The Caloron Correspondence and Odd Differential K-theory

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Contents

Abstract

The caloron correspondence (introduced in [32] and generalised in [25, 33, 41]) is a tool that gives an equivalence between principal G-bundles based over the manifold $M \times S^1$ and principal LG-bundles on M, where LG is the Fréchet Lie group of smooth loops in the Lie group G. This thesis uses the caloron correspondence to construct certain differential forms called *string potentials* that play the same role as Chern-Simons forms for loop group bundles. Following their construction, the string potentials are used to define degree 1 differential characteristic classes for $\Omega U(n)$-bundles.

The notion of an Ω *vector bundle* is introduced and a caloron correspondence is developed for these objects. Finally, string potentials and Ω vector bundles are used to define an Ω bundle version of the structured vector bundles of [38]. The Ω *model* of odd differential K-theory is constructed using these objects and an elementary differential extension of odd K-theory appearing in [40].

Signed statement

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In Erinnerung an meinen Opa,

Ermut Geldmacher

(20. Juni 1923 bis 24. Februar 2009)

einen geselligen und freundlichen Mann,
dessen Kreativität mich nach wie vor inspiriert.

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Introduction

The caloron correspondence appeared initially in the guise of a bijection between certain isomorphism classes of periodic instantons, or *calorons*, on $\mathbb{R}^4$ and isomorphism classes of monopoles on $\mathbb{R}^3$.

Considering monopoles for loop groups and their twistor theory, Garland and Murray established in [16] a correspondence between $SU(n)$-calorons on $\mathbb{R}^4$ and monopoles on $\mathbb{R}^3$ with structure group $LSU(n)$, the Fréchet Lie group of smooth loops in $SU(n)$. By virtue of being periodic, a caloron on $\mathbb{R}^4$ may be naturally viewed as an instanton on $\mathbb{R}^3 \times S^1$, thus the work of Garland and Murray may be viewed as establishing a relationship between certain geometric data on $SU(n)$-bundles over $\mathbb{R}^3 \times S^1$ and similar data on $LSU(n)$-bundles on $\mathbb{R}^3$.

The underlying principle of this original *caloron correspondence*—as it was first described by Murray and Stevenson [32]—is that, for any compact Lie group G and manifold M, there is a bijective correspondence between G-bundles over $M \times S^1$ and LG-bundles over M, with LG the Fréchet Lie group of smooth loops in G. This procedure gives a sort of fake dimensional reduction, whereby the circle direction of the G-bundle $P \to M \times S^1$ is hidden in the fibres to obtain an LG-bundle $\mathsf{P} \to M$ and vice-versa.

The caloron correspondence may be thought of as the bundle-theoretic generalisation of the following simple observation. If X, Y and Z are sets, then denoting by Y^X the set of all functions $X \to Y$, there is a bijection

$$c\colon Z^{X \times Y} \xrightarrow{\sim} \left(Z^Y\right)^X \tag{I.1}$$

given by sending $f \mapsto \check{f}$ with

$$\check{f}(x)(y) := f(x, y)$$

for $x \in X$ and $y \in Y$. In the case that X, Y and Z are finite-dimensional manifolds, let $\mathrm{Map}(X, Y)$ be the set of all smooth maps $X \to Y$. If Y is compact then $\mathrm{Map}(Y, Z)$ becomes a (smooth) Fréchet manifold. The map (I.1) now gives a method by which one may study smooth maps from X into the infinite-dimensional manifold $\mathrm{Map}(Y, Z)$ by instead studying smooth maps from $X \times Y$ into Z. In fact, in the case that $X = M$, $Y = S^1$ and $Z = G$, the map (I.1) gives a bijective correspondence between the space of sections of the trivial G-bundle over $M \times S^1$ and the space of sections of the trivial LG-bundle over M. For general (non-trivial) G-bundles, the caloron correspondence is a twisted version of this equivalence.

The caloron correspondence outlined thus far gives a means by which one may more easily study LG-bundles, which are necessarily infinite-dimensional manifolds, by instead studying their finite-dimensional G-bundle counterparts. Perhaps more importantly, the caloron correspondence may be extended to incorporate geometric data. In [32] it was shown that a G-bundle over $M \times S^1$ equipped with a G-connection is equivalent to an LG-bundle over M equipped with an LG-connection and an extra geometric datum called a *Higgs field*, which is essentially the component of a connection on a G-bundle over $M \times S^1$ in the S^1 direction.

Using this *geometric* caloron correspondence together with the machinery of bundle gerbes, Murray and Stevenson developed a useful generalisation of string classes. String classes first appeared in the work of Killingback [27] on string structures; the string theory versions of the spin structures that are important in quantum field theory. Taking a compact Lie group G one may ask whether a given LG-bundle $\mathsf{P} \to M$ admits a lifting of the structure group to the Kac-Moody group $\widehat{LG}$, which is a central extension of LG by S^1 (see [36], for example). The obstruction to such a lift is a class in the degree three integral cohomology of M. In the case that $\mathsf{P} = LQ \to LM$ is given by taking smooth loops in a G-bundle $Q \to M$, Killingback showed that this obstruction is given by transgressing the first Pontryagin class $p_1(Q)$ of Q. Thus the *string class* is

$$s(\mathsf{P}) = \widehat{\int_{S^1}} \mathrm{ev}^* \, p_1(Q) \in H^3(LM)$$

where $\mathrm{ev}\colon S^1 \times LM \to M$ is the evaluation map and $\widehat{\int_{S^1}}$ denotes integration over the fibre in integral cohomology. The string class of P measures the obstruction to P having *string structure*; i.e. a lifting to an $\widehat{LG}$-bundle. String structures are important in string theory because, as the work of Killingback shows, the loop bundle $LQ \to LM$ has a Dirac-Ramond operator if and only if LQ has a string structure.

Murray and Stevenson used the caloron correspondence to extend the work of Killingback by first defining the string class for all LG-bundles $\mathsf{P} \to M$ and showing that it satisfies

$$s(\mathsf{P}) = \widehat{\int_{S^1}} p_1(P)$$

where $p_1(P)$ is the first Pontryagin class of the caloron transform P of P. They also showed, using bundle gerbes, that a de Rham representative for the string class is given by

$$-\frac{1}{4\pi^2} \int_{S^1} \langle \mathsf{F}, \nabla\Phi \rangle$$

where F is the curvature of a chosen LG-connection on P, $\nabla\Phi$ is the covariant derivative of a chosen Higgs field Φ on P and $\langle \cdot, \cdot \rangle$ is the (normalised) Killing form.

In his PhD thesis [41] and together with Murray in [33], Vozzo generalised the caloron correspondence to principal bundles with structure group ΩG; the Fréchet Lie group of smooth loops in G based at the identity. The key innovation here is the use of framings to establish a correspondence between ΩG-bundles over M and G-bundles over $M \times S^1$ equipped with a distinguished section over $M \times \{0\}$. As before, this correspondence generalises to incorporate connective data, which must necessarily be compatible with the framing data on the G-bundle side.

Murray and Vozzo also defined (higher) string classes, which are characteristic classes for ΩG-bundles that live in odd integral cohomology. Fixing the ΩG-bundle $\mathsf{P} \to M$ and choosing an ΩG-connection A and Higgs field Φ, explicit de Rham representatives for these characteristic classes called *string forms* are given by

$$s_f(\mathsf{A}, \Phi) = k \int_{S^1} f(\nabla\Phi, \underbrace{\mathsf{F}, \dots, \mathsf{F}}_{k-1 \text{ times}}),$$

where f is an ad-invariant symmetric polynomial on the Lie algebra $\mathfrak{g}$ of G of degree k and F, $\nabla\Phi$ are as above. If A denotes the corresponding connection on the caloron transform P of P, it turns out that the string forms satisfy

$$s_f(\mathsf{A}, \Phi) = \widehat{\int_{S^1}} cw_f(A),$$

where $cw_f(A)$ is the Chern-Weil form associated to f and A. The higher string classes give a version of Chern-Weil theory for loop group bundles different from the more analytic approach of [35]. In addition, by considering the path fibration $PG \to G$ which is a smooth model for the universal ΩG-bundle, Murray and Vozzo show that the construction of the higher string classes provides a geometric interpretation of Borel's transgression map $\tau\colon H^{2k}(BG) \to H^{2k-1}(G)$ (see [2] for details).

This thesis grew out of the attempt to answer a natural question that arises when one contrasts the theory of string classes for loop group bundles to the familiar Chern-Weil theory. In Chern-Weil theory, differential form representatives for characteristic classes of the G-bundle $P \to X$ are given in terms of the curvature of a chosen connection A on P. Whilst the characteristic cohomology classes of the bundle P are necessarily independent of the choice of A, the differential form representatives are not. There are well-known differential forms, the Chern-Simons forms introduced in [11], that measure the dependence of the Chern-Weil forms on the choice of connection. It is natural to ask, therefore, whether such forms exists in the context of loop group bundles and string classes.

The first part of this thesis deals with the construction of the *string potentials*, which are the analogues of Chern-Simons forms for loop group bundles. Like Chern-Simons forms, the string potentials come in two different flavours: one has *relative* string potentials, which live on the base space of a loop group bundle and encode the dependence of the string forms on the choice of connection and Higgs field; and *total* string potentials, which live on the total space and carry secondary geometric data associated to a particular choice of connection and Higgs field.

Within the framework of the differential characters of Cheeger and Simons [10], the total Chern-Simons forms become *differential* characteristic classes (characteristic classes valued in differential cohomology). This thesis hints at a similar interpretation for the total string potentials by constructing such classes in a limited setting.

The interpretation of the relative string potential forms is more involved and proceeds by analogy with the codification of relative Chern-Simons forms given by Simons and Sullivan in [38]. In that paper, the authors use relative Chern-Simons forms to define an equivalence relation on the space of connections on a given vector bundle. The space of isomorphism classes of *structured vector bundles*, i.e. vector bundles equipped with such an equivalence class of connections, determines a functor from the category of compact manifolds with corners to the category of abelian semi-rings. Passing to the Grothendieck group completion, one obtains a multiplicative differential extension of the even-degree part of topological K-theory. By a result of Bunke and Schick [6] this differential extension, denoted here by $\check{\mathcal{K}}^0$, is isomorphic to any other differential extension of even K-theory via a unique isomorphism.

The Simons-Sullivan model of even differential K-theory is built upon vector bundles rather than principal bundles[1], since the even topological K-theory of a compact manifold M has a natural construction in terms of vector bundles over M. Topological K-theory is a generalised cohomology theory and as such has a 'homotopy-invariant' representation as homotopy classes of maps into a spectrum KU. By the well-known Bott Periodicity Theorem this spectrum is 2-periodic and in fact

$$K^0(M) \cong [M, BU \times \mathbb{Z}] \quad \text{and} \quad K^{-1}(M) \cong [M, U],$$

where $U = \varinjlim U(n)$ is the stabilised unitary group and BU is its classifying space. Using this representation, it is clear why even K-theory $K^0(M)$ may be represented by vector bundles over M. In fact, as M is taken to be a smooth manifold, one may define $K^0(M)$ using only smooth vector bundles.

The odd K-theory of M is a little more subtle and is usually defined in terms of vector bundles over ΣM^+, the reduced suspension of $M^+ := M \sqcup \{*\}$. This is problematic when attempting to construct a differential extension after the fashion of Simons-Sullivan as ΣM^+ is rarely a smooth manifold so it is not clear how to incorporate differential form data. The homotopy-theoretic model for $K^{-1}(M)$ gives a clue as to how to resolve this issue: by pulling back the path fibration $PU \to U$ one may construct odd K-theory using ΩU-bundles, or rather their associated vector bundles, over M. The benefits of this are two-fold since the building blocks of the theory are bundles over M that may additionally be taken to be smooth without loss of generality.

The latter part of this thesis introduces Ω vector bundles, which are the associated vector bundles of $\Omega GL_n(\mathbb{C})$- and $\Omega U(n)$-bundles. As with their frame bundles, there is a caloron correspondence for Ω vector bundles that may be extended to incorporate the appropriate connective data. A model for odd topological K-theory is given in terms of Ω vector bundles and the odd Chern character is computed in this model in terms of characteristic classes of the underlying Ω vector bundles. Using the relative string potentials to define an equivalence relation on connective data, this model is refined to give a differential extension of odd K-theory: the Ω model. Using the work of Bunke and Schick [5, 6, 7] and Tradler, Wilson and Zeinalian [40] it is shown that the Ω model is isomorphic to the odd part of differential K-theory, thereby giving the desired codification of relative string potentials.

An outline of this thesis is:

Chapter 1. This chapter gives a detailed review of the construction of the caloron correspondence as formulated by Murray and Stevenson [32] for free loop groups and Murray and Vozzo [33] for based loop groups. Following this, an in-depth exposition of the construction of string forms and string classes is presented.

Chapter 2. This chapter describes the construction of the relative and total string potential forms for loop group bundles and collects some facts about these objects used in subsequent chapters. Following a brief review of differential cohomology, in particular Cheeger-Simons differential characters, the total string potentials are used to construct degree 1 differential characteristic classes for $\Omega U(n)$-bundles.

[1] though, of course, the two are naturally related by the frame bundle and associated vector bundle functors.

Chapter 3. This chapter focusses on the introduction of Ω vector bundles. These objects are Fréchet vector bundles with typical fibre LV and structure group ΩG for some complex vector space V and matrix group $G \subseteq GL(V)$ with its standard action on V. A caloron correspondence is developed relating Ω vector bundles over M to framed vector bundles over $M \times S^1$ that respects the frame bundle functor and principal bundle caloron correspondence. A version of the Serre-Swan theorem is proved for Ω vector bundles, which shows that every Ω vector bundle over compact M may be regarded as a smoothly-varying family of modules for the ring $L\mathbb{C}$ over M. This module structure is used to define connective data (*module connections* and *vector bundle Higgs fields*) on Ω vector bundles, which fit into a geometric caloron correspondence for vector bundles.

After introducing the analogue of Hermitian structures for Ω vector bundles, together with an associated caloron correspondence, a model for odd K-theory is defined by applying the Grothendieck group completion to the abelian semi-group of isomorphism classes of Ω vector bundles. The odd Chern character is computed in this model of odd K-theory in terms of string forms of the underlying Ω vector bundles.

Chapter 4. Based on the results of Chapters 2 and 3 and following a review of the Simons-Sullivan construction of [38], a differential extension of odd K-theory is constructed in terms of Ω vector bundles. This construction uses the relative string potential forms to generate an equivalence relation on the space of module connections and Higgs fields of a given Ω vector bundle, an equivalence class of which is called a *string datum*. The Ω model is given by applying the Grothendieck group completion device to the abelian semi-group of (a certain collection of) isomorphism classes of *structured Ω vector bundles*; Ω vector bundles equipped with string data.

Bunke and Schick showed in [6] that differential extensions of odd K-theory are non-unique and that additional structure is required in order to obtain differential K-theory, which is unique up to unique isomorphism. Nevertheless, by relating the Ω model to a differential extension appearing in a recent paper of Tradler, Wilson and Zeinalian [40], the caloron transform is used to show that the Ω model defines the odd part of differential K-theory. The effect of this it two-fold, as it provides a sort of homotopy-theoretic interpretation of the Ω model as well as a proof that the TWZ differential extension defines odd differential K-theory, a result not previously obtained.

Appendices. Appendix A provides background material on Fréchet spaces and Fréchet manifolds, a proof that the path fibration $PG \to G$ gives a model for the universal ΩG-bundle and some results on direct limits of directed systems of manifolds. Appendix B discusses the integration over the fibre operations on differential forms and in singular cohomology. Appendix C records the Bunke-Schick definition of differential extensions together with some results that are required in this thesis.

Remark on conventions. Unless stated otherwise all smooth finite-dimensional manifolds are taken to be paracompact and Hausdorff (so that they admit smooth partitions of unity) and all maps between smooth manifolds are smooth. All unadorned cohomology groups $H^\bullet$ represent integer-valued singular cohomology and $\Omega_{d=0}(M)$ denotes the space of closed differential forms on the smooth manifold M. The symbol G shall usually denote a smooth connected finite-dimensional Lie group, with Θ its (left-invariant) Maurer-Cartan form and $\mathfrak{g} = \mathrm{Lie}(G)$ its Lie algebra. The terms 'G-bundle' and 'principal G-bundle' are used interchangeably. The circle group S^1 is regarded as the quotient of $\mathbb{R}$ modulo the equivalence relation $x \sim y \Leftrightarrow x = y + 2k\pi$ for some $k \in \mathbb{Z}$ and the equivalence class of 0 defines a distinguished basepoint for S^1, which is also denoted 0. The integration over the fibre operation $\widehat{\int}_{S^1}$ is always taken with respect to the canonical orientation on S^1 inherited from $\mathbb{R}$. The Fréchet Lie group of smooth maps $S^1 \to G$ with pointwise group operations is denoted by LG and the subgroup of those maps sending $0 \in S^1$ to the identity in G is denoted ΩG.