Deformation retractions from spaces of continuous maps onto spaces of holomorphic maps

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Contents

Abstract

A fundamental property of an Oka manifold Y is that every continuous map from a Stein manifold X to Y can be deformed to a holomorphic map. In a recent paper, Lárusson [19] considers the natural question of whether it is possible to simultaneously deform all continuous maps f from X to Y to holomorphic maps, in a way that depends continuously on f and does not change f if f is holomorphic to begin with. In other words, is $\mathscr{O}(X,Y)$ a deformation retract of $\mathscr{C}(X,Y)$? Lárusson provided a partial answer to this question. In this thesis we further develop the work of Lárusson on the topological relationship between spaces of continuous maps and spaces of holomorphic maps from Stein manifolds to Oka manifolds, mainly in the context of domains in $\mathbb{C}$. The main tools we use come from complex analysis, Oka theory, algebraic topology and the theory of absolute neighbourhood retracts. One of our main results provides a large supply of infinitely connected domains X in $\mathbb{C}$ such that $\mathscr{O}(X,\mathbb{C}^*)$ is a deformation retract of $\mathscr{C}(X,\mathbb{C}^*)$.

Signed Statement

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In memory of Grant Williams.

Chapter 1

Introduction

1.1 Context

Oka theory is a recent subfield of complex analysis and complex geometry that studies a close relationship between topology and analysis in a geometric setting. For an overview of Oka theory, see the survey article [11]. The concepts of Stein manifolds and Oka manifolds are fundamental in Oka theory.

Stein manifolds were introduced by K. Stein [26] in 1951. They generalise the notion of a domain of holomorphy. Roughly speaking, Stein manifolds are complex manifolds X with many holomorphic maps $X \to \mathbb{C}$. More precisely, a complex manifold X is called Stein if:

1. holomorphic functions on X separate points, that is, if x, $y \in X$ and $x \neq y$, then there exists a holomorphic function $f : X \to \mathbb{C}$ with $f(x) \neq f(y)$, and

2. X is holomorphically convex, that is, if $Y \subset X$ is not relatively compact, then there exists a holomorphic function $f : X \to \mathbb{C}$ so that $f|_Y$ is unbounded.

The class of Oka manifolds originates from a seminal paper by M. Gromov [13] in 1989. Say that a complex manifold Y has the convex interpolation property (CIP) if whenever T is a contractible closed complex submanifold of $\mathbb{C}^n$ and $f : T \to Y$ is a holomorphic map, f extends to a holomorphic map $\mathbb{C}^n \to Y$. A complex manifold Y is called Oka if it has the convex interpolation property. There are at least a dozen non-trivially equivalent ways to define an Oka manifold. It is know, from Grauert's [12] work in 1957, that all complex Lie groups are Oka. In particular, $\mathbb{C}$ and $\mathbb{C}^*$ are Oka.

A fundamental property of an Oka manifold Y that follows, quite non-trivially, from CIP, called the basic Oka property (BOP), is that every continuous map from a Stein manifold X to Y can be deformed to a holomorphic map. In fact, a family of continuous maps indexed by a compact parameter space $P \subset \mathbb{R}^n$ can be so deformed with continuous dependence on the parameter and in such a way that a map remains fixed during the deformation if it is holomorphic to begin with. This latter property is called the parametric Oka property (see Definition 2.37).

In a recent paper by Lárusson [19], the question of whether this deformation can be done with respect to the ultimate parameter space, that is, the whole space of all continuous maps $X \to Y$, is considered. In other words, is the space $\mathscr{O}(X, Y)$ of all holomorphic maps from X to Y a (strong) deformation retract of the space $\mathscr{C}(X, Y)$ of all continuous maps from X to Y? (Following [21], we use the term deformation retract for what is sometimes referred to as a strong deformation retract.) Lárusson uses tools from Oka theory and absolute neighbourhood retract theory to provide a partial answer to this question.

Absolute retracts (AR's) and absolute neighbourhood retracts (ANR's) were introduced by Borsuk [2], [3] in the 1930s. A metrisable space X is called an absolute retract if whenever X is embedded in a metrisable space Y, X is a retract of Y. A metrisable space X is called an absolute neighbourhood retract if whenever X is embedded in a metrisable space Y, X is a neighbourhood retract of Y, that is, there exists a neighbourhood U of X in Y so that X is a retract of U. ANR's have many nice properties. They are locally contractible and being an ANR is a local property. There exists a combinatorial characterisation of ANR's which is due to Lefschetz [20] and Dugundji [8]. Following [19] we will call it the Dugundji-Lefschetz characterisation. The following characterisation of ANR's came much later; it is due to Cauty [5]: A metrisable space is an ANR if and only if every open subset has the homotopy type of a CW complex.

Suppose that X is Stein and Y is Oka. Lárusson's [19] key observation is that if $\mathscr{C}(X,Y)$ and $\mathscr{O}(X,Y)$ are ANR's, then $\mathscr{O}(X,Y)$ is a deformation retract of $\mathscr{C}(X,Y)$. Lárusson uses the Dugundji-Lefschetz characterisation to prove that $\mathscr{O}(X,Y)$ is an ANR if X has a strictly plurisubharmonic Morse exhaustion with finitely many critical points. A sufficient condition on X for $\mathscr{C}(X,Y)$ to be an ANR was already known, namely that X is finitely dominated, formulated as [19, Proposition 8]. Note that if X has a strictly plurisubharmonic Morse exhaustion with only finitely many critical points, then X is finitely dominated. Lárusson's main result follows from the above.

Theorem 1.1 (Lárusson's [19] main theorem)**.** *Let Y be an Oka manifold and X be a Stein manifold with a strictly plurisubharmonic Morse exhaustion with finitely many critical points. Then $\mathscr{O}(X,Y)$ is a deformation retract of $\mathscr{C}(X,Y)$.*

The hypothesis on X is not necessary: for example $\mathscr{O}(X,\mathbb{C}^n)$ and $\mathscr{C}(X,\mathbb{C}^n)$ are ANR's for any Stein manifold X. The spaces $\mathscr{O}(X,Y)$ and $\mathscr{C}(X,Y)$ are not ANR's in general. For example $\mathscr{O}(\mathbb{C}\backslash\mathbb{N},\mathbb{C}^*)$ and $\mathscr{C}(\mathbb{C}\backslash\mathbb{N},\mathbb{C}^*)$ do not even have the homotopy type of an ANR (hence these spaces do not have the homotopy type of a CW complex, see [5]). Yet for this very particular example, Lárusson was able to prove that $\mathscr{O}(\mathbb{C}\backslash\mathbb{N},\mathbb{C}^*)$ is a deformation retract of $\mathscr{C}(\mathbb{C}\backslash\mathbb{N},\mathbb{C}^*)$. His proof starts with the fact that for a general Stein manifold X, the space of null-homotopic holomorphic maps $\mathscr{O}_0(X,\mathbb{C}^*)$ and the space of null-homotopic continuous maps $\mathscr{C}_0(X,\mathbb{C}^*)$ are ANR's. He then shows that it is sufficient to find a section of the divisor map $\mathscr{C}(\mathbb{C}\backslash\mathbb{N},\mathbb{C}^*)\to\mathbb{Z}^{\mathbb{N}}$, that is, the map which sends a continuous map to its sequence of winding numbers about each punture, with image in $\mathscr{O}(\mathbb{C}\backslash\mathbb{N},\mathbb{C}^*)$. He constructs such a section using Weierstrass products.

1.2 Overview of research

In this thesis we further develop the work of Lárusson [19] on the topological relationship between spaces of continuous maps and spaces of holomorphic maps from Stein manifolds to Oka manifolds, mainly in the context of domains in $\mathbb{C}$.

Suppose that X is a Stein manifold. Then $\mathscr{O}(X,\mathbb{C})$ is a deformation retract of $\mathscr{C}(X,\mathbb{C})$. We give a proof of this fact that is different from that of [19] and is more explicit. In fact, we prove that $\mathscr{O}(X,\mathbb{C})$ is a deformation retract of $\mathscr{C}(X,\mathbb{C})$ for any complex manifold X. The key to our proof is the Dugundji extension theorem. We use the fact that $\mathscr{O}(X,\mathbb{C})$ and $\mathscr{C}(X,\mathbb{C})$ are Fréchet spaces and that finding a continuous retraction $\mathscr{C}(X,\mathbb{C}) \to \mathscr{O}(X,\mathbb{C})$ is the same as finding an extension of the identity map $\mathscr{O}(X,\mathbb{C}) \to \mathscr{O}(X,\mathbb{C})$ to $\mathscr{C}(X,\mathbb{C})$. The Dugundji extension theorem tells us that the spaces $\mathscr{O}(X,\mathbb{C})$ and $\mathscr{C}(X,\mathbb{C})$ are AR's. A retraction between AR's is the same as a deformation retraction and so we are done. Our proof can easily be adapted to show that $\mathscr{O}(X,\mathbb{D})$ is a deformation retract of $\mathscr{C}(X,\mathbb{D})$. An interesting consequence of our proof is that there is a deformation retraction $\mathscr{C}(\mathbb{C},\mathbb{C}) \to \mathscr{O}(\mathbb{C},\mathbb{C})$ that maps every non-holomorphic continuous map to a polynomial.

Again, suppose that X is a Stein manifold. We want $\mathscr{O}(X,\mathbb{C}^*)$ to be a deformation retract of $\mathscr{C}(X,\mathbb{C}^*)$. Say that X is *good* if this is the case. A sufficient condition for $\mathscr{O}(X,\mathbb{C}^*)$ to be a deformation retract of $\mathscr{C}(X,\mathbb{C}^*)$ is that the quotient map $\mathscr{C}(X,\mathbb{C}^*) \to \mathscr{C}(X,\mathbb{C}^*)/\mathscr{C}_0(X,\mathbb{C}^*)$ has a continuous section with image in $\mathscr{O}(X,\mathbb{C}^*)$. For $X = \mathbb{C}\backslash\mathbb{N}$, this is equivalent to finding a section of the divisor map with image in $\mathscr{O}(\mathbb{C}\backslash\mathbb{N},\mathbb{C}^*)$. An ostensibly weaker sufficient condition is that the map $\cdot_* : \mathscr{C}(X,\mathbb{C}^*) \to \mathrm{Hom}(\pi_1(X,x),\mathbb{Z})$, $f \mapsto f_*$, has a continuous section with image in $\mathscr{O}(X,\mathbb{C}^*)$ for some (equivalently all) $x \in X$. For $X = \mathbb{C}\backslash\mathbb{N}$, the map $\cdot_*$ is essentially the divisor map. We do not know whether these two conditions are equivalent nor do we know whether either is necessary. The latter condition is equivalent to $\mathscr{O}(X,\mathbb{C}^*)$ being a trivial $\mathscr{O}_0(X,\mathbb{C}^*)$-bundle over $\mathrm{Hom}(\pi_1(X,x),\mathbb{Z})$, and $\mathscr{C}(X,\mathbb{C}^*)$ being a trivial $\mathscr{C}_0(X,\mathbb{C}^*)$-bundle also over $\mathrm{Hom}(\pi_1(X,x),\mathbb{Z})$. The former condition has a similar interpretation. The latter sufficient condition is better because it seems easier to to find a section of $\cdot_*$ than it is to find a section of the quotient map directly. If a section of $\cdot_*$ with image in $\mathscr{O}(X, \mathbb{C}^*)$ exists, then we say that X is *very good*.

It is rather easy to prove that if $\pi_1(X)$ is finitely generated, then X is good. If, moreover, $\dim X = 1$, then X is very good. The reason we introduce the ostensibly stronger property of being very good is that we can prove that certain domains in $\mathbb{C}$ are very good and it is relatively easy to construct new very good domains from old, but it seems hard or even impossible to do the same for good domains in a non-trivial way, that is, without the new domain being biholomorphic to the original one. The first 'new from old' result that we have is Proposition 3.20, which says that the very good property travels up holomorphic homotopy equivalences.

The most important of the other such results is Proposition 3.34, which says that if $X \subset \mathbb{C}$ is a very good domain, then you can remove a countable family of discs and points, which do not accumulate 'too much', and the remaining domain is still very good. The key technical lemma in proving this result is the construction of continuous sections using Weierstrass products. We also need a good deal of planar topology.

He and Schramm [15] proved in 1993 that every domain in the Riemann sphere $\mathbb{P}_1$ with countably many boundary components is biholomorphic to a circle domain, that is, a domain whose boundary components are either circles or points. He and Schramm's result is a generalisation of the Riemann mapping theorem and a partial answer to the conjecture, made by Koebe [18] in 1908, that every planar domain is biholomorphic to a circle domain. This conjecture was proved by Koebe for finitely connected domains.

We use He and Schramm's big theorem, as well as Proposition 3.34 and the examples of very good domains that we already have, to prove our main theorem (Theorem 3.37). The gist of this theorem is that we can produce many infinitely connected domains in $\mathbb{C}$ that are very good. The only previously known example was $\mathbb{C}\backslash\mathbb{N}$.

1.3 Further directions

It is of interest whether an arbitrary domain in $\mathbb{C}$ is good and furthermore whether every non-compact Riemann surface is good. It does not seem possible to generalise the Weierstrass product idea further. So a new idea would be needed.

In this thesis we have only considered the targets $\mathbb{C}$ and $\mathbb{C}^*$. An extension of what we have done would be to consider the other 1-dimensional Oka targets, namely the Riemann sphere $\mathbb{P}_1$ and tori. The case of tori would probably be similar to the case of target $\mathbb{C}^*$. We expect that the case of $\mathbb{P}_1$ would be more difficult.

Say that a pair (X, Y) of Riemann surfaces is a Winkelmann pair if every continuous map $X \to Y$ can be deformed to a holomorphic map. Winkelmann [28] in 1993 determined all such pairs. These are the pairs (X, Y) such that:

1. X or Y is biholomorphic to $\mathbb{C}$ or $\mathbb{D}$.

2. X is biholomorphic to $\mathbb{P}_1$ and Y is not.

3. X is non-compact and Y is isomorphic to $\mathbb{P}_1$, $\mathbb{C}^*$ or a torus.

4. Y is isomorphic to $\mathbb{D}^*$ and $X = Z \setminus \bigcup_{j \in J} D_j$, where Z is a compact Riemann surface, J is finite and non-empty and $(D_j)_{j \in J}$ is a family of disjoint closed subsets of Z biholomorphic to non-degenerate closed discs.

For all other pairs (X, Y), there is a continuous map $X \to Y$ that cannot be deformed to a holomorphic map, so $\mathscr{O}(X, Y)$ is not a deformation retract of $\mathscr{C}(X, Y)$. It would be interesting to know whether $\mathscr{O}(X, Y)$ is a deformation retract of $\mathscr{C}(X, Y)$ for all Winkelmann pairs (X, Y), that is, whether the only obstruction to $\mathscr{O}(X, Y)$ being a deformation retract of $\mathscr{C}(X, Y)$ in the 1-dimensional case is that there exists a continuous map which cannot be deformed to a holomorphic map.

Chapter 2

Background

2.1 Riemann surfaces

The reader is assumed to be familiar with the definition of a complex manifold and what it means for a continuous map between two complex manifolds to be holomorphic. Thus we will not give these definitions here. However, it should be noted that we take connectedness as part of the definition of a complex manifold.

A Riemann surface is a 1-dimensional complex manifold.

Theorem 2.1 (The Runge Approximation Theorem)**.** *Suppose X is a non-compact Riemann surface and Y is an open subset whose complement contains no compact connected component. Then every holomorphic function on Y can be approximated uniformly on every compact subset of Y by holomorphic functions on X.*

Proof. See [9, Theorem 25.5]. □

Theorem 2.2 (The Weierstrass Theorem)**.** *Suppose X is a non-compact Riemann surface. If $A \subset X$ is a discrete subset of X, then every function $A \to \mathbb{C}$ can be extended to a holomorphic function $X \to \mathbb{C}$.*

Proof. See [9, Theorem 26.7]. □

Definition 2.3. *A complex manifold X is said to be* Stein *if the following hold.*

1. *Given any two points $x, y \in X$ with $x \neq y$, there exists a holomorphic function $f : X \to \mathbb{C}$ such that $f(x) \neq f(y)$.*

2. *If K is a compact subset of X, then its $\mathscr{O}(X)$-hull*

$$\hat{K} = \{x \in X : |f(x)| \leq \max_K |f| \text{ for all } f \in \mathscr{O}(X)\}$$

is also compact.

Theorem 2.4. *Every non-compact Riemann surface is Stein.*

Proof. See [9, Corollary 26.8]. □

Let X be a Riemann surface. Call X *elliptic* if the universal covering space of X is the Riemann sphere $\mathbb{P}_1$, *parabolic* if the universal covering space is $\mathbb{C}$ and *hyperbolic* if the universal covering space is the unit disc.

Theorem 2.5. *The Riemann sphere $\mathbb{P}_1$ is elliptic. The complex plane $\mathbb{C}$, the punctured plane $\mathbb{C}^*$ and all tori $\mathbb{C}/\Gamma$, where Γ is a lattice in $\mathbb{C}$, are parabolic. Every other Riemann surface is hyperbolic.*

Proof. See [9, Theorem 27.12]. □

2.2 Function spaces

Let X and Y be topological spaces. We denote the set of all continuous maps from X to Y by $\mathscr{C}(X, Y)$. There is a natural topology that we can give to $\mathscr{C}(X, Y)$ called the *compact-open topology*. The compact-open topology can be defined as follows. If $K \subset X$ is compact and $U \subset Y$ is open, define the set

$$[K, U] = \{f \in \mathscr{C}(X, Y) : f(K) \subset U\}.$$

The compact-open topology is the coarsest topology on $\mathscr{C}(X, Y)$ containing all sets of the form $[K, U]$ so that $K \subset X$ is compact and $U \subset Y$ is open.

If X and Y are complex manifolds, then the collection of holomorphic functions from X to Y, which we denote $\mathscr{O}(X,Y)$, is a subset of $\mathscr{C}(X,Y)$. We therefore give $\mathscr{O}(X,Y)$ the subspace topology.

Recall that a metric d on a topological space X is called admissible if d induces the topology on X. A topological space X is called metrisable if there exists an admissible metric on X.

In this section we have tried to present results about $\mathscr{C}(X,Y)$ and $\mathscr{O}(X,Y)$ with minimal assumptions on the source and the target. We make use of the properties Lindelöf, separable and second countable. In general these properties are not equivalent. Second countable implies separable and second countable implies Lindelöf, but the converses to these statements need not hold. However, if the space is metrisable, then these properties are equivalent. Not all spaces in this section are metrisable, therefore we make the distinction between these three properties. Of course we could take spaces to be second countable whenever we need one of these conditions, as this would imply separable and Lindelöf, but this is often more restrictive than necessary.

Note that the assumptions on X and Y in the results about $\mathscr{C}(X,Y)$ that we present here are satisfied if X and Y are manifolds. For suppose X is a manifold. By definition X is second countable which implies X is Lindelöf and separable. Clearly X is locally compact, that is, every point $x \in X$ has a compact neighbourhood K. A Hausdorff, locally compact space is regular [17, p. 267]. Therefore, X is second countable and regular. Hence by the Urysohn metrisation theorem, X is metrisable [23, p. 215].

Lemma 2.6. *Suppose X, Y and Z are topological spaces. If Y is locally compact and Hausdorff, then the map*

$$T : \mathscr{C}(X,Y) \times \mathscr{C}(Y,Z) \to \mathscr{C}(X,Z), \ (f,g) \mapsto g \circ f,$$

is continuous.

Proof. Suppose $A \subset \mathscr{C}(X,Z)$ is open. Without loss of generality, suppose that $A = [K,W]$, where W is an open subset of Z and K is a compact subset of X. Let $h \in [K,W]$ and suppose $h = g \circ f$, where $f \in \mathscr{C}(X,Y)$ and $g \in \mathscr{C}(Y,Z)$. Note that $f(K) \subset g^{-1}(W)$.

A Hausdorff, locally compact space is regular [17, p. 267], so Y is regular. Therefore, for each $y \in f(K)$ there exist open neighbourhoods U_y and V_y of $\{y\}$ and $Y \backslash g^{-1}(W)$ respectively such that $U_y \cap V_y = \emptyset$. Since $f(K)$ is compact, there exist $y_1, \ldots, y_n \in f(K)$ such that $U_{y_1}, \ldots, U_{y_n}$ cover $f(K)$. Set $C = \bigcup_{j=1}^{n} Y \backslash V_{y_j}$. Note that C is closed. One can easily show that $f(K) \subset C^\circ \subset C \subset g^{-1}(W)$.

There exists a compact subset $L' \subset Y$ such that $f(K) \subset (L')^\circ \subset L'$ since Y is locally compact and $f(K)$ is compact. Therefore $L = C \cap L'$ is a compact subset of Y and $L^\circ = C^\circ \cap (L')^\circ$, so $f(K) \subset L^\circ \subset L \subset g^{-1}(W)$. Hence $[K, L^\circ] \times [L, W]$ is an open neighbourhood of (f,g) in $T^{-1}([K,W])$. Therefore T is continuous. □

Lemma 2.7. *Suppose X, Y, Z are topological spaces and Y is locally compact and Hausdorff. If $F : X \to Y$ is a continuous map, then the pullback map $F^* : \mathscr{C}(Y,Z) \to \mathscr{C}(X,Z)$, $f \mapsto f \circ F$ is continuous. In particular, if $X \subset Y$, then the restriction map $\mathscr{C}(Y,Z) \to \mathscr{C}(X,Z)$ is continuous.*

Proof. Immediate from Lemma 2.6. □

Corollary 2.8. *If $(G, \cdot)$ is a locally compact, Hausdorff topological group and X is a topological space, then $\mathscr{C}(X,G)$ with pointwise multiplication $\cdot$ is a topological group.*

Proof. Let $m : G \times G \to G$, $(g,h) \mapsto g \cdot h$, and $i : G \to G$, $g \mapsto g^{-1}$. Since G is a topological group, $m \in \mathscr{C}(G \times G, G)$ and $i \in \mathscr{C}(G,G)$. If $f, g : X \to G$ are continuous, then $\varphi : X \to G \times G$, $x \mapsto (f(x), g(x))$, is continuous, so $m \circ \varphi = f \cdot g \in \mathscr{C}(X,G)$. Therefore pointwise multiplication $\cdot$ is a binary operation on $\mathscr{C}(X,G)$. Clearly $\cdot$ is associative, $\mathscr{C}(X,G)$ has an identity element (namely the constant map e, where e is the identity of G), and if $f \in \mathscr{C}(X,G)$, then f has an inverse $f^{-1} \in \mathscr{C}(X,G)$ (namely $i \circ f$). So $(\mathscr{C}(X,G), \cdot)$ is a group.

Let $\mu : \mathscr{C}(X,G) \times \mathscr{C}(X,G) \to \mathscr{C}(X,G)$ and $\iota : \mathscr{C}(X,G) \to \mathscr{C}(X,G)$ be the multiplication and inversion map respectively. Note that $\iota(f) = i \circ f$ for all $f \in \mathscr{C}(X,G)$. Hence ι is continuous by Lemma 2.6. We may write μ as the composition

$$\mathscr{C}(X,G) \times \mathscr{C}(X,G) \xrightarrow{(\cdot,\cdot)} \mathscr{C}(X, G \times G) \xrightarrow{m\circ\cdot} \mathscr{C}(X,G).$$

The map $(\cdot,\cdot)$ is clearly continuous, and the map $m \circ \cdot$ is continuous by Lemma 2.6. Therefore, μ is continuous. Hence $\mathscr{C}(X,G)$ is a topological group. $\square$

Suppose X is a metric space, $a \in X$ and $r > 0$. Let $B(a,r)$ denote the open ball in X of radius $r > 0$ centred at a.

Next we prove that if X and Y are separable metric spaces, then $\mathscr{C}(X,Y)$ is second countable. In particular, if X and Y are complex manifolds, then $\mathscr{C}(X,Y)$ is second countable. It follows that the space $\mathscr{O}(X,Y)$ of holomorphic maps from X to Y is second countable and hence separable. This fact is needed in the proof of Proposition 3.7.

Proposition 2.9. *Suppose that X and Y are separable metric spaces (equivalently, X and Y are Lindelöf metric spaces). If X is locally compact, then $\mathscr{C}(X,Y)$ is second countable.*

Proof. For each $j \in \mathbb{N} = \{1,2,3,\dots\}$, let $\mathscr{K}^j$ be a countable cover of X by compact sets each with diameter less than $\frac{2}{j}$. This is possible because X is Lindelöf and each $x \in X$ has arbitrarily small compact neighbourhoods since X is Hausdorff and locally compact [23, Theorem 29.2].

Let $\mathscr{K} = \bigcup_{j\in\mathbb{N}} \mathscr{K}^j$. Clearly $\mathscr{K}$ is countable also. Since Y is separable, we can find a countable dense subset A of Y. Let

$$\mathscr{B}' = \{[K, B(a,1/n)] : K \in \mathscr{K}, a \in A, n \in \mathbb{N}\}. \tag{2.1}$$

Clearly $\mathscr{B}'$ is a countable collection of open subsets of $\mathscr{C}(X,Y)$. Therefore, the collection $\mathscr{B}$ of all finite intersections of elements of $\mathscr{B}'$ is countable. We will now show that $\mathscr{B}$ is a base for the compact-open topology on $\mathscr{C}(X,Y)$. To this end, suppose that V is an open subset of $\mathscr{C}(X,Y)$ and that f is an arbitrary element of V. It suffices to show that there exists $W \in \mathscr{B}$ so that $f \in W$ and $W \subset V$.

It follows from the definition of the compact-open topology that V is a union of finite intersections of sets of the form $[K,U]$, where $K \subset X$ is compact and $U \subset Y$ is open. Therefore, if $f \in V$, then

$$f \in \bigcap_{j=1}^{n} [K_j, U_j] \subset V,$$

where $n \in \mathbb{N}$ and for each $j \in \{1,\ldots,n\}$, K_j is a compact subset of X and U_j is an open subset of Y. Let $r = \min_{j\in\{1,\ldots,n\}} d(f(K_j), Y\backslash U_j) > 0$. Choose $n_0 \in \mathbb{N}$ sufficiently large such that $\frac{1}{n_0} < r$. Let $r' = \frac{1}{n_0}$. For each $x \in \bigcup_{j=1}^{n} K_j$, choose $a_x \in A$ such that

$$d(a_x, f(x)) < \frac{r'}{6}, \tag{2.2}$$

and $C_x \in \mathscr{K}$ such that $x \in C_x$ and

$$C_x \subset f^{-1}(B(f(x), \frac{r'}{6})). \tag{2.3}$$

Then $\{C_x^\circ : x \in \bigcup_{j=1}^{n} K_j\}$ is an open cover of $\bigcup_{j=1}^{n} K_j$ and therefore has a finite subcover $\{C_{x_1},\ldots,C_{x_m}\}$. Let

$$W = \bigcap_{j=1}^{m} [C_{x_j}, B(a_{x_j}, r'/3)].$$

Observe that W is an element of $\mathscr{B}$. We claim that W is a neighbourhood of f in V. It follows easily from (2.2) and (2.3) that $f \in W$. We now show that $W \subset V$. If $g \in W$ and $x \in K_j$ for some $j \in \{1,2,\ldots,n\}$, then $x \in C_{x_k}$ for some $k \in \{1,2,\ldots,m\}$. Then by (2.2) and (2.3) and because $g \in W$,

$$\begin{aligned} d(f(x), g(x)) &\leq d(f(x), f(x_k)) + d(f(x_k), a_{x_k}) + d(a_{x_k}, g(x))), \\ &\leq \frac{r'}{6} + \frac{r'}{6} + \frac{r'}{3} < r' < r. \end{aligned}$$

Hence, by the definition of r, it follows that $g(x) \in U_j$. Because $x \in K_j$ was taken arbitrarily it follows that $g \in [K_j, U_j]$, and because $j \in \{1,2,\ldots,n\}$ was arbitrary it follows that

$$g \in \bigcap_{j=1}^{n} [K_j, U_j] \subset V,$$

as required. □

Lemma 2.10. *Let X be a Hausdorff topological space, Y a metrisable topological space with admissible metric d and $f \in \mathscr{C}(X,Y)$. If $K \subset X$ is a compact subset of X and $\delta > 0$, then there exists an open neighbourhood W of f in $\mathscr{C}(X,Y)$ such that*

$$\sup_{z \in K} d(f(z), g(z)) < \delta$$

for every $g \in W$.

Proof. Let $\mathscr{U} = \{f^{-1}(B(a, \delta/3)) : a \in Y\}$. Clearly $\mathscr{U}$ is an open cover of X. In particular $\mathscr{U}$ is an open cover of K. Let

$$\mathscr{V} = \{f^{-1}(B(a_1, \delta/3)), \ldots, f^{-1}(B(a_n, \delta/3))\}$$

be a finite subcover of K, where $a_1, \ldots, a_n \in Y$. For each $j = 1, \ldots, m$, let $V_j = f^{-1}(B(a_j, \delta/3))$ and $C_j = f^{-1}\left(\overline{B(a_j, \delta/3)}\right) \cap K$. Each C_j is the intersection of a closed subset of X with a compact subset of X and is therefore compact. If $z \in K$, then $z \in V_j \cap K \subset C_j$ for some $j \in \{1, \ldots, m\}$. Hence $C_1, \ldots, C_m$ cover K. We claim that

$$W = \bigcap_{j=1}^{n} [C_j, B(a_j, \delta/2)]$$

is the desired open neighbourhood of f. Clearly W is open and $f \in W$. Suppose $g \in W$. If $z \in K$, then $z \in C_j$ for some $j \in \{1, \ldots, m\}$. It follows that

$$d(f(z), g(z)) \leq d(f(z), a_j) + d(a_j, g(z)) < \delta/3 + \delta/2 < \delta.$$

Since $z \in K$ was arbitrary, $\sup_{z \in K} d(f(z), g(z)) < \delta$. □

By the previous lemma, convergence in the compact-open topology on $\mathscr{C}(X, Y)$ implies uniform convergence on compact subsets of X. It is easy to check that uniform convergence on compact subsets of X implies convergence in the compact-open topology on $\mathscr{C}(X, Y)$. So these two modes of convergence are the same.

Proposition 2.11. *If X is a topological space, then $\mathscr{C}(X, \mathbb{C})$ is a topological vector space over $\mathbb{C}$.*

Proof. Let $\mathscr{C} = \mathscr{C}(X, \mathbb{C})$. Clearly $\mathscr{C}$ is a vector space, so all that is left for us to show is that both addition and scalar multiplication are continuous. Addition is continuous by Corollary 2.8 (since $(\mathbb{C}, +)$ is a locally compact, Hausdorff topological group). It remains to show that scalar multiplication is continuous. To this end, let $\theta : \mathbb{C} \times \mathscr{C} \to \mathscr{C}$, $(k, f) \mapsto kf$.

Suppose $U \subset \mathscr{C}$ is open. Without loss of generality suppose that $U = [K, V]$, where K is a compact subset of X and V is an open subset of $\mathbb{C}$. Suppose that $(k, f) \in \theta^{-1}(U)$. Let $r = d(kf(K), \mathbb{C}\backslash V) > 0$ (where d is the usual metric on $\mathbb{C}$, that is $d(z, w) = |z - w|$), $M = \max_{z \in K} |f(z)|$, and $t = \max\{|k|, 1\}$. Let

$$W_1 = B\left(k, \frac{r}{3} \cdot \frac{1}{M + 2r/(3t)}\right) \subset \mathbb{C}$$

and let W_2 be an open neighbourhood of f in $\mathscr{C}$ such that $\sup_{z \in K} d(f(z), h(z)) < \dfrac{r}{3t}$ for all $h \in W_2$ (which exists by Lemma 2.10). Define $W = W_1 \times W_2$. Clearly W is open neighbourhood of (k, f). If $(k', f') \in W$ and $x \in K$, then

$$\begin{aligned}
&|k'f'(x) - kf(x)| \\
&\quad = |(k' - k)f'(x) - k(f(x) - f'(x))| \\
&\quad \leq |k' - k||f'(x)| + |k|(|f(x) - f'(x)| \\
&\quad \leq \frac{r}{3} \cdot \left(\frac{1}{M + 2r/(3t)}\right) \cdot \left(M + \frac{r}{3t}\right) + |k|\frac{r}{3t} \\
&\quad \leq \frac{r}{3} + \frac{r}{3} < r.
\end{aligned}$$

Therefore, it follows from the definition of r that $k'f'(x) \in V$. Because $x \in K$ was arbitrary, it follows that $k'f' \in [K, V]$. Therefore W is an open neighbourhood of (k, f) contained in $\theta^{-1}(U)$. $\square$

Proposition 2.12. *Suppose X and Y are topological spaces. If X is Lindelöf and locally compact, and Y is separable and metrisable, then the compact-open topology on $\mathscr{C}(X, Y)$ is metrisable.*

Proof. Let $\mathscr{C} = \mathscr{C}(X, Y)$ and $(K_j)_{j\in\mathbb{N}}$ be a sequence of compact subsets of X so that $X = \bigcup_{j\in\mathbb{N}} K_j^\circ$. Let d be an admissible metric on Y. For each $j \in \mathbb{N}$ define

$$d_j(f, g) = \sup_{x\in K_j} d(f(x), g(x)), \quad f, g \in \mathscr{C}(X, Y).$$

Let $D : \mathscr{C}(X, Y) \times \mathscr{C}(X, Y) \to \mathbb{R}$,

$$D(f, g) = \sum_{j=1}^{\infty} 2^{-j} \frac{d_j(f, g)}{1 + d_j(f, g)}.$$

We now show that D is an admissible metric for $\mathscr{C}(X, Y)$ and therefore $\mathscr{C}(X, Y)$ is metrisable. To show that D is an admissible metric we first prove that D is a metric, and then we show that D induces the correct topology on $\mathscr{C}(X, Y)$.

Step 1: Showing that D is a metric.

Clearly $D(f, g) = D(g, f)$ and $D(f, g) \geq 0$ for all $f, g \in \mathscr{C}$. If $f, g \in \mathscr{C}$ such that $D(f, g) = 0$, then $f = g$. For suppose $f, g \in \mathscr{C}$ with $D(f, g) = 0$. Then $d_j(f, g) = 0$ for every $j \in \mathbb{N}$. Therefore, $f(x) = g(x)$ on K_j for every $j \in \mathbb{N}$. So $f = g$ since $X = \bigcup_{j\in\mathbb{N}} K_j^\circ$.

The map D satisfies the triangle inequality. For suppose $f, g, h \in \mathscr{C}$. Then

$$\begin{aligned} d_j(f, h) = \sup_{x\in K_j} d(f(x), h(x)) &\leq \sup_{x\in K_j} d(f(x), g(x)) + \sup_{x\in K_j} d(g(x), h(x)) \\ &= d_j(f, g) + d_j(g, h), \end{aligned}$$

for $i \in \mathbb{N}$. Since the function $F : [0, \infty) \to \mathbb{R}$, $F(x) = \dfrac{x}{1+x}$ is increasing,

$$\frac{d_j(f,h)}{1+d_j(f,h)} \leq \frac{d_j(f,g)+d_j(g,h)}{1+d_j(f,g)+d_j(g,h)}.$$

Hence,

$$\begin{aligned} D(f,h) &= \sum_{j=1}^{\infty} 2^{-j} \frac{d_j(f,h)}{1+d_j(f,h)} \\ &\leq \sum_{j=1}^{\infty} 2^{-j} \frac{d_j(f,g)+d_j(g,h)}{1+d_j(f,g)+d_j(g,h)} \\ &\leq \sum_{j=1}^{\infty} 2^{-j} \left(\frac{d_j(f,g)}{1+d_j(f,g)} + \frac{d_j(g,h)}{1+d_j(g,h)} \right) = D(f,g) + D(g,h), \end{aligned}$$

as required.

Step 2: Showing that D is an admissible metric.

Let τ_C be the compact-open topology and τ_D be the topology induced by D. Recall that τ_C is the coarsest topology containing all sets of the form $[K, U]$, where $K \subset X$ is compact and $U \subset Y$ is open. First we will show that τ_D contains all sets of this form and hence τ_D is finer than τ_C. Then we will show that τ_D is also coarser than τ_C and so $\tau_C = \tau_D$, as required.

Claim 1: The topology τ_D induced by D contains all sets of the form $[K, U]$, where $K \subset X$ is compact and $U \subset Y$ is open.

If $K \subset X$ is compact, $U \subset Y$ is open and $f \in [K, U]$, then there exists $n_0 \in \mathbb{N}$ such that $K \subset K_1^\circ \cup K_2^\circ \cup \cdots \cup K_{n_0}^\circ$. Let $r = d(f(K), Y \backslash U) > 0$. We prove that

$$B\left(f, \frac{1}{2^{n_0}} \cdot \frac{r}{1+r}\right) \subset [K, U],$$

and therefore $[K, U] \in \tau_D$. Suppose $g \in \mathscr{C}$ and $D(f,g) < \dfrac{1}{2^{n_0}} \cdot \dfrac{r}{1+r}$. Then if $j \in \{1, \ldots, n_0\}$,

$$2^{-j} \frac{d_j(f,g)}{1+d_j(f,g)} \leq D(f,g) < 2^{-n_0} \frac{r}{1+r},$$

and therefore

$$\frac{d_j(f,g)}{1+d_j(f,g)} < \frac{r}{1+r},$$

from which it follows that $d_j(f,g) < r$ since on $[0,\infty)$, $\dfrac{x}{1+x}$ is strictly increasing. Now let $x \in K$. Then $x \in K_j$ for some $j \in \{1,\ldots,n_0\}$, and therefore $d(f(x),g(x)) < d_j(f,g) < r$, and hence by the definition of r, $g(x) \in U$. Hence $g \in [K,U]$ as required.

Claim 2: The topology τ_D is coarser than the compact-open topology τ_C.

Suppose that $V \in \tau_D$. Let f be an arbitrary element of V and suppose $\epsilon > 0$ is such that $B(f,\epsilon) \subset V$. We will show that there exists $W \in \tau_C$ so that $f \in W \subset B(f,\epsilon)$ and therefore that V can be written as the union of sets in τ_C and hence $U \in \tau_C$.

Let $n_1 \in \mathbb{N}$ be so large that

$$\sum_{i=n_1+1}^{\infty} 2^{-i} < \frac{\epsilon}{2}.$$

By Lemma 2.10 there exists $W \in \tau_C$ such that $f \in W$ and

$$\sup\left\{d(f(z),g(z)) : z \in \bigcup_{i=1}^{n_1} K_i\right\} < \frac{\epsilon}{2}$$

whenever $g \in W$. We now claim that $W \subset B(f,\epsilon)$. If $g \in W$, then

$$\frac{d_i(f,g)}{1+d_i(f,g)} \le d_i(f,g) = \sup_{x\in K_i} d(f(x),g(x)) < \frac{\epsilon}{2}$$

for $i = 1,\ldots,n_1$. Hence,

$$D(f,g) = \sum_{i=1}^{\infty} \frac{2^{-i}d_i(f,g)}{1+d_i(f,g)} = \sum_{i=1}^{n_1} \frac{2^{-i}d_i(f,g)}{1+d_i(f,g)} + \sum_{i=n_1+1}^{\infty} \frac{2^{-i}d_i(f,g)}{1+d_i(f,g)}$$
$$< \frac{\epsilon}{2}\sum_{i=1}^{n_1} 2^{-i} + \sum_{i=n_1+1}^{\infty} 2^{-i} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Therefore g is contained in $B(f,\epsilon)$. So $W \subset B(f,\epsilon)$.

Therefore $\mathscr{C}$ is metrisable as required. □

Recall that a topological vector space is called locally convex if the origin has a neighbourhood basis whose members are convex. A Fréchet space is a locally convex space which is complete with respect to a translation invariant metric.

Proposition 2.13. *Let X be a Lindelöf, locally compact topological space. Then $\mathscr{C}(X,\mathbb{C})$ can be given a Fréchet space structure.*

Proof. Let $\mathscr{C} = \mathscr{C}(X,\mathbb{C})$ and $(K_j)_{j\in\mathbb{N}}$ be a sequence of compact subsets of X with the property that $\bigcup_{j\in\mathbb{N}} K_j^\circ = X$. Such a sequence exists because X is Lindelöf and locally compact. Let d be the usual metric on $\mathbb{C}$, that is, $d : \mathbb{C}\times\mathbb{C} \to \mathbb{R}$, $d(x, y) = |x - y|$. Define an admissible metric D on $\mathscr{C}(X,\mathbb{C})$ as in Proposition 2.12, that is, define $D : \mathscr{C}(X,\mathbb{C}) \times \mathscr{C}(X,\mathbb{C}) \to \mathbb{R}$,

$$D(f, g) = \sum_{j=1}^{\infty} 2^{-j} \frac{d_j(f, g)}{1 + d_j(f, g)},$$

where $d_j(f, g) = \sup_{x\in K_j} d(f(x), g(x))$. Since d is translation invariant, it follows that D is translation invariant also.

We saw in Proposition 2.11 that $\mathscr{C}$ is a topological vector space. A neighbourhood basis for the origin is

$$\left\{ \left[\bigcup_{j=1}^{n} K_j, B(0, 1/n) \right] : n \in \mathbb{N} \right\}.$$

Each element of this basis is convex since $B(0, 1/n)$ is convex in $\mathbb{C}$.

We now show that $\mathscr{C}(X,\mathbb{C})$ is complete with respect to D. To this end, suppose that $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathscr{C}(X,\mathbb{C})$ with respect to D. Given $j \in \mathbb{N}$ and $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$2^{-j} \frac{d_j(f_m, f_n)}{1 + d_j(f_m, f_n)} \le D(f_m, f_n) < 2^{-j} \frac{\epsilon}{1 + \epsilon}$$

whenever $m, n \ge N$. Since $\dfrac{x}{1 + x}$ is strictly increasing it follows that $d_j(f_m, f_n) < \epsilon$ whenever $m, n \ge N$. Therefore given $j \in \mathbb{N}$ and $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $d_j(f_m, f_n) < \epsilon$ whenever $m, n \geq N$. If $x \in X$, then $x \in K_j$ for some $j \in \mathbb{N}$, so $(f_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$ and therefore converges. Let $f : X \to \mathbb{C}$ be the pointwise limit of $(f_n)_{n\in\mathbb{N}}$.

If $j \in \mathbb{N}$, then $f_n \to f$ uniformly on K_j. For suppose $j \in \mathbb{N}$ and $\epsilon > 0$ are given. Choose $N \in \mathbb{N}$ so large that $d_j(f_m, f_n) < \frac{\epsilon}{2}$ whenever $m, n \geq N$. Then if $x \in K_j$ and $n \geq N$,

$$d(f(x), f_n(x)) \leq d(f(x), f_m(x)) + d(f_m(x), f_n(x)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

where m is sufficiently large such that $m \geq N$ and $d(f(x), f_m(x)) < \frac{\epsilon}{2}$. Therefore $f_n \to f$ uniformly on K_j. Hence f is continuous on K_j°. Since $X = \bigcup_{n=1}^{\infty} K_n^\circ$, it follows that $f \in \mathscr{C}$. Note that $f_n \to f$ uniformly on compact subsets of X, because $f_n \to f$ uniformly on K_j for each $j \in \mathbb{N}$ and $X = \bigcup_{n=1}^{\infty} K_n^\circ$. Therefore f_n converges to f in the compact-open topology. Hence $\mathscr{C}$ is complete. □

Suppose that X is a domain in $\mathbb{C}$. If $(f_n)_{n\in\mathbb{N}}$ is a sequence of holomorphic functions $X \to \mathbb{C}$ converging uniformly to a function $f : X \to \mathbb{C}$ on every compact subset of X, then f is holomorphic on X [25, p. 53]. It easily follows that if X and Y are arbitrary complex manifolds and $(f_n)_{n\in\mathbb{N}}$ is a sequence of holomorphic maps $X \to Y$ converging uniformly to a map f on every compact subset of X, then f is holomorphic on X. Hence we have the following result.

Proposition 2.14. *If X and Y are complex manifolds, then $\mathscr{O}(X, Y)$ is closed in $\mathscr{C}(X, Y)$.*

The following is an immediate consequence of Propositions 2.13 and 2.14, since a closed subspace of a Fréchet space is a Fréchet space.

Corollary 2.15. *If X is a complex manifold, then $\mathscr{O}(X, \mathbb{C})$ can be given a Fréchet space structure.*

We conclude this section with a fundamental lemma about the space of continuous maps from a topological space X to $\mathbb{C}^*$. This lemma is important to §3.2.2. In particular, we need it to prove Proposition 3.15.

Lemma 2.16. *If X is a topological space, $n \in \mathbb{N}$, and $\gamma : [0,1] \to X$ is a loop in X, then the set*

$$\{f \in \mathscr{C}(X, \mathbb{C}^*) : \textit{the winding number of } f \circ \gamma \textit{ about } 0 \textit{ is } n\}$$

is open in $\mathscr{C}(X, \mathbb{C}^*)$.

Proof. Suppose that f wraps γ around the puncture in $\mathbb{C}^*$ a number n times. The image of γ is a compact subset of X. Therefore, by Lemma 2.10 there exists a neighbourhood W of f such that if $g \in W$, then $\sup_{t\in[0,1]} d(f \circ \gamma(t), g \circ \gamma(t)) < \delta$, where $\delta = \frac{1}{2} \min_{t\in[0,1]} |f \circ \gamma(t)|$. Then $H : [0,1] \times [0,1] \to \mathbb{C}^*$,

$$H(s,t) = tg \circ \gamma(s) + (1-t) f \circ \gamma(s)$$

is a homotopy from $f \circ \gamma(t)$ to $g \circ \gamma(t)$. Therefore $f \circ \gamma(t)$ and $g \circ \gamma(t)$ have the same winding number about 0. □

2.3 Weierstrass products

Weierstrass products give us a way of constructing a holomorphic function with prescribed zeros. We will use them in §3.2.3. The following definitions and theorems are taken from [6, p. 168–169].

Definition 2.17. *For $p \in \mathbb{N}$ define $E_p : \mathbb{C} \to \mathbb{C}$,*

$$E_p(z) = (1-z) \exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right).$$

Call these functions elementary factors.

Lemma 2.18. *If $|z| \leq 1$ and $p \geq 1$, then $|1 - E_p(z)| \leq |z|^{p+1}$.*

Proof. See [6, Lemma 5.11]. □

Theorem 2.19. *If* $(a_n)_{n\in\mathbb{N}}$ *is a sequence in* $\mathbb{C}$ *such that* $\lim_{n\to\infty} |a_n| = \infty$ *and* $a_n \neq 0$ *for all* $n \geq 1$*, then*

$$f(z) = \prod_{n=1}^{\infty} E_n(z/a_n) \tag{2.1}$$

converges in $\mathscr{O}(\mathbb{C})$*. The function* f *is an entire function with zeros only at the points* a_n*. If* z_0 *occurs in the sequence* $(a_n)_{n\in\mathbb{N}}$ *exactly* m *times, then* f *has a zero at* z_0 *of multiplicity* m*.*

Proof. See [6, Theorem 5.12] □

2.4 Algebraic topology

Following [14] and [21] we define a deformation retraction to be what is sometimes referred to as a strong deformation retraction.

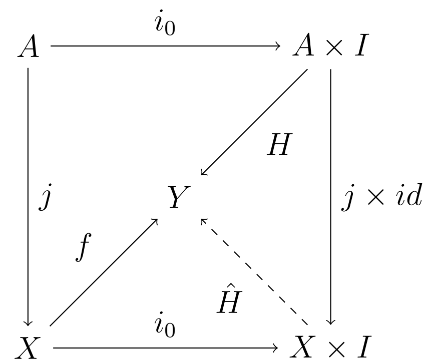
Definition 2.20. *Suppose* X *is a topological space and* A *is a subspace of* X*. A retraction of* X *onto* A *is a continuous map* $r : X \to A$ *such that* $r|A = \mathrm{id}_A$*. If* r *is homotopic to* id_X *relative to* A*, then* r *is called a deformation retraction of* X *onto* A *and we say that the space* A *is a deformation retract of* X*.*

We take the following definition from [1, p. 90]. From now on let I denote the closed unit interval $[0, 1]$.

Definition 2.21. *Suppose* A *and* X *are topological spaces. A continuous map* $j : A \to X$ *is a* cofibration *if it satisfies the homotopy extension property (HEP). This means that for every topological space* Y *and every map* $f : X \to Y$ *and every homotopy* $H : A \times I \to Y$ *satisfying* $H(a, 0) = f(j(a))$ *for* $a \in A$*, there exists a homotopy* $\hat{H} : X \times I \to Y$ *such that* $\hat{H}(j(a), t) = H(a, t)$ *for* $a \in A$ *and* $t \in I$ *and such that* $\hat{H}(x, 0) = f(x)$ *for* $x \in X$*.*

If $j : A \to X$ is a cofibration, then A is a subspace of X and j is the inclusion [1, Proposition 4.16]. Moreover, if X is Hausdorff, then A is closed in X.

This definition can be illustrated with the following diagram, where $i_0(x) = (x, 0)$ [21, p. 41].



2.5 Absolute (neighbourhood) retracts

Suppose X is a topological space and Y is a subspace of X. We say that Y is a neighbourhood retract of X if there exists an open neighbourhood U of Y in X so that Y is a retract of U.

Let $\mathscr{SM}$ denote the class of all separable metrisable topological spaces (or equivalently second countable metrisable topological spaces). Note that if X is in $\mathscr{SM}$, then every subspace of X is also in $\mathscr{SM}$.

Definition 2.22. *Let Y be a space in $\mathscr{SM}$. We say that Y is an absolute retract (AR) if Y is a retract of every space X in $\mathscr{SM}$ containing Y as a closed subspace. We say that Y is an absolute neighbourhood retract (ANR) if Y is a neighbourhood retract of every space X in $\mathscr{SM}$ containing Y as a closed subspace.*

The following characterisation of AR's and ANR's will prove to be very useful.

Theorem 2.23. *Let X be a space in $\mathscr{SM}$. The following statements are equivalent.*

1. *The space X is an A(N)R.*

2. *For every space Y in $\mathscr{SM}$ and for every closed subspace A of Y, every continuous function $f : A \to X$ can be extended to Y (to an open neighbourhood (depending on f) of A in Y).*

Proof. See [4, Theorem 4.2]. □

Theorem 2.24 (Borsuk homotopy extension theorem)**.** *Let A be a closed subspace of a space X in $\mathscr{SM}$, let Z be an ANR and let $H : A \times I \to Z$ be a homotopy such that H_0 is extendable to a function $f : X \to Z$. Then there is a homotopy $F : X \times I \to Z$ such that $F_0 = f$ and $F_t|A = H_t$ for every $t \in I$.*

Proof. See [27, Theorem 1.6.3]. □

Theorem 2.25 (Dugundji extension theorem)**.** *Let X be a metric space, A a closed subset of X, L a locally convex topological vector space and $f : A \to L$ a continuous map. Then there exists an extension $F : X \to L$ of f with $F(X) \subset \operatorname{conv} f(A)$.*

Proof. See [7, Theorem 4.1]. □

By Theorems 2.23 and 2.25, it follows that every locally convex topological vector space in $\mathscr{SM}$ is an AR.

A space X is called locally contractible at $x \in X$ if for every open neighbourhood U of x in X there is an open neighbourhood $V \subset U$ of x and a homotopy $H : V \times I \to U$ such that H_1 is the inclusion $V \hookrightarrow U$ and H_0 is constant. A space X is called locally contractible if X is locally contractible at every point of X.

Proposition 2.26. *If X is an ANR, then X is locally contractible.*

Proof. See [4, p. 87]. □

Theorem 2.27. *Let X be a space in $\mathscr{SM}$. Then X is an AR if and only if X is a contractible ANR.*

Proof. See [27, Theorem 5.2.15]. □

Theorem 2.28. *Let X be an ANR and let U be an open subspace of X. Then U is an ANR.*

Proof. See [27, Theorem 5.4.1]. □

Theorem 2.29. *Let X be a space and suppose that X admits an open cover consisting of ANR's. Then X is an ANR.*

Proof. See [27, Theorem 5.4.5]. □

Theorem 2.30. *Suppose that X is an ANR and $A \subset X$ is closed. Then A is an ANR if and only if the inclusion $A \hookrightarrow X$ is a cofibration.*

Proof. See [1, Theorem 4.2.15]. □

A *deformation retraction in the weak sense* of a topological space X onto a subspace $A \subset X$ is a continuous map $r : X \to A$ so that $r|_A = \mathrm{id}_A$ and r is homotopic to id_X. If such a map exists, then we say that A is a deformation retract of X in the weak sense.

Theorem 2.31. *If X is an ANR and A is a subspace of X, then the following two statements are equivalent:*

1. *A is a deformation retract of X.*

2. *A is a deformation retract of X in the weak sense.*

Proof. See [16, p. 199]. □

2.6 Oka manifolds and Oka theory

In this section we give a very brief introduction to Oka manifolds. For a comprehensive discussion of Oka theory see the survey article [11].

There are generalisations of the Weierstrass theorem and the Runge approximation theorem to Stein manifolds called the *Oka-Weil approximation theorem* and the *Cartan extension theorem* respectively. These theorems can be thought of as expressing some properties of the target $\mathbb{C}$. Reformulating these theorems for a general target X leads us to the *interpolation property* (IP) and *approximation property* (AP).

Definition 2.32. *A complex manifold Y has the interpolation property if for every Stein manifold X with a subvariety S, a holomorphic map $S \to Y$ has a holomorphic extension $X \to Y$ if it has a continuous extension.*

Suppose X and Y are complex manifolds and $K \subset X$ is compact. A continuous map $f : X \to Y$ is said to be holomorphic on K, if there exists an open neighbourhood of K on which f is holomorphic.

Definition 2.33. *A complex manifold Y has the approximation property if for every Stein manifold X with a holomorphically convex compact subset K, a continuous map $X \to Y$ that is holomorphic on K can be uniformly approximated on K by holomorphic maps $X \to Y$.*

One can then define ostensibly weaker conditions called the convex approximation property (CAP) and the convex interpolation property (CIP).

Definition 2.34. *A complex manifold Y has CIP if whenever T is a closed submanifold of $\mathbb{C}^n$ which is biholomorphic to a convex domain in some $\mathbb{C}^k$ and $f : T \to Y$ is a holomorphic map, f extends to a holomorphic map $\mathbb{C}^n \to Y$.*

Definition 2.35. *A complex manifold Y has CAP if every holomorphic map $f : K \to Y$ from a compact convex set $K \subset \mathbb{C}^n$ can be approximated uniformly on K by holomorphic maps $\mathbb{C}^n \to Y$.*

Definition 2.36. *A complex manifold Y is Oka if Y satisfies CAP.*

It was proved by Forstnerič that if X is a Stein manifold and Y is an Oka manifold, then every continuous map $X \to Y$ can be deformed to a holomorphic map. In fact, Forstnerič proved that CAP implies something much stronger, namely the *parametric Oka property with respect to approximation and interpolation* (POPAI). The property POPAI implies IP and AP mentioned above. Also POPAI implies the *basic Oka property with approximation* BOPA and *parametric Oka property* POP (defined below).

Definition 2.37. *A complex manifold Y satisfies the parametric Oka property if whenever X is a Stein manifold and $Q \subset P$ are compact subsets of $\mathbb{R}^n$ the following is true. For every continuous map $f : X \times P \to Y$ such that $f(\cdot, q) : X \to Y$ is holomorphic for every $q \in Q$, there is a continuous map $F : X \times P \times I \to Y$ such that:*

1. $F(\cdot, \cdot, 0) = f$,
2. $F(\cdot, q, t) = f(\cdot, q)$ *for every* $q \in Q$ *and* $t \in I$,
3. $F(\cdot, p, 1)$ *is holomorphic for every* $p \in P$.

Definition 2.38. *Suppose X is a complex manifold. A compact subset $K \subset X$ is* $\mathscr{O}(X)$-convex *if $K = \hat{K}$, where $\hat{K}$ is the* $\mathscr{O}(X)$-hull *of K.*

Definition 2.39. *A complex manifold Y satisfies the basic Oka property with approximation if every continuous map $f : X \to Y$ from a Stein manifold X that is holomorphic on a compact $\mathscr{O}(X)$-convex subset $K \subset X$ can be deformed to a holomorphic map $g : X \to Y$ by a homotopy of maps that are holomorphic near K and arbitrary close to f on K.*

Theorem 2.40. *If X is a Stein mainfold and Y is an Oka manifold, then the inclusion $\mathscr{O}(X, Y) \hookrightarrow \mathscr{C}(X, Y)$ is a weak homotopy equivalence.*

Proof. This follows by appying POP with the pairs $\emptyset \hookrightarrow *$, $\{0, 1\} \hookrightarrow [0, 1]$, $* \hookrightarrow S^k$ and $S^k \hookrightarrow B^{k+1}$, where B^{k+1} is the $(k + 1)$-dimensional ball and $S^k = \partial B^{k+1}$ is the k-dimensional sphere for every $k \geq 0$ [11, p. 19]. □

The following gives us a rich source of examples of Oka manifolds.

Proposition 2.41. *Every complex Lie group is Oka.*

Proof. See [10, Proposition 5.5.1]. □

In particular the complex plane $\mathbb{C}$, the punctured plane $\mathbb{C}^*$ and all tori $\mathbb{C}/\Gamma$ are Oka manifolds. The only other example of a Riemann surface that is Oka is the Riemann sphere $\mathbb{P}_1$. For suppose X is an Oka manifold and X is not one of the aforementioned examples. Then X is hyperbolic by Theorem 2.5. Therefore the universal covering $\tilde{X}$ of X is isomorphic to the unit disc $\mathbb{D}$. So any holomorphic map $\mathbb{C} \to X$ can be lifted to a holomorphic map $\mathbb{C} \to \mathbb{D}$. But Liouville's theorem tells us that every holomorphic map $\mathbb{C} \to \mathbb{D}$ is constant. Therefore every holomorphic map $\mathbb{C} \to X$ is constant. So X does not have IP and hence X is not Oka.

Chapter 3

Research

If X is a Stein manifold and Y is an Oka manifold, then every continuous map $X \to Y$ can be deformed to a holomorphic map $X \to Y$. The question is whether this can be done for all continuous maps at once in a way that depends continuously on f and leaves f fixed if f is holomorphic to begin with. In other words is $\mathscr{O}(X,Y)$ a deformation retract of $\mathscr{C}(X,Y)$? Following [21] we use the term *deformation retract* for what is sometimes referred to as a *strong deformation retract*.

This question is resolved in the affirmative when X is Stein, Y is Oka and $\mathscr{O}(X,Y)$ and $\mathscr{C}(X,Y)$ are absolute neighbourhood retracts (see [19, Proposition 5]). Suppose that X is Stein and Y is Oka. Then $\mathscr{C}(X,Y)$ and $\mathscr{O}(X,Y)$ are absolute neighbourhood retracts if:

- X has a strictly plurisubharmonic Morse exhaustion with finitely many critical points (see [19, Theorem 9] and [19, Proposition 7]),
- X is simply connected and Y is the quotient of $\mathbb{C}^n$ by a discrete subgroup (see [19, Proposition 11]), or
- $Y = \mathbb{C}^n$ (see [19, p. 1164]).

A deformation retraction is not constructed explicitly in the proof of [19, Proposition 5]. The goal of Section §3.1 is to present a more elementary proof that $\mathscr{O}(X,\mathbb{C})$

is a deformation retract of $\mathscr{C}(X,\mathbb{C})$ and to define an explicit deformation retraction.

In Section §3.2 we turn our attention to the next simplest target $\mathbb{C}^*$. We will see that the spaces $\mathscr{O}(\mathbb{C}\backslash\mathbb{N},\mathbb{C}^*)$ and $\mathscr{C}(\mathbb{C}\backslash\mathbb{N},\mathbb{C}^*)$ are not absolute neighbourhood retracts and so [19, Proposition 5] does not apply. Yet it can still be proved that $\mathscr{O}(\mathbb{C}\backslash\mathbb{N},\mathbb{C}^*)$ is a deformation retract of $\mathscr{C}(\mathbb{C}\backslash\mathbb{N},\mathbb{C}^*)$ by a hands-on calculation [19, p. 1165–1166]. We extend arguments of [19, p. 1165–1166] from $\mathbb{C}\backslash\mathbb{N}$ to a bigger collection of domains in $\mathbb{C}$, to most of which the above results do not apply.

3.1 Defining a retraction explicitly

The construction of the explicit retraction presented here is motivated by [27, Exercise 1.4.4], which is essentially an explicit direct proof of the Dugundji extension theorem. I believe that such a proof first appeared in [7, p. 358] where a similar formula is used.

The following is a simple observation about retracts.

Lemma 3.1. *Let X be a first countable topological space and $Y \subset X$ be a retract of X. Then Y is closed in X.*

Proof. Suppose that $r : X \to Y$ is a retraction of X onto Y. If $(x_j)_{j\in\mathbb{N}}$ is a sequence of points in Y converging to $x \in X$, then by continuity of r, $x_j = r(x_j) \to r(x) \in Y$ as $j \to \infty$. Therefore by uniqueness of limits $x = r(x) \in Y$. Hence Y is closed. □

The following lemma greatly simplifies the problem of finding a deformation retraction. It tells us that in the special setting where X and Y are absolute retracts, finding a deformation retraction is no harder than finding a retraction.

Lemma 3.2. *If X is an AR and Y is a subspace of X, then a continuous map $r : X \to Y$ is a deformation retraction if and only if r is a retraction.*

Proof. Suppose that r is a retraction. Then Y is closed in X by Lemma 3.1. A retract of an AR is an AR (this follows easily from Theorem 2.23). So Y is an AR.

Let $A = X \times I$ and

$$B = (X \times \{0\}) \cup (Y \times I) \cup (X \times \{1\}),$$

where $I = [0,1]$. Then B is a closed subspace of A. The spaces $X \times \{0\}$, $Y \times I$, $X \times \{1\}$, $Y \times \{0\}$ and $Y \times \{1\}$ are AR's by [27, Proposition 1.5.4]. Note that

$$Y \times \{0\} = (X \times \{0\}) \cap (Y \times I),$$
$$Y \times \{1\} = ((X \times \{0\}) \cup (Y \times I)) \cap (X \times \{1\}).$$

Since $X \times \{0\}$, $Y \times I$ and $(X \times \{0\}) \cap (Y \times I)$ are AR's, $(X \times \{0\}) \cup (Y \times I)$ is an AR by [27, Theorem 1.5.9]. Then since $(X \times \{0\}) \cup (Y \times I)$, $X \times \{1\}$ and $((X \times \{0\}) \cup (Y \times I)) \cap (X \times \{1\})$ are AR's, B is an AR using [27, Theorem 1.5.9] again. Since A contains B as a closed subspace and B is an AR, there exists a retraction $\rho : A \to B$ of A onto B.

Define $G : B \to X$,

$$G(x,t) = \begin{cases} r(x) & \text{if } (x,t) \in X \times \{0\}, \\ x & \text{if } (x,t) \in (Y \times I) \cup (X \times \{1\}). \end{cases}$$

The map G is continuous. For $X \times \{0\}$, $(Y \times I) \cup (X \times \{1\})$ are closed subspaces of B and $r(x) = x$ for all $(x,t) \in (X \times \{0\}) \cap ((Y \times I) \cup (X \times \{1\}))$.

One can easily check that $G \circ \rho : X \times I \to X$ is the desired homotopy from r to the identity on X relative to Y. □

Lemma 3.3. *Let (X,d) be a metric space, $B \subset X$ a closed subspace and $A \subset B$ a dense subset of B. For each $a \in A$ let*

$$U_a = \{x \in X : d(x,a) < 2d(x,B)\},$$
$$V_a = \{x \in X : d(x,a) < \frac{3}{2}d(x,B)\}.$$

Then $\mathscr{V} = \{V_a : a \in A\}$ and $\mathscr{U} = \{U_a : a \in A\}$ are open covers of $X \backslash B$.

Proof. For each $a \in A$ the set U_a is open because

$$U_a = \{x \in X : 0 < 2d(x, B) - d(x, a)\}$$

and $f_a : X \to \mathbb{R}$, $f_a(x) = 2d(x, B) - d(x, a)$, is continuous. Similarly, for each $a \in A$ the set V_a is open.

Suppose that there exists $x \in X \backslash B$ such that $x \notin V_a$ for every $a \in A$. Then $d(x, a) \geq \frac{3}{2} d(x, B)$ for every $a \in A$. Since A is dense in B it follows that $d(x, b) \geq \frac{3}{2} d(x, B)$ for every $b \in B$. Therefore we have

$$d(x, B) = \inf_{b \in B} d(x, b) \geq \frac{3}{2} d(x, B),$$

which is a contradiction since $x \in X \backslash B$ and therefore $d(x, B) > 0$. So $\mathscr{V}$ covers $X \backslash B$. For each $a \in A$ we have $V_a \subset U_a$. Hence $\mathscr{U}$ is also a cover for $X \backslash B$. $\square$

We say that a cover $\mathscr{U}$ of a topological space X is locally finite if for every point $x \in X$ there exists a neighbourhood of x meeting only finitely many elements of $\mathscr{U}$.

Lemma 3.4. *Suppose (X, d) is a metric space, $B \subset X$ is a closed subspace, and $(a_n)_{n \in \mathbb{N}}$ is a dense sequence in B. If for each $n \in \mathbb{N}$ we define*

$$W_n = U_{a_n} \backslash \left(\bigcup_{m < n} \bar{V}_{a_m} \right)$$

with V_{a_n} and U_{a_n} as above, then $\mathscr{W} = \{W_n : n \in \mathbb{N}\}$ is a locally finite open refinement of $\mathscr{U} = \{U_{a_j} : j \in \mathbb{N}\}$ with $W_n \subset U_{a_n}$ for every $n \in \mathbb{N}$.

Proof. It is obvious that W_n is open for every $n \in \mathbb{N}$. It is also clear that $W_n \subset U_{a_n}$ for every $n \in \mathbb{N}$. For each $x \in X \backslash B$, let $n(x) = \min\{n \in \mathbb{N} : x \in U_{a_n}\}$. Observe that $\bar{V}_{a_m} \subset U_{a_m} \cup \{a_m\}$ for all $m \in \mathbb{N}$. So if $m < n(x)$, then $x \notin V_{a_m}$ since $x \notin U_{a_m}$ by definition of $n(x)$ and $x \neq a_m$ because $x \notin B$. Therefore $x \in W_{n(x)}$. Hence $\mathscr{W}$ is an open refinement of $\mathscr{U}$.

It remains to be seen that $\mathscr{W}$ is locally finite. If $x \in X \backslash B$, let $m(x) = \min\{n \in \mathbb{N} : x \in V_{a_n}\}$. Then V_{a_m} is an open neighbourhood of x that intersects only finitely many elements of $\mathscr{W}$. Therefore $\mathscr{W}$ is locally finite. $\square$

A family $(f_j)_{j\in J}$ of continuous functions $X \to [0,1]$ is called a partition of unity on X if each $x \in X$ has an open neighbourhood U_x so that $f_j|_{U_x} = 0$ for all but finitely many $j \in J$ and $\sum_{j\in J} f_j = 1$.

Let (X, d) be a metric space and $\mathscr{U}$ a locally finite open cover for X. For each $U \in \mathscr{U}$ define $\kappa_U : X \to \mathbb{R}$,

$$\kappa_U(x) = \frac{d(x, X\backslash U)}{\sum_{V\in\mathscr{U}} d(x, X\backslash V)}.$$

Note that the sum $\sum_{V\in\mathscr{U}} d(x, X\backslash V)$ is finite and non-zero since $\mathscr{U}$ is a locally finite open cover of X.

These functions appear in [7, p. 355], in which the following fundamental fact about them is proved (without actually using the term partition of unity). Following [27, p. 27] we call these functions *κ-functions with respect to the cover $\mathscr{U}$.*

Lemma 3.5. *Let (X, d) be a metric space and $\mathscr{U}$ a locally finite open cover for X. Then the family $(\kappa_U)_{U\in\mathscr{U}}$ of κ-functions with respect to the cover $\mathscr{U}$ is a partition of unity on X.*

Proof. Each κ_U is continuous. For suppose $U \in \mathscr{U}$. It is easy to check that κ_U is continuous on any open set W intersecting only finitely many elements of $\mathscr{U}$. Because $\mathscr{U}$ is a locally finite open cover we can cover X by such sets. Therefore κ_U is continuous.

Suppose that $x \in X$. Let W be a neighbourhood of x intersecting only finitely many elements of $\mathscr{U}$. Suppose $V \in \mathscr{U}$ and $V \cap W = \emptyset$. Then $d(x, X\backslash V) = 0$ for all $x \in W$ and hence $\kappa_V = 0$ on W. So $\kappa_V|_W = 0$ for all $V \in \mathscr{U}$ not intersecting W. Hence $\kappa_V|_W = 0$ for all but finitely many $V \in \mathscr{U}$. It is clear from the formula defining κ_U that $\sum_{U\in\mathscr{U}} \kappa_U = 1$. □

Proposition 3.6. *Suppose that X is a locally convex topological vector space, d is an admissible metric on X, $B \subset X$ is a closed vector subspace, and $(a_n)_{n\in\mathbb{N}}$ is a dense sequence in B. Define $\mathscr{W} = \{W_n : n \in \mathbb{N}\}$ as in Lemma 3.4, and let $(\kappa_W)_{W\in\mathscr{W}}$*

be the family of κ-functions (defined on $X\backslash B$) with respect to the cover (of $X\backslash B$) $\mathscr{W}$. The map $r : X \to B$ defined by

$$r(x) = \begin{cases} x & \text{if } x \in B, \\ \sum_{n=1}^{\infty} \kappa_{W_n}(x) \cdot a_n & \text{if } x \in X\backslash B, \end{cases}$$

is a retraction of X onto B.

Proof. Since r is the identity on B, all that remains for us to prove is that r is continuous. Note that $(\kappa_W)_{W \in \mathscr{W}}$ is a partition of unity on $X\backslash B$ by Lemma 3.5. So clearly r is continuous on $X\backslash B$.

Suppose now that $x \in B$ and that $\epsilon > 0$ is given. By local convexity of X there exists a convex neighbourhood C of $r(x) = x$ contained in $B(x, \epsilon)$. Let $r > 0$ be so small that $B(x, r) \subset C$ and $r < \epsilon$. Let $\delta = r/4$. We will prove that if $y \in X$ and $d(x, y) < \delta$, then $d(r(x), r(y)) < \epsilon$. To this end, let $y \in X$ be such that $d(x, y) < \delta$.

If $y \in B$, then $d(r(x), r(y)) = d(x, y) < \delta < \epsilon$, as required.

Suppose that $y \in X\backslash B$. Note that $d(y, B) < \delta$ since $x \in B$. Let $n \in \mathbb{N}$ be such that $y \in W_n$. Since $W_n \subset U_{a_n}$, $d(y, a_n) < 2d(y, B) < 2\delta$. Hence $d(x, a_n) \leq d(x, y) + d(y, a_n) < \delta + 2\delta < r$. Therefore $a_n \in C$. It follows that

$$r(y) = \sum_{j=1}^{\infty} \kappa_{W_j}(y) \cdot a_j = \sum_{j \in J} \kappa_{W_j}(y) \cdot a_j$$

is a convex combination of elements of C, where $J = \{m \in \mathbb{N} : y \in W_m\}$ (note that J is finite). So $r(y) \in C \subset B(x, \epsilon)$. Hence, $d(r(x), r(y)) = d(x, r(y)) < \epsilon$, as required. □

Proposition 3.7. *If X is a complex manifold, then $\mathscr{O}(X, \mathbb{C})$ is a deformation retract of $\mathscr{C}(X, \mathbb{C})$.*

Proof. Let $\mathscr{O} = \mathscr{O}(X, \mathbb{C})$ and $\mathscr{C} = \mathscr{C}(X, \mathbb{C})$. Note that $\mathscr{O}$ is a closed subspace of $\mathscr{C}$ by Proposition 2.14. The space $\mathscr{C}$ is a locally convex topological space by Proposition 2.13. It follows from Proposition 2.9 that $\mathscr{O}$ is separable. Let A be a countable dense subset of $\mathscr{O}$. Let $a : \mathbb{N} \to A$ be a surjection. Let d be an admissible metric on $\mathscr{C}$, which we know to exist since $\mathscr{C}$ is metrisable (see Proposition 2.12). For each $n \in \mathbb{N}$ let

$$\begin{aligned} U_n &= \{f \in \mathscr{C} : d(f, a_n) < 2d(f, \mathscr{O})\}, \\ V_n &= \{f \in \mathscr{C} : d(f, a_n) < \frac{3}{2} d(f, \mathscr{O})\}, \\ W_n &= U_n \backslash \left(\bigcup_{m<n} \bar{V}_m \right). \end{aligned}$$

By Lemma 3.4, $\mathscr{W} = \{W_n : n \in \mathbb{N}\}$ is a locally finite open cover of $\mathscr{C} \backslash \mathscr{O}$. Let $(\kappa_W)_{W \in \mathscr{W}}$ be the family of κ-functions with respect to $\mathscr{W}$. Then by Proposition 3.6, it follows that the map $r : \mathscr{C} \to \mathscr{O}$ defined by

$$r(f) = \begin{cases} f & \text{if } f \in \mathscr{O}, \\ \sum_{n=1}^{\infty} \kappa_{W_n}(f) \cdot a_n & \text{if } f \in \mathscr{C} \backslash \mathscr{O}, \end{cases}$$

is a retraction.

Note that $\mathscr{C}$ is metrisable and second countable by Propositions 2.12 and 2.9 respectively. By Proposition 2.13, $\mathscr{C}$ is a locally convex topological vector space. A locally convex topological vector space in $\mathscr{SM}$ is an AR (recall the comment immediately after Theorem 2.25). So $\mathscr{C}$ is an AR. Hence r is a deformation retraction by Lemma 3.2. □

We are able to say more about the nature of this retraction.

Corollary 3.8. *There exists a deformation retraction $\mathscr{C}(\mathbb{C}, \mathbb{C}) \to \mathscr{O}(\mathbb{C}, \mathbb{C})$ so that each non-holomorphic continuous function $\mathbb{C} \to \mathbb{C}$ is mapped to a polynomial.*

Proof. Let $\mathscr{O} = \mathscr{O}(\mathbb{C}, \mathbb{C})$, $\mathscr{C} = \mathscr{C}(\mathbb{C}, \mathbb{C})$ and d be some admissible metric on $\mathbb{C}$. Let $A = \mathbb{Q}(i)[z] \subset \mathscr{O}$ be the subspace of polynomial functions with coefficients in $\mathbb{Q}(i) = \{a + bi : a, b \in \mathbb{Q}\}$ and suppose $(a_n)_{n \in \mathbb{N}}$ is a listing of A, that is, an injective sequence with image A. By considering Taylor series it is easy to show that A is dense in $\mathscr{O}$. Then define a retraction in the same way as in Proposition 3.7. Note that if $f \in \mathscr{C}\backslash\mathscr{O}$, then $\kappa_{W_j}(f) \neq 0$ for only finitely many $j \in \mathbb{N}$, from which it follows that $r(f) \in \mathbb{C}[z]$. □

We conclude this section with a comment about the related topic of Bergman projections. Suppose that X is a domain in $\mathbb{C}^n$. We can consider the Hilbert space $L^2(X)$ of square integrable functions $X \to \mathbb{C}$. This space contains the space $L^2_{\mathscr{O}}(X)$ of all square integrable holomorphic functions as a closed subspace. The orthogonal projection $L^2(X) \to L^2_{\mathscr{O}}(X)$ is called the *Bergman projection*. It is, in particular, a retraction of $L^2(X)$ onto $L^2_{\mathscr{O}}(X)$. However, Bergman projections are not relevant here because our focus is on the space $\mathscr{C}(X, \mathbb{C})$, which is quite different from $L^2(X)$.

3.2 The spaces $\mathscr{O}(X, \mathbb{C}^*)$ and $\mathscr{C}(X, \mathbb{C}^*)$

We now consider the case where the target is $\mathbb{C}^*$ rather than $\mathbb{C}$. Call a complex manifold X *good* if $\mathscr{O}(X, \mathbb{C}^*)$ is a deformation retract of $\mathscr{C}(X, \mathbb{C}^*)$.

If X was a Stein manifold and $\mathscr{C}(X, \mathbb{C}^*)$ and $\mathscr{O}(X, \mathbb{C}^*)$ were ANR's, then X would be good by [19, Proposition 5]. However, as we shall see, the spaces $\mathscr{C}(X, \mathbb{C}^*)$ and $\mathscr{O}(X, \mathbb{C}^*)$ need not even have the homotopy type of an ANR.

A topological space X is called semi-locally contractible if it has a basis $\mathscr{U}$ of open subsets such that $U \hookrightarrow X$ is null-homotopic for all $U \in \mathscr{U}$. It follows that the path components of X are open. Suppose X and Y are topological spaces. It is easy to see that if X is semi-locally contractible and Y is homotopy equivalent to X, then Y is semi-locally contractible. Recall that an ANR is locally contractible, so in particular an ANR is semi-locally contractible. Hence it follows from the next proposition that $\mathscr{O}(\mathbb{C}\backslash\mathbb{N}, \mathbb{C}^*)$ and $\mathscr{C}(\mathbb{C}\backslash\mathbb{N}, \mathbb{C}^*)$ do not have the homotopy type of an ANR, which is the same (see [22, Theorem 1]) as to say that these spaces do not have the homotopy type of a CW complex.

Proposition 3.9. *The path components of $\mathscr{O}(\mathbb{C}\backslash\mathbb{N}, \mathbb{C}^*)$ and $\mathscr{C}(\mathbb{C}\backslash\mathbb{N}, \mathbb{C}^*)$ have no interior points. So $\mathscr{O}(\mathbb{C}\backslash\mathbb{N}, \mathbb{C}^*)$ and $\mathscr{C}(\mathbb{C}\backslash\mathbb{N}, \mathbb{C}^*)$ do not have the homotopy type of* *an ANR or a CW complex.*

Proof. Let $U \subset \mathscr{C}(\mathbb{C}\backslash\mathbb{N}, \mathbb{C}^*)$ be an arbitrary open neighbourhood of the constant map $1 \in \mathscr{C}(\mathbb{C}\backslash\mathbb{N}, \mathbb{C}^*)$. Because U is open in the compact-open topology, U is the union of finite intersections of elements of the form $[K, V]$, where $K \subset \mathbb{C}\backslash\mathbb{N}$ is compact and $V \subset \mathbb{C}^*$ is open. So there exist compact sets $K_1, \ldots, K_n \subset \mathbb{C}\backslash\mathbb{N}$ and open sets $V_1, \ldots, V_n \subset \mathbb{C}^*$ so that $1 \in \bigcap_{j=1}^{n} [K_j, V_j] \subset U$. Let $K = \bigcup_{j=1}^{n} K_j$ and $V = \bigcap_{j=1}^{n} V_j$. Note that $1 \in V_j$ for $j = 1, \ldots, n$, so $1 \in V$ and hence the constant map 1 lies in $[K, V]$. Clearly $[K, V] \subset \bigcap_{j=1}^{n} [K_j, V_j] \subset U$. We may assume that K is $\mathscr{O}(\mathbb{C}\backslash\mathbb{N})$-convex. For if this were not the case, then we could replace K by its $\mathscr{O}(\mathbb{C}\backslash\mathbb{N})$-hull $\hat{K}$, which is compact since $\mathbb{C}\backslash\mathbb{N}$ is Stein.

Choose $a \in \mathbb{N}$ such that $a > 3 + \sup_{z \in K} |z|$. Let $\psi : \mathbb{C}\backslash\mathbb{N} \to \mathbb{C}$ be a continuous function such that $\psi(z) = 1$ if $|z - a| \leq 1$ and $\psi(z) = 0$ if $|z - a| \geq 2$. Define $f : \mathbb{C}\backslash\mathbb{N} \to \mathbb{C}^*$,

$$f(x) = \begin{cases} 1 + \dfrac{1}{z-a} & \text{if } z \in \mathbb{C}\backslash\mathbb{N} \text{ and } |z-a| \leq 1, \\ \exp\left(\psi(z) \log\left(1 + \dfrac{1}{z-a}\right)\right) & \text{if } z \in \mathbb{C}\backslash\mathbb{N} \text{ and } 1 < |z-a| < 2, \\ 1 & \text{if } z \in \mathbb{C}\backslash\mathbb{N} \text{ and } 2 \leq |z-a|, \end{cases}$$

where log is a holomorphic branch of the logarithm defined on $B(1,1) \subset \mathbb{C}$. Note that the only zero of the function $\mathbb{C} \to \mathbb{C}$, $1 - \dfrac{1}{z-a}$, is $a - 1 \in \mathbb{N}$, so f indeed maps into $\mathbb{C}^*$. The map f is in $[K, V]$ since $f|_K = 1$. However f is not homotopic to 1 since f has a pole of order 1 at a (and therefore f wraps a little circle about a around the puncture in $\mathbb{C}^*$ once).

Since f is holomorphic on $(\mathbb{C}\backslash\mathbb{N})\backslash\bar{B}(a, 2)$, which is an open neighbourhood of K, BOPA (recall Definition 2.39) implies that there is a holomorphic map $g \in [K, V] \subset U$ which is homotopic to f. So $g \in U$ and g is not homotopic to 1.

This shows that the path components of $\mathscr{O}(\mathbb{C}\backslash\mathbb{N}, \mathbb{C}^*)$ and $\mathscr{C}(\mathbb{C}\backslash\mathbb{N}, \mathbb{C}^*)$ have no interior points. □

3.2.1 Lárusson's sufficient condition

In this section we present a sufficient condition for a Stein manifold X to be good (meaning that $\mathscr{O}(X, \mathbb{C}^*)$ is a deformation retract of $\mathscr{C}(X, \mathbb{C}^*)$) and prove its sufficiency. The condition we present here appears in [19, p. 1165] for $X = \mathbb{C}\backslash\mathbb{N}$.

Suppose X is a complex manifold. Let $\mathscr{C}_0(X, \mathbb{C}^*)$ be the path component of $\mathscr{C}(X, \mathbb{C}^*)$ consisting of the null-homotopic maps. Define $\mathscr{O}_0(X, \mathbb{C}^*)$ similarly. ANR theory is still relevant (even though $\mathscr{O}(X, \mathbb{C}^*)$ and $\mathscr{C}(X, \mathbb{C}^*)$ need not be ANR's) because of the following lemma (see [19, p. 1165]).

Lemma 3.10. *Let X be a complex manifold, then $\mathscr{C}_0(X, \mathbb{C}^*)$ and $\mathscr{O}_0(X, \mathbb{C}^*)$ are ANR's.*

Proof. In what follows we will show that $\mathscr{C}_0(X, \mathbb{C}^*)$ is an ANR (the proof that $\mathscr{O}_0(X, \mathbb{C}^*)$ is an ANR is analogous).

Recall that $\mathscr{C}(X, \mathbb{C})$ is a Fréchet space (see Proposition 2.13). Therefore $\mathscr{C}(X, \mathbb{C})$ is a locally convex topological vector space. Hence $\mathscr{C}(X, \mathbb{C})$ is an ANR by Theorems 2.23 and 2.25. By Theorems 2.28 and 2.29, to show that $\mathscr{C}_0(X, \mathbb{C}^*)$ is an ANR it suffices to find a surjective local homeomorphism $\mathscr{C}(X, \mathbb{C}) \to \mathscr{C}_0(X, \mathbb{C}^*)$. We claim that $\exp : \mathscr{C}(X, \mathbb{C}) \to \mathscr{C}_0(X, \mathbb{C}^*)$ is such a map. A continuous null-homotopic map $X \to \mathbb{C}^*$ can be lifted via $\exp : \mathbb{C} \to \mathbb{C}^*$, so $\exp : \mathscr{C}(X, \mathbb{C}) \to \mathscr{C}_0(X, \mathbb{C}^*)$ is surjective. The map exp is continuous by Lemma 2.6. We now show that exp is open.

Let $(K_j)_{j\in\mathbb{N}}$ be a sequence of connected compact sets in X such that

$$K_1^\circ \subset K_1 \subset K_2^\circ \subset K_2 \subset \cdots$$

and $X = \bigcup_{j\in\mathbb{N}} K_j$. Define $V_j = \left[-\frac{1}{j}, \frac{1}{j}\right] \times i \left[-\frac{1}{j}, \frac{1}{j}\right]$ for $j \in \mathbb{N}$. The collection $\mathscr{U} = \{[K_j, V_j] : j \in \mathbb{N}\}$ is an open neighbourhood basis for $0 \in \mathscr{C}(X, \mathbb{C})$. Fix $j \in \mathbb{N}$ and $x_0 \in K_j$. Let $\log : \exp(V_j) \to V_j \subset \mathbb{C}$ be (the restriction of) the principal branch of the logarithm. Clearly

$$\exp([K_j, V_j]) \subset [K_j, \exp(V_j)] \cap \mathscr{C}_0(X, \mathbb{C}^*).$$

If $f \in [K_j, \exp(V_j)] \cap \mathscr{C}_0(X, \mathbb{C}^*)$, then f is null-homotopic, so f can be lifted to a continuous function $\tilde{f} : X \to \mathbb{C}$ with

$$\tilde{f}(x_0) = \log(f(x_0)) \tag{3.1}$$

via $\exp : \mathbb{C} \to \mathbb{C}^*$. But then $\log \circ (f|_{K_j})$ and $\tilde{f}|_{K_j}$ are both liftings of $f|K_j \to \mathbb{C}^*$, so by Equation 3.1 and the uniqueness of lifting, $\tilde{f}|_{K_j} = \log \circ (f|_{K_j})$. So $\tilde{f}(K_j) \subset V_j$, that is, $\tilde{f} \in [K_j, V_j]$. Hence $f \in \exp([K_j, V_j])$ and therefore

$$\exp([K_j, V_j]) = [K_j, \exp(V_j)] \cap \mathscr{C}_0(X, \mathbb{C}^*).$$

Hence $\exp([K_j, V_j])$ is open for all $j \in \mathbb{N}$. So $\mathscr{U}$ is an open neighbourhood basis of $0 \in \mathscr{C}(X, \mathbb{C})$ such that $\exp(U)$ is open for all $U \in \mathscr{U}$. Since $\mathscr{C}(X, \mathbb{C})$ is a topological group, every $f \in \mathscr{C}(X, \mathbb{C})$ has an open neighbourhood basis $\mathscr{U}_f$ such that $\exp(U)$ is open for all $U \in \mathscr{U}_f$. Therefore $\exp : \mathscr{C}(X, \mathbb{C}) \to \mathscr{C}_0(X, \mathbb{C}^*)$ is open.

Finally, we show that $\exp : \mathscr{C}(X, \mathbb{C}) \to \mathscr{C}_0(X, \mathbb{C}^*)$ is a local homeomorphism. Suppose $f \in \mathscr{C}(X, \mathbb{C})$. Let $x_1 \in X$ and $U = [\{x_1\}, B(f(x_1), 1)]$. Clearly U is an open neighbourhood of f, so $\exp(U)$ is open in $\mathscr{C}_0(X, \mathbb{C})$. Suppose that $g, h \in U$ are such that $\exp(g) = \exp(h)$. In particular, $\exp(g(x_1)) = \exp(h(x_1))$. So $g(x_1) = h(x_1) + 2n\pi i$ for some $n \in \mathbb{N}$. But $g(x_1), h(x_1) \in B(f(x_1), 1)$, therefore $|g(x_1) - h(x_1)| < 2$, so $n = 0$. Then g and h are both liftings of $\exp \circ g = \exp \circ h$ with $h(x_1) = g(x_1)$. Therefore by uniqueness of liftings, $g = h$. So exp is injective on U and hence

$$\exp|_U : U \to \exp(U) \tag{3.2}$$

is a homeomorphism. Since f was arbitrary, exp is a local homeomorphism. □

Proposition 3.11. *If X is a Stein manifold, then $\mathscr{O}_0(X, \mathbb{C}^*)$ is a deformation retract of $\mathscr{C}_0(X, \mathbb{C}^*)$.*

Proof. Let $\mathscr{O}_0 = \mathscr{O}_0(X, \mathbb{C}^*)$ and $\mathscr{C}_0 = \mathscr{C}_0(X, \mathbb{C}^*)$. The space $\mathscr{O}_0$ is a closed subset of $\mathscr{C}_0$. For if a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathscr{O}_0(X, \mathbb{C}^*)$ converges to $f \in \mathscr{C}_0(X, \mathbb{C}^*)$, then f

is holomorphic (since convergence in the compact-open topology is uniform convergence on compact sets). Therefore by Lemma 3.10 and Theorem 2.30 the inclusion $\mathscr{O}_0 \hookrightarrow \mathscr{C}_0$ is a cofibration.

Recall that $\mathbb{C}^*$ is an Oka manifold. Therefore if X is Stein, $\mathscr{O}(X, \mathbb{C}^*) \hookrightarrow \mathscr{C}(X, \mathbb{C}^*)$ is a weak homotopy equivalence (see Theorem 2.40). Hence the inclusion $i : \mathscr{O}_0(X, \mathbb{C}^*) \hookrightarrow \mathscr{C}_0(X, \mathbb{C}^*)$ is a weak homotopy equivalence. An ANR has the homotopy type of a CW complex [5]. Let A and B be CW complexes which are homotopy equivalent to $\mathscr{O}_0(X, \mathbb{C}^*)$ and $\mathscr{C}_0(X, \mathbb{C}^*)$ respectively. Let $f_A : \mathscr{O}_0(X, \mathbb{C}^*) \to A$ and $f_B : \mathscr{C}_0(X, \mathbb{C}^*) \to B$ be a homotopy equivalences. Let g_A and g_B be homotopy inverses for f_A and f_B respectively. Then $f_B \circ i \circ g_A : A \to B$ is a weak homotopy equivalence between CW complexes. By Whitehead's theorem (see [21, p. 74]) the map $f_B \circ i \circ g_A$ is a homotopy equivalence. Therefore $g_B \circ f_B \circ i \circ g_A \circ f_A$ is a homotopy equivalence. Therefore $i : \mathscr{O}_0(X, \mathbb{C}^*) \to \mathscr{C}_0(X, \mathbb{C}^*)$ is a homotopy equivalence since $g_B \circ f_B \circ i \circ g_A \circ f'_A$ is homotopic to i. From now on if $f, g : X \to Y$ are continuous maps we will write $f \simeq g$ to mean f is homotopic to g.

In what follows we use the fact that $i : \mathscr{O}_0(X, \mathbb{C}^*) \hookrightarrow \mathscr{C}_0(X, \mathbb{C}^*)$ is a homotopy equivalence and a cofibration to prove that $\mathscr{O}_0(X, \mathbb{C}^*)$ is a deformation retract of $\mathscr{C}_0(X, \mathbb{C}^*)$. Suppose g' is a homotopy inverse of i. Then $g' \circ i \simeq \mathrm{id}_{\mathscr{O}_0}$, that is, $g'|_{\mathscr{O}_0} \simeq \mathrm{id}_{\mathscr{O}_0}$. By HEP g' is homotopic to a continuous map $g : \mathscr{C}_0 \to \mathscr{O}_0$ with $g|_{\mathscr{O}_0} = \mathrm{id}_{\mathscr{O}_0}$. Note that g is a deformation retraction in the weak sense and therefore it follows from Theorem 2.31 that $\mathscr{O}_0$ is a deformation retract of $\mathscr{C}_0$. □

The following gives us a sufficient condition for X to be good.

Proposition 3.12. *Suppose X is a Stein manifold. Give $\mathscr{C}(X, \mathbb{C}^*)/\mathscr{C}_0(X, \mathbb{C}^*)$ the quotient topology. If the quotient map $\mathscr{C}(X, \mathbb{C}^*) \to \mathscr{C}(X, \mathbb{C}^*)/\mathscr{C}_0(X, \mathbb{C}^*)$, $f \mapsto [f]$, has a continuous section s with image in $\mathscr{O}(X, \mathbb{C}^*)$, then X is good.*

Note that $\mathscr{C}_0(X, \mathbb{C}^*)$ is a normal subgroup of $\mathscr{C}(X, \mathbb{C}^*)$ and $\mathscr{C}(X, \mathbb{C}^*)$ is a topological group, so the topological space $\mathscr{C}(X, \mathbb{C}^*)/\mathscr{C}_0(X, \mathbb{C}^*)$ inherits the structure of a topological group, namely the quotient group. Let e denote the identity of $\mathscr{C}(X,\mathbb{C}^*)/\mathscr{C}_0(X,\mathbb{C}^*)$. Note that $e = [1]$, where 1 denotes the constant map $1 : X \to \mathbb{C}^*$. The quotient map $\mathscr{C}(X,\mathbb{C}^*) \to \mathscr{C}(X,\mathbb{C}^*)/\mathscr{C}_0(X,\mathbb{C}^*)$ is a continuous homomorphism.

Proof. If $f \in \mathscr{C}(X,\mathbb{C}^*)$, then $\dfrac{f}{s([f])} \in \mathscr{C}_0(X,\mathbb{C}^*)$. For

$$\left[\frac{f}{s([f])}\right] = [f]\cdot[s([f])]^{-1} = [f]\cdot[f]^{-1} = e = [1].$$

The space $\mathscr{O}_0(X,\mathbb{C}^*)$ is a deformation retract of $\mathscr{C}_0(X,\mathbb{C}^*)$ by Proposition 3.11. Let $r : \mathscr{C}_0(X,\mathbb{C}^*) \to \mathscr{O}_0(X,\mathbb{C}^*)$ be a deformation retraction. Define $R : \mathscr{C}(X,\mathbb{C}^*) \to \mathscr{O}(X,\mathbb{C}^*)$ by

$$R(f) = s([f])\cdot r\left(\frac{f}{s([f])}\right).$$

The map R can be written as a composition of continuous maps, so R is continuous (recall that multiplication and inversion are continuous maps because $\mathscr{C}(X,\mathbb{C}^*)$ is a topological group by Proposition 2.8). If $f \in \mathscr{O}(X,\mathbb{C}^*)$, then $\dfrac{f}{s([f])} \in \mathscr{O}_0(X,\mathbb{C}^*)$, so

$$R(f) = s([f])\cdot r\left(\frac{f}{s([f])}\right) = s([f])\cdot\frac{f}{s([f])} = f.$$

Therefore, R is a retraction of $\mathscr{C}(X,\mathbb{C}^*)$ onto $\mathscr{O}(X,\mathbb{C}^*)$. We now show that R is a deformation retraction of $\mathscr{C}(X,\mathbb{C}^*)$ onto $\mathscr{O}(X,\mathbb{C}^*)$. To this end, let $H : \mathscr{C}(X,\mathbb{C}^*)\times I \to \mathscr{C}(X,\mathbb{C}^*)$ be defined by

$$H(f,t) = s([f])\cdot h\left(\frac{f}{s([f])}, t\right),$$

where $h : \mathscr{C}_0(X,\mathbb{C}^*)\times I \to \mathscr{C}_0(X,\mathbb{C}^*)$ is a homotopy of r to the identity relative to $\mathscr{O}_0(X,\mathbb{C}^*)$. The map H can be written as the composition of continuous functions and is therefore continuous. Clearly $H(\cdot,0) = R(\cdot)$, and $H(\cdot,1) = \mathrm{id}_{\mathscr{C}(X,\mathbb{C}^*)}$.

Suppose $t \in [0,1]$ and $f \in \mathscr{O}(X,\mathbb{C}^*)$. Then $\dfrac{f}{s([f])} \in \mathscr{O}_0(X,\mathbb{C}^*)$, so

$$h\left(\frac{f}{s([f])}, t\right) = \frac{f}{s([f])}.$$

Hence $H(f,t) = f$. Since $t \in [0,1]$ and $f \in \mathscr{O}(X,\mathbb{C}^*)$ were arbitrary, $H(f,t) = f$ for every $f \in \mathscr{O}(X,\mathbb{C}^*)$ and $t \in [0,1]$. Therefore R is a deformation retraction of $\mathscr{C}(X,\mathbb{C}^*)$ onto $\mathscr{O}(X,\mathbb{C}^*)$. □

3.2.2 'Very good' sources

In this section we define the property 'very good' which is an ostensibly stronger version of good. Proposition 3.12 will be used to show that a Stein manifold X is very good only if X is good. The reason we introduce this property is that the calculations we do in Lemma 3.21 and Proposition 3.31 will show that certain domains are very good and not just good. This is worth noting since there are ways of constructing new very good complex manifolds from old that we do not have at our disposal in the context of complex manifolds which are merely good.

Suppose X is a topological space and $x \in X$. For each $f \in \mathscr{C}(X,\mathbb{C}^*)$, the map $w_{f(x)} : \pi_1(\mathbb{C}^*, f(x)) \to \mathbb{Z}$, which takes an element of $\pi_1(\mathbb{C}^*, f(x))$ to its winding number about 0, is an isomorphism. So we will denote the composition $w_{f(x)} \circ f_*$, where $f_* : \pi_1(X,x) \to \pi_1(\mathbb{C}^*, f(x))$ is the homomorphism induced by f, simply by f_*. Hence, we have a map $\cdot_* : \mathscr{C}(X,\mathbb{C}^*) \to \mathrm{Hom}(\pi_1(X,x),\mathbb{Z})$, $f \mapsto f_*$. This is of course a slight abuse of notation.

Lemma 3.13. *Suppose that X is a connected, locally path connected topological space and that $x \in X$. If $f, g : X \to \mathbb{C}^*$ are continuous maps, then f is homotopic to g if and only if the homomorphisms $f_*, g_* : \pi_1(X,x) \to \mathbb{Z}$ are equal.*

Proof. If $f, g : X \to \mathbb{C}^*$ are continuous maps and f is homotopic to g, then it is easily shown that for all loops γ in $\mathbb{C}^*$, $f \circ \gamma$ and $g \circ \gamma$ have the same winding number as each other. Hence $f_* = g_*$.

Suppose $f_* = g_*$. Without loss of generality suppose that $f(x) = g(x)$. If this were not the case, then we could replace g by $\dfrac{f(x)}{g(x)} g \simeq g$. Let $h = f/g$. Then $h_* = (f/g)_* = f_* - g_* = 0$. Therefore, by [23, Lemma 79.1], h can be lifted to a map

$X \to \mathbb{C}$ by $\exp : \mathbb{C} \to \mathbb{C}^*$. Therefore h is null-homotopic, and hence f is homotopic to g. □

Proposition 3.14. *If X is a connected, locally path connected topological space and $x \in X$, then the map*

$$\Phi : \mathscr{C}(X, \mathbb{C}^*)/\mathscr{C}_0(X, \mathbb{C}^*) \to \mathrm{Hom}(\pi_1(X, x), \mathbb{Z}),\ [f] \mapsto f_*,$$

is an injective group homomorphism.

Proof. The map Φ is well defined and injective by Lemma 3.13. We now show that Φ is a homomorphism. For each loop ψ in $\mathbb{C}^*$ based at 1, let $\tilde{\psi} : I \to \mathbb{C}$ be the unique lifting of ψ by exp with $\tilde{\psi}(0) = 0$. Note that $w(\psi) = \mathrm{Im}(\tilde{\psi}(1))$, where $w(\psi)$ is the winding number of ψ. Suppose that $[f]$, $[g] \in \mathscr{C}(X, \mathbb{C}^*)/\mathscr{C}_0(X, \mathbb{C}^*)$ and without loss of generality suppose that $f(x) = g(x) = 1$. We want to show that $\Phi([f] \cdot [g]) = \Phi([f]) + \Phi([g])$. To this end, suppose $[\gamma]$ is an arbitrary element of $\pi_1(X, x)$. We have

$$\Phi([f] \cdot [g])([\gamma]) = \Phi([f \cdot g])([\gamma]) = w((f \cdot g) \circ \gamma) = \mathrm{Im}(\widetilde{(f \cdot g) \circ \gamma}(1)). \tag{3.3}$$

Now, observe that $\widetilde{f \circ \gamma} + \widetilde{g \circ \gamma}$ is also a lifting of $(f \cdot g) \circ \gamma$ by exp with $0 \mapsto 0$. Hence, by uniqueness of liftings $\widetilde{(f \cdot g) \circ \gamma} = \widetilde{f \circ \gamma} + \widetilde{g \circ \gamma}$. So by Equation 3.3 we have

$$\begin{aligned}\Phi([f] \cdot [g])([\gamma]) = \mathrm{Im}(\widetilde{f \circ \gamma}(1) + \widetilde{g \circ \gamma}(1)) &= w(f \circ \gamma) + w(g \circ \gamma)\\ &= (\Phi(f) + \Phi(g))([\gamma]),\end{aligned}$$

as required. □

Note that the quotient map $p : \mathscr{C}(X, \mathbb{C}^*) \to \mathscr{C}(X, \mathbb{C}^*)/\mathscr{C}_0(X, \mathbb{C}^*)$ is a homomorphism. Therefore $\cdot_* : \mathscr{C}(X, \mathbb{C}^*) \to \mathrm{Hom}(\pi_1(X, x), \mathbb{Z})$ is a homomorphism (since $\cdot_*$ can be written as the composition $\Phi \circ p$).

Let X be a path connected topological space and let $x \in X$. Give $\pi_1(X, x)$ and $\mathbb{Z}$ the discrete topology. The space $\mathscr{C}(\pi_1(X, x), \mathbb{Z})$ with pointwise addition is a topological group by Corollary 2.8. Note that $\mathrm{Hom}(\pi_1(X,x),\mathbb{Z})$ is a subgroup of $\mathscr{C}(\pi_1(X,x),\mathbb{Z})$. So $\mathrm{Hom}(\pi_1(X,x),\mathbb{Z})$ is a topological group when given the subspace topology. From now on $\mathrm{Hom}(\pi_1(X,x),\mathbb{Z})$ will always be given this topology.

Proposition 3.15. *If X is a path connected topological space and $x \in X$, then the map $\cdot_* : \mathscr{C}(X,\mathbb{C}^*) \to \mathrm{Hom}(\pi_1(X,x),\mathbb{Z})$ is continuous.*

Proof. Suppose $U \subset \mathrm{Hom}(\pi_1(X,x),\mathbb{Z})$ is open. Without loss of generality we may suppose

$$U = [\{[\gamma]\},\{b\}] \cap \mathrm{Hom}(\pi_1(X,x),\mathbb{Z}),$$

where $[\gamma] \in \pi_1(X,x)$ and $b \in \mathbb{Z}$. The preimage of U under $\cdot_*$ is then precisely the set of all continuous functions $X \to \mathbb{C}^*$ which wrap γ about the puncture in the target b times. This set is open by Lemma 2.16. Hence $\cdot_*$ is continuous. □

Proposition 3.16. *If X is a Stein manifold, x is a point in X and $\cdot_* : \mathscr{C}(X,\mathbb{C}^*) \to \mathrm{Hom}(\pi_1(X,x),\mathbb{Z})$ has a continuous section with image in $\mathscr{O}(X,\mathbb{C}^*)$, then X is good.*

Proof. Define Φ as in Proposition 3.14. It follows from Proposition 3.15 that Φ is continuous. Suppose $\cdot_*$ has a continuous section. Then $\cdot_*$ is surjective. So Φ is surjective and hence a group isomorphism.

Let $s : \mathrm{Hom}(\pi_1(X,x),\mathbb{Z}) \to \mathscr{C}(X,\mathbb{C}^*)$ be a continuous section of $\cdot_*$ with image in $\mathscr{O}(X,\mathbb{C}^*)$. Then $s \circ \Phi$ is a continuous section of the projection map $p : \mathscr{C}(X,\mathbb{C}^*) \to \mathscr{C}(X,\mathbb{C}^*)/\mathscr{C}_0(X,\mathbb{C}^*)$ with image in $\mathscr{O}(X,\mathbb{C}^*)$. For $p \circ (s \circ \Phi) = \Phi^{-1} \circ \cdot_* \circ s \circ \Phi = \Phi^{-1} \circ \Phi = \mathrm{id}$. Hence X is good by Proposition 3.12. □

The reason we switch from the quotient map $\mathscr{C}(X,\mathbb{C}^*) \to \mathscr{C}(X,\mathbb{C}^*)/\mathscr{C}_0(X,\mathbb{C}^*)$ to $\cdot_*$ is because it seems easier to find a section of $\cdot_*$ than to find a section of the quotient map directly.

Call a complex manifold X *very good* if for some $x \in X$, the map $\cdot_* : \mathscr{C}(X,\mathbb{C}^*) \to \mathrm{Hom}(\pi_1(X,x),\mathbb{Z})$ possesses a continuous section with image in $\mathscr{O}(X,\mathbb{C}^*)$. Note that the choice of $x \in X$ is not important. For suppose $\cdot_{*'} : \mathscr{C}(X,\mathbb{C}^*) \to \mathrm{Hom}(\pi_1(X,y),\mathbb{Z})$ is the analogous map defined for a different element $y \in X$, then $\cdot_{*'}$ factors as follows

$$\begin{array}{ccc} & \mathrm{Hom}(\pi_1(X,x),\mathbb{Z}) & \\ {\cdot_*}\nearrow & & \searrow{\phi} \\ \mathscr{C}(X,\mathbb{C}^*) & \xrightarrow{\cdot_{*'}} & \mathrm{Hom}(\pi_1(X,y),\mathbb{Z}) \end{array}$$

where ϕ is the pullback of an isomorphism $\pi_1(X,x) \to \pi_1(X,y)$ that conjugates an element of $\pi_1(X,x)$ by a fixed path from y to x. The map ϕ is a homeomorphism by Lemma 2.7. So $\cdot_*$ has a continuous section if and only if $\cdot_{*'}$ does.

The above proposition tells us that if X is Stein, then a sufficient condition for X to be good is that X is very good. We do not know if this condition is necessary.

We will now show that if X is a non-compact Riemann surface so that $\pi_1(X,x)$ is finitely generated, then X is very good. Since a non-compact Riemann surface is Stein this implies that X is good. First we need the following lemma.

Lemma 3.17. *If X is a non-compact Riemann surface, then $\cdot_* : \mathscr{C}(X,\mathbb{C}^*) \to \mathrm{Hom}(\pi_1(X,x),\mathbb{Z})$ is surjective for all $x \in X$.*

Proof. By the above discussion, it suffices to prove that the map $\cdot_* : \mathscr{C}(X,\mathbb{C}^*) \to \mathrm{Hom}(\pi_1(X,x),\mathbb{Z})$ is surjective for some convenient choice of $x \in X$.

Since X is a non-compact Riemann surface, X has the homotopy type of a CW complex with cells of dimension at most one [24, Theorem 2.2]. That is, X is homotopy equivalent to a connected graph. A connected graph is homotopy equivalent to a bouquet of circles [21, §4.3], so X is homotopy equivalent to a bouquet of circles. Let Y be a bouquet of circles with basepoint $y \in Y$ homotopy equivalent to X and $F : Y \to X$ be a homotopy equivalence with homotopy inverse $G : X \to Y$. Let $\mathscr{S}$ be a free generating set for $\pi_1(Y,y)$ consisting of one generator for each circle (see [21, §2.8]). Then $F_*\mathscr{S}$ is a free generating set for $\pi_1(X,x)$, where $x = F(y)$.

Given a family of integers $(n_s)_{s\in\mathscr{S}}$ we can always find a continuous function $f : Y \to \mathbb{C}^*$ so that $f_*(s) = n_s$ for all $s \in \mathscr{S}$. Hence the map $\cdot_* : \mathscr{C}(Y,\mathbb{C}^*) \to \mathrm{Hom}(\pi_1(Y,y),\mathbb{Z})$ is surjective. If $b \in \mathrm{Hom}(\pi_1(X,x),\mathbb{Z})$, then the composition $b \circ F_*$

is in $\mathrm{Hom}(\pi_1(Y,y),\mathbb{Z})$, so there exists $f \in \mathscr{C}(Y,\mathbb{C}^*)$ such that $f_* = b \circ F_*$. Therefore $g = f \circ G \in \mathscr{C}(X,\mathbb{C}^*)$ is such that $g_* = b$. Hence $\cdot_* : \mathscr{C}(X,\mathbb{C}^*) \to \mathrm{Hom}(\pi_1(X,x),\mathbb{Z})$ is surjective. □

Proposition 3.18. *If X is a non-compact Riemann surface and $\pi_1(X)$ is finitely generated, then X is very good.*

Proof. Suppose that $x \in X$ and that $\mathscr{A}$ is a finite subset of $\pi_1(X,x)$ that generates $\pi_1(X,x)$. Then the restriction map $\mathrm{Hom}(\pi_1(X),\mathbb{Z}) \to \mathbb{Z}^{\mathscr{A}}$ is a continuous injection (see Lemma 2.7). The space $\mathbb{Z}^{\mathscr{A}}$ has the discrete topology since $\mathbb{Z}$ has the discrete topology and $\mathscr{A}$ is finite. Therefore $\mathrm{Hom}(\pi_1(X,x),\mathbb{Z})$ has the discrete topology.

For each $\lambda \in \mathrm{Hom}(\pi_1(X,x),\mathbb{Z})$ let $f_\lambda : X \to \mathbb{C}^*$ be a continuous map with $(f_\lambda)_* = \lambda$. This is possible because $\cdot_*$ is surjective (see Proposition 3.17). For $\lambda \in \mathrm{Hom}(\pi_1(X,x),\mathbb{Z})$ let $g_\lambda : X \to \mathbb{C}^*$ be a holomorphic map homotopic to f_λ (such g_λ exists by the basic Oka property). Note that $(g_\lambda)_* = (f_\lambda)_* = \lambda$ for $\lambda \in \mathrm{Hom}(\pi_1(X,x),\mathbb{Z})$. Therefore $\mathrm{Hom}(\pi_1(X,x),\mathbb{Z}) \to \mathscr{C}(X,\mathbb{C})$, $\lambda \mapsto g_\lambda$, is a section of $\cdot_*$ with image in $\mathscr{O}(X,\mathbb{C}^*)$. This section is continuous because $\mathrm{Hom}(\pi_1(X,x),\mathbb{Z})$ has the discrete topology. □

This proposition serves to give us some examples of very good complex manifolds. Note that we needed $\cdot_*$ to be surjective. If we simply wanted to prove that a Stein manifold X with $\pi_1(X)$ finitely generated was good, then we could avoid this complication by using Proposition 3.12 directly as follows.

Proposition 3.19. *If X is a Stein manifold so that $\pi_1(X)$ is finitely generated, then X is good.*

Proof. Let $\mathscr{C} = \mathscr{C}(X,\mathbb{C}^*)$ and $\mathscr{C}_0$ be the path component of $\mathscr{C}$ consisting of the null-homotopic maps. Let $x \in X$ and $H \subset \mathrm{Hom}(\pi_1(X,x),\mathbb{Z})$ be the image of $\mathscr{C}(X,\mathbb{C}^*)$ under $\cdot_* : \mathscr{C}(X,\mathbb{C}^*) \to \mathrm{Hom}(\pi_1(X,x),\mathbb{Z})$. Since $\pi_1(X,x)$ is finitely generated, $\mathrm{Hom}(\pi_1(X,x),\mathbb{Z})$ has the discrete topology (as we saw in the proof of Proposition 3.18). So H has the discrete topology.

For each $\lambda \in H$ let $f_\lambda : X \to \mathbb{C}^*$ be a continuous map with $(f_\lambda)_* = \lambda$ and let $g_\lambda : X \to \mathbb{C}^*$ be a holomorphic map homotopic to f_λ (which exists by the basic Oka property). Note that $(g_\lambda)_* = (f_\lambda)_* = \lambda$ for $\lambda \in H$. Define $s : H \to \mathscr{C}(X, \mathbb{C}^*)$, $\lambda \mapsto g_\lambda$. Note that s has image in $\mathscr{O}(X, \mathbb{C}^*)$ and $\cdot_* \circ s = \mathrm{id}_H$. This map s is continuous because H has the discrete topology.

Define $F : \mathscr{C}/\mathscr{C}_0 \to H$, $[f] \mapsto f_*$. The map F is continuous because $\cdot_*$ is continuous. The map F is injective by Lemma 3.13 and surjective by definition of H. The composition $s \circ F$ is a continuous section of the projection map $p : \mathscr{C} \to \mathscr{C}/\mathscr{C}_0$ with image in $\mathscr{O}(X, \mathbb{C}^*)$, since $p = F^{-1} \circ \cdot_*$. Therefore X is good by Proposition 3.12. □

In the next section we will see some examples of domains in $\mathbb{C}$ that are very good and whose fundamental groups are not finitely generated. We do not know how to prove directly that these examples are good. The following result will be helpful.

Proposition 3.20. *Suppose X, Y are complex manifolds, x is a point in X and $i : X \to Y$ is a holomorphic map so that $i_* : \pi_1(X, x) \to \pi_1(Y, i(x))$ is an isomorphism. If Y is very good, then X is very good.*

Proof. Since i_* is an isomorphism of topological groups, the pullback

$$(i_*)^* : \mathrm{Hom}(\pi_1(Y, i(x)), \mathbb{Z}) \to \mathrm{Hom}(\pi_1(X, x), \mathbb{Z})$$

is an isomorphism of topological groups. Suppose s_Y is a continuous section of $\cdot_* : \mathscr{C}(Y, \mathbb{C}^*) \to \mathrm{Hom}(\pi_1(Y, i(x)), \mathbb{Z})$ with image in $\mathscr{O}(Y, \mathbb{C}^*)$. Let

$$s_X : \mathrm{Hom}(\pi_1(X, x), \mathbb{Z}) \to \mathscr{C}(X, \mathbb{C}^*), \ \lambda \mapsto s_Y(\lambda \circ ({i_*}^{-1})) \circ i.$$

Note that s_X is continuous since s_X is the composition of continuous maps. The map s_X has image in $\mathscr{O}(X, \mathbb{C}^*)$ since i is holomorphic and s_Y has image in $\mathscr{O}(Y, \mathbb{C}^*)$.

If $\lambda \in \mathrm{Hom}(\pi_1(X,x),\mathbb{Z})$, then

$$\begin{aligned}(s_X(\lambda))_* &= (s_Y(\lambda \circ ({i_*}^{-1})) \circ i)_* \\ &= (s_Y(\lambda \circ ({i_*}^{-1})))_* \circ i_* \\ &= \lambda \circ ({i_*}^{-1}) \circ i_* = \lambda.\end{aligned}$$

Therefore s_X is a continuous section of $\cdot_* : \mathscr{C}(X,\mathbb{C}^*) \to \mathrm{Hom}(\pi_1(X),\mathbb{Z})$ with image in $\mathscr{O}(X,\mathbb{C}^*)$, as required. □

Let X, Y and i be as above. We do not know whether X is very good implies Y is very good.

In the case of open Riemann surfaces a continuous map which induces an isomorphism of fundamental groups is a weak homotopy equivalence, and Whitehead's theorem tells us that a weak homotopy equivalence between CW complexes is a homotopy equivalence. So if X and Y are open Riemann surfaces, then i is a holomorphic homotopy equivalence, where by a holomorphic homotopy equivalence we simply mean a homotopy equivalence which is holomorphic. A homotopy inverse of a holomorphic homotopy equivalence need not be holomorphic. For example the map $\mathbb{D}^* \hookrightarrow \mathbb{C}^*$ is a holomorphic homotopy equivalence without a holomorphic homotopy inverse because every holomorphic map $\mathbb{C}^* \to \mathbb{D}^*$ is constant by Liouville's theorem and the identity on $\mathbb{C}^*$ is not null-homotopic.

Note that it was relatively easy to show that if Y is very good and $X \to Y$ is a holomorphic homotopy equivalence, then X is very good. It seems hard or impossible to prove the analogous statement for the good property. Suppose Y is good and $f : X \to Y$ is a holomorphic homotopy equivalence with a continuous homotopy inverse $g : Y \to X$. Let us try to show that X is good. Let $r : \mathscr{C}(Y,\mathbb{C}^*) \to \mathscr{O}(Y,\mathbb{C}^*)$ be a deformation retraction. We want to define a deformation retraction $\mathscr{C}(X,\mathbb{C}^*) \to \mathscr{O}(X,\mathbb{C}^*)$. First let us try to define a retraction. We could define $\rho = f^* \circ r \circ g^* : \mathscr{C}(X,\mathbb{C}^*) \to \mathscr{O}(X,\mathbb{C}^*)$. However, there does not appear to be any reason to expect $r(h \circ g) \circ f = h$ for all holomorphic maps $h : X \to \mathbb{C}^*$. We are not aware of any way to use a holomorphic homotopy equivalence from X to a good complex manifold to prove that X is good. This is why we have introduced the very good property.

3.2.3 Further examples of very good sources

Suppose A and B are topological spaces. We write B^A to mean the set of all maps from A to B. If $\lambda \in B^A$ and $a \in A$, then we write λ_a to mean $\lambda(a)$.

If A and B are discrete spaces, then every map from A to B is continuous. So $B^A = \mathscr{C}(A, B)$ (as sets). We can then give B^A the compact-open topology. Another way to think about B^A is as the product $\prod_A B$. It is therefore natural to give B^A the product topology. It is easy to verify that these two topologies are the same.

We say that A is a discrete subset of B if A is closed in B, and the subspace topology on A is the discrete topology on B. A discrete subset of $\mathbb{C}$ is countable, for otherwise it will have a point of accumulation in $\mathbb{C}$.

A domain in the Riemann sphere $\mathbb{P}_1$ is called a *circle domain* if every connected component of its boundary in $\mathbb{P}_1$ is either a circle or a point. It was proved by He and Schramm [15, Theorem 0.1] that every domain in $\mathbb{C}$, whose boundary has countably many components, is biholomorphic to a circle domain. This is why we spend so much time proving that a certain class of circle domains is very good in Proposition 3.31. He and Schramm's result is a generalisation of the Riemann mapping theorem and a partial answer to the conjecture, made by Koebe in 1908, that every planar domain is biholomorphic to a circle domain. This conjecture was proved by Koebe for finitely connected domains.

The following important lemma will be needed for the proof of Proposition 3.29. This lemma has a lengthy proof which uses Weierstrass products. Our construction of the continuous map $\mathbb{Z}^J \to \mathscr{O}(X \backslash A, \mathbb{C}^*)$ below is similar to the construction of the continuous map $\mathbb{Z}^{\mathbb{N}} \to \mathscr{O}(\mathbb{C} \backslash \mathbb{N}, \mathbb{C}^*)$ in [19, p. 1165–1166].

Lemma 3.21. *Suppose $X \subset \mathbb{C}$ is a domain and A is a discrete subset of X. If $J \subset \mathbb{N}$*

and $a : J \to A$ *is a bijection, then there exists a continuous map* $\mathbb{Z}^J \to \mathscr{O}(X\backslash A, \mathbb{C}^*)$, $\lambda \mapsto f_\lambda$*, so that for each* $\lambda \in \mathbb{Z}^J$*,* f_λ *is the restriction of a meromorphic function* $\tilde{f}_\lambda$ *on* X *with* $\operatorname{ord}_{a_j} \tilde{f}_\lambda = \lambda_j$ *for every* $j \in J$*.*

Note that $\mathbb{Z}$ and J are given the discrete topology and $\mathbb{Z}^J$ is given the topology desribed above.

The reader may wonder why we introduce $J \subset \mathbb{N}$ rather than just consider sequences of distinct points. The reason we introduce J is so that we can deal with the finite case at the same time as the infinite case.

Proof. If J is finite, then $\mathbb{Z}^J$ has the discrete topology and so $\mathbb{Z}^J \to \mathscr{O}(X\backslash A, \mathbb{C}^*)$, $\lambda \mapsto \prod_{j\in J} (z - a_j)^{\lambda_j}$, is a continuous map with the desired property.

Suppose that J is infinite. Without loss of generality suppose that $J = \mathbb{N}$. It suffices to find a continuous map $s : (\mathbb{N} \cup \{0\})^{\mathbb{N}} \to \mathscr{O}(X\backslash A, \mathbb{C}^*)$ so that for each $\lambda \in (\mathbb{N} \cup \{0\})^{\mathbb{N}}$, $s(\lambda)$ is the restriction of a holomorphic map $\widetilde{s(\lambda)} : X \to \mathbb{C}$ with $\operatorname{ord}_{a_j} \widetilde{s(\lambda)} = \lambda_j$ for every $j \in J$. For given a sequence $\lambda \in \mathbb{Z}^{\mathbb{N}}$ we can decompose λ as $\lambda = \lambda_+ - \lambda_-$, where $\lambda_+ = \max(0, \lambda)$ and $\lambda_- = \max(0, -\lambda)$. Then the map $\mathbb{Z}^{\mathbb{N}} \to \mathscr{O}(X\backslash A, \mathbb{C}^*)$, $\lambda \mapsto \dfrac{s(\lambda_+)}{s(\lambda_-)}$, is the desired map.

Step 1: Defining a map $s : (\mathbb{N} \cup \{0\})^{\mathbb{N}} \to \mathscr{O}(X\backslash A, \mathbb{C}^*)$.

Let $a_0 \in X\backslash A$ and $r_0 > 0$ be such that $B(a_0, r_0) \subset X\backslash A$. Define $F : X\backslash\{a_0\} \to \mathbb{C}$, $z \mapsto (z - a_0)^{-1}$. Let $Y = F(X\backslash\{a_0\})$ and $b_j = F(a_j)$ for $j \in \mathbb{N}$. Observe that $\{z \in \mathbb{C} : |z| > r_0^{-1}\} \subset Y$ and $|b_j| \leq r_0^{-1}$ for all $j \in \mathbb{N}$. Clearly $B = \{b_j : j \in \mathbb{N}\}$ has no points of accumulation in Y, therefore it follows from boundedness of B that $Y \neq \mathbb{C}$. Note that $\mathbb{C}\backslash Y \neq \emptyset$ is compact since it is a closed bounded subset of $\mathbb{C}$. For each $n \in \mathbb{N}$, let c_n be a point in $\mathbb{C}\backslash Y$ such that $|b_n - c_n| = d(b_n, \mathbb{C}\backslash Y)$. Note that because B is discrete in Y and bounded by r_0^{-1}, $|b_n - c_n| \to 0$ as $n \to \infty$.

For each $\lambda \in (\mathbb{N}\cup\{0\})^{\mathbb{N}}$, let $N_\lambda = \sum_{j=1}^{\infty} \lambda_j$ if λ is eventually zero; if λ is not eventually zero, let $N_\lambda = \infty$. For each $\lambda \in (\mathbb{N} \cup \{0\})^{\mathbb{N}}$ and $j \in \mathbb{N}$ such that $1 \leq j < N_\lambda + 1$, let $z_{j,\lambda}$ be the j-th term in the possibly finite sequence $\underbrace{b_1,\dots,b_1}_{\lambda_1 \text{ times}},\underbrace{b_2,\dots,b_2}_{\lambda_2 \text{ times}},\dots$

For each $j \in \mathbb{N}$ and $\lambda \in (\mathbb{N}\cup\{0\})^{\mathbb{N}}$, let $w_{j,\lambda} = c_m$, where $m \in \mathbb{N}$ is such that $z_{j,\lambda} = b_m$. Define $s : (\mathbb{N}\cup\{0\})^{\mathbb{N}} \to \mathscr{O}(X\backslash A, \mathbb{C}^*)$,

$$s(\lambda)(z) = \begin{cases} \prod_{n=1}^{N_\lambda} E_n\left(\dfrac{z_{n,\lambda}-w_{n,\lambda}}{F(z)-w_{n,\lambda}}\right) & \text{if } z \in X\backslash(A\cup\{a_0\}),\\ 1 & \text{if } z = a_0,\end{cases}$$

where the E_n are the elementary factors in Definition 2.17. Note that this map is well defined because $f_\lambda : Y \to \mathbb{C}$,

$$f_\lambda(z) = \prod_{n=1}^{N_\lambda} E_n\left(\frac{z_{n,\lambda}-w_{n,\lambda}}{z-w_{n,\lambda}}\right),$$

is holomorphic on Y, non-zero on $Y\backslash B$ (with a zero of order λ_j at b_j) and $\lim_{z\to\infty} f_\lambda(z) = 1$ [6, p. 170-173].

Step 2: Showing that for each $\lambda \in (\mathbb{N}\cup\{0\})^{\mathbb{N}}$, $s(\lambda)$ is the restriction of a holomorphic map $\widetilde{s(\lambda)} : X \to \mathbb{C}$ with $\mathrm{ord}_{a_j}\, \tilde{s(\lambda)} = \lambda_j$ for every $j \in \mathbb{N}$.

Fix $\lambda \in (\mathbb{N}\cup\{0\})^{\mathbb{N}}$. Suppose $j \in \mathbb{N}$. Since f_λ has a zero of order λ_j at b_j, it is easy to check that $f_\lambda \circ F$ has a zero of order λ_j at a_j. Also, $f_\lambda \circ F$ can be extended to a_0 holomorphically because $\lim_{z\to\infty} f_\lambda(z) = 1$. So clearly $f_\lambda \circ F$ extended to a_0 is the desired map.

Step 3: Proving that the map $s : (\mathbb{N}\cup\{0\})^{\mathbb{N}} \to \mathscr{O}(X\backslash A, \mathbb{C}^*)$ is continuous.

Suppose $U \subset \mathscr{O}(X\backslash A, \mathbb{C}^*)$. Without loss of generality suppose $U = [K, V]$, where $K \subset X\backslash A$ is compact and $V \subset \mathbb{C}^*$ is open. Suppose $\lambda \in s^{-1}(U)$.

Let $\epsilon = d(s(\lambda)(K), \mathbb{C}\backslash V) > 0$. Let $\delta_1 \in (0,1)$ be so small that $|1-\exp(z)| < \epsilon/3$ whenever $z \in \bar{B}(0,\delta_1)$. Let $0 < r_1 < r_2 < r_3$ be such that

$$B(a_0, r_3) \subset (X\backslash A) \cap (s(\lambda))^{-1}(B(1,\epsilon/3)), \text{ and} \tag{3.4}$$

$$|F(z)| > \frac{2}{\delta_1}\sup_{n\in\mathbb{N}}|b_n - c_n| + \sup_{n\in\mathbb{N}}|c_n| \tag{3.5}$$

for all $z \in B(a_0, r_2)\backslash\{a_0\}$. Let $K_1 = K\backslash B(a_0, r_1)$ and $K_2 = K \cap \bar{B}(a_0, r_2)$. Note that $K = K_1 \cup K_2$ and hence $U = [K_1, V] \cap [K_2, V]$. Let $R_\lambda > 1$ be such that

$$\left\| \prod_{n=1}^{m} E_n \left(\frac{z_{n,\lambda} - w_{n,\lambda}}{z - w_{n,\lambda}} \right) \right\|_{F(K_1)} < R_\lambda \tag{3.6}$$

for every $m \in \mathbb{N}$ such that $1 \le m < N_\lambda + 1$. It is obvious why R_λ exists if N_λ is finite. If $N_\lambda = \infty$, then R_λ exists because the product

$$\prod_{n=1}^{\infty} E_n \left(\frac{z_{n,\lambda} - w_{n,\lambda}}{z - w_{n,\lambda}} \right)$$

converges uniformly on compact subsets of Y and hence on $F(K_1)$ (see [6, p. 172]). Let $\delta_2 \in (0, 1)$ be so small that $|1 - \exp(z)| < \dfrac{\epsilon}{3R_\lambda}$ whenever $z \in \bar{B}(0, \delta_2)$. Choose M_1 so large that if $j \ge M_1$, then $\dfrac{3}{\delta_2}|b_j - c_j| \le |\xi - c_j|$ for $\xi \in F(K_1)$ (this is possible because $|b_j - c_j| \to 0$ as $j \to \infty$ and $d(F(K_1), \mathbb{C}\backslash Y) > 0$). We will prove that

$$W = \{\lambda_1\} \times \{\lambda_2\} \times \cdots \times \{\lambda_{M_1}\} \times \mathbb{Z} \times \mathbb{Z} \times \cdots$$

is an open neighbourhood of λ in $s^{-1}(U)$. To this end, let $\lambda' \in W$. We will prove that $s(\lambda') \in U = [K_1, V] \cap [K_2, V]$.

Claim 1: $s(\lambda') \in [K_2, V]$.

If $K_2 = \emptyset$, then it trivially follows that $s(\lambda') \in [K_2, V]$. Suppose $K_2 \ne \emptyset$. If $K_2 = \{a_0\}$, then it trivially follows that $s(\lambda') \in [K_2, V]$ because $s(\lambda')(a_0) = s(\lambda)(a_0) = 1$. So suppose $K_2\backslash\{a_0\}$ is non-empty.

Let $z \in K_2\backslash\{a_0\}$. Then

$$\left| 1 - E_n \left(\frac{z_{n,\lambda'} - w_{n,\lambda'}}{F(z) - w_{n,\lambda'}} \right) \right| < \left| \frac{z_{n,\lambda'} - w_{n,\lambda'}}{F(z) - w_{n,\lambda'}} \right|^{n+1}$$

by (3.5) and Lemma 2.18 (it may be helpful for the reader to recall the definition of $w_{n,\lambda'}$). So

$$\left|1 - E_n\left(\frac{z_{n,\lambda'} - w_{n,\lambda'}}{F(z) - w_{n,\lambda'}}\right)\right| < \left|\frac{\sup_{n\in\mathbb{N}}|b_n - c_n|}{\frac{2}{\delta_1}\sup_{n\in\mathbb{N}}|b_n - c_n| + \sup_{n\in\mathbb{N}}|c_n| - |w_{n,\lambda'}|}\right|^{n+1}$$
$$\leq (\delta_1/2)^{n+1} < \frac{2}{3}(1/2)^n\delta_1.$$

Now $|\log(1+\zeta)| \leq \frac{3}{2}|\zeta|$ if $|\zeta| \leq \frac{1}{2}$. Therefore

$$\left|\log\left(E_n\left(\frac{z_{n,\lambda'} - w_{n,\lambda'}}{F(z) - w_{n,\lambda'}}\right)\right)\right| \leq \frac{3}{2}\cdot\frac{2}{3}\left(\frac{1}{2}\right)^n\delta_1 = \left(\frac{1}{2}\right)^n\delta_1, \tag{3.7}$$

and so

$$\left|\sum_{n=1}^{N_{\lambda'}}\log\left(E_n\left(\frac{z_{n,\lambda'} - w_{n,\lambda'}}{F(z) - w_{n,\lambda'}}\right)\right)\right| \leq \sum_{n=1}^{N_{\lambda'}}\left|\log\left(E_n\left(\frac{z_{n,\lambda'} - w_{n,\lambda'}}{F(z) - w_{n,\lambda'}}\right)\right)\right| \leq \delta_1. \tag{3.8}$$

Hence

$$\left|1 - \prod_{n=1}^{N_{\lambda'}} E_n\left(\frac{z_{n,\lambda'} - w_{n,\lambda'}}{F(z) - w_{n,\lambda'}}\right)\right| < \frac{\epsilon}{3}.$$

by our choice of δ_1. Therefore,

$$|s(\lambda)(z) - s(\lambda')(z)| < |s(\lambda)(z) - 1| + |1 - s(\lambda')(z)| < \epsilon/3 + \epsilon/3 = \epsilon,$$

since $K_2 \subset B(a_0, r_3)$. Hence, $s(\lambda')(z) \in V$ for $z \in K_2\backslash\{a_0\}$ by definition of ϵ.

If $z = a_0$, then $s(\lambda)(z) = s(\lambda')(z) = 1$. So (irrespective of whether $a_0 \in K$) $s(\lambda') \in [K_2, V]$, since $s(\lambda) \in [K_2, V]$ and $s(\lambda')(K\backslash\{a_0\}) \subset V$.

Claim 2: $s(\lambda') \in [K_1, V]$.

As above we may assume that K_1 is non-empty. Let $M_2 = \sum_{j=1}^{M_1}\lambda_j \leq N_\lambda$. Note that $M_2 \leq N_{\lambda'}$ since $\lambda_j = \lambda'_j$ for $j = 1, \ldots, M_1$. Since $\lambda_j = \lambda'_j$ for $j = 1, \ldots, M_1$ it follows (from how $z_{j,\lambda}$ and $z_{j,\lambda'}$ were defined in Step 1) that $z_{j,\lambda} = z_{j,\lambda'}$ for $j = 1, \ldots, M_2$. Therefore for $z \in K_1$,

$$s(\lambda')(z) = \prod_{n=1}^{N_{\lambda'}} E_n\left(\frac{z_{n,\lambda'} - w_{n,\lambda'}}{F(z) - w_{n,\lambda'}}\right)$$
$$= \prod_{n=1}^{M_2} E_n\left(\frac{z_{n,\lambda} - w_{n,\lambda}}{F(z) - w_{n,\lambda}}\right)\cdot\prod_{n=M_2+1}^{N_{\lambda'}} E_n\left(\frac{z_{n,\lambda'} - w_{n,\lambda'}}{F(z) - w_{n,\lambda'}}\right),$$

where $\prod_{n=M_2+1}^{N_{\lambda'}} E_n\left(\frac{z_{n,\lambda'}-w_{n,\lambda'}}{F(z)-w_{n,\lambda'}}\right)$ is defined to be 1 if $M_2+1 > N_{\lambda'}$ (which happens only if $M_2 = N_{\lambda'}$ since $M_2 \leq N_{\lambda'}$). Similarly,

$$s(\lambda)(z) = \prod_{n=1}^{M_2} E_n\left(\frac{z_{n,\lambda}-w_{n,\lambda}}{F(z)-w_{n,\lambda}}\right) \cdot \prod_{n=M_2+1}^{N_\lambda} E_n\left(\frac{z_{n,\lambda}-w_{n,\lambda}}{F(z)-w_{n,\lambda}}\right),$$

where $\prod_{n=M_2+1}^{N_\lambda} E_n\left(\frac{z_{n,\lambda}-w_{n,\lambda}}{F(z)-w_{n,\lambda}}\right)$ is defined to be 1 if $M_2+1 > N_\lambda$ (which happens only if $M_2 = N_\lambda$ since $M_2 \leq N_\lambda$). So

$$\begin{aligned}\|s(\lambda')-s(\lambda)\|_{K_1} \leq{}& \left\|\prod_{n=1}^{M_2} E_n\left(\frac{z_{n,\lambda}-w_{n,\lambda}}{F(z)-w_{n,\lambda}}\right)\right\|_{K_1} \\ &\times \left\|\prod_{n=M_2+1}^{N_\lambda} E_n\left(\frac{z_{n,\lambda}-w_{n,\lambda}}{F(z)-w_{n,\lambda}}\right) - \prod_{n=M_2+1}^{N_{\lambda'}} E_n\left(\frac{z_{n,\lambda'}-w_{n,\lambda'}}{F(z)-w_{n,\lambda'}}\right)\right\|_{K_1}.\end{aligned}$$

Note that $\left\|\prod_{n=1}^{M_2} E_n\left(\frac{z_{n,\lambda}-w_{n,\lambda}}{F(z)-w_{n,\lambda}}\right)\right\|_{K_1} < R_\lambda$ by definition of R_λ. Therefore, to show that $\|s(\lambda')-s(\lambda)\|_{K_1} < \epsilon$ and hence $s(\lambda') \in [K_1, V]$ it suffices to show

$$\left\|1 - \prod_{n=M_2+1}^{N_{\lambda'}} E_n\left(\frac{z_{n,\lambda'}-w_{n,\lambda'}}{F(z)-w_{n,\lambda'}}\right)\right\|_{K_1} < \frac{\epsilon}{2R_\lambda} \tag{3.9}$$

and the analogous inequality for λ. We show this inequality for λ', the proof for λ is similar.

If $M_2+1 > N_{\lambda'}$, then (3.9) is obvious. Suppose $M_2+1 \leq N_{\lambda'}$. Let $n \in \mathbb{N}$ be such that $M_2+1 \leq n < N_{\lambda'}+1$. Then $z_{n,\lambda'} = b_j$ for some $j \geq M_1$ because $\lambda_k = \lambda'_k$ for $k = 1, \ldots, M_1$. So by the definition of $w_{n,\lambda'}$ in Step 1, $w_{n,\lambda'} = c_j$. Hence, because of the way M_1 was chosen, $\frac{3}{\delta_2}|z_{n,\lambda'} - w_{n,\lambda'}| \leq |\xi - w_{n,\lambda'}|$ for every $\xi \in F(K_1)$. So

$$\begin{aligned}\left|1 - E_n\left(\frac{z_{n,\lambda'}-w_{n,\lambda'}}{F(z)-w_{n,\lambda'}}\right)\right| &\leq \left|\frac{z_{n,\lambda'}-w_{n,\lambda'}}{F(z)-w_{n,\lambda'}}\right|^{n+1} \\ &\leq (\delta_2/3)^{n+1} \leq \frac{1}{3}(1/2)^n\delta_2 < \frac{2}{3}\delta_2(1/2)^n,\end{aligned}$$

whenever $z \in K_1$. In particular, if $z \in K_1$, then

$$\left|1 - E_n\left(\frac{z_{n,\lambda'}-w_{n,\lambda'}}{F(z)-w_{n,\lambda'}}\right)\right| < \frac{1}{2}.$$

Again, $|\log(1+w)| \leq \frac{3}{2}|w|$ if $|w| \leq \frac{1}{2}$. Therefore if $z \in K_1$,

$$\begin{aligned}\left|\sum_{n=M_2+1}^{N_{\lambda'}} \log\left(E_n\left(\frac{z_{n,\lambda'}-w_{n,\lambda'}}{F(z)-w_{n,\lambda'}}\right)\right)\right| &\leq \sum_{n=M_2+1}^{N_{\lambda'}} \left|\log\left(E_n\left(\frac{z_{n,\lambda'}-w_{n,\lambda'}}{F(z)-w_{n,\lambda'}}\right)\right)\right| \\ &\leq \frac{3}{2}\sum_{n=M_2+1}^{N_{\lambda'}} \left|E_n\left(\frac{z_{n,\lambda'}-w_{n,\lambda'}}{F(z)-w_{n,\lambda'}}\right)-1\right| \\ &\leq \frac{3}{2}\cdot\frac{2}{3}\delta_2 \sum_{n=M_2+1}^{N_{\lambda'}} (1/2)^n \leq \delta_2.\end{aligned}$$

Note that

$$\exp\left(\sum_{n=M_2+1}^{N_{\lambda'}} \log\left(E_n\left(\frac{z_{n,\lambda'}-w_{n,\lambda'}}{F(z)-w_{n,\lambda'}}\right)\right)\right) = \prod_{n=M_2+1}^{N_{\lambda'}} E_n\left(\frac{z_{n,\lambda'}-w_{n,\lambda'}}{F(z)-w_{n,\lambda'}}\right).$$

This equation is obvious if $N_\lambda < \infty$ and if $N_\lambda = \infty$, then this follows easily from continuity of exp. So

$$\left|1-\prod_{n=M_2+1}^{N_{\lambda'}} E_n\left(\frac{z_{n,\lambda'}-w_{n,\lambda'}}{F(z)-w_{n,\lambda'}}\right)\right| < \frac{\epsilon}{3R_\lambda}$$

because of the way δ_2 was chosen. Then (3.9) follows since $z \in K_1$ was arbitrary. Therefore, $s(\lambda') \in [K_1, V] \cap [K_2, V]$ which implies $s(\lambda') \in [K, V]$ as required. □

We want to use the previous lemma to prove Proposition 3.29, which states that a very good domain with a discrete subset removed is still very good. First we need to better understand the fundamental group of a domain with a discrete set removed. We do this through the following sequence of lemmas.

Lemma 3.22. *If $X \subset \mathbb{C}$ is a domain, $A \subset X$ is discrete and $x_0 \in X\backslash A$, then the map $i : \pi_1(X\backslash A, x_0) \to \pi_1(X, x_0)$, induced by the inclusion $X\backslash A \hookrightarrow X$, is surjective.*

Proof. Suppose $\sigma \in \pi_1(X, x_0)$ and $\gamma : I \to X$ is a loop in X based at x_0 such that $\sigma = [\gamma]$. For each $a \in A$ let $D_a \subset X$ be a disc centred at a so small that $\bar{D}_a \cap A = \{a\}$. Note that the sets $X\backslash A$ and D_a for $a \in A$ form an open cover for X. There exists a partition $0 = s_0 < s_1 < \cdots < s_n = 1$ such that each subinterval $[s_k, s_{k+1}]$ is mapped into one of these open sets by γ. Without loss of generality we may suppose that $\gamma(s_k) \notin A$ for $k = 1, \ldots, n-1$. For if $\gamma(s_k) \in A$ for some $k = 1, \ldots, n-1$, then remove s_k from the partition. Note that $\gamma(s_0) = \gamma(s_n) = x_0 \notin A$. If $\gamma(s) = a \in A$ for some $s \in I$, then $s \in (s_k, s_{k+1})$, for some $k \in \{0, \ldots, n-1\}$, and $\gamma([s_k, s_{k+1}]) \subset D_a$. Now by dealing with γ on one subinterval at a time, it is clear that we can deform γ to a loop γ' which is contained in $X \backslash A$. Then $\sigma = i([\gamma'])$. So i is surjective as required. □

Suppose G is a group and $S \subset G$. We denote the subgroup of G that S generates by $\langle S \rangle$.

Lemma 3.23. *Suppose that G, H are groups and that $f : G \to H$ is a homomorphism. If S, T are subsets of G such that* $\ker(f) \subset \langle S \rangle$ *and* $\langle f(T) \rangle = \operatorname{im}(f)$, *then $S \cup T$ generates G.*

Proof. Let $g \in G$. Then $f(g) = f(t_1 \ldots t_n)$, where $t_1, \ldots, t_n \in T$. So $g(t_1 \ldots t_n)^{-1} \in \ker(f)$. Hence $g(t_1 \ldots t_n)^{-1} = s_1 \ldots s_m$, where $s_1, \ldots, s_m \in S$. Therefore $g = s_1 \ldots s_m t_1 \ldots t_n$, as required. □

Suppose that Y is a topological space and $\gamma : I \to Y$ is a path in Y. Define the *inverse path* $\gamma^- : I \to Y$, $\gamma^-(t) = \gamma(1-t)$.

Let X be a domain and A a discrete subset of X. For convenience, we define an (X, A)-circle to be a loop $\sigma : I \to X \backslash A$, $\sigma(t) = a + \epsilon e^{2\pi i t}$, where $a \in A$ and $\epsilon > 0$ is so small that $\bar{B}(a, \epsilon) \backslash \{a\} \subset X \backslash A$.

Lemma 3.24. *Suppose $X \subset \mathbb{C}$ is a domain, $A \subset X$ is finite, $x \in X \backslash A$ and $i : \pi_1(X \backslash A, x) \to \pi_1(X, x)$ is the homomorphism induced by the inclusion $X \backslash A \hookrightarrow X$. Then* $\ker(i)$ *is generated by the collection S_1 of all elements of the form $[\gamma \cdot \sigma \cdot \gamma^-]$, where σ is an (X, A)-circle and γ is a path in $X \backslash A$ from x to $\sigma(0)$.*

Proof. Observe that $S_1 \subset \ker(i)$. It remains to be seen that S_1 generates $\ker(i)$.

If $A = \emptyset$, then there is nothing to prove.

If A contains only one element a, then we can cover X by $X\backslash A$ and D_a, where D_a is a disc centred at a in X. Let σ be an (X, A)-circle contained in D_a and let $x' = \sigma(0)$. The fundamental group $\pi_1(D_a\backslash\{a\}, x')$ is generated by $[\sigma]$. By the van Kampen theorem [14, Theorem 1.20], the kernel of the map $i' : \pi_1(X\backslash A, x') \to \pi_1(X, x')$, induced by the inclusion $X\backslash A \hookrightarrow X$, is the normal subgroup in $\pi_1(X\backslash A, x')$ generated by $[\sigma]$. So $\ker(i')$ is generated (as a subgroup) by $\{\zeta \cdot [\sigma] \cdot \zeta^{-1} : \zeta \in \pi_1(X\backslash A, x')\}$. It follows by conjugating by $[\gamma_0]$, for some path γ_0 from x to x', that $\ker(i)$ is generated by

$$\{[\gamma \cdot \sigma \cdot \gamma^-] : \gamma \text{ is a path in } X\backslash A \text{ from } x \text{ to } x' = \sigma(0)\} \subset S_1.$$

So S_1 generates $\ker(i)$, as required.

Suppose that this lemma is true for all $A \subset X$ with $|A| = n$, for some $n \in \mathbb{N}$.

Let $A = \{a_0, a_1, \ldots, a_n\}$ be a subset of X with $n + 1$ elements. Define $A^- = \{a_1, \ldots, a_n\}$. Let $i_1 : \pi_1(X\backslash A, x) \to \pi_1(X\backslash A^-, x)$ be the homomorphism induced by the inclusion $X\backslash A \hookrightarrow X\backslash A^-$ and $i_2 : \pi_1(X\backslash A^-, x) \to \pi_1(X, x)$ be the homomorphism induced by inclusion $X\backslash A^- \hookrightarrow X$. By assumption $\ker(i_2)$ is generated by the collection S_2 of all elements of the form $[\gamma \cdot \sigma \cdot \gamma^-]$, where σ is an (X, A^-)-circle and γ is a path in $X\backslash A^-$ from x to $\sigma(0)$. Let $\mathscr{A}$ be a subset of $\pi_1(X, x)$ that freely generates $\pi_1(X, x)$, and for each $\alpha \in \mathscr{A}$, let $\psi_\alpha : I \to X$ be a representative of α contained in $X\backslash A$. It is possible to find such a set $\mathscr{A}$ because the fundamental group of a non-compact Riemann surface is free (see the proof of Lemma 3.17). Then the set $S_2 \cup T_2$ generates $\pi_1(X\backslash A^-, x)$, where $T_2 = \{[\psi_\alpha] : \alpha \in \mathscr{A}\}$, by Lemma 3.23.

Note that $\ker(i_1)$ is generated by elements of the form $[\gamma \cdot \sigma \cdot \gamma^-]$, where σ is an $(X\backslash A^-, \{a\})$-circle (and hence an (X, A)-circle) and γ is a path in $X\backslash A$ from x to $\sigma(0)$. This is simply the case of $|A| = 1$ with X replaced by $X\backslash A^-$. So $\ker(i_1) \subset \langle S_1\rangle$.

For each $s \in S_2$, there exists an element $s' \in S_1 \subset \pi_1(X\backslash A, x)$ with $s = i(s')$. Let $T_1 = \{[\psi_\alpha] : \alpha \in \mathscr{A}\} \subset \pi_1(X\backslash A, x)$. Observe that $i_1(S_1 \cup T_1) \supset S_2 \cup T_2$. So since $S_2 \cup T_2$ generates $\pi_1(X\backslash A^-, x) = \operatorname{im}(i_1)$ it follows that $\langle i_1(S_1 \cup T_1)\rangle = \operatorname{im}(i_1)$.

Hence $\pi_1(X\backslash A, x)$ is generated by $S_1 \cup T_1$ by Lemma 3.23.

We claim that $\ker(i) \subset \pi_1(X\backslash A, x)$ is generated by S_1. Suppose $\zeta \in \ker(i)$. Suppose that $x_1, \ldots, x_n \in T_1 \cup S_1$ are such that $x_j \neq x_{j-1}^{-1}$ for $j = 2, \ldots, n$ and $\zeta = x_1 x_2 \ldots x_n$.

We can assume that there does not exist a pair $(j, k) \in \{1, \ldots, n-1\} \times \{1, \ldots, n\}$ so that $k > j+1$, the elements x_j, x_k are contained in T_1, $x_k = x_j^{-1}$ and $x_{j+1}, \ldots, x_{k-1} \in S_1$. For if there did exist such a pair, then replace the original $x_{j+1}, \ldots, x_{k-1} \in S_1$ by $x_j x_{j+1} x_j^{-1}, \ldots, x_j x_{k-1} x_j^{-1} \in S_1$. So

$$\zeta = x_1 \ldots x_{j-1} x_{j+1} \ldots x_{k-1} x_{k+1} \ldots x_n \tag{3.10}$$

Relabel $x_1, \ldots, x_{n-2}$ so that x_j is the j-th term in the product in Equation 3.10. Hence $\zeta = x_1 \ldots x_{n-2}$. Replace n by $n-2$, so $\zeta = x_1 \ldots x_n$. One can repeat the above procedure a finite number of times (because n is decreasing by 2 each time) until there does not exist such a pair (j, k).

Let $m = |\{\nu \in \{1, \ldots, n\} : x_\nu \in T_1\}|$. Suppose that $m > 0$. Let $r_1, r_2, \ldots, r_m \in \{1, \ldots, n\}$ be such that $r_1 < r_2 < \cdots < r_m$ and $x_{r_m} \in T_1$. By definition of T_1, i is injective on T_1. Because $\zeta \in \ker(i)$,

$$e = i(\zeta) = i(x_1) \ldots i(x_n) = i(x_{r_1}) \ldots i(x_{r_m}), \tag{3.11}$$

where e is the identity in $\pi_1(X, x)$. If $m = 1$, then Equation 3.11 contradicts $\mathscr{A}$ freely generating $\pi_1(X, x)$ since $i(x_{r_1}) \in \mathscr{A}$. Suppose $m \geq 2$. Note that we have assumed $x_{r_{j+1}} \neq {x_{r_j}}^{-1}$ for $j = 1, \ldots, m-1$, so $i(x_{r_j}) \neq i(x_{r_{j+1}})^{-1}$ for $j = 1, \ldots, n-1$, by injectivity of i. So again Equation 3.11 contradicts $\mathscr{A}$ freely generating $\pi_1(X, x)$, since $i(x_{r_j}) \in \mathscr{A}$ for $j = 1, \ldots, n$. Hence we have a contradiction if $m > 0$. Therefore $m = 0$ and hence S_1 generates K. $\square$

Lemma 3.25. *If $X \subset \mathbb{C}$ is a domain, A is a discrete subset of X and $x \in X\backslash A$, then $\ker(\pi_1(X\backslash A, x) \to \pi_1(X, x))$ is generated by elements of the form $[\gamma \cdot \sigma \cdot \gamma^-]$, where σ is an (X, A)-circle and γ is a path in $X\backslash A$ from x to $\sigma(0)$.*

Proof. This lemma is true for the case where A is finite (see Lemma 3.24). Suppose that A is infinite. Let $X_1, X_2, \dots$ be a sequence of domains in X so that $x \in X_1$, $X_1 \subset\subset X_2 \subset\subset X_3 \subset\subset \dots$ and $\bigcup_{j\in\mathbb{N}} X_j = X$. For $j \in \mathbb{N}$, let $A_j = X_j \cap A$. Note that A_j is finite for $j \in \mathbb{N}$ since X_j is relatively compact in X. Suppose $[\gamma] \in \ker(i)$, where $\gamma : I \to X\backslash A$ is a loop in $X\backslash A$ based at x. Let $H : I \times I \to X$ be a homotopy from γ to the constant loop x. This homotopy exists because $[\gamma]$ is in the kernel of i. The image of H is compact and is therefore contained in X_m for some $m \in \mathbb{N}$. Let $i_m : \pi_1(X_m\backslash A_m, x) \to \pi_1(X_m, x)$. Then $[\gamma]'$ is contained in the kernel of i_m. So γ is homotopic (in $X_m\backslash A_m$ and hence in $X\backslash A$) to a product of loops of the desired form by Lemma 3.24. □

Suppose X is a topological space and $\gamma_1, \gamma_2 : I \to X$ are paths in X with $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1(1) = \gamma_2(1)$. We write $\gamma_1 \simeq_{\partial I} \gamma_2$ to mean γ_1 is homotopic to γ_2 relative to endpoints.

Lemma 3.26. *Suppose $X \subset \mathbb{C}$ is a domain, $A \subset X$ is a discrete subset and $x \in X\backslash A$. Let S be the collection of all elements of the form $[\gamma \cdot \sigma \cdot \gamma^-]$, where σ is an (X, A)-circle and γ is a path from x to $\sigma(0)$ in $X\backslash A$. Then there exists a subset $T \subset \pi_1(X\backslash A, x)$ such that $S \cup T$ generates $\pi_1(X\backslash A, x)$, i is injective on T and $i(T)$ freely generates $\pi_1(X, x)$, where $i : \pi_1(X\backslash A, x) \to \pi_1(X, x)$ is the map induced by the inclusion $X\backslash A \hookrightarrow X$.*

Proof. Let $\mathscr{A}$ be a subset of $\pi_1(X, x)$ that freely generates $\pi_1(X, x)$. For each $\alpha \in \mathscr{A}$ choose an element $\rho(\alpha)$ in the preimage of α under i. This is possible since the map $i : \pi_1(X\backslash A, x) \to \pi_1(X, x)$ is surjective. We claim that $T = \{\rho(\alpha) : \alpha \in \mathscr{A}\}$ has the desired properties. By construction i is injective on $\mathscr{A}$ and $i(\mathscr{A})$ freely generates $\pi_1(X, x)$. All that remains to prove is that $S \cup T$ generates $\pi_1(X\backslash A, x)$. Note that since $i(T)$ generates the image $\pi_1(X, x)$, it suffices to show that $\ker(i)$ is generated by S. This has been done already in Lemma 3.24. □

Lemma 3.27. *Suppose $X \subset \mathbb{C}$ is a domain, $A \subset X$ is a discrete subset and $x \in X\backslash A$. Let $a \in A$ and let S_a be the collection of all elements in $\pi_1(X\backslash A, x)$ of* *the form* $[\gamma \cdot \sigma \cdot \gamma^-]$*, where* σ *is an* (X, A)*-circle centred at* a *and* γ *is a path in* $X \backslash A$ *from* x *to* $\sigma(0)$*. If* $\lambda : \pi_1(X \backslash A, x) \to \mathbb{Z}$ *is a homomorphism, then* λ *is constant on* S_a*.*

Proof. Let σ_a be an (X, A)-circle centred at a. Note that each element ξ of S_a can be represented by $\gamma \cdot \sigma_a \cdot \gamma^-$ for some path γ from x to $\sigma_a(0)$. For if σ is another (X, A)-circle, then $\sigma \simeq_{\partial I} \psi \cdot \sigma_a \cdot \psi^-$, where $\psi : I \to X \backslash A$ is a parameterisation of the straight line from $\sigma(0)$ to $\sigma_a(0)$. So if $\xi \in \pi_1(X \backslash A, x)$ can be represented by $\tau \cdot \sigma \cdot \tau$, where τ is a path from x to $\sigma(0)$, then $\tau \cdot \sigma \cdot \tau \simeq_{\partial I} \tau \cdot \psi \cdot \sigma_a \cdot \psi^- \cdot \tau^-$ and hence setting $\gamma = \tau \cdot \psi$ we have $\xi = [\gamma \cdot \sigma_a \cdot \gamma^-]$.

Suppose $\xi_1, \xi_2 \in S_a$. By the above, there exist paths $\gamma_1, \gamma_2 : I \to X \backslash A$ from x to $\sigma_a(0)$ such that $\xi_1 = [\gamma_1 \cdot \sigma_a \cdot \gamma_1^-]$ and $\xi_2 = [\gamma_2 \cdot \sigma_a \cdot \gamma_2^-]$. Then $[\gamma_2 \cdot \sigma_a \cdot \gamma_2^-] = [\gamma_2 \cdot \gamma_1^-] \cdot [\gamma_1 \cdot \sigma_a \cdot \gamma_1^-] \cdot [\gamma_1 \cdot \gamma_2^-]$ and therefore, since $\mathbb{Z}$ is abelian and λ is a homomorphism, $\lambda([\gamma_1 \cdot \sigma_a \cdot \gamma_1^-]) = \lambda([\gamma_2 \cdot \sigma_a \cdot \gamma_2^-])$. □

Lemma 3.28. *Suppose* $X \subset \mathbb{C}$ *is a domain,* $A \subset \mathbb{C}$ *is a discrete subset and* $x \in X \backslash A$*. Let* $a \in A$*. Define* S_a *as in Lemma 3.27. If* $f : X \backslash A \to \mathbb{C}^*$ *is a holomorphic map with a zero of order* n *at* $a \in A$*, then* $f_*([\zeta]) = n$ *whenever* $\zeta \in S_a$*.*

Proof. Since f has a zero of order n at a, there exists an open disc U centred at a and a nowhere vanishing holomorphic function $g : U \to \mathbb{C}$ such that $f(z) = (z - a)^n g(z)$ for all $z \in U$ (we define $f(a) = 0$). Let ξ be an arbitrary element of S_a. Let σ_a be an (X, A)-circle centred at a and contained in U. As in the proof of Lemma 3.27 there exists a path γ from x to $\sigma_a(0)$ such that $\xi = [\gamma \cdot \sigma_a \cdot \gamma^-]$. Now,

$$\begin{aligned} f_*(\xi) = w(f(\gamma \cdot \sigma_a \cdot \gamma^-)) &= w(f(\gamma) \cdot f(\sigma_a) \cdot f(\gamma^-)) = w(f(\sigma_a)) \\ &= w((\sigma_a - a)^n g(\sigma_a)) \\ &= w((\sigma_a - a)^n) + w(g(\sigma_a)) \\ &= n + 0 = n. \end{aligned}$$

since σ_a is null-homotopic relative to endpoints in U and g does not vanish on U. □

Proposition 3.29. *If X is a very good domain in $\mathbb{C}$ and A is a discrete subset of X, then $X\backslash A$ is very good.*

Proof. Let $i : \pi_1(X\backslash A, x) \to \pi_1(X, x)$ be the map induced by the inclusion $X\backslash A \hookrightarrow X$. Let S be the collection consisting of all elements of the form $[\gamma \cdot \sigma \cdot \gamma^-]$, where σ is an (X, A)-circle and γ is a path in $X\backslash A$ from x to $\sigma(0)$. For each $a \in A$, let S_a be as defined in Lemma 3.27. By Lemma 3.26, there exists a subset $T \subset \pi_1(X\backslash A, x)$ so that $i(T)$ freely generates $\pi_1(X, x)$, i is injective on T and $S \cup T$ generates $\pi_1(X\backslash A, x)$. Let $f : T \to i(T)$, $\alpha \mapsto i(\alpha)$. Note that f is a bijection since i is injective.

For each $\lambda \in \mathbb{Z}^{i(T)}$, let $E(\lambda)$ be the unique extension of λ to a homomorphism $\pi_1(X, x) \to \mathbb{Z}$. The map $E : \mathbb{Z}^{i(T)} \to \mathrm{Hom}(\pi_1(X, x), \mathbb{Z})$, $\lambda \mapsto E(\lambda)$, is continuous. For suppose $U \subset \mathrm{Hom}(\pi_1(X, x), \mathbb{Z})$ is open. Without loss of generality, suppose $U = [\{\sigma\}, \{n\}] \cap \mathrm{Hom}(\pi_1(X, x), \mathbb{Z})$, where $\sigma \in \pi_1(X, x)$ and $n \in \mathbb{Z}$. Suppose $\sigma = \sigma_1\sigma_2 \ldots \sigma_m$, where $m \in \mathbb{N}$ and $\sigma_j \in i(T)$ for $j = 1, 2, \ldots, m$. If $\lambda \in E^{-1}(U)$, then $\bigcap_{j=1}^{m} [\{\sigma_j\}, \{\lambda(\sigma_j)\}]$ is an open neighbourhood of λ in $E^{-1}(U)$. Therefore E is continuous.

Let $J \subset \mathbb{N}$ and $J \to A$, $j \mapsto a_j$, be an enumeration of A. For each $j \in J$, let ξ_j be an element of $S_{a_j} \subset \pi_1(X\backslash A, x)$. Let $k : J \to S$, $j \mapsto \xi_j$. Since X is very good, there exists a continuous section s_1 of $\cdot_* : \mathscr{O}(X, \mathbb{C}^*) \to \mathrm{Hom}(\pi_1(X, x), \mathbb{Z})$. Let $s_2 : \mathbb{Z}^J \to \mathscr{O}(X\backslash A, \mathbb{C}^*)$ be a continuous map with $\mathrm{ord}_{a_j} s_2(\lambda) = \lambda_j$ for all $\lambda \in \mathbb{Z}^J$ and $j \in J$. Define $s_3 : \mathrm{Hom}(\pi_1(X\backslash A, x), \mathbb{Z}) \to \mathscr{O}(X\backslash A, \mathbb{C}^*)$, $\lambda \mapsto s_1(E(\lambda \circ f^{-1}))|_{X\backslash A}$ and $s_4 : \mathrm{Hom}(\pi_1(X\backslash A, x), \mathbb{Z}) \to \mathscr{O}(X\backslash A, \mathbb{C}^*)$, $\lambda \mapsto s_2(\lambda \circ k)$. The maps s_3 and s_4 are compositions of continuous maps and are therefore continuous.

Let $s : \mathrm{Hom}(\pi_1(X\backslash A, x), \mathbb{Z}) \to \mathscr{O}(X\backslash A, \mathbb{C}^*)$, $\lambda \mapsto s_3(\lambda - s_4(\lambda)_*)s_4(\lambda)$. Let λ be an arbitrary element of $\mathrm{Hom}(\pi_1(X\backslash A, x), \mathbb{Z})$. We want to show that $s(\lambda)_* = \lambda$. It suffices to show that $s(\lambda)_* = \lambda$ on $S \cup T$ because $S \cup T$ generates $\pi_1(X\backslash A, x)$. Note that

$$\mathrm{ord}_{a_m} s(\lambda) = \mathrm{ord}_{a_m} s_3(\lambda - s_4(\lambda)_*)s_4(\lambda) = \mathrm{ord}_{a_m} s_4(\lambda) = \lambda \circ k(m) = \lambda(\xi_m) \quad (3.12)$$

for $m \in J$. Suppose $\xi \in S$. Then $\xi \in S_{a_m}$ for some $m \in J$. So $\lambda(\xi_m) = \lambda(\xi)$ by Lemma 3.27. By Equation 3.12 and Lemma 3.28, $s(\lambda)_*(\xi) = \lambda(\xi_m) = \lambda(\xi)$ as required. Let $\tau = \lambda - s_4(\lambda)_*$. If $\alpha \in T$, then

$$s_3(\tau)_*(\alpha) = (s_1(E(\tau \circ f^{-1}))|_{X\backslash A})_*(\alpha) = s_1(E(\tau \circ f^{-1}))_* \circ i(\alpha)$$

because restriction to $X\backslash A$ is simply pulling back by the inclusion $X\backslash A \hookrightarrow X$. Note that $s_1(E(\tau \circ f^{-1}))_* = E(\tau \circ f^{-1})$ by definition of s_1, and $i(\alpha) \in i(T)$, so

$$s_1(E(\tau \circ f^{-1}))_* \circ i(\alpha) = \tau(\alpha).$$

Hence

$$\begin{aligned} s(\lambda)_*(\alpha) &= (s_3(\tau)s_4(\lambda))_*(\alpha) \\ &= s_3(\tau)_*(\alpha) + s_4(\lambda)_*(\alpha) = \tau(\alpha) + s_4(\lambda)(\alpha) = \lambda(\alpha), \end{aligned}$$

as required. □

Let X be a domain in $\mathbb{P}_1$. Let $J \subset \mathbb{N}$ and $(D_j)_{j\in J}$ be a family of closed discs in X (which are allowed to be points). We say that $(D_j)_{j\in J}$ is a *family of isolated closed discs* if for each $j \in J$ there is an open neighbourhood of D_j which does not intersect D_k for $k \neq j$.

Proposition 3.30. *Let $\Omega \subset \mathbb{P}_1$ be a domain and suppose $(D_j)_{j\in J}$ is a countable family of isolated closed discs such that $\bigcup_{j\in J} D_j$ is closed in Ω. Then $\Omega\backslash \bigcup_{j\in J} D_j$ is connected.*

The proof below is similar to the proof of Lemma 3.22.

Proof. Without loss of generality suppose that $J = \{n \in \mathbb{N} : n < N\}$ for some $N \in \mathbb{N} \cup \{\infty\}$. For each $j \in J$, let U''_j be an open neighbourhood of D_j not intersecting D_k for $k \neq j$. Let U'_j be an open neighbourhood of D_j such that $\overline{U'_j} \subset U''_j$. Define $U_j = U'_j \backslash \bigcup_{k<j} \overline{U'_k}$. Note that $(U_j)_{j\in J}$ is a family of mutually disjoint open sets and $D_j \subset U_j$ for each $j \in J$. Moreover, for each $j \in J$, by shrinking U_j

if necessary, we may assume that U_j is an open disc with the same centre as D_j. Hence for each $j \in J$, $U_j \backslash D_j$ is an open annulus and therefore is path connected.

Now let $x, y \in \Omega \backslash \bigcup_{j \in J} D_j$ and let γ be a path in Ω from x to y. Note that the sets $\Omega \backslash \bigcup_{j \in J} D_j$ and U_j for $j \in J$ form an open cover for Ω. There exists a partition $0 < s_0 < s_1 < \cdots < s_n = 1$ such that each subinterval $[s_k, s_{k+1}]$ is mapped into one of these open sets by γ. Without loss of generality suppose that $\gamma(s_k) \notin \bigcup_{j \in J} D_j$ for $k = 1, \ldots, n-1$. If $\gamma(s_k)$ was in $\bigcup_{j \in J} D_j$ for some $k \in \{1, \ldots, n-1\}$, then we could remove s_k from the partition. Note that $\gamma(s_0) = x, \gamma(s_n) = y \notin \bigcup_{j \in J} D_j$.

If $\gamma(s) \in \bigcup_{j \in J} D_j$ for some $s \in I$, then $s \in (s_k, s_{k+1})$ for some $k \in \{0, \ldots, n-1\}$ and $\gamma([s_k, s_{k+1}]) \subset U_j$ for some $j \in J$. By assumption $\gamma(s_k), \gamma(s_{k+1}) \notin \bigcup_{j \in J} D_j$, so $\gamma(s_k), \gamma(s_{k+1}) \in U_j \backslash D_j$. From the simple geometry of the situation, it is evident that we can deform $\gamma|[s_k, s_{k+1}]$ to a path $\gamma_k : [s_k, s_{k+1}] \to U_j \backslash D_j$ from x to y in $U_j \backslash D_j$ relative to endpoints. Doing this for each subinterval containing such an s yields a homotopic curve in $\Omega \backslash \bigcup_{j \in J} D_j$ from x to y. □

Proposition 3.31. *If X is a very good domain in $\mathbb{C}$, $J \subset \mathbb{N}$ and $(D_j)_{j \in J}$ is a family of isolated closed discs contained in X so that $\bigcup_{j \in J} D_j$ is closed in X, then the domain $X \backslash \bigcup_{j \in J} D_j$ is very good.*

As in Lemma 3.21, the reason we do not simply take $J = \mathbb{N}$ is so that we can deal with the finite case and the infinite case at the same time.

Proof. Let z_j be the centre of D_j for $j \in J$. Clearly $\{z_j : j \in J\}$ is a discrete subet of X, so $X \backslash \{z_j : j \in J\}$ is very good by Proposition 3.29.

For each $j \in J$, let U_j' be an open neighbourhood of D_j in X such that $U_j' \cap \bigcup_{k \neq j} D_k = \emptyset$ and let $\epsilon_j > 0$ be so small that $B(D_j, \epsilon_j) \subset U_j'$, where $B(D_j, \epsilon_j)$ is the ball of radius ϵ_j around D_j in $\mathbb{C}$. Define $U_j = B(D_j, \epsilon_j/3)$ for $j \in J$. Note that $U_j \cap U_k = \emptyset$ for $j \neq k$. If this were not the case, then there exists $z \in U_j \cap U_k$. Then there exist $\zeta_j \in D_j$ and $\zeta_k \in D_k$ so that $d(\zeta_j, z) \leq \epsilon_j/3$ and $d(\zeta_k, z) \leq \epsilon_k/3$. Without loss of generality suppose $\epsilon_k \geq \epsilon_j$. Then $d(\zeta_j, \zeta_k) \leq 2\epsilon_k/3 < \epsilon_k$. So

$\zeta_j \in B(D_k, \epsilon_k) \cap D_j$, which contradicts $U'_k \cap D_j = \emptyset$.

Define $R : X\backslash \bigcup_{j\in J}\{z_j\} \to X\backslash \bigcup_{j\in J} U_j$,

$$R(z) = \begin{cases} z & \text{if } z \in X\backslash \bigcup_{j\in J} U_j \\ z_* & \text{if } z \in U_j \text{ for some } j \in J, \end{cases}$$

where for $z \in U_j$, z_* is the point where the ray from z_j to z intersects ∂U_j in $\mathbb{C}$. Clearly R is continuous. It is easy to check that R and $R|_{X\backslash \bigcup_{j\in J} D_j}$ are homotopy inverses for $X\backslash \bigcup_{j\in J} U_j \hookrightarrow X\backslash \bigcup_{j\in J}\{z_j\}$ and $X\backslash \bigcup_{j\in J} U_j \hookrightarrow X\backslash \bigcup_{j\in J} D_j$ respectively. It follows that $X\backslash \bigcup_{j\in J} D_j \hookrightarrow X\backslash \bigcup_{j\in J}\{z_j\}$ is a homotopy equivalence. Therefore $X\backslash \bigcup_{j\in J} D_j$ is very good by Proposition 3.20. □

Suppose Y is a topological space and $Z \subset Y$ is a subspace. We say that a point $y \in Y$ is a point of accumulation of Z if whenever U is an open neighbourhood of y, $Z \cap U\backslash\{y\} \neq \emptyset$. Let $\mathscr{D}_Y : \mathcal{P}(Y) \to \mathcal{P}(Y)$, $Z \mapsto Z'$, where $\mathcal{P}(Y)$ is the power set of Y and Z' is the derived set of Z, that is,

$$Z' = \{z \in Y : z \text{ is an accumulation point of } Z\}.$$

Note that $\mathscr{D}_Y(Z)$ is closed in Y for every $Z \subset Y$. In particular, $\mathscr{D}_Y^n(Y)$ is closed in Y for every $n \in \mathbb{N}$. We take $\mathscr{D}_Y^0(Z)$ to be Z itself. It is easily shown that if $Z \subset Y$ is closed, then

$$\mathscr{D}_Y(Z) = Z\backslash\{y \in Z : y \text{ is an isolated point of } Z\}.$$

In what follows we will omit the subscript in $\mathscr{D}_Y$ because we will always take Y to be either $S_{\mathcal{F}}$ or $B(X)$ (to be defined) and it will be clear from the context which it is.

Let X be a domain in $\mathbb{C}$. Let $J \subset \mathbb{N}$ and $\mathcal{F} = (D_j)_{j\in J}$ be a family of mutually disjoint closed discs in X indexed by J. Define an equivalence relation $\sim$ on $\bigcup_{j\in J} D_j$ by $\zeta \sim \xi$ if ζ and ξ belong to the same disc. Let $S_{\mathcal{F}} = \bigcup_{j\in J} D_j/\sim$. Give $S_{\mathcal{F}}$ the quotient topology (as a set, $S_{\mathcal{F}}$ is simply J, but the topology on $S_{\mathcal{F}}$ need not be discrete). Note that $S_{\mathcal{F}} \supset \mathscr{D}(S_{\mathcal{F}}) \supset \mathscr{D}^2(S_{\mathcal{F}}) \supset \dots$ and $\mathscr{D}^{m-1}(S_{\mathcal{F}})\backslash\mathscr{D}^m(S_{\mathcal{F}})$ is the set of isolated points in $\mathscr{D}^{m-1}(S_{\mathcal{F}})$ for $m \in \mathbb{N}$. A disc D is isolated in $\mathscr{D}^{m-1}(S_{\mathcal{F}})$ if and only if there exists an open neighbourhood of D in $\mathbb{C}$ not intersecting any of the other elements of $\mathscr{D}^{m-1}(S_{\mathcal{F}})$.

Lemma 3.32. *Suppose X is a topological space, $U \subset X$ is a closed subspace and $V \subset U$ is closed in the subspace topology on U. Then V is closed in X*

Proof. Clearly $U\backslash V = W \cap U$, where W is open in X. Without loss of generality suppose that $X\backslash U \subset W$. If this were not the case, then we could replace W by $W \cup (X\backslash U)$. Then

$$X\backslash V = (X\backslash U) \cup (U\backslash V) = ((X\backslash U) \cap W) \cup (U \cap W) = W.$$

So V is closed in X. □

Lemma 3.33. *Suppose that $X \subset \mathbb{P}_1$ is a domain. Let $\mathcal{F} = (D_j)_{j\in J}$ be a countable family of mutually disjoint closed discs in X with $\bigcup \mathcal{F}$ closed in X and $\mathscr{D}^n(S_{\mathcal{F}}) = \emptyset$ for some $n \in \mathbb{N}$. Then $X\backslash \bigcup_{j\in J} D_j$ is connected.*

Proof. Suppose $n \in \mathbb{N}$ is such that $\mathscr{D}^n(S_{\mathcal{F}}) = \emptyset$ and $\mathscr{D}^{n-1}(S_{\mathcal{F}}) \neq \emptyset$. For $j = 0, \dots, n$, let $\Omega_j = X\backslash \bigcup \mathscr{D}^{n-j}(S_{\mathcal{F}})$. Note that for $j = 0, 1, \dots, n-1$, Ω_{j+1} is obtained from Ω_j by removing a countable family of isolated closed discs, namely $\mathscr{D}^{n-j-1}(S_{\mathcal{F}})\backslash\mathscr{D}^{n-j}(S_{\mathcal{F}})$. It is easy to show, using the assumption that $\bigcup \mathcal{F}$ is closed, that $\bigcup \mathscr{D}^{n-j-1}(S_{\mathcal{F}})\backslash\mathscr{D}^{n-j}(S_{\mathcal{F}})$ is closed in Ω_j. So for each $j = 0, 1, 2, \dots, n-1$, by Proposition 3.30, Ω_{j+1} is a domain if Ω_j is a domain. Hence Ω_n is a domain since $\Omega_0 = X$ is a domain. □

Proposition 3.34. *If X is a very good domain in $\mathbb{C}$, $J \subset \mathbb{N}$ and $\mathcal{F} = (D_j)_{j\in J}$ is a family of mutually disjoint closed discs contained in X so that $\bigcup \mathcal{F}$ is closed in X and $\mathscr{D}^n(S_{\mathcal{F}}) = \emptyset$ for some $n \in \mathbb{N}$, then $X\backslash \bigcup \mathcal{F}$ is very good.*

Proof. Note that $X\backslash \bigcup \mathcal{F}$ is a domain by Lemma 3.33. If $J = \emptyset$, then there is nothing to prove. So suppose $J \neq \emptyset$. Let $n \in \mathbb{N}$ be such that $\mathscr{D}^n(S_{\mathcal{F}}) = \emptyset$ and

$\mathscr{D}^{n-1}(S_{\mathcal{F}}) \neq \emptyset$. For $m \in \mathbb{N} \cup \{0\}$, define $Z_m = X \backslash \bigcup \mathscr{D}^m(S_{\mathcal{F}})$. Note that Z_{n-1} is very good by Proposition 3.31. If Z_{n-k} is very good for some $1 \leq k \leq n-1$, then Z_{n-k-1} is very good. For $\mathscr{D}^{n-k-1}(S_{\mathcal{F}}) \backslash \mathscr{D}^{n-k}(S_{\mathcal{F}})$ is a collection of isolated closed discs in Z_{n-k} and $\bigcup(\mathscr{D}^{n-k-1}(S_{\mathcal{F}}) \backslash \mathscr{D}^{n-k}(S_{\mathcal{F}}))$ is closed in Z_{n-k} since its complement Z_{n-k-1} is open. So

$$Z_{n-k-1} = Z_{n-k} \backslash \bigcup (\mathscr{D}^{n-k-1}(S_{\mathcal{F}}) \backslash \mathscr{D}^{n-k}(S_{\mathcal{F}}))$$

is very good by Proposition 3.31. It follows that Z_0 is very good, as required. □

We now give a few examples of domains in $\mathbb{C}$ which are very good by the above proposition.

Recall that domains with finitely generated fundamental groups are very good by Proposition 3.18. So in particular $\mathbb{C}$ is very good. We already knew that $X_1 = \mathbb{C} \backslash \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ was very good because X_1 is biholomorphic to $X_1' = \mathbb{C} \backslash \{1, 2, 3, \ldots\}$ and X_1' is very good by Proposition 3.29. Note that the trick was to biholomorphically send the point of accumulation to ∞ thus leaving us with a discrete subset of $\mathbb{C}$. The same trick will not work if the sequence also accumulates at ∞, for example $X_2 = \mathbb{C} \backslash \{\ldots, 2, 1, 0, \frac{1}{2}, \frac{1}{3}, \ldots\}$. It seems that the results up to and including Proposition 3.29 could not deal with this domain. Yet X_2 is very good by Proposition 3.34.

The domain X_2 is the complement of a countable closed set in $\mathbb{P}_1$ with two points of accumulation. Proposition 3.34 allows us to deal with complements of some countable closed sets with infinitely many points of accumulation. For example, $\mathbb{C} \backslash A$, where

$$A = \{x + iy : x, y \in \{\ldots, 2, 1, 0, \frac{1}{2}, \frac{1}{3}, \ldots\}\},$$

is very good.

Suppose that X is a domain in $\mathbb{C}$. Let $B(X)$ denote the collection of all connected components of ∂X in $\mathbb{P}_1$. We can give $B(X)$ a topology as follows. Define an equivalence relation $\sim$ on ∂X (the boundary of X in $\mathbb{P}_1$) by $x \sim y$ if x and y are in the same connected component of ∂X. Then $B(X) = \partial X/\sim$, so we can give $B(X)$ the quotient topology.

Lemma 3.35. *Suppose $X \subset \mathbb{C}$ is a domain and $C \subset B(X)$. For every $c \in C\backslash\mathscr{D}(C)$ there exists an open neighbourhood W of c in $\mathbb{P}_1$ which does not intersect c' for all $c' \in C\backslash\mathscr{D}(C)$ with $c' \neq c$.*

Proof. By definition $\mathscr{D}(C)$ is the set of all points of accumulation of C in $B(X)$, so

$$C\backslash\mathscr{D}(C) = \{c \in C : c \text{ is not an accumulation point of } C\}.$$

Suppose $c \in C\backslash\mathscr{D}(C)$. Then there exists an open neighbourhood U of c in $B(X)$ which does not intersect $C\backslash(\mathscr{D}(C)\cup\{c\})$. Let $V = q^{-1}(U)$, where $q : \partial X \to B(X)$ is the quotient map. The set V is clearly open and does not intersect c' for $c' \in C\backslash\mathscr{D}(C)$ with $c' \neq c$. For if such a c' did intersect $V = q^{-1}(U)$, then $U = q(q^{-1}(U))$ would contain $c' \in C\backslash(\mathscr{D}(C) \cup \{c\})$, which is a contradiction. Then $V = W \cap \partial X$ for some open subset $W \subset \mathbb{C}$, by definition of the topology on ∂X. Clearly W has the desired property. □

Lemma 3.36. *If X is a domain in $\mathbb{C}$ with $\mathscr{D}^n(B(X)) = \emptyset$ for some $n \in \mathbb{N}$, then ∂X has countably many connected components, that is, $B(X)$ is countable.*

Proof. Note that

$$B(X) = (B(X)\backslash\mathscr{D}(B(X))) \cup (\mathscr{D}(B(X))\backslash\mathscr{D}^2(B(X))) \cup \cdots \\ \cup(\mathscr{D}^{n-2}(B(X))\backslash\mathscr{D}^{n-1}(B(X))) \cup \mathscr{D}^{n-1}(B(X)).$$

If $B(X)$ is uncountable, then for some $m \in \{1, \ldots, n\}$, $\mathscr{D}^{m-1}(B(X))\backslash\mathscr{D}^m(B(X))$ is countable. For each $c \in \mathscr{D}^{m-1}(B(X))\backslash\mathscr{D}^m(B(X))$ let z_c be a point in c. By Lemma 3.35 the collection

$$S = \{z_c : c \in \mathscr{D}^{m-1}(B(X))\backslash\mathscr{D}^m(B(X))\} \subset \mathbb{P}_1 \tag{3.13}$$

given the subspace topology is discrete. But S is a subspace of a second countable space and is therefore second countable, which contradicts S being uncountable and discrete. So $B(X)$ is countable. □

We now come to our main result. Here we invoke He and Schramm's big theorem [15, Theorem 0.1] .

Theorem 3.37. *If X is a domain in $\mathbb{C}$ with $\mathscr{D}^n(B(X)) = \emptyset$ for some $n \in \mathbb{N}$, then X is very good.*

Proof. Since $\mathscr{D}^n(B(X)) = \emptyset$, X has at most countably many boundary components. A homeomorphism $Y \to Z$ of domains in $\mathbb{P}_1$ induces a homeomorphism $B(Y) \to B(Z)$ (see [15, Fact 1.1]). So by [15, Theorem 0.1] we may assume without loss of generality that X is a circle domain in $\mathbb{P}_1$, not $\mathbb{P}_1$ itself, with countably many boundary components.

Suppose that $(c_j)_{j\in J}$ is a family of mutually disjoint circles in $\mathbb{P}_1$ such that $B(X) = \{c_j : j \in J\}$. For each $j \in J$, let D_j be the closed disc in $\mathbb{P}_1$ with $\partial D_j = c_j$ and $D_j \cap X = \emptyset$. Then $\mathcal{F} = (D_j)_{j\in J}$ is a family of mutually disjoint closed discs in $\mathbb{P}_1$.

Claim: $S_{\mathcal{F}}$ is homeomorphic to $B(X)$.

Let $f : B(X) \to S_{\mathcal{F}}$, $c_j \mapsto D_j$. We will prove that f is a homeomorphism. Clearly f is a bijection. Define $f' : \partial X \to S_{\mathcal{F}}$, $z \mapsto [z]$. Note that f' is the restriction of the quotient map $q : \bigcup_{j\in J} D_j \to S_{\mathcal{F}}$ to ∂X and so f' is continuous. Observe that $f' = f \circ p$, where $p : \partial X \to B(X)$ is the quotient map. Therefore f is continuous since f' is continuous.

All that remains to prove is that f is closed. To this end, suppose that $V \subset B(X)$ is closed. Let $J_0 \subset J$ be such that $V = \{c_j : j \in J_0\}$. Then $p^{-1}(V) = \bigcup_{j\in J_0} c_j$ is closed in ∂X and hence in $\mathbb{P}_1$. We want to show that $f(V) = \{D_j : j \in J_0\}$ is closed in $S_{\mathcal{F}}$. Since $\bigcup_{j\in J_0} D_j$ is saturated with respect to q, it suffices to show that $\bigcup_{j\in J_0} D_j$ is closed in $\bigcup_{j\in J} D_j$. This follows if we can show that $\bigcup_{j\in J_0} D_j$ is closed in $\mathbb{P}_1$. Let x be an arbitrary element of $\mathbb{P}_1 \setminus \bigcup_{j\in J_0} D_j$ and let W be a connected open neighbourhood of x contained in the open set $\mathbb{P}_1 \setminus \bigcup_{j\in J_0} c_j$. Since W is connected,

W does not intersect D_j° for $j \in J_0$. Hence W is an open neighbourhood of x contained in $\mathbb{P}_1 \backslash \bigcup_{j \in J_0} D_j$. Therefore $\bigcup_{j \in J_0} D_j$ is closed in $\mathbb{P}_1$.

If $V = B(X)$ in the above, then $V = \{c_j : j \in J\}$. Hence we have proved that $\bigcup_{j \in J} D_j$ is closed in $\mathbb{P}_1$. The claim implies that $\mathscr{D}^n(S_{\mathcal{F}}) = \emptyset$. So by Lemma 3.33, $X' = \mathbb{P}_1 \backslash \bigcup \mathcal{F}$ is connected, that is, a domain.

We will prove that $X = X'$. Clearly $X \subset X'$. Suppose $x \in X'$ and $x \notin X$. Choose a point $y \in X$ and a path γ from x to y in X'. Since $x \notin X$ and $y \in X$, there exists $s \in [0, 1)$ such that $\gamma(s) \in \partial X$. But this is a contradiction since $\gamma(s) \in \mathbb{P}_1 \backslash \bigcup_{j \in J} D_j$ and $\partial X \subset \bigcup_{j \in J} D_j$. Therefore $X' \subset X$ and so $X = X'$.

Without loss of generality suppose that $\infty \in \bigcup_{j \in J} D_j$. Let $j_0 \in J$ be such that $\infty \in D_{j_0}$. Note that $\mathbb{P}_1 \backslash D_{j_0}$ is a very good domain in $\mathbb{C}$ because $\mathbb{P}_1 \backslash D_{j_0}$ is simply connected (see Proposition 3.18). Let $\mathcal{F}' = (D_j)_{j \in J \backslash \{j_0\}}$. Note that $\mathcal{F}'$ is a family of mutually disjoint closed discs with $\bigcup \mathcal{F}'$ closed in $\mathbb{P}_1 \backslash D_{j_0}$. Clearly $\mathscr{D}^n(S_{\mathcal{F}'}) = \emptyset$ since $\mathscr{D}^n(S_{\mathcal{F}}) = \emptyset$. Therefore X is very good by Proposition 3.34. $\square$

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