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Observer-based H_∞ Control on Nonhomogeneous Markov Jump Systems with Nonlinear Input

Yanyan Yin [a,b], Peng Shi [c,d*], Fei Liu [a], Kok Lay Teo [b]

[a] Key Laboratory of Advanced Process Control for Light Industry (Ministry of Education)
Institute of Automation, Jiangnan University, Wuxi, 214122, China

[b] Department of Mathematics and Statistics, Curtin University, Perth, Western Australia, 6102

[c] School of Engineering and Science, Victoria University, Melbourne, Vic 8001, Australia

[d] School of Electrical and Electronic Engineering, The University of Adelaide, Adelaide, SA 5005, Australia

[*] Corresponding author. E-mail: peng.shi@vu.edu.au Tel: +61 3 99196273

Abstract: This paper considers the problem of observer-based H_∞ controller design for a class of discrete-time nonhomogeneous Markov jump systems with nonlinear input. Actuator saturation is considered to be a nonlinear input of such system and the time-varying transition probability matrix in the system is described as a polytope set. Furthermore, a mode-dependent and parameter-dependent Lyapunov function is investigated and a sufficient condition is derived to design observer-based controllers such that the resulting error dynamical system is stochastically stable and a prescribed H_∞ performance is achieved. Finally, estimation of attraction domain of such nonhomogeneous Markov jump systems is also made. A simulation example shows the effectiveness of developed techniques.

Keywords: observer-based, H_∞ control, nonhomogeneous Markov jump parameters, nonlinear input.

1 Introduction

Markov jump systems (MJSs) [1] have very strong ability to model lots of practical systems , which evolves in Markov process or Markov chain, such as manufacturing systems, economic systems, electrical systems and communication systems. MJSs are also appropriate and reasonable to describe systems subject to abrupt variation in their structures or parameters, which caused by failures of subsystems, sudden environmental changes, and system noises. It is a fact that transition probability plays an important role in the performance of such systems, under the assumption that the transition probabilities of MJSs are time invariant, some problems have been studied, such as system analysis [2, 3], stochastic stability and stabilization [4], control [5-12], fault detection and filtering [13-17], fault tolerant and estimation [18, 19] etc. Some work has also been done on systems with partially known or uncertain transition probability (see, e.g., [20-24] and the references therein). However, in many practical systems, the transition probability is not a constant matrix, but a time-varying and time-depended matrix. One example is the failures and repairs of subsystems on discrete-time MJSs, which depends on system age and working time deeply. On another point, one can only obtain some measured values of the transition probability, these estimation values of probability bring in some time-varying uncertainty inherently in transition matrix, which leads the nonhomogeneous Markov jump systems [25]. Another example is networked control systems, as it is well known that packet dropout and stochastic delays in the systems can be expressed by Markov process or Markov chain. But in practice, delay or packet dropouts are changing in different periods, which results in the time-varying transition probabilities, so it is a challenge to study such nonhomogeneous Markov jump systems, and the investigation of control problem on MJSs subject to nonhomogeneous Markov process or Markov chain would be more useful and important. These motivate our work in this paper.

On another research front line, we also interested in Markov jump systems with actuator saturation nonlinear input. Actually, in some practical manufacture systems, one can not open the valve of input unlimitedly, in lots of situations, a bound is required, however, this limitation will change a linear system into a nonlinear one. It is also a well recognized fact that this nonlinearity degrades system performance and even leads a stable system into an instable one. Therefore, actuator saturation is probably the most dangerous nonlinearity in many manufacture systems. Saturation nonlinearity has received increasing attention in recent years, and the researchers have done some attempt on control problems of stochastic systems with actuator saturation [26, 27].

In this paper, we will design observer-based H_∞ controller for a class of nonhomogeneous MJSs subject to actuator nonlinearity. This paper is organized as follows: Problem statement and preliminaries of this paper are given in Section 2 and a series of definitions of actuator nonlinearity and stochastic systems are introduced. In Section 3, stochastic stability analysis of the aforementioned systems are made in terms of LMIs. In Section 4, observer-based H_∞ controllers for the nonhomogeneous Markov jump systems are designed. The estimation of attraction domain is made in Section 5. A numerical example is given to illustrate the effectiveness of our approach in Section 6. Finally, some concluding remarks are given in Section 7.

In the sequel, the notation $\mathbb{R}^n$ stands for a n-dimensional Euclidean space, the transpose of a matrix is denoted by A^{T}, $E\{\cdot\}$ denotes the mathematical statistical expectation of the stochastic process or vector, $L_2^n[0,\infty]$ stands for the space of n-dimensional square integrable function vector over $[0,\infty]$, a positive-definite matrix is denoted by $P>0$, I is the unit matrix with appropriate dimensions, and $*$ means the symmetric term in a symmetric matrix, $\sigma(\cdot)$ is the standard saturation function with appropriate dimensions.

2 Problem Statement and Preliminaries

Consider a probability space (M,F,P) where M, F and P represent the sample space, the algebra of events and the probability measure defined on F, respectively, then the following discrete-time Markov jump systems (MJSs) with actuator nonlinearity are considered in this paper:

$$\begin{cases} x_{k+1} = A(r_k)x_k + B(r_k)\sigma(u_k) + D(r_k)w_k \\ y_k = C_1(r_k)x_k \\ z_k = C_2(r_k)x_k \end{cases} \tag{2.1}$$

where $x_k \in \mathbb{R}^n$ is the state vector of the system, $u_k \in \mathbb{R}^m$ is the input vector of the system, $y_k \in \mathbb{R}^p$ is the output vector of the system, $z_k \in \mathbb{R}^p$ is the controlled output vector of the system, $w_k \in L_2^q[0,\infty]$ is the external disturbance vector of the system. $\sigma(u_k) = \begin{bmatrix} \sigma(u_{1k}) & \sigma(u_{2k}) & \dots & \sigma(u_{mk}) \end{bmatrix}^{\mathrm{T}}$ and $\sigma(u_{lk}) = \{sign(u_{lk})min\{1, |u_{lk}|\}, l = 1, \dots, m\}$, $\{r_k, k \geq 0\}$ is the concerned time-discrete Markov stochastic process which takes values in a finite state set $\Lambda = \{1, 2, 3, \dots, N\}$, and r_0 represents the initial mode, the transition probability matrix is defined as $\Pi(k) = \{\pi_{ij}(k)\}$, i, $j \in \Lambda$, $\pi_{ij}(k) = P(r_{k+1} = j|r_k = i)$ is the transition probability from mode i at time k to mode j at time $k+1$, which satisfies $\pi_{ij}(k) \geq 0$ and $\sum\limits_{j=1}^{N} \pi_{ij}(k) = 1$.

For given vertices $\Pi^s(k)$, $s = 1, \dots, w$, the time varying transition matrix $\Pi(k)$ of the nonhomogeneous Markov jump systems is constructed as

$$\Pi(k) = \sum_{s=1}^{w} \alpha_s(k)\Pi^s(k)$$

where

$$0 \leq \alpha_s(k) \leq 1, \quad \sum_{s=1}^{w} \alpha_s(k) = 1$$

Hence, time-varying transition probability matrix of system (2.1) belongs to a polytope which described by several vertices.

We design an observer-based controller for system (2.1) of the form

$$\begin{cases} \bar{x}_{k+1} = A(r_k)\bar{x}_k + B(r_k)\sigma(u_k) + H(r_k)(y_k - \bar{y}_k) \\ \bar{y}_k = C_1(r_k)\bar{x}_k \\ u_k = K(r_k)\bar{x}_k \end{cases} \tag{2.2}$$

where $\bar{x}_k$ and $\bar{y}_k$ are the estimated state and output, $H(r_k)$ is the gain matrix of the designed observer, and $K(r_k)$ is the gain matrix of the feedback controller.

For simplicity, when $r_k = i$, $i \in \Lambda$, the matrices $A(r_k)$, $B(r_k)$, $C_1(r_k)$, $C_2(r_k)$, $D(r_k)$, $H(r_k)$ and $K(r_k)$ are denoted as $A(i)$, $B(i)$, $C_1(i)$, $C_2(i)$, $D(i)$, $H(i)$ and $K(i)$.

It is worth mentioning that if $\Pi(k)$ is a constant matrix, we call the systems as homogeneous Markov jump systems. In this paper, we consider the problem of observer-based H_∞ controller design for nonhomogeneous Markov jump systems, in which $\Pi(k)$ is a time varying matrix and evolves in a polytope.

Before proceeding with the study, some concepts and Lemmas are given below:

Definition 2.1 *Given a symmetric matrix $P(i) > 0$ for system (2.1), one can define a series of ellipsoid sets as follows:*

$$\varepsilon(P(i)) = \{x_k \in \mathbb{R}^n : x_k^{\mathrm{T}} P(i) x_k \leq 1\}$$

Definition 2.2 *Given a matrix $F(i)$ for system state in (2.1), one can denote $f_q(i)$ as the qth row of the matrix $F(i)$, and then, a symmetric polyhedron set is defined as follows:*

$$\Theta(F(i)) = \{x_k \in R^n : |f_q(i)x_k| \leq 1, q = 1, 2, \ldots, m\}$$

Lemma 2.1 *[28] For given symmetric matrices $R(i) > 0$, and appropriate dimensioned matrices W_p, if $0 \leq \varepsilon_p \leq 1$ and $\sum\limits_{p=1}^{h} \varepsilon_p = 1$, then*

$$(\sum_{p=1}^{h} \varepsilon_p W_p)^{\mathrm{T}} R(i) (\sum_{p=1}^{h} \varepsilon_p W_p) \leq \sum_{p=1}^{h} \varepsilon_p W_p^{\mathrm{T}} R(i) W_p$$

Lemma 2.2 *Consider R_1 and R_2 as positive definite symmetric matrices, then it holds*

$$R_1 + R_1^{\mathrm{T}} - R_2 \leq R_1 R_2^{-1} R_1^{\mathrm{T}}$$

Proof: As R_2 is a positive definite symmetric matrix, we have

$$(R_1 - R_2) R_2^{-1} (R_1 - R_2)^{\mathrm{T}} \geq 0$$

subsequently, the following inequality is derived

$$R_1 R_2^{-1} R_1^{\mathrm{T}} - R_1 - R_1^{\mathrm{T}} + R_2 \geq 0$$

then, the proof is completed.

Lemma 2.3 *[28] Given matrices u_k and v_k for system (2.2), if $|v_k| < 1$, then, $\sigma(u_k) = \sum\limits_{t=1}^{2^m} \theta_t (M_t u_k + M_t^{-} v_k)$, where $0 \leq \theta_t \leq 1$, $\sum\limits_{t=1}^{2^m} \theta_t = 1$, M_t is a $m \times m$ diagonal matrix whose diagonal elements are either 1 or 0, and $M_t^{-} = I - M_t$.*

Remark 2.1 *Given matrices $v_k = F(i)\bar{x}_k$ for system (2.2), if $\bar{x}_k \in \Theta(F(i))$, that is $|v_k| < 1$, then, $\sigma(K(i)\bar{x}_k) = \sum\limits_{t=1}^{2^m} \theta_t(M_t K(i) + M_t^- F(i))\bar{x}_k$. Obviously, $\{M_t K(i) + M_t^- F(i) : i \in [1, 2^m]\}$ is the set formed by matrices, and these matrices are formed by choosing some rows from $K(i)$ and the remaining from $F(i)$.*

Recalling Lemma 2.3, one can denote $\bar{x}_k \in \Theta(F(i))$, then, we have $\sigma(K(i)\bar{x}_k) = \sum_{t=1}^{2^m} \theta_t(M_t K(i) + M_t^- F(i))\bar{x}_k$. Augmenting system (2.1) to include the states of the observer-based controller (2.2), we obtain the resulting closed-loop system:

$$\begin{cases} \hat{x}_{k+1} = \breve{A}(i)\hat{x}_k + \breve{D}(i)w_k \\ z_k = \breve{C}(i)\hat{x}_k \end{cases} \tag{2.3}$$

where $e_k = x_k - \bar{x}_k$, $\hat{x}_k^{\mathrm{T}} = \begin{bmatrix} e_k^{\mathrm{T}} & x_k^{\mathrm{T}} \end{bmatrix}$, $\breve{A}(i) = \sum\limits_{t=1}^{2^m} \theta_t \begin{bmatrix} A(i) - H(i)C_1(i) & 0 \\ -B(i)\hat{M}_t(i) & A(i) + B(i)\hat{M}_t(i) \end{bmatrix}$ $\hat{M}_t(i) = M_t K(i) + M_t^- F(i)$, $\breve{D}(i) = \begin{bmatrix} D(i) \\ D(i) \end{bmatrix}$, $\breve{C}(i) = \begin{bmatrix} 0 & C_2(i) \end{bmatrix}$

Definition 2.3 *For any initial mode r_0 and a given initial state $\hat{x}_0$, system (2.3) (with $K(i) = 0$, $F(i) = 0$ and $w_k = 0$) is stochastically stable if*

$$\lim_{m\to\infty} E\{\sum_{k=0}^{m} \hat{x}_k^{\mathrm{T}}\hat{x}_k|\hat{x}_0, r_0\} < \infty \tag{2.4}$$

Definition 2.4 *If there exists a positive scalar λ and a positive number $\tilde{N}(\hat{x}_0, r_0)$ such that system (2.3) satisfies condition (2.5) and condition (2.6), then, system (2.3) is stochastically stable and satisfies a H_∞ performance index*

$$\lim_{m\to\infty} E\{\sum_{k=0}^{m} \hat{x}_k^{\mathrm{T}}\hat{x}_k|\hat{x}_0, r_0\} < \tilde{N}(\hat{x}_0, r_0) \tag{2.5}$$

$$E\left\{\sum_{k=0}^{\infty} z_k^{\mathrm{T}} z_k\right\} \leq \lambda^2 E\left\{\sum_{k=0}^{\infty} w_k^{\mathrm{T}} w_k\right\} \tag{2.6}$$

We now state formally the purpose of the paper as follows: consider system (2.1) with time-varying jump transition probabilities. Design a mode-dependent observer-based controller (2.2), such that the resulting closed-loop system (2.3) is stochastically stable and satisfies a prescribed H_∞ performance index.

3 Stochastic Stability

Let us first discuss the stochastic stability of the closed-loop system (2.3), in which the transition probability is a time-varying matrix.

Theorem 3.1 *For a given initial condition, system (2.3) (with $K(i) = 0$, $F(i) = 0$ and $w_k = 0$) is stochastically stable, if there exist a set of positive definite symmetric matrices $P_{1s}(i)$, $P_{2s}(i)$, $P_{1q}(j)$ and $P_{2q}(j)$ such that*

$$\Xi = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ * & a_{22} & a_{23} & a_{24} \\ * & * & a_{33} & a_{34} \\ * & * & * & a_{44} \end{bmatrix} < 0 \qquad \forall i, j \in \Lambda \tag{3.1}$$

where

$$a_{11} = -\sum_{s=1}^{w} \alpha_s(k) P_{1s}(i)$$

$$a_{22} = -\sum_{s=1}^{w} \alpha_s(k) P_{2s}(i)$$

$$a_{13} = (A(i) - H(i)C_1(i))^{\mathrm{T}}$$

$$a_{14} = 0, \quad a_{24} = A^{\mathrm{T}}(i)$$

$$a_{33} = -(\sum_{j=1}^{N}\sum_{s=1}^{w}\sum_{q=1}^{w} \alpha_s(k)\beta_q(k)\pi_{ij}^{s} P_{1q}(j))^{-1}$$

$$a_{44} = -(\sum_{j=1}^{N}\sum_{s=1}^{w}\sum_{q=1}^{w} \alpha_s(k)\beta_q(k)\pi_{ij}^{s} P_{2q}(j))^{-1}$$

$$a_{12} = 0, \quad a_{23} = 0, \quad a_{34} = 0$$

$$0 \le \alpha_s(k) \le 1, \quad \sum_{s=1}^{w} \alpha_s(k) = 1$$

$$0 \le \beta_q(k) \le 1, \quad \sum_{q=1}^{w} \beta_q(k) = 1$$

Proof: Under conditions $K(i) = 0$, $F(i) = 0$ and $w_k = 0$, system (2.3) is reduced to

$$\begin{cases} \hat{x}_{k+1} = \hat{A}(i)\hat{x}_k \\ z_k = \breve{C}(i)\hat{x}_k \end{cases} \tag{3.2}$$

where

$$\hat{A}(i) = \begin{bmatrix} A(i) - H(i)C_1(i) & 0 \\ 0 & A(i) \end{bmatrix}$$

A parameter-dependent and mode-depended Lyapunov function for system (3.2) is constructed by using symmetric positive definite matrices $\hat{P}_s(i)$ as follows:

$$V(\hat{x}_k, i) = \hat{x}_k^{\mathrm{T}} \sum_{s=1}^{w} \alpha_s(k) \hat{P}_s(i) \hat{x}_k \quad (i \in \Lambda)$$

where

$$\hat{P}_s(i) = \begin{bmatrix} P_{1s}(i) & 0 \\ 0 & P_{2s}(i) \end{bmatrix}$$

Then, we have

$$\begin{aligned} \Delta V(\hat{x}_k, i) &= E\{V(\hat{x}_{k+1}, i)\} - V(\hat{x}_k, i) \\ &= \hat{x}_k^{\mathrm{T}} [\hat{A}^{\mathrm{T}}(i) \sum_{j=1}^{N} \sum_{s=1}^{w} \sum_{s=1}^{w} \alpha_s(k) \alpha_s(k+1) \pi_{ij}^{s} \hat{P}_s(j) \hat{A}(i)] \hat{x}_k \\ &- \hat{x}_k^{\mathrm{T}} \sum_{s=1}^{w} \alpha_s(k) \hat{P}_s(i) \hat{x}_k \end{aligned}$$

Define

$$\sum_{s=1}^{w} \alpha_s(k+1) \hat{P}_s(j) = \sum_{q=1}^{w} \beta_q(k) \hat{P}_q(j)$$

One has

$$\begin{aligned} \Delta V(\hat{x}_k, i) &= \hat{x}_k^{\mathrm{T}} [\hat{A}^{\mathrm{T}}(i) (\sum_{j=1}^{N} \sum_{s=1}^{w} \sum_{q=1}^{w} \alpha_s(k) \beta_q(k) \pi_{ij}^{s} \hat{P}_q(j)) \hat{A}(i)] \hat{x}_k \\ &- \hat{x}_k^{\mathrm{T}} \sum_{s=1}^{w} \alpha_s(k) \hat{P}_s(i) \hat{x}_k \\ &= \hat{x}_k^{\mathrm{T}} \Xi \hat{x}_k \end{aligned}$$

For system (3.2), condition (3.1) implies

$$\Delta V(\hat{x}_k, i) < 0 \quad \forall i \in \Lambda$$

On the other hand, let

$$\mu = \min_{k} \{\lambda_{min}(-\Xi)\} \quad \forall i \in \Lambda$$

where $\lambda_{min}(-\Xi)$ is the minimal eigenvalue of $-\Xi$

then, it follows

$$\Delta V(\hat{x}_k, i) \leq -\mu \hat{x}_k^{\mathrm{T}} \hat{x}_k$$

Hence, we have

$$\begin{aligned} E\{\sum_{k=0}^{T} \Delta V(\hat{x}_k, i)\} &= E\{V(\hat{x}_{T+1}, i)\} - V(\hat{x}_0, i) \\ &\leq -\mu E\{\sum_{k=0}^{T} \|\hat{x}_k\|^2\} \end{aligned}$$

and the following inequality holds

$$\begin{aligned} E\{\sum_{k=0}^{T} \|\hat{x}_k\|^2\} &\leq -\tfrac{1}{\mu}(E(V(\hat{x}_{T+1}, i) - V(\hat{x}_0, i))) \\ &\leq \tfrac{1}{\mu}(E(V(\hat{x}_0, i) - V(\hat{x}_{T+1}, i))) \end{aligned}$$

which implies

$$\lim_{T\to\infty} E\{\sum_{k=0}^{T} \|\hat{x}_k\|^2\} \leq \frac{1}{\mu} V(\hat{x}_0, i)$$

From Definition 2.3, system (3.2) is stochastically stable with $K(i) = 0$, $F(i) = 0$ and $w_k = 0$, and this concludes the proof.

Denote $Q_s(i) = P_s^{-1}(i)$, then, a sufficient condition for stochastic stability of system (3.2) can be developed as below.

Theorem 3.2 *For a given initial condition $\hat{x}_0 = 0$, system (2.3) (with $K(i) = 0$, $F(i) = 0$ and $w_k = 0$) is stochastically stable, if there exist a set of positive definite symmetric matrices $G_s(i)$, $Q_{1s}(i)$, $Q_{2s}(i)$, $Q_{1q}(j)$ and $Q_{2q}(j)$ such that*

$$
\Phi_1 = \begin{bmatrix}
b_{11} & 0 & \sqrt{\pi_{i1}^s} G_s^{\mathrm{T}}(i) \tilde{A}^{\mathrm{T}} & \dots & \sqrt{\pi_{iN}^s} G_s^{\mathrm{T}}(i) \tilde{A}^{\mathrm{T}} & 0 & \dots & 0 \\
* & b_{22} & 0 & 0 & 0 & b_{23} & \dots & b_{24} \\
* & * & -Q_{1q}(1) & 0 & 0 & 0 & 0 & 0 \\
* & * & * & \ddots & 0 & 0 & 0 & 0 \\
* & * & * & * & -Q_{1q}(N) & 0 & 0 & 0 \\
* & * & * & * & * & -Q_{2q}(1) & 0 & 0 \\
* & * & * & * & * & * & \ddots & 0 \\
* & * & * & * & * & * & * & -Q_{2q}(N)
\end{bmatrix} < 0 \quad (3.3)
$$

where

$$
\begin{gathered}
\tilde{A} = A(i) - H(i)C_1(i) \\
b_{11} = -G_s^{\mathrm{T}}(i) - G_s(i) + Q_{1s}(i), \quad Q_{1s}(i) = P_{1s}^{-1}(i), \quad Q_{1q}(j) = P_{1q}^{-1}(j) \\
b_{22} = -G_s^{\mathrm{T}}(i) - G_s(i) + Q_{2s}(i), \quad Q_{2s}(i) = P_{2s}^{-1}(i), \quad Q_{2q}(j) = P_{2q}^{-1}(j) \\
b_{23} = \sqrt{\pi_{i1}^s} G_s^{\mathrm{T}}(i) A^{\mathrm{T}}(i) \\
b_{24} = \sqrt{\pi_{iN}^s} G_s^{\mathrm{T}}(i) A^{\mathrm{T}}(i)
\end{gathered}
$$

Proof: First note that a sufficient condition for stochastic stability of system (2.3) is that all the vertices of the polytope satisfy the desired stable requirements.

From Theorem 3.1, we have

$$
\Phi_2 = \begin{bmatrix}
c_{11} & 0 & \sqrt{\pi_{i1}^s} \tilde{A}^{\mathrm{T}} & \dots & \sqrt{\pi_{iN}^s} \tilde{A}^{\mathrm{T}} & 0 & \dots & 0 \\
* & c_{22} & 0 & 0 & 0 & c_{23} & \dots & c_{24} \\
* & * & -Q_{1q}(1) & 0 & 0 & 0 & 0 & 0 \\
* & * & * & \ddots & 0 & 0 & 0 & 0 \\
* & * & * & * & -Q_{1q}(N) & 0 & 0 & 0 \\
* & * & * & * & * & -Q_{2q}(1) & 0 & 0 \\
* & * & * & * & * & * & \ddots & 0 \\
* & * & * & * & * & * & * & -Q_{2q}(N)
\end{bmatrix} < 0 \quad (3.4)
$$

where

$$c_{11} = -P_{1s}(i), \qquad c_{22} = -P_{2s}(i)$$
$$c_{23} = \sqrt{\pi_{i1}^s} A^{\mathrm{T}}(i)$$
$$c_{24} = \sqrt{\pi_{iN}^s} A^{\mathrm{T}}(i)$$

Multiply Φ_2 by $\hat{G}_s^{\mathrm{T}}(i)$ and $\hat{G}_s(i)$ on the left hand side and right hand side respectively where

$$\hat{G}_s(i) = diag\left\{ \begin{array}{cccccccc} G_s(i) & G_s(i) & I & \cdots & I & I & \cdots & I \end{array} \right\}$$

then, we have

$$\Phi_3 = \begin{bmatrix} d_{11} & 0 & \sqrt{\pi_{i1}^s} G_s^{\mathrm{T}}(i)\tilde{A}^{\mathrm{T}} & \dots & \sqrt{\pi_{iN}^s} G_s^{\mathrm{T}}(i)\tilde{A}^{\mathrm{T}} & 0 & \dots & 0 \\ * & d_{22} & 0 & 0 & 0 & b_{23} & \dots & b_{24} \\ * & * & -Q_{1q}(1) & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \ddots & 0 & 0 & 0 & 0 \\ * & * & * & * & -Q_{1q}(N) & 0 & 0 & 0 \\ * & * & * & * & * & -Q_{2q}(1) & 0 & 0 \\ * & * & * & * & * & * & \ddots & 0 \\ * & * & * & * & * & * & * & -Q_{2q}(N) \end{bmatrix} < 0 \quad (3.5)$$

where $d_{11} = -G_s^{\mathrm{T}}(i)P_{1s}(i)G_s(i)$, $d_{22} = -G_s^{\mathrm{T}}(i)P_{2s}(i)G_s(i)$.
Recalling Lemma 2.2, it follows that

$$G_s^{\mathrm{T}}(i)P_{1s}(i)G_s(i) \geq G_s^{\mathrm{T}}(i) - Q_{1s}(i) + G_s(i)$$
$$G_s^{\mathrm{T}}(i)P_{2s}(i)G_s(i) \geq G_s^{\mathrm{T}}(i) - Q_{2s}(i) + G_s(i)$$

Therefore, it can be shown $\Phi_1 < 0$ guarantees $\Phi_3 < 0$, which completes the proof.

Remark 3.1 *Note that the results obtained in Theorem 3.2 are sufficient conditions for the stochastic stability of system (2.3) (with $K(i) = 0$, $F(i) = 0$ and $w_k = 0$), and they are given in terms of linear matrix inequalities.*

4 Observer-based H_∞ Controller Design

In this section, observer-based controllers will be designed for system (2.3) such that the system is stochastically stable with a given H_∞ performance index λ.

Theorem 4.1 *For given matrices M_t, M_t^- and $w(t) = 0$, system (2.3) is stochastically stable under controller u_k, if there exist a set of positive definite symmetric matrices $P_{1s}(i)$, $P_{2s}(i)$, $P_{1q}(j)$ and $P_{2q}(j)$ such that*

$$\Psi = \begin{bmatrix} e_{11} & e_{12} & e_{13} & e_{14} \\ * & e_{22} & e_{23} & e_{24} \\ * & * & e_{33} & e_{34} \\ * & * & * & e_{44} \end{bmatrix} < 0 \qquad \forall i \in \Lambda \tag{4.1}$$

$$\varepsilon(\hat{P}_s(i)) \in \Theta(\tilde{F}(i)) \tag{4.2}$$

where

$$e_{11} = -\sum_{s=1}^{w} \alpha_s(k) P_{1s}(i)$$

$$e_{22} = -\sum_{s=1}^{w} \alpha_s(k) P_{2s}(i)$$

$$e_{13} = (A(i) - H(i)C_1(i))^{\mathrm{T}}$$

$$e_{14} = \sum_{t=1}^{2^m} \theta_t [-B(i)(M_t K(i) + M_t^- F(i))]^{\mathrm{T}}$$

$$e_{24} = \sum_{t=1}^{2^m} \theta_t (A(i) + B(i)(M_t K(i) + M_t^- F(i)))^{\mathrm{T}}$$

$$e_{33} = -(\sum_{j=1}^{N} \sum_{s=1}^{w} \sum_{q=1}^{w} \alpha_s(k) \beta_q(k) \pi_{ij}^s P_{1q}(j))^{-1}$$

$$e_{44} = -(\sum_{j=1}^{N} \sum_{s=1}^{w} \sum_{q=1}^{w} \alpha_s(k) \beta_q(k) \pi_{ij}^s P_{2q}(j))^{-1}$$

$$e_{12} = 0, \quad a_{23} = 0, \quad e_{34} = 0$$

$$0 \le \alpha_s(k) \le 1, \quad \sum_{s=1}^{w} \alpha_s(k) = 1$$

$$0 \le \beta_q(k) \le 1, \quad \sum_{q=1}^{w} \beta_q(k) = 1$$

$$\hat{P}_s(i) = \begin{bmatrix} P_{1s}(i) & 0 \\ 0 & P_{2s}(i) \end{bmatrix}, \qquad \tilde{F}(i) = \begin{bmatrix} F(i) & 0 \\ 0 & F(i) \end{bmatrix}$$

Moreover, a suitable controller u_k for system (2.3) is constructed as

$$u_k = \sum_{t=1}^{2^m} \theta_t (M_t K(i) + M_t^- F(i))\bar{x}_k$$

Proof: Under condition (4.2), by Lemma 2.3, system (2.3) (with $w_k = 0$) can be written as

$$\begin{cases} \hat{x}_{k+1} = \breve{A}(i)\hat{x}_k \\ z_k = \breve{C}(i)\hat{x}_k \end{cases} \tag{4.3}$$

A parameter-dependent Lyapunov-Krasovskii function for system (4.3) is constructed by using symmetric positive definite matrices $\hat{P}_s(i)$ as follows:

$$V(\hat{x}_k, i) = \hat{x}_k^{\mathrm{T}} \sum_{s=1}^{w} \alpha_s(k)\hat{P}_s(i)\hat{x}_k \quad (i \in \Lambda)$$

where

$$\hat{P}_s(i) = \begin{bmatrix} P_{1s}(i) & 0 \\ 0 & P_{2s}(i) \end{bmatrix}$$

then, we have

$$\begin{aligned} \Delta V(\hat{x}_k, i) &= E\{V(\hat{x}_{k+1}, i)\} - V(\hat{x}_k, i) \\ &= \hat{x}_k^{\mathrm{T}}[\breve{A}^{\mathrm{T}}(i) \textstyle\sum_{j=1}^{N} \sum_{s=1}^{w} \sum_{s=1}^{w} \alpha_s(k)\alpha_s(k+1)\pi_{ij}^s \hat{P}_s(j)\breve{A}(i)]\hat{x}_k \\ &\quad -\hat{x}_k^{\mathrm{T}} \textstyle\sum_{s=1}^{w} \alpha_s(k)\hat{P}_s(i)\hat{x}_k \end{aligned}$$

where

$$\breve{A}(i) = \sum_{t=1}^{2^m} \theta_t \begin{bmatrix} A(i) - H(i)C_1(i) & 0 \\ -B(i)\hat{M}_t(i) & A(i) + B(i)\hat{M}_t(i) \end{bmatrix}$$

$$\hat{M}_t(i) = M_t K(i) + M_t^- F(i)$$

We define

$$\sum_{s=1}^{w} \alpha_s(k+1)\hat{P}_s(j) = \sum_{q=1}^{w} \beta_q(k)\hat{P}_q(j)$$

From Lemma 2.1, we have the following

$$
\begin{aligned}
\Delta V(\hat{x}_k, i) &= \hat{x}_k^{\mathrm{T}}[\sum_{t=1}^{2^m} \theta_t \bar{A}^{\mathrm{T}}(i)(\sum_{j=1}^{N} \sum_{s=1}^{w} \sum_{q=1}^{w} \alpha_s(k)\beta_q(k)\pi_{ij}^{s}\hat{P}_q(j)) \sum_{t=1}^{2^m} \theta_t \bar{A}(i)]\hat{x}_k \\
&\quad -\hat{x}_k^{\mathrm{T}} \sum_{s=1}^{w} \alpha_s(k)\hat{P}_s(i)\hat{x}_k \\
&= \hat{x}_k^{\mathrm{T}} \Psi \hat{x}_k
\end{aligned}
$$

where

$$
\bar{A}(i) = \begin{bmatrix} A(i) - H(i)C_1(i) & 0 \\ -B(i)\hat{M}_t(i) & A(i) + B(i)\hat{M}_t(i) \end{bmatrix}
$$

Under condition (4.2), for system (4.3), condition (4.1) implies

$$
\Delta V(\hat{x}_k, i) < 0 \qquad \forall t \in [1, 2^m]
$$

We denote

$$
\eta = \min_k \{\lambda_{min}(-\Psi)\} \quad \forall i \in \Lambda
$$

where $\lambda_{min}(-\Psi)$ is the minimal eigenvalue of $-\Psi$
then

$$
\Delta V(\hat{x}_k, i) \leq -\eta \hat{x}_k^{\mathrm{T}} \hat{x}_k
$$

Hence, we have

$$
\begin{aligned}
E\{\sum_{k=0}^{T} \Delta V(\hat{x}_k, i)\} &= E\{V(\hat{x}_{T+1}, i)\} - V(\hat{x}_0, i) \\
&\leq -\eta E\{\sum_{k=0}^{T} \|\hat{x}_k\|^2\}
\end{aligned}
$$

and it follows:

$$
\begin{aligned}
E\{\sum_{k=0}^{T} \|\hat{x}_k\|^2\} &\leq -\tfrac{1}{\eta}(E(V(\hat{x}_{T+1}, i) - V(\hat{x}_0, i))) \\
&\leq \tfrac{1}{\eta}(E(V(\hat{x}_0, i) - V(\hat{x}_{T+1}, i)))
\end{aligned}
$$

which implies

$$\lim_{T\to\infty} E\{\sum_{k=0}^{T} \|\hat{x}_k\|^2\} \le \frac{1}{\eta} V(\hat{x}_0, i)$$

From Definition 2.4, system (2.3) is stochastically stable with a suitable controller u_k under condition (4.2), and this concludes the proof.

Remark 4.1 *In order to decrease the influences of disturbances, H_∞ performance is considered in the following, and the controllers designed will make system (2.3) stochastically stable and satisfies a given H_∞ performance index.*

Theorem 4.2 *For given matrices M_t, M_t^- ($\forall t \in [1, 2^m], i \in \Lambda$) and a positive scalar λ, system (2.3) is stochastically stable, and it also satisfies condition (2.6) if there exist a set of positive definite symmetric matrices $G_s(i)$, $Q_{1s}(i)$, $Q_{2s}(i)$, $Q_{1q}(j)$ and $Q_{2q}(j)$ such that*

$$\Gamma_1 = \begin{bmatrix} g_{11} & 0 & 0 & \sqrt{\pi_{i1}^s} G_s^{\mathrm{T}}(i)\tilde{A}^{\mathrm{T}} & \dots & \sqrt{\pi_{iN}^s} G_s^{\mathrm{T}}(i)\tilde{A}^{\mathrm{T}} & g_{13} & \dots & g_{14} & 0 \\ * & g_{22} & 0 & 0 & 0 & 0 & g_{23} & \dots & g_{24} & g_{25} \\ * & * & -\lambda^2 I & \sqrt{\pi_{i1}^s} D^{\mathrm{T}}(i) & \dots & \sqrt{\pi_{iN}^s} D^{\mathrm{T}}(i) & \sqrt{\pi_{i1}^s} D^{\mathrm{T}}(i) & \dots & \sqrt{\pi_{iN}^s} D^{\mathrm{T}}(i) & 0 \\ * & * & * & -Q_{1q}(1) & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & \ddots & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -Q_{1q}(N) & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & -Q_{2q}(1) & 0 & 0 & 0 \\ * & * & * & * & * & * & * & \ddots & 0 & 0 \\ * & * & * & * & * & * & * & * & -Q_{2q}(N) & 0 \\ * & * & * & * & * & * & * & * & * & -I \end{bmatrix} < 0 \tag{4.4}$$

$$\varepsilon(\hat{P}_s(i)) \in \Theta(\tilde{F}(i)) \tag{4.5}$$

Moreover, a suitable controller u_k for system (2.3) is constructed as

$$u_k = \sum_{t=1}^{2^m} \theta_t (M_t K(i) + M_t^- F(i)) \bar{x}_k$$

where

$$\tilde{A} = A(i) - H(i)C_1(i)$$

$$g_{11} = -G_s^{\mathrm{T}}(i) - G_s(i) + Q_{1s}(i), \quad Q_{1s}(i) = P_{1s}^{-1}(i), \quad Q_{1q}(j) = P_{1q}^{-1}(j)$$

$$g_{22} = -G_s^{\mathrm{T}}(i) - G_s(i) + Q_{2s}(i), \quad Q_{2s}(i) = P_{2s}^{-1}(i), \quad Q_{2q}(j) = P_{2q}^{-1}(j)$$

$$g_{13} = \sqrt{\pi_{i1}^{s}} G_s^{\mathrm{T}}(i)(-B(i)\hat{M}_t(i))^{\mathrm{T}}$$

$$g_{14} = \sqrt{\pi_{iN}^{s}} G_s^{\mathrm{T}}(i)(-B(i)\hat{M}_t(i))^{\mathrm{T}}$$

$$g_{23} = \sqrt{\pi_{i1}^{s}} G_s^{\mathrm{T}}(i)(A(i) + B(i)\hat{M}_t(i))^{\mathrm{T}}$$

$$g_{24} = \sqrt{\pi_{iN}^{s}} G_s^{\mathrm{T}}(i)(A(i) + B(i)\hat{M}_t(i))^{\mathrm{T}}$$

$$g_{25} = G_s^{\mathrm{T}}(i) C_2^{\mathrm{T}}(i)$$

Proof:

Introduce the following cost function for system (2.3) (with $w_k \neq 0$) as $k > 0$

$$J(\infty) = E\left\{\sum_{k=0}^{\infty} z_k^{\mathrm{T}} z_k\right\} - \lambda^2 E\left\{\sum_{k=0}^{\infty} w_k^{\mathrm{T}} w_k\right\} \tag{4.6}$$

Under zero initial condition, index $J(T)$ can be rewritten as

$$J(T) \leq E\left\{\sum_{k=0}^{T} [z_k^{\mathrm{T}} z_k - \lambda^2 w_k^{\mathrm{T}} w_k + \Delta V(\hat{x}_k, i)]\right\} \tag{4.7}$$

Recalling Theorem 4.1, under condition (4.5), it follows that

$$\begin{aligned}
J(T) &\leq E\left\{\sum_{k=0}^{T} [z_k^{\mathrm{T}} z_k - \lambda^2 w_k^{\mathrm{T}} w_k + \Delta V(\hat{x}_k, i)]\right\} \\
&= E\left\{\sum_{k=0}^{T} \{[C_2(i)x_k]^{\mathrm{T}}[C_2(i)x_k] - \lambda^2 w_k^{\mathrm{T}} w_k + \Delta V(\hat{x}_k, i)\}\right\} \\
&\leq E\left\{\sum_{k=0}^{T} \{[C_2(i)x_k]^{\mathrm{T}}[C_2(i)x_k] - \lambda^2 w_k^{\mathrm{T}} w_k\}\right\} \\
&+ E\left\{\sum_{k=0}^{T} [\sum_{t=1}^{2^m} \theta_t \bar{A}^{\mathrm{T}}(i)(\sum_{j=1}^{N}\sum_{s=1}^{w}\sum_{q=1}^{w} \alpha_s(k)\beta_q(k)\pi_{ij}^{s}\hat{P}_q(j))\bar{A}(i)]\right\} \\
&- E\left\{\sum_{k=0}^{T} \hat{x}_k^{\mathrm{T}} \sum_{s=1}^{w} \alpha_s(k)\hat{P}_s(i)\hat{x}_k\right\}
\end{aligned}$$

A sufficient condition for system (2.3) to be stable is that all the vertices of the polytope satisfy the desired stable requirements.

From Theorem 4.1, and recalling Schur Complement, we have

$$J(T) \leq \tilde{x}_k^{\mathrm{T}} \Gamma_2 \tilde{x}_k$$

where

$$\tilde{x}_k^{\mathrm{T}} = \begin{bmatrix} e_k^{\mathrm{T}} & x_k^{\mathrm{T}} & w_k^{\mathrm{T}} \end{bmatrix}$$

$$\Gamma_2 = \begin{bmatrix} h_{11} & 0 & 0 & \sqrt{\pi_{i1}^s}\tilde{A}^{\mathrm{T}} & \dots & \sqrt{\pi_{iN}^s}\tilde{A}^{\mathrm{T}} & h_{13} & \dots & h_{14} \\ * & h_{22} & 0 & 0 & 0 & 0 & h_{23} & \dots & h_{24} \\ * & * & -\lambda^2 I & \sqrt{\pi_{i1}^s}D^{\mathrm{T}}(i) & \dots & \sqrt{\pi_{iN}^s}D^{\mathrm{T}}(i) & \sqrt{\pi_{i1}^s}D^{\mathrm{T}}(i) & \dots & \sqrt{\pi_{iN}^s}D^{\mathrm{T}}(i) \\ * & * & * & -Q_{1q}(1) & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & \ddots & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -Q_{1q}(N) & 0 & 0 & 0 \\ * & * & * & * & * & * & -Q_{2q}(1) & 0 & 0 \\ * & * & * & * & * & * & * & \ddots & 0 \\ * & * & * & * & * & * & * & * & -Q_{2q}(N) \end{bmatrix} < 0 \tag{4.8}$$

and

$$h_{11} = -P_{1s}(i), \qquad h_{22} = -P_{2s}(i) + C_2^{\mathrm{T}}(i)C_2(i)$$
$$h_{13} = \sqrt{\pi_{i1}^s}(-B(i)\hat{M}_t(i))^{\mathrm{T}}$$
$$h_{14} = \sqrt{\pi_{iN}^s}(-B(i)\hat{M}_t(i))^{\mathrm{T}}$$
$$h_{23} = \sqrt{\pi_{i1}^s}(A(i) + B(i)\hat{M}_t(i))^{\mathrm{T}}$$
$$h_{24} = \sqrt{\pi_{iN}^s}(A(i) + B(i)\hat{M}_t(i))^{\mathrm{T}}$$

Multiply Γ_2 by $\bar{G}_s^{\mathrm{T}}(i)$ and $\bar{G}_s(i)$ on the left hand side and right hand side respectively where

$$\bar{G}_s(i) = diag\left\{ \begin{array}{cccccccc} G_s(i) & G_s(i) & I & I & \cdots & I & I & \cdots & I \end{array} \right\}$$

then, we have

$$
\Gamma_3 = \begin{bmatrix}
l_{11} & 0 & 0 & \sqrt{\pi_{i1}^s} G_s^{\mathrm{T}}(i)\tilde{A}^{\mathrm{T}} & \dots & \sqrt{\pi_{iN}^s} G_s^{\mathrm{T}}(i)\tilde{A}^{\mathrm{T}} & g_{13} & \dots & g_{14} & 0 \\
* & l_{22} & 0 & 0 & 0 & 0 & g_{23} & \dots & g_{24} & g_{25} \\
* & * & -\lambda^2 I & \sqrt{\pi_{i1}^s} D^{\mathrm{T}}(i) & \dots & \sqrt{\pi_{iN}^s} D^{\mathrm{T}}(i) & \sqrt{\pi_{i1}^s} D^{\mathrm{T}}(i) & \dots & \sqrt{\pi_{iN}^s} D^{\mathrm{T}}(i) & 0 \\
* & * & * & -Q_{1q}(1) & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & \ddots & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & -Q_{1q}(N) & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & -Q_{2q}(1) & 0 & 0 & 0 \\
* & * & * & * & * & * & * & \ddots & 0 & 0 \\
* & * & * & * & * & * & * & * & -Q_{2q}(N) & 0 \\
* & * & * & * & * & * & * & * & * & -I
\end{bmatrix} < 0 \tag{4.9}
$$

where $l_{11} = -G_s^{\mathrm{T}}(i) P_{1s}(i) G_s(i)$ and $l_{22} = -G_s^{\mathrm{T}}(i) P_{2s}(i) G_s(i)$.

Recalling Lemma 2.2, it follows that

$$
G_s^{\mathrm{T}}(i) P_{1s}(i) G_s(i) \geq G_s^{\mathrm{T}}(i) - Q_{1s}(i) + G_s(i)
$$
$$
G_s^{\mathrm{T}}(i) P_{2s}(i) G_s(i) \geq G_s^{\mathrm{T}}(i) - Q_{2s}(i) + G_s(i)
$$

Thus, $\Gamma_1 < 0$ guarantees $\Gamma_3 < 0$, and under condition (4.5), $\Gamma_1 < 0$ can be reduced to inequality (4.1) by denoting $w(t) = 0$. On the other hand, for $T \to \infty$, $\Gamma_1 < 0$ results in $J(\infty) < -V(x_\infty, i) < 0$, that is

$$
E\left\{\sum_{k=0}^{\infty} z_k^{\mathrm{T}} z_k\right\} \leq \lambda^2 E\left\{\sum_{k=0}^{\infty} w_k^{\mathrm{T}} w_k\right\} \tag{4.10}
$$

Therefore, system (2.3) is stochastically stable, and it also satisfies the H_∞ performance (2.6), which concludes the proof.

Our next result deals with observer-based controller design for system (2.3) in terms of linear matrix inequalities.

Theorem 4.3 *For given matrices M_t, M_t^-, initial state $\hat{x}_0$ and a scalar $\lambda > 0$, system (2.3) is stochastically stable in the region $\bigcap_{s=1}^{w} \bigcap_{i=1}^{N} \varepsilon(\hat{P}_s(i))$, and it also satisfies condition (2.6) if there exist a set of positive definite symmetric matrices $G_s(i)$, $P_s(i)$, $Q_{1s}(i)$, $Q_{2s}(i)$, $Q_{1q}(i)$ and $Q_{2q}(j)$, and matrices $\hat{K}(i)$, $\hat{F}(i)$ and $\hat{H}(i)$ such that*

$$
\Omega = \begin{bmatrix}
g_{11} & 0 & 0 & m_{12} & \dots & m_{13} & m_{14} & \dots & m_{15} & 0 \\
* & g_{22} & 0 & 0 & 0 & 0 & m_{23} & \dots & m_{24} & g_{25} \\
* & * & -\lambda^2 I & \sqrt{\pi_{i1}^s} D^{\mathrm{T}}(i) & \dots & \sqrt{\pi_{iN}^s} D^{\mathrm{T}}(i) & \sqrt{\pi_{i1}^s} D^{\mathrm{T}}(i) & \dots & \sqrt{\pi_{iN}^s} D^{\mathrm{T}}(i) & 0 \\
* & * & * & -Q_{1q}(1) & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & \ddots & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & -Q_{1q}(N) & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & -Q_{2q}(1) & 0 & 0 & 0 \\
* & * & * & * & * & * & * & \ddots & 0 & 0 \\
* & * & * & * & * & * & * & * & -Q_{2q}(N) & 0 \\
* & * & * & * & * & * & * & * & * & -I
\end{bmatrix} < 0 \tag{4.11}
$$

$$
\tilde{f}_q^{\mathrm{T}}(i)\tilde{f}_q(i) \le \hat{P}_s(i) \tag{4.12}
$$

An observer-based controller u_k for system (2.3) can be constructed as

$$
u_k = \sum_{t=1}^{2^m} \theta_t (M_t K(i) + M_t^- F(i))\bar{x}_k
$$

where $\tilde{f}_q(i)$ is the qth row of matrix $\tilde{F}(i)$, $\quad q = 1, 2, \dots m$

$$
\hat{K}(i) = K(i)G_s(i)
$$

$$
\hat{F}(i) = F(i)G_s(i)
$$

$$
\hat{H}(i) = H(i)C_1(i)G_s(i)
$$

$$
m_{12} = \sqrt{\pi_{i1}^s}(A(i)G_s - \hat{H}(i))^{\mathrm{T}}
$$
$$
m_{13} = \sqrt{\pi_{iN}^s}(A(i)G_s - \hat{H}(i))^{\mathrm{T}}
$$

$$
m_{14} = \sqrt{\pi_{i1}^s} G_s^{\mathrm{T}}(i)(-B(i)M_t\hat{K}(i) - B(i)M_t^-\hat{F}(i))^{\mathrm{T}}
$$
$$
m_{15} = \sqrt{\pi_{iN}^s} G_s^{\mathrm{T}}(i)(-B(i)M_t\hat{K}(i) - B(i)M_t^-\hat{F}(i))^{\mathrm{T}}
$$

$$m_{23} = \sqrt{\pi_{i1}^s} G_s^{\mathrm{T}}(i)(A(i) + B(i)M_t\hat{K}(i) + B(i)M_t^- \hat{F}(i))^{\mathrm{T}}$$
$$m_{24} = \sqrt{\pi_{iN}^s} G_s^{\mathrm{T}}(i)(A(i) + B(i)M_t\hat{K}(i) + B(i)M_t^- \hat{F}(i))^{\mathrm{T}}$$

Proof:

From Theorem 4.2, we have $\Gamma_1 < 0$ is equivalent to $\Omega < 0$

where

$$K(i) = \hat{K}(i)G_s^{-1}(i), \quad F(i) = \hat{F}(i)G_s^{-1}(i)$$

On the other hand, a state ellipsoid set is given as follows

$$\hat{x}_k^{\mathrm{T}} \sum_{s=1}^{w} \alpha_s(k)\hat{P}_s(i)\hat{x}_k \leq 1$$

From Lemma 2.3, we define the system state evolves in the following set

$$\tilde{f}_q^{\mathrm{T}}(i)\tilde{f}_q(i) \leq \sum_{s=1}^{w} \alpha_s(k)\hat{P}_s(i)$$

Hence, system (2.3) is stochastically stable and satisfies a H_∞ performance index. This completes the proof.

5 Estimation of Attraction Domain

In this section, we estimate the attraction domain of system (2.3) with input nonlinearity and the largest one is also given.

Theorem 5.1 *For given initial condition $\hat{x}_0$, the attraction domain of system (2.3) (with $w_k = 0$) is $\bigcap_{s=1}^{w}\bigcap_{i=1}^{N} \varepsilon(\hat{P}_s(i))$, if there exist a set of positive definite symmetric matrices $\hat{P}_s(i)$ such that*

$$\begin{bmatrix} g_{11} & 0 & \sqrt{\pi_{i1}^s}G_s^{\mathrm{T}}(i)\tilde{A}^{\mathrm{T}} & \dots & \sqrt{\pi_{iN}^s}G_s^{\mathrm{T}}(i)\tilde{A}^{\mathrm{T}} & g_{13} & \dots & g_{14} \\ * & g_{22} & 0 & 0 & 0 & g_{23} & \dots & g_{24} \\ * & * & -Q_{1q}(1) & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \ddots & 0 & 0 & 0 & 0 \\ * & * & * & * & -Q_{1q}(N) & 0 & 0 & 0 \\ * & * & * & * & * & -Q_{2q}(1) & 0 & 0 \\ * & * & * & * & * & * & \ddots & 0 \\ * & * & * & * & * & * & * & -Q_{2q}(N) \end{bmatrix} < 0 \qquad (5.1)$$

$$\varepsilon(\hat{P}_s(i)) \in \Theta(\tilde{F}(i)) \tag{5.2}$$

Proof: A parameter-dependent Lyapunov-Krasovskii function for system (2.3) can be constructed by using a symmetric positive definite matrices $\hat{P}_s(i)$ as follows:

$$V(\hat{x}_k, i) = \hat{x}_k^{\mathrm{T}} \sum_{s=1}^{w} \alpha_s(k)\hat{P}_s(i)\hat{x}_k \quad (i \in \Lambda)$$

then, it follows

$$\begin{aligned}
\Delta V(\hat{x}_k, i) &= E\{V(\hat{x}_{k+1}, i)\} - V(\hat{x}_k, i) \\
&= \hat{x}_k^{\mathrm{T}}[\breve{A}^{\mathrm{T}}(i) \sum_{j=1}^{N} \sum_{s=1}^{w} \sum_{s=1}^{w} \alpha_s(k)\alpha_s(k+1)\pi_{ij}^s \hat{P}_s(j)\breve{A}(i)]\hat{x}_k \\
&-\hat{x}_k^{\mathrm{T}} \sum_{s=1}^{w} \alpha_s(k)\hat{P}_s(i)\hat{x}_k
\end{aligned}$$

From Theorem 4.1, system (2.3) is stochastically stable in the region $\bigcap_{s=1}^{w}\bigcap_{i=1}^{N} \varepsilon(\hat{P}_s(i))$, and this concludes the proof.

Next, the largest attraction domain of system (2.3) is given in the following theorem.

Theorem 5.2 *For a given positive definite symmetric matrix S, the largest attraction domain of (2.3) (with $w_k = 0$) is $\bigcap_{s=1}^{w}\bigcap_{i=1}^{N} \varepsilon(\hat{P}_s(i))$, if there exist a set of positive definite symmetric matrices $\hat{P}_s(i)$, $Q_q(j)$ and a number $\gamma > 0$ such that*

$$sup \quad \gamma$$

$$s.t. \quad LMIs \quad (5.3), \quad (5.4), \quad (5.5)$$

where

$$\begin{bmatrix}
g_{11} & 0 & \sqrt{\pi_{i1}^s}G_s^{\mathrm{T}}(i)\tilde{A}^{\mathrm{T}} & \dots & \sqrt{\pi_{iN}^s}G_s^{\mathrm{T}}(i)\tilde{A}^{\mathrm{T}} & g_{13} & \dots & g_{14} \\
* & g_{22} & 0 & 0 & 0 & g_{23} & \dots & g_{24} \\
* & * & -Q_{1q}(1) & 0 & 0 & 0 & 0 & 0 \\
* & * & * & \ddots & 0 & 0 & 0 & 0 \\
* & * & * & * & -Q_{1q}(N) & 0 & 0 & 0 \\
* & * & * & * & * & -Q_{2q}(1) & 0 & 0 \\
* & * & * & * & * & * & \ddots & 0 \\
* & * & * & * & * & * & * & -Q_{2q}(N)
\end{bmatrix} < 0 \tag{5.3}$$

$$\varepsilon(\hat{P}_s(i)) \in \Theta(\tilde{F}(i)) \tag{5.4}$$

$$\gamma\varepsilon(S) \in \Theta(\tilde{F}(i)) \tag{5.5}$$

Proof: In order to enlarge the attraction domain of system (2.3), condition (5.5) is proposed. Under conditions (5.4) and (5.5), combining Theorem 5.1, one can obtain the stable condition for each vertices of system (2.3). Therefore, system (2.3) is stochastically stable in region $\bigcap\limits_{s=1}^{w}\bigcap\limits_{i=1}^{N}\varepsilon(\hat{P}_s(i))$. This completes the proof.

Remark 5.1 *Note that in order to get the largest domain of attraction for system (2.3) subject to actuator saturation and disturbance noises, for a given positive definite matrix S, Theorem 5.2 can be transformed into an optimization problem as follows*

$$sup \quad \gamma$$

$$s.t. \quad LMIs \quad (5.6), \quad (5.7), \quad (5.8)$$

$$\begin{bmatrix} g_{11} & 0 & 0 & m_{12} & \dots & m_{13} & m_{14} & \dots & m_{15} & 0 \\ * & g_{22} & 0 & 0 & 0 & 0 & m_{23} & \dots & m_{24} & g_{25} \\ * & * & -\lambda^2 I & \sqrt{\pi_{i1}^s}D^{\mathrm{T}}(i) & \dots & \sqrt{\pi_{iN}^s}D^{\mathrm{T}}(i) & \sqrt{\pi_{i1}^s}D^{\mathrm{T}}(i) & \dots & \sqrt{\pi_{iN}^s}D^{\mathrm{T}}(i) & 0 \\ * & * & * & -Q_{1q}(1) & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & \ddots & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -Q_{1q}(N) & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & -Q_{2q}(1) & 0 & 0 & 0 \\ * & * & * & * & * & * & * & \ddots & 0 & 0 \\ * & * & * & * & * & * & * & * & -Q_{2q}(N) & 0 \\ * & * & * & * & * & * & * & * & * & -I \end{bmatrix} < 0 \tag{5.6}$$

$$\varepsilon(\hat{P}_s(i)) \in \Theta(\tilde{F}(i)) \tag{5.7}$$

$$\gamma\varepsilon(S) \in \varepsilon(\hat{P}_s(i)) \tag{5.8}$$

and condition (5.8) can be transformed into the following inequality.

$$\frac{S}{\gamma^2} > \hat{P}_s(i) \qquad \forall i \in \Lambda \tag{5.9}$$

We denote $\gamma^{-2} = \eta$, then, (5.9) becomes

$$\begin{bmatrix} -\eta S & I \\ * & -\hat{Q}_s(i) \end{bmatrix} < 0 \qquad \forall i \in \Lambda \tag{5.10}$$

where

$$\hat{P}_s(i) = \hat{Q}_s^{-1}(i)$$

Therefore, the largest attraction domain of system (2.3) is $\bigcap\limits_{s=1}^{w} \bigcap\limits_{i=1}^{N} \varepsilon(\hat{P}_s(i))$.

Remark 5.2 *Note that attraction domain is a set of system state, system states start from it will remain in it. In order to enlarge the initial state set, we proposed Theorem 5.2 and Remark 5.1, and we can obtain the largest feasible region. Then, in simulation part, we can choose the initial state region in this set, such that the system is stochastically stable and satisfies a given H_∞ performance index under the controller we designed.*

6 Simulation Results

First, we consider a numerical example, the jump parameters are aggregated into 2 modes:

$$A(1) = \begin{bmatrix} 1.3 & -0.45 \\ 0 & 1.1 \end{bmatrix}, \quad A(2) = \begin{bmatrix} 0 & -0.29 \\ 0.9 & 1.5 \end{bmatrix}$$

$$B(1) = \begin{bmatrix} 0.5 \\ 1.1 \end{bmatrix}, \quad B(2) = \begin{bmatrix} 0.6 \\ 1.4 \end{bmatrix}$$

$$C_1(1) = \begin{bmatrix} 0.1 & 0.2 \end{bmatrix}, \quad C_1(2) = \begin{bmatrix} 0.3 & 0.1 \end{bmatrix}$$

$$C_2(1) = \begin{bmatrix} 0.2 & 0.2 \end{bmatrix}, \quad C_2(2) = \begin{bmatrix} 0.3 & 0.1 \end{bmatrix}$$

$$D(1) = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}, \quad D(2) = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}$$

The vertices of the time-varying transition probability matrix are given by

$$\Pi^1(k) = \begin{bmatrix} 0.2 & 0.8 \\ 0.35 & 0.65 \end{bmatrix}, \quad \Pi^2(k) = \begin{bmatrix} 0.55 & 0.45 \\ 0.48 & 0.52 \end{bmatrix}$$

$$\Pi^3(k) = \begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{bmatrix}, \quad \Pi^4(k) = \begin{bmatrix} 0.4 & 0.6 \\ 0.9 & 0.1 \end{bmatrix}$$

Given $\lambda = 0.5$ and $x_0 = \begin{bmatrix} 5 & -3 \end{bmatrix}^{\mathrm{T}}$ and by Theorem 4.3, the state trajectories of stochastic system, observer and the jumping modes are obtained, see Figures 1-3. Obviously, the states of the system are stable under such observer-based controller.

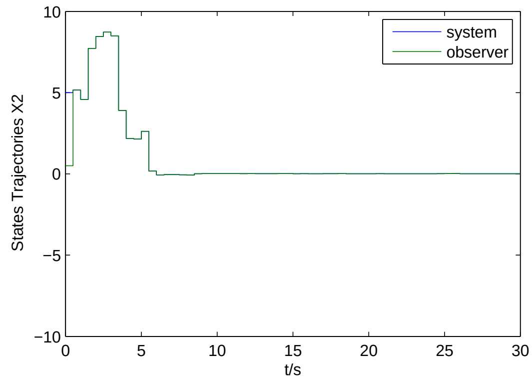

Figure 1: Trajectory of system state x_1

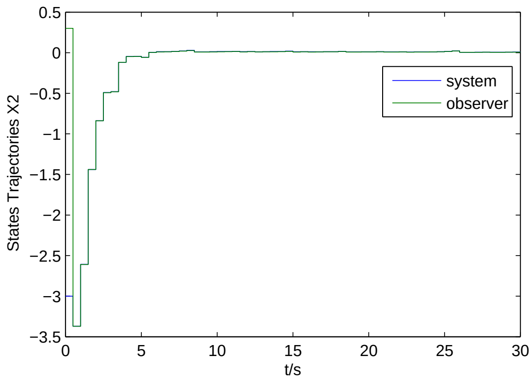

Figure 2: Trajectory of system state x_2

Next, we consider a nonhomogeneous economic system [29], which are aggregated into 3 modes:

$$\begin{cases} x_{k+1} = A(i)x_k + B(i)\sigma(u_k) + D(i)w_k \\ y_k = C_1(i)x_k \\ z_k = C_2(i)x_k \end{cases} \tag{6.1}$$

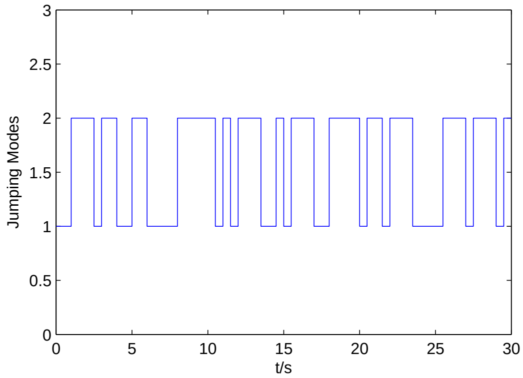

Figure 3: Jumping Modes

where

$$A(i) = \begin{bmatrix} 0 & 1 \\ -w(i) & 1 - s(i) + w(i) \end{bmatrix}, \quad B(i) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C_1(1) = \begin{bmatrix} 0.1 & 0 \end{bmatrix}, \quad C_1(2) = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}, \quad C_1(3) = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}$$

$$C_2(1) = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad C_2(2) = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad C_2(3) = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$D(1) = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}, \quad D(2) = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}, \quad D(3) = \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix}$$

x_k represents the national income, and u_k represents the investment of the government. The coefficients $s(i)$ and $w(i)$ were computed and given as follows:

Mode	Terminology	Description
1	Norm	s (or w) in mid-range
2	Boom	s in low range (or w in high)
3	Slump	s in high range (or w in low)

Table 1

i	$s(i)$	$w(i)$
1	2.5	0.3
2	43.7	-0.7
3	-5.3	0.9

Table 2

And the vertices of the time-varying transition probability matrix are given below:

$$\Pi^1(k) = \begin{bmatrix} 0.2 & 0.7 & 0.1 \\ 0.35 & 0.2 & 0.45 \\ 0.1 & 0.4 & 0.5 \end{bmatrix}, \quad \Pi^2(k) = \begin{bmatrix} 0.55 & 0.3 & 0.15 \\ 0.48 & 0.22 & 0.3 \\ 0.3 & 0.2 & 0.5 \end{bmatrix}$$

$$\Pi^3(k) = \begin{bmatrix} 0.67 & 0.17 & 0.16 \\ 0.3 & 0.47 & 0.23 \\ 0.26 & 0.1 & 0.64 \end{bmatrix}, \quad \Pi^4(k) = \begin{bmatrix} 0.4 & 0.2 & 0.4 \\ 0.8 & 0.1 & 0.1 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}$$

The initial state is given as $x_0 = \begin{bmatrix} -0.5 & -1 \end{bmatrix}^{\mathrm{T}}$

Given $\lambda = 0.1$, by solving LMIs (4.11) - (4.12), the state trajectory of stochastic system and jumping modes are obtained, see Figures 4-6. Obviously, the state of the system is stable under such observer-based controller.

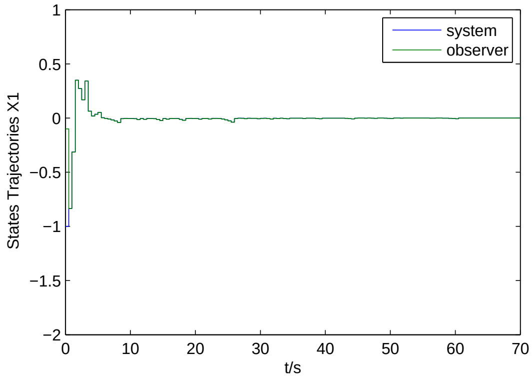

Figure 4: Trajectory of system state x_1

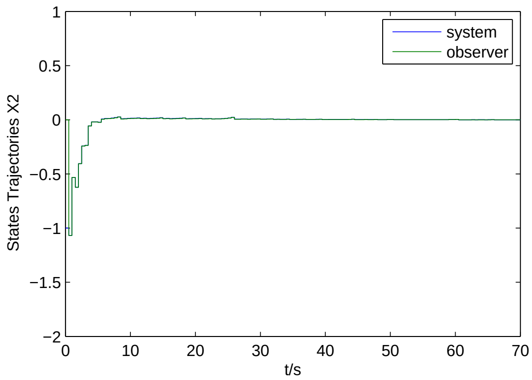

Figure 5: Trajectory of system state x_2

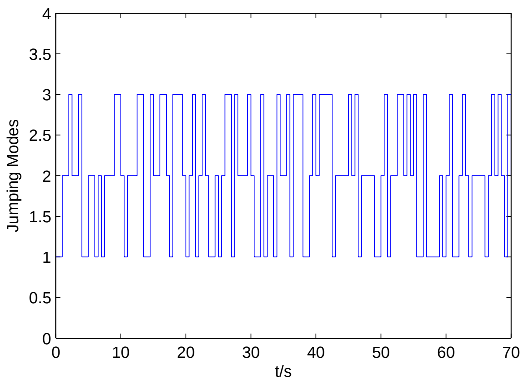

Figure 6: Jumping Modes

7 Conclusions

In this paper, the issue on observer-based H_∞ controller design for a class of nonhomogeneous Markov jump systems is addressed. A polytope is used to express time-varying transition probability matrices, in which values of vertices are given, and then, actuator saturation nonlinearity is expressed in terms of linear constraints on system states. An observer-based controller is designed such that the resulting closed-loop dynamic system is stochastically stable and satisfies a prescribed H_∞ performance. Furthermore, the largest domain of attraction is also given. The simulation result shows the effectiveness of the proposed techniques.

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