrm{Tr}(e^{-tP}) \text{ as } t \to 0 \text{ contains term of the form } t^{\beta}(\log t)^{\ell}$$
$$\Updownarrow$$
$$\Gamma(s)\zeta(P,s) \text{ contains pole of the form } (s+\beta)^{-(\ell+1)}$$

We state the full result without proof.

Proposition 10 ([20], 5.1)**.** *Let $f(t)$ be a function that satisfies the following,*

1. *$f(t)$ is holomorphic in a sector $V_{\theta_0} = \{re^{i\theta} \in \mathbb{C} : r > 0, |\theta| < \theta_0\}$*

2. *$f(t)$ decreases exponentially as $|t| \to \infty$*

3. *$f(t) = O(|t|^{\alpha}), \alpha \in \mathbb{R}$ as $t \to 0$ for $t \in V_{\delta}, \delta < \theta_0$.*

Then the Mellin transform

$$g(s) = \mathcal{M}\left[f(t); s\right] := \int_0^{\infty} t^{s-1} f(t) dt$$

is defined and holomorphic for $\Re(s) > -a$ and $g(c+i\xi)$ is $O(e^{-\delta|\xi|})$ for $|\xi| \to \infty$, and this estimate is uniform for c in compact intervals of $(-a, \infty)$. Furthermore, the asymptotic behaviour of $f(t)$ as $t \to 0$ determines the pole structure of $g(s)$. Specifically, if

$$f(t) \sim \sum_{j=0}^{\infty} \sum_{\ell=0}^{m_j} a_{j,\ell} t^{\beta_j} (\log t)^{\ell}, \quad \beta_j \nearrow \infty, \; m_j \in \mathbb{N}_0$$

uniformly in $t \in V_{\delta}, \delta < \theta_0$, then

$$g(s) \sim \sum_{j=0}^{\infty} \sum_{\ell=0}^{m_j} \frac{(-1)^{\ell} \ell! a_{j,\ell}}{(s+\beta_j)^{\ell+1}}$$

Remark 1. *The above results hold without much modification if $f(t)$ takes values in a Banach space, c.f. [20]*

We can now finally arrive at the definition of the zeta function of our generalised Laplacians.

Definition 8. *Let $Q : C^{\infty}(E) \to C^{\infty}(E)$ be a generalised Laplacian, and $P : L^2(E) \to L^2(E)$ the projection onto its kernel. The* zeta function *of Q is defined by*

$$\Gamma(s)\zeta(Q,s) := \mathcal{M}\left[\mathrm{Tr}(e^{-tQ} - P); s\right]$$

The estimates (2, 1.1.14) show that the function $f(t) = \mathrm{Tr}(e^{-tQ} - P)$ satisfies the requirements of the Mellin transform theorem.

Importantly, we can differentiate inside the transform to easily show the desired amount of differentiability at $s = 0$

Lemma 8. *$\zeta(Q,s)$ extends to a meromorphic function on $\mathbb{C}$, which is finite and differentiable at $s=0$, where*

$$\zeta(Q,0) = \begin{cases} (4\pi)^{-n/2} a_{n/2} - \dim\ker Q & \text{if } n \text{ even} \\ -\dim\ker Q & \text{if } n \text{ odd} \end{cases} \tag{1.2.16}$$

Proof. Using our asymptotic expansion of the heat kernel trace $\mathrm{Tr}(e^{-tD^2}) = (4\pi t)^{-n/2}\sum_{r=0}^{\infty} a_r t^r$, and our theorem on the Mellin transform (Thm 10) we have

$$\mathcal{M}\left[\mathrm{Tr}(e^{-tQ} - P); s\right] \sim (4\pi)^{-n/2} \sum_{r=0}^{\infty} \frac{a_r}{s + (r - n/2)} - s^{-1}\,\mathrm{tr}(P)$$

The possible poles coming from the terms in the sum are at the points $\{n/2 - r\}_{r=0}^{\infty}$. We can see a pole only exists at $s=0$ is $n/2$ is a positive integer, i.e. n is even. For n even, since $\Gamma(s)^{-1} = s + O(s^2)$, the pole from the the $a_{n/2}$ is canceled by the zero of the reciprocal gamma function, so the zeta function is finite at $s=0$. The extra regularity comes from the ability to differentiate inside the Mellin integral. □

With a little extra work, we can show that this definition agrees with the 'power sum' definition of the zeta function that we mentioned in the introduction. We mention this result only to complete the analogy between the finite and infinite dimensional cases, we do not use it in the following chapters. Note that the pole with the largest real part lies at $s = n/2$. Thus for every $c > n/2$, the zeta function is holomorphic in the half plane $\mathrm{Re}(s) \geq c$.

Lemma 9. *For $\mathrm{Re}(s) > n/2$, we have*

$$\zeta(Q,s) = \sum_i \lambda_i^{-s}$$

where λ_i is the set of the non-zero eigenvalues of Q.

Proof. See [37]. □

1.2.2 Regularised determinants

With the zeta function now defined and proven to be regular at zero, we are able to define our determinants

Definition 9. *The (zeta)-*regularised determinant *of an elliptic operator Q, is defined to be*

$$\det{}'(Q) = \exp\left(-\partial_s \zeta(Q,s)\right)_{s=0}$$

This expression should be thought of as a *regularised product* of the divergent set of non-zero eigenvalues of Q. We move straight on to an important basic example.

Example: The Laplacian on Functions on the Circle

Here we compute the regularised determinant of the scalar Laplacian on the circle of radius R.

The Laplacian $\Delta = -\frac{\partial^2}{\partial\theta^2}$, acts on the eigenvectors $f_n = e^{in\theta/R}, n \in \mathbb{Z}$, with eigenvalues $\lambda_n = (n/R)^2, n \in \mathbb{Z}$. The heat kernel in this case is given by a modified Jacobi theta function

$$\vartheta_R(x,t) = 2\sum_{n=0}^{\infty} e^{-tn^2/R^2}$$

The Mellin transform of which is the famous Riemann zeta function, which we can see from the large $\Re(s)$ expression

$$\begin{aligned} \zeta(D,s) &= 2\sum_{n>0}(n/R)^{-2s} \\ &= 2R^{2s}\sum_{n>0} n^{-2s} \\ &= 2R^{2s}\zeta(2s) \end{aligned}$$

The Riemann zeta function has a well known meromorphic expansion to the whole complex plane, which is analytic near the origin, with

$$\zeta(0) = -\tfrac{1}{2}, \quad \zeta'(0) = -\tfrac{1}{2}\log 2\pi$$

So we confirm that $\zeta(D,s)$ is finite at the origin and

$$\det{}'(D) = \exp(-\zeta'(D,0)) = (2\pi R)^2$$

Thus we see that in the case where the full spectrum is known, we can possibly produce an explicit value for the regularised determinant. We will come back to this example later when we consider small analytic deformations of the Laplacian, and see how that effects the determinant.

1.3 Heat Kernel Expansions with Auxiliary Operators.

So far have shown how to obtain asymptotic expansions for the heat kernel along the diagonal

$$k(x,x) \sim \sum_{k=-N/2}^{\infty} A_k(x)t^k$$

In the next chapter, we will show that this result will allow us to define the analytic torsion for $\mathbb{Z}$-graded complexes, as was first done by Ray and Singer. However for $\mathbb{Z}_2$ graded case we need to find a more general type of asymptotic expansion, the study of which references the key works of index theory.

In [2], the index problem for elliptic operators on manifolds with boundary was investigated. It was shown that to guarantee that the Dirac operator involved was Fredholm, the domain would need to be restricted to satisfy a certain spectral boundary conditions, known as *Atiyah-Patodi-Singer* boundary conditions. If an elliptic operator Q is acting on some Hilbert space of sections $\mathcal{H}$, we denote by Q_B the same operator restricted to act on the space of sections $\mathcal{H}_B \subset \mathcal{H}$ satisfying some boundary condition B. Then the analysis of the index problem leads to studying the asymptotic behaviour of heat kernel traces of the form

$$\operatorname{tr}_{\mathcal{H}_B} e^{-tQ_B} = \operatorname{tr}_{\mathcal{H}} P_B e^{-tQ}$$

where P_B is the projection operator onto the subspace $\mathcal{H}_A$. We call this a *heat kernel trace with auxiliary operator* P_B. In general, we allow the projection P_B to be replaced by an arbitrary pseudo-differential auxiliary operator A, and take Q to be an elliptic operator order $m \in \mathbb{Z}$. The presence of this auxiliary operator complicates the calculation of the asymptotic expansion. To study these vastly more general type of heat kernel trace expansions, we follow the the approach of Grubb [18], where detailed analysis of the operator resolvent is used. We aim to outline the key steps in the proof of the crucial asymptotic expansion:

Proposition 11 ([18, 19]). *Let A be a* ΨDO *of order $\omega \in \mathbb{R}$ on a smooth manifold M^n, and P be a elliptic differential operator of order $m \in \mathbb{Z}$. Then there is an asymptotic expansion*

$$\operatorname{Tr}(Ae^{-tP}) \sim \sum_{j \in \mathbb{N}_0} c_j t^{\frac{j-\omega-n}{m}} + \sum_{k \in \mathbb{N}_0} (-c'_k \log t + c''_k) t^k$$

as $t \to 0^+$.

Remark 2. *The remarkable new feature in this expansion is the appearance of the logarithmic term $-c'_0 \log t$, which because of (10) implies certain* double *poles in the Mellin transform of this trace, particularly at $s = 0$. If the coefficient of this double-pole term does not vanish, there will be an obstruction to defining a regularised determinant.*

We will outline the proof of this in several steps, referencing several key results from [19] and classical results from the calculus of pseudodifferential operators, many of which can be found in the book of Shubin [41]. Instead of working with with the heat kernel techniques developed earlier in this chapter, we will prove an asymptotic expansion for the resolvent-like operator $A(P-\lambda)^{-N}$, from which we can deduce heat kernel asymptotics via a different integral transform. In the subsequent chapter, we will need to use this more sophisticated result to show that the determinants of the $\mathbb{Z}_2$-graded operators that we will be studying have well defined determinants. The proof here is very local in nature, using calculations in a certain pseudo-differential calculus. At present the author does not know of a proof of this result that has the geometric nature of the earlier proof due to Roe of the case without the auxiliary factor given, it is for this reason that we include both approaches.

1.3.1 Definitions and Symbol Classes

For the sake of brevity, we refer to Shubin [41] for the basic theory of pseudo-differential calculus, and we try to only highlight the unique elements found in the proof of theorem 1.3.21. We recall that a function $p \in C^\infty(\mathbb{R}^\nu \times \mathbb{R}^m)$ belongs to the classical symbol class $S^m(\mathbb{R}^\nu, \mathbb{R}^m)$ if satisfies the growth condition

$$\partial_x^\beta \partial_\xi^\alpha p(\xi, x) = O(\langle \xi \rangle^{m-|\alpha|})$$

Such a symbol is called *classical* (polyhomogeneous) of degree m if there exists a series of functions $p_j(x, \xi)$, with p_j homogenous in ξ of degree $m - j$, such that

$$p - \sum_{0 \leq j < J} p_j \in S^{m-J}$$

for each J. The method deployed in [19] was to describe the behaviour of the symbol of the resolvent of an operator Q, i.e. the symbol of $R(Q, \lambda) := (Q - \lambda)^{-1}$ defined for λ outside the spectrum of Q. By the ΨDO calculus, the symbol $r(x, \xi)$ of the resolvent has leading term

$$r_0(x, \xi) = (q(x, \xi) - \lambda)^{-1}$$

i.e. for the Laplace-type operator $q(x, \xi) = |\xi|^2$ we have

$$r_0(x, \xi) = (|\xi|^2 - \lambda)^{-1} = -\lambda^{-1}(1 + \tfrac{\lambda}{|\xi|^2} + \tfrac{\lambda^2}{|\xi|^4} + \dots)$$

which we see is polyhomogeneous in ξ, but each term in the expansion contains an *increasing* powers of λ. A symbol class was described in [19] that can handle this type of parameter dependent symbol. We use the notational convieniences $\langle \xi \rangle = (1 + |\xi|)^{1/2}$, and $(\xi, \mu) = (|\xi| + |\mu|)^{1/2}$.

Definition 10. *Let $\Gamma \subset \mathbb{C}/\{0\}$, be a sector of the form $\{re^{i\theta} : r > 0, \theta \in I \subset [0, 2\pi]\}$. The symbol space $S^{m,0}(\mathbb{R}^\nu, \mathbb{R}^n, \Gamma)$ is defined to consist of functions $p(x, \xi, \mu) \in C^\infty(\mathbb{R}^\nu \times \mathbb{R}^n \times \Gamma)$ that satisfy the following properties*

- *$p(x, \xi, \mu)$ is holomorphic in μ on the interior of Γ, for all $(\xi, \mu) \geq \varepsilon > 0$.*
- *For $z^{-1} \in \Gamma$, we have $\partial_z^j p(\cdot, \cdot, z^{-1}) \in S^{m+j}$ for each $j \in \mathbb{Z}$*

We also set $S^{m,d} = \mu^d S^{m,0}$, i.e. $p \in S^{m,d}$ satisfies $\partial_z^j (z^d p(\cdot, \cdot, z^{-1})) \in S^{m+j}$. Let $p_j \in S^{m_j,d}$, $j \in \mathbb{N}$ be a sequence of symbols, where $m_j \to -\infty$. Then we write

$$p \sim \sum_{p \in \mathbb{N}} p_j \text{ in } S^{\infty,d}$$

to mean that $p - \sum_{0 \leq j < J} p_j \in S^{m_J,d}$ for all J. Consequentially, if $p_j = 0$ for all $j > M$, then $p \in S^{M,d}$.

With these definitions, we can easily show the following rule for point-wise products

$$S^{m,d}(\mathbb{R}^\nu, \mathbb{R}^n, \Gamma) \cdot S^{m',d'}(\mathbb{R}^{\nu'}, \mathbb{R}^n, \Gamma) \subset S^{m+m',d+d'}(\mathbb{R}^{\nu+\nu'}, \mathbb{R}^n, \Gamma)$$

We will need some basic analytical results from [19] describing the properties of this symbol class.

Lemma 10 ([19])**.** • *The spaces $S^{m,d}$ are Frechet under the family of semi-norms*

$$p \to \sup\{\langle\xi\rangle^{-m-j+|\alpha|}|\partial_x^\beta \partial_\xi^\alpha \partial_z^j (z^d p(x,\xi,z^{-1}))| : x \in K \text{ compact in } \mathbb{R}^\nu, \xi \in \mathbb{R}^n, z^{-1} \in \Gamma\}$$

- $S^m(\mathbb{R}^\nu, \mathbb{R}^n) \subset S^{m,0}(\mathbb{R}^\nu, \mathbb{R}^n, \mathbb{C})$

We can now define the type of symbols that will be of interest to us.

Definition 11. *A symbol $p \in S^{\infty,d}$ is said to be* (weakly) polyhomogeneous *(wphg) if there exists a sequence of symbols $p_j \in S^{m_j-d,d}$, with p_j homogeneous in (ξ,μ) for $|\xi| > 1$ of degree $m_j \searrow -\infty$, such that $p \sim \sum_{j=0}^\infty p_j \in S^{\infty,d}$.*

For a symbol p of the above type, we write $p \in S^{\infty,d}_{whpg}(\mathbb{R}^\nu, \mathbb{R}^n, \Gamma)$ with degrees $\{m_j\}$ and μ-exponent d.

These symbols will be used to model the type of parameter dependence that we saw in our basic resolvent-like symbol expansions. The key observation of [19], is that these symbols posess a *second* kind of asymptotic expansion.

Theorem 1 ([19], Thm 1.12)**.** *Let $p \in S^{m,d}(\mathbb{R}^\nu, \mathbb{R}^n, \Gamma)$, then there exists an expansion in powers of μ, such that for each $N \in \mathbb{Z}$*

$$p(x,\xi,\mu) - \sum_{0 \le k < N} \mu^{d-k} p^{(k)}(x,\xi) \in S^{m+N,d-N}(\mathbb{R}^\nu, \mathbb{R}^n, \Gamma)$$

where

$$p^{(k)}(x,\xi) = \tfrac{1}{k!}\partial_z^k (z^d p(x,\xi,z^{-1}))|_{z=0}$$

We can now state the asymptotic expansion theorem for the operator kernels of polyhomogeneous symbols

Theorem 2 ([19] Thm 2.1)**.** *Let $p \sim \sum_{j\in\mathbb{N}} p_j \in S^{\infty,d}_{wphg}(\mathbb{R}^n, \mathbb{R}^n, \Gamma)$ with degrees $\{m_j\}$ and μ-exponent d. Furthermore, assume that all p_j with $m_j - d \ge -n$ are in some $S^{m',d'}$, form some $m' < -n, d' \in \mathbb{R}$. Then the operator $OP(p)$ has a kernel $K_p(x,y,\mu)$ with asymptotic expansion along the diagonal*

$$K_p(x,x,\mu) \sim \sum_{j=0}^\infty c_j(x)\mu^{m_j+n} + \sum_{k=0}^\infty [c'_k(x)\log\mu + c''_k(x)]\mu^{d-k}$$

for $|\mu| \to \infty$ uniformly for μ in closed subsectors of Γ.

Remark 3. *We will later show the the coefficients $c_j(x)$ and $c'_j(x)$ for all j are* locally *determined from the homogeneous terms in the symbol expansion p_j, where as the terms $c''_k(x)$ are non-local.*

Proof. We follow [19] closely. First, define $p'(x,\xi,\mu) = \mu^{-d}p(x,\xi\,\mu)$, so $p' \in S^{\infty,d}_{wphg}(\mathbb{R}^n,\mathbb{R}^n,\Gamma)$ with degrees $m_j - d$ and μ-exponent 0. Its easy to see that it suffices to who the theorem for p', and the resulting kernel expansion for p differs just by an overall factor of μ^d. So assume $d = 0$. The hypothesis placed on the symbols ensures that each of the p_j are integrable in ξ for each μ, and so they define continuous kernels

$$K_{p_j}(x,y,\mu) = \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} p_j(x,\xi,\mu)d\xi$$

Similarly, we have the remainder symbol $r_J = p - \sum_{j<J} p_J$, and its associated kernel K_{r_J}. We first consider this remainder term. We use the μ-expansion of theorem (1) to express this remainder symbol as

$$r_J = \sum_{0\le k<N} s_k(x,\xi)\mu^{-k} + O(\langle\xi\rangle^{m_J+N}\mu^{-N})$$

for any given N. We take J large enough so that $m_J + N < -n$, so that we can ensure integrability of the error term. When we restrict to the diagonal, the remainder kernel satisfies

$$\begin{aligned} K_{r_J}(x,x,\mu) &= \int_{\mathbb{R}^n} r_J(x,\xi,\mu)d\xi \\ &= \sum_{0\le k<N}\left(\int_{\mathbb{R}^n} s_k(x,\xi)d\xi\right)\mu^{-k} + O(\mu^{-N}) \end{aligned}$$

These terms will be contributing to the c''_k terms in our expansion.

Now we analyse the contribution of a homogenous term p_j in our symbol expansion. As usual, consider $p_j \in S^{m_j,0}$ a term homogeneous in (ξ,μ) for $|\xi| \ge 1$ of degree m_j. For the kernel restricted to the diagonal, we split the contribution in to three terms

$$\begin{aligned} K_{p_j}(x,x,\mu) &= \int_{\mathbb{R}^n} p_j(x,\xi,\mu)d\xi \\ &= \left(\int_{|\xi|\le 1} + \int_{1<|\xi|\le|\mu|} + \int_{|\xi|>|\mu|}\right) p_j(x,\xi,\mu)d\xi \end{aligned}$$

We first take the $|\xi| > |\mu|$ term. Recall that p_j was homogeneous for $|\xi| > 1$, so for $|\mu| > 1$, we find a contribution of the form

$$\begin{aligned} \int_{|\xi|>|\mu|} p_j(x,\xi,\mu)d\xi &= \int_{|\xi|>|\mu|} |\mu|^{m_j} p_j(x,\xi/|\mu|,\mu/|\mu|)d\xi \\ &= \mu^{m_j+n}\int_{|\xi'|>1} (\mu/|\mu|)^{-m_j-n} p_j(x,\xi',\mu/|\mu|)d\xi' \end{aligned}$$

A priori, The integrand in the above formula depends on $\mu/|\mu|$, but a short argument using the analyticity of p_j in μ, shows that in fact it is independent of $\arg\mu$. Thus this large ξ term contributes a coefficient of the form μ^{m_j+n}. For the small ξ term, $|\xi|<1$, we again use the expansion theorem (1) to write

$$p_j(x,\xi,\mu) = \sum_{0\le k<M} \mu^{-k} p_j^{(k)}(x,\xi) + R_M(x,\xi,\mu) \tag{1.3.17}$$

where $p_{j,k}(x,\xi) = \frac{1}{k!}\partial_z^k p_j(x,\xi,z^{-1})|_{z=0} \in S^{m_j+k}$ is homogeneous in ξ for $|\xi|>1$, and $R_M = O(\langle\xi\rangle^{m_j+M}\mu^{-M})$. So we have contribution

$$\int_{|\xi|<1} p_j(x,\xi,\mu)d\xi = \sum_{0\le k<M} \mu^{-k}\left(\int_{|\xi|<1} p_j^{(k)}(x,\xi)d\xi\right) + O(\mu^{-M})$$

For the choices of N, J so far, we only need to take $M>N$ for each p_j with $j<J$.

The third integral over the intermediate region, $1\le|\xi|<|\mu|$, gives a more complicated contribution. We are still in the homogeneous range, $|\xi|>1$, and so each term in the expansion (1.3.17) as well as the remainder term are also homogeneous of degree m_j. For each $p_{j,k}$ in the expansion we have a contribution of the form

$$\begin{aligned}
\mu^{-k}\int_{1\le|\xi|<|\mu|} p_j^{(k)}(x,\xi)d\xi &= \int_{1\le|\xi|<|\mu|} p_j^{(k)}(x,\xi/|\xi|)|\xi|^{m_j+k}d\xi \\
&= \mu^{-k}\int_{|\xi|=1} p_j^{(k)}(x,\xi)d\xi \int_{1\le r<|\mu|} r^{m_j+k+n-1}dr
\end{aligned}$$

performing the final integral gives

$$\int_{1\le r<|\mu|} r^{m_j+k+n-1}dr = \begin{cases} \frac{|\mu|^{m_j+k+n}-1}{m_j+k+n} & \text{if } m_j+k+n\neq 0, \\ \log|\mu| & \text{if } m_j+k+n=0, \end{cases}$$

Thus, up to factors that depend on $\mu/|\mu|$, we have the contributions

$$\begin{aligned}
\int_{1\le|\xi|<|\mu|} p_j(x,\xi,\mu)d\xi &= \sum_{0\le k<M}\mu^{-k}\left(\int_{1\le|\xi|<|\mu|} p_j^{(k)}(x,\xi)d\xi\right) + O(\mu^{-M}) \\
&= (\mu^{m_j+n}\log|\mu|)\int_{|\xi|=1} p_{j,-m_j-n}(x,\xi)d\xi \\
&+ \sum_{0\le k<M,\, k\neq -m_j-n} \frac{\mu^{m_j+n}-\mu^{-k}}{m_j+k+n}\int_{|\xi|=1} p_j^{(k)}(x,\xi)d\xi
\end{aligned}$$

Where the first term in this sum only exists if $m_j + n \in \mathbb{Z}$. Lastly, for the remainder term

$$\begin{aligned} R_M(x,\xi,\mu) &= |\mu|^{m_j} R_M(x,\xi/|\xi|,\mu/|\xi|) && (1.3.18) \\ &= |\mu|^{m_j} O((\mu/|\xi|)^{-M}) && (1.3.19) \\ &= O(|\mu|^{m_j+M}\mu^{-M}) && (1.3.20) \end{aligned}$$

Combining these three separate contributions, we find

$$\int_{\mathbb{R}^n} p_j(x,\xi,\mu)d\xi = c_j(x)\mu^{m_j+n} + c_j'(x)\mu^{m_j+n}\log\mu + \sum_{k=0}^{M-1} c_{j,k}(x)\mu^{-k} + O(\mu^{-M})$$

where

$$\begin{aligned} c_j(x) &= \int_{|\xi'|>1} (\mu/|\mu|)^{-m_j-n} p_j(x,\xi',\mu/|\mu|)d\xi' \\ &\quad + \sum_{0\le k<M,\, k\neq -m_j-n} \frac{1}{m_j+k+n}\int_{|\xi|=1} p_j^{(k)}(x,\xi)d\xi \\ c_j'(x) &= \int_{|\xi|=1} p_{j,-m_j-n}(x,\xi)d\xi \\ c_{j,k}''(x) &= -\frac{1}{m_j+k+n}\int_{|\xi|=1} p_j^{(k)}(x,\xi)d\xi \\ &\quad + \int_{|\xi|<1} p_j^{(k)}(x,\xi)d\xi \\ &\quad + \int_{\mathbb{R}^n} s_k(x,\xi)d\xi \end{aligned}$$

Here we can see that the coefficients c_j and c_j' are determined from p_j for $|\xi| > 1$ and are thus *local*. Since the remainder terms s_k contribute to $c_{j,k}''$ it is *non-local*. □

1.3.2 Resolvent Kernel Expansions

The above kernel expansion theorem was for an arbitrary wphg symbol, we now specialise to the case of the resolvent, to which this formalism was applied in detail in [18].

Lemma 11. *Let P be a pseudo-differential (differential) of order $m \in \mathbb{Z}$. Then the symbol of $(P + \mu^m)^{-N}$ is weakly (strongly) polyhomogenous in (ξ,μ) i.e.*

$$\sigma((P+\mu^m)^{-N}) \in S^{-mN,0} \cap S^{0,-mN}$$

Proof. This result was the motivating reason for the construction of such a symbol class. We refer to the full proof in [18] □

We now allow a pseudo-differential factor A of order $\nu \in \mathbb{R}$, and consider the operator $Q(\mu) = A(P + \mu^m)^{-N}$.

As usual, we localise the symbol of $Q(\mu)$ so we only need to consider locally supported symbols. As we described previously, the symbol $q(x,\xi,\mu) = \sigma(Q(\mu)) \in S^{\nu-mN,0} \cap S^{\nu,-mN}$ has an expansion

$$q(x,\xi,\mu) \sim \sum_{k\geq 0} q^{(mk)}(x,\xi)\mu^{-m(N+k)} + O(\langle\xi\rangle^{\nu-mL}\mu^{-m(N+L)})$$

where the $q^{(mk)}(x,\xi)$ are homogeneous of degree $\nu - mk$, for $|\xi| > 1$. This expansion in powers of μ corresponds to the operator expansion

$$\begin{aligned} A(P+\mu^m)^{-N} &= \mu^{mN} A(I+\mu^{-m}P)^{-N} \\ &= A\sum_{0\leq k<L} \binom{-N}{l} \mu^{-m(N+k)}P^k + O(\mu^{-m(N+L)}) \end{aligned}$$

which is why only powers of μ^m appear. Now perform the usual homogeneous expansion

$$q(x,\xi,\mu) = \sum_{j\geq\mathbb{N}} q_{\nu-mN-j}(x,\xi,\mu)$$

where $q_{\nu-mN-j}$ is homogeneous in (ξ,μ) of degree $\nu - mN - j$. Now apply theorem (2) to this symbol to yield

Theorem 3 (Grubb, Seeley [18, 19]). *The diagonal of the localised kernel $K(Q(\mu),x,y)$ for the resolvent type operator $Q(\mu) = A(P+\mu^m)^{-N} = A(P-\lambda)^{-N}$ has an asymptotic expansion for $N > (\nu+n)/m$*

$$K(Q(\mu),x,x) \sim \sum_{j=0}^{\infty} \tilde{c}_j(-\lambda)^{\frac{\nu+n-j}{m}-N} + \sum_{k=0}^{\infty}[\tilde{c}'_k \log(-\lambda) + \tilde{c}''_k](-\lambda)^{-k-N}$$

We have thus arrived at a useful asymptotic expansion for a resolvent type operator $A(P+\mu^m)^{-N}$. Recall that we were interested in obtaining an expansion for the heat kernel Ae^{-tP}. We will see that these two expansions are related by a simple integral transform, much like how the Mellin transform relates the heat-kernel expansion, and the zeta-function expansion.

1.3.3 The Resolvent Trace and the Zeta Function

So far, we have seen an asymptotic expansion for the trace of the resolvent-type expression

$$Q(\mu) = \mathrm{Tr}(A(P-\lambda)^{-N})$$

What we required for our definition of the regularised determinant was an asymptotic expansion of the Zeta function. Luckily, there is an easy transition between the This transition was done in [GS2]. Let A, P be operators as in theorem (3) above, we have shown that there exists an asymptotic expansion (for $N > (n+\nu)/m$)

$$\mathrm{Tr}(A(P-\lambda)^{-N}) \sim \sum_{j=0}^{\infty} \tilde{c}_j (-\lambda)^{\frac{\nu+n-j}{m}-N} + \sum_{k=0}^{\infty} [\tilde{c}'_k \log(-\lambda) + \tilde{c}''_k](-\lambda)^{-k-N} \tag{1.3.21}$$

Proposition 12 ([19]). *The expansion (1.3.21) is equivalent to the following expansion for the zeta function.*

$$\Gamma(s)\zeta(A,P,s) \quad \sim \quad \sum_{j\geq 0} \frac{c_j}{s+\frac{j-\nu-n}{m}} - \sum_{k\geq 0}\left(\frac{c''_k}{(s+k)} + \frac{c'_k}{(s+k)^2}\right) \tag{1.3.22}$$

where for each k, the coefficient c_k are proportional to the coefficient $\tilde{c}_k$ by a universal non-zero constant, and the pair $\{c'_k, c''_k\}$ are linearly related to the pair $\{\tilde{c}'_k, \tilde{c}''_k\}$.

Proof. We need to show that a term of the kind $(-\lambda)^{k-N}\log^{\sigma}(-\lambda)$ in the expansion of the expansion of $\mathrm{Tr}(A(P-\lambda)^{-N})$ leads to a term of the form $(s-k)^{-\sigma}$ in the expansion of $\Gamma(s)\zeta(A,P,s)$. We prove this in the case $N=1$, for example $P = d^2/d\theta^2$ acting on L^2 functions on the circle, and the extension to all N follows via differentiation. Suppose that P is a closed operator such that the resolvent $R_P(\lambda) = (P-\lambda)^{-1}$ that is meromorphic at $\lambda = 0$, and holomorphic in some sector $-\alpha < \arg(-\lambda) < \alpha$, furthermore we require $\|R_P(\lambda)\| = O(|\lambda|^{-1})$. We consider the following operator valued contour integral, which can be used as a definition for the power function P^s.

$$P^s = -(2\pi i)^{-1} \int_C \lambda^{-s} R_P(\lambda) d\lambda$$

where C is the curve

$$\begin{aligned} C \quad &= \quad C_{\theta,r_0} = \{re^{i\theta} : \infty > r > r_0\} \cup \{r_0 e^{i\theta'} : \theta \geq \theta' \geq -\theta\} && (1.3.23)\\ &\cup\{re^{i(2\pi-\theta)} : r_0 \leq r < \infty\} && (1.3.24)\end{aligned}$$

where θ is chosen so that $\pi - \alpha < \theta \leq \pi$, and r_0 large enough so that $R_Q(\lambda)$ is holomorphic for $0 < |\lambda| <\leq r_0$. The required identifies follow easily from evaluating the contour integrals on each of the terms that appear in the asymptotic exapansion. As we mentioned, to extend to general N, we just differentiate these expressions, using the obvious identities $d/d\lambda(P-\lambda)^{-N} = -N(P-\lambda)^{-(N+1)}$ □

Thus we have determined that the zeta function $\zeta(A,P,s)$ has the following Laurent expansion at $s=0$

Corollary 3.

$$\zeta(A,P,s) = \Gamma(s)^{-1}\mathcal{M}\left[\mathrm{tr}(Ae^{-tP}); s\right] = c'_0 s^{-1} + (c_{\nu+n} + c''_0) + O(s) \tag{1.3.25}$$

As expected, we find that the logarithmic term in the heat kernel expansions lead to the possibility of a singularity in the zeta function at $s = 0$. Next we will show that the coefficient of this singular term has some very interesting properties, and in the case we consider, a pseudodifferential projection, we show that it vanishes and thus the zeta function is non-singular.

1.3.4 The Non-commutative Wodzicki-Guillemin Residue

We have seen that a logarithmic term appears in the trace expansion when we include a pseudodifferential prefactor. This term was investigated by Wodzicki where it was observed that it behaved like a trace on the space of pseudo-differential operators, but did not extend the usual trace on trace class operators. This term was independently investigated by V. Guillemin [21] at the same time.

Theorem 4 ([25, 45])**.** *Let A be a pseudo-differential operator, and P an elliptic differential operator. The* Non-commutative residue *of A*

$$\operatorname{res} A := \deg(P)\operatorname{Res}_{s=0}\zeta(A, P, s) = c_0' \tag{1.3.26}$$

is independent of the choice of elliptic operator P.

It was also shown that the residue is a trace (cyclic cocycle) on the algebra of pseudo-differential operators.

Theorem 5 ([45])**.** *For pseudo-differential operators A, B, we have*

$$\operatorname{res}[A, B] = 0 \tag{1.3.27}$$

Proof. for a proof of this, see [9]. □

In the examples we will be considering, A will be a ΨDO projection, i.e. $A^2 = A$.

Theorem 6 ([9, 45])**.** *If A is a pseudodifferential projection of order 0, then*

$$\operatorname{res} A = 0 \tag{1.3.28}$$

This theorem was stated in Wodzicki's thesis [45] without proof, here we follow the proof given in [9].

Proof. Here we prove this in the case where A is also self adjoint, the general case follows from a small perturbation argument c.f. [9]. Firstly, we want to show that there exists a first-order classical elliptic pseudodifferential operator P of order 1, such that

$$A = \frac{1}{2}(\operatorname{sgn} P - I)$$

where

$$(\operatorname{sgn} P)x = \begin{cases} |P|^{-1}Px & x \in (\ker P)^{\perp} \\ 0 & x \in \ker P \end{cases}$$

is the sign operator of P. Choose any positive first-order elliptic ΨDO Q with principal symbol $\sigma_1(Q)(\xi) = |\xi|$. Set

$$\tilde{P} = AQA - (I - A)Q(I - A)$$

Notice $\tilde{P}$ is elliptic and commutes with A. We now modify $\tilde{P}$ on its finite dimensional kernel as follows: define P by

$$Px = \begin{cases} \tilde{P}x & x \in (\ker P)^{\perp} \\ x, & x \in \ker P \cap \operatorname{im} A \\ -x, & x \in \ker P \cap \ker A \end{cases}$$

Then we check

$$\operatorname{sgn} P = \begin{cases} |AQ|^{-1}AQx & x \in (\ker P)^{\perp} \cap \operatorname{im} A \\ -|(A-I)Q|^{-1}(A-I)Qx & x \in (\ker P)^{\perp} \cap \ker A \\ x, & x \in \ker P \cap \operatorname{im} A \\ -x, & x \in \ker P \cap \ker A \end{cases}$$

Now since Q is positive, we have $|AQ|^{-1}AQx = x$ on $(\ker P)^{\perp} \cap \operatorname{im} A$. so we have

$$\operatorname{sgn} P = \begin{cases} x & x \in \operatorname{im} A \\ -x & x \in \ker A \end{cases}$$

Since A is idempotent, we see $\operatorname{sgn} P = 2A - I$, which is $A = \frac{1}{2}(\operatorname{sgn} P + I)$. We now construct the zeta function for the pair (A, P^2), and by using the linearity of the Mellin transform we find

$$\zeta(A, P^2, s) \quad = \quad \tfrac{1}{2}\left(\zeta(\operatorname{sgn} P, P^2, s) + \zeta(I, P^2, s)\right)$$

We already know the the second zeta function is regular at $s = 0$, so we find $\operatorname{res} A = \frac{\deg P^2}{2} \operatorname{res} \operatorname{sgn} P =$

res sgn P. To show that the first zeta function is regular, we write it out for large Re s

$$\begin{aligned}
\zeta(\operatorname{sgn} P, P^2, s) &= \Gamma(s)^{-1}\mathcal{M}\left[\operatorname{tr}((\operatorname{sgn} P)e^{-tP^2}); s\right] \\
&= \sum_{\lambda\in \operatorname{Spec} A/\{0\}} (\operatorname{sgn}\lambda)|\lambda|^{-2s} \\
&= \sum_{\lambda\in \operatorname{Spec} A/\{0\}} \lambda|\lambda|^{-(2s+1)} \\
&= \Gamma(s+\tfrac{1}{2})^{-1}\mathcal{M}\left[\operatorname{tr}(Pe^{-tP^2}); s+\tfrac{1}{2}\right] \\
&= \zeta(P, P^2, s+\tfrac{1}{2})
\end{aligned}$$

So the residue of $\zeta(\operatorname{sgn} P, P^2, s)$ at $s = 0$ is equal to the residue of $\zeta(P, P^2, s)$ at $s = \frac{1}{2}$. Historically, the zeta functions of this type are known as eta functions

$$\begin{aligned}
\eta(P, s) &:= \zeta(P, P^2, (s+1)/2) \\
&= \Gamma((s+1)/2)^{-1}\mathcal{M}\left[\operatorname{tr}(Pe^{-tP^2}); (s+1)/2\right] \\
&= \sum_{\lambda\in \operatorname{Spec} P/\{0\}} (\operatorname{sgn}\lambda)|\lambda|^{-s}
\end{aligned}$$

So, we have $\operatorname{res} A = \operatorname{Res}_{s=0}\eta(P, 2s) = \frac{1}{2}\operatorname{Res}_{s=0}\eta(P, s)$. So all that is left to do is to show that the eta function is regular at zero for elliptic pseudodifferential operators of positive order, the proof of which we omit, with a reference to the original paper. □

Theorem 7 (Atiyah-Bott-Patodi [3]). *For an elliptic pseudodifferential operator P of order $d \in \mathbb{N}$, we have*

$$\operatorname{Res}_{s=0}\eta(P, s) = 0 \tag{1.3.29}$$

Proof. For a detailed exposition of this result, see the monograph of Gilkey [16]. □

Combining all these facts, we arrive at the main result of this chapter

Corollary 4. *If A is a pseudodifferential projection of order 0, the zeta function $\zeta(A, P, s)$ is finite and differentiable at $s = 0$.*

This powerful result allows us to define regularised determinants for elliptic operators acting on the image of projection operators. We will see later that this is the essential analytical ingredient used by Mathai-Wu [28] to construct analytic torsion for $\mathbb{Z}_2$-graded complexes.

Chapter 2

Analytic Torsion

The study of torsion invariants of chain complexes began with Reidemiester, who was interested in classifying Lens spaces both up to homotopy and homeomorphism. This lead to the discovery of a certain homological invariants of finite dimensional acyclic chain complexes, now called the *Reidemeister torsion* and its slight generalisation, the *R-torsion*. These invariants, when computed for an acyclic simplicial model of the Lens spaces, were shown to provide such a classification. This work eventually lead to the discovery of the *Whitehead torsion*, a more refined torsion invariant, which was used by several authors to great success, c.f. [31]. Torsion invariants seek to generalise the notion of the determinant of a linear isomorphism, to the case of chain maps of simplicial complexes.

In [34], building on the work of Minakshisundaram [32], Ray and Singer proposed a smooth analogue of the R-torsion. They showed that many of the properties that characterised the R-torsion held for this analytical invariant, and the two torsions were later shown to be equal [6].

2.1 Algebraic Preliminaries

We begin here with a discussion of some of the background needed to define the R-torsion. The objects we will be studying are co-chain complexes and their invariants under chain homotopies.

For a co-chain complex $(C^\bullet, d)$, we can compute the cohomology groups $H^i(C,d)$. We think of these groups as sitting in a co-chain complex with all differentials given by the zero map. The dimensions of the cohomology groups are called the *Betti numbers*, $b_k = \dim H^k(C,d)$. These are the most basic examples of chain homotopy invariants. We can form the Poincaré polynomial $p_C(t) = \sum_k t^k b_k$, which neatly contains all of this information in a way that makes it compatible with products, i.e $P_{C\times C'}(t) = P_C(t)P_{C'}(t)$. The *Euler characteristic* of the complex is defined to be $\chi(C,d) := p(-1) = \sum_{i=0}^n (-1)^k b_k$, which is again a classical homological invariant.

Chain complexes are $\mathbb{Z}$-graded objects, each term in the complex has a integer degree. In the situations we will be studying we will need a slightly different concept: 2-periodic complexes. These are unbounded complexes of the form

$$\cdots \longrightarrow C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^0 \xrightarrow{d^0} C^1 \longrightarrow \cdots$$

which are naturally $\mathbb{Z}_2$-graded objects. We represent the complex given above as

$$\begin{tikzcd} C^0 \arrow[r, bend left, "d^0"] & C^1 \arrow[l, bend left, "d^1"] \end{tikzcd}$$

We can define the Euler characteristic of the folded complex to be

$$\chi(C^\bullet, D) := \dim H^0(C, D) - \dim H^1(C, D)$$

It is quite natural (c.f. [1]) to *fold up* a $\mathbb{Z}$-graded chain complex $(C^\bullet, d)$, by forming the 2-periodic complex $(\mathscr{C}^\bullet, D)$

$$\begin{tikzcd}[row sep=small] \mathscr{C}^0 \arrow[r, bend left, "D_0"] & \mathscr{C}^1 \arrow[l, bend left, "D_1"] \\ \| & \| \\ C^0 \arrow[dr, "d_0"] & \\ \oplus & C^1 \arrow[dl, "d_1"'] \\ C^2 \arrow[dr, "d_2"] & \oplus \\ \oplus & C^3 \\ \vdots & \vdots \end{tikzcd}$$

i.e

$$\mathscr{C}^0 = \oplus_i C^{2i}, \quad \mathscr{C}^1 = \oplus_i C^{2i+1}$$

$$D^0 = \oplus_i d_{2i}, \quad D_1 = \oplus_i d_{2i+1}$$

We occasionally write the differential as the odd endomorphism in matrix form.

$$D := \begin{pmatrix} 0 & D_0 \\ D_1 & 0 \end{pmatrix} \in \mathrm{End}^-(C)$$

We can see that with these definitions, the Euler characterisitc of the folder complex $\chi(\mathscr{C}, S)$ is clearly equal to the Euler characteristic of unfolded complex $\chi(C, d)$. Thus the Euler characteristic has a definition that extends naturally, i.e. is invariant under folding up, to $\mathbb{Z}_2$-graded complexes.

We will be using the language of superalgebra which is designed for the $\mathbb{Z}_2$-graded setting, a good exposition of which is found in [5]. We use boldface to indicate supertraces, superdeterminants etc. i.e.

$$\mathbf{tr} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \mathrm{tr}(A) - \mathrm{tr}(D)$$

2.1.1 The Determinant Line of a Complex

We follow [8]. All vector spaces are over $\mathbb{C}$. Given a finite dimensional vector space V, we define the *determinant line* as $\det V := \wedge^n V$, where $n = \dim V$. This has the property that for a linear map $f : V \to W$, the induced map $\det(f) : \det V \to \det W$ agrees with the usual definition of the determinant for $W = V$. Given a cochain complex $(C^\bullet, d)$ of finite dimensional complex vector spaces,

$$0 \longrightarrow C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{N-1}} C^N \longrightarrow 0$$

We define the determinant line as

$$\det C^\bullet := \bigotimes_{i=0}^{N} (\det C^i)^{(-1)^i}$$

Where $(\det V)^{-1} := (\det V)^*$. We write elements of the determinant line as

$$c_\bullet := \otimes_{i=0}^{N} c_j^{(-1)^i} \in \det C^\bullet$$

If we consider the cohomology $H^\bullet(C, d)$ as a complex with trivial differential

$$0 \longrightarrow H^0(C, d) \xrightarrow{0} H^1(C, d) \xrightarrow{0} \cdots \xrightarrow{0} H^N(C, d) \longrightarrow 0$$

then we can construct the determinant line of the cohomology $\det H^\bullet(C, d)$. We also need to use the canonical *fusion* isomorphisms

$$\mu_{V,W} : \det V \otimes \det W \to \det(V \oplus W)$$

$$\mu_{V,W} : (\wedge_i v_i) \otimes (\wedge_j w_j) \mapsto (\wedge_i v_i) \wedge (\wedge_j w_j)$$

and this construction is associative, i.e.

$$\mu(U, V \oplus W) \circ (1_{\det U} \otimes \mu(V, W)) = \mu(U \oplus V, W) \circ (\mu(U, V) \otimes 1_{\det W})$$

i.e. the map

$$\mu_{U,V,W} : \det U \otimes \det V \otimes \det W \to \det(U \oplus V \oplus W)$$

is well defined.

2.1.2 The Knudsen-Mumford Map

By construction, a cochain complex $(C^\bullet, \partial)$ and its cohomology $(H^i(C), 0)$ are chain homotopy equivalent as complexes, but not canonically as there is a choice of basis involved. To remove dependence on such a choice, we pass to the determinant line. The Knudsen-Mumford map is a *canonical* isomorphism

$\kappa : \det C^\bullet \to \det H^\bullet$, which we will now construct in detail. Choose a chain inclusion $\iota : H^\bullet \hookrightarrow C^\bullet$, i.e. $d \circ \iota = 0$, and a surjective chain map $P : C^\bullet \to H^\bullet$. i.e. $P \circ d = 0$, such that $P\iota = \mathbf{1}_H$. Then $Q := \iota P : C^\bullet \to C^\bullet$ is a projection, and so $C^\bullet \cong Q(C^\bullet) \oplus (\mathbf{1}_C - Q)(C^\bullet)$, and $Q(C^\bullet) \cong H^\bullet$, via P.

Now choose a chain homotopy T between Q and $\mathbf{1}_C$, i.e a linear map $T : C^\bullet \to C^{\bullet-1}$ such that $\mathbf{1}_C - Q = dT + Td$. Now its easy to see that each of Td, dT are disjoint projections. This data gives us an isomorphism

$$C^\bullet \cong Q(C^\bullet) \oplus dTC^\bullet \oplus TdC^\bullet$$

With $Q(C^\bullet) = H^\bullet$, and $dTC^\bullet \cong \operatorname{im} d$ and such that $d : TdC^\bullet \to dTC^{\bullet+1}$ is an isomorphism. We use the compact notation $C^\bullet \cong H^\bullet \oplus B^\bullet \oplus A^\bullet$. With such a homological decomposition of our chain complex, we now construct the Knudsen-Mumford map. Choose a sequence of volume elements $h_j \in \det(H^j)$, and a sequence of elements $a_j \in \det A^j$. We have shown that $\det(d^j) : \det A^j \to \det B^{j+1}$ is an isomorphism of lines, thus $\det(d^{j-1})a_{j-1} \in \det B^j$. The wedge product determines a map

$$\mu_j : \det B^j \otimes \det H^j \otimes \det A^j \to \det C^j$$

and thus

$$\mu_j((\det(d^{j-1})a_{j-1}) \otimes h_j \otimes a_j) \in \det C^j$$

and so if we choose an arbitrary element $c_j \in \det C^j$, there must be a linear relation of the form

$$v_j \times \mu_j((\det(d^{j-1})a_{j-1}) \otimes h_j \otimes a_j) = c_j, \quad v_j \in \mathbb{R}$$

i.e. if $\{a_{j,k}\}$ was a basis form A^j, $\{h_{j,k}\}$ for H^j, and $\{c_{j,k}\}$ for C^j, then v_j is scaling of the volume element induced by the change of basis $\{d^j a_{j-1,k}, h_{j,k}, a_{j,k}\}$ to $\{c_{j,k}\}$. In Milnor's notation [31], this is denoted $v_j = [d^j a_{j-1}, h_j, a_j/c_j]$. So if we let

$$c_\bullet = c_0 \wedge c_1^{-1} \wedge c_2 \wedge \ldots \in \det C^\bullet$$

and

$$vh_\bullet = v_0 h_0 \wedge (v_1 h_1)^{-1} \wedge v_2 h_2 \wedge \ldots \in \det H^\bullet$$

we define the *Knudsen-Mumford* map as

$$\kappa : c_\bullet \mapsto vh_\bullet \in \det H^\bullet \tag{2.1.1}$$

This map is named after its initial appearance in [23]. We can easily show that this map is independent of the choice of a_j. For if we had chosen a different element $a'_j = w_j a_j \in \det A^j$, then this has the effect of shifting $v_i \to w_{j-1}^{-1} v_i w_j$, which leaves $vh_\bullet$ invariant. For an acyclic complex $(C^\bullet, d)$, we have $\det H^\bullet = \mathbb{R}$. Thus in the acyclic case, for an element $c_j \in \det C^j$ we define the *torsion* of the complex to be

$$\kappa(c_\bullet) \in \mathbb{R}$$

Description Via Hodge Theory

Instead of choosing an arbitrary injection $\iota : H^\bullet \to C^\bullet$ and a chain homotopy T to define the Knudsen-Mumford map, we can also use a Hodge-theoretic approach to construct this map. Pick a basis for each of the spaces C^q, which defines an inner product on them. With respect these bases the differentials, $d_q : C_q \to C_{q-1}$, are represented by real matrices. Let $d_q^* : C^{q-1} \to C^q$ be their adjoints with respect to this inner product. We then define the combinatorial Laplacian $\Delta_q^{(c)} : C^q \to C^q$ by

$$\Delta_q^{(c)} = -(d_q^* d_q + d_{q-1} d_{q-1}^*)$$

This is a non-negative, self-adjoint operator, and we obtain a Hodge decomposition

$$C^q = \mathcal{H}_q \oplus \operatorname{im} d_{q+1}^* \oplus \operatorname{im} d_{q-1}$$

Where $\mathcal{H}_q = \ker \Delta_q^{(c)}$ is the space of harmonic elements. This is our desired cohomological splitting of $C^\bullet$. If we choose an unit volume harmonic basis of $h_\bullet \in \det \mathcal{H}_q$, and an unit volume element of $c_\bullet \in \det C^\bullet$, (i.e. the wedge product of an orthonormal basis), then we have

Proposition 13. *The Knudsen-Mumford map is given by the formula*

$$\kappa(c_\bullet) = \prod_k \left(\det{}'(d_k^* d_k)^{\frac{1}{2}(-1)^k} \right) h_\bullet \in \det \mathcal{H}^\bullet \tag{2.1.2}$$

where $\det{}'(d_k^* d_k)$ *indicates the product of the non-zero eigenvalues of the endomorphism* $d_k^* d_k : C^k \to C^k$

Proof. It suffices to show that $v_k = \det{}'(d_k^* d_k)^{\frac{1}{2}}$. When restricted to $\operatorname{im} d_{q-1}$, $\Delta_q^{(c)} = d_{q-1} d_{q-1}^*$, choose an orthonormal basis of eigenvectors $\{b_{q,i}\}$ for this operator, i.e. $d_{q-1} d_{q-1}^* b_{q,i} = \lambda_{q,i} b_{q,i}$. Then set $\tilde{b}_{q-1,i} = \lambda_{q,i}^{-1} d_{q-1}^* b_{q,i}$, and so $d_{q-1}\tilde{b}_{q-1,i} = b_{q,i}$. Let A^q be the span of the $\tilde{b}_{q,i}$. Then we have $C^q = A^q \oplus H^q \oplus B^q$. Because $\|b_{q,i}\| = 1$, we find $\|\tilde{b}_{q-1,i}\|^2 = \lambda_{q,i}^{-2} \|d_{q-1}^* b_{q-i}\| = \lambda_{q-1,i}^{-1}$. So we find $v_k = \left(\prod_{i=0}^{r_q} \|b_{q,i}\|\right) = \prod_{i=0}^{r_q - 1} (\lambda_{q-1,i})^{-1/2}$ □

Lemma 12. *The Knudsen-Mumford map is also given by the formula*

$$\kappa(c_\bullet) = \left(\prod_{k=0}^{N} (\det{}'\Delta_k^{(c)})^{\frac{1}{2}k(-1)^{k+1}} \right) h_\bullet \tag{2.1.3}$$

Proof. We can see that if $d_q^* d_q v = \lambda v$, then $(d_q d_q^*) d_q v = \lambda d_q v$, and thus $\operatorname{Spec}(d_q^* d_q) = \operatorname{Spec}(d_q d_q^*)$. Then we see that $\operatorname{Spec}(\Delta_q) = \operatorname{Spec}(d_q^* d_q) \cup \operatorname{Spec}(d_{q-1} d_{q-1}^*)$ because of the orthogonality of their domains. So we find

$$\det{}'(\Delta_q) = \det{}'(d_q^* d_q) \det{}'(d_{q-1}^* d_{q-1})$$

Subsituting this into 2.1.3, we see

$$\begin{aligned}\prod_{k=0}^{N}(\det{}'\Delta_k^{(c)})^{-\frac{1}{2}k(-1)^k} &= \prod_{k=0}^{N}(\det{}'(d_k^*d_k)\det{}'(d_{k-1}^*d_{k-1}))^{-\frac{1}{2}k(-1)^k} \\ &= \prod_{k=0}^{N}(\det{}'(d_k^*d_k))^{-\frac{1}{2}(k(-1)^k+(k+1)(-1)^{k+1})}\end{aligned}$$

□

Now if we assume that $C^\bullet$ is acyclic, then we see that $\Delta_q^{(c)}$ is non-singular for each q. We can then take the usual determinants $\det\Delta_q^{(c)}$, and then we have

$$\kappa(c_\bullet) = \sum_{k=0}^{N}(\det\Delta_k^{(c)})^{\frac{1}{2}k(-1)^{k+1}} \in \mathbb{R} \tag{2.1.4}$$

So to an acyclic chain complex, we can assign a real number $\kappa(c_\bullet)$ which is independent of the choice of inner products used. This is the definition which historically appeared first in the works of Reidemeister.

2.1.3 Reidemeister Torsion

Let K be a triangulation of a smooth manifold W, with a compatible triangulation $\tilde{K}$, of the universal cover $\tilde{W}$. $\tilde{K}$ then inherits an action of $\pi_1(W)$ as deck transformations and subsequently the simplicial co-chain groups $C^j(\tilde{K},\mathbb{R})$ are left modules over the group ring $\mathbb{Z}(\pi_1(W))$. Given an orthogonal representation $\rho : \pi_1(W) \to O(n)$, we form the ρ-*twisted* simplicial chain complex of K

$$C^j(K,\rho) := C^j(\tilde{K},\mathbb{R}) \otimes_\rho \mathbb{R}^n \tag{2.1.5}$$

i.e. $\gamma c \otimes v \sim c \otimes \rho(\gamma^{-1})v$, for $\gamma \in \pi_1(W)$. In his original work, Reidemeister studied the situation when this complex is acyclic.

Definition 12. *For an acyclic complex $C^\bullet(K,\rho)$, then we define the R*-torsion *to be*

$$\tau_R(K,\rho) := \kappa(C^\bullet(K,\rho)) \in \mathbb{R}$$

This quantity allowed for the classification of lens spaces up to homeomorphism, c.f. [31, 35]. The Ray-Singer analytic torsion is its smooth analogue.

2.2 Analytic Torsion

Starting with the same data used previously: an orthogonal representation $\rho : \pi_1(M) \to O(n)$, we describe the smooth analogs of the constructions that lead to the definition of the R-torsion, and use them to define the analytic torsion.

2.2.1 The de-Rham Complex Twisted by Monodromy

Given a monodromy representation $\pi_1(M) \to O(n)$, we can construct a flat, real vector bundle V_ρ over M, as a quotient of the universal covering space

$$V_\rho = \tilde{M} \times_\rho \mathbb{R}^n$$

Associated to this bundle, there is a canonical flat connection that comes from the descent of the regular de Rham differential

$$d_\rho : \Gamma(V_\rho) \to \Omega^1(V_\rho)$$

The flatness implies that $d_\rho^2 = 0$, which allows us to form the de Rham cochain complex of V_p-*twisted* differential forms

$$0 \longrightarrow \Omega^0(V_\rho) \xrightarrow{d} \Omega^1(V_\rho) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^N(V_\rho) \longrightarrow 0$$

This is the proposed smooth analogue of the twisted simplicial complex $C^\bullet(K,\rho)$, (2.1.5). We first compare the cohomology of this complex compares to its finite dimensional analogue.

Theorem 8 (Twisted de-Rham theorem)**.** *We have isomorphisms*

$$H^k(\Omega^\bullet(V_\rho), d) \cong H^k(C^\bullet(K,\rho), d)$$

for each k.

Thus, the same cohomological information is captured by both the simplicial and de Rham complexes. This motivates our search for an analytical counterpart of the R-torsion built out of this de Rham complex. Since the de Rham complex is infinite dimensional, we cant simply apply the Knudsen-Mumford map to obtain an R-torsion. However, we can mimic the definitions using our results from Chapter one on regularised determinants of elliptic operators.

2.2.2 Ray-Singer Analytic Torsion

Motivated by the formula for the R-torsion in terms of determinants of the combinatorial Laplacian (2.1.4), Ray and Singer [34] proposed a smooth analogue of the Reidemeister torsion. If we equip the manifold M and the bundle V_ρ with metrics, we can form the adjoints to the flat connection and the corresponding Hodge Laplacians $\Delta_{k,\rho} = d_\rho^* d_\rho + d_\rho d_\rho^*$ are elliptic operators. Thus, using 9, we are able to define the zeta-regularised determinants $\det'(\Delta_k)$.

Definition 13. *For an representation ρ such that the complex of twisted differential forms is acyclic,* *the* Ray-Singer analytic torsion*,* $\tau_{RS}(M,\rho)$*, is defined to be the positive real number*

$$\tau_{RS}(M,\rho) = \prod_{k=0}^{n} (\det{}'\Delta_{k,\rho})^{\frac{1}{2}k(-1)^{k+1}} \tag{2.2.6}$$

In terms of the zeta functions involved we have

$$\begin{aligned} \log \tau_{RS}(M,\rho) &= \tfrac{1}{2}\sum_{k=0}^{n} k(-1)^{k+1} \log \det{}'(\Delta_{k,\rho}) \\ &= \tfrac{1}{2}\sum_{k=0}^{n} k(-1)^{k} \zeta'(\Delta_{k,\rho},0) \end{aligned}$$

This definition easily extends to non-acyclic elliptic complexes, where τ_{RS} is now an element of the determinant line of the cohomology. This extension was explored in detail in [40].

Definition 14. *Let* (E,d) *elliptic complex. The* analytic torsion *is given by*

$$\tau(E,d) = \left(\prod_{k=0}^{n} (\det{}'\Delta_k)^{\frac{1}{2}k(-1)^{k+1}} \right) \bigotimes_{i=0}^{n} h_i^{(-1)^i} \in \det\ker\Delta$$

where h_i *is an unit element in* $\det\ker\Delta_k$*.*

In [40], motivated by formula (2.1.2), there is an attempt to use the definition

$$\log \tau(M,d_\rho) = \tfrac{1}{2}\sum_{k=0}^{n} (-1)^k \log\det{}'(d^\dagger d) \tag{2.2.7}$$

Where the regularised determinants here are defined via the meromorphic extension of the spectral zeta functions $\zeta(d^\dagger d,s) = \sum_{\lambda\in \mathrm{Spec}\, d^\dagger d} \lambda^{-s}$ for large positive $\Re(s)$. However, since the operators $d^\dagger d$ are *non-elliptic*, we do not-know that these zeta functions are non-singular at $s=0$. So, a priori, this quantity is not defined. However, we know that $\zeta(\Delta_q,s) = \zeta(d_q^\dagger d_q,s) + \zeta(d_{q-1}^\dagger d_{q-1},s)$, Starting from $\zeta(\Delta_0,s) = \zeta(d_0^\dagger d_0,s)$, which we know is finite at $s=0$, and using $\mathrm{Spec}(d_k^\dagger d_k) = \mathrm{Spec}(d_{k-1}d_{k-1}^\dagger)$, we can inductively guarantee that each of the $\zeta(d_q^\dagger d_q,s)$ are also finite at the origin. Thus the definition (2.2.7) makes sense since there are only a finite number of terms. It is this argument, first noted in a paper of Schwarz [40], which breaks down when we try to extend this theory over to the $\mathbb{Z}_2$ graded case. Once we have introduced this generalisation, we will show that the analytic torsion is independent of the metric chosen.

2.2.3 The Cheeger-Müller Theorem

Due to the similarities of the definitions of combinatorial and analytic torsion, Ray and Singer conjectured that they should indeed be equal, and gave evidence to support their claim. This equivalence was later proven independently by Cheeger [10], and Müller [33]. Motivated by the cobordism arguments involved in the proof of the Hirzebruch signature theorem, Cheeger used surgery techniques to reduce the proof to that of Lens spaces, for which the result was already well known. Müller uses the combinatorial Hodge theory of Dodziuk and Patodi [11], to study families of refinements of triangulations of a manifold, and the limits of their simplicial geometry. This theorem was later re-proven by Bismut-Zhang [7] in more generality.

Theorem 9 (Cheeger [10], Müller [33], Bismut-Zhang [7])**.** *Let K be a triangulation of a smooth manifold M, $\rho : \pi_1(M) \to O(n)$ an orthogonal representation, the we have the equality of torsions*

$$\tau_{RS}(M,\rho) = \tau_R(C^\bullet(K,\rho)) \in \det H^\bullet(M,\rho)$$

This theorem really demonstrates the nature of this of invariant, it can be computed from either a simplicial model or the analytical/spectral properties of the space.

2.3 Torsion For 2-Periodic Complexes

In [27], it was shown that the definition of analytic torsion could be extended to elliptic 2-periodic complexes, where the usual notion of ellipticity applies. Given such a complex $(C^\bullet, d)$ equipped with metrics, we can form adjoints and and construct the corresponding partial Laplacians. Consider the composite operator $D := \begin{pmatrix} 0 & d_0 \\ d_1 & 0 \end{pmatrix} \in \mathrm{End}^-(C)$ and its adjoint $D^\dagger = \begin{pmatrix} 0 & d_1^\dagger \\ d_0^\dagger & 0 \end{pmatrix}$. Then the Laplacian is $\Delta := DD^\dagger + D^\dagger D = \begin{pmatrix} \Delta_0 & 0 \\ 0 & \Delta_1 \end{pmatrix} \in \mathrm{End}^+(C)$. Let $P := \begin{pmatrix} P_0 & 0 \\ 0 & P_1 \end{pmatrix}$ be the projection onto the closure of the image of $D^\dagger$. On the image of D, the operators Δ and $DD^\dagger$ are equal and invertible, and so $P := D^\dagger(DD^\dagger)^{-1}D = D^\dagger(\Delta)^{-1}D$. By the pseudodifferential calculus (c.f. [41]) this means that P is a pseudodifferential operator of order 0, which is also a projection. Since $D^\dagger D = P\Delta = \Delta P$, we have

$$(D^\dagger D)^{-s} = P(\Delta)^{-s}$$

This is precisely the type of expression we described in Chapter one: P a pseudo-differential projection, and Δ a second order elliptic differential operator. Thus, following Mathai-Wu, we form the graded-zeta function for $D^\dagger D$, i.e.

$$\begin{aligned} \boldsymbol{\zeta}(D^\dagger D, s) &:= \mathrm{Tr}((d_0^\dagger d_0)^{-s}) - \mathrm{Tr}((d_1^\dagger d_1)^{-s}) \\ &= \mathrm{Tr}(P_0\Delta_0^{-s}) - \mathrm{Tr}(P_1\Delta_1^{-s}) \\ &= \mathbf{Tr}(P\Delta^{-s}) \end{aligned}$$

We know from equation (1.3.25) that each of the two graded components of this zeta function are finite at $s = 0$, which we now use to define the extension of analytic torsion to $\mathbb{Z}_2$ graded complex, as first defined in [27].

Definition 15. *The* analytic torsion *for a 2-periodic elliptic complex* $(C^\bullet, D)$ *is the element*

$$(\,\mathbf{det}'\, D^\dagger D)^{1/2}\,\eta_0 \otimes \eta_1^{-1} \in \det H^\bullet(C) \tag{2.3.8}$$

where η_i *is an unit element in the determinant line* $\det\ker\Delta_i$*. Here we are using* $\log \mathbf{det}'(A) := -\boldsymbol{\zeta}'(A,0)$.

If $(C^\bullet, D)$ is the folding up of a $\mathbb{Z}$-graded complex $(\tilde{C}^\bullet, d)$, we have already shown (c.f. formulas 2.1.2, 2.2.7) that $(\,\mathbf{det}'\, D^\dagger D)^{1/2} = \prod_{k=0}^{n}(\det{}'\, d_k^\dagger d_k)^{1/2(-1)^k}$. Then using $\zeta(\Delta_i, s) = \zeta(d_i^\dagger d_i, s) + \zeta(d_{i-1}^\dagger d_{i-1}, s)$, we easily see that the analytic torsion of the folded up complex agrees with the usual definition of the torsion. Thus we have defined a full extension of analytic torsion to $\mathbb{Z}_2$ graded complexes.

2.3.1 Variation Under Change of Metric

We will now show how the torsion varies under a smooth change in the metrics involved. Let $(C = \Omega^*(M,E), D)$ be a $\mathbb{Z}_2$ graded complex on an oriented manifold M, and let $g_u, u \in (-\epsilon, \epsilon)$ be a one parameter family of metrics on M, and h_u a one parameter family of hermitian metrics on E. These determine families of Hodge dual operators $*_u$, and anti-linear isomorphisms $\phi_u : E \to E^*$. The inner product on $\Omega^*(M,E)$ is $(\omega, \omega')_u = \int_M \omega \wedge *_u \phi_u \omega' := \int_M \omega \wedge \Gamma_u \omega'$. Let $D_u^\dagger$ be the one parameter of adjoints of D.

Variation Of The Determinant

Using the expansion (1.3.25), we use the following symbols for the leading terms in the s expansion of the zeta-function.

$$\boldsymbol{\zeta}(A,s) =: \boldsymbol{\chi}'(A) + \boldsymbol{\zeta}'(A)s + O(s^2) \tag{2.3.9}$$

Proposition 14. *Under a smooth change* g_u*, of metric, the following hold*

$$\tfrac{\partial}{\partial u}\boldsymbol{\chi}'(D_u^\dagger D) = 0 \tag{2.3.10}$$

$$\tfrac{\partial}{\partial u}\boldsymbol{\zeta}'(D_u^\dagger D) = \mathbf{Tr}(\alpha_u\,(A_{n/2} - Q)) \tag{2.3.11}$$

where $\alpha = \Gamma_u^{-1}\frac{\partial}{\partial u}\Gamma_u \in \Gamma(M, \mathrm{End}(\Lambda^* M \otimes E))$.

Proof. We have $\frac{\partial}{\partial u} D_u^\dagger = -[\alpha_u, D_u^\dagger]$. Recall

$$\Gamma(s)\boldsymbol{\zeta}(D_u^\dagger D, s) = \mathcal{M}\left[\,\mathbf{Tr}(e^{-tD_u^\dagger D}P_u); s\right]$$

Then using Duhamel's formula (1.1.15)

$$\tfrac{\partial}{\partial u}\,\mathbf{Tr}(e^{-tD_u^\dagger D}P_u) = \mathbf{Tr}\left(t[\alpha, D^\dagger]De^{-tD^\dagger D} + e^{-tD^\dagger D}\tfrac{\partial P}{\partial u}\right)$$

Now we rearrange,

$$
\begin{aligned}
\mathbf{Tr}\left([\alpha, D^\dagger] D e^{-tD^\dagger D}\right) &= \mathbf{Tr}\left(\alpha [D^\dagger, D e^{-tD^\dagger D}]\right) \\
&= \mathbf{Tr}\left(\alpha D^\dagger D e^{-tD^\dagger D} + \alpha D e^{-tD^\dagger D} D^\dagger\right) \\
&= \mathbf{Tr}\left(\alpha e^{-tD^\dagger D} D^\dagger D + \alpha e^{-tDD^\dagger} D D^\dagger\right) \\
&= \mathbf{Tr}\left(\alpha e^{-t\Delta}\Delta\right) \\
&= -\,\mathbf{Tr}\left(\alpha \tfrac{\partial}{\partial t}\left(e^{-t\Delta} - Q\right)\right)
\end{aligned}
$$

Because P is a projection, we have $P^2 = P$, so $\frac{\partial P}{\partial u} = P\frac{\partial P}{\partial u} + \frac{\partial P}{\partial u}P$ and thus $P\frac{\partial P}{\partial u}P = 0$. Thus

$$
\begin{aligned}
\mathbf{Tr}\left(e^{-tD^\dagger D}\tfrac{\partial P}{\partial u}\right) &= \mathbf{Tr}\left(e^{-tD^\dagger D}(P\tfrac{\partial P}{\partial u} + \tfrac{\partial P}{\partial u}P)\right) \\
&= \mathbf{Tr}\left(e^{-tD^\dagger D}(P^2\tfrac{\partial P}{\partial u} + \tfrac{\partial P}{\partial u}P^2)\right) \\
&= 2\,\mathbf{Tr}\left(e^{-tD^\dagger D}(P\tfrac{\partial P}{\partial u}P)\right) \\
&= 0
\end{aligned}
$$

since $[P, D^\dagger D] = 0$. So we continue

$$
\begin{aligned}
\Gamma(s)\tfrac{\partial}{\partial u}\boldsymbol{\zeta}(D_u^\dagger D, s) &= -\mathcal{M}\left[t\tfrac{\partial}{\partial t}\,\mathbf{Tr}\left(\alpha(e^{-t\Delta} - Q)\right); s\right] \\
&= s\mathcal{M}\left[\,\mathbf{Tr}\left(\alpha(e^{-t\Delta} - Q)\right); s\right]
\end{aligned}
$$

Here we see that the variation formula reveals another similar heat kernel trace, however this one involved the heat kernel of the full Laplacian, with a multiplicative pre-factor, α. It can easily be shown that a factor of this form does not effect the form of the series expansion, c.f. [5]. Thus by the arguments in Chapter one, we find that $\mathcal{M}\left[\,\mathbf{Tr}\left(\alpha(e^{-t\Delta} - Q)\right); s\right]$ has a simple pole at $s = 0$, with residue $\mathbf{Tr}(\alpha(A_{n/2} - Q))$, so we find that

$$
s\mathcal{M}\left[\,\mathbf{Tr}\left(\alpha(e^{-t\Delta} - Q)\right); s\right] = \mathbf{Tr}(\alpha(A_{n/2} - Q)) + O(s)
$$

So using $\Gamma(s)^{-1} = s + O(s^2)$, we have

$$
\tfrac{\partial}{\partial u}\boldsymbol{\zeta}(D_u^\dagger D, s) = s\,\mathbf{Tr}(\alpha(A_{n/2} - Q)) + O(s^2)
$$

Comparing with 2.3.9, we are done. □

As usual, if M is odd dimensional then $A_{n/2} = 0$, and we have

$$\tfrac{\partial}{\partial u}\boldsymbol{\zeta}'(D_u^\dagger D) = -\,\mathbf{Tr}(\alpha Q_u) \tag{2.3.12}$$

From now on, we only consider elliptic complexes over odd dimensional manifolds to avoid this issue with the term $A_{n/2}$. However, in the even dimensional case, there is a relative version of analytic torsion in which the effect of this term is canceled out by comparing two torsions simultaneously.

Variation of the Volume Form

The formula for the analytic torsion comprised of two terms, a determinant and a volume form. We have shown how the former varies under a smooth change of the metric, now we consider the latter.

Proposition 15 ([28]). *The volume elements η_0, η_1 can be chosen so that along a one-parameter deformation of the metrics (g_u, h_u), we have*

$$\partial_u(\eta_0 \otimes \eta_1^{-1}) = -\tfrac{1}{2}\,\mathbf{tr}(\alpha_u Q)(\eta_0 \otimes \eta_1^{-1})$$

Proof. Choose an orthonormal basis of $\ker \Delta_u^{\bar{k}}$ consisting of $v_{\bar{k},i} \equiv v_{\bar{k},i}(u) \in \ker \Delta_u^{\bar{k}}$, we then compute

$$\begin{aligned} 0 &= \partial_u \|v_{\bar{k},i}\|^2 && (2.3.13)\\ &= \left(\partial_u v_{\bar{k},i}, v_{\bar{k},i}\right) + \left(v_{\bar{k},i}, \partial_u v_{\bar{k},i}\right) + \int_M \langle v_{\bar{k},i} \wedge (\partial_u \Gamma) v_{\bar{k},i}\rangle dx && (2.3.14)\\ &= 2\Re\left(\partial_u v_{\bar{k},i}, v_{\bar{k},i}\right) + \left(v_{\bar{k},i}, \alpha_u v_{\bar{k},i}\right) && (2.3.15)\end{aligned}$$

We can consistently modify the phase of each $v_{\bar{k},i}$, so that $\left(\partial_u v_{\bar{k},i}, v_{\bar{k},i}\right)$ is real. Thus

$$\begin{aligned} \partial_u \eta_{\bar{k}} &= \partial_u \bigwedge_{i=0}^{b_{\bar{k}}} v_{\bar{k},i}\\ &= \sum_{i=0}^{b_{\bar{k}}} v_{\bar{k}_0} \wedge \ldots \wedge (\partial_u v_{\bar{k},i}) \wedge \ldots \wedge v_{\bar{k},b_{\bar{k}}}\end{aligned}$$

We have $\partial_u v_{\bar{k},i} = \sum_{j=0}^{b_{\bar{k}}} (\partial_u v_{\bar{k},i}, v_{\bar{k},j}) v_{\bar{k},j}$. Upon inserting this formula into the above, all of the off diagonal terms are killed off, and we have

$$\begin{aligned} \partial_u \eta_{\bar{k}} &= \sum_{i=0}^{b_{\bar{k}}} v_{\bar{k}_0} \wedge \ldots \wedge (\partial_u v_{\bar{k},i}, v_{\bar{k},i}) v_{\bar{k},i} \wedge \ldots \wedge v_{\bar{k},b_{\bar{k}}}\\ &= -\frac{1}{2}\left(\sum_{i=0}^{b_{\bar{k}}} (v_{\bar{k},i}, \alpha_u v_{\bar{k},i})\right) \eta_{\bar{k}}\end{aligned}$$

The bracketed term is the definition of the trace $\mathrm{tr}(\alpha_u Q_{\bar{k}})$. The result follows. □

We now have the desired invariance

Corollary 5. *The $\mathbb{Z}_2$ graded analytic torsion, on an odd dimensional manifold, is invariant under smooth changes in the metric.*

Proof. If we choose the volume elements as in proposition (2.3.1), then the variational formula (2.3.12) cancels the variation in the volume element

$$\begin{aligned}\frac{\partial}{\partial u}\left(\mathbf{det}'(D_u^\dagger D)^{1/2}\eta_{\bar{0}}\otimes\eta_{\bar{1}}^{-1}\right) &= (-\tfrac{1}{2}\tfrac{\partial}{\partial u}\boldsymbol{\zeta}'(D_u^\dagger D) - \tfrac{1}{2}\,\mathbf{tr}(\alpha_u Q))\left(\mathbf{det}'(D_u^\dagger D)^{1/2}\eta_{\bar{0}}\otimes\eta_{\bar{1}}^{-1}\right)\\ &= 0\end{aligned}$$

□

Derived Euler Characteristic

We now examine the leading term in the expansion (2.3.9), which we will show can be computed easily in terms of its untwisted analogue.

Definition 16 ([7])**.** *For a 2-periodic elliptic complex $(C^\bullet, D)$, the number $\boldsymbol{\chi}'(D) := \boldsymbol{\zeta}(D^\dagger D, 0)$ is called the* derived Euler characteristic*.*

One of our objectives in the next Chapter will be to compute this number in several cases. Its important to note that this term is closely related to the pseudo-trace of Kontsevich-Vishik [18, 24], but we will not explore this connection in this thesis.

Proposition 16. *In the case of a folded up $\mathbb{Z}$-graded elliptic complex, we have the equality*

$$\boldsymbol{\chi}'(C) = \sum_{k=0}^{n}(-1)^k k\, b_k$$

i.e.

$$\boldsymbol{\chi}'(C) = -p'(-1)$$

where $p(t) = \sum_k t^k\, b_k$ is the Poincare polynomial of M.

Proof. From lemma (8), we know $\zeta(\Delta_i, 0) = -b_i$. Since $\zeta(\Delta_i, s) = \zeta(d_i^\dagger d_i, s) + \zeta(d_{i-1}^\dagger d_{i-1}, s)$, we deduce

$\zeta(d_i^\dagger d_i, 0) = \sum_{k=0}^{i}(-1)^{k-i} b_k$. Then we have

$$\begin{aligned}
\boldsymbol{\zeta}(D^\dagger D, 0) &= \sum_{i=0}^{n}(-1)^i \zeta(d_i^\dagger d_i, 0) \\
&= \sum_{i=0}^{n}\sum_{k=0}^{i}(-1)^k b_k \\
&= \sum_{k=0}^{n}\sum_{i=0}^{k}(-1)^k b_k \\
&= \sum_{k=0}^{n}(-1)^k k\, b_k
\end{aligned}$$

□

So for the $\mathbb{Z}$-graded case, we see that $\boldsymbol{\chi}'(D) \in \mathbb{Z}$. In the next chapter we will show that the derived Euler characteristic for the twisted de-Rham complex is also an integer.

We also have the following interesting interpretation of the derived Euler characteristic. For a finite rank operator $A \in \mathrm{End}(V)$, we have

$$\det{}'(tA) = t^{\mathrm{rk}(A)} \det{}'(A)$$

where here $\det{}'$ is just the product of the non-zero eigenvalues. Now for elliptic operators D, we have

$$\det{}'(tD) = t^{\zeta(D,0)} \det{}'(D)$$

So we can interpret $\chi'(D) = \zeta(D, 0)$ as the *regularised rank* of D. It is this interpretation that makes an appearance in regularization techniques in quantum field theory, c.f. [40, 43]. The paper of Mathai-Wu develops a topological field theory involving the flux-twisted de-Rham complex, in which the twisted analytic torsion appears in the partition function. It would be interesting to investigate this connection further, and see if the formulas we produce later in this thesis have any connection with dimensional phenomena in QFT.

2.3.2 Flat Superconnections

Ray and Singer originally defined analytic torsion for flat connections, but now that their formulation has been extended to the $\mathbb{Z}_2$ graded setting, we can consider a more general type of connective structure, superconnections.

Definition 17. *A* superconnection *on a $\mathbb{Z}_2$-graded vector bundle $\mathscr{E} = \mathscr{E}^+ \oplus \mathscr{E}^-$ on M, is an odd parity, first order differential operator $\mathbb{A} \in \mathrm{End}^-(\mathscr{A}(M, \mathscr{E}))$, satisfying the Leibniz rule*

$$[\mathbb{A}, \alpha] = d\alpha \quad \textit{for } \alpha \in \mathscr{A}(M, \mathscr{E}),$$

where we do not distinguish between the form α and the operator $\alpha\wedge$.

Note that if we fold up a $\mathbb{Z}$-graded complex with a connection, we get a superconnection. We can extend $\mathbb{A}$ to act on $\mathscr{A}(M, \operatorname{End}\mathscr{E})$ by $\mathbb{A}\alpha := [\mathbb{A}, \alpha]$ for $\alpha \in \mathscr{A}(M, \operatorname{End}\mathscr{E})$. Using these definitions, we find that the *curvature* $\operatorname{curv}(\mathbb{A}) := \mathbb{A}^2$ is given by exterior multiplication by a form, $F_{\mathbb{A}} \in \mathscr{A}^+(M, \operatorname{End}\mathscr{E})$, i.e. $[\mathbb{A}^2, f]s = 0$, where f is a scalar section. If we decompose the connection into homogenous components $\mathbb{A} = \sum_{i=0}^n \mathbb{A}_{[i]}$, where $\mathbb{A}_{[i]} : \Gamma(M, \mathscr{E}) \to \mathscr{A}^i(M, \mathscr{E})$, we see that $\mathbb{A}_{[1]}$ is a connection on the bundle $\mathscr{E}$ that preserves the $\mathbb{Z}_2$ grading, and that each other $\mathbb{A}_{[i]}$ is given by exterior multiplication with some form $\mathbb{A}_{[i]}\alpha = \omega_{[i]} \wedge \alpha$, where $\omega_{[i]} \in \mathscr{A}^{i,-}(M, \operatorname{End}\mathscr{E})$.

We then find that the curvature decomposes as

$$F_{\mathbb{A}} = \sum_{i=0}^{n} (F_{\mathbb{A}})_{[i]}$$

where

$$\begin{aligned}
(F_{\mathbb{A}})_{[0]} &= \mathbb{A}_{[0]}^2 \\
(F_{\mathbb{A}})_{[1]} &= [\mathbb{A}_{[0]}, \mathbb{A}_{[1]}] \\
(F_{\mathbb{A}})_{[2]} &= [\mathbb{A}_{[0]}, \mathbb{A}_{[2]}] + \mathbb{A}_{[1]}^2 \\
&\vdots \\
(F_{\mathbb{A}})_{[k]} &= [\mathbb{A}_{[0]}, \mathbb{A}_{[k]}] + [\mathbb{A}_{[1]}, \mathbb{A}_{[k-1]}] + \dots
\end{aligned}$$

We say a superconnection is *flat*, if its curvature vanishes identically, $F_{\mathbb{A}} = 0$. Note that if a superconnection $\mathbb{A}$ is flat, it does not necessarily mean that the connection $\mathbb{A}_{[1]}$ is also flat. Given a flat superconnection, $\mathbb{A}$, we can form the 2-periodic complex

$$\begin{tikzcd} \mathscr{A}^+(M, \mathscr{E}) \arrow[r, shift left, "\mathbb{A}"] & \mathscr{A}^-(M, \mathscr{E}) \arrow[l, shift left, "\mathbb{A}"] \end{tikzcd}$$

The Leibnitz property ensures that this complex is elliptic, and so we can define an analytic torsion

$$\tau(M, \mathscr{E}, \mathbb{A}) \in \det H^\bullet(\mathscr{A}^\bullet(M, \mathscr{E}), \mathbb{A})$$

Since the connection $\mathbb{A}_{[1]}$, is not flat, there is not any associated torsion invariant. However, if we examine the first three flatness constraints

$$\begin{aligned}
0 &= \mathbb{A}_{[0]}^2 \\
0 &= [\mathbb{A}_{[0]}, \mathbb{A}_{[1]}] \\
0 &= [\mathbb{A}_{[0]}, \mathbb{A}_{[2]}] + \mathbb{A}_{[1]}^2
\end{aligned}$$

we see that first condition implies that the bundle map $d = \mathbb{A}_{[0]}$ squares to zero, i.e. $(\mathscr{E}, d)$ is a $\mathbb{Z}_2$-graded complex of vector bundles. The second flatness condition implies that the connection $\nabla = \mathbb{A}_{[1]}$ commutes with d. The third condition implies that the curvature $F_\nabla = \nabla^2$, is d-chain homotopically trivial, i.e. vanishes in the d-cohomology. We write this in more familiar language, using $\phi = \mathbb{A}_{[2]}$.

$$\begin{aligned} d^2 &= 0 \\ d\nabla &= \nabla d \\ F_\nabla - 0 &= d\phi - \phi d \end{aligned}$$

Now, if we assume that $\ker d \subset \mathscr{E}$ is a constant-rank vector bundle, then $\operatorname{im} d \subset \ker d$ is a constant-rank vector sub-bundle, and so the quotient vector bundle $H(\mathscr{E}, d) = \ker d / \operatorname{im} d$ is also constant rank. In this case, the flatness conditions for $\mathbb{A}$ imply the following

Corollary 6. *If the quotient bundle $H(\mathscr{E}, d)$ is suitably well defined, then the connection ∇ descends to a (super)-connection on the quotient $\tilde{\nabla}$. Furthermore, this quotient connection is flat.*

This flat connection is sometimes known as the *Gauss-Manin* connection. We can then form the analytic torsion element

$$\tau_{GM} = \tau(M, H(\mathscr{E}, d), \tilde{\nabla}) \in \det H(\mathscr{A}(M, H(\mathscr{E}, d)), \tilde{\nabla})$$

We have shown that from a flat superconnection, we can construct two torsion elements, the analytic torsion as defined by Mathai-Wu, and the same construction applied to the Gauss-Manin connection. The broader question that goes beyond the scope of this thesis is how are these two torsions related? We aim to investigate the case of the flux-twisted de Rham complex, where the connection is $d + H$, thus $\mathbb{A}_{[0]} = 0$, and we will find that this allows us to compare the two associated torsions.

Gauge Transformations

Given an element $g \in \mathscr{A}^+(M, \operatorname{End} \mathscr{E})$ that is point-wise invertible, we can form the gauge-transformed superconnection

$$\mathbb{A}_g = g^{-1} \mathbb{A} g$$

If $\mathbb{A}$ was flat, then so is $\mathbb{A}_g$. Given a one parameter family of such gauge elements $g_t \in \mathscr{A}^+(M, \operatorname{End} \mathscr{E})$, we have the variation formulas

$$\tfrac{\partial}{\partial t} \mathbb{A}_{g_t} = [\beta_t, \mathbb{A}_{g_t}], \quad \tfrac{\partial}{\partial t} \mathbb{A}_{g_t}^\dagger = -[\beta_t^\dagger, \mathbb{A}_{g_t}^\dagger] \quad \text{where } \beta_t = g_t^{-1} \tfrac{\partial}{\partial t} g_t$$

Then following the same method as the proof of proposition 14, we find

$$\begin{aligned}\tfrac{\partial}{\partial t}\,\mathbf{Tr}(e^{-z\mathbb{A}_{g_t}^\dagger \mathbb{A}_{g_t}}P_t) &= -z\,\mathbf{Tr}\left((-[\beta_t^\dagger,\mathbb{A}_{g_t}^\dagger]\mathbb{A}_{g_t}+\mathbb{A}_{g_t}^\dagger[\beta_t,\mathbb{A}_{g_t}])e^{-z\mathbb{A}_{g_t}^\dagger \mathbb{A}_{g_t}}\right)\\ &= z\,\mathbf{Tr}\left((\beta_t+\beta_t^\dagger)e^{-z\Delta_t}\Delta_t\right)\end{aligned}$$

But since $\mathbf{Tr}(A) = \overline{\mathbf{Tr}(A^\dagger)}$, and so

$$\tfrac{\partial}{\partial t}\zeta'(\mathbb{A}_{g_t}^\dagger \mathbb{A}_{g_t}) = -2\,\mathbf{Tr}(\Re(\beta_t)(A_{n/2}-Q_t))$$

From now on we will only be considering a small range of gauge transformations, and only with respect to our main complex of interest, the flux-twisted de Rham complex.

Flux-Twisted Torsion

Here we consider flat superconnections on the trivial $\mathbb{Z}_2$-graded real line bundle $\mathscr{E} = \mathscr{E}^0 = \mathbb{R}$. A flat superconnection must be of the form $\mathbb{A} = d + H$, where $H \in \mathscr{A}^-(M,\mathbb{R})$ is a closed, odd degree form. The following analysis can easily be generalised to the case where $\mathscr{E} = \mathscr{E}^0 = \mathcal{L}$ is a flat line bundle, but for simplicity, we only consider the trivial bundle. Now a gauge transformation of the type $g_t = \exp(tB)$, where $B \in \mathscr{A}^+(M,\mathbb{R})$, gives $\mathbb{A}_{g_t} = \mathbb{A} + tdB$, and hence $\frac{\partial}{\partial t}\mathbb{A}_{g_t} = dB$. So a transformation of this type has the effect of changing $H \to H + dB$. Since H, was closed, this gauge transformation leaves the cohomology class $[H] \in H^-(M,\mathbb{R})$ invariant.

If we consider the gauge transformation $g_t = \exp(t\beta)$, where β is a real form, we have shown that

$$\tfrac{\partial}{\partial t}\zeta'(\mathbb{A}_{g_t}^\dagger \mathbb{A}_{g_t})_{t=0} = 2\,\mathbf{Tr}(\beta Q_t)$$

Notice that when looking at the variation of the volume elements along this path, we have to map the varying determinant lines back to a fixed reference point. This is encapsulated in the following theorem that demonstrates *flux* independence (variation in cohomology class) of the twisted analytic torsion, not just metric independence.

Proposition 17 ([28] pp315)**.** *Let X be a compact oriented manifold of odd dimension, equipped with a flat vector bundle $\mathscr{E}$. Let H be a closed differential form of odd degree, and B an even form. We then have an equality of analytic torsions*

$$(\det e^B)^{-1}\tau(X,\mathscr{E},d+H-dB) = \tau(X,\mathscr{E},d+H)$$

where e^B is acting on a fixed volume form $\eta_\bullet$.

We can now state the relationship with the Gauss-Manin connection. For the super-connection $\mathbb{A} = d + H$, we have $\mathbb{A}_0 = 0$, so the the complex is equal to its own cohomology under this trivial differential. Thus the two torsions we described are

$$\tau(C, d + H) \in \det H(C, d + H)$$

and

$$\tau_{GM} = \tau(C, d) \in \det H(C, d)$$

Our primary aim is to show that these two torsions are related through an *analytical deformation* that provides a mechanism to compare these torsions, and in certain cases we can prove equality.

Chapter 3

Deformations of Elliptic Complexes

For an elliptic complex, the analytic torsion is an invariant of the underlying geometry. We now study what happens to the torsion as we deform the differentials of the complex. Our motivation comes from the desire to relate the analytic torsion for a particular complex, where it might be difficult to compute, to a related complex where the calculation is well known. Our main example is the flux-twisted de Rham operator $d+H$. We show that in some cases we can reduce the calculation of its analytic torsion to that of the regular de Rham differential.

The torsion consists of two components: a regularised determinant and a volume element. We begin with the description of the determinant under a deformation, and then in the next chapter we will discuss the behaviour of the volume term. In the work of Farber [12], the behaviour of determinants under a deformation of the complex is studied. Here we explore Farber's results and extend them to the case of 2-periodic complexes, and use these to prove our results about the torsion of the twisted de Rham complex.

3.1 Deformations of Elliptic Operators

We begin by considering a very basic example which exhibits the phenomena that we wish to describe. Consider the family of elliptic operators $d_t = d + (t/R)id\theta$ acting on functions on the circle of radius R. If we use the metric $d\theta^2$, the spectrum of the Laplacian $\Delta_t = d_t^* d_t : C^\infty(S^1_R) \to C^\infty(S^1_R)$ is

$$\begin{aligned}\mathrm{Spec}(\Delta_t) &= \{R^{-2}(n+t)^2 : n \in \mathbb{Z}\} \\ &= \{(t/R)^2\} \cup \{R^{-2}(n+t)^2 : 0 \neq n \in \mathbb{Z}\}\end{aligned}$$

Since we know the full spectrum explicitly, we can avoid the use of heat kernels and move straight to the zeta function.

$$\begin{aligned}\zeta(\Delta_t, s) &= (t/R)^{-2s} + R^{2s}\left(\sum_{n=1}^{\infty}(n+t)^{-2s} + \sum_{n=1}^{\infty}(n-t)^{-2s}\right) \\ &= (t/R)^{-2s} + R^{2s}(\zeta(2s, 1+t) + \zeta(2s, 1-t))\end{aligned}$$

where $\zeta(s,t)$ is the Hurwitz zeta function, which has a well known meromorphic extension (in s) to the whole complex plane. It is finite and analytic at the origin, where we have

$$\zeta(0,a) = \zeta(0) - a, \quad \tfrac{\partial}{\partial s}\zeta(s,a)|_{s=0} = \zeta'(0) + \ln\Gamma(a)$$

which tells us how the zeta function for $a \neq 0$ differs from the case $a = 0$. Thus we find the zeta regularised determinant is

$$\det{}'(\Delta_t) = (t/R)^2 f(t)\det{}'(\Delta_0), \quad f(t) = \left(\frac{\sin\pi t}{\pi t}\right)^2$$

Here we see the first signs of a more general phenomenon. We perturb the elliptic operator d^*d, by adding a small 1-form to d and when $t > 0$ find that the kernel vanishes. The space of constant functions $v_0(\theta) = 1$ has eigenvalue $(t/R)^2$ and lies in the kernel only for $t = 0$. For this reason we say it constitutes the *unstable* kernel. This jump in the kernel at $t = 0$, implies that the regularised determinant has a zero of order 2 as $t \to 0$. In [12], it is shown that this is the generic behaviour, and we will explore their results through out the course of this chapter. In theorem (12) we show that if Q_t an *analytic* deformation of a self-adjoint elliptic operator Q_0, then there exists an $\epsilon > 0$, such that the zeta-regularised determinant satisfies

$$\det{}'(Q_t) = \theta\, t^{\alpha}\det{}'(Q_0) + O(t^{\alpha+1})$$

for $0 \neq |t| < \epsilon$ and where $\theta \in \mathbb{R}, \alpha \in \mathbb{Z}^{\geq 0}$ are computable from the structure of the germ of the deformation at $t = 0$. Before we get to the discussion of this result, we need to determine precisely what kinds of perturbations we will be considering.

3.1.1 Deformations of Differential Operators

There are several notions of analyticity for for maps from a region of the complex plane into a topological vector space V. Here we will be thinking of V as one of the following: $\mathbb{C}$, $C^\infty(\mathscr{E})$, $\mathrm{Diff}_\ell(\mathcal{E},\mathcal{F})$, $B(\mathcal{H}^k(\mathcal{E}), \mathcal{H}^{k-\ell}(\mathcal{F}))$.

We choose to work with the strongest notion of analyticity for such functions (c.f [39]).

Definition 18. *A continuous map $f : \Omega \to V$, where Ω is an open subset of $\mathbb{C}$, and V is a topological* *vector space, is called* strongly holomorphic *if the limits*

$$\lim_{w \to z} \frac{f(w) - f(z)}{w - z}$$

exist (in the topology on V) for every $z \in \Omega$.

In a practical sense, the only families and functions we will be using are vectors in V with coefficients that are real analytic functions. In full generality, we consider curves $\phi : (a, b) \to V$, such that there exists a open region $\Omega \subset \mathbb{C}$ with $(a, b) \subset \Omega$, and a strongly holomorphic map $\tilde{\phi} : \Omega \to V$ that extends ϕ.

Definition 19. *A curve of smooth sections $\phi : (a, b) \to C^\infty(\mathscr{E})$ is*

- weakly analytic*, if it admits strongly holomorphic extension $\tilde{\phi} : \Omega \to L^2(\mathscr{E})$ in the L^2 topology.*
- real analytic*, if for each $\ell \geq 0$ it admits strongly holomorphic extension $\tilde{\phi}_\ell : \Omega_\ell \to \mathcal{H}^\ell(E)$ in the Sobolev $\mathcal{H}^\ell$ topology.*

We will need the weaker notion of analyticity to prove results extending elliptic regularity to real analytic curve of elliptic operators.

Given two vector bundles E, F over a smooth manifold M, let $D : (a, b) \to \mathrm{Diff}_k(E, F)$ be a family of differential operators of order k acting between smooth sections of the bundles E and F. According to a result of Palais (see [12]), for each $t \in (a, b)$, we can consider D_t as a smooth section of the bundle $\hom(J^k(E), F)$, where $J^k(E)$ is the bundle of k-jets of smooth sections of E.

Definition 20. *The family of operators $D : (a, b) \to C^\infty(\hom(J^k(E), F)$, is called an* analytic family *if there is a complex neighbourhood $\Omega \subset \mathbb{C}$ that contains (a, b) and a strongly holomorphic map $\tilde{D} : \Omega \to \hom(J^k(E), F)$ that extends D in the Frechet topology on $C^\infty(\hom(J^k(E), F)$.*

The motivation for this definition is that a real analytic family D_t of differential operators of order k, induces an family of bounded linear operators $D_t : \mathcal{H}^\ell(E) \to \mathcal{H}^{\ell-k}(F)$, for each $\ell > k$, which is strongly holomorphic in the operator norm topology of $B(\mathcal{H}^\ell(E), \mathcal{H}^{\ell-k}(F))$.

We now have the result we need

Corollary 7. *If $\phi : (a, b) \to C^\infty(E)$ is a real analytic curve, and $D : (a, b) \to \mathrm{Diff}_k(E, F)$ is an analytic family, then $D\phi : (a, b) \to C^\infty(F)$ is a real analytic curve.*

This allows us to proceed with the analysis of these families.

3.1.2 Analytic Deformation of Elliptic Self-Adjoint Operators

We now consider analytic families of elliptic operators acting on certain spaces of sections of a vector bundle $\mathscr{E}$. The following classical theorem is of fundamental importance to the study of self-adjoint elliptic operators.

Theorem 10 (Spectral Theorem)**.** *Let T be an unbounded self-adjoint elliptic linear operator of order k acting on $H = L^2(\mathscr{E})$ with domain $D = \mathcal{H}^k(\mathscr{E})$. The space H decomposes into countably many orthogonal subspaces $H = \oplus_\lambda H_\lambda$, where each H_λ is a finite dimensional vector space with a basis consisting smooth sections of $\mathscr{E}$, each of which is an eigenvector for T with eigenvalue λ. Furthermore, the eigenvalues form a discrete subset of $\mathbb{R}$.*

Now given an analytic family of such operators, we would like to know the behaviour of the set of eigenvalues. In general, the eigenvalues of a family might not be well behaved at all: we have no guarantee that the eigenvalues vary continuously, let alone smoothly or analytically. However, for analytic families, Kato [22] shows that we have a somewhat ideal situation.

Definition 21. *A* self-adjoint holomorphic family[1] *is an analytic family T of operators, such that $T(z)^* = T(\bar{z})$, for all $z \in \Omega$, and the domain $D(T(z))$ is fixed for all z. Clearly, $T(z)$ is self-adjoint for real z.*

Theorem 11 (Kato [22] Theorem 3.9, pp 392)**.** *Let $T(z)$ be a self-adjoint holomorphic family acting on a Hilbert space V, for z in a neighbourhood of an interval I of the real axis, such that the resolvent $R(T)$ is a compact operator for some $z \in I$. Then there is a sequence of real-analytic functions $\mu_n : I \to \mathbb{C}$ and a sequence of vector valued real-analytic functions $\varphi_n : I \to V$, such that for each $z \in I$, the $\mu_n(z)$ represent all the repeated eigenvalues of $T(z)$ and the set $\{\varphi_n(z)\}$ form a complete orthonormal family of the associated eigenvectors of $T(z)$.*

Remark 4. *Each individual eigenfunction φ_n is real analytic on the interval I, and thus extends to a holomorphic function on a neighbourhood U_n of I. However, this neighbourhood generically depends on n, and thus we might not be able to find a neighbourhood $U \supset I$ such that every eigenfunction is holomorphic on U.*

Applying this to a self-adjoint holomorphic family such that for each real z, the operator $T(z)$ is elliptic, we arrive at

Corollary 8. *If $Q : \Omega \to \mathrm{Diff}_\ell(\mathscr{E})$ is a self-adjoint holomorphic family, such that $Q(z)$ is elliptic for each $z \in I$, then there exists a countable sequence of real analytic functions $\lambda_n : I \to \mathbb{C}$, and a sequence of real analytic curves of smooth sections $\varphi_n : I \to C^\infty(\mathscr{E})$, such that they form a complete set of eigenvalues for each $z \in I$.*

Such a family is called an *elliptic self-adjoint holomorphic family*, and these constitute the main subject of our study.

Definition 22. *An elliptic self-adjoint holomorphic family , $D : (-\epsilon, \epsilon) \to \mathrm{Diff}_\ell(\mathscr{E})$ is called a* analytic deformation *of the operator D_0, if for each $t \in (-\epsilon, \epsilon)$, the operator $D_t - D_0$ is of order $< \ell$.*

[1] This definition is called a 'family of type A' in Kato's book

We always have in mind the examples given by the differential $d_z = d + zH$ acting on the $\mathbb{Z}_2$ graded complex of forms, where H is a degree 3 closed form. The Laplacian of this operator is an analytic deformation of the Laplacian for the unperturbed operator d_0. We consider this and other operators with polynomial dependence on z in the next section.

3.1.3 Germs of Sections

Analytic functions are locally determined by their series expansions at a specific point, so we translate the description of analytic deformations to the study of their germs at $t = 0$. For an analytic deformation $Q : (-\epsilon, \epsilon) \to \mathrm{Diff}_\ell(\mathscr{E})$ Consider the space of germs of real analytic curves of smooth sections

$$\mathcal{O}C^\infty(\mathscr{E}) := \{ \text{ germs at } t = 0 \text{ of real analytic maps } I \to C^\infty(\mathscr{E})\}$$

This is a module over the ring of germs of real analytic functions, $\mathcal{O}$. The germ of a curve $f(t)$ is often also denoted f, when there is no possible confusion. We define the germ extended operator $\tilde{Q} : \mathcal{O}C^\infty(\mathscr{E}) \to \mathcal{O}C^\infty(\mathscr{E})$

$$(\tilde{Q}f)(t) = Q_t f(t)$$

for a germ $f \in \mathcal{O}C^\infty(\mathscr{E})$. We know this map is indeed a map of real analytic curves because of corollary 7. With this operator, we can express corollary 8 in the following useful way

Proposition 18. *Let Q be an analytic deformation of a self-adjoint elliptic operator Q_0, and $\tilde{Q}$, the corresponding germ operator. The spectrum of the operator $\tilde{Q}$ consists of real analytic functions $\lambda \in \mathcal{O}$, each of which belong to one of the following 'types'*

1. $\lambda_n(t) \equiv 0,$ *for* $1 \leq n \leq N_0$ *(The* stable kernel*)*

2. $\lambda_n(t) = t^{\nu_n}\bar{\lambda}_n(t), \nu_n \geq 1, \bar{\lambda}_n(0) \neq 0,$ *for* $N_0 < n \leq N$ *(The* unstable kernel*)*

3. $\lambda_n(0) \neq 0,$ *for* $n > N$

Proof. We look at how each of the eigenvalues $\lambda(t)$ of Q_t deform as we move away from zero. Since we know the eigenvalues are analytic at zero, one of the following three conditions on the series expansion of $\lambda(t)$ is satisfied: (1) vanishes to all orders, (2) vanishes up to some finite order $\nu > 0$, (3) has non-zero constant term. From general elliptic operator theory, for any t, the values $\lambda_n(t)$ must tend to infinity as n does. Thus, there can only be finitely many $\tilde{\lambda}_n$ with $\tilde{\lambda}_n(0) = 0$. □

Now, we wish to describe the decomposition of the space of germs of sections under the operator $\tilde{Q}$, recall that for a single self-adjoint elliptic operator Q_0, elliptic regularity gives us a decomposition

$$C^\infty(\mathcal{E}) = \ker Q_0 \oplus \operatorname{im} Q_0$$

as an orthogonal direct sum in $L^2(\mathcal{E})$. For families, we will see a slight variation on this result, and to describe it we will need a definition

Definition 23. *Given an inclusion of $\mathcal{O}$-modules $N \hookrightarrow M$, we define the* pure module *of N, $p(N)$ to be the sub-module of M given by*

$$p(N) = \{m \in M : t^k m \in N\}$$

We can now state the main piece of analysis in [12], that shows why the germ decomposition is particularly useful.

Proposition 19. *(Farber) For an analytic deformation Q_t of an elliptic operator Q_0, we have a decomposition*

$$\mathcal{O}C^\infty(\mathscr{E}) = \ker(\tilde{Q}) \oplus p(\operatorname{im}(\tilde{Q}))$$

where $\tilde{Q}$ is the extension of the family Q_t acting on germs of sections

Thus, this result is a extension of elliptic regularity to analytic families.

Proof. Take an element $f \in \mathcal{O}C^\infty(E)$. From Kato's result (Thm 11) and the previous result on eigenvalues, prop (18), we know that we have a decomposition

$$f = \sum_{1 \leq n \leq N_0} f_n(t)\phi_n(t) + \sum_{N_0 < n \leq N} f_n(t)\phi_n(t) + \sum_{N < n < \infty} f_n(t)\phi_n(t)$$

where the $\phi_n(t)$ are the orthogonal eigenfunction germs corresponding to the eigenvalue 'types' described in (18), and the $f_n(t) \in \mathcal{O}$. Clearly, $\tilde{\Delta} f = 0$ iff $f_n = 0$ for all $n > N_0$, and the corresponding $\phi_n(t)$ form a free $\mathcal{O}$-basis of $\ker \tilde{\Delta}$. Now instead suppose that f belongs to the image of $\tilde{\Delta}$. The curve f then has the following properties. $(f, \phi_n) = 0$, for any $\phi_n \in \ker \tilde{\Delta}$, where we are using the $\mathcal{O}$-linear induced inner product on $\mathcal{O}C^\infty(E)$. For each type 2 eigenfunction, we have (f, ϕ_n) must be divisible by t^{ν_n} for each $N_0 < n \leq N$.

Suppose now that a different function f also satisfies these properties, then we aim to show that it too lies in $\operatorname{im} \tilde{\Delta}$. Since the type 3 eigenvalues are invertible in $\mathcal{O}$, we can form the sum

$$g(t) = \sum_{N_0 < n \leq N} \frac{f_n(t)}{\lambda_n(t)}\phi_n(t) + \sum_{N < n < \infty} \frac{f_n(t)}{\lambda_n(t)}\phi_n(t)$$

Clearly $\tilde{\Delta} g = f$. We aim to show that $g \in \mathcal{O}C^\infty(E)$. Since the first sum is finite, and we have shown the required divisibility, it is holomorphic. Using the divergence of the $\lambda_n(t)$, we can show that the infinite sum term of $g(t)$ converges at least as quickly as the infinite sum term of f in any of the Sobolev spaces $H^k(E)$, and thus, g is a curve with values in $C^\infty(E)$.

We need a technical result now: we want to show that if $Q_t\phi(t) = \psi(t)$ for all $t \in I$, where ψ is a real analytic curve, and ϕ is a weakly analytic curve, then ϕ is also real-analytic in some neighbourhood of $t = 0$. We consider the modified operator, $R_t = Q_t + P_0$, where $P_0 : \mathcal{H}^\ell(E) \to \mathcal{H}^\infty(E)$ is projection onto the kernel of Q_0 for some $\ell > 0$. Thus R_t is invertible at $t = 0$, and so it is also invertible in a neighbourhood $U = (-\delta, \delta)$. We then have $R_t\phi(t) = \psi(t) + P_0\phi_t$. The remainder term $P_0\phi(t)$ lies in a finite dimensional subspace of $\mathcal{H}^\infty(E)$, and is certainly a weakly analytic curve, it is also real analytic curve because all linear topologies on a finite dimension vector space are equivalent, so there is a unique notion of analyticity. We now know that $R_t\phi(t)$ is a real analytic curve for $t \in U$, and R_t invertible in this domain, and thus by the open mapping theorem the maps R_t constitute *real-analytic* linear homeomorphism $\mathcal{H}^{k+\ell}(E) \to \mathcal{H}^k(E)$ for each k, and thus $R_t^{-1}\psi(t)$ is analytic as a curve into each $\mathcal{H}^k$, and this is real-analytic.

Thus, modulo a finite dimensional perturbation for the kernel of $\tilde{\Delta}$, we know that $g \in \mathcal{O}C^\infty(E)$, as we desired. Thus, we have shown that a function f that is in the orthogonal complement of $\ker \tilde{\Delta}$, and that has a sufficient amount of t-divisibility in the range of the type 2 eigenfunctions, must be in the image of $\tilde{\Delta}$, i.e. $(\ker \tilde{\Delta})^\perp = p(\operatorname{im} \tilde{\Delta})$.

□

3.1.4 Behaviour of the Regularised Determinant

Now that we know that we can essentially diagonalise the operator $\tilde{Q}$, we can begin to discuss the behaviour of the zeta functions of Q_t as $t \to 0$. Using the spectral decomposition from the previous section, we can compute

$$\begin{aligned}
\zeta(Q_t, s) &= \sum_{\lambda(t) \text{ of type 2}} \lambda(t)^{-s} + \sum_{\lambda(t) \text{ of type 3}} \lambda(t)^{-s} \\
&=: \zeta_2(Q_t, s) + \zeta_3(Q_t, s)
\end{aligned}$$

Since there are only finitely many unstable eigenvalues, we have

$$\exp\left(-\zeta_2'(Q_t, 0)\right) = \prod_{n > N_0}^{N} t^{\nu_n} \bar{\lambda}_n(t)$$

There is a countable number of stable eigenvalues $\lambda(t)$, and we know that the values $\lambda(0)$ capture all the non-zero eigenvalues of Q_0.

Proposition 20 ([15], Corollary 1, pp 365, [12]). *The zeta function for the stable eigenvalues is a real analytic function of* t*, and furthermore*

$$\lim_{t \to 0} \zeta_3(Q_t, s) = \zeta(Q_0, s)$$

Putting this together, we arrive at one of the main analytical results of [12], which was outlined in the introduction

Theorem 12 (Farber [12]). *For an analytic deformation Q_t of a self-adjoint elliptic operator Q_0, we have*

$$\det{}'(Q_t) = t^{\alpha}\theta \det{}'(Q_0) + o(t^{\alpha+1})$$

as $t \to 0$, where

$$\alpha = \sum_{N_0 < N \leq N} \nu_n, \quad \theta = \prod_{N_0 < N \leq N} \bar{\lambda}_n(0)$$

and $\{t^{\nu_n}\bar{\lambda}_n(t)\}$ are the type 2 eigenvalues of the operator $\tilde{Q}$.

In the next chapter, we will show that the values α, θ have a cohomological interpretation, provided by a spectral sequence associated to the deformation.

3.2 Deformations of Elliptic Complexes

Now we know how the regularised determinant of a single elliptic operator behaves under analytic deformations, we move on to the study deformations of elliptic complexes. Let $(C^\infty(\mathscr{E}), d)$ be an elliptic complex. We say that an analytic curve of complexes $d_t, t \in (-\epsilon, \epsilon)$ is an *analytic deformation* of d if for each t, $(C^\infty(\mathscr{E}), d_t)$ is an elliptic complex and $d_0 = d$. This means that if we equip the complex with a metric, that the Laplacians $\Delta_t = d_t^\dagger d_t + d_t d_t^\dagger$ form an analytic deformation of the operator Δ_0.

Our goal is to determine how the cohomology of the complex changes as we deform the differential and also to determine how the analytic torsion varies.

3.2.1 Parametrised Hodge Decomposition

We first look at how the Hodge decomposition varies as we move along the deformation. Let $(C^\infty(\mathscr{E}), d_t)$ be an analytic deformation of an elliptic 2-periodic complex.

Definition 24. *The* germ complex *of the deformation $(C^\infty(\mathscr{E}), d_t)$, is the complex of $\mathcal{O}$ modules $(\mathcal{O}C^\infty(\mathscr{E}), \tilde{d})$. We often use the notation $C^{\bar{k}} := \mathcal{O}C^\infty(\mathscr{E}^{\bar{k}})$*

We then have the *germ Laplacian*, $\tilde{\Delta} = \tilde{d}^\dagger \tilde{d} + \tilde{d}\tilde{d}^\dagger$, to which we employ proposition (19) to obtain

$$\mathcal{O}C^\infty(\mathscr{E}^{\bar{k}}) = \ker(\tilde{\Delta}^{\bar{k}}) \oplus p(\operatorname{im}(\tilde{\Delta}^{\bar{k}}))$$

The parametrised spectral theorem tells us that the torsion $\mathcal{O}$-module $\Theta^{\bar{k}} := p(\operatorname{im}(\tilde{\Delta}^{\bar{k}}))/\operatorname{im}(\tilde{\Delta}^{\bar{k}})$ has finite dimension $\dim_{\mathbb{C}} \Theta^{\bar{k}} = \sum_{\ell > N_0}^{N} \nu_\ell^{\bar{k}}$, i.e. for each eigenvalue $t^\nu \bar{\lambda}(t)$ in the unstable kernel of $\tilde{\Delta}$, we obtain a torsion submodule of Θ of the form $\mathcal{O}/t^\nu\mathcal{O} \cong \mathbb{C}^\nu$. Further more, we know that

$$\operatorname{Spec}(\tilde{\Delta}) = \operatorname{Spec}(\tilde{d}^\dagger \tilde{d}) \cup \operatorname{Spec}(\tilde{d}\tilde{d}^\dagger)$$

So the decomposition of the spectrum of $\tilde{\Delta}$ into types 1, 2 and 3 can be refined as follows: the non-zero spectrum of $\tilde{d}^\dagger \tilde{d}$ will be types $2a, 3a$ and likewise we call the non-zero eigenvalues of $\tilde{d}\tilde{d}^\dagger$ types $2b, 3b$.

Since we are interested in computing the determinant of the operator $d^\dagger d$, only the eigenvalues of type $2a/3a$ will contribute.

For the ordinary Laplacian, we have the Hodge decomposition

$$C^\infty(\mathscr{E}^{\bar{k}}) = \ker(\Delta^{\bar{k}}) \oplus \operatorname{im} d_{\overline{k-1}} \oplus \operatorname{im} d^\dagger_{\bar{k}}$$

which is of fundamental importance to understanding the underlying geometry. For the germ Laplacian, we have the analogous *parameterised Hodge decomposition*

Theorem 13 (Farber [12]). *For an analytic deformation of an elliptic complex (E,d), we have*

$$\mathcal{O}C^\infty(E^{\bar{k}}) = \ker(\tilde{\Delta}_{\bar{k}}) \oplus p(\operatorname{im} \tilde{d}_{\overline{k-1}}) \oplus p(\operatorname{im} \tilde{d}^\dagger_{\bar{k}})$$

Proof. As usual, let $C^{\bar{k}} = \mathcal{O}C^\infty(E^{\bar{k}})$. Starting from the splitting from proposition 19, we only need to show that $p(\operatorname{im}(\tilde{\Delta})) \cong p(\operatorname{im} \tilde{d}_{\overline{k-1}}) \oplus p(\operatorname{im} \tilde{d}^\dagger_{\bar{k}})$. Let $f(t) \in p(\operatorname{im} \tilde{\Delta}))$, then $t^\ell f = \tilde{\Delta} g$ for some $g \in C^{\bar{k}}$. We then know that $\tilde{d}\tilde{d}^\dagger g$ and $\tilde{d}^\dagger \tilde{d} g$ are orthogonal to each other, and

$$t^\ell f = \tilde{d}\tilde{d}^\dagger g + \tilde{d}^\dagger \tilde{d} g$$

Which means that each there exists $h, h' \in C^{\bar{k}}$ such that $\tilde{d}\tilde{d}^\dagger g = t^j h$, and $\tilde{d}^\dagger \tilde{d} g = t^j h'$ and $f = h + h'$. Thus $p(\operatorname{im}(\tilde{\Delta})) \subset p(\tilde{d} C^{\overline{k-1}}) \oplus p(\tilde{d}^\dagger C^{\overline{k+1}})$. To prove the reverse inclusion, choose $f \in p(dC^{\overline{k-1}})$, such that $t^\ell f = dg$, with $g \in C^{\bar{k}}$. We have shown (theorem 19) that we can write $g = h + g'$, where h is harmonic and $g' \in p(\operatorname{im} \tilde{\Delta})$, ie. $t^r g' = \tilde{\Delta} g''$. Thus we have $t^{\ell+r} f = t^r(dg') = d(t^r g') = d\tilde{\Delta} g'' = \tilde{\Delta} d g''$. Thus $f \in p(\operatorname{im} \tilde{\Delta})$, i.e. $p(dC^{\overline{k-1}}) \subset p(\operatorname{im} \tilde{\Delta})$. We can prove the other inclusion $p(\tilde{d}^\dagger C^{\overline{k+1}}) \subset p(\operatorname{im} \tilde{\Delta})$ is proven similarly. The result follows.

□

We can now compute the cohomology of the germ complex, the so-called *germ cohomology.*

Corollary 9. *The cohomology groups of the germ complex are given by*

$$H^{\bar{k}}(\mathcal{O}C^\infty(\mathscr{E}), \tilde{d}) := \frac{\ker \tilde{d}_{\bar{k}} : C^{\bar{k}} \to C^{\overline{k+1}}}{\operatorname{im} \tilde{d}_{\overline{k-1}} : C^{\overline{k-1}} \to C^{\bar{k}}} \cong \ker(\tilde{\Delta}_{\bar{k}}) \oplus \tau^{\bar{k}}$$

where $\tau^{\bar{k}}$ is the torsion $\mathcal{O}$-module $p(\operatorname{im} \tilde{d}_{\overline{k-1}})/\operatorname{im} \tilde{d}_{\overline{k-1}}$.

We will show that these $\mathcal{O}$-modules capture the cohomology of both the undeformed complex, and the deformed complex for small t. We will make use of the following torsion $\mathcal{O}$-modules

$$\begin{aligned} X^{\bar{k}} &:= p(\operatorname{im} \tilde{d}_{\overline{k-1}})/\operatorname{im} \tilde{d}_{\overline{k-1}} d^\dagger_{\overline{k-1}} \\ Y^{\bar{k}} &:= p(\operatorname{im} \tilde{d}^\dagger_{\bar{k}})/\operatorname{im} \tilde{d}^\dagger_{\bar{k}} \tilde{d}_{\bar{k}} \end{aligned}$$

We write $\mu(M)$ to denote the number of cyclic summands of a torsion $\mathcal{O}$-module M, i.e. $\mu(\oplus_{i=1}^{n}\mathcal{O}/t^{r_i}\mathcal{O}) = n$. With these definitions, we can state one of the main results of Farber's paper [12]

Proposition 21 (Farber)**.** *The cohomology groups of the deformed complex for small $t \neq 0$ satisfy*

$$\dim_{\mathbb{C}} H^{\bar{k}}(C^{\infty}(\mathscr{E}), d_t) = \operatorname{rk}_{\mathcal{O}} H^{\bar{k}}(\mathcal{O}C^{\infty}(\mathscr{E}), \tilde{d})$$

Furthermore, the jump in the cohomology groups as $t \to 0$ are calculated by

$$\lim_{t\to 0}\Big(\dim_{\mathbb{C}} H^{\bar{k}}(C^{\infty}(\mathscr{E}), d_0) - \dim_{\mathbb{C}} H^{\bar{k}}(C^{\infty}(\mathscr{E}), d_t)\Big) = \mu(X^{\bar{k}}) + \mu(Y^{\bar{k}})$$

Proof. It's clear that the number of harmonic forms that leave the kernel for $t > 0$ is given by $\mu(\Theta^{\bar{k}})$, where $\Theta^{\bar{k}} = p(\operatorname{im}(\tilde{\Delta}^{\bar{k}}))/\operatorname{im}\tilde{\Delta}^{\bar{k}}$ as before, i.e. one for each eigenvalue of type 2. Using the parametrised hodge decomposition 13, we easily determine that $\Theta^{\bar{k}} \cong X^{\bar{k}} \oplus Y^{\bar{k}}$ and the result follows. $\square$

To compute the numbers $\mu(X^{\bar{k}}), \mu(Y^{\bar{k}})$ in terms of more easily computable objects, and in such a way that it becomes clear that these numbers are independent of the metric used, we need some structure theorems about these torsion modules.

Along with the torsion module that appears in the germ cohomology

$$\tau^{\bar{k}} \quad = \quad p(\operatorname{im}\tilde{d}_{\overline{k-1}})/\operatorname{im}\tilde{d}_{\overline{k-1}}$$

We introduce a similar module

$$\varrho^{\bar{k}} \quad = \quad p(\operatorname{im}\tilde{d}^{\dagger}_{\bar{k}})/\operatorname{im}\tilde{d}^{\dagger}_{\bar{k}}$$

and then we can state

Proposition 22 ([12] Prop 4.2)**.** *The following sequences of torsion $\mathcal{O}$-modules*

$$0 \longrightarrow \varrho^{\overline{k-1}} \xrightarrow{\tilde{d}^{\bar{k}}} X^{\bar{k}} \xrightarrow{\beta} \tau^{\bar{k}} \longrightarrow 0$$

$$0 \longrightarrow \tau^{\overline{k+1}} \xrightarrow{\tilde{d}^{\dagger}_{\bar{k}}} Y^{\bar{k}} \xrightarrow{\beta'} \varrho^{\bar{k}} \longrightarrow 0$$

are exact, where β, β' are the quotient maps.

Proof. We prove the statement for the first sequence, the second is proved in an almost identical fashion. Note that the kernel of β is $d^{\dagger}C^{\overline{k-1}}/d^{\dagger}dC^{\bar{k}} \subset X^{\bar{k}}$. We are thus left to show that d maps $\varrho^{\overline{k+1}}$ bijectively onto $dC^{\overline{k-1}}/dd^{\dagger}C^{\bar{k}} \subset X^{\bar{k}}$. To show injectivity, take $x \in C^{\overline{k-1}}$ such that $t^{\ell}x = d^{\dagger}y$ for some $y \in C\bar{k}, \ell > 0$, and suppose $dx = dd^{\dagger}z$, for some $z \in C^{\bar{k}}$, i.e. x is in the kernel of the induced map $\varrho^{\overline{k+1}} \to dC^{\overline{k-1}}/dd^{\dagger}C^{\bar{k}}$. Then we have $dd^{\dagger}(y - t^{\ell}z) = 0$, which implies $d^{\dagger}(y - t^{\ell}z) = 0$, and thus

$d^\dagger y = t^\ell d^\dagger z$, i.e. $t^\ell x = t^\ell d^\dagger z$, and so the coset $[x] \in \varrho^{\overline{k+1}}$ is zero, and thus the kernel of the induced map $\varrho^{\overline{k+1}} \to dC^{\overline{k-1}}/dd^\dagger C^{\bar{k}}$ is trivial. The surjectivity of this map follows quite easily from the parametrised Hodge theorem (Thm 13).

□

In Farber, it is shown that the type of the extensions given by the exact sequences are in a sense 'maximally' non-trivial

Proposition 23 ([12]). *The following equalities hold*

$$\mu(X^{\bar{k}}) = \mu(\tau^{\bar{k}}) = \mu(\varrho^{\overline{k-1}}), \qquad \mu(Y^{\bar{k}}) = \mu(\tau^{\overline{k+1}}) = \mu(\varrho^{\bar{k}})$$

In our case, we aren't so concerned with the number of cyclic submodules, but rather the dimension over $\mathbb{C}$ of these torsion submodules, which trivially distributes over exact sequences. i.e.

$$\dim_{\mathbb{C}}(Y^{\bar{k}}) = \dim_{\mathbb{C}}(\tau^{\overline{k+1}}) + \dim_{\mathbb{C}}(\varrho^{\bar{k}})$$

and a similar result holds for the other exact sequence. For each $\bar{k}$, we will show that there is a canonical non-degenerate pairing

$$\{\,,\,\} : \tau^{\bar{k}} \times \varrho^{\overline{k-1}} \to \mathcal{M}/\mathcal{O}$$

which is $\mathcal{O}$-linear in the first argument, and $\mathcal{O}$-antilinear in the second. This was constructed in [12], and shows that $\tau^{\bar{k}} \cong \varrho^{\overline{k-1}}$ as $\mathcal{O}$-modules. The form is constructed as follows: given $x \in p(dC^{\overline{k-1}}) \subset C^{\bar{k}}$ and $y \in p(d^\dagger C^{\bar{k}}) \subset C^{\overline{k-1}}$. We then have that $t^\ell x = dz$ for some $z \in C^{\overline{k-1}}$. Define

$$\{\,,\,\} : p(dC^{\overline{k-1}}) \times p(d^\dagger C^{\bar{k}}) \to \mathcal{M}/\mathcal{O}$$

by

$$\{x, y\} = t^{-\ell}(z, y)$$

We have left to prove that his map descends to $\tau^{\bar{k}} \times \varrho^{\overline{k-1}}$, is well defined, and is non-degenerate, all which follow from simple algebraic manipulations, so for brevity we defer to the original proof in [12].

3.2.2 Behaviour of the Regularised Determinant

We now relate all these cohomological quantities to the behaviour of the regularised determinant of a family of elliptic complexes. Consider again a $\mathbb{Z}_2$-graded elliptic complex $(C^\bullet, d)$

$$\begin{tikzcd} C^0 \arrow[r, bend left, "d_{\bar{0}}"] & C^1 \arrow[l, bend left, "d_{\bar{1}}"] \end{tikzcd}$$

For our purposes, we want to study the regularised super-determinant of the partial laplacian $\mathbf{det}'(d^\dagger d) = \det{}'(d_{\bar{0}}^\dagger d_{\bar{0}})/\det{}'(d_{\bar{1}}^\dagger d_{\bar{1}})$. For an analytic deformation $(C^\bullet, d_t), |t| < \epsilon$ of this complex, proposition 12 tells us that as $t \to 0$,

$$\mathbf{det}'(d_t^\dagger d_t) \sim t^{\alpha_{\bar{0}}-\alpha_{\bar{1}}}(\theta_{\bar{0}}/\theta_{\bar{1}})\,\mathbf{det}'(d_0^\dagger d_0)$$

Thus we see that when $\alpha_{\bar{1}} > \alpha_{\bar{0}}$, $\mathbf{det}'(d_t^\dagger d_t)$ is singular as $t \to 0$. One of the objectives of [12] was to investigate this singularity. We can now relate the integers $\alpha_{\bar{k}}$ to the torsion modules in the parameterised hodge theory

Proposition 24.

$$\alpha_{\bar{k}} = \dim_{\mathbb{C}} Y^{\bar{k}} = 2\dim_{\mathbb{C}} \tau^{\overline{k+1}}$$

Proof. We have shown that for each type 2a eigenvalue $t^\nu\lambda(t)$, $Y^{\bar{k}}$ contains a torsion submodule of the form $\mathcal{O}/t^\nu\mathcal{O} \cong \mathbb{C}^\nu$, and thus the first equality follows easily. The exact sequence 22 shows that $\dim_{\mathbb{C}} Y^{\bar{k}} = \dim_{\mathbb{C}} \tau^{\overline{k+1}} + \dim_{\mathbb{C}} \varrho^{\bar{k}}$. The non-degenerate form constructed at the end of the last section showed that $\tau^{\bar{k}} \cong \varrho^{\overline{k-1}}$ as $\mathcal{O}$-modules, and thus the result follows. □

Notice that since $\tau^{\overline{k+1}}$ was independent of the metric on the bundle E, the order of the singularity is also independent of this choice. Combining all these results, we arrive at

Corollary 10. *For a deformation of 2-periodic elliptic complex $(C^\infty(\mathscr{E}), d_t)$, we have*

$$\begin{aligned}
\mathbf{det}'(d_t^\dagger d_t) &= t^{\alpha_0-\alpha_1}\,\mathbf{det}'(d_0^\dagger d_0)f(t) && (3.2.1)\\
&= t^{-2(\dim\tau^{\bar{0}}-\dim\tau^{\bar{1}})}\,\mathbf{det}'(d_0^\dagger d_0)f(t) && (3.2.2)
\end{aligned}$$

This gives us a formula for the singular behaviour of the regularised determinant in terms of the germ cohomology groups. We will later produce an alternate description for this behaviour, which upon comparison with this approach will reveal information about the derived Euler characteristic and the regularised determinants.

3.2.3 The Flux Twisted de Rham Complex

The the main focus of our study is the flux-twisted de Rham complex, equivalently, the complex associated to a flat superconnection on the trivial $\mathbb{Z}_2$-graded bundle, $\mathscr{C} = \mathscr{C}^0 = \mathbb{R}$. We then had $\mathbb{A} = d + H$, where $H \in \Omega^-_{cl}(M)$. For simplicity we assume that H has no 1-form component, however generalising to this case, as well as taking $\mathscr{C}^0$ to be a non-trivial line bundle, is a straightforward extension of this work. For this complex there is a very interesting deformation that in some cases allows us to compute the analytic torsion of $\mathbb{A}$. Consider the following deformation of the standard de Rham complex $(\Omega^\bullet(\mathscr{C}), d)$.

$$D_t = d + \sum_{i=1}^{n/2} t^i H_{2i+1}$$

Then clearly $D_0 = d$ and $D_1 = \mathbb{A}$. This family is particularly special because it comes from a filtration of the complex, an aspect that we will explore in detail in the next chapter. Let N be the grading operator of the de Rham complex and let $\rho_t = t^{N/2}$. A small algebraic computation (c.f. [14]) reveals the important relation

$$t^{1/2} D_t = \rho_t \mathbb{A} \rho_t^{-1}, \quad t > 0 \tag{3.2.3}$$

Now, if we choose a metric g, we can compute the adjoint operator

$$D_t^\dagger = d^\dagger + \sum_{i=1}^{n/2} t^i H_{2i+1}^\dagger$$

and then we find

Lemma 13 ([14])**.** *The adjoint of the family D_t is proportional to the adjoint of $\mathbb{A}$ in the scaled metric, given by the following relation*

$$t^{1/2} D_t^\dagger = \rho_t \mathbb{A}^{\dagger_t} \rho_t^{-1}, \quad t > 0$$

where $\dagger_t$ indicates the adjoint taken with respect to the scaled metric $g_t = tg$.

This says that for $t > 0$ the family D_t is related to $\mathbb{A}$ by a conjugation and a scaling. This implies that the cohomology groups $H^\bullet(C, D_t)$ and $H^\bullet(C, \mathbb{A})$ are isomorphic for $t > 0$, however, there could be a jump in the cohomology at $t = 0$. The isomorphism is given by

$$H^\bullet(C, D_t) \ni u \mapsto \rho_t^{-1} u \in H^\bullet(C, \mathbb{A})$$

If we consider harmonic forms, we have

$$t^{1/2}(D_t + D_t^\dagger) = \rho_t (\mathbb{A} + \mathbb{A}^{\dagger_t}) \rho_t^{-1}$$

so $\rho_t^{-1} : \ker(D_t + D_t^\dagger) \to \ker(\mathbb{A} + \mathbb{A}^{\dagger_t})$. i.e, the D_t-harmonic forms in the metric g, are mapped to the $\mathbb{A}$-harmonic forms, in the metric tg.

We can also use these identities to compute some of the spectral invariants of the family.

Lemma 14. *For the family of elliptic complexes $(C^\bullet, D_t)$, $t > 0$, the derived Euler characteristic is independent of t, i.e.*

$$\boldsymbol{\chi}'(\mathbb{A}^\dagger \mathbb{A}) = \boldsymbol{\chi}'(D_t^\dagger D_t) \tag{3.2.4}$$

Proof. since conjugation by an invertible operator doesn't effect the spectrum, we have

$$\begin{aligned} \boldsymbol{\zeta}(\mathbb{A}^{\dagger_t}\mathbb{A}, s) &= \boldsymbol{\zeta}(\rho_t \mathbb{A}^{\dagger_t}\mathbb{A}\rho_t^{-1}, s) \\ &= \boldsymbol{\zeta}(t D_t^\dagger D_t, s) \\ &= t^{-s} \boldsymbol{\zeta}(D_t^\dagger D_t, s) \end{aligned}$$

So we find

$$\boldsymbol{\zeta}(\mathbb{A}^{\dagger_t}\mathbb{A},0)=\boldsymbol{\zeta}(D_t^\dagger D_t,0)$$

and we have already shown that the derived Euler characteristic is independent of the choice of metric, so we are free to take $t=1$ on the left hand side. $\square$

Note that g_t is not a metric at $t=0$, so in general $\boldsymbol{\chi}'(\mathbb{A}^\dagger\mathbb{A})\neq\boldsymbol{\chi}'(d^\dagger d)$. Later, in proposition 14, we give a formula for the difference $\boldsymbol{\chi}'(\mathbb{A}^\dagger\mathbb{A})-\boldsymbol{\chi}'(d^\dagger d)$. We often just write $\boldsymbol{\chi}'(\mathbb{A}):=\boldsymbol{\chi}'(\mathbb{A}^\dagger\mathbb{A})$, since it is independent of t for $t>0$. We now produce a variation formula, similar to the one produced in the last chapter.

Lemma 15. *For the family D_t, we have*

$$\tfrac{\partial}{\partial t}\boldsymbol{\zeta}(tD_t^\dagger D_t,s)=s\Gamma(s)^{-1}t^{-1}\mathcal{M}\left[\mathbf{Tr}(N(e^{-z\Delta_t}-P_t));z:s\right],$$

where P_t is the projection onto the kernel of $\Delta_t:=D_t^\dagger D_t+D_tD_t^\dagger$.

Proof. We begin with a formula for the derivative of the scaled differential

$$\tfrac{\partial}{\partial t}(t^{1/2}D_t)=[N/2t,\rho_t\mathbb{A}\rho_t^{-1}]=t^{-1/2}[N/2,D_t]$$

and

$$\tfrac{\partial}{\partial t}(t^{1/2}D_t^\dagger)=-t^{-1/2}[N/2,D_t^\dagger]$$

so we see

$$\tfrac{\partial}{\partial t}(tD_t^\dagger D_t)=(-[N/2,D_t^\dagger]D_t+D_t^\dagger[N/2,D_t])$$

Now, using Duhamel's formula 1.1.15

$$\begin{aligned}\tfrac{\partial}{\partial t}\,\mathbf{tr}(e^{-ztD_t^\dagger D_t}-P_t) &= -z\,\mathbf{tr}((-[N/2,D_t^\dagger]D_t+D_t^\dagger[N/2,D_t])e^{-ztD_t^\dagger D_t})\\ &= z\,\mathbf{tr}(Ne^{-zt\Delta_t}\Delta_t)\end{aligned}$$

So

$$\begin{aligned}\tfrac{\partial}{\partial t}\boldsymbol{\zeta}(tD_t^\dagger D_t,s) &= \Gamma(s)^{-1}\mathcal{M}\left[z\,\mathbf{tr}(Ne^{-zt\Delta_t}\Delta_t);z:s\right]\\ &= \Gamma(s)^{-1}t^{-(s+1)}\mathcal{M}\left[z\,\mathbf{tr}(Ne^{-z\Delta_t}\Delta_t);z:s\right]\\ &= -\Gamma(s)^{-1}t^{-(s+1)}\mathcal{M}\left[z\tfrac{\partial}{\partial z}\,\mathbf{tr}(N(e^{-z\Delta_t}-P_t));z:s\right]\\ &= s\Gamma(s)^{-1}t^{-(s+1)}\mathcal{M}\left[\,\mathbf{tr}(N(e^{-z\Delta_t}-P_t));z:s\right]\end{aligned}$$

and we arrive at the result. ☐

Using this result, we can calculate the variation of the determinant

Corollary 11. *The derivative of the zeta function of the family D_t has the following t-dependence*

$$\tfrac{\partial}{\partial t}\boldsymbol{\zeta}'(D_t^\dagger D_t) = t^{-1}\boldsymbol{\chi}'(\mathbb{A}) + t^{-1}\,\mathbf{Tr}(N(A_{n/2} - P_t)) \tag{3.2.5}$$

Proof. Clearly, we have

$$\begin{aligned}\tfrac{\partial}{\partial t}\boldsymbol{\zeta}(tD_t^\dagger D_t, s) &= \tfrac{\partial}{\partial t}(t^{-s}\boldsymbol{\zeta}(D_t^\dagger D_t, s)) \\ &= -st^{-(s+1)}\boldsymbol{\zeta}(D_t^\dagger D_t, s) + t^{-s}\tfrac{\partial}{\partial t}\boldsymbol{\zeta}(D_t^\dagger D_t, s)\end{aligned}$$

and so we arrive at

$$\tfrac{\partial}{\partial t}\boldsymbol{\zeta}(D_t^\dagger D_t, s) = st^{-1}\boldsymbol{\zeta}(D_t^\dagger D_t, s) + s\Gamma(s)^{-1}t^{-1}\mathcal{M}\left[\,\mathbf{Tr}(N(e^{-z\Delta_t} - P_t)); z : s\right]$$

taking the derivative w.r.t s and evaluating at $s = 0$, we recover the result. ☐

The main difference between this variation formula and the one produced in the last chapter is that the family of metrics g_t behaves poorly near $t = 0$, whereas the family of operators D_t does not. For example, the harmonic forms in the metric g_t might be unbounded as $t \to 0$. Because we instead varied the differentials, we can analyse the behaviour of $\mathbf{Tr}(NP_t)$ near $t = 0$. From now on we assume M is odd-dimensional, so we can ignore the effects of the term $\mathbf{Tr}(NA_{n/2})$.

Proposition 25. *As $t \to 0$,*

$$\mathbf{Tr}(NP_t) = \boldsymbol{\chi}_0'(d) + O(t)$$

where $\boldsymbol{\chi}_0'(d)$ is the derived Euler characteristic of only the type 1 eigenvectors of $\Delta = d^\dagger d + dd^\dagger$.

We know that the complex (C, d) is $\mathbb{Z}$-graded, so the usual formula $\boldsymbol{\chi}_0'(d) = \sum_k (-1)^k k n_k$ applies, where n_k is the number of type 1 eigenvalues of $dd^\dagger + d^\dagger d$ of degree k.

Proof. We need to show that there exists an orthonormal basis $\{\phi_i(t)\}$ for the stable kernel of $\tilde{\Delta}$ such that each of the forms $\phi_i(0)$ are homogenous in the $\mathbb{Z}$-grading, not just the $\mathbb{Z}_2$ grading. The existence of such a basis follows quite simply from ([14] Thm 6). We will prove this in the next chapter (corollary 13), once we have introduced the spectral sequence of a deformation. Assuming such a basis exists, then we have

$$\begin{aligned}\mathbf{Tr}(NP_t) &= \sum_i (\phi_i(t), (-1)^N N\phi_i(t)) \\ &= \sum_i (-1)^{d_i} d_i(\phi_i(0), \phi_i(0)) + O(t) \\ &=: \boldsymbol{\chi}_0'(d) + O(t)\end{aligned}$$

where $d_i = \deg \phi_i(0)$. □

Using the above we arrive at

Proposition 26.
$$\mathbf{det}'(D_t^\dagger D_t) = t^{\boldsymbol{\chi}'_0(d) - \boldsymbol{\chi}'(\mathbb{A})}\,\mathbf{det}'(\mathbb{A}^\dagger\mathbb{A}) g(t)$$

where $g(0) \neq 0$.

Proof. for small t, we have

$$\begin{aligned}
\boldsymbol{\zeta}'(D_t^\dagger D_t) &= \boldsymbol{\zeta}'(\mathbb{A}^\dagger\mathbb{A}) - \int_t^1 \left(\tfrac{\partial}{\partial s}\boldsymbol{\zeta}'(D_s^\dagger D_s)\right) ds \\
&= \boldsymbol{\zeta}'(\mathbb{A}^\dagger\mathbb{A}) - \int_t^1 \left(s^{-1}\boldsymbol{\chi}'(\mathbb{A}) - s^{-1}\,\mathbf{Tr}(NP_s)\right) ds \\
&= \boldsymbol{\zeta}'(\mathbb{A}^\dagger\mathbb{A}) + (\boldsymbol{\chi}'(\mathbb{A}) - \boldsymbol{\chi}'_0(d))\log t + \int_t^1 s^{-1}\left(\,\mathbf{Tr}(NP_s) - \boldsymbol{\chi}'_0(d)\right) ds
\end{aligned}$$

but because of prop (25), we know that $\mathbf{Tr}(NP_s) - \boldsymbol{\chi}'_0(d)$ is of order $O(s)$, so the integral is finite as $t \to 0$. This implies

$$\begin{aligned}
\mathbf{det}'(D_t^\dagger D_t) &= \exp(-\boldsymbol{\zeta}'(D_t^\dagger D_t)) \\
&= t^{\boldsymbol{\chi}'_0(d) - \boldsymbol{\chi}'(\mathbb{A})}\,\mathbf{det}'(\mathbb{A}^\dagger\mathbb{A}) e^{-f(t)}
\end{aligned}$$

where
$$f(t) = \int_t^1 (\,\mathbf{Tr}(NP_s) - \boldsymbol{\chi}'_0(d)) s^{-1} ds.$$
We set $g(t) = e^{-f(t)}$, and the result follows. □

By comparing the singularity in t in the limiting behaviour as $t \to 0$ given by proposition (26),

$$\mathbf{det}'(D_t^\dagger D_t) = t^{\boldsymbol{\chi}'_0(d) - \boldsymbol{\chi}'(\mathbb{A})}\,\mathbf{det}'(\mathbb{A}^\dagger\mathbb{A}) e^{-f(t)}$$

with the behaviour computed from the germ complex prop (10),

$$\mathbf{det}'(D_t^\dagger D_t) \sim t^{\alpha_{\bar{0}} - \alpha_{\bar{1}}}(\theta_{\bar{0}}/\theta_{\bar{1}})\,\mathbf{det}'(d^\dagger d)$$

we arrive at one of our original results, a formula for $\boldsymbol{\chi}'(\mathbb{A})$ in terms of the cohomology of the germ complex.

Theorem 14. *The derived Euler characteristic $\boldsymbol{\chi}'(\mathbb{A})$ of the flux-twisted de-Rham complex is given by* *the following formula*

$$\begin{aligned}\boldsymbol{\chi}'(\mathbb{A}) &= \boldsymbol{\chi}'_0(d) - \alpha_{\bar{0}} + \alpha_{\bar{1}} \\ &= \boldsymbol{\chi}'_0(d) + 2(\dim \tau^{\bar{0}} - \dim \tau^{\bar{1}})\end{aligned}$$

Furthermore, the determinants are related by

$$\mathbf{det}'(\mathbb{A}^\dagger \mathbb{A}) = (\theta_{\bar{0}}/\theta_{\bar{1}})\ \mathbf{det}'(d^\dagger d)\, e^{f(0)}$$

where $\theta_{\bar{k}} := \prod_i \bar{\lambda}_i(0)$ *is the product is over the type* 2 *eigenvalues* $\lambda(t) = t^\nu \tilde{\lambda}(t)$ *of* $\mathbb{A}^\dagger_{\bar{k}} \mathbb{A}_{\bar{k}}$*, and*

$$f(0) = \int_0^1 (\,\mathbf{Tr}(NP_s) - \boldsymbol{\chi}'_0(d))\, s^{-1} ds$$

In the next chapter, we will give an alternative description of the term $(\theta_{\bar{0}}/\theta_{\bar{1}})$ of the spectral sequence of the deformation. If we can show that $\frac{\partial}{\partial t}\mathbf{Tr}(NP_t) = 0$, then the 'correction' term $f(0)$ in the above formula is identically zero. There are several interesting cases in which this happens.

Lemma 16. *If* $H^\bullet(C, \mathbb{A}) = 0$*, then*

$$\boldsymbol{\chi}'(\mathbb{A}) = -\alpha_{\bar{0}} + \alpha_{\bar{1}}$$

and

$$\mathbf{det}'(\mathbb{A}^\dagger \mathbb{A}) = (\theta_{\bar{0}}/\theta_{\bar{1}})\ \mathbf{det}'(d^\dagger d)$$

Proof. If the twisted cohomology vanishes for all $t > 0$, then $\mathbf{Tr}(NP_t) \equiv 0$ and so $\boldsymbol{\chi}'_0(d) = 0$ and $g = 1$. □

Thus in the case where the twisted cohomology vanishes there is a simple formula for the regularised determinant of the twisted de Rham complex. In the next chapter, we will show how to compute the twisted cohomology from the untwisted cohomology.

In [27], the case where H is of top degree was studied

Proposition 27. *If* M^n *is an odd dimensional, compact, oriented Riemannian manifold,* H *is a multiple of the volume form and let* $\mathbb{A} = d + H$*, then*

$$\boldsymbol{\chi}'(\mathbb{A}) = \boldsymbol{\chi}'(d) + (n-2)b_0$$

where b_0 *is the* 0*th Betti number, and furthermore*

$$\mathbf{det}'(\mathbb{A}^\dagger \mathbb{A}) = \|H\|^{2b_0}\, \mathbf{det}'(d^\dagger d)$$

Proof. For $t > 0$, The operator $D_t = d + tH$, although it is $\mathbb{Z}_2$ graded, preserves the $\mathbb{Z}$ grading on forms except in the highest and lowest few degrees. $D_t : C^0 \oplus C^{n-1} \to C^1 \oplus C^n$ has kernel C^{n-1}_{cl}, since

$H\wedge : C^0 \to C^n$ is a bijection. Now $D_t = d$ when acting on forms of non-zero degree, so the D_t-closed forms are just d-closed forms in this range. The D_t-exact forms are given by d-exact forms of degree > 1, as well as the composite forms $(d\tau, tH \wedge \tau) \in C^1 \oplus C^n$. So the cohomology groups of $\mathbb{A}$ are the same as those for d, except possibly in degrees $0, 1$ and n. Consider a closed form $\beta \in C^n_{cl}$. Since $H^n(C, d) = \mathbb{R}[H]$, choose a form $\gamma \in C^{n-1}$, so that $\beta - d\gamma = \lambda t H, \lambda \in C^0_{cl}$, i.e $\beta = D_t(\lambda + \gamma)$, thus β is D_t-exact. Now, take a closed form $\alpha \in C^1_{cl}$. α is cohomologus to $(\alpha + d\tau, tH \wedge \tau)$ and since $tH \wedge \tau$ is closed, it is also exact by the previous argument. Thus $\alpha \sim \alpha + d\tau$, and if α is d-harmonic, then it is also D_t-harmonic. Thus we find

$$H^\bullet(C, D_t) \cong \bigoplus_{k=1}^{n-1} H^k(C, d)$$

We can also see this by noting that $H(C, \mathbb{A}) = H(H(C, d), H\wedge)$, but its not obvious that we can choose homogenous harmonic representatives for our cohomology classes. This means that the D_t-harmonic forms are $\mathbb{Z}$-graded, and so we have

$$\mathbf{Tr}(NP_t) = \boldsymbol{\chi}_0'(d) = \sum_{k=1}^{n-1} (-1)^k k b_k = \boldsymbol{\chi}'(d) + n b_n = \boldsymbol{\chi}'(d) + n b_0$$

We need to look at the behavior of the unstable kernel under this deformation. The locally constant functions have a basis $\{v_i\}_{i=1}^{b_0}$, where v_i is non-vanishing only on the ith connected component. These trivially satisfy $d^\dagger d v_i = 0$, and we find $D_t^\dagger D_t v_i = t^2 \|H\|^2 v_i$, since H is co-closed. This tells us each v_i is a type $2a$ eigenvector of $D_t^\dagger D_t$, with eigenvalue $\lambda(t) = t^2 \|H\|^2$, and these are the only even forms in the unstable kernel. Thus we find

$$\theta_0 = \|H\|^{2b_0}, \quad \alpha_0 = 2b_0$$

The n-forms $v_i H$ are harmonic and are in the unstable kernel, however, these are of type $2b$, since $D_t D_t^\dagger H = t^2 \|H\|^2 H$. The result follows once we recall that only the type $2a$ eigenvalues contribute to the zeta function and hence the derived Euler characteristic. $\square$

In their paper [27], Mathai and Wu prove the part of this result about the regularised determinant using a subtle factorization argument of regularised determinants developed by Kontsevich and Vishik [24]. The main difference in our proof is that we only have to focus on the contributions of the finite collection of unstable eigenvalues, as suggested by Farber's work, and not the entire spectrum of the twisted differential. In the next chapter, we will show that that the volume form deforms in such a way that the analytic torsion is unchanged.

Chapter 4

The Spectral Sequence of a Deformation

We have shown that when there is a jump in the cohomology groups of a deformation d_t at $t = 0$, Farber [12] gives a formula for this jump in terms of the germ cohomology. In practice, the germ cohomology is hard to compute and thus it is also difficult to compute this size of this jump. There is a method to compute this cohomology provided by a certain spectral sequence that successively approximates the kernel of the operator D_t for small t, starting from the kernel of the undeformed operator D_0.

The idea is easily illustrated for the twisted de Rham operators $D_t = d + tH_3$: Let $v_k \in \Omega^k$ be a d-closed form, then we have $D_t v_k = tH_3 v_k = O(t)$. So all d-closed forms are D_t-closed to first order in t. If we can find a $k + 2$ form v_{k+2} such that $dv_{k+2} = -Hv_k$, then the curve $v(t) = v_k + tv_{k+2}$ is D_t-closed to second order, $D_t v(t) = O(t^2)$. If we can't find such a $(k+2)$-form, then v_k cannot possibly by the leading term of a D_t-closed form. We then repeat this process and search for a $(k + 4)$-form v_{k+4} such that $dv_{k+2} = -Hv_{k+2}$, and if we can find one then $v(t) = v_k + tv_{k+2} + t^2 v_{k+4}$ is D_t-closed to third order in t. The k-forms v_k such that we can repeat this process up to any order in t are precisely those which are the lowest degree terms of D_t-closed forms.

We give two constructions of this spectral sequence, and reference a result of Forman that states they are isomorphic. Once we have constructed this spectral sequence, we can extract information about the eigenvalues of the deformed complex, and we can also gain an insight into how the analytic torsion varies along a deformation.

4.1 The Spectral Sequence for Flat Superconnections

We begin with a preliminary introduction to the Leray spectral sequence for a filtered complex.

Let $(V^\bullet, d)$ be a $\mathbb{Z}_2$-graded complex. Suppose that $V^\bullet$ has a *decreasing filtration* of supspaces,

$$V^{\bar{k}} = U_0^{\bar{k}} \supset U_1^{\bar{k}} \supset \ldots \supset U_{N-1}^{\bar{k}} \supset U_N^{\bar{k}} = 0$$

such that each $(U_i^\bullet, d)$ is a subcomplex, i.e. $dU_l^{\bar{k}} \subset U_l^{\overline{k+1}}$. Now, define

$$\begin{aligned} Z_{j,l}^{\bar{k}} &= \{v \in U_l^{\bar{k}} : dv \in U_{l+j}^{\overline{k+1}}\} \\ B_{j,l}^{\bar{k}} &= U_l^{\bar{k}} \cap dU_{l-j}^{\overline{k-1}} \end{aligned}$$

Then define the *pages* of the spectral sequence

$$E_{j,l}^{\bar{k}} = Z_{j,l}^{\bar{k}} / (Z_{j-1,l+1}^{\bar{k}} + B_{j-1,l}^{\bar{k}})$$

We can now check that the differential, d, descends to a map ∂_j on this quotient, which is also a differential. We now state

Proposition 28 (Leray). *The quotient differential ∂_j maps $E_{j,l}^{\bar{k}}$ into $E_{j,l+j}^{\overline{k+1}}$, and we have*

$$E_{j+1,l}^{\bar{k}} \cong H^{\bar{k}}(E_{j,l}^\bullet, \partial_j)$$

furthermore, under certain hypothesis, the sequence stabilises after finitely many pages, $E_{M,l}^{\bar{k}} \cong E_{M+1,l}^{\bar{k}} \cong \ldots \cong E_{\infty,l}^{\bar{k}}$, and we have

$$\oplus_l E_{\infty,l}^{\bar{k}} \cong H^n(V, d)$$

We now show that for the flat superconnection that we have been studying, there is a natural filtration that allows us to apply this proposition.

4.1.1 Flat Superconnections

Let $E \to M$, be a super-vector bundle over a compact manifold M, equipped with a flat superconnection $\mathbb{A}$. Following [14, 30, 38], we show that there is an iterative procedure for computing the cohomology groups $H^\bullet(\mathscr{A}(M, E), \mathbb{A})$.

The sequence we construct here is the Leray spectral sequence associated to a decreasing filtration of $\mathscr{A}(M, E)$. Define the filtration by form degree

$$\mathscr{A}^{i,\bar{k}}(M, E) = \bigoplus_{j \geq i} \Omega^j(M, E^{\overline{k-j}})$$

i.e. $\mathscr{A}^{2,\bar{1}}(M, E) = \Omega^2(M, E^{\bar{1}}) \oplus \Omega^3(M, E^{\bar{0}}) \oplus \cdots$. It is clear that these spaces filter $\mathscr{A}^{\bar{k}}(M, E)$

$$\mathscr{A}^{\bar{k}}(M, E) = \mathscr{A}^{0,\bar{k}}(M, E) \supset \mathscr{A}^{1,\bar{k}}(M, E) \supset \ldots \mathscr{A}^{n,\bar{k}}(M, E)$$

and that $\mathbb{A} : \mathscr{A}^{i,\bar{k}}(M,E) \to \mathscr{A}^{i,\overline{k+1}}(M,E)$. Thus the superconnection is compatible with this filtration.

It is the Leray spectral sequence for the $\mathbb{Z}_2$-graded complex given by a flat superconnection which is of principal importance for us. For the case we have already studied, namely flat superconnection on the trivial $\mathbb{Z}_2$-graded line bunlde $E = E_0 = \mathbb{R}$, we have $\mathbb{A}_0 = 0$, and our spectral sequence has $E_2^{\bar{k}} = H^{\bar{k}}(E, \mathbb{A}_1)$. If we wish to consider higher rank bundles, the second page of the spectral sequence is built out of sections of E that are in the kernel of $\mathbb{A}_0$.

We turn again to our simple example $D = d + H$, where H is a three form. We argued in the introduction to this chapter that the first few pages of this spectral sequence are described by the following space

$$\begin{aligned} E_1^{\bar{k}} &= C^k(E) \\ E_2^{\bar{k}} &= H^{\bar{k}}(E,d) \\ E_3^{\bar{k}} &= \{v_0 \in E_2^{\bar{k}} : \exists v_2 \in E_1^{\bar{k}} \text{ with } H \wedge v_0 = -dv_2\} \end{aligned}$$

The condition for E_3 can be rephrased as $[H] \cup [v_0] = 0$ in $H^{\bar{k}}(E,d)$. We continue on

$$E_4^{\bar{k}} = \{v_0 \in E_3^{\bar{k}}, H \wedge v_0 = -dv_2 : \exists v_4 \in E_1^{\bar{k}} \text{ with } H \wedge v_2 = -dv_4\}$$

The higher differentials in the Leray spectral sequence of the flux-twisted differential $d+H$, where H is a total odd form, have an alternative description in terms of higher operations in cohomology. These operations are known as *Massey Products*, c.f. [4, 38, 42], but we will not explore their topological significance here.

4.2 The Adiabatic Spectral Sequence

It was shown in [14] that the Leray spectral sequence for a (finite dimensional) filtered complex has an alternative description in terms of Hodge theory. To extend these results to infinite dimensions and to the $\mathbb{Z}_2$-graded setting, we use the analyticity theorem of Kato [22], following Farber [12], whereas Forman uses a result of Rellich [36] to guarantee that certain eigenvectors and eigenvalues have appropriate power series expansions. Most of the algebra, however, remains largely unchanged.

First, equip the complex $(V^\bullet, d)$ with a metric g: the essential ingredient of Hodge theory. Since $U_{i+1}^{\bar{k}} \subset U_i^{\bar{k}}$, let $V_i^{\bar{k}}$ be the orthogonal complement of $U_{i+1}^{\bar{k}}$ in $U_i^{\bar{k}}$, and let $g_{V_i^{\bar{k}}}$ be the restriction of the metric to this subspace. This process yields and orthogonal decomposition

$$V^{\bar{k}} = \oplus_i V_i^{\bar{k}} \quad \text{with} \quad U_j^{\bar{k}} = \oplus_{j>i} V_i^{\bar{k}}$$

Now let $\rho_t : V^\bullet \to V^\bullet$ be the operator which acts as multiplication by t^j on $V_j^{\bar{k}}$. Forman considers the one parameter family of metrics $g_t = \rho_t^* g = \sum t^{2j} g_{V_j^{\bar{k}}}$, with the goal of studying the dependence of the spectrum of the associated Laplacians on the parameter t. This analysis is based on the following observations: the Laplacian will be written $\Delta_t = dd^{*_t} + d^{*_t}d$, where $*_t$ indicates adjoint with respect to the metric g_t. If we order by increasing magnitude, the k^{th} eigenvalue of Δ_t can be written in the variational form

$$\lambda_k(t) = \sup_{v_1,\ldots,v_{k-1}\in V^n} \inf_{v_k \perp \{v_1 \ldots v_{k-1}\}} \frac{|dv_k|_t^2 + |d^{*_t}v_k|_t^2}{|v_k|_t^2} \tag{4.2.1}$$

Conjugating with the operator ρ_t, we have $|dv_k|_t = |d_t\rho_t v_k|$, where $d_t := \rho_t d \rho_t^{-1}$. We can decompose the operator d_t as follows, for $v \in V_j^{\bar{k}}$ we write

$$dv =: \sum_{\ell \geq 0} d_\ell v \qquad \text{where } d_\ell v \in V_{j+\ell}^{\overline{k+1}}$$

since d preserves the filtration. The conjugated operator is then $d_t = \sum_{\ell\geq 0} t^\ell d_\ell$. We also have $|d^{*_t}v_k|_t = |d_t^* \rho_t v_k|$, where $d_t^* := \rho_t d^{*_t} \rho_t^{-1}$, and it is not hard to check that in fact $d_t^* = (d_t)^*$. Thus the spectrum of the Laplacian can be written as a variational form of the expression $|d_t v_k| + |d_t^* v_k|$. The conjugated Laplacian $\Delta_t' := \rho_t \Delta_t \rho_t^{-1} = d_t d_t^* + d_t^* d_t$ is therefore a operator valued polynomial in t.

Motivated by minimising the quotient in (4.2.1), we introduce the spaces

$$E_j^{\bar{k}} = \{v_0 \in V^{\bar{k}} | \exists v_t = v_0 + tv_1 + \ldots + t^j v_j,\, d_t v_t \in t^j V^{\overline{k+1}}[t], d_t^* v_t \in t^j V^{\overline{k+1}}[t]\}$$

I.e, if $v \in E_j^{\bar{k}}$ then it is the leading term of a power series in t that is d_t closed and co-closed to order j. This means that for each $v_0 \in E_j^{\bar{k}}$, there is a $v_t \in V^{\bar{k}}[t]$ such that $(\Delta_t v_t, v_t) = |d_t v|^2 + |d_t^* v|^2 = O(t^{2j})$. Because of this, we will show that these spaces approximate the kernel of the Laplacian. Consider the eigenvalue equation

$$\Delta_t w(t) = \lambda(t) w(t), \qquad \text{with } \lambda(t) = O(t^{2j}) \tag{4.2.2}$$

The dimension of the space of solutions to this equation is computed by the spaces $E_j^{\bar{k}}$.

Proposition 29 (Forman [14])**.**

$$\#\{\lambda(t) \textit{ such that (4.2.2) holds and } \lambda(t) = O(t^{2j})\} = \dim E_j^{\bar{k}}$$

$$\mathrm{Span}\{w(t) \in V^{\bar{k}} \textit{ such that (4.2.2) holds}\} = \rho_t(E_j^{\bar{k}} + O(t))$$

This shows that the sections which are both d_t closed and co-closed to order j do actually represent the sections which vanish under Δ_t to order $2j$.

Proof. By definition, for each v_0 in $E_j^{\bar{k}}$ there is a $v(t) = v_0 + O(t) \in V^{\bar{k}}[t]$ such that the quotient in the variational form is $O(t^{2j})$. These sections span a one parameter subspace of $V^{\bar{k}}$, on which the variational formula for the eigenvalues yields $\lambda_i^{\bar{k}}(t) = O(t^{2j})$ for $i < \dim E_j^{\bar{k}}$. Now, the spectrum of the Laplacian Δ_t is equal to that of the conjugated Laplacian $\Delta_t' = \rho_t \Delta_t \rho_t^{-1} = d_t d_t^* + d_t^* d_t$. This operator is a finite power series in t, and is self adjoint for real t. This gives rise to a analytic deformation (in the sense of the last chapter) of the operator Δ'_0. Thus, appealing to the theorem of Kato [22], for each eigenvalue $\lambda_i(t) = O(t^{2j})$ of Δ'_t there exists a corresponding *analytic* eigenvector $v(t) = v_0 + O(t)$. Terminating this series expansion at some term of sufficiently high degree, we find v_0 in $E_j^{\bar{k}}$, and thus the dimension of $E_j^{\bar{k}}$ measures the number of eigenvalues that are $O(t^{2j})$. Correspondingly, for each eigenvector $v(t)$ of Δ'_t, the leading term lies in $E_j^{\bar{k}}$, and thus $\rho_t v(t)$ is an eigenvalue of Δ_t, and so its leading term is in $\rho_t E_j^{\bar{k}}$. □

Now that we have shown how these spaces relate to the spectrum of the Laplacian, we discuss the extra structure that allows us to compute their dimensions. For $v_0 \in E_j^{\bar{k}}$, such that $v(t) = v_0 + O(t)$ is both closed and coclosed to order j. We then know that $t^{-j} d_t v(t) \in V^{\overline{k+1}}[t]$ is an analytic power series. Let π_j denote orthogonal projection onto $E_j^{\overline{k+1}}$.

Lemma 17. *The map $\partial_j : E_j^{\bar{k}} \to E_j^{\overline{k+1}}$, given by $\partial_j v_0 = \pi_j \lim_{t\to 0} t^{-j} d_t v(t)$, where $v(t)$ is any choice of closed/co-closed series with $t^{-j} d_t v(t) \in V^{\overline{k+1}}[t]$ and $v(0) = v_0$, is well defined. Furthermore, $\partial_j^2 = 0$.*

Proof. We need to show that if $v(t), w(t)$ are two such extensions of v_0, then $\pi_j \lim_{t\to 0} t^{-j} d_t(v(t) - w(t)) = 0$. This is equivalent to saying that for every $u(t) \in V_j^{\overline{k+1}}$ both closed and co-closed to order j, we have $g(u(t), t^{-j} d_t(v(t) - w(t)) = O(t)$, i.e. $g(u(t), d_t(v(t) - w(t))) = O(t^{j+1})$. We have $g(u(t), d_t(v(t) - w(t)) = g(d_t^* u(t), v(t) - w(t))$. Now, by definition, we have $d_t^* u(t) = O(t^j)$, and $v(t) - w(t) = O(t)$, since they both have the same leading term. Thus the inner product of these two terms is $O(t^{j+1})$. To show that $\partial_j^2 = 0$, observe that $\partial_j(\partial_j v_0) = \partial_j(\pi_j \lim_{t\to 0} t^{-j} d_t v(t)) = \pi_j \lim_{t\to 0} t^{-2j}(d_t^2 v(t)) = 0$, since $\pi_j = t^{-j} d_t v(t)$ is an extension of $\partial_j v_0$. □

The goal of was to approximate the eigenspaces of the Laplacian, and now we can see that these spaces do just that.

Lemma 18. *The eigenvalues $\lambda(t)$ of $\Delta_t^{\bar{k}}$ that are $O(t^{2j})$ are of the form*

$$\lambda(t) = \lambda_{2j} t^{2j} + O(t^{2j+2})$$

where λ_{2j} is an eigenvalue of $\partial_j \partial_j^* + \partial_j^* \partial_j : E_j^{\bar{k}} \to E_j^{\bar{k}}$.

Proof. Given a solution to $\Delta'_t v(t) = \lambda(t) v(t)$, with $\lambda(t) = O(t^{2j})$, i.e $v_0 := v(0) \in E_j^{\bar{k}}$, we have $(d_t^* d_t + d_t^* d_t) v(t) \in t^{2j} V^{\bar{k}}[t]$. Since $\operatorname{im} d_t^* d_t \perp \operatorname{im} d_t d_t^*$, we must have each of $d_t^* d_t v(t), d_t^* d_t v(t) \in t^{2j} V^{\bar{k}}[t]$, and thus $d_t^* v(t), d_t v(t) \in t^j V^{\bar{k}}[t]$. Let $d_t v(t) = t^j w(t)$, Clearly, $d_t w(t) = 0$, and $d_t^* w(t) \in t^j V^{\overline{k+1}}[t]$, so by definition $w_0 := w(0) \in E_j^{\bar{k}}$, i.e. $w_0 = \partial_j v_0$. We then have $\partial_j^* \partial_j v_0 = \partial_j^* w_0 = \pi_j \lim_{t\to 0} t^{-j} d_t^* w(t)$. We then substitue $w(t) = t^{-j} d_t v(t)$ into this last expression to yield $\partial_j^* \partial_j v_0 = \pi_j \lim_{t\to 0} t^{-2j} d_t^* d_t v(t)$.

After similar analysis for $\partial_j \partial_j^* v_0$, we find

$$\begin{aligned}(\partial_j \partial_j^* + \partial_j^* \partial_j) v_0 &= \pi_j \lim_{t \to 0} t^{-2j} (d_t^* d_t + d_t^* d_t) v(t) \\ &= \pi_j \lim_{t \to 0} t^{-2j} \lambda(t) v(t) \\ &= \pi_j \lim_{t \to 0} (\lambda_{2j} + O(t)) v(t) \\ &= \lambda_{2j} v_0\end{aligned}$$

The result follows. □

This now leads us to one of the main conclusions of Forman's work.

Corollary 12. *The spaces $\{E_j^{\bar{k}}, \partial_j\}$ form a spectral sequence, known as the* adiabatic spectral sequence. *In particular*

$$\ker(\partial_j \partial_j^* + \partial_j^* \partial_j : E_j^{\bar{k}} \to E_j^{\bar{k}}) = E_{j+1}^{\bar{k}}$$

Proof. From the last proposition, we know that

$$\dim \ker(\partial_j \partial_j^* + \partial_j^* \partial_j : E_j^{\bar{k}} \to E_j^{\bar{k}}) = \#\{\lambda(t) \in \mathrm{Spec}(\Delta_t) : \lambda(t) = O(t^{2(j+1)})\}$$

We just need to show that each of these eigenvalues corresponds to a eigenvector $v(t)$, with $v(0) \in E_{j+1}^{\bar{k}}$. This is easily seen since $v(0) \in E_{j+1}^{\bar{k}} \subset E_j^{\bar{k}}$ guarantees $d_t^* v(t)$ and $d_t v(t) \in t^{2(j+1)} V^{\overline{k+1}}[t]$, and thus $\partial_j^* v(0) = \partial_j v(0) = 0$, and so $v(0) \in \ker \partial_j \partial_j^* + \partial_j^* \partial_j$. □

There is also a natural $\mathbb{Z}$-grading on these spaces, which we have made reference to before.

Proposition 30 (Forman)**.** *The differential ∂_j is compatible with the $\mathbb{Z}$ grading on $E_j^{\bar{k}}$ given by $E_{j,l}^{\bar{k}} = E_j^{\bar{k}} \cap V_l^{\bar{k}}$, i.e.*

1) $E_j^{\bar{k}} = \bigoplus_{\ell \geq 0} E_{j,l}^{\bar{k}}$

2) $\partial_j E_{j,l}^{\bar{k}} \subset E_{j,\ell+j}^{\overline{k+1}}$

3) $\partial_j^* E_{j,l}^{\bar{k}} \subset E_{j,\ell-j}^{\overline{k+1}}$

Proof. We follow Forman's original proof closely. We prove all three statements simultaneously by induction, starting with $E_0^{\bar{k}} = V^{\bar{k}}$. For $v_0 \in E_{0,\ell} = V_\ell$, we have $\partial_0 v_0 = \lim_{t \to 0} d_t v(t) = d_0 v_0 \in V_\ell^{\overline{k+1}}$, where $v(t)$ is an arbitrary extension of v_0. Clearly the initial statement hold holds. For the inductive step, pick $v \in E_j^{\bar{k}}$, write $v = \sum_\ell v_\ell$ where $v_\ell \in V_\ell^{\bar{k}}$, we need to show that each $v_\ell \in E_j^{\bar{k}}$ to prove the first part. By the inductive hypothesis, and since $E_j^{\bar{k}} \subset E_{j-1}^{\bar{k}}$, we know that each $v_\ell \in E_{j-1}^{\bar{k}}$. In the last proposition, we showed that $E_j = \ker(\partial_j \partial_j^* + \partial_j^* \partial_j : E_{j-1}^{\bar{k}} \to E_{j-1}^{\bar{k}})$, which implies $0 = \partial_{j-1} v = \partial_{j-1}^* v$. So by linearity $0 = \sum_\ell \partial_{j-1} v_\ell$. Now, again by the inductive hypothesis, $\partial_j v_\ell \in E_{j-1,\ell+(j-1)}^{\overline{k+1}}$. Since the space $E_{j-1,\ell+(j-1)}$ are orthogonal for varying ℓ, we must have $\partial_{j-1} v_\ell = 0$, and similarly for $\partial_j^* v_\ell$. Being both ∂_j closed and coclosed, we have $v_\ell \in E_j^{\bar{k}}$. This proves part (1). To prove part (2), we need to show for $v \in E_{j,l}$, we have $\partial_j v \in E_{j,\ell+j}$. For such a v, choose an extension $v(t) = v + O(t)$ that is closed and co-closed to order j. By definition, we have $d_t v_t = t^j(\partial_j v) + O(t^{j+1})$. Now write

$$v(t) = \{v + \sum_{i>0} t^i v_{i,\ell+i}\} + \{\sum_{i>0, c\in\mathbb{Z}, c\neq i} t^i v_{i,\ell+c}\}$$

where $v_{i,r}$ is the orthogonal projection on v_i onto $V_r^{\bar{k}}$. Label this decomposition at $v(t) = v_1(t) + v_2(t)$. Then $d_t v(t) = d_t v_1(t) + d_t v_2(t) =: \left(\sum_i t^i w_{1,i}\right) + \left(\sum_i t^i w_{2,i}\right)$, and $\partial_j v = \pi_j w_{1,j} + \pi_j w_{2,j}$, which we will show corresponds to the decomposition $V^{\bar{k}} = V_{\ell+j}^{\bar{k}} \oplus (V_{\ell+j}^{\bar{k}})^{\perp}$.

We will show that $\pi_j w_{1,j} \in E_{j,\ell+j}$ as required, and $\pi_j w_{2,j} = 0$. Notice that for each k, we have $w_{1,k} = \sum_{m\geq 0} d_{k-m} v_{m,\ell+m} \in V_{\ell+k}$, and in particular, $\pi_j u_j \in E_j^{\bar{k}} \cap V_{\ell+j} = E_{j,\ell+j}^{\bar{k}}$. To show that the second term in the splitting of vanishes $\partial_j v$, notice that again for each k, we have

$$w_{2,k} = \sum_{m\geq 0, c\in\mathbb{Z}, c\neq m} d_{k-m} v_{m,\ell+c} \in \bigoplus_{m\geq 0, c\neq m} V_{\ell+k+(c-m)} = (V_{l+k})^{\perp}$$

Now, since $w_i = w_{1,i} + w_{2,i} = 0$ for $i < j$, and we know this splitting is orthogonal, we have $w_{1,i} = w_{2,i} = 0$. Thus, this guarantees there is no mixing of terms (at least for $i < j$) in the decomposition

$$d_t(v_1(t) + v_2(t)) = \left(\sum_i t^i w_{1,i}\right) + \left(\sum_i t^i w_{2,i}\right)$$

i.e. $d_t v_2(t) = t^j w_{2,j} + O(t^{j+1})$, and thus $w_{2,j} = \partial v_2(0)$. Now, choose any $u \in E_j$, and then check $g(w_{2,j}, u) = \lim_{t\to 0} t^{-j} g(d_t v_2(t), u)$. Then we just recall $v_2(t) = O(t)$ and $d_t^* u(t) = O(t^j)$, where $u(t)$ is an extension of u. This shows that $w_{2,j} \in (E_j^{\bar{k}})^{\perp}$, i.e. $\pi_j w_{2,j} = 0$. This proves part (2), and part (3) is proven almost identically. □

The compatibility of the leading terms of this spectral sequence with the grading yields an important result that we used in the previous chapter

Corollary 13. *For the twisted de-Rham complex, $(C^\bullet, d_t = d + tH)$, the space of harmonic forms $H^\bullet$ has a basis in which every element $v(t) = v_0 + O(t)$ has homogenous leading term v_0 in the natural $\mathbb{Z}$ grading on forms.*

Now that we have explored this spectral sequence, we will soon see how it fits into our analysis of analytic torsion.

4.2.1 Comparison of the Two Approaches

So far, we have constructed two spectral sequences out of a filtered complex. First, the Leray spectral sequence, a classical construction which successively computes the cohomology by looking at how the differential acts on certain quotients. Secondly, the adiabatic spectral sequence requires a metric and then used hodge theory to define the pages, and a suitable differential was found constructed. It is then somewhat surprising that the second major focus of Forman's work was to show that these spectral sequences were in fact isomorphic

Proposition 31 (Forman [14], Thm 7)**.** *Given a filtered complex, let $\{e^{\bar{k}}_{j,l}, \tilde{d}\}$ be the associated Leray spectral sequence, and $\{E^{\bar{k}}_{j,l}, \partial_j\}$ be the adiabatic spectral sequence. There is a sequence of bijections*

$$\phi_j : E^{\bar{k}}_{j,l} \to e^{\bar{k}}_{j,l}$$

which are compatible with the differentials.

Proof. We refer to the proof in the original paper [14] □

Since the Leray spectral sequence did not involve a metric, this theorem shows that the numbers $\dim E^{\bar{k}}_j$ are independent of the metric chosen, and thus we see that the spectral dependence of Δ_t on t, i.e. the number of eigenvalues that are $O(t^{2j})$, is also independent of this choice.

4.2.2 Determinant Lines

We can now apply the Knudsen-Mumford isomorphisms discussed earlier to the adiabatic spectral sequence. Since $E^{\bar{k}}_{j+1} \cong H(E^{\bar{k}}_j, \partial_j)$, we have a canonical isomorphism $\det E^{\bar{k}}_{j+1} \cong \det E^{\bar{k}}_j$, which can be described as follows: let $\{v_i\}_{i=1}^{N^{\bar{k}}}$ be an orthonormal basis for for $E^{\bar{k}}_j$, such that $\{v_i\}_{i=1}^{M^{\bar{k}}}$, $M^{\bar{k}} \leq N^{\bar{k}}$ form such a basis for $E^{\bar{k}}_{j+1} \subset E^{\bar{k}}_j$. Recall that the v_i extend to eigenvectors $v_i(t)$ of Δ'_t, with eigenvalues that are $O(t^{2j})$. The isomorphism $\kappa_j : \det(E^{\bullet}_j) \to \det(E^{\bullet}_{j+1})$ is given by

$$\kappa_j : \left(\bigwedge_{i=1}^{N^{\bar{k}}} v^{\bar{k}}_i\right) \wedge \left(\bigwedge_{i=1}^{N^{\overline{k+1}}} v^{\overline{k+1}}_i\right)^{-1} \quad \mapsto \quad \mathbf{det}'(\partial^*_j\partial_j : E^{\bullet}_j \to E^{\bullet}_j) \left(\bigwedge_{i=1}^{M^{\bar{k}}} v^{\bar{k}}_i\right) \wedge \left(\bigwedge_{i=1}^{M^{\overline{k+1}}} v^{\overline{k+1}}_i\right)^{-1}$$

We know that $\mathrm{Spec}(d^*_t d_t + d_t d^*_t) = \mathrm{Spec}(d^*_t d_t) \cup \mathrm{Spec}(d_t d^*_t)$, and so the proof of lemma 18 reveals that

$$\mathrm{Spec}(d^*_t d_t | \lambda(t) = O(t^{2j})) = t^{2j}\,\mathrm{Spec}(\partial^*_j \partial_j) + O(t^{2j+1})$$

Thus the l.h.s of the above is

$$\frac{\prod \lambda^i_{2j}}{\prod \lambda^i_{2j}} \left(\bigwedge_{i=1}^{M^{\bar{k}}} v^{\bar{k}}_i\right) \wedge \left(\bigwedge_{i=1}^{M^{\overline{k+1}}} v^{\overline{k+1}}_i\right)^{-1}$$

By composing all of the Knudsen-Mumford isomorphisms for the various pages of the adiabatic spectral sequence, we find

Corollary 14. *For an analytic deformation, there is a canonical isomorphism*

$$\kappa : \det H(C, d_0) \to \det H(C, D_t)$$

for small t.

Proof. The spectral sequence constructed above begins at $E_1 = H(C, d_0)$, and then proceeds $E_{r+1} = H(E_r, \partial_r)$, using the Knudsen-Mumford isomorphisms (2.1.1), we get

$$\kappa_r : \det E_r \xrightarrow{\sim} \det E_{r+1}$$

Composing all these isomorphisms up to the terminal stage $E_N = H(C, d_t)$, we get

$$\kappa := \kappa_N \cdots \kappa_2 \kappa_1 : \det E_1 \xrightarrow{\sim} \det E_N$$

that is

$$\kappa : \det H(C, d_0) \xrightarrow{\sim} \det H(C, d_t)$$

□

4.3 Application To Analytic Torsion

We can now apply this analysis of the adiabatic spectral sequence to the analytic torsion for a $\mathbb{Z}_2$-graded complex, specifically, the flux twisted torsion. In this case, we are considering the family $d_t = td + \sum t^{2i+1} H^{2i+1}$. This operator is closely related to the operator $D_t = d + \sum t^i H^{2i+1}$ considered in chapter 2, $d_t = tD_{2t}$, which is an analytic deformation of the untwisted exterior derivative $d = D_0$. Recall we had two elements

$$\tau(C^\bullet, d) \in \det(H(C^\bullet, d))$$

$$\tau(C^\bullet, D_1) \in \det(H(C^\bullet, D_1))$$

defined by

$$\tau(C^\bullet, D_t) = \mathbf{det}'(D_t^* D_t)^{1/2} \left(\bigwedge_{i=1}^{b^{\bar{k}}} e_i(t)^{\bar{k}} \right) \otimes \left(\bigwedge_{i=1}^{b^{\overline{k+1}}} e_i(t)^{\overline{k+1}} \right)^{-1}$$

where $\{e_i^{\bar{k}}(t)\}$ form an orthonormal basis of the $\Delta_t^{\bar{k}}$-harmonic sections. One of our main theorems (14) was to show that for the flux twisted torsion, we have

$$\mathbf{det}'(D_1^* D_1) = (\theta_{\bar{0}}/\theta_{\bar{1}})\, \mathbf{det}'(d_t^* d_t) e^{f(0)}$$

The argument in the last section showed that

$$(\theta_{\bar{0}}/\theta_{\bar{1}}) = \prod_{j=1}^{\infty} \mathbf{det}'(\partial_j^* \partial_j : E_j^\bullet \to E_j^\bullet)$$

Thus, the leading coefficient in our expression for the flux twisted torsion is precisely the factor accrued by performing the successive Knudsen-Mumford isomorphisms to the determinant line as we move through the pages of the spectral sequence.

After all this work, we now arrive at the main theorem of this thesis.

Theorem 15. *Upon applying the Knudsen-Mumford isomorphism, the flux twisted analytic torsion is related to the Ray-Singer torsion by the following*

$$\kappa(\tau(C^\bullet, d)) = e^{f(0)}\tau(C^\bullet, D_1) \in \det(H(C^\bullet, D_1))$$

where

$$f(0) = \int_0^1 \left(\mathbf{Tr}(NP_s) - \boldsymbol{\chi}_0'(d)\right) s^{-1} ds$$

P_s is the projection onto the kernel of $\Delta_s = D_s^* D_s + D_s D_s^*$, and $\boldsymbol{\chi}_0'(d)$ is the number of unstable eigenvalues of D_t at $t = 0$.

We now make the observation that since both the twisted and untwisted analytic torsions were independent of the metric chosen, the constant $f(0)$ must also be independent of such a choice. If it could be shown that $f(0)$ vanishes, it would reveal a quite remarkable results: That the analytic torsions of two quite different elliptic operators, d_0 and $d = d_0 + H$, are only related via a simple canonical cohomological isomorphism. Further investigation of the s-dependence of the term $\mathbf{Tr}(NP_s)$ might shed light on this conjecture.

4.3.1 Example: Lie Groups

An interesting example of twisted analytic torsion comes from the case of compact Lie groups. Several unique properties of Lie groups simplify the computations involved, we will first explore these interesting properties, and then process to compute the twisted torsion in the case of a twist by a certain canonical 3-form. Firstly, because of the group structure, the cohomology ring of a Lie group is an exterior algebra [17]. For the simple Lie group $G = U_n$, we have

$$H^\bullet(U_n, \mathbb{Z}) = \Lambda_{\mathbb{Z}}(\alpha_1, \alpha_3, \alpha_5, \dots \alpha_{2n-1}) \tag{4.3.3}$$

The classes α_j have G-invariant representatives as differential forms given by the expressions

$$\alpha_j = b_j \operatorname{tr} \theta^j, \quad \theta = g^{-1}dg \in \Omega^1(G, \mathfrak{g})$$

Where the $b_j \in \mathbb{C}$, in particular

$$\alpha_3 = \tfrac{1}{24\pi^2} \operatorname{tr} \theta^3$$

Rohm and Witten [38] noticed that the flux-twisted cohomology of $D = d + \alpha_3$ vanishes, which can be shown by using a spectral sequence argument which follows from the adiabatic spectral sequence as follows: We have $E_2^{\bar{k}} = H^{\bar{k}}(G,\mathbb{R}) = \bigwedge_{\mathbb{Z}}\{\alpha_i\}$. Because of this algebra structure, the map $\alpha_3\wedge : H^{\bar{0}}(G,\mathbb{R}) \to H^{\bar{1}}(G,\mathbb{R})$ is an isomorphism, and hence $E_4^{\bar{k}} = H^{\bar{k}}(H(G,\mathbb{R}),\alpha_3\wedge) = 0$, and the spectral sequence vanishes at this page. The second important property of Lie groups is that there is a canonical left-invariant metric given by the Killing form, once we have trivialised the tangent bundle using left multiplication $TG \cong G \times \mathfrak{g}$.

Proposition 32 ([17])**.** *With respect to this metric, the space of harmonic forms is isomorphic to the space of G-invariant forms. Furthermore, $d^\dagger$ is an odd derivation of the algebra of differential forms.*

Because of this, we have the important fact

Corollary 15. *On a Lie group G, let $H = H_{2k+1}$ be a closed, G-invariant, odd form. With $D = d + H$, we find*

$$\Delta_H := D^\dagger D + DD^\dagger = \Delta + H^\dagger H + HH^\dagger$$

i.e. Δ_H preserves the $\mathbb{Z}$-grading on forms.

Proof. We have to check that the off diagonal terms vanish. They are $(d^\dagger H + Hd^\dagger) : C^\bullet \to C^{\bullet+2k}$ and its adjoint. These operations are identically zero because $d^\dagger$ is an odd derivation and H is coclosed □

Because of this fact, we can choose a basis of the eigenfunctions of Δ_H that are all homogeneous forms. To compute the regularised determinant, we will need some preliminary computations

Lemma 19. *Let $H = \mu\alpha_3$ The action of $H^\dagger H$ on $H^\bullet(G)$ is given by multiplication by μ^2 on $(\operatorname{im} H)^\perp$, and 0 otherwise.*

Proof. From the ring structure, it is obvious that $\ker H = \operatorname{im} H$ and that $H^\dagger H\alpha = \pm *(H \wedge *(H\wedge\alpha)) \sim \alpha$, for any $\alpha \in H^\bullet(G)$. Then $(H^\dagger H\alpha,\alpha) = (H\alpha, H\alpha) = \|H\|^2(\alpha,\alpha)$. □

This result tells us the action of the operator $H^\dagger H$ on the invariant forms, but not the full complex of forms. We have shown that we only need to compute the leading non-vanishing terms of the spectrum of Δ_t.

Proposition 33. *The type 2 eigenvalues (unstable) of the germ Laplacian for the family $D_t = d + t\mu\alpha_3$ are of the form*

$$\lambda(t) = t^2\mu^2 + O(t^3)$$

Proof. We use our result, Theorem 18, that the leading terms of these eigenvalues are given by the eigenvalues of the partial Laplacian of the differentials in the adiabatic spectral sequence, and $H^\dagger H$ is the partial Laplacian of the differential $\partial_3 = H$. □

Now, because of the structure of the cohomology ring, we have $\Omega(G)^G = \dim\ker H \oplus (\dim\ker H)^\perp$, and $H\wedge : (\dim\ker H)^\perp \to \dim\ker H$ is an isomorphism. Using this information, and the theorems presented in this chapter, we have the following calculation of the twisted analytic torsion for Lie groups

Proposition 34. *On $G = U_n$, for the twisted De Rham operator $D = d + \lambda\alpha_3$, we have*

$$\boldsymbol{\chi}'(D) = 0$$

$$\mathbf{det}'(D^\dagger D) = \mathbf{det}'(d^\dagger d)$$

and thus

$$\kappa(\tau(G, d)) = \tau(G, d + \lambda\alpha_3) \in \mathbb{R}$$

Proof. Because the twisted cohomology vanishes, we use theorem (16). Because each type 2 unstable eigenvalue is of the form given in proposition (33), each harmonic form leaves the kernel with the coefficient $t^2\|H\|^2$. Since $H^\bullet(G)$ is an exterior algebra on the generators $\alpha_1 \ldots \alpha_{2n-1}$ we have $\alpha_{\bar{k}} = 2\dim H^{\bar{k}}(G)$, and $\theta_{\bar{k}} = \|H\|^{2\dim H^{\bar{k}}(G)}$. The result follows one we recall that the structure of the cohomology implies that $H^{\bar{0}}(G) \cong H^{\bar{1}}(G)$. □

So we have found that in this very specialised case of Lie groups, the twisted analytic torsion computation is quite simple, and just reduces to that of the regular regularized determinant as an element of the trivial line. This is indeed a peculiar example, the complex is $\mathbb{Z}_2$-graded, so are the partial Laplacians, and the full Laplacian preserves the $\mathbb{Z}$ grading. This is precisely a situation suited for the machinery developed in this thesis. We continue the search for more examples that display this grading compatibility phenomenon.

Chapter 5

Conclusions and Future Work

We have shown that with twisted analytic torsion, as defined by Mathai and Wu, we can extend Ray-Singer torsion for flat connections to the case of flat superconnections. The example we focused on was that of a flat superconnection on the trivial $\mathbb{Z}_2$ graded bundle $E = E_0 = \mathbb{R}$, namely the twisted de Rham complex. We found that if we consider an associated analytic deformation of this complex, we see that the torsion varies in a calculable way. Specifically, we can determine the order of a singularity at $t = 0$ from information found in the parameterized hodge theory of Farber. We then use the extensions of the classical variation formulas of the analytic torsion to recompute the order of this singularity from large t as it approaches 0. Comparison of these two computed orders yields a formula for the derived Euler characteristic in terms of the deformation orders of the unstable eigenvalues. If the twisted cohomology vanishes, then we showed that the twisted and untwisted torsions are related by a canonical cohomological map, the Knudsen-Mumford map.

The work of Farber [12] was central to our analysis of the flux-twist analytic torsion. We considered the family $D_t = d + tH$ near the point $t = 0$, which provided us with a method to compare the torsions of $D_0 = d$, the untwisted differential, and D_1, the flux-twisted differential. We propose that this approach can be applied to the recent work of Mathai-Benameur [26], concerning a flux-twisted version of the signature complex. There is already relevant research done by Farber-Levine [13] looking at how the signature invariants change under analytical deformations of a complex, which is very much a follow on from the work in [12]. It is our hope to apply this work to relate the twisted signature invariants of Mathai-Benameur with their classic untwisted twisted analogues, in a similar fashion that was employed in this thesis.

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